MODELS FOR SPATIAL WEIGHTS: A SYSTEMATIC LOOK

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Abstract: General properties of spatial weights models, in particular Markovian properties, are systematically investigated. The role of stationary spatial distribution, interpretable as an importance-centrality or prominence index, is emphasized. Spatial interaction models, and among them the gravity model, are classified with respect to the time reversal and aggregation invariance properties obeyed by the associated spatial weights. Nine examples, involving connectivity, flows and distance decay analysis, integral geometry and Dirichlet-Voronoi tessellations illustrate the main concepts, with a particular geometrical emphasis, and show how traditional, heuristic ingredients aimed at defining spatial weights can be recovered form general models.

Keywords: aggregation invariance, balanced flows, Dirichlet-Voronoi cells, integral geometry, interaction models, gravity model, marginal homogeneity, Markov chains, prominence index, quasi-symmetry, shopping behavior models, spatial weights, time reversal duality.

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1 Introduction

Places influence each other, and the simplest and most used measure of influence are spatial weights. Since the pioneering works of Moran [1948], Geary [1954] Cliff and Ord [1973] [1981] and many others, the problem of properly specifying the spatial weight coefficients has largely been recognized as difficult and controversial. Whereas workers generally agree that spatial weights should decrease with some generalized distance, increase with the the length of their common boundary, or, more generally, reflect their general accessibility (see e.g. Anselin [1988] for a review of models), a univocal specification is yet still lacking. On one hand, this state of affairs is not surprising, for there is no such thing as "true", "universal" spatial weights, optimal in all situations: good candidates must reflect the properties of the particular phenomenon, properties which are bound to differ from field to field (Cliff and Ord 1973; Arora and Brown 1977; Cressie 1991). On the other hand, this difficulty should not impede a more systematic investigation of *models* for spatial weights, starting with the question "which classes of models yield specified families of spatial weights, and what are the properties of the latter?" This is the approach we shall follow here.

This paper concentrates upon some *theoretical issues* related to spatial weights: we intend to deal with empirical issues subsequently in a forthcoming paper; we are aware of, and apologize for the fact that this omission ignores the question of formal modelling, and makes the comparison with other approaches difficult. Also, theoretical issues related to spatial autocorrelation, spatial regression, and spatial interpolation, have not been addressed despite their obvious relevance to our topic. We furthermore have restrained from discussing other spatial interaction models, such as those based upon the concept of spatial stochastic explorers. Beside length constraints, the main motivation for these omissions is the author's feeling that, in contrast to statistical concepts, geometric concepts are possibly underexploited in Geography, thus justifying a little more emphasis on the latter, as attempted here.

In section 2, we define the class of spatial weights under consideration and emphasize their markovian properties. In particular, we concentrate upon the associated stationary distribution which we propose to interpret as an importance-centrality or *prominence* index.

Section 3 draws a classification of spatial interactions as well as associated "export" and "import" weights and stationary distributions. The gravity model emerges as the class of interactions satisfying certain time-reversibility conditions. The role of marginally homogeneous interactions is underlined, in particular regarding their spatial aggregation invariance properties.

Section 4 concentrates on symmetric distance decay interactions. Short- and long-distance expansions yield spatial weights expressions containing traditional ingredients such as the relative area, the spatial dispersion, the relative distance, the common boundary and the connectivity.

Section 5 exemplifies models of spatial weights in the setup of Integral Geometry, and section 6 shows how next-nearest neighbors in a Dirichlet-Voronoi tesselation can serve to define spatial weights, of possible relevance in shopping behavior modelling.

To the best of our knowledge, most of the material presented here is new, in particular regarding the stationary distribution associated with spatial weights (the prominence index), the structural classes of interaction, and the time reversal or aggregation invariance properties. This, added to the absence of empirical material, as well as the use of some possibly lesser known parts of mathematics, might be a source of difficulty for the reader. We have tried to alleviate this difficulty by systematically presenting the material in a series of examples illustrating the concepts.

2 Spatial weights as Markov chains

Consider a set of m places. Those places can consist of points $x_1, ...x_m$, representing for instance the location of towns, or of regions $A_1, ..., A_m$ partitioning the total area Ω of interest.

Definition 1 Let $S = \{1, ..., m\}$ be a set of places. A *spatial weight matrix* is a $m \times m$ matrix W of components w_{jk} satisfying
a) $w_{jk} \geq 0$
b) $\sum_{k=1}^{m} w_{jk} = 1$ (for all $j = 1, ..., m$)

The terms "contiguity", "connectivity", "adjacency" or "association" matrices can also be found, with roughly similar connotations. The simple normalization above makes different choices of weights immediately comparable and nothing forbids the use of an explicitly distinct multiplicative factor for a given application (as e.g. in spatial autoregression (Hordijk and Nijkamp 1977; Huang 1984; Anselin 1988), where the multiplicative factor has to be estimated independently).

Definition 1 does not forbid the existence of non-zero diagonal weights: w_{jj} can be thought of as the self-influence of place j upon itself, and measures the inertia or autonomy of the place:

the greater the w_{jj}, the more place j resists to neighbor's changes (relative to the phenomenon under investigation). Thus non-zero diagonal weights should be allowed, and represent indeed the main contribution in the "short time" or "long distance limit" models, as shown below.

Basically, w_{jk} is meant to quantify the relative spatial influence of place k on place j. Thus, non-symmetric weights offer greater flexibility and realism. Also, non zero w_{jk} should not, in general, be restricted to spatially contiguous pairs, even if one-step accessibility considerations suggest it: for two-step (and higher order) accessibility modelling is bound to produce non-zero non-contiguous weights (the neighbors of our neighbors are not necessarily our neighbors). The same reasonning also shows the need for allowing non-zero diagonal weights.

In summary, we require non-negative, normalized, asymmetric, non-contiguous and non-zero diagonal spatial weights: this makes the set of spatial weights w_{jk} identical to the set of the Markov chains transition matrices on S (an introduction to this topic and its basic properties can be found e.g. in Feller [1966], Kemeny and Snell [1967] or Collins [1974]). This immediately opens up a wealth of perspectives. First, the set of spatial weights matrices is a convex cone closed under integer power raising:

a) If $W = (w_{jk})$ is a spatial weight matrix, so is W^n, for nonnegative integer n. In particular, $W^0 = I$, the identity matrix (whose components are noted δ_{jk}).

b) If W_1 and W_2 are spatial weight matrices, so are $W(\lambda) = pW_1 + (1-p)W_2$ for all $p \in [0, 1]$.

In particular, any combination of the form $\sum_{n\geq 0} p_n W^n$, where $\{p_n\}$ denotes a probability distribution, is a weight matrix. So:

c) If W is a spatial weight matrix, so are
- $W_1 := (pW + (1-p)I)^N$ (binomial convolution)
- $W_2 := ((1-p)(I - pW)^{-1})^N$ (negative binomial convolution)
- $W_3 := \exp(-\lambda)\exp(\lambda W)$ (Poisson convolution)

where $p \in [0, 1]$, N is a positive integer and $\lambda > 0$. For instance, if W happens to describe a typical pattern of destinations conditional upon the occurence of some emigration process, W_3 might be suitable in describing an unconditional destination pattern if $1/\lambda$ is the average occurrence time of emigration.

Secondly, assuming the chain to be *ergodic* (which holds e.g. if $(W^n)_{jk} > 0$ for some $n > 0$ and all j, k), we get a unique *stationary distribution* $\pi_j \geq 0$ ($\sum_j \pi_j = 1$) solution of $W'\pi = \pi$, i.e. $\sum_j \pi_j w_{jk} = \pi_k$. Equivalently, π is a left eigenvector of W with eigenvalue 1.

Whereas w_{jk} is best thought of as a measure of the relative influence of place k on place j (the total influence *on* j is 100% by normalisation), π_j can be interpreted as the total influence *of* place j on the total area under consideration. π_j will be further referred to as an *influence-centrality* or *prominence* index.

A nice feature of the prominence index is its invariance when W is replaced by W^n, W_1, W_2 or W_3 above. In particular, π is left unchanged under the diagonal change $W \to pW + (1-p)I$: thus the question of diagonal weights is not critical with respect to the value of the prominence index; this is a consequence of the long-term, stationary nature of the latter. Note incidentally that symmetric weights $W = W'$ have a corresponding uniform stationary distribution $\pi_j = 1/m$.

Example 1: define the spatial weights in figure 1 as proportional to the common boundary between regions (with proportions 2:1:1:1)

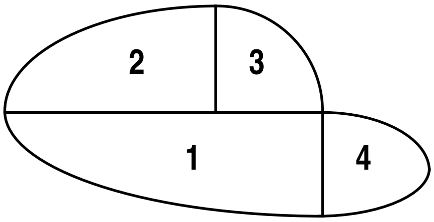

Figure 1

Then

$$W = \begin{pmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 2/3 & 0 & 1/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad (1)$$

is obtained, with prominence indices $\pi_1 = 0.4$, $\pi_2 = 0.3$, $\pi_3 = 0.2$ and $\pi_4 = 0.1$.

Example 2: Let the peripheral regions $j = 1, ..., n$ in figure 2 be connected only to a central region $j = 0$ by links of the same importance.

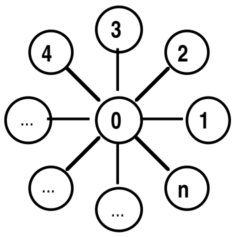

Figure 2

Then

$$W = \begin{pmatrix} 0 & 1/n & . & . & 1/n \\ 1 & 0 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & 0 & . & . & 0 \end{pmatrix} \qquad (2)$$

is obtained, with prominence indices $\pi_0 = 1/2$ and $\pi_1 = \pi_2 = ... = \pi_n = 1/(2n)$. From this point of view, and irrespective of n, central region and *total* periphery are exactly equally prominent.

We now turn to the question (implicit in examples 1 and 2) of the *determination* of spatial weights. Models of spatial autocorrelation (to be addressed in a forthcoming paper) constitute a possible source for identifying spatial weights. Models of spatial interaction, developed below, constitute another.

3 Defining spatial weights from spatial interaction

Let n_{jk} denote positive *fluxes* (i.e. number of goods, persons, or units of any kind in motion) or, more generally, *spatial interaction* from place j to place k. In this context, j represents an origin and k a destination. In what follows, we restrict ourselves to the case of identical sets of origins and destinations. A spatial interaction *model* consists of expressing n_{jk} as a function of spatial (e.g. intrinsic geometry or separation distances) and/or non-spatial (e.g. economic or climatic attractiveness) attributes of both places (Fotheringham and O'Kelly 1989; Sen and Smith 1995). As said earlier, no inferential procedure is considered in this paper; thus the n_{jk} can either be thought as the observed transition count during a certain period, or as a theoretical quantity such as e.g. the expectation of flow in a Poisson model.

Let $n_{j\bullet} := \sum_{l=1}^{m} n_{jl}$ be the total flux leaving place j, $n_{\bullet j} := \sum_{l=1}^{m} n_{lj}$ the total flux entering place j and $n := \sum_{j=1}^{m} n_{j\bullet} = \sum_{j=1}^{m} n_{\bullet j}$ the total flux. There are two obvious ways of defining spatial weights from n_{jk}:

Definition 2 Let n_{jk} be a spatial interaction matrix. One defines associated **export** weights w_{jk} and **import** weights w^*_{jk} as

$$w_{jk} := \frac{n_{jk}}{n_{j\bullet}} \qquad\qquad w^*_{jk} := \frac{n_{kj}}{n_{\bullet j}} \tag{3}$$

As $n_{jk} \neq n_{kj}$ in general, the export π and import π^* prominence indices (defined as the stationary distribution of W, respectively W^*) generally *differ*: the prominence of a region depends on the point of view (export or import) adopted. In the particular case of the *gravity model* for spatial interactions (see below), export (resp. import) weights are formally identical with the so-called *conditional destination* (resp. *origin*) *probabilities* (Sen and Smith 1995).
Of course, in some particular context, one might wish to overcome the export/import distinction, and to that effect use symmetrized fluxes, as $n^{as}_{jk} := \frac{1}{2}(n_{jk} + n_{kj})$ or $n^{gs}_{jk} := \sqrt{n_{jk}\, n_{kj}}$. Note in passing that tests of symmetry ($H_0 : n_{jk} = n_{kj}$) yield n^{as}_{jk} as the adapted decision variable in the classical maximum likelihood approach, and n^{gs}_{jk} in the minimum discrimination approach (see e.g. Bishop and al. [1975] and references therein).

3.1 Structural classes of interactions

Carrying out spatial weights properties analysis requires additional structural requirements on spatial interactions; in the following definition, α_j, a_j, β_j, b_k denote positive quantities, and γ_{jk}, c_{jk}, d_{jk} denote symmetric positive matrices.

Definition 3

a) Spatial interaction is *quasi symmetric* if it is of the form $n_{jk} = \alpha_j \beta_k \gamma_{jk}$ with $\gamma_{jk} = \gamma_{kj}$ (or equivalently $n_{jk} = a_j c_{jk}$ with $c_{jk} = c_{kj}$ or $n_{jk} = b_k d_{jk}$ with $d_{jk} = d_{kj}$).

b) Spatial interaction is *marginally homogeneous* or *globally balanced* if $n_{j\bullet} = n_{\bullet j}$.

c) Spatial interaction is *symmetric* or *locally balanced* if $n_{jk} = n_{kj}$.

d) Spatial interaction is *independent* if $n_{jk} = \alpha_j \beta_k$.

e) Spatial interaction is *trivial* if $n_{jk} = \alpha_j \alpha_k$.

(the above conditions are understood to hold for all possible $j, k = 1, ..., m$).

Obviously, requiring quasi-symmetry exactly amounts to postulating a *gravity model* for spatial interaction: see Fotheringham and O'Kelly [1989] p. 22, and Sen and Smith [1995] p. 50, which constitute strongly advisable contemporary extensive reviews of theory and applications of the gravity models. In this context, α_j is referred to as the origin-dependent *propulsiveness*, β_k as the destination-dependent *attractiveness*, and γ_{jk} as the (generalized) *distance deterrence function.* The equivalence mentioned in a) can be constructed as $c_{jk} := \gamma_{jk}\beta_j\beta_k$ and $a_j := \alpha_j/\beta_j$, as well as $d_{jk} := \gamma_{jk}\alpha_j\alpha_k$ and $b_j := \beta_j/\alpha_j$ (at least for non-zero coefficients). Gravity models are equivalently characterized by $n_{jk}n_{kl}n_{lj} = n_{jl}n_{lk}n_{kj}$ for all (repeated or not) triples (j,k,l) (Caussinus 1966). Violations of the latter have been used to detect failures of the gravity model in the case of migration streams (Smith and Clayton 1978; Goodchild and Smith 1980).

Marginal homogeneity reflects balanced *total* exports and imports. For instance, seasonally averaged commuting traveler fluxes are marginally homogeneous, but consumption goods fluxes are not in general. As developed below, marginal homogeneity is a highly desirable feature regarding spatial aggregation invariance.

Symmetry is too strong a requirement to possibly constitute a realistic model of spatial interaction in general. However, its simplicity makes it appealing as a purely spatial "model of order 0", neglecting in the gravity model aspatial differences in origin propulsiveness and destination attractiveness. As presented in example 4, symmetric models allow fairly convenient investigations of distance-decay effects.

Independence models constitute another caricature of gravity models, obtained when distance deterrence is simply ignored, allowing only distinctions in origin propulsiveness and destination attractiveness. As shown below (theorem 2), independence and globally balanced models are the only ones to satisfy good invariance properties w.r.t. aggregation.

For completeness sake, a *trivial* model, independent and symmetric, has also been defined.

3.2 Time reversal properties

The relationship between the export and import weights W and W^* in (3) is somewhat reminiscent, but *distinct* from the relationship between a Markov chain W and its *dual* $\hat{W}$. Recall that the dual chain is defined as $\hat{w}_{jk} := w_{kj}\pi_k/\pi_j$, where π is the stationary distribution of W (Feller 1966). $\hat{W}$ has the interpretation of the process W evolving backwards (time reversal). By construction, $\hat{W}$ also possesses π as stationary distribution, and $\hat{\hat{W}} = W$. When $W = \hat{W}$, the chain is called *reversible.*

Theorem 1

a) Export weights are reversible ($W = \hat{W}$) $\Leftrightarrow$ import weights are reversible ($W^* = \hat{W}^*$) $\Leftrightarrow$ spatial interaction is quasi-symmetric ($n_{jk} = \alpha_j\beta_k\gamma_{jk}$). In this case, the export and import stationary distributions are

$$\pi_j = \frac{\sum_{k=1}^m \gamma_{jk}\beta_j\beta_k}{\sum_{j',k=1}^m \gamma_{j'k}\beta_{j'}\beta_k} \qquad\qquad \pi_j^* = \frac{\sum_{k=1}^m \gamma_{jk}\alpha_j\alpha_k}{\sum_{j',k=1}^m \gamma_{j'k}\alpha_{j'}\alpha_k} \tag{4}$$

b) Export and imports weights are the dual of each other ($\hat{W} = W^*$ and $\hat{W}^* = W$) $\Leftrightarrow$ spatial interaction is globally balanced. In this case, the export and import stationary distributions coincide: $\pi_j = \pi_j^* = n_{\bullet j}/n = n_{j\bullet}/n$.

c) Export and import weights are identical ($W = W^*$) $\Leftrightarrow$ spatial interaction is symmetric $\Leftrightarrow$ spatial interaction is quasisymmetric *and* globally balanced. In this case, both weights are also reversible, with stationary distribution $\pi_j = \pi^*_j = n_{\bullet j}/n = n_{j\bullet}/n$.

The equivalence "quasi-symmetry + marginal homogeneity $\Leftrightarrow$ symmetry" has been known for a long time (Caussinus 1966) in the statistical analysis of contingency tables (Bishop and al. 1975). The connection of the latter concepts to duality considerations appears to be little or not known, however. In particular, the characterization of the class of gravity models as coinciding with the class of interactions yielding time reversible spatial weights might be of some interest: gravity modelling is adequate if and only if forward and backward fluxes are indistinguishable.

Let us prove the first part of point a) (the proof of the other points is fairly straightforward): suppose $W = \hat{W}$ i.e. $\pi_j\ w_{jk} = \pi_k\ w_{kj}$ where π is the stationary distribution of W. It also follows from (3) that $n_{jk} = q_j w_{jk}$ for some positive vector q satisfying $q_\bullet = n$. Defining $a_j := q_j/\pi_j$, the interaction can be written as $n_{jk} = a_j c_{jk}$ where $c_{jk} := \pi_j\ w_{jk}$ is symmetric by hypothesis, thus making n_{jk} quasi symmetric. Conversely, it is easy to demonstrate by direct substitution the validity of (4) and then to show that quasi symmetry of n_{jk} implies reversibility of w_{jk}. $\Box$

Prominence indices for gravity models can explictly be written down as in (4), a fact noticeable in itself. In accordance with intuition, export prominence is independent of origin propulsiveness α_j, and import prominence is independent of destination attractiveness β_j. In the special case of independent spatial interaction $n_{jk} = \alpha_j\beta_k$, the spatial weights degenerate into a zero-order chain: $w_{jk} = \pi_k = \beta_k/\beta_\bullet$ and $w^*_{jk} = \pi^*_k = \alpha_k/\alpha_\bullet$.

Example 3: let a neighborhood relation hold among m places. An adjacency or contiguity interaction model consists of a symmetric matrix n_{jk} such that $n_{jk} = 1$ if j and k are distinct neighbors, and $n_{jk} = 0$ otherwise (examples 1 and 2 above fit into this framework). Let $c_j := n_{j\bullet}$ denote the total number of neighbors of place j; $< c > := (\sum_j c_j)/m$ is the average number of neighbors. We suppose the system to be connected, ensuring ergodicity of the corresponding weight Markov matrix $w_{jk} = n_{jk}/c_j$. The unique stationary distribution is then $\pi_j = c_j/(m < c >)$.
For instance, the Go board (Lasker 1060; Lichtenstein and Sipser 1980) is a regular $m = 19\times19 = 361$ square grid, with a natural neighborhood structure yielding $c_j = 4$, 3 or 2 for $j =$ "inner", "edge" and "corner" positions respectively, with $< c >= 3.79$. In this approach, an inner position can be said to be twice as prominent as a corner position.

3.3 Aggregation for spatial weights

Suppose the m regions $\mathcal{A} := \{A_j\}_{j=1,\ldots,m}$ partitioning Ω are aggregated into $\tilde{m} < m$ regions $\mathcal{B} := \{B_\alpha\}_{\alpha=1,\ldots,\tilde{m}}$. The new partition $\mathcal{B}$ is coarser than $\mathcal{A}$, or, equivalently, $\mathcal{A}$ is a refinement of $\mathcal{B}$. Let $[\alpha]$ denote the set of initial indices j whose aggregation yields B_α, i.e. $B_\alpha = \cup_{j\in[\alpha]} A_j$. The aggregated fluxes are

$$N_{\alpha\beta} = \sum_{j\in[\alpha],k\in[\beta]} n_{jk} \qquad (5)$$

The latter yields by (3) new export and import weights $W_{\alpha\beta}$ and $W^*_{\alpha\beta}$, which in turn yield new stationary distributions Π_α and Π^*_α, to be interpreted as a prominence index of the region B_α. The following question now arises: is the prominence index of B_α equal to the sum of the prominence indices of its constituents, i.e. are the following equalities

$$\Pi_\alpha = \sum_{j\in[\alpha]} \pi_j \qquad\qquad \Pi^*_\alpha = \sum_{j\in[\alpha]} \pi^*_j \tag{6}$$

always true? If yes, spatial interaction is said to be *aggregation invariant.* The following characterization theorem yields necessary and sufficent conditions for aggregation invariance, ensuring the value of the importance index of a region to be independent of the partitioning chosen:

Theorem 2

Spatial interaction is aggregation invariant iff one (or both) of the following conditions holds:

- the interaction is independent, i.e. $n_{jk} = \alpha_j \beta_k$
- the interaction is globally balanced, i.e. $n_{j\bullet} = n_{\bullet j}$

Proof: The aggregation $\mathcal{A} \to \mathcal{B}$ can be obtained as the composition of $\tilde{m} - m$ pair aggregations. Consider the case where two formerly distinct regions A_j and $A_{j'}$ are aggregated into the single region $A_{[j\cup j']} := A_j \cup A_{j'}$. Rewriting definitions (corresponding to the export case), one gets aggregation invariance if and only if the following holds

$$\frac{n_{jk} + n_{j'k}}{n_{j\bullet} + n_{j'\bullet}}(\pi_j + \pi_{j'}) = \frac{n_{jk}}{n_{j\bullet}}\pi_j + \frac{n_{j'k}}{n_{j'\bullet}}\pi_{j'} \tag{7}$$

for all $k \neq j, j'$. Multiplying the latter by denominators and after simplification, one verifies (7) to hold if and only if one of the following condition (or both) hold:

- $n_{jk} n_{j'\bullet} = n_{j'k} n_{j\bullet}$ (independence)
- $n_{j\bullet} \pi_{j'} = n_{j'\bullet} \pi_j$ (global balance)

Those conditions are invariant by transposition of n_{jk}, thus solving the problem for the import case too. Also, the conditions themselves are aggregation invariant, thus solving the problem for any aggregation $\mathcal{A} \to \mathcal{B}$. □

Note that gravity models are generally not aggregation invariant. This difficulty (recognized at the interaction level rather than the prominence index level) has inspired many workers (Schwab and Smith, 1985, and references therein). Breaking of aggregation invariance does not ruin the concept of prominence π_j of a region, but simply makes it dependent upon the partitioning chosen. In contrast, aggregation invariant interactions permit us, proceeding backwards, to define prominence *densities* $\pi(x)$ by considering a sequence of increasingly fine partitions: taking the limit always yields the same $\pi(x)$, independently of the details of the sequence. The prominence index of any region $A_j \subset \Omega$ is then simply $\pi_j = \int_{A_j} dx\ \pi(x)$.

The following diagram summarizes the structural relationships described in theorems 1 and 2.

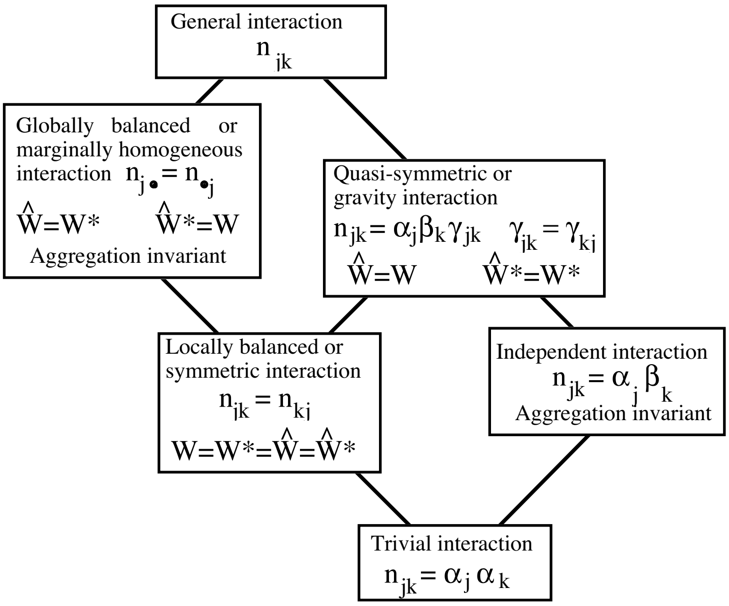

Figure 3

Models are increasingly structured in a downwards direction. Bottom models inherit the properties of linked top models. Also, the diagram emphasizes the equivalences "marginal homogeneity + quasi-symmetry = symmetry" and "independent + symmetric interaction = trivial interaction".

4 Symmetric distance decay models

4.1 Distance decay models for points

Example 4: let for convenience D stand for the *square* of the euclidian distance, and consider a positive, decreasing, smooth function $f(D)$, defining *symmetric* fluxes between points x_j and x_k as:

$$n_{jk} := f(D_{jk})(1-\delta_{jk}) \qquad \text{with} \qquad D_{jk} = \sum_{\alpha=1}^{2}(x_{j\alpha}-x_{k\alpha})^2 \tag{8}$$

The factor $(1-\delta_{jk})$ precludes self-interaction, thus diagonal weights vanish; but as far as $f(0)$ is finite, self-interaction could be allowed, and would play little role in what follows. (8) represents a purely spatial caricature of the gravity model where all propulsiveness and attractiveness are constant. By theorem 1c), stationary weights are

$$\pi_j = \frac{\sum_{k=1;k\neq j}^{m} f(D_{jk})}{\sum_{k,l=1;k\neq l}^{m} f(D_{lk})} \tag{9}$$

If no *characteristic intrinsic length* (such as the maximum length of a non-stop trip, or the indifference trip length between walking and driving) exists, it makes sense to require the stationary distribution to be invariant w.r.t. the scaling $D_{jk} \to \lambda^2 D_{jk}$, thus yielding the algebraic decay $f(D) = cD^{-r}$ with $c, r > 0$. If cost differences are the only thing that matters, and if costs are quadratic in euclidean distances, then (9) should remain invariant w.r.t. $D_{jk} \to D_{jk} + \lambda$, thus yielding the gaussian decay $f(D) = c\exp(-\beta D)$ with $c, \beta > 0$. Algebraic, exponential or logistic forms for $f(D)$ are common proposals in the literature (Taylor 1975). In the maximum entropy approach (Wilson 1971), β defined above is interpretable as an inverse temperature. Long (resp. small) distance limit considered below can thus also be interpreted as a high (resp. low) temperature limit, or as a cheap (resp. expensive) travel limit.

It is instructive to identify the position x_j maximizing π_j, the positions of other points x_k, $k \neq j$ being fixed. By (9), the sought for coordinates are

$$x_{j\alpha} = \frac{\sum_{k;k\neq j}^{m} f'(D_{jk})\ x_{k\alpha}}{\sum_{k;k\neq j}^{m} f'(D_{jk})} \tag{10}$$

Therefore, π_j is maximum when x_j is a weighted average of other places' coordinates, with positive weights $-f'(D_{jk})$, in complete conformity with the centrality or prominence index interpretation of π_j.

4.2 Aggregation: distance decay models for regions

Example 5 : consider a positive decreasing smooth decay function $f(D)$, normalized for convenience as $\int_0^\infty dD\ f(D) = 1$, thus interpretable as an interaction density at (squared euclidian) distance D. Significantly, the size invariant function $f(D) = cD^{-r}$ is non-normalizable.
Consider also the normalized rescaled density $f_\tau(D) := \tau f(\tau D)$, where $\tau > 0$ represents an inverse characteristic distance. Let the total region of interest to be partioned in m regions A_j, $j = 1, ..., m$.
One then defines additive, symmetric fluxes between regions $A_j \subset \Omega$ and $A_k \subset \Omega$ as

$$n_{jk}(\tau) := \int_{A_j} dx \int_{A_k} dy\ f_\tau((x-y)^2) \tag{11}$$

Denote by $\bar{x}_k$, resp. $\bar{x}_\Omega$ the gravity center of A_k, resp. Ω. We also define regional, resp. total spatial dispersions, as

$$s_k^2 := \frac{1}{|A_k|}\int_{A_k} dx\ (x-\bar{x}_k)^2 \qquad\qquad s_\Omega^2 := \frac{1}{|\Omega|}\int_{\Omega} dx\ (x-\bar{x}_\Omega)^2 \tag{12}$$

Assuming smoothness of $f(D)$ around $D = 0$, small τ (i.e. large characteristic distances) expansion reads

$$n_{jk}(\tau) = \tau|A_j||A_k|\{f(0) + f'(0)\tau[s_j^2 + s_k^2 + (\bar{x}_j - \bar{x}_k)^2]\} + 0(\tau^3) \tag{13}$$

where $|A_j|$ denotes the area of A_j. By symmetry, import and export weights coefficients (3) coincide. They are

$$w_{jk}(\tau) = \frac{|A_k|}{|\Omega|}\{1 + [s_k^2 - s_\Omega^2 + (\bar{x}_j - \bar{x}_k)^2 - (\bar{x}_j - \bar{x}_\Omega)^2]\frac{f'(0)}{f(0)}\tau\} + 0(\tau^2) \tag{14}$$

with associated stationary distribution

$$\pi_j(\tau) = \frac{|A_j|}{|\Omega|}\{1 + [s_j^2 - s_\Omega^2 + (\bar{x}_j - \bar{x}_\Omega)^2]\frac{f'(0)}{f(0)}\tau\} + 0(\tau^2) \tag{15}$$

To the lowest order in τ, prominence indices are proportional to the area of the region. First-order corrections *increase* this weight (recall $f'(0) \leq 0$):

- for regions of *small* spatial dispersion s_j^2, thus favoring approximately *circular* regions against elongated ones of same area

- for *central* regions ($(\bar{x}_j - \bar{x}_\Omega)^2$ small).

On average, those corrections cancel, as shown by the familiar variance decomposition $s_\Omega^2 = \sum_j \frac{|A_j|}{|\Omega|}[s_j^2 + (\bar{x}_j - \bar{x}_\Omega)^2]$. Also, (15) can be shown to be aggregation invariant ($\pi_{[j\cup j']} = \pi_j + \pi_{j'}$), as it must from theorem 2 and the symmetry of $n_{jk}(\tau)$.

Before tackling large τ expansion, relevant for small characteristic distances, let us introduce the following definitions: ∂A_{jk} denotes the common boundary between A_j and A_k. By construction, $\partial A_{jk} \neq \emptyset$ iff A_j and A_k are contiguous ($\partial A_{jj} = \emptyset$ by convention). Then $\partial A_j^{\text{int}} := \cup_{k=1}^m A_{jk}$ denotes the *internal* boundary of A_j, i.e. that part of the boundary shared with some other region A_k, whereas $\partial A_j^{\text{ext}}$ denotes the external boundary, i.e. that part of the boundary shared with $\partial\Omega$. The total boundary of A_j then obtains as $\partial A_j = \partial A_j^{\text{int}} \cup \partial A_j^{\text{ext}}$. Finally, $|\partial B|$ denotes the measure of the length (perimeter) of ∂B. For instance, $|A_j^{\text{int}}| = \sum_{k=1}^m |A_{jk}|$. Also, $|\partial A_j| = |\partial A_j^{\text{int}}| + |\partial A_j^{\text{ext}}|$ and $\sum_{j=1}^m |\partial A_j^{\text{ext}}| = |\partial\Omega|$

As $\lim_{\tau\to\infty} f_\tau((x-y)^2) = \delta((x-y)^2) = \pi\delta(x-y)$, fluxes are purely diagonal to the leading order. The next-order term is more tricky, and is non-zero for off-diagonal components iff A_j and A_k are adjacent. Performing integrations with local coordinates, parallel and orthogonal to ∂A_{jk}, and taking for convenience $f(D)$ as an arbitrary superposition of decreasing exponentials in D, one finally gets, hopefully without loss of generality:

$$n_{jk}(\tau) = \begin{cases} \pi|A_j| - |\partial A_j| L\tau^{-1/2} + 0(\tau^{-1}) & \text{if } j = k \\ |\partial A_{jk}| L\tau^{-1/2} + 0(\tau^{-1}) & \text{otherwise} \end{cases} \tag{16}$$

where $L := <\sqrt{D}> = \int_0^\infty dD\ \sqrt{D}\ f(D)$ is the unscaled average interaction distance. Now weights coefficients become

$$w_{jk}(\tau) = \delta_{jk} + \frac{L\tau^{-1/2}}{\pi|A_j|}\{|\partial A_{jk}| - \delta_{jk}|\partial A_j^{\text{int}}|\} + 0(\tau^{-1}) \tag{17}$$

with associated stationary distribution

$$\pi_j(\tau) = \frac{|A_j|}{|\Omega|} + \frac{L\tau^{-1/2}}{\pi|\Omega|}\{\frac{|A_j|}{|\Omega|}|\partial\Omega| - |\partial A_j^{\text{ext}}|\} + 0(\tau^{-1}) \tag{18}$$

To the lowest order in τ^{-1}, prominence indices are proportional to the area of the region. First-order corrections increase this weight for *large* ($|A_j|$ large), *non boundary* ($|\partial A_j^{\text{ext}}|$ small) regions. On average, those corrections cancel. Also, the aggregation invariance of (18) is obvious.

Example 5 makes it possible to introduce, in a minimal, unified setting, many traditional geometric ingredients aimed at the construction of spatial weights, previously proposed in the literature: the relative area $|A_j|/|\Omega|$ (at both ends of the long-short distance spectrum); the dispersion s_j^2 and the relative distance $(\bar{x}_j - \bar{x}_k)^2$, relevant for the long distance limit; the common boundary (and, hence, the connectivity) $|\partial A_{jk}|$ and the distinction internal/external boundary $|\partial A_j^{\text{int/ext}}|$, relevant for the short distance limit.

For the sake of concreteness, consider a 12×12 (arbitrary units) square universe Ω divided into 4 regions $A_1, \ldots, A_4$, as depicted in figure 4.

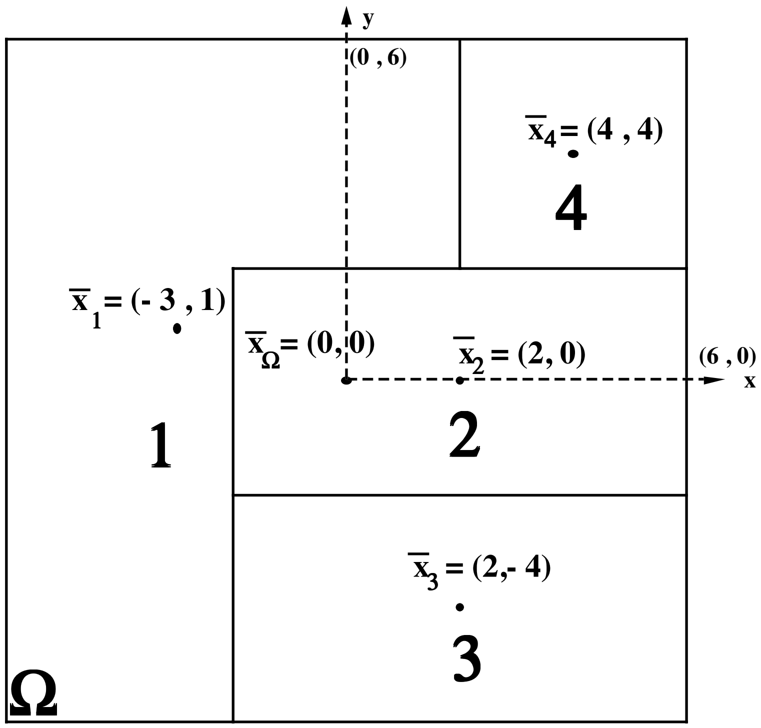

Figure 4

One verifies the corresponding areas, inner and outer boundaries, gravity centers and spatial dispersions to be:

$\lvert A_1\rvert = 64$	$\lvert\partial A_1{}^{\text{int}}\rvert = 16$	$\lvert\partial A_1{}^{\text{ext}}\rvert = 24$	$\lvert\partial A_1\rvert = 40$	$\bar{x}_1 = (-3, 1)$	$s_1^2 = 50/3$
$\lvert A_2\rvert = 32$	$\lvert\partial A_2{}^{\text{int}}\rvert = 20$	$\lvert\partial A_2{}^{\text{ext}}\rvert = 4$	$\lvert\partial A_2\rvert = 24$	$\bar{x}_2 = (2, 0)$	$s_2^2 = 20/3$
$\lvert A_3\rvert = 32$	$\lvert\partial A_3{}^{\text{int}}\rvert = 12$	$\lvert\partial A_3{}^{\text{ext}}\rvert = 12$	$\lvert\partial A_3\rvert = 24$	$\bar{x}_3 = (2, -4)$	$s_3^2 = 20/3$
$\lvert A_4\rvert = 16$	$\lvert\partial A_4{}^{\text{int}}\rvert = 8$	$\lvert\partial A_4{}^{\text{ext}}\rvert = 8$	$\lvert\partial A_4\rvert = 16$	$\bar{x}_4 = (4, 4)$	$s_4^2 = 8/3$
$\lvert\Omega\rvert = 144$		$\lvert\partial\Omega\rvert = 48$		$\bar{x}_\Omega = (0, 0)$	$s_\Omega^2 = 24$

Let $c := -f'(0)/f(0) \geq 0$. Then, in the limit $\tau \to 0$, one gets
$\pi_1(\tau) = 4/9 - (32/27)c\,\tau$
$\pi_2(\tau) = 2/9 + (80/27)c\,\tau$
$\pi_3(\tau) = 2/9 - (16/27)c\,\tau$
$\pi_4(\tau) = 1/9 - (32/27)c\,\tau$.
On average, the $0(\tau)$ corrections cancel; these corrections increase the relative prominence of region A_2. This was to be expected, the smallness of the ratio $|\partial A_2{}^{\text{ext}}|/|A_2|$ reflecting the centrality of region A_2.

Let $\tilde{c} := L/(\pi|\Omega|) \geq 0$. Then, in the limit $\tau \to \infty$, one gets
$\pi_1(\tau) = 4/9 - (8/3)\tilde{c}\,\tau^{-1/2}$
$\pi_2(\tau) = 2/9 + (20/3)\tilde{c}\,\tau^{-1/2}$
$\pi_3(\tau) = 2/9 - (4/3)\tilde{c}\,\tau^{-1/2}$
$\pi_4(\tau) = 1/9 - (8/3)\tilde{c}\,\tau^{-1/2}$

On average, the $0(\tau^{-1/2})$ corrections cancel; again, these corrections increase the relative prominence of region A_2, but for reasons differing from the preceeding: the centrality of region A_2 is now reflected by the centrality of its gravity center as well as by its relatively small spatial dispersion.

Corrections stand in proportions $-2 : 5 : -1 : -2$ for both limits; this constitutes a pure coincidence, somewhat favored by the simplicity of the geometry of figure 4. Needless to say, this simplicity is hardly representative of most partitionings encountered in geography; but still, the previous theoretical considerations might help discriminating between good and poor approximations aimed at simplifying a real situation: for instance, a complicated border between regions A_1 and A_2, as depicted by the solid line in figure 5, can be approximated by the indicated dotted polygonal line, provided $|a| + |c| + |e| = |b| + |d|$: this condition insures the invariance of the areas $|A_1|$ and $|A_2|$, and thus leaves in the first approximation the corresponding prominence indices unchanged.

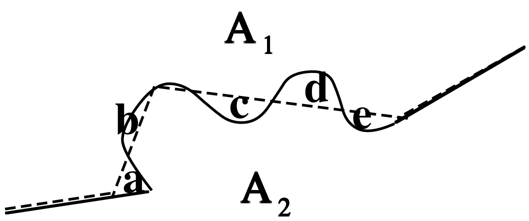

Figure 5

5 Integral geometric models

A potential source of relevant results towards a geometric definition of spatial weights is constituted by *Integral geometry* (Santalo 1976, Ambartzumian 1990), dealing with random measurements of a fixed geometric set with "solid" apparatus, as in Buffon's needle paradigmatic problem.

Consider an infinite straight line G in the plane together with a given regular curve C (closed or not) of length $L(C)$, and let $N(G, C)$ be the number of intersections of G with C. Let dG denote the infinitesimal measure element of all possible positions of G in the plane, differing by uniform *parallel translations* and rotations. dG plays for lines a role analog to the Lesbesgue measure dx for points, and can be written as $dG = dr\ d\phi$, where r is the distance of G with some fixed origin and $\phi \in [0, 2\pi]$ its direction (see figure 6). Then Poincaré's formula holds:

$$\int dG\ N(G, C) = 4L(C) \qquad (19)$$

By construction, $N(G, C) \geq 1$ iff $G \cap C \neq \emptyset$. Thus the integration region is de facto restricted to intersecting positions. Formula (19) has been used for approximatively measuring the length of curves by counting intersections with a lattice of equidistant parallels (Steinhaus 1954, Moran 1966).

Example 6 : a purely geometric spatial interaction model can be obtained by throwing randomly a long (actually infinite) stick G on a geographical map partitioned in m regions A_j, and to define symmetric fluxes as $n_{jk} := \int dG\ N(G, \partial A_{jk})$, where, as in the previous section, ∂A_{jk} denotes the boundary common to A_j and A_k.

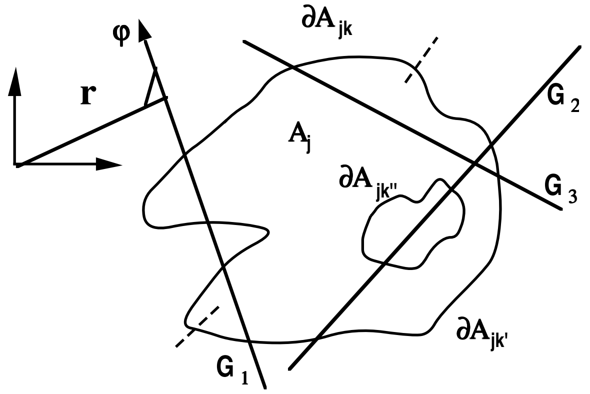

Figure 6

For instance, consider in figure 6 the region A_j, interior to Ω and contiguous to the regions A_k, $A_{k'}$ and $A_{k''}$. Thus $N(G_1, \partial A_{jk'}) = 1$, $N(G_1, \partial A_{jk}) = 3$, $N(G_1, \partial A_{jk''}) = 0$, $N(G_2, \partial A_{jk'}) = 2$, $N(G_2, \partial A_{jk}) = 0$, $N(G_2, \partial A_{jk''}) = 2$, $N(G_3, \partial A_{jk'}) = 1$, $N(G_3, \partial A_{jk}) = 1$, $N(G_3, \partial A_{jk''}) = 0$. Note in general $\sum_k N(G, \partial A_{jk}) = N(G, \partial A_j^{\text{int}})$ to be even with probability 1 if $\partial A_j^{\text{int}}$ is closed, i.e. if $\partial A_j^{\text{ext}} = \emptyset$, as realized here. By (19) and theorem 1, the associated spatial weights and prominence indices are

$$w_{jk} = w^*_{jk} = \frac{|\partial A_{jk}|}{|\partial A_j^{\text{int}}|} \qquad\qquad \pi_j = \pi^*_j = \frac{|\partial A_j^{\text{int}}|}{\sum_{k=1}^m |\partial A_k^{\text{int}}|} \tag{20}$$

Therefore, stick throwing yields familiar spatial weights, namely ones proportional to the length of the common border. In this setup, the prominence of a region is proportional to its total interregional boundary. Note that replacing the measuring stick by any other fixed bounded curve K of length $L(K)$ does not change the spatial weights (20), since

$$\int dK\ N(K, C) = 4L(K)L(C) \tag{21}$$

and factor $4L(K)$ cancels out in (20). In (21), the *kinematic density* dK runs over all *bidimensional translations* and rotations of the curve K.

Consider the aggregation $A_{[j\cup j']} := A_j \cup A_{j'}$ with $j \neq j'$. Then

$$\pi_{[j\cup j']} = \frac{|\partial A_j^{\text{int}}| + |\partial A_{j'}^{\text{int}}| - 2|\partial A_{jj'}|}{(\sum_{k=1}^m |\partial A_k^{\text{int}}|) - 2|\partial A_{jj'}|} \neq \pi_j + \pi_{j'} \tag{22}$$

unless $\partial A_{jj'} = \emptyset$. This apparent aggregation-invariance breaking seems to contradict theorem 2, since the fluxes n_{jk} are symmetric and thus globally balanced. Actually, the merging $A_{[j\cup j']} := A_j \cup A_{j'}$ is not an aggregation in the sense of paragraph 3.3, but rather an aggregation followed by the annihilation of those fluxes previously crossing the border $\partial A_{jj'}$, disappearing after merging. For instance, $n_{[j\cup j'],[j\cup j']} = 0 \leq n_{jj} + n_{jj'} + n_{j'j} + n_{j'j'} = 2n_{jj'}$.

Example 7 : let an organism randomly land on Ω with uniform (Lesbegue) density dx. The organism then moves in a random direction ϕ uniformly selected in $[0, 2\pi]$, until it crosses some internal border. Let the interaction n_{jk} be proportional to the probability to land in A_j and then reach region A_k.

In the integral geometric setup, this model may be expressed as $n_{jk} = \int dG\ \sigma_j(G)\ N(G, \partial A_{jk})$, where $\sigma_j(G)$ is the length of the chord $G \cap A_j$. To get some feeling for the formula, imagine the organism to land somewhere on $G_1 \cap A_j$ in figure 6. Choosing randomly the sense of motion (the unsigned direction being already fixed), the odds of meeting A_k against $A_{k'}$ are $N(G_1, \partial A_{jk})/N(G_1, \partial A_{jk'}) = 3/1$. On the other hand, the probability of landing on $G_1 \cap A_j$ (conditional upon landing in A_j) is proportional to $\sigma_j(G_1)$.

The interaction n_{jk}, although well defined, does not possess strong structural features in the sense of theorem 1, thus making the computation of the weights and the stationary distribution difficult. Let us however mention the property $n_{j\bullet} = 2\pi|A_j^{\text{int}}|$, valid if $\partial A_j^{\text{int}}$ is a convex closed curve.

Example 8 : the quantity

$$D(\partial A_j, \partial A_k) := \frac{1}{2} \int dG\ |N(G, \partial A_j) - N(G, \partial A_k)| \tag{23}$$

behaves as a metric index of dissimilarity between borders ∂A_j and ∂A_k (Santalo 1976). For instance, $D(\partial A_j, \partial A_k) = |\partial A_j| - |\partial A_k|$ if the convex closed curve ∂A_k lies inside the convex closed curve ∂A_j. The symmetric interaction $n_{jk} := f(D(\partial A_j, \partial A_k))$ (with $f(D)$ positive decreasing) yields large prominence values π_j for regions A_j whose borders have a small average dissimilarity with other borders. A variant of this model is obtained when considering inner borders $\partial A_j^{\text{int}}$ only. The same caveat met in example 6 about aggregation applies to examples 7 and 8 as well.

6 Spatial weights for Dirichlet-Voronoi tessellations

Example 9: let $x_1, ..., x_m$ be m distinct points (e.g. cities or shops) in some bounded planar set Ω. The Dirichlet-Voronoi cells $A_j := \{x \in \Omega \mid d(x, x_j) \leq d(x, x_l)\ \forall l \neq j\}$, where $d(x, y)$ denotes the euclidean distance, constitute a partition of Ω (see e.g. Brassel and Reif 1979, Møller 1994). Consider also the sets $A_{jk} := \{x \in \Omega \mid d(x, x_j) \leq d(x, x_k) \leq d(x, x_l)\ \forall l \neq j, k\}$ if $j \neq k$, and $A_{jj} := \emptyset$. The set A_j consists of the points of Ω with x_j as nearest neighbor among the m points, and A_{jk} is the part of A_j with x_k as second nearest neighbor, so that if x_j were removed, A_k would be enlarged by adding A_{jk} to it. Note that $|A_{jk}| \neq |A_{kj}|$, although both values are zero or strictly positive together (in the latter case, x_j and x_k are said to be neighbors).
In the context of distance-minimizing customers, A_{jk} can be thought of as the area primarily served by shop x_j, and otherwise by shop x_k when the former is closed. Let us consider the interaction model $n_{jk} := |A_{jk}|$. The associated export and import weights

$$w_{jk} := \frac{|A_{jk}|}{|A_j|} \qquad\qquad w^*_{jk} := \frac{|A_{kj}|}{\sum_l |A_{lj}|} \tag{24}$$

possess strict screening properties: they vanish if j and k are not neighbors. Export weights w_{jk} enjoy a remarkable property discovered by Sibson (1980), namely $\sum_k w_{jk}\ x_k = x_j$ for all inner x_j (i.e. satisfying $A_j \cap \partial\Omega = \emptyset$). This property opens up the possibility of a "natural neighbor interpolation", besides better known methods such as kriging or splines techniques (Cressie 1991).

Two- and higher-dimensional configurations do not possess analytical solutions in general. In one dimension, however, the problem can be solved fairly completely: consider the ordered $m+2$ points $x_0 \leq x_1 \leq ... \leq x_m \leq x_{m+1}$. Non zero export weights are $w_{01} = w_{m+1,m} = 1$ and

$$w_{j,j-1} = \frac{x_{j+1} - x_j}{x_{j+1} - x_{j-1}} \qquad w_{j,j+1} = \frac{x_j - x_{j-1}}{x_{j+1} - x_{j-1}} \tag{25}$$

for $1 \leq j \leq m$. Others weights are zero (screening), and one verifies the distribution

$$\pi_j = \begin{cases} \frac{\kappa}{x_{j+1}-x_j} + \frac{\kappa}{x_j - x_{j-1}} & = \frac{\kappa}{d_j} + \frac{\kappa}{d_{j-1}} & \text{if } 1 \leq j \leq m \\ \frac{\kappa}{x_1 - x_0} & = \frac{\kappa}{d_0} & \text{if } j = 0 \\ \frac{\kappa}{x_{m+1}-x_m} & = \frac{\kappa}{d_m} & \text{if } j = m+1 \end{cases} \tag{26}$$

to be solution of the stationarity equation $\sum_{j=0}^{m+1} \pi_j w_{jk} = \pi_k$, where the $d_j := x_{j+1} - x_j$ $(j = 0, \ldots, m)$ denote the interpoint distances. The normalization constant $\kappa > 0$ is determined by $\sum_j \pi_j = 1$, that is

$$\kappa = \frac{1}{2 \sum_{j=0}^{m} \frac{1}{d_j}} \tag{27}$$

On the other hand, non zero one-dimensional Dirichlet-Voronoi import weights for $x_0 \leq x_1 \leq ... \leq x_m \leq x_{m+1}$ are $w^*_{01} = w^*_{m+1,m} = 1$,

$$w^*_{10} = \frac{x_1 - x_0}{x_3 - x_2 + x_1 - x_0} \qquad w^*_{12} = \frac{x_3 - x_2}{x_3 - x_2 + x_1 - x_0} \tag{28}$$

$$w^*_{m,m-1} = \frac{x_{m-1} - x_{m-2}}{x_{m+1} - x_m + x_{m-1} - x_{m-2}} \qquad w^*_{m,m+1} = \frac{x_{m+1} - x_m}{x_{m+1} - x_m + x_{m-1} - x_{m-2}} \tag{29}$$

and

$$w^*_{j,j-1} = \frac{x_{j-1} - x_{j-2}}{x_{j+2} - x_{j+1} + x_{j-1} - x_{j-2}} \qquad w^*_{j,j+1} = \frac{x_{j+2} - x_{j+1}}{x_{j+2} - x_{j+1} + x_{j-1} - x_{j-2}} \tag{30}$$

for $2 \leq j \leq m-1$. The corresponding import prominence index turns out to be

$$\pi^*_j = \begin{cases} \kappa^*(d_{j-2}\, d_{j-1}\, d_j + d_{j-1}\, d_j\, d_{j+1}) & \text{if } 2 \leq j \leq m-1 \\ \kappa^* d_0^2\, d_1 & \text{if } j = 0 \\ \kappa^*\, (d_0^2\, d_1 + d_0\, d_1\, d_2) & \text{if } j = 1 \\ \kappa^*\, (d_{m-2}\, d_{m-1}\, d_m + d_{m-1}\, d_m^2) & \text{if } j = m \\ \kappa^*\, d_{m-1}\, d_m^2 & \text{if } j = m+1 \end{cases} \tag{31}$$

with the normalization

$$\kappa^* = \frac{1}{2 \sum_{j=0}^{m} d_{j-1}\, d_j\, d_{j+1}} \tag{32}$$

with $d_j = x_{j+1} - x_j$ $(j = 0, \ldots, m)$, $d_{-1} := d_0 = x_1 - x_0$ and $d_{m+1} := d_m = x_{m+1} - x_m$.
Figure 7 and 8 illustrate prominence indexes in both cases, for the same configuration with interpoint spacings in simple proportions $1 : 2 : 4 : 6$. In the distance-minimizing customers interpretation, shop x_j $(1 \leq j \leq m)$ inherits almost the totality of customers primarily attached to shops $x_{j\pm1}$ if the distance $|x_j - x_{j\pm1}|$ is small. As a result, export prominence indices are *large* for places possessing close neighbors. In the limit $d_j \to 0$ of two coinciding points x_j and x_{j+1}, the mass of the distribution actually becomes entirely concentrated on π_j and $\pi_{j+1} = 1 - \pi_j$: places x_j and x_{j+1} act as an *interaction trap*, for $w_{j,j+1} \to 1$ and $w_{j+1,j} \to 1$ when $d_j \to 0$ (one can think of a consumer wandering from one shop to the next neareast one). Conversely, one

can show $\pi_j(x_j)$ to be *minimal at the local equidistant location* $x_j = \frac{1}{2}(x_{j+1} + x_{j-1})$ (the other coordinates x_k, $k \neq j$, being fixed).

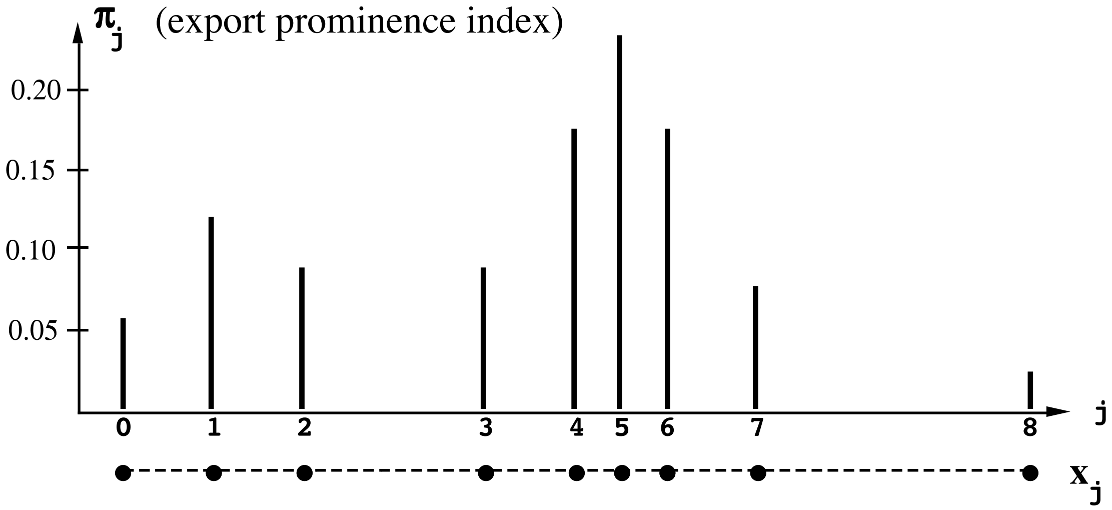

Figure 7

On the other hand, a shop x_j located in a dense area possesses a small basin of customers attached to it. As a consequence, when the shop x_j is closed, the added flux of customers towards its neighbors $x_{j\pm1}$ is small too: thus, in the import view, prominence indices are *small* for places possessing close neighbors. Conversely, one can show $\pi_j(x_j)$ $(2 \leq j \leq m-1)$ to be *maximal near the local equidistant location* (the other coordinates x_k, $k \neq j$, being fixed), the maximum being attained at $x_j = \frac{1}{2}(x_{j+1} + x_{j-1}) + a$, where $|a| << d_j + d_{j-1}$, and $a > 0$ iff $d_{j+1}d_{j+2} > d_{j-3}d_{j-2}$. For instance, $d_6d_7 > d_2d_3$ in figure 8, thus displacing x_5 by a small amount to the right would increase π_5.

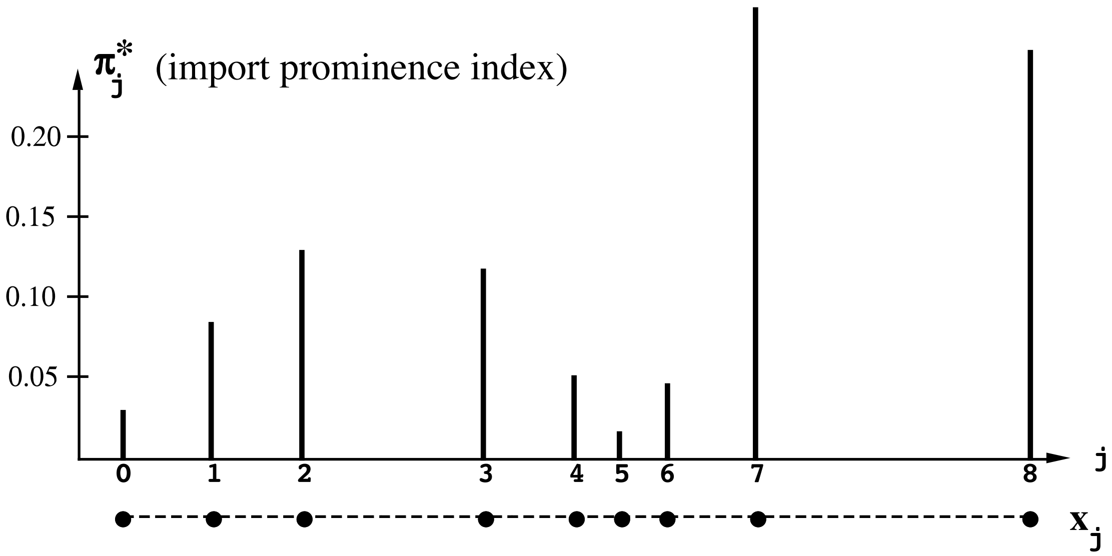

Figure 8

The strong contrast exhibited by export and import quantities (whose prominence indexes are, among other things, not aggregation invariant) in this simple model is made possible by the absence of structural constraints for the fluxes: the interaction $|A_{jk}|$ is indeed not globally balanced nor quasi-homogeneous. This again underlines the necessity of the existence of additional constraints (such as those described in theorems 1 and 2, among which gravity constraints are the most familiar in geography) for allowing tractable modelling in general.

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