REPRESENTATIONS OF MAX-STABLE PROCESSES VIA EXPONENTIAL TILTING

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Abstract: The recent contribution [1] obtained representations of max-stable stationary Brown-Resnick process $\zeta_Z(t), t \in \mathbb{R}^d$ with spectral process Z being Gaussian. With motivations from [1] we derive for general Z, representations for ζ_Z via exponential tilting of Z. Our findings concern *Dieker-Mikosch representations* of max-stable processes, two-sided extensions of stationary max-stable processes, inf-argmax representation of max-stable distributions, and new formulas for generalised Pickands constants. Our applications include conditions for the stationarity of ζ_Z, a characterisation of Gaussian distributions and an alternative proof of Kabluchko's characterisation of Gaussian processes with stationary increments.

Key Words: Max-stable process; spectral tail process; Brown-Resnick stationary; Dieker-Mikosch representation; inf-argmax representation; Pickands constants; tilt-shift formula; extremal index.
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1. Introduction

A random process $\zeta(t), t \in \mathcal{T}$ is max-stable if all its finite dimensional distributions (fidi's) are max-stable. For simplicity we shall assume hereafter that $\zeta(t)$ has unit Gumbel distribution $e^{-e^{-x}}, x \in \mathbb{R}$ for all $t \in \mathcal{T}$ and shall consider $\mathcal{T} = \mathbb{R}^d$ or $\mathcal{T} = \mathbb{Z}^d, d \geq 1$. In view of [2] any stochastically continuous max-stable process $\zeta(t), t \in \mathcal{T}$ satisfies (below $\stackrel{fdd}{=}$ means equality of all fidi's)

$$\zeta \stackrel{fdd}{=} \zeta_Z,$$

with

$$\zeta_Z(t) \quad = \quad \max_{i \geq 1}\big(P_i + Z_i(t)\big), \quad t \in \mathcal{T}, \tag{1.1}$$

where $Z(t), t \in \mathcal{T}$ is a random process taking values in $[-\infty, \infty)$ with $\mathbb{E}\{e^{Z(t)}\} = 1, t \in \mathcal{T}$ and $\Pi = \sum_{i=1}^{\infty} \varepsilon_{P_i}$ is a Poisson point process (PPP) on $\mathbb{R}$ with intensity $e^{-x}dx$. Further, Z_i's are independent copies of Z being also independent of Π; see for more details [3–13].

We shall refer to ζ_Z as the *associated max-stable* process of Z; commonly Z is referred to as the *spectral process*. For convenience, we shall write Z as

$$Z(t) = B(t) - \ln \mathbb{E}\big\{e^{B(t)}\big\}, \quad t \in \mathcal{T}, \tag{1.2}$$

with $B(t), t \in \mathcal{T}$ a random process satisfying $\mathbb{E}\{e^{B(t)}\} < \infty, t \in \mathcal{T}$. Consequently, $\mathbb{E}\{e^{Z(t)}\} = 1, t \in \mathcal{T}$ implying that the marginal distribution functions (df's) of ζ_Z are unit Gumbel.

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One canonical instance is the classical Brown-Resnick construction with B being a centred Gaussian process with covariance function r and thus $2\ln \mathbb{E}\{e^{B(t)}\} = r(t,t) =: \sigma^2(t), t \in \mathcal{T}$. In view of [14] the law of ζ_Z is determined by the incremental variance function $\gamma(s,t) = Var(B(t)-B(s)), s,t \in \mathcal{T}$. This fact can be shown by utilising the *tilted spectral process* $\Xi_h Z, h \in \mathcal{T}$ defined by

$$\Xi_h Z(t) \quad = \quad B(t) - B(h) - \gamma(h,t)/2, \quad t \in \mathcal{T}.$$

The law of $\Xi_h Z$ is uniquely determined by the following conditions: $\Xi_h Z$ is Gaussian, $\Xi_h Z(h) = 0$ almost surely (a.s.) and the incremental variance function of $\Xi_h Z$ is γ. Note that these conditions do not involve σ^2.
Next, setting $Z^{[h]}(t) = B(t) - \sigma^2(t)/2 + r(h,t)$ we have

$$\Xi_h Z(t) \quad = \quad Z^{[h]}(t) - Z^{[h]}(h), \qquad t \in \mathcal{T}. \tag{1.3}$$

In view of Lemma 6.1 below $Z^{[h]}$ is the exponential tilt of Z by $Z(h)$ i.e.,

$$\mathbb{P}\{Z^{[h]} \in A\} = \mathbb{E}\{e^{Z(h)}\mathbb{I}\{Z \in A\}\}, \quad \forall A \in \mathcal{B}(\mathbb{R}^{\mathcal{T}}),$$

where $\mathcal{B}(\mathbb{R}^{\mathcal{T}})$ is the σ-field generated by all evaluation maps. The representation (1.1) implies that (see e.g., [1, 15])

$$\begin{aligned} -\ln \mathbb{P}\{\zeta_Z(t_i) \le x_i, 1 \le i \le n\} &= \mathbb{E}\left\{\max_{1\le i\le n} e^{Z(t_i)-x_i}\right\} = \mathbb{E}\left\{e^{Z(h)} \max_{1\le i\le n} e^{Z(t_i)-Z(h)-x_i}\right\} \\ &= \mathbb{E}\left\{\max_{1\le i\le n} e^{\Xi_h Z(t_i)-x_i}\right\} \end{aligned} \tag{1.4}$$

holds for $t_i \in \mathcal{T}, x_i \in \mathbb{R}, i \le n$ i.e.,

$$\zeta_Z \stackrel{fdd}{=} \zeta_{\Xi_h Z}. \tag{1.5}$$

Since as mentioned above the process $\Xi_h Z$ can be characterised without making reference to σ^2, by (1.5) it follows that the law of ζ_Z depends on γ only!
Observe that we can define $Z^{[h]}$ via exponential tilting for any random process Z such that $\mathbb{E}\{e^{Z(h)}\} = 1$. Furthermore, the calculation of the fidi's of ζ_Z via (1.4) does not refer to the Gaussianity of Z, but only to the representation (1.1) and the fact that

$$\mathbb{P}\{Z(h) > -\infty\} \quad = \quad 1. \tag{1.6}$$

Consequently, under (1.6) we have that (1.5) is valid for a general *spectral process* Z with values in $\mathbb{R}$. Since we assume (1.6), then by (1.3)

$$\Xi_h Z(h) = Z^{[h]}(h) - Z^{[h]}(h) = 0 \tag{1.7}$$

almost surely, which in view of [16][Lemma 4.1] is a crucial uniqueness condition.
The change of measure technique, or in our case the exponential tilting has been utilised in the context of max-stable processes in [6, 8, 17, 18]. In this contribution we present some further developments and applications that are summarised below:

A) According to [19] the *spectral process* Z is called *Brown-Resnick stationary*, if the associated max-stable process ζ_Z is stationary i.e., $\zeta_Z \overset{fdd}{=} L^h \zeta_Z$ for any $h \in \mathcal{T}$, where L is the lag (backshift) operator with L^h its hth iterate.
For a positive σ-finite measure μ on $\mathcal{T}$, let $\Pi_\mu = \sum_{i=1}^{\infty} \varepsilon_{(P_i,T_i)}$ be a PPP on $\mathbb{R} \times \mathcal{T}$ with intensity $e^{-p}dp{\cdot}\mu(dt)$ being independent of anything else. If Z is a *Brown-Resnick stationary* and sample continuous Gaussian process on $\mathcal{T} = \mathbb{R}^d$, in view of [1][Th. 2.1] (see also [20][Th. 2]) the following *Dieker-Mikosch representation*

$$\zeta_Z(t) \overset{fdd}{=} \max_{i\ge 1}\left(P_i + Z_i(t-T_i) - \ln \int_{\mathcal{T}} e^{Z_i(s-T_i)}\,\mu(ds)\right), \quad t \in \mathcal{T} \tag{1.8}$$

is valid, provided that μ is a probability measure and a.s. $Z(0_{\mathcal{T}}) = 0$ with $0_{\mathcal{T}}$ the origin of $\mathcal{T}$. For notational simplicity hereafter we shall write simply 0 instead of $0_{\mathcal{T}}$.
We shall show that (1.8) given in terms of the *tilted spectral processes* holds for general non-Gaussian Z and some positive σ-finite measure μ on $\mathcal{T}$, see Theorem 2.2 and Theorem 6.5 below. Motivated by [1] we present some useful conditions for the stationarity of ζ_Z. As a by-product we derive a new characterisation of Gaussian df's and give a new proof of Kabluchko's characterisation of Gaussian processes with stationary increments, see Theorem 2.7 and Theorem 2.8 in Section 2.

B) An interesting class of stationary max-stable processes $\zeta_Z(t), t \ge 0$ is constructed by Stoev in [21], where $B(t), t \ge 0$ is a real-valued Lévy process with $\mathbb{E}\{e^{B(1)}\} < \infty$ and Z is specified via (1.2). We show in Theorem 3.1 that for general Z a two-sided extension of ζ_Z can be defined in terms of some *spectral process* Y determined by $\Xi_h Z, h \in \mathcal{T}$.

C) If (1.7) does not hold we modify the definition of $\Xi_h Z$, see Lemma 4.1. Such a modification shows that the *tilted spectral processes* have a component which is identifiable and moreover it determines the law of ζ_Z. Specifically, for any $t_1, \ldots, t_n \in \mathcal{T}$ and H the df of $(\zeta_Z(t_1), \ldots, \zeta_Z(t_n))$, we derive the following (referred to as the *inf-argmax representation*)

$$-\ln H(x) = \sum_{h=1}^{n} e^{-x_h} \Psi_h(x), \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \tag{1.9}$$

with Ψ_h's determined by the identifiable part of $\Xi_h Z$, see below Theorem 4.2. In the special case that H is continuously differentiable (1.9) is a consequence of Euler's homogeneous function theorem, see e.g., [22][Eq. (9.11)].

D) For $\zeta_Z(t), t \in \mathbb{R}^d, d = 1$ being max-stable and stationary [23] introduced the generalised Pickands constant $\mathcal{H}_Z^\delta$ defined by

$$\mathcal{H}_Z^\delta = \lim_{T\to\infty} \frac{1}{T^d} \mathbb{E}\left\{ \sup_{t \in \delta\mathbb{Z}^d \cap [0,T]^d} e^{Z(t)} \right\}, \quad \delta \ge 0, \tag{1.10}$$

with the convention that $0\mathbb{Z}^d = \mathbb{R}^d$. We show in Section 5 sufficient conditions that imply the positivity of $\mathcal{H}_Z^\delta$ for $\delta \ge 0, d \ge 1$ and derive further two new representations for $d \ge 1$ and $\mathbb{P}\{Z(t) = -\infty\} \ge 0, t \in \mathcal{T}$ in terms of the so-called *spectral tail process* defined in [24]. Our new formulas for Pickands type constants are important due to the connection with the extremal index of stationary times series, see (6.19) below.

Organisation of the rest of the paper. In Section 2 we establish the *Dieker-Mikosch representation* of ζ_Z if Z satisfies (1.6) for any $h \in \mathcal{T}$, and discuss further some new conditions for the stationarity of ζ_Z. We continue with an application in Section 3 where we show how to construct a two-sided extension of ζ_Z. Section 4 is concerned with the general case that Z takes values in $[-\infty, \infty)$. New formulas for $\mathcal{H}_Z^\delta$ are displayed in Section 5 followed by discussions and further results in Section 6. All the proofs are relegated to Section 7.

2. Max-stable processes with real-valued Z

Let $Z, Z_i, i \ge 1, \Pi_\mu$ be as in the Introduction and suppose that for some $h \in \mathcal{T}$ the random variable (rv) $Z(h)$ satisfies (1.6) (the case $\mathbb{P}\{Z(h) = -\infty\} > 0$ will be discussed in Section 4). Let in the following $\mathcal{F}_{-\infty}$ denote the set of functions on $\mathcal{T}$ with values in $[-\infty, \infty)$ excluding the function f equal to $-\infty$ and write $\mathcal{B}(\mathcal{F}_{-\infty})$ for the σ-field generated by all evaluation maps. As in (1.3) we define $\Xi_h Z$ by $\Xi_h Z(t) = Z^{[h]}(t) - Z^{[h]}(h), t \in \mathcal{T}$ with $Z^{[h]}$ the exponential tilt of Z by $Z(h)$ i.e., $\mathbb{P}\{Z^{[h]} \in A\} = \mathbb{E}\{e^{Z(h)}\mathbb{I}\{Z \in A\}\}, \forall A \in \mathcal{B}(\mathcal{F}_{-\infty})$.
By (1.4), if η is a rv with values in $[-\infty, \infty)$ being independent of Z satisfying $\mathbb{E}\{e^\eta\} = 1$, then

$$\zeta_Y \stackrel{fdd}{=} \zeta_Z, \tag{2.1}$$

with ζ_Y the max-stable process associated to $Y(t) = \eta + Z(t), t \in \mathcal{T}$. Although Y and Z are completely different processes, we have that $\Xi_h Y \stackrel{fdd}{=} \Xi_h Z$. Surprisingly, as shown below this fact holds for a general Y satisfying (2.1); see also its extension in Lemma 8.1.

Lemma 2.1. *([16][Lemma 4.1]) If (2.1) holds and a.s. $Y(h) = Z(h) = 0$ for some $h \in \mathcal{T}$, then $Y \stackrel{fdd}{=} Z$.*

The claim of Lemma 2.1 is included in [8] and [25]; a direct proof is mentioned in [26] which is elaborated in [27][Lemma 1.1]. We present yet another proof in Section 7.
Since when $\mathbb{P}\{Z(h) > -\infty\} = 1$ we have $\Xi_h Z(h) = 0$ almost surely, then Lemma 2.1 proves the uniqueness of $\Xi_h Z$ (in the sense therein). This implies that $\Xi_h Z$ can be determined directly in terms of ζ_Z.
Our next result below confirms this. Moreover we show that ζ_Z possesses a *Dieker-Mikosch representation* determined by $\Xi_h Z$ and some positive σ-finite measure μ on $\mathcal{T}$, provided that $\mathcal{S}_Z = \int_{\mathcal{T}} e^{Z(s)} \mu(ds)$ is a rv satisfying

$$\mathbb{P}\{\mathcal{S}_Z < \infty\} = 1. \tag{2.2}$$

Note that the assumption $\mathbb{E}\{e^{Z(t)}\} = 1, t \in \mathcal{T}$ implies that (2.2) holds for any probability measure μ on $\mathcal{T}$. Throughout in the following H stands for the df of $(\zeta_Z(t_1), \ldots, \zeta_Z(t_n))$ for some distinct $t_1, \ldots, t_n \in \mathcal{T}$ and denote by $W = (W_1, \ldots, W_n)$ an n-dimensional random vector with df G given by

$$G(x) = \frac{1}{\ln H(0)} \ln\Big(\frac{H(\min(x_1, 0), \ldots, \min(x_n, 0))}{H(x)}\Big), \quad x = (x_1, \ldots, x_n) \in [-\infty, \infty]^n. \tag{2.3}$$

Since H is associated, see e.g., [28, 29], then $H(0) > 0$. In view of [30][p. 278], see also [31][Eq. (2.6)] the df's G corresponding to different t_i's are the so-called generalised Pareto df's, here referred to as the *associated GPD's* of ζ_Z. Set below

$$(W_1^{(h)}, \dots, W_n^{(h)}) := W^{(h)} \stackrel{d}{=} W|(W_h > 0), \quad h \in \{1, \dots, n\}$$

and note that $W_h^{(h)}$ is a unit exponential rv; here we write $\stackrel{d}{=}$ for equality of df's.

Theorem 2.2. *i) If $\mathbb{P}\{Z(h) > -\infty\} = 1$ for some $h \in \mathcal{T}$, then for distinct $t_1 = h, t_2, \dots, t_n \in \mathcal{T}$*

$$\Big(\Xi_h Z(t_2), \dots, \Xi_h Z(t_n)\Big) \stackrel{d}{=} \Big(W_2^{(h)} - W_1^{(h)}, \dots, W_n^{(h)} - W_1^{(h)}\Big). \tag{2.4}$$

ii) If μ is a positive σ-finite measure on $\mathcal{T}$ satisfying (2.2) with $\Pi_\mu = \sum_{i=1}^\infty \varepsilon_{(P_i,T_i)}$ a PPP on $\mathbb{R}\times\mathcal{T}$ with intensity $e^{-p}dp \cdot \mu(dt)$ being independent of anything else we have

$$\zeta_Z(t) \stackrel{fdd}{=} \max_{i\ge1}\left(P_i + \Xi_{T_i} Z_i(t) - \ln \int_{\mathcal{T}} e^{\Xi_{T_i} Z_i(s)}\,\mu(ds)\right), \quad t \in \mathcal{T}. \tag{2.5}$$

Example 2.3. *Consider $Z(t) = B(t) - r(t,t)/2, t \in \mathcal{T}$ with B a centred, sample path continuous Gaussian process with stationary increments and covariance function r. Setting $\sigma^2(t) = r(t,t)$ we have*

$$\Xi_R Z(t) \stackrel{fdd}{=} B(t-R) - \sigma^2(t-R)/2, \quad t \in \mathcal{T} \tag{2.6}$$

for any real-valued rv R independent of B. Hence (2.5) reduces to [1][Th. 2.1] when μ is a probability measure.

Remark 2.4. *If $\zeta_Z(t), t \in \mathbb{R}^d, d = 1$ is stationary and a.s. $Z(0) = 0$, then by (2.6) the representation in (2.5) agrees with the finding of [23][Th. 4].*

Several contributions have investigated the stationarity of max-stable processes and particle systems, see e.g., [6, 14–16, 19, 32–35]. The main result of this section displays three criteria for the stationarity of ζ_Z. Below we define $L^b \Xi_a Y := L^b(\Xi_a Y)$ by

$$L^b \Xi_a Y(t) = L^b(Y^{[a]}(t) - Y^{[a]}(a)) = Y^{[a]}(t-b) - Y^{[a]}(a) \tag{2.7}$$

for any $a, t-b, t \in \mathcal{T}$ and some random process Y such that $\mathbb{E}\{e^{Y(a)}\} = 1$ with $Y^{[a]}$ the tilted process by $Y(a)$. Recall that in our notation $\mathcal{T} = \mathbb{R}^d$ or $\mathcal{T} = \mathbb{Z}^d$ and 0 is the origin in $\mathcal{T}$.

Theorem 2.5. *Let $\zeta_Z(t), t \in \mathcal{T}$ be a max-stable process with unit Gumbel marginals and spectral process Z defined via (1.1) and let for some σ-finite measure μ on $\mathcal{T}$ the PPP Π_μ be as in the Introduction. If (1.6) holds for any $h \in \mathcal{T}$, then the following are equivalent:*
a) ζ_Z is stationary i.e., $\zeta_Z \stackrel{fdd}{=} \zeta_{L^h Z}$ for any $h \in \mathcal{T}$.
b) For any positive σ-finite measure μ on $\mathcal{T}$ we have that (2.5) holds with $L^{T_i}\Xi_0 Z_i$ instead of $\Xi_{T_i} Z_i$, provided that $\mathcal{S}_Z = \int_{\mathcal{T}} e^{Z(t)}\mu(dt)$ is a positive finite rv.
c) For any functional $\Gamma : \mathcal{F}_{-\infty} \to [0,\infty)$ which is $\mathcal{B}(\mathcal{F}_{-\infty})/\mathcal{B}(\mathbb{R})$ measurable, such that $\Gamma(f+c) = \Gamma(f), c \in \mathbb{R}, f \in \mathcal{F}_{-\infty}$ holds, we have

$$\mathbb{E}\big\{e^{Z(a+h)}\Gamma(Z)\big\} = \mathbb{E}\big\{e^{Z(a)}\Gamma(L^h Z)\big\}, \quad \forall h \in \mathcal{T}, \tag{2.8}$$

provided that the expectations exist.
d) For any $a, a+h \in \mathcal{T}$

$$\Xi_{a+h} Z \stackrel{fdd}{=} L^h \Xi_a Z. \tag{2.9}$$

Remark 2.6. *i) If Z is as in Example 2.3, then statement a) $\implies$ c) in Theorem 2.5 has been shown in [1][Lemma 5.2], whereas the non-Gaussian case is derived in [23][Lemma 1] under the restriction that a.s. $Z(0) = 0$.*
ii) If statement c) and d) in Theorem 2.5 hold for any $h \in \mathcal{T}$ and $a = 0$ being the origin, then ζ_Z is max-stable and stationary.

We present next two applications, a third one is displayed in Section 3.
Motivated by [36][Th. 1] we derive below a new characterisation of multivariate Gaussian df's. Hereafter $(\cdot, \cdot)$ stands for the scalar product in $\mathbb{R}^d$.

Theorem 2.7. *Let X be a d-dimensional random vector with non-degenerate components and define $Z(t) = (t, X) - \kappa(t), t \in \mathbb{R}^d$, with κ some measurable function satisfying $\kappa(0) = 0$. Suppose that the associated max-stable process $\zeta_Z(t), t \in \mathbb{R}^d$ has unit Gumbel marginals and set $\zeta_Z^\delta(t) = \zeta_Z(t), t \in \delta\mathbb{Z}^d$. If for any $\delta \in (0, \infty), h \in \delta\mathbb{Z}^d$*

$$\Xi_h Z(t) \stackrel{fdd}{=} L^h Z(t), \quad t \in \delta\mathbb{Z}^d, \tag{2.10}$$

then X is Gaussian $N_d(\mu, \Sigma)$ and $\kappa(t) = (t, \mu) + (t, \Sigma t)/2, t \in \mathbb{R}^d$.

Our second application is a different proof of Kabluchko's characterisation of Gaussian random fields with stationary increments stated in [32][Th. 1.1].

Theorem 2.8. *Let $B(t), t \in \mathbb{R}^d$ be a centred Gaussian process with non-zero variance function σ^2 such that $\sigma(0) = 0$. The max-stable process ζ_Z associated to $Z(t) = B(t) - \sigma^2(t)/2, t \in \mathbb{R}^d$ is stationary if and only if B has stationary increments.*

3. Two-sided stationary max-stable processes

Consider $\zeta_Z(t), t \ge 0$ defined via (1.1), where $Z(t) = B(t) - t/2, t \ge 0$ with $B(t), t \in \mathbb{R}$ a two-sided standard Brownian motion. The seminal article [37] showed that ζ_Z is max-stable and one-sided stationary. In view of [19], in order to define $\zeta_Z(t)$ also for $t < 0$ i.e., to define a two-sided stationary max-stable process ζ_Z, we can take $Z(t) = B(t) - |t|/2, \forall t \in \mathbb{R}$. This construction is fundamental since B is both a centred Gaussian process with stationary increments and also a Lévy process. Stoev showed in [21] that if $B(t), t \ge 0$ is a real-valued Lévy process with Laplace exponent $\Phi(\theta) = \mathbb{E}\{e^{\theta B(1)}\}$ being finite for $\theta = 1$, then $\zeta_Z(t), t \ge 0$ defined by (1.1) with $Z(t) = B(t) - \Phi(1)t$ is both max-stable and stationary. The recent contribution [38] is primarily motivated by the question of how to define directly $Z(t), t < 0$ such that $\zeta_Z(t), t \in \mathbb{R}$ is both max-stable and stationary. In Theorem 1.2 therein a two-sided version of Z and thus of ζ_Z is constructed. Specifically, as in [38] define $Z(t), t < 0$ by setting $Z(t) = Z^-(-t), \quad t < 0$, where $Z^-(t), t \ge 0$ is independent of $Z(t), t \ge 0$ such that $-Z^-(t)$ is the exponential tilt of Z at t i.e., in our notation since a.s. $Z(0) = 0$, then for any $t > 0$

$$Z^-(t) = \Xi_t Z(-t) = Z^{[t]}(0) - Z^{[t]}(t) = -Z^{[t]}(t).$$

Hence, in view of [39][Theorem 3.9] (see also [40]) Z^- is a Lévy process with Laplace exponent $\ln \mathbb{E}\{e^{\theta Z^-(1)}\} = \Phi(1-\theta) - (1-\theta)\Phi(1)$. Our next result is not restricted to the particular cases of Z being a Lévy or a Gaussian process.

Theorem 3.1. *Let $\zeta_Z(t), t \geq 0$ be a max-stable and stationary process determined by Z as in (1.1) with $\mathbb{E}\{e^{Z(t)}\} = 1, t \geq 0$. If (1.6) holds for any $h \geq 0$, then there exists a random process $Y(t), t \in \mathbb{R}$ such that for distinct $t_1, \ldots, t_n \in \mathbb{R}$*

$$(3.1)\quad \Big(Y(t_1), \ldots, Y(t_n)\Big) \stackrel{d}{=} \Big(\Xi_h Z(t_1+h), \ldots, \Xi_h Z(t_n+h)\Big), \quad h := -\min\Big(0, \min_{1\leq j\leq n} t_j\Big)$$

and $\zeta_Y(t) \stackrel{fdd}{=} \zeta_Z(t), t \geq 0$. Moreover $Y(t), t \in \mathbb{R}$ is Brown-Resnick stationary.

Example 3.2. *(Brown-Resnick process) Let $\zeta_Z(t), t \geq 0$ be a max-stable process associated to $Z(t) = B(t) - \sigma^2(t)/2, t \geq 0$ with $B(t), t \in \mathbb{R}$ a centred Gaussian process with stationary increments and variance function σ^2. If $\sigma(0) = 0$, by Example 2.3 and (3.1) it follows easily that $Y \stackrel{fdd}{=} Z^*$, where $Z^*(t) = B^*(t) - \sigma^2(|t|)/2, t \in \mathbb{R}$ with $B^*(t), t \in \mathbb{R}$ a centred Gaussian process with covariance function $(\sigma^2(|t|) + \sigma^2(|s|) - \sigma^2(|t-s|))/2$. Since B^* has stationary increments, then by Theorem 2.8 ζ_Z is stationary.*

Example 3.3. *(Lévy-Brown-Resnick process) Suppose that $Z(t), t \geq 0$ is a Lévy process with $\mathbb{E}\{e^{Z(t)}\} = 1, t \geq 0$. According to [21] the max-stable process $\zeta_Z(t), t \geq 0$ associated to Z is stationary. Hence we are in the setup of Theorem 3.1, which ensures that $\zeta_Y(t), t \in \mathbb{R}$ is a max-stable stationary extension of ζ_Z. Further for $s \leq t < 0$ by (3.1) and $Z(0) = 0$*

$$(3.2)\qquad \big(Y(s) - Y(t), Y(t)\big) \stackrel{d}{=} \big(-Z^{[-s]}(t-s), Z^{[-s]}(t-s) - Z^{[-s]}(-s)\big).$$

Since $Z(t), t \geq 0$ is a Lévy process, then it follows easily that Y agrees with the definition of [38].

4. General Spectral Processes

In this section we assume that $Z(h) = -\infty$ for some $h \in \mathcal{T}$ with non-zero probability. Write next (set below $0 \cdot \infty = 0$)

$$(4.1)\qquad Z \stackrel{fdd}{=} J_h V_h + (1 - J_h) W_h,$$

where J_h is a Bernoulli rv with

$$\mathbb{P}\{J_h = 1\} = \mathbb{P}\{Z(h) > -\infty\} \in (0, 1]$$

and

$$V_h \stackrel{fdd}{=} Z|(Z(h) > -\infty), \quad W_h \stackrel{fdd}{=} Z|(Z(h) = -\infty).$$

Furthermore, J_h, V_h, W_h are mutually independent and

$$\mathbb{P}\{V_h(h) > -\infty\} = \mathbb{P}\{W_h(h) = -\infty\} = 1.$$

For $V_h^{[h]}$ given via exponential tilting as

$$\mathbb{P}\{V_h^{[h]} \in A\} = \mathbb{E}\{e^{V_h(h) - \ln \mathbb{E}\{e^{V_h(h)}\}} \mathbb{I}\{V_h \in A\}\}, \quad A \in \mathcal{B}(\mathcal{F}_{-\infty})$$

define the *tilted spectral process* $\Xi_h Z$ by

$$\Xi_h Z(t) = J_h \Theta_h(t) + (1-J_h)[W_h(t) - V_h^{[h]}(h)] - \ln \mathbb{P}\{J_h = 1\}, \quad t \in \mathcal{T}, \tag{4.2}$$

where

$$\Theta_h(t) := \Xi_h V_h(t) = V_h^{[h]}(t) - V_h^{[h]}(h). \tag{4.3}$$

Hereafter, we shall consider $V_h^{[h]}$ to be independent of J_h and W_h.
The next result establishes the counterpart of (1.5). Further, we give a representation of ζ_Z which is motivated by [7][Th. 2].

Lemma 4.1. *For any $h \in \mathcal{T}$ we have $\zeta_Z \stackrel{fdd}{=} \zeta_{\Xi_h Z}$. Moreover, for any probability measure μ on $\mathcal{T}$ we have $\zeta_Z \stackrel{fdd}{=} \eta$ where $\eta(t) = \max_{i\ge 1}(P_i + \Xi_{T_i} Z_i(t)), t \in \mathcal{T}$ with (P_i, T_i)'s the points of a PPP on $\mathbb{R}\times\mathcal{T}$ with intensity $e^{-p}dp\cdot\mu(dt)$ being independent of $Z_i, i \ge 1$.*

In view of Lemma 8.1 in Appendix $\Theta_h, h \in \mathcal{T}$ is the identifiable part of the family of *tilted spectral processes* $\Xi_h Z, h \in \mathcal{T}$. Moreover, as shown below $\Theta_h, h \in \mathcal{T}$ determines the law of ζ_Z.

Theorem 4.2. *(Inf-argmax representation) For any distinct $t_i \in \mathcal{T}, i \le n$ the df H of $(\zeta_Z(t_1), \ldots, \zeta_Z(t_n))$ is given by*

$$-\ln H(x) \;=\; \sum_{k=1}^n e^{-x_k}\Psi_k(x), \quad \textit{with } \Psi_k(x) = \mathbb{P}\Big\{\inf \operatorname{argmax}_{1\le i\le n}\big(\Theta_{t_k}(t_i) - x_i\big) = k\Big\}$$

for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

We conclude this section with an extension of Theorem 2.5.

Theorem 4.3. *Let $Z(t), t \in \mathcal{T}$ be a random process with values in $[-\infty, \infty)$. If $\mathbb{E}\{e^{Z(t)}\} = 1, t \in \mathcal{T}$ and ζ_Z is given by (1.1), then the following are equivalent:*
a) ζ_Z is max-stable and stationary.
b) For any Γ as in Theorem 2.5 statement c)

$$\mathbb{E}\big\{e^{Z(a+h)}\Gamma(Z)\big\} \;= \mathbb{E}\big\{e^{Z(a)}\Gamma(L^h Z)\big\} = \mathbb{E}\big\{\Gamma(L^h\Theta_a)\big\}, \quad a, a+h \in \mathcal{T}. \tag{4.4}$$

c) For any $a, h \in \mathcal{T}$ and Z with representation (4.1) we have

$$\Theta_{a+h} \stackrel{fdd}{=} L^h \Theta_a. \tag{4.5}$$

Otherwise specified, hereafter we set

$$\Theta = \Theta_0, \quad x = (x_1, \ldots, x_n).$$

Remark 4.4. *If Γ, ζ_Z are as in Theorem 4.3, then (4.4) is equivalent with*

$$\mathbb{E}\big\{\mathbb{I}\{Z(-h) > -\infty\} e^{Z(-h)}\Gamma(L^h Z)\big\} \;= \mathbb{E}\big\{\Gamma(\Theta)\big\}, \quad h \in \mathcal{T}. \tag{4.6}$$

Hence the inf-argmax representation in (4.4) simplifies to

$$-\ln H(x) \;=\; \sum_{k=1}^n e^{-x_k}\mathbb{P}\Big\{\inf \operatorname{argmax}_{1\le i\le n}\big(L^{t_k}\Theta(t_i) - x_i\big) = k\Big\}, \quad x \in \mathbb{R}^n \tag{4.7}$$

and thus we conclude that the fidi's of ζ_Z are given in terms of those of Θ.

5. Generalised Pickands Constants

Given $Z(t), t \in \mathbb{R}^d, d \geq 1$ with representation (1.2) we define for any $\delta > 0$ the generalised Pickands constant $\mathcal{H}_Z^\delta$ as in (1.10) i.e.,

$$\mathcal{H}_Z^\delta = \lim_{T\to\infty} \frac{1}{T^d} \mathbb{E}\Big\{ \sup_{t \in \delta\mathbb{Z}^d \cap [0,T]^d} e^{Z(t)} \Big\}.$$

A canonical example here is the Brown-Resnick stationary case with $Z(t) = \sqrt{2}B_\alpha(t) - |t|^\alpha, t \in \mathbb{R}$, where $B_\alpha, \alpha \in (0,2]$ is a standard fractional Brownian motion with Hurst index $\alpha/2 \in (0,1]$. For this case $\mathcal{H}_Z^\delta$ is the classical Pickands constant, see e.g., [41–46] for its properties.
The recent contribution [23] investigates $\mathcal{H}_Z^\delta$ under the assumption that a.s. $Z(0) = 0$ and $d = 1$. In this section we shall assume that $\zeta_Z(t), t \in \mathcal{T}$ is max-stable, stationary and has unit Gumbel marginals. In order to show the positivity of $\mathcal{H}_Z^\delta$, we shall suppose further that

$$\mathbb{P}\Big\{ \int_{\mathbb{R}^d} e^{Z(t)} \lambda(dt) < \infty \Big\} = 1, \tag{5.1}$$

where λ is the Lebesgue measure on $\mathbb{R}^d$. In light of [47][Th. 2] (see also [48]) (5.1) is equivalent with

$$\mathbb{P}\Big\{ \lim_{\|t\|\to\infty} Z(t) = -\infty \Big\} = 1. \tag{5.2}$$

Under (5.1), as in [23][Th. 1] if a.s. $Z(0) = 0$, then for any $\delta > 0$ and $d = 1$

$$\mathcal{H}_Z^\delta = \mathbb{E}\left\{ \frac{\max_{t \in \delta\mathbb{Z}^d} e^{Z(t)}}{\delta^d \int_{\mathbb{R}^d} e^{Z(s)} \mu_\delta(ds)} \right\} \in (0,\infty), \tag{5.3}$$

where μ_δ denotes the counting measure on $\delta\mathbb{Z}^d$. If $\mathbb{P}\{Z(0) = 0\} < 1$ the expression in (5.3) needs to be modified, since by Lemma 7.1 in Appendix, for any $T > 0$ and $d \geq 1$ we have

$$\begin{aligned} \mathcal{H}_Z^\delta &= \lim_{T\to\infty} \int_{[0,1]^d} \mathbb{E}\left\{ \frac{\max_{t \in \delta\mathbb{Z}^d \cap [-hT,(1-h)T]^d} e^{\Theta(t)}}{\int_{\delta\mathbb{Z}^d \cap [-hT,(1-h)T]^d} e^{\Theta(s)} \mu_\delta(ds)} \right\} \mu^T(dh) \\ &=: \lim_{T\to\infty} \int_{[0,1]^d} \eta_T(h) \mu^T(dh), \end{aligned} \tag{5.4}$$

where $\mu^T(dh) = \mu_\delta(T dh)/T^d$. In applications, often Pickands-type constants corresponding to $\delta = 0$ appear. In order to define $\mathcal{H}_Z^0$, we shall suppose further that ζ_Z has cadlag sample paths. This is equivalent with $\mathbb{E}\{\sup_{t\in K} e^{Z(t)}\} < \infty$ for any compact set $K \subset \mathbb{R}^d$, see [49]. The definition of $\mathcal{H}_Z^0$ is exactly as in (6.9) where we interpret $0\mathbb{Z}^d$ as $\mathbb{R}^d$ i.e., $\mathcal{H}_Z^0 = \lim_{T\to\infty} \frac{1}{T^d}\mathbb{E}\{\sup_{t\in[0,T]^d} e^{Z(t)}\}$. The existence and the finiteness of $\mathcal{H}_Z^0$ follow by the stationarity and max-stability of ζ_Z. As in the case $\delta > 0$ of interest is the positivity of $\mathcal{H}_Z^0$, derivation of tractable expressions and sharp bounds for $\mathcal{H}_Z^\delta, \delta \geq 0$. For Z Gaussian or Lévy and $d = 1, Z(0) = 0$ a.s. [23] shows that under some weak restrictions

$$\mathcal{H}_Z^0 = \mathbb{E}\left\{ \frac{\sup_{t\in\mathbb{R}^d} e^{Z(t)}}{\int_{\mathbb{R}^d} e^{Z(s)} \lambda(ds)} \right\} \in (0,\infty).$$

Since the aforementioned results cover only the case $Z(0) = 0$ a.s. and $d = 1$, below we shall derive two formulas for $\mathcal{H}_Z^\delta, \delta > 0$ for the general case $d \ge 1$ and $\mathbb{P}\{Z(0) = -\infty\} \ge 0$. If $\delta = 0$, we give a positive lower bound for $\mathcal{H}_Z^0$.

Theorem 5.1. *Let $Z(t), t \in \mathbb{R}^d, d \ge 1$ be such that the associated max-stable process ζ_Z is stationary with unit Gumbel marginals and suppose that (5.1) holds.*
i) For any $\delta > 0$

$$\mathcal{H}_Z^\delta = \mathbb{E}\left\{ \frac{\max_{t \in \delta \mathbb{Z}^d} e^{\Theta(t)}}{\delta^d \int_{\mathbb{R}^d} e^{\Theta(s)} \mu_\delta(ds)} \right\} =: C^\delta \in (0, \infty). \tag{5.5}$$

Moreover, if further the fidi's of Θ are absolutely continuous, then

$$\mathcal{H}_Z^\delta = \frac{1}{\delta^d} \mathbb{P}\Big\{ \max_{t \in \delta \mathbb{Z}^d} \Theta(t) = 0 \Big\}. \tag{5.6}$$

ii) If $\delta = 0$, then

$$\mathcal{H}_Z^0 \ge \mathbb{E}\left\{ \frac{\sup_{t \in \mathbb{R}^d} e^{\Theta(t)}}{\int_{\mathbb{R}^d} e^{\Theta(s)} \lambda(ds)} \right\} =: C^0 \in (0, \infty). \tag{5.7}$$

Remark 5.2. *i) If Z is as in Example 3.2, then $Z = \Theta$ and thus (5.6) follows from [23][Th. 1] combined with [7][Th. 8 and Remark 9]. A direct proof for B being a standard fractional Brownian motion is given in [50][Prop. 4]. The lower bound $\mathcal{H}_Z^0 \ge C^0$ is derived in [23][Th. 1] for $d = 1, Z(0) = 0$. It is of interest (and open question) to know general tractable conditions that yield $\mathcal{H}_Z^0 = C^0$.*
ii) If $Z(t), t \in \mathbb{R}^d$ is as in Theorem 5.1, then for any $\delta \ge 0$ we have (see for related results [1, 51, 52])

$$\lim_{T \to \infty} \mathbb{P}\Big\{ \sup_{t \in \delta \mathbb{Z}^d \cap [0,T]^d} \zeta_Z(t) \le x + d \ln T \Big\} = e^{-\mathcal{H}_Z^\delta e^{-x}}, \quad \forall x \in \mathbb{R}. \tag{5.8}$$

Consequently, $\delta^d \mathcal{H}_Z^\delta \in (0, 1], \delta > 0$ is the extremal index of the stationary random field $\zeta_Z(t), t \in \delta \mathbb{Z}^d$.
iii) Generalised Pickands constants have appeared also in the non-Gaussian setup, see e.g., [53][Lemma 5.16] and [54].

6. Discussions & Further Results

6.1. **Tilted processes.** If $Z(t), t \in \mathcal{T}$ is a random process with finite $\varphi(h) = \ln \mathbb{E}\{e^{Z(h)}\}, h \in \mathcal{T}$, then we define $Z^{[h]}$ by

$$\mathbb{P}\{Z^{[h]} \in A\} = \mathbb{E}\{e^{Z(h) - \varphi(h)} \mathbb{I}\{Z \in A\}\}, \quad A \in \mathcal{B}(\mathcal{F}_{-\infty}).$$

The exponential tilting of df's in the exponential family can be calculated explicitly. In particular, for the Gaussian case, the tilted process is again Gaussian, with the same covariance function, but modified mean, see [55][p. 130] or [20][Lemma 1].

Lemma 6.1. *Let $Z(t), t \in \mathcal{K}$ be a real-valued Gaussian process defined on some set $\mathcal{K}$ with covariance function r. For any $h \in \mathcal{K}$ the random process $Z^{[h]}$ is again Gaussian and moreover*

$$Z^{[h]}(t) \stackrel{fdd}{=} Z(t) + r(h, t), \quad t \in \mathcal{K}. \tag{6.1}$$

Conversely, if $Z^{[h]}(t), t \in \mathcal{K}$ is for some $h \in \mathcal{K}$ a Gaussian process with covariance function r and mean $r(h,t)$, then Z is a centred Gaussian process with covariance function r.

Example 6.2. *Consider $Z(t) = B(t) - r(t,t)/2, t \in \mathcal{K}$ with B a centred Gaussian process with covariance function r. For any $h \in \mathcal{K}$ by Lemma 6.1 $Z^{[h]}(t) \stackrel{fdd}{=} B(t) - r(t,t)/2 + r(t,h)$ implying that*

$$\Xi_h Z(t) = Z^{[h]}(t) - Z^{[h]}(h) \stackrel{fdd}{=} B(t) - B(h) - Var(B(t) - B(h))/2, \quad t \in \mathcal{K}, \tag{6.2}$$

which agrees with the definition of $\Xi_h Z$ given in (1.3).

Example 6.3. *For ζ_Z defined in Example 3.2, by (6.2)*

$$\Theta_{t_k}(t) = \Xi_{t_k} Z(t) \stackrel{fdd}{=} B(t) - B(t_k) - Var(B(t) - B(t_k))/2, \quad t, t_k \in \mathcal{T}.$$

Hence (1.9) holds with Ψ_h the df of the Gaussian random vector $(\Theta_k(t_1), \ldots, \Theta_k(t_n))_{-h}$ (the subscript $_{-h}$ means that the hth component is dropped). Such a representation of max-stable Hüsler-Reiss df has been derived by another approach in [56], see also [57].

Remark 6.4. *If a.s. $Z(h) > -\infty$, then $\Xi_h Z = \Theta_h$ and thus it can be also calculated using (2.4). If $\mathbb{P}\{Z(h) > -\infty\} > 0$, then in view of Lemma 8.1, we have that (2.4) holds with Θ_h instead of $\Xi_h Z$. Hence when the df's of the associated GPD's of ζ_Z are known, we can calculate Θ_h using the right-hand side of (2.4).*

6.2. **Dieker-Mikosch representation.** In view of our findings in Section 2 we have the representation (recall (2.5))

$$\zeta_Z(t) \stackrel{fdd}{=} \max_{i \ge 1} \big(P_i + F_i(t, T_i)\big), \quad t \in \mathcal{T}, \tag{6.3}$$

where (P_i, T_i)'s are the points of a PPP Π_μ on $\mathbb{R} \times \mathcal{T}$ with intensity $e^{-p} dp \cdot \mu(dt)$ being further independent of F_i's which are independent copies of a random shape function F defined by

$$F(t,h) = \Xi_h Z(t) - \ln \int_{\mathcal{T}} e^{\Xi_h Z(s)} \mu(ds), \quad h, t \in \mathcal{T}, \tag{6.4}$$

with μ a positive σ-finite measure on $\mathcal{T}$ (recall Z_i's are independent of the points of Π_μ).
Next, we shall assume that $\mathcal{M}_Z = \sup_{t \in \mathcal{T}} e^{Z(t)}$ and $\mathcal{S}_Z = \int_{\mathcal{T}} e^{Z(t)} \mu(dt)$ are non-negative and finite rv's. The representation (2.5) of ζ_Z is shown under the assumption that $\mathcal{S}_Z$ is a.s. positive. Since we assume that $\mathbb{E}\{e^{Z(t)}\} = 1, t \in \mathcal{T}$, if μ is a probability measure, then the finiteness of $\mathcal{S}_Z$ is guaranteed also for general *spectral processes* Z with values in $[-\infty, \infty)$. However, $\mathcal{S}_Z$ can be equal to zero with non-zero probability. Therefore, in this section the *Dieker-Mikosch representation* for Z with values in $[-\infty, \infty)$ will be shown under the following restriction

$$\mathbb{P}\{\mathcal{M}_Z > 0, \mathcal{S}_Z = 0\} = 0. \tag{6.5}$$

If μ possesses a positive probability mass function $p(t), t \in \mathcal{T} = \{t_1, \ldots, t_n\}$, then

$$\mathbb{P}\{\mathcal{M}_Z > 0, \mathcal{S}_Z = 0\} = \mathbb{P}\Big\{\mathcal{M}_Z > 0, \sum_{k=1}^{n} p(t_k) e^{Z(t_k)} = 0\Big\} = 0,$$

hence (6.5) is valid for such μ. Similarly, (6.5) holds also for μ the counting measure on $\mathcal{T} = \mathbb{Z}^d$ and we do not need further conditions on Z to show that $\mathcal{S}_Z$ is a rv.

Theorem 6.5. *Let $\zeta_Z(t), t \in \mathcal{T}$ be a max-stable process with representation (1.1) and spectral process Z satisfying (4.1) being both measurable and separable. If μ is a positive σ-finite measure on $\mathcal{T}$ such that (6.5) is valid with $\mathcal{M}_Z, \mathcal{S}_Z$ being two non-negative finite rv's, then (6.3) holds with random shape function F given by*

$$F(t,h) = \Theta_h(t) - \ln \int_{\mathcal{T}} e^{\Theta_h(s)} \mu(ds), \quad h,t \in \mathcal{T}. \tag{6.6}$$

For the case that Z is *Brown-Resnick stationary* we have the following mixed-moving-maxima (M3) Dieker-Mikosch representation:

Corollary 6.6. *Under the assumptions of Theorem 6.5, if further ζ_Z is stationary, then (6.3) holds with random shape function F given by*

$$F(t,h) = L^h\Theta(t) - \ln \int_{\mathcal{T}} e^{L^h\Theta(s)} \mu(ds), \quad h,t \in \mathcal{T}. \tag{6.7}$$

Conversely, if for some measurable and separable random process $\Theta(t), t \in \mathcal{T}$ we have that $\int_{\mathcal{T}} e^{\Theta(s)}\mu(ds)$ is a positive rv with μ the Lebesgue measure on $\mathcal{T}$, then $\zeta(t), t \in \mathcal{T}$ with representation given by the right-hand side of (6.3) and random shape function F given in (6.7) is max-stable and stationary.

Remark 6.7. *In the special case $d = 1$ and $\mathbb{P}\{Z(0) = 0\} = 1$ the representation (6.7) is stated in [23][Thm. 3.1].*

6.3. **Max-stable processes with Fréchet marginals.** Our results derived under the assumption of Gumbel marginals hold with minor adjustments when the marginals are assumed to be Fréchet or Weibull. Since quite often in applications max-stable processes $\zeta_Z(t), t \in \mathcal{T}$ with Fréchet marginals are considered, see e.g., [58–63] we shall discuss these processes in some details. Specifically, we define

$$\zeta_Z(t) \overset{fdd}{=} \max_{i \ge 1} P_i Z_i(t), \quad t \in \mathcal{T}, \tag{6.8}$$

where $\sum_{i=1}^\infty \delta_{P_i}$ is a PPP on $(0,\infty)$ with intensity $x^{-2}dx$ being independent of $Z_i, i \ge 1$ which are independent copies of a non-negative random process $Z(t), t \in \mathcal{T}$ with $\mathbb{E}\{Z(t)\} = 1, t \in \mathcal{T}$. Let $V_h \overset{fdd}{=} Z|(Z(h) > 0)$ and recall that $1 = \mathbb{E}\{Z(h)\} = \mathbb{E}\{V_h(h)\}\mathbb{P}\{Z(h) > 0\}$. As in the Gumbel case, the *tilted spectral processes* $\Theta_h := \Xi_h V_h, h \in \mathcal{T}$ are defined by (interpret below 0/0 as 0)

$$\begin{aligned}\mathbb{P}\{\Theta_h \in A\} &= \mathbb{E}\Big\{\frac{V_h(h)}{\mathbb{E}\{V_h(h)\}}\mathbb{I}\{V_h/V_h(h) \in A\}\Big\} = \mathbb{E}\Big\{Z(h)\mathbb{I}\{Z(h) > 0\}\mathbb{I}\{Z/Z(h) \in A\}\Big\}\\ &= \int_{\mathcal{T}} f(h)\mathbb{I}\{f(h) > 0\}\mathbb{I}\{f/f(h) \in A\}\nu(df), \quad A \in \mathcal{B}(\mathcal{F}_0), \end{aligned} \tag{6.9}$$

where $\mathcal{F}_0$ is the set of non-negative functions on $\mathcal{T}$ excluding the zero function endowed with the σ-field $\mathcal{B}(\mathcal{F}_0)$ generated by all evaluation maps and ν stands for the law of Z.

If H denotes the df of $(\zeta_Z(t_1),\ldots,\zeta_Z(t_n))$, then its marginals are unit Fréchet and moreover its *inf-argmax representation* is given by

$$
\begin{aligned}
-\ln H(x) &= \sum_{k=1}^{n}\frac{1}{x_k}\mathbb{P}\Big\{\max_{1\le i<k}\Theta_{t_k}(t_i)<\frac{x_i}{x_k},\max_{k\le i<n}\Theta_{t_k}(t_i)\le\frac{x_i}{x_k}\Big\}\\
&= \sum_{k=1}^{n}\frac{1}{x_k}\mathbb{P}\Big\{\inf\operatorname{argmax}_{1\le i\le n}\Big(\frac{\Theta_{t_k}(t_i)}{x_i}\Big)=k\Big\},\quad x\in(0,\infty)^n. \qquad (6.10)
\end{aligned}
$$

Note in passing that if ζ_Z is stationary, then by (4.5)

$$
\Theta_{t_k}\stackrel{fdd}{=}L^{t_k}\Theta,
$$

hence (6.10) is determined only by Θ in this case (recall that we set $\Theta=\Theta_0$).
Next, as in [47] define the *functional PPP* $\Phi=\{\phi_i,i\ge1\}$ on $\mathcal{F}_0$ with $\phi_i=P_iZ_i,i\ge1$ and intensity measure q determined by

$$
q(A)=\int_A q(df)=\int_0^\infty\mathbb{P}\{uZ\in A\}u^{-2}du,\quad A\in\mathcal{B}(\mathcal{F}_0). \qquad (6.11)
$$

We have that H is determined by q as follows

$$
-\ln H(x)=q\{f\in\mathcal{F}_0: f(t_i)>x_i,\ \text{for some } i=1,\ldots,n\},\quad x\in(0,\infty)^n. \qquad (6.12)
$$

Remark 6.8. *A direct implication of (6.11),(6.12) and Lemma 2.1 is that for any two random processes $Y(t),Z(t),t\in\mathcal{T}$ such that $\mathbb{E}\{Z(t)\}\in(0,\infty),t\in\mathcal{T}$ and*

$$
\int_0^\infty\mathbb{P}\{uZ\in A\}u^{-2}du=\int_0^\infty\mathbb{P}\{uY\in A\}u^{-2}du,\quad A\in\mathcal{B}(\mathcal{F}_0), \qquad (6.13)
$$

then $Z\stackrel{fdd}{=}Y$, provided that a.s. $Z(h)=Y(h)=1$ for some $h\in\mathcal{T}$. Note further that in view of Lemma 8.1 the relation (6.13) implies $\Theta_h\stackrel{fdd}{=}\Xi_hU_h$ with $U_h\stackrel{fdd}{=}Y|(Y(h)>0)$.

Denote by $\phi_h^+,h\in\mathcal{T}$ the extremal function at h (see [17, 20, 64] for details) i.e., this is the set of functions $\phi\in\Phi$ such that $\phi(h)=Z(h)$.

It follows that $\Phi_h=\Phi\cap\{f\in\mathcal{F}_0: f(h)>0\},h\in\mathcal{T}$ is a PPP with intensity

$$
q_h(A)\int_0^\infty\mathbb{P}\{u\Theta_h\in A\}u^{-2}du,\quad A\in\mathcal{B}(\mathcal{F}_0), \qquad (6.14)
$$

which is a minor extension of [20][Prop. 2] where additionally ζ_Z is assumed to have continuous sample paths. The properties of Φ_h can be utilised to give an alternative proof that a) implies d) in Theorem 4.3.
To this end, we note that if ζ_Z has continuous sample paths, in view of [17][Prop. 4.2] the expression in (6.9) yields

$$
\Theta_h\stackrel{fdd}{=}\phi_h^+/Z(h),\quad h\in\mathcal{T}. \qquad (6.15)
$$

As suggested by a reviewer, (6.15) can be utilised to show that Theorem 4.3 d) follows by the stationarity of ζ_Z (when ζ_Z has continuous sample paths).

6.4. **Tilt-shift formula.** Let $X(t), t \in \mathbb{Z}^d$ be a real-valued stationary time series. Commonly, X is called jointly regularly varying with index $\alpha > 0$, if the random vectors $(X(t_1), \ldots, X(t_n)), t_i \in \mathbb{Z}^d, i \le n$ are for any $n \in \mathbb{N}$ regularly varying with index α. For such X, as shown in [24] there exists the so-called *spectral tail process* (STP) $\Theta(t), t \in \mathbb{Z}^d$ with $\Theta(0) = 1$ a.s. that satisfies the *time-change formula* of [24], which in our context reads

$$\mathbb{E}\Big\{|\Theta(-h)|^\alpha \, \Gamma(L^h\Theta)\Big\} \; = \mathbb{E}\Big\{\Gamma(\Theta)\Big\}, \quad h \in \mathbb{Z}^d \tag{6.16}$$

for any 0-homogeneous integrable functional $\Gamma : [0,\infty)^{\mathbb{Z}} \to \mathbb{R}$ that vanishes for $x \in [0,\infty)^{\mathbb{Z}}, x_0 = 0$. If $\zeta_Z(t), t \in \mathbb{R}^d$ is a max-stable process with marginals $\Phi_\alpha(x) = e^{-1/x^\alpha}, x > 0$, then clearly $(\zeta_Z(t_1), \ldots, \zeta_Z(t_n)), t_i \in \mathbb{Z}^d, i \le n$ are for all $n \in \mathbb{N}$ regularly varying with index α, and therefore $\zeta_Z(t), t \in \mathbb{Z}^d$ has a STP which we denote by Θ. Below we specify Θ in terms of Z utilising i) tilting (change of measure) for $V_0 \stackrel{fdd}{=} Z|(Z(0) > 0)$ and ii) the *tilt-shift formula* (6.17).

Theorem 6.9. *If $\zeta_Z(t), t \in \mathbb{R}^d$ is a stationary max-stable process with unit marginals $\Phi_\alpha, \alpha > 0$, then $\zeta_Z(t), t \in \mathbb{Z}^d$ has STP $\Theta(t) = \Xi_0 V_0(t) \ge 0, t \in \mathbb{Z}^d$ and for any 0-homogeneous functional $\Gamma : \mathcal{F}_0 \to [0,\infty)$ which is $\mathcal{B}(\mathcal{F}_0)/\mathcal{B}(\mathbb{R})$ measurable we have*

$$\mathbb{E}\Big\{Z^\alpha(-h)\Gamma(L^h Z)\Big\} = \mathbb{E}\big\{Z^\alpha(0)\Gamma(Z)\big\} = \mathbb{E}\Big\{\Gamma(\Theta)\Big\}, \quad h \in \mathbb{R}^d, \tag{6.17}$$

provided that the expectations exist.

By the above clearly the STP of ζ_Z is non-negative, and if $\mathbb{P}\{Z(h) > 0\} = 1, h \in \mathbb{Z}^d$, then

$$\Theta \stackrel{fdd}{=} Z.$$

Consequently, (6.17) reduces to (6.16). Note that $\Theta(t) = \Xi_0 V_0(t)$ is defined for any $t \in \mathbb{R}^d$ and satisfies (6.17) for all $t \in \mathbb{R}^d$, whereas the *time-change formula* of [24] is stated only for discrete stationary time series.

To this end, for $\zeta_Z(t), t \in \mathbb{R}$ max-stable and stationary as above, we give the formula for Pickands constant $\mathcal{H}_Z^1$ which is simply the extremal index of the stationary time series $\zeta_Z(t), t \in \mathbb{Z}$. If almost surely

$$\Theta(t) \to 0, \quad |t| \to \infty, \tag{6.18}$$

then by Theorem 5.1 the Pickands constant $\mathcal{H}_Z^1$ is given by

$$\mathcal{H}_Z^1 = \mathbb{E}\Big\{\frac{\max_{t \in \mathbb{Z}} |\Theta(t)|^\alpha}{\sum_{t \in \mathbb{Z}} |\Theta(t)|^\alpha}\Big\} \in (0,1]. \tag{6.19}$$

We write absolute values in the rhs of (6.19), since in view of [66] this formula also holds for the candidate extremal index of a multivariate regularly varying (with index α) time series with STP Θ, provided that (6.18) holds.

7. Proofs

Proof of Lemma 2.1 For any $k > 1, t_i \in \mathcal{T}, x_i \in \mathbb{R}, i \le n, x_{n+1} > 0$, the assumption $Y(h) = 0$ a.s. implies (set $c_k = 1/(1 - e^{-1/k}), K = \{1, \ldots, n+1\}, t_{n+1} = h, Y_i = Y(t_i)$)

$$\mathbb{P}\Big\{\forall j \in K : \zeta_Y(t_j) \le x_j + \ln k \Big| \zeta_Y(h) > \ln k\Big\}$$

$$
\begin{aligned}
&= c_k\Big[e^{-\int_{\mathbb{R}}\mathbb{P}\{\exists j\in K: Y_j>x_j+\ln k-y\}e^{-y}\,dy}-e^{-\int_{\mathbb{R}}\mathbb{P}\{Y_{n+1}>\ln k-y,\text{ or }\exists j\in K: Y_j>x_j+\ln k-y\}e^{-y}\,dy}\Big]\\
&= c_k\Big[e^{-\frac{1}{k}\int_{\mathbb{R}}\mathbb{P}\{\exists j\in K: Y_j>x_j-y\}e^{-y}\,dy}-e^{-\frac{1}{k}\int_{\mathbb{R}}\mathbb{P}\{Y_{n+1}>-y,\text{ or }\exists j\in K: Y_j>x_j-y\}e^{-y}\,dy}\Big]\\
&\to \int_{\mathbb{R}}\Big[\mathbb{P}\{y>0,\ \text{or }\exists j\in K: Y_j>x_j-y\}-\mathbb{P}\{\exists j\in K: Y_j>x_j-y\}\Big]e^{-y}\,dy,\quad k\to\infty\\
&= \int_{\mathbb{R}}\Big[\mathbb{P}\{\forall j\in K: Y_j\le x_j-y\}-\mathbb{P}\{y\le 0,\forall j\in K: Y_j\le x_j-y\}\Big]e^{-y}\,dy\\
&= \int_0^\infty \mathbb{P}\{\forall j\in K: Y_j\le x_j-y\}e^{-y}\,dy\\
&= \mathbb{P}\{\forall j\in K: Y_j+\mathcal{E}\le x_j\},
\end{aligned}
$$

where $\mathcal{E}$ has a unit exponential df being independent of Y. Consequently, since a.s. $Y_h=Y(h)=0$ we have the convergence in distribution as $k\to\infty$

$$
\Big(\zeta_Y(t_1)-\zeta_Y(h),\dots,\zeta_Y(t_n)-\zeta_Y(h),\zeta_Y(h)\Big)\Big|(\zeta_Y(h)>\ln k)\xrightarrow{d}(Y_1,\dots,Y_n,\mathcal{E}),\tag{7.1}
$$

hence the proof is complete. $\square$

Proof of Theorem 2.2 *i)* Let H be the df of $(\zeta_Z(t_1),\dots,\zeta_Z(t_n))$. For W with df G given in (2.3) set $(W_1^{(h)},\dots,W_n^{(h)})=W|(W_h>0)$. If $h\in\{t_1,\dots,t_n\}$, then by [30][Eq. (8.67)]

$$
\Big(\zeta_Z(t_1)-\ln k,\dots,\zeta_Z(t_n)-\ln k\Big)\Big|\Big(\zeta_Z(h)>\ln k\Big)\xrightarrow{d}(W_1^{(h)},\dots,W_n^{(h)}),\quad k\to\infty
$$

holds, see also [63][Eq. (13)]. (Note that $W_k^{(h)},k\neq h$ can be equal to $-\infty$ with non-zero probability). Hence the claim follows by (7.1).

ii) First note that $\mathbb{E}\{e^{Z(t)}\}=1,t\in\mathcal{T}$ implies for μ a probability measure on $\mathcal{T}$ (recall $\mathcal{S}_Z=\int_{\mathcal{T}}e^{Z(s)}\mu(ds)$)

$$
\mathbb{E}\{\mathcal{S}_Z\}=\int_{\mathcal{T}}\mathbb{E}\{e^{Z(s)}\}\mu(ds)=1
$$

and a.s. $\mathcal{S}_Z<\infty$, which is assumed to hold if μ is a positive σ-finite measure. Since a.s. $Z(h)>-\infty,h\in\mathcal{T}$, then a.s. $\mathcal{S}_Z\in(0,\infty)$. Hence, for any $h,t_i\in\mathcal{T},x_i\in\mathbb{R},i\le n$ by Fubini-Tonelli theorem

$$
\begin{aligned}
-\ln H(x) &= \mathbb{E}\Big\{\frac{\mathcal{S}_Z}{\mathcal{S}_Z}\max_{1\le k\le n}e^{-x_k+Z(t_k)}\Big\}\\
&= \int_{\mathcal{T}}\mathbb{E}\Big\{e^{Z(h)}\frac{\max_{1\le k\le n}e^{-x_k+Z(t_k)-Z(h)}}{\int_{\mathcal{T}}e^{Z(s)-Z(h)}\mu(ds)}\Big\}\mu(dh)\\
&= \int_{\mathcal{T}}\mathbb{E}\Big\{\max_{1\le k\le n}e^{-x_k+\Xi_h Z(t_k)-\ln\left(\int_{\mathcal{T}}e^{\Xi_h Z(s)}\mu(ds)\right)}\Big\}\mu(dh).
\end{aligned}\tag{7.2}
$$

Now, if $\sum_{i\ge1}\varepsilon_{(P_i,T_i,Z_i)}$ is a PPP on $\mathbb{R}\times\mathcal{T}\times\mathbb{R}^{\mathcal{T}}$ with intensity $e^{-p}dp\cdot\mu(dt)M(dz)$ where M is the law of Z, then setting

$$
\eta(t)=\max_{i\ge1}\left(P_i-\ln\int_{\mathcal{T}}e^{\Xi_{T_i}Z_i(s)-\Xi_{T_i}Z_i(t)}\,\mu(ds)\right)
$$

we have

$$-\ln \mathbb{P}\{\eta(t_i) \le x_i, 1 \le i \le n\} \;=\; \mathbb{E}\Big\{ \int_{\mathcal{T}} \max_{1\le k\le n} e^{-x_k + \Xi_h Z(t_k) - \ln\left(\int_{\mathcal{T}} e^{\Xi_h Z(s)} \mu(ds)\right)} \, \mu(dh) \Big\},$$

hence the claim follows. □

PROOF OF THEOREM 2.5 $a)$ implies $d)$:
Let $t_i \in \mathcal{T}, i \le n$ be given. As mentioned in the Introduction for the validity of (1.5), since a.s. both $Z(a) > -\infty$ and $Z(a+h) > -\infty$ hold, then we have

$$\zeta_Z \stackrel{fdd}{=} \zeta_{\Xi_a Z} \stackrel{fdd}{=} \zeta_{L^h \Xi_a Z} \stackrel{fdd}{=} \zeta_{\Xi_{a+h} Z}.$$

By our definition in (2.7)

$$L^h \Xi_a Z(t) = L^h(Z^{[a]}(t) - Z^{[a]}(a)) = Z^{[a]}(t-h) - Z^{[a]}(a), \quad t \in \mathcal{T}$$

and (1.7) we have a.s.

$$L^h \Xi_a Z(a+h) = \Xi_{a+h} Z(a+h) = 0.$$

and thus statement d) follows by Lemma 2.1.
$d)$ implies $c)$:
First note that by the shift-invariance of Γ

$$\mathbb{E}\{\Gamma(\Xi_{a+h} Z)\} \;=\; \mathbb{E}\big\{e^{Z(a+h)} \Gamma\big(Z - Z(a+h)\big)\big\} = \mathbb{E}\big\{e^{Z(a+h)} \Gamma(Z)\big\}$$

and

$$\mathbb{E}\big\{\Gamma(L^h \Xi_a Z)\big\} \;=\; \mathbb{E}\big\{e^{Z(a)} \Gamma\big(L^h Z - Z(a)\big)\big\} = \mathbb{E}\big\{e^{Z(a)} \Gamma(L^h Z)\big\}.$$

By statement $d)$ we have that the fidi's of $\Xi_{a+h} Z$ and $L^h \Xi_a Z$ are the same, which together with the measurability of Γ implies $\Gamma\big(\Xi_{a+h} Z\big) \stackrel{d}{=} \Gamma\big(L^h \Xi_a Z\big)$. Consequently, we obtain

$$\mathbb{E}\big\{\Gamma\big(\Xi_{a+h} Z\big)\big\} \;=\; \mathbb{E}\big\{\Gamma\big(L^h \Xi_a Z\big)\big\} = \mathbb{E}\big\{e^{Z(a+h)} \Gamma(Z)\big\} = \mathbb{E}\big\{e^{Z(a)} \Gamma(L^h Z)\big\}$$

and thus the claim follows.
$c)$ implies $b)$:
If μ is a probability measure on $\mathcal{T}$, for any $h, t_i \in \mathcal{T}, x_i \in \mathbb{R}, i \le n$ by (2.9), Fubini-Tonelli theorem, statement $c)$ and (7.2) yield

$$\begin{aligned} -\ln \mathbb{P}\{\zeta_Z(t_i) \le x_i, i = 1, \ldots, n\} \;&=\; \int_{\mathcal{T}} \mathbb{E}\Big\{ e^{Z(h)} \frac{\max_{1\le k\le n} e^{-x_k + Z(t_k)}}{\int_{\mathcal{T}} e^{Z(s)} \mu(ds)} \Big\} \mu(dh) \\ &=: \int_{\mathcal{T}} \mathbb{E}\big\{e^{Z(h)} \Gamma(Z)\big\} \, \mu(dh) \\ &= \int_{\mathcal{T}} \mathbb{E}\big\{\Gamma(L^h \Xi_0 Z)\big\} \, \mu(dh) \\ &= \mathbb{E}\big\{\Gamma(L^T \Xi_0 Z)\big\}, \end{aligned}$$

with T a copy of T_1, which is independent of $\Xi_0 Z$ (recall $0 = (0, \ldots, 0)$). Consequently,

$$-\ln \mathbb{P}\{\zeta_Z(t_i) \le x_i, i = 1, \ldots, n\} \;=\; \mathbb{E}\Big\{ \max_{1\le k\le n} e^{-x_k + L^T \Xi_0 Z(t_k) - \ln \int_{\mathcal{T}} e^{L^T \Xi_0 Z(s)} \mu(ds)} \Big\}.$$

The case that μ is a positive σ-finite measure on $\mathcal{T}$ such that (2.2) holds follows with similar arguments.

b) implies a):

Let μ be the Dirac measure at h. We have that ζ_Z has the same law as $\max_{i\ge1}(P_i+\Xi_0 Z_i(t-h)), t\in\mathcal{T}$ and by (1.5), this implies that $\zeta_Z \stackrel{fdd}{=} \zeta_{L^h Z}$, hence ζ_Z is stationary. □

Proof of Theorem 2.7 Let $\varphi(t)=\ln \mathbb{E}\{e^{(t,X)}\}, t\in\mathbb{R}^d$ denote the cumulant generating function of X. As shown in [36][Th. 1] ζ_Z is stationary in $\mathbb{R}^d$ i.e., in our setup $Z(t)=(t,X)-\kappa(t), t\in\mathbb{R}^d$ is Brown-Resnick stationary, if and only if X is Gaussian with mean μ, covariance matrix Σ and further $\kappa(t)=\varphi(t), t\in\mathbb{R}^d$. Our assumption is slightly weaker since we assume the stationarity of ζ_Z restricted on $\delta\mathbb{Z}^d$ for any $\delta>0$ and not its stationarity on $\mathbb{R}^d$.

As in the proof of Theorem 1 therein, our assumption is equivalent with

$$\varphi\Big(\sum_{i=1}^n u_i t_i\Big)-\sum_{i=1}^n u_i\varphi(t_i)=\varphi\Big(h+\sum_{i=1}^n u_i t_i\Big)-\sum_{i=1}^n u_i\varphi(h+t_i) \tag{7.3}$$

for any $h,t_i\in\delta\mathbb{Z}^d, u_i\in\mathbb{R}, i\le n$ where $\sum_{i=1}^n u_i=1$. Write next $[a]=([a_1],\dots,[a_d])$ for any $a\in\mathbb{R}^d$ with $[a_i]$ the largest integer smaller than a_i. By (7.3) for any $\lambda\in(0,1),\delta>0$ and any $t_0,t_1,t_2\in\mathbb{R}^d$ (set $h=\delta t_0, z_i=\delta[t_i/\delta], v=\lambda z_1+(1-\lambda)z_2$)

$$\varphi(v)-\lambda\varphi(z_1)-(1-\lambda)\varphi(z_2) \;=\; \varphi(h+v)-\lambda\varphi(h+z_1)-(1-\lambda)\varphi(h+z_2).$$

Letting $\delta\to0$ we obtain

$$\nabla\varphi\big(\lambda t_1+(1-\lambda)t_2\big)=\lambda\nabla\varphi(t_1)+(1-\lambda)\nabla\varphi(t_2),$$

where $\nabla\varphi(t)$ denotes the gradient of φ at $t\in\mathbb{R}^d$. Consequently, with the same arguments as in [36] it follows that X is Gaussian with mean μ and covariance matrix Σ and further $\kappa(t)=\varphi(t), t\in\mathbb{R}^d$. Hence the proof follows by Remark 2.6 ii). □

Proof of Theorem 2.8 If B has stationary increments, then by (2.6)

$$\Xi_h Z \stackrel{fdd}{=} L^h Z, \quad \forall h\in\mathcal{T}, \tag{7.4}$$

hence statement d) in Theorem 2.5 implies that ζ_Z is stationary. Note that (7.4) is previously shown in [6, Prop. 2]. Conversely, if ζ_Z is stationary, then by statement d) in Theorem 2.5 we have that (7.4) holds, which combined with (6.2) yields for any $h\in\mathcal{T}$

$$B(t-h)-\sigma^2(t-h)/2 \stackrel{fdd}{=} B(t)-B(h)-Var(B(t)-B(h))/2, \quad t\in\mathcal{T},$$

hence B has stationary increments and thus the claim follows. □

Proof of Theorem 3.1 The proof is based on the result of Theorem 2.5, which is also valid for $\mathcal{T}=[0,\infty)$. First, we show that Y can be defined using the Kolmogorov's consistency theorem, see e.g., [67][Th. 1.1]. It suffices to consider in the following only $t_1<\dots<t_n\in\mathbb{R}$ such that $t_1<0$. For any permutation π of $t_1,\dots,t_n$ we have that

$$Y_{\pi(t_1),\dots,\pi(t_n)}=\Big(Y(\pi(t_1)),\dots,Y(\pi(t_n))\Big)\stackrel{d}{=}\Big(\Xi_h(\pi(t_1)),\dots,\Xi_h(\pi(t_n))\Big), \quad h=-t_1$$

implying $Y_{\pi(t_1),\ldots,\pi(t_n)} \stackrel{d}{=} Y_{t_1,\ldots,t_n}$ since h is independent of the chosen permutation.
The consistency of the family of fidi's follows if we can further show that for any non-empty $I \subset \{1,\ldots,n\}$ (write $J := \{1,\ldots,n\} \setminus I$)

$$\mathbb{P}\{Y(t_i) \le x_i, i \in I, Y(t_j) \in \mathbb{R}, j \in J\} \;=\; \mathbb{P}\{Y(t_i) \le x_i, i \in I\}$$

for any $x_i \in \mathbb{R}, i \in I$. If $1 \in I$, the above follows immediately by the definition of Y. Suppose next that $1 \in J$ and assume for notation simplicity that $J = \{1\}$. We need to show that

$$\begin{aligned}\mathbb{P}\{Y(t_1) \in \mathbb{R}, Y(t_i) \le x_i, 2 \le i \le n\} &= \mathbb{P}\{L^{t_1}\Xi_{-t_1}Z(t_i) \le x_i, 2 \le i \le n\}\\ &= \mathbb{P}\{L^{t_2}\Xi_{-t_2}Z(t_i) \le x_i, 2 \le i \le n\}\\ &= \mathbb{P}\{Y(t_i) \le x_i, 2 \le i \le n\}\end{aligned}$$

for any $x_i \in \mathbb{R}, 2 \le i \le n$, which follows directly by (2.9). Hence since the conditions of [67][Th. 1.1] are satisfied, then $Y(t), t \in \mathbb{R}$ exists. By (3.1) for any $t \in \mathbb{R}$

$$Y(t) \stackrel{d}{=} \Xi_{-t}Z(0) = Z^{[-t]}(0) - Z^{[-t]}(-t)$$

implying that $\mathbb{E}\{e^{Y(t)}\} = 1$, hence $\zeta_Y(t), t \in \mathbb{R}$ associated to Y (as in (1.1) with Z substituted by Y and $\mathcal{T} = \mathbb{R}$) has unit Gumbel marginals and is max-stable. The stationarity of ζ_Y follows easily, we omit the proof. □

Proof of Lemma 4.1 For notational simplicity write J, V, W instead of J_h, V_h, W_h. Since J is independent of V and a.s. $W(h) = -\infty$, the assumption that $\mathbb{E}\{e^{Z(h)}\} = 1$ implies (recall $0 \cdot \infty = 0$ and set $p = \mathbb{P}\{J = 1\}$)

$$1 = \mathbb{E}\{e^{Z(h)}\} = \mathbb{E}\{e^{JV(h)+(1-J)W(h)}\} = \mathbb{E}\{\mathbb{I}\{J=1\}e^{JV(h)}\} = \mathbb{E}\{e^{V(h)+\ln p}\}. \tag{7.5}$$

For any $t_1 = h, \ldots, t_n \in \mathcal{T}, x \in \mathbb{R}^n$ we have (set $\bar{J} = 1 - J$)

$$\begin{aligned}-\ln H(x) &= \mathbb{E}\Big\{\max_{1\le j\le n} e^{JV(t_j)+\bar{J}W(t_j)-x_j}\Big\}\\ &= \mathbb{E}\Big\{e^{V(h)+\ln p}\max_{1\le j\le n} e^{J[V(t_j)-V(h)]+\bar{J}[W(t_j)-V(h)]-\ln p-x_j}\Big\}\\ &= \mathbb{E}\Big\{\max_{1\le j\le n} e^{J[V^{[h]}(t_j)-V^{[h]}(h)]+\bar{J}[W(t_j)-V^{[h]}(h)]-\ln p-x_j}\Big\},\end{aligned}$$

where we used the fact that J, V, W are mutually independent. Hence we have $\zeta_Z \stackrel{fdd}{=} \zeta_{\Xi_h Z}$.
Next, consider the PPP $\sum_{i\in N}\varepsilon_{(P_i,T_i,Z_i)}$ on $\mathbb{R}\times\mathcal{T}\times\mathbb{R}^{\mathcal{T}}$. Using the void probability formula and $\zeta_Z \stackrel{fdd}{=} \zeta_{\Xi_h Z}$, for any $t_i \in \mathcal{T}, x_i \in \mathbb{R}, i \le n$ we obtain

$$\begin{aligned}-\ln\mathbb{P}\{\eta(t_i) \le x_i, 1 \le i \le n\} &= \int_{\mathcal{T}} -\ln\mathbb{P}\{\zeta_{\Xi_h Z}(t_i) \le x_i, 1 \le i \le n\}\mu(dh)\\ &= -\int_{\mathcal{T}} \ln\mathbb{P}\{\zeta_Z(t_i) \le x_i, 1 \le i \le n\}\mu(dh)\\ &= -\ln\mathbb{P}\{\zeta_Z(t_i) \le x_i, 1 \le i \le n\},\end{aligned}$$

establishing the proof. □

PROOF OF THEOREM 4.2 Define next for a random process $X(s), s \in \mathcal{T}$

$$Q_{t,x}(X) = \inf \operatorname{argmax}_{1 \le j \le n} e^{X(t_j)-x_j}, \quad t=(t_1,\ldots,t_n) \in \mathcal{T}^n, \quad x \in \mathbb{R}^n.$$

Hereafter write $\mathbb{E}\{K;B\} := \mathbb{E}\{K\mathbb{I}\{B\}\}$ for K some random element and B an event. Recall that $W_h(h) = -\infty$ a.s. and set

$$Y_h(t) := J_h V_h(t) + (1-J_h) W_h(t) \stackrel{fdd}{=} Z(t), \quad h \in \mathcal{T}. \tag{7.6}$$

For any $x \in \mathbb{R}^n$ we have (recall that a.s. $W_h(h) = -\infty$ and the indicator rv's J_{t_k} are independent of V_{t_k}, W_{t_k})

$$\begin{aligned}
&-\ln H(x)\\
&= \mathbb{E}\left\{\sum_{k=1}^n \mathbb{I}\{Q_{t,x}(Z)=k\} \max_{1\le j\le n} e^{Z(t_j)-x_j}\right\}\\
&= \sum_{k=1}^n e^{-x_k}\Big[\mathbb{E}\Big\{\mathbb{I}\{Q_{t,x}(Y_{t_k})=k\}e^{Y_{t_k}(t_k)}; J_{t_k}=1\Big\} + \mathbb{E}\Big\{\mathbb{I}\{Q_{t,x}(Y_{t_k})=k\}e^{Y_{t_k}(t_k)}; J_{t_k}=0\Big\}\Big]\\
&= \sum_{k=1}^n \mathbb{P}\{J_{t_k}=1\}\mathbb{E}\Big\{\mathbb{I}\{Q_{t,x}(V_{t_k})=k\}e^{V_{t_k}(t_k)}\Big\}\\
&= \sum_{k=1}^n e^{-x_k}\mathbb{E}\Big\{\mathbb{I}\{Q_{t,x}(V_{t_k})=k\}e^{V_{t_k}(t_k)+\ln \mathbb{P}\{J_{t_k}=1\}}\Big\}\\
&= \sum_{k=1}^n e^{-x_k}\mathbb{E}\{\mathbb{I}\{Q_{t,x}(V_{t_k}^{[t_k]})=k\}\} \qquad (7.7)\\
&= \sum_{k=1}^n e^{-x_k}\mathbb{P}\Big\{\max_{1\le i<k}\Big(V_{t_k}^{[t_k]}(t_i)-x_i\Big) < V_{t_k}^{[t_k]}(t_k)-x_k, \max_{n\ge i>k}\Big(V_{t_k}^{[t_k]}(t_i)-x_i\Big) \le V_{t_k}^{[t_k]}(t_k)-x_k\Big\}\\
&= \sum_{k=1}^n e^{-x_k}\mathbb{P}\Big\{\inf \operatorname{argmax}_{1\le i\le n}\Big(\Theta_{t_k}(t_i)-x_i\Big)=k\Big\},
\end{aligned}$$

where $\Theta_h(t) := \Xi_h V_h = V_h^{[h]}(t) - V_h^{[h]}(h)$ and $\max_{1\le j<1}(\cdot) = \max_{n\ge i>n}(\cdot) =: -\infty$, hence the claim follows. ☐

PROOF OF THEOREM 4.3 $a)$ implies $c)$:

Let $\Xi_h Z$ be given from (4.2). As in the proof of Theorem 2.5 for any $a,h \in \mathcal{T}$ we have $\zeta_{\Xi_{a+h}Z} \stackrel{fdd}{=} \zeta_{L^h \Xi_a Z}$, hence by Lemma 8.1 in Appendix

$$\Theta_{a+h} := \Xi_{a+h}V_{a+h} \stackrel{fdd}{=} L^h \Xi_a V_a =: L^h \Theta_a$$

establishing the claim.

$c)$ implies $b)$:

First note that by the shift-invariance of Γ, for any $h \in \mathcal{T}$ we have

$$\Gamma(V_h^{[h]}) = \Gamma(V_h^{[h]} - V^{[h]}(h)) = \Gamma(\Theta_h).$$

Further, since J_h is independent of V_h and W_h, (recall $W_h(h) = -\infty$ a.s.), then using the shift-invariance of Γ yields

$$\begin{aligned}\mathbb{E}\Big\{e^{Z(h)}\Gamma(Z)\Big\} &= \mathbb{E}\Big\{e^{J_h V_h(h)+(1-J_h)W_h(h)}\Gamma(J_h V_h + (1-J_h)W_h)\Big\}\\ &= \mathbb{E}\Big\{e^{V_h(h)+\ln \mathbb{P}\{J_h=1\}}\Gamma(V_h)\Big\} = \mathbb{E}\{\Gamma(V_h^{[h]})\} = \mathbb{E}\{\Gamma(\Theta_h)\}.\end{aligned}$$

Consequently, since $\Theta_{a+h} \stackrel{fdd}{=} L^h\Theta_a$ is valid for any $a, a+h \in \mathcal{T}$, then

$$\mathbb{E}\Big\{e^{Z(a+h)}\Gamma(Z)\Big\} = \mathbb{E}\Big\{\Gamma(\Theta_{a+h})\Big\} = \mathbb{E}\Big\{\Gamma(L^h\Theta_a)\Big\} = \mathbb{E}\Big\{e^{Z(a)}\Gamma(L^h Z)\Big\}$$

establishing the claim.

$b)$ implies $a)$:

Given $t_1, \ldots, t_n \in \mathcal{T}$ distinct and $h \in \mathcal{T}$ by (7.7) for any $x \in \mathbb{R}^n$

$$-\ln H(x) = -\ln \mathbb{P}\{\zeta_Z(t_i) \le x_i, 1 \le i \le n\} = \sum_{k=1}^n e^{-x_k}\mathbb{E}\Big\{\Gamma_k(\Theta_{t_k})\Big\},$$

where

$$\Gamma_k(\Theta_{t_k}) := \mathbb{I}\Big\{\inf \operatorname{argmax}_{1\le i\le n}\big(\Theta_{t_k}(t_i) - x_i\big) = k\Big\}.$$

The functional Γ_k is shift-invariant, hence from statement $b)$

$$\mathbb{E}\{\Gamma_k(\Theta_{t_k})\} = \mathbb{E}\Big\{\Gamma_k(L^h\Theta_{t_k-h})\Big\}$$

implying (set $t_i^* = t_i - h$)

$$\begin{aligned}-\ln H(x) &= \sum_{k=1}^n e^{-x_k}\mathbb{E}\Big\{\Gamma_k(L^h\Theta_{t_k-h})\Big\}\\ &= \sum_{k=1}^n e^{-x_k}\mathbb{E}\Big\{\mathbb{I}\Big\{\inf \operatorname{argmax}_{1\le i\le n}\big(\Theta_{t_k-h}(t_i-h) - x_i\big) = k\Big\}\Big\}\\ &= \sum_{k=1}^n e^{-x_k}\mathbb{E}\{\Gamma_k(\Theta_{t_k^*})\} = -\ln \mathbb{P}\{\zeta_Z(t_i - h) \le x_i, 1 \le i \le n\},\end{aligned}$$

which proves the stationarity of ζ_Z, hence the claim follows. ☐

The next result extends Lemma 2 in [23] formulated for the case $d = 1$ under the assumption that a.s. $Z(0) = 0$.

Lemma 7.1. *Let Z be as in Theorem 4.3. If μ is the counting measure on $(k\delta)\mathbb{Z}^d, d \ge 1$ with $k \in \mathbb{N}, \delta > 0$, then for any $T > 0$*

$$\mathbb{E}\Big\{\max_{t\in\delta\mathbb{Z}^d\cap[0,T]^d} e^{Z(t)}\Big\} = T^d \int_{[0,1]^d} \mathbb{E}\left\{\frac{\max_{t\in\delta\mathbb{Z}^d\cap\times_{i=1}^d[-h_iT,(1-h_i)T]} e^{\Theta(t)}}{\int_{\delta\mathbb{Z}^d\cap\times_{i=1}^d[-h_iT,(1-h_i)T]} e^{\Theta(s)}\mu(ds)}\right\}\mu^T(dh), \tag{7.8}$$

with $\Theta = \Xi_0 V_0, \mu^T(dh) = \mu(Tdh)/T^d$ and $h = (h_1, \ldots, h_d)$.

Proof of Lemma 7.1 Since μ is a counting measure on $(k\delta)\mathbb{Z}^d \cap [0,T]$, then (set $\mathcal{E}_\delta := \delta\mathbb{Z}^d \cap [0,T]^d$)

$$\mathbb{P}\Big\{\int_{\mathcal{E}_\delta} e^{Z(s)}\mu(ds) = 0, \max_{t\in\mathcal{E}_\delta} e^{Z(t)} > 0\Big\} = 0.$$

Consequently, with Y_h defined in (7.6) we have (recall $W_h(h) = -\infty$ a.s.)

$$\begin{aligned}
\mathbb{E}\Big\{\max_{t\in\mathcal{E}_\delta} e^{Z(t)}\Big\} &= \int_{[0,T]^d} \mathbb{E}\Big\{ e^{Z(h)} \frac{\max_{t\in\mathcal{E}_\delta} e^{Z(t)}}{\int_{\mathcal{E}_\delta} e^{Z(s)}\mu(ds)}; \max_{t\in\mathcal{E}_\delta} e^{Z(t)} > 0 \Big\}\mu(dh) \\
&= \int_{[0,T]^d} \mathbb{E}\Big\{ e^{Y_h(h)} \frac{\max_{t\in\mathcal{E}_\delta} e^{Y_h(t)}}{\int_{\mathcal{E}_\delta} e^{Y_h(s)}\mu(ds)}; \max_{t\in\mathcal{E}_\delta} e^{Y_h(t)} > 0 \Big\}\mu(dh) \\
&= \int_{[0,T]^d} \Big[\mathbb{E}\Big\{ e^{Y_h(h)} \frac{\max_{t\in\mathcal{E}_\delta} e^{Y_h(t)}}{\int_{\mathcal{E}_\delta} e^{Y_h(s)}\mu(ds)}; \max_{t\in\mathcal{E}_\delta} e^{Y_h(t)} > 0; J_h = 1 \Big\} \\
&= \int_{[0,T]^d} \mathbb{E}\Big\{ e^{V_h(h)} \frac{\max_{t\in\mathcal{E}_\delta} e^{V_h(t)-V_h(h)}}{\int_{\mathcal{E}_\delta} e^{V_h(s)-V_h(h)}\mu(ds)}; J_h = 1 \Big\}\mu(dh) \\
&= \int_{[0,T]^d} \mathbb{E}\Big\{ e^{V_h(h)-\ln \mathbb{E}\{V_h(h)\}} \frac{\max_{t\in\mathcal{E}_\delta} e^{V_h(t)-V_h(h)}}{\int_{\mathcal{E}_\delta} e^{V_h(s)-V_h(h)}\mu(ds)} \Big\}\mu(dh) \\
&= \int_{[0,T]^d} \mathbb{E}\Big\{ \frac{\max_{t\in\mathcal{E}_\delta} e^{\Theta_h(t)}}{\int_{\mathcal{E}_\delta} e^{\Theta_h(s)}\mu(ds)} \Big\}\mu(dh) \\
&= \int_{[0,T]^d} \mathbb{E}\Big\{ \frac{\max_{t\in\mathcal{E}_\delta} e^{\Theta(t-h)}}{\int_{\mathcal{E}_\delta} e^{\Theta(s-h)}\mu(ds)} \Big\}\mu(dh),
\end{aligned}$$

where the last equality follows from $\Theta_h \stackrel{fdd}{=} L^h\Theta = L^h\Theta_0$, which is a consequence of Theorem 4.3. Alternatively, using directly (4.4) and omitting few details, we obtain

$$\begin{aligned}
\mathbb{E}\Big\{\max_{t\in\mathcal{E}_\delta} e^{Z(t)}\Big\} &= \int_{[0,T]^d} \mathbb{E}\Big\{ e^{Z(h)} \frac{\max_{t\in\mathcal{E}_\delta} e^{Z(t)}}{\int_{\mathcal{E}_\delta} e^{Z(s)}\mu(ds)}; \max_{t\in\mathcal{E}_\delta} e^{Z(t)} > 0 \Big\}\mu(dh) \\
&=: \ \mathbb{E}\big\{ e^{Z(h)}\Gamma(Z)\big\} = \int_{[0,T]^d} \mathbb{E}\Big\{ \frac{\max_{t\in\mathcal{E}_\delta} e^{L^h\Theta(t)}}{\int_{\mathcal{E}_\delta} e^{L^h\Theta(s)}\mu(ds)} \Big\}\mu(dh).
\end{aligned}$$

Hence the claim follows using further the fact that μ is translation invariant. □

Remark 7.2. *Suppose that $Z(t), t \in \mathbb{R}^d$ has cadlag sample paths. If for any compact $K \subset \mathbb{R}^d$ we have that $\mathbb{E}\{\sup_{t\in K} e^{Z(t)}\} < \infty$, then as in the proof above it follows that for any $T > 0$*

$$\mathbb{E}\Big\{\sup_{t\in[0,T]^d} e^{Z(t)}\Big\} = T^d \int_{[0,1]^d} \mathbb{E}\left\{ \frac{\sup_{\times_{i=1}^d [-h_iT,(1-h_i)T]} e^{\Theta(t)}}{\int_{\times_{i=1}^d [-h_iT,(1-h_i)T]} e^{\Theta(s)}\mu(ds)} \right\} \mu^T(dh),$$

with $\mu^T(dh) = \lambda(Tdh)/T^d$ where λ is the Lebesgue measure on $\mathbb{R}^d$.

Proof of Theorem 5.1 *i)* With the same notation as in (5.4), the assumption (5.1) implies for any $\delta > 0$ that

$$\lim_{T\to\infty} \frac{\eta_T(h)}{\delta^d} = \mathbb{E}\left\{ \frac{\max_{t\in\delta\mathbb{Z}^d} e^{\Theta(t)}}{\delta^d \int_{\mathbb{R}^d} e^{\Theta(s)}\mu_\delta(ds)} \right\} < \infty,$$

where μ_δ denotes the counting measure on $\delta\mathbb{Z}^d$. Since $\delta^d\mu^T(dh)=\delta^d\mu_\delta(Tdh)/T^d$ converges weakly as $T\to\infty$ to the Lebesgue measure $\lambda(dh)$, then by (5.4), as in [23] we obtain

$$\lim_{T\to\infty}\int_{[0,1]^d}\eta_T(h)\mu^T(dh) \;=\; \mathbb{E}\left\{\frac{\max_{t\in\delta\mathbb{Z}^d}e^{\Theta(t)}}{\delta^d\int_{\mathbb{R}^d}e^{\Theta(s)}\mu_\delta(ds)}\right\}>0,$$

hence the first claim follows. Next, let $\Gamma_k(f)=\mathbb{I}\{\inf \operatorname{argmax}_{s\in\delta\mathbb{Z}^d\cap[0,T]^d}f(s)=k\}$ for some $k\in\delta\mathbb{Z}^d$. Clearly, $\Gamma_k(f+c)=\Gamma_k(f)$ for any constant c. Applying Lemma 7.1 we obtain (below we set $u=(u_1,\ldots,u_d)$ and $1-u=(1-u_1,\ldots,1-u_d)$)

$$\begin{aligned}
\frac{1}{T^d}\mathbb{E}\Big\{\max_{t\in\delta\mathbb{Z}^d\cap[0,T]^d}e^{Z(t)}\Big\} &= \frac{1}{T^d}\sum_{k\in\delta\mathbb{Z}^d\cap[0,T]^d}\mathbb{E}\big\{e^{Z(k)}\Gamma_k(Z)\big\}\\
&= \frac{1}{T^d}\sum_{k\in\delta\mathbb{Z}^d\cap[0,T]^d}\mathbb{E}\big\{\Gamma_k(L^k\Theta))\big\} \qquad (7.9)\\
&= \frac{1}{T^d}\sum_{k\in\delta\mathbb{Z}^d\cap[0,T]^d}\mathbb{P}\Big\{\max_{s\in\delta\mathbb{Z}^d\cap[0,T]^d}\Theta(s-k)=0\Big\} \qquad (7.10)\\
&= \frac{1}{\delta^d}\int_{[0,1]^d}\mathbb{P}\Big\{\max_{t\in\delta\mathbb{Z}^d\cap[-uT,(1-u)T]}\Theta(t)=0\Big\}\delta^d\mu^T(du)\\
&\to \frac{1}{\delta^d}\int_{[0,1]^d}\mathbb{P}\Big\{\max_{t\in\delta\mathbb{Z}^d}\Theta(t)\le 0\Big\}\lambda(du),\quad T\to\infty \qquad (7.11)\\
&= \frac{1}{\delta^d}\mathbb{P}\Big\{\max_{t\in\delta\mathbb{Z}^d}\Theta(t)=0\Big\},
\end{aligned}$$

where (7.9) follows by statement $c)$ of Theorem 4.3 and (7.10) is a consequence of the assumption that Θ has absolutely continuous fidi's. Note that by (5.1) we have the almost sure convergence $\max_{\|t\|>T,t\in\delta\mathbb{Z}^d}\Theta(t)\overset{a.s.}{\to}-\infty$ as $T\to\infty$, which implies the convergence in probability

$$\max_{t\in\delta\mathbb{Z}^d\cap[-u_TT,(1-u_T)T]}\Theta(t)\overset{p}{\to}\max_{t\in\delta\mathbb{Z}^d}\Theta(t)\ge\Theta(0)=0,\quad T\to\infty$$

for any u_T such that $\lim_{T\to\infty}u_T=u\in(0,1)^d$ and thus

$$\lim_{T\to\infty}\mathbb{P}\Big\{\max_{t\in\delta\mathbb{Z}^d\cap[-u_TT,(1-u_T)T]}\Theta(t)\le 0\Big\}=\mathbb{P}\Big\{\max_{t\in\delta\mathbb{Z}^d}\Theta(t)\le 0\Big\}.$$

Since further $\delta^d\mu^T(dh)$ converges weakly as $T\to\infty$ to $\lambda(dh)$, then (7.11) is justified from the validity of (8.4) below.

$ii)$ If $\delta=0$, then by Remark 7.2 the proof follows using further Lemma 8.2. □

Proof of Lemma 6.1 If a.s. $Z(h)=0$, then the claim is clear. Suppose therefore that $Z(h)$ has positive variance $\sigma^2(h)>0$. For any distinct $t_1=h,t_2,\ldots,t_n\in\mathcal{T}$ the df of $Z^{[h]}=(Z^{[h]}(t_1),\ldots,Z^{[h]}(t_n))$ denoted by F_h is specified by

$$F_h(dx) \;=\; F(dx)e^{x_1-\varphi(h)},\quad x\in\mathbb{R}^n, \qquad (7.12)$$

where F is the df of $Z=(Z(t_1),\ldots,Z(t_n))$. For any $a\in\mathbb{R}^n$ the df of the rv $(a,Z^{[h]})$ is obtained by tilting the Gaussian rv (a,Z). Hence from here it follows that $Z^{[h]}$ is Gaussian with the same covariance matrix as Z. We calculate next $\mathbb{E}\{Z^{[h]}(t)\}$. For any $t \in \mathcal{T}$

$$\mathbb{E}\{Z^{[h]}(t)\} = \mathbb{E}\{e^{Z(h)-r(h,h)/2}Z(t)\} = \mathbb{E}\{e^{Z(h)-r(h,h)/2}\}[r(t,h)+\mathbb{E}\{Z(t)\}] = r(t,h)+\mathbb{E}\{Z(t)\},$$

where the second equality follows by Stein's Lemma, see e.g., (3.4) in [68]. The converse follows easily by (7.12) and is therefore omitted. □

Proof of Theorem 6.5 Define $\mathcal{M}_Z = \sup_{t\in\mathcal{T}} e^{Z(t)}, \mathcal{S}_Z = \int_{\mathcal{T}} e^{Z(s)}\mu(ds)$, which by our assumption are rv's. Since $\mathbb{E}\{e^{Z(t)}\} = 1, t \in \mathcal{T}$, then a.s. $\mathcal{M}_Z > 0$. For Y_h defined in (7.6) being both measurable and separable, by (6.5)

$$\mathbb{P}\{\mathcal{M}_{Y_h} > 0, \mathcal{S}_{Y_h} = 0\} = \mathbb{P}\{\mathcal{M}_Z > 0, \mathcal{S}_Z = 0\} = 0. \tag{7.13}$$

We can assume without loss of generality that $\mathbb{P}\{J_h = 1\} > 0$. Given distinct $t_i \in \mathcal{T}, i \le n$, using (7.13) together with the fact that a.s. $V_h(h) > -\infty$ and $W_h(h) = -\infty$, for any $x \in \mathbb{R}^n$ we have

$$\begin{aligned}
&-\ln H(x)\\
&= \mathbb{E}\left\{\frac{\mathcal{S}_Z}{\mathcal{S}_Z}\max_{1\le k\le n} e^{-x_k+Z(t_k)}; \mathcal{M}_Z > 0\right\}\\
&= \int_{\mathcal{T}} \mathbb{E}\left\{\frac{e^{Z(h)}}{\mathcal{S}_Z}\max_{1\le k\le n} e^{-x_k+Z(t_k)}; \mathcal{M}_Z > 0\right\}\mu(dh)\\
&= \int_{\mathcal{T}} \mathbb{E}\left\{\frac{e^{Y_h(h)}}{\mathcal{S}_{Y_h}}\max_{1\le k\le n} e^{-x_k+Y_h(t_k)}; \mathcal{M}_{Y_h} > 0\right\}\mu(dh)\\
&= \int_{\mathcal{T}} \mathbb{E}\left\{\frac{e^{Y_h(h)}}{\mathcal{S}_{Y_h}}\max_{1\le k\le n} e^{-x_k+Y_h(t_k)}; \mathcal{M}_{Y_h} > 0, J_h = 1\right\}\mu(dh)\\
&= \int_{\mathcal{T}} \mathbb{E}\left\{\frac{e^{V_h(h)}}{\int_{\mathcal{T}} e^{V_h(s)}\mu(ds)}\max_{1\le k\le n} e^{-x_k+V_h(t_k)}; \max_{t\in\mathcal{T}} e^{V_h(t)} > 0, J_h = 1\right\}\mu(dh)\\
&= \int_{\mathcal{T}} \mathbb{E}\left\{\frac{e^{V_h(h)}}{\int_{\mathcal{T}} e^{V_h(s)-V_h(h)}\mu(ds)}\max_{1\le k\le n} e^{-x_k+V_h(t_k)-V_h(h)}, J_h = 1\right\}\mu(dh)\\
&= \int_{\mathcal{T}} \mathbb{E}\left\{e^{V_h(h)+\ln\mathbb{P}\{J_h=1\}}\max_{1\le k\le n} e^{-x_k+V_h(t_k)-V_h(h)-\ln\left(\int_{\mathcal{T}} e^{V_h(s)-V_h(h)}\mu(ds)\right)}\right\}\mu(dh)\\
&= \mathbb{E}\left\{\int_{\mathcal{T}}\max_{1\le k\le n} e^{-x_k+\Theta_h(t_k)-\ln\left(\int_{\mathcal{T}} e^{\Theta_h(s)}\mu(ds)\right)}\mu(dh)\right\}, \qquad (7.14)
\end{aligned}$$

hence the proof follows. □

Proof of Corollary 6.6 In view of statement c) in Theorem 4.3 $L^h\Xi_0 V_0 \stackrel{fdd}{=} \Xi_h V_h$, hence the claim follows by (6.7). If μ is the Lebesgue measure on $\mathcal{T}$, then it follows by (7.14) and the translation invariance of μ that ζ_Z is stationary. □

Proof of Theorem 6.9 By the definition of the STP in [24] (see also [69, 70]) and Lemma 8.1 (see (8.2)) we have that

$$\Theta = \Xi_0 V_0, \quad V_0 \stackrel{fdd}{=} Z|(Z(0) > 0).$$

Under this setup it is easy to see that (6.17) is a re-formulation of (4.4) in terms of STP. □

8. Appendix

Let $\zeta_Z(t), t\in \mathcal{T}$ be as in Section 4 where Z has representation (4.1) for some $h \in \mathcal{T}$, and let Y be a random process given by

$$Y(t)=J_hA(t)+(1-J_h)B(t)-\ln \mathbb{P}\{J_h=1\}, \quad t\in \mathcal{T}, \tag{8.1}$$

with A,B,J_h being mutually independent and $\mathbb{P}\{A(h)=0\}=\mathbb{P}\{B(h)=-\infty\}=1$. Denote by $\zeta_Y(t), t\in \mathcal{T}$ the max-stable process associated to Y.

Lemma 8.1. *If $\zeta_Y \stackrel{fdd}{=} \zeta_Z$, then $A \stackrel{fdd}{=} \Theta_h$.*

Proof of Lemma 8.1 For notational simplicity we suppress the subscript h writing simply J instead of J_h. Let $t_1,\ldots,t_{n+1}\in \mathcal{T}, t_{n+1}=h$ be distinct and set

$$c_k=\frac{1}{1-e^{-1/k}}, \quad p:=\mathbb{P}\{J=1\}>0, \quad A_j:=A(t_j), \quad B_j:=B(t_j), \quad K=\{1,\ldots,n+1\}.$$

With the same arguments as in the proof of Theorem 2.2, using the fact that $A(h)=0$ and $B(h)=-\infty$ almost surely, we obtain for $k>1$ and $x_1,\ldots,x_n\in\mathbb{R}$

$$\begin{aligned}
&\mathbb{P}\Big\{\forall j\in K: \zeta_Y(t_j)\le x_j+\ln k \Big| \zeta_Y(t_h)>\ln k\Big\}\\
&= c_k\Big[e^{-\int_{\mathbb{R}}\Big[\mathbb{P}\{\exists j\in K: A_j>x_j+\ln(k/p)-y, J=1\}+\mathbb{P}\{\exists j\in K: B_j>x_j+\ln(k/p)-y, J=0\}\Big]e^{-y}\,dy}\\
&\quad -e^{-\frac{1}{kp}\int_{\mathbb{R}}\Big[\mathbb{P}\{A_{n+1}>-y, \text{ or } \exists j\in K: A_j>x_j-y, J=1\}+\mathbb{P}\{B_{n+1}>-y, \text{ or } \exists j\in K: B_j>x_j-y, J=0\}\Big]e^{-y}\,dy}\Big]\\
&= c_k\Big[e^{-\frac{1}{kp}\int_{\mathbb{R}}[p\mathbb{P}\{\exists j\in K: A_j>x_j-y\}+(1-p)\mathbb{P}\{\exists j\in K: B_j>x_j-y\}]e^{-y}\,dy}\\
&\quad -e^{-\frac{1}{kp}\int_{\mathbb{R}}[p\mathbb{P}\{0>-y, \text{ or } \exists j\in K: A_j>x_j-y\}+(1-p)\mathbb{P}\{-\infty>-y, \text{ or } \exists j\in K: B_j>x_j-y\}]e^{-y}\,dy}\Big]\\
&\to \int_{\mathbb{R}}\Big[\mathbb{P}\{y>0, \text{ or } \exists j\in K: A_j>x_j-y\}-\mathbb{P}\{\exists j\in K: A_j>x_j-y\}\Big]e^{-y}\,dy, \quad k\to\infty\\
&= \mathbb{P}\{\forall j\in K: A_j+\mathcal{E}\le x_j\}, \qquad (8.2)
\end{aligned}$$

with $\mathcal{E}$ a unit exponential rv being independent of A, hence since a.s. $A(h)=0$ the proof follows. $\square$

Finally, we discuss the asymptotics of $\int_{\mathbb{R}^d} f_n(x)\nu_n(dx)$ as n tends to infinity.

Lemma 8.2. *Let $\nu_n, n\ge 1$ be positive finite measures on $\mathbb{R}^d, d\ge 1$ which converge weakly as $n\to\infty$ on each set $[-k,k]^d, k\in\mathbb{N}$ to some finite measure ν. If $f, f_n, n\ge 1$ is a sequence of measurable functions on $\mathbb{R}^d$, then for any $k\in\mathbb{N}$ we have*

$$\liminf_{n\to\infty}\int_{[-k,k]^d} f_n(x)\nu_n(dx) \ \ge\ \int_{[-k,k]^d}\liminf_{n\to\infty, v\to x} f_n(v)\nu(dx). \tag{8.3}$$

Assume that for any $u_n\in\mathbb{R}^d, n\in\mathbb{N}$ such that $\lim_{n\to\infty}u_n=u\in B$ and $\nu(\mathbb{R}^d\setminus B)=0$, we have $\lim_{n\to\infty} f_n(u_n)=f(u)$. If further $f_n, n\in\mathbb{N}$ is uniformly bounded on compacts of $\mathbb{R}^d$, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^d} f_n(x)\nu_n(dx)=\int_{\mathbb{R}^d} f(x)\nu(dx), \tag{8.4}$$

provided that

$$\lim_{k\to\infty}\sup_{n\ge 1}\int_{\mathbb{R}^d\setminus[-k,k]^d} f_n(x)\nu_n(dx)=0. \tag{8.5}$$

Proof of Lemma 8.2 The first claim in (8.3) is a special case of [71][Th. 1.1]. In light of (8.5) the claim in (8.4) can be established if we show that for any integer k

$$\lim_{n\to\infty}\int_{[-k,k]^d} f_n(x)\nu_n(dx)=\int_{[-k,k]^d} f(x)\nu(dx)<\infty,$$

which follows directly by [72][Lemma 4.2], see also [73][Lemma 6.1]. ☐

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