

Self-avoiding Random Walks and Olbers' Paradox

Yuanan Diao

Department of Mathematics
University of North Carolina at Charlotte
Charlotte, NC 28223, USA
ydiao@uncc.edu

Andrzej Stasiak¹

Laboratory of Ultrastructural Analysis
Department of Biology and Medicine
University of Lausanne, Lausanne, CH 1015, Switzerland
Andrzej.Stasiak@unil.ch

Abstract. In this paper, we prove that a self-avoiding walk of infinite length provides a structure that would resolve Olbers' paradox. That is, if the stars of a universe were distributed like the vertices of an infinite random walk with each segment length of about a parsec, then the night sky could be as dark as actually observed on the Earth. Self-avoiding random walk structure can therefore resolve the Olbers' paradox even in a static universe.

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In 1721 Halley discussed what he called "Methaphysical paradox" stating that "if the number of Fixt stars were more than finite, the whole superficies of their apparent sphere would be luminous" [11]. In 1831 Olbers proposed that the fact that the entire sky does not glow like the surface of the Sun is due to interstellar absorption of starlight [15]. Although Olbers' explanation was shown by Hershel to be fallacious [13] the riddle of the dark night sky is known today as Olbers' paradox [12]. Present day explanations of Olbers' paradox rely on the finite model of expanding universe where the number of visible

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stars is finite and is much too small to completely cover our celestial sphere [12, 14]. In fact Olbers' paradox is frequently used as a proof against a static steady state infinite model of the Universe [14]. Halley's reasoning was based on an intuitive assumption that stars are distributed uniformly. However, this assumption is not supported by present day observations. Interestingly, it was already suggested that the solution of the Olbers' paradox could be provided by the fractal structure of the universe by Hershel in a private letter (discussed in [1]). This idea was developed further by Charlier [2] who defined the criterion of the hierarchical structure of the universe, i.e., its fractal dimensions that would resolve the Olbers' paradox. Until today the exact fractal dimensions of the universe are not known and are subject of intense discussions [8]. Therefore, it is difficult to perform rigorous calculations leading to the estimation of the fraction of celestial sphere that is covered by the stellar discs. However, there are fundamental physical systems well known from polymer science that could serve as a model for such calculations.

There are three classical distance distributions investigated thoroughly in the polymer field [4, 5, 6, 9, 10]. Segments of a long polymer in a solution can attract each other, repulse each other or be neutral to each other. Attraction between segments can overcome entropic costs and lead to the collapse of a polymer chain. Collapsed polymers increase their overall dimensions with the cubic root of their length. Neutral thin polymers represent so called theta state where the entropic contribution and the fact that independent segments do not "feel" each other cause the overall dimensions of the polymer chains (such as their radius of gyration) to increase with the square root of the polymer length. Lastly, polymer chains in which the segments repulse each behave as so-called self-avoiding random walks whose overall dimensions scale with the power 0.588 with respect to their chain length.

For the purpose of discussion, let us now assume that we deal with an infinite random walk in which each step has a unit length (such random walks are called equilateral random walks). We further assume that every vertex is occupied by a sphere with a radius a . The radii of these spheres should be chosen to reflect the actual ratio between the average size of a star and the average distance between two nearby stars (segment length in a walk). If an observer located near one of the spheres looks around, what would he or she see? Would he or she see in every possible viewing direction overlapping images of these spheres (meaning the entire sky will be very bright) or see the images of these spheres covering only a certain percentage of the observers celestial sphere (meaning the sky could be rather dark)?

Let us first consider the case in which the stars follow the distance distribution of a collapsed random walk (of infinite length). An intuitive way of thinking about the collapsed random walk is thinking of it as a string of beads being packed tightly. In this case, it is not hard to see that the stars are actually rather uniformly distributed in the space with a near constant density.

Clearly, in this case Olbers' reasoning would apply and one would reach the Paradox: the sky would shine like the surface of the sun! Apparently, this is not the case we are interested in. In the following, we will concentrate on the case in which the stars follow the distance distribution of a self-avoiding random walk (of infinite length).

Let W be a self-avoiding random walk of infinite length so that its vertices may be labelled as $X_0, X_1, X_{-1}, X_2, X_{-2}, \dots$, and so on, such that X_k and X_{k+1} are adjacent to each other (hence the distance between them is 1). This unit distance could be interpreted as a cosmologic distance such as one parsec. Let the position of the observer be the origin and assume that the observer is nearest to the star (sphere) represented by (centered at) X_0 . Since the radius of gyration of a self-avoiding (equilateral) random walk of length n is of the order of n^μ with $\mu \approx 0.588$, we will assume that the mean distance between X_0 and X_k is of the order of k^μ . For the sake of discussion, we will further assume that the distribution of X_k can be approximated by the classical Gaussian distribution (this is also supported by experimental data, see [16] for instance). That is, the density function of X_k may be approximated by the function

$$f(X_k) \approx \left(\frac{1}{\sqrt{2\pi}\sigma_k}\right)^3 \exp\left(-\frac{|X_k|^2}{2\sigma_k^2}\right).$$

Notice that $|X_k|$ is the distance between X_k (star number k) and the origin (the observer's position) and that the mean of $|X_k|$ is of the order $|k|^\mu$. It is necessary that the standard deviation σ_k is also of the order $|k|^\mu$. The observed area of the star at X_k from the observer is $\frac{4\pi a_k^2}{|X_k|^2}$ where a_k is the radius of the sphere at X_k . Let $d_0 = |X_0|$ be the distance between the observer (at the origin) and the nearest star (at X_0). The mean contribution of X_k to the total observed area of the observer can be approximated by

$$\begin{aligned} & \int \int \int_{|X_k| \geq d_0} \frac{4\pi a_k^2}{|X_k|^2} \left(\frac{1}{\sqrt{2\pi}\sigma_k}\right)^3 \exp\left(-\frac{|X_k|^2}{2\sigma_k^2}\right) dX_k \\ &= 16\pi^2 a_k^2 \int_{d_0}^\infty \left(\frac{1}{\sqrt{2\pi}\sigma_k}\right)^3 \exp\left(-\frac{r^2}{2\sigma_k^2}\right) dr \\ &\leq \frac{8\pi^2 a_k^2}{\sigma_k^2} \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{r^2}{2}\right) dr \\ &= \frac{4\pi^2 a_k^2}{\sigma_k^2}. \end{aligned}$$

Thus the mean of the total observed area without considering the nearest star at X_0 is bounded above by

$$\sum_{k=\pm 1, \pm 2, \dots} \frac{4\pi^2 a_k^2}{\sigma_k^2}.$$

Notice that the above series is convergent since each a_k is very small and σ_k^2 is of the order $|k|^{2\mu} \approx |k|^{1.176}$. In fact, if we substitute a_k by the largest star size a (which is still very small comparing to 1) and σ_k by $b|k|^\mu$, where $\mu \approx 0.588$ and $b > 0$ is a constant determined by the random walk model estimated to be at least 10^{-1} in [7], we can bound the series by (some tedious calculations are omitted here)

$$\frac{10\pi a^2}{0.17b^2} \leq \frac{10^3\pi a^2}{0.17}.$$

If we substitute a with the radius of the largest known star (in terms of radius), which is about 9.04×10^{-6} , then the above bound becomes 1.51×10^{-6} , which is still much smaller than the observer's sky area (the area of the unit sphere). We may thus interpret this result as the following: if the stars follow the typical (or average) distribution of the vertices of a self-avoiding equilateral random walk of infinite length, then the total observed area covered by stellar discs can be quite small when an observer is not facing the nearest star (night sky situation). Here, the space is assumed to be unlimited and the number of stars is infinite.

We have shown above that if the distribution of distances between stars in the Universe have followed the principle of self-avoiding walks (of infinite length) then even if the Universe would have been infinite and would contain infinite number of stars the sky we see could look just as our night sky. The question arises if principles of self-avoiding walks that operate in case of polymer chains, for example, can be applied to "celestial mechanics". In self-avoiding chains the connectivity along the chain assures that the chains do not blow apart but still the distant segments can repulse each other. Interestingly, the interactions between the stars that are close to each other are dominated by attractive gravitational forces that would correspond to the stabilizing interactions along individual segments of the chains. As we know now the interactions between very distant portions of the universe are dominated by repulsive forces that in a way act similarly to the repulsion between distant segments in self-repulsing polymers [3].

The second question concerning the self-avoiding random walk model of the Universe is whether it satisfies the Copernican principle, i.e., from any two points of observation in the Universe, any two sufficiently big regions (of the same size) centered at these points contain a similar number of stars and generally look similar to each other. If the considered self-avoiding walk would be infinite then in fact every portion of such a walk would essentially look as any other portion since such a walk is circular in nature (there is no beginning or ending).

In conclusion, we have shown above that a self-avoiding random walk provides us with a model of infinite and isotropic universe in which the sky could look just like the one we see every night. As we point out at the beginning of

the paper, this is not the only known static model (in fact, any infinite universe with a fractal dimension smaller than 2 could also have a dark sky even if there are infinitely many stars in such a universe [1, 2, 14]). Our example demonstrates that it is possible to have infinite static systems that would also resolve the Olbers' paradox. In our discussion we did not consider factors such as the age of the Universe, its rate of expansion and the speed of light. When these factors are adequately introduced into the picture the Paradox can be resolved even if the stars are distributed uniformly [15]. But that is not an issue of interest in this paper.

REFERENCES

- [1] Y. Baryshev and P. Teerikorpi, Fractal Approach to Large-Scale Galaxy Distribution, *arXiv:astro-ph/0505185* v1.
- [2] C. Charlier, *Arkiv f. matematik, astronomi och fysik Band 16*(1922), 1–1.
- [3] P. Coles, The state of the Universe, *Nature* **433** (2005), 248–256.
- [4] Y. Diao, A. Dobay, R. Kusner, K. Millett and A. Stasiak, The Average Crossing Number of Equilateral Random Polygons, *J. Physics A* **36**(2003), 11561–11574.
- [5] A. Dobay, Y. Diao, J. Dubochet and A. Stasiak, Scaling of the average crossing number in equilateral random walks, knots and proteins, *Physical and Numerical Models in Knot Theory Including Applications to the Life Sciences* (eds Calvo J A, Millett K C, Rawdon E J and Stasiak A). World Scientific Publishing, Singapore (2005), 219–231.
- [6] M. Doi and S. F. Edwards, *The Theory of Polymer Dynamics*. Oxford University Press, 1986.
- [7] A. Dobay, J. Dubochet, K. Millett, P. E. Sottas and A. Stasiak, Scaling behavior of random knots, *PNAS* **100**(2003), 5611–5.
- [8] J-P. Eckmann, Esa Järvenpää, M. Järvenpää and I. Procaccia, On the Fractal Dimension of the Visible Universe, *arXiv:astro-ph/0301034* v1.
- [9] J. P. Flory, *Principles of Polymer Chemistry*. Cornell University Press, 1953.
- [10] P. G. de Gennes, *Scaling Concepts in Polymer Physics*. Cornell University Press, 1979.
- [11] E. Halley, *Philos. Trans.* **31**(1721), 22–22.
- [12] E. R. Harrison, The Dark Night-Sky Riddle: A “Paradox” That Resisted Solution, *Science* **226**(1984), 941–941.
- [13] J. F. W. Herschel, *Edinburgh Rev.* **87**(1848), 170–170.
- [14] M. Livio, *The Accelerating Universe*. John Wiley & Sons, Inc, New York, 2000.
- [15] H. W. M. Olbers, *Astronomisches Jahrbuch fuer das Jahr 1826*(1823), Ed. JE Bode. Berlin, 133–141.
- [16] A. Yao, H. Tsukahara, T. Deguchi and T. Inami, Distribution of the Distance between Opposite Nodes of Random Polygons with a Fixed Knot, *J. Physics A* **37**(2004), 7993-8006.

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