A note on moments of dividends

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Abstract

We reconsider a formula for arbitrary moments of expected discounted dividend payments in a spectrally negative Lévy risk model that was obtained in Renaud and Zhou [3] and in Kyprianou and Palmowski [4] and extend the result to stationary Markov processes that are skip-free upwards.

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In two recent papers, Renaud and Zhou [3] and Kyprianou and Palmowski [4] independently showed that the kth moment of the expected discounted dividend payments

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in a spectrally negative Lévy risk model with initial surplus u and horizontal dividend barrier $b \geq u$ is given by

$$V_k(u;b) = k!\,\frac{W_{k\delta}(u)}{W_{k\delta}(b)}\prod_{i=1}^{k}\frac{W_{i\delta}(b)}{W'_{i\delta}(b)}. \tag{1}$$

Here $\delta \geq 0$ is a constant discount rate and $W_q(x)$ is the scale function of the underlying Lévy process (with Laplace exponent ψ) defined through its Laplace transform $\int_0^\infty e^{-\lambda x} W_q(x)\,\mathrm{d}x = 1/(\psi(\lambda)-q)$.

In this short note we show that formula (1) can be established in a more general framework by direct probabilistic reasoning. Concretely, assume that the surplus process $U(t)$ is a stationary Markov process that has no jumps upwards and has the strong Markov property. Let $T_{(0,a)} = \inf\{t \geq 0 \,|\, U(t) \notin (0,a)\}$ and define for $0 \leq u_1 \leq u_2$ the function $C_\delta(u_1,u_2) = \mathbb{E}_{u_1}[e^{-\delta T_{(0,u_2)}}; U(T_{(0,u_2)}) = u_2]$, which is the Laplace transform of the upper exit time out of the interval $(0,u_2)$ when starting in u_1, i.e. $U(0) = u_1$. As discussed in Gerber et al. [1], one immediately deduces from the absence of upward jumps and the strong Markov property that

$$C_\delta(u_1,u_3) = C_\delta(u_1,u_2)\,C_\delta(u_2,u_3), \qquad 0 \leq u_1 \leq u_2 \leq u_3.$$

Thus, there exists a positive increasing function $h_\delta(x)$ such that

$$C_\delta(u_1,u_2) = \frac{h_\delta(u_1)}{h_\delta(u_2)} \quad \text{for } 0 \leq u_1 \leq u_2$$

(note that in the particular situation where $U(t)$ is a spectrally negative Lévy process, $h_\delta(x)$ can be identified with the scale function $W_\delta(x)$). Since the function $h_\delta(x)$ is unique only up to a constant factor, we can choose u_0 and set $h_\delta(u_0) = 1$, giving

$$h_\delta(u) = \begin{cases} C_\delta(u,u_0), & u < u_0, \\ 1/C_\delta(u_0,u), & u > u_0. \end{cases}$$

Dividends are paid according to the barrier strategy with horizontal barrier b, that is, any potential excess of the surplus beyond b is paid as dividends. Let $D_u(t)$ denote the aggregate dividends paid up to time t, and let τ be the time of ruin. Then the present value of all dividends up to ruin is

$$D_u = \int_0^\tau e^{-\delta t}\, \mathrm{d}D_u(t).$$

The kth moment of D_u is denoted by $V_k(u;b) = \mathbb{E}_u(D_u^k)$.
Analogously to Proposition 2 of Renaud and Zhou [3], it immediately follows from $(e^{-\delta t} D_u)^k = e^{-k\delta t} D_u^k$ and the strong Markov property of $U(t)$ applied at the upper exit time of the interval $(0,b)$ that

$$V_k(u;b) = C_{k\delta}(u;b)\, V_k(b;b) = \frac{h_{k\delta}(u)}{h_{k\delta}(b)}\, V_k(b;b), \qquad 0 \le u \le b. \tag{2}$$

Related to an idea of Gerber and Shiu [2], consider next the difference between the total discounted dividends when starting in $U(0) = b$ and $U(0) = b - \epsilon$, respectively, for a sufficiently small $\epsilon > 0$. If $U(0) = b - \epsilon$, then the dividend barrier will be reached "shortly". At that time, the process that starts at b has led to a total dividend of ϵ, and after this time the trajectories of the two processes are identical. Hence we have the approximate relationship $D_b - D_{b-\epsilon} \approx \epsilon$ and subsequently $D_b^k - D_{b-\epsilon}^k \approx D_b^k - (D_b - \epsilon)^k = \epsilon k D_b^{k-1} + o(\epsilon)$. Taking expectations and the limit $\epsilon \to 0$, we arrive at

$$\left.\frac{\mathrm{d}V_k(u;b)}{\mathrm{d}u}\right|_{u=b^-} = k\, V_{k-1}(b;b). \tag{3}$$

From (2) and (3) we obtain the recursive formula

$$V_k(b;b) = k\, \frac{h_{k\delta}(b)}{h'_{k\delta}(b)}\, V_{k-1}(b;b).$$

From this and $V_0(u;b) = 1$ we obtain

$$V_k(b;b) = k! \prod_{i=1}^{k} \frac{h_{i\delta}(b)}{h'_{i\delta}(b)}.$$

Substitution in (2) yields

$$V_k(u;b) = k!\, \frac{h_{k\delta}(u)}{h_{k\delta}(b)} \prod_{i=1}^{k} \frac{h_{i\delta}(b)}{h'_{i\delta}(b)}, \qquad 0 \le u \le b,$$

which extends (1) to stationary Markov processes that are skip-free upwards.

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