Boundary non-crossing probabilities for fractional Brownian motion with trend

Enkelejd Hashorva[1], Yuliya Mishura[2], and Oleg Seleznjev[3]

January 27, 2015

Abstract: In this paper we investigate the boundary non-crossing probabilities of a fractional Brownian motion considering some general deterministic trend function. We derive bounds for non-crossing probabilities and discuss the case of a large trend function. As a by-product we solve a minimization problem related to the norm of the trend function.

Key Words: boundary crossings; Cameron-Martin-Girsanov theorem; reproducing kernel Hilbert space; large deviation principle; Molchan martingale; fractional Brownian motion.

1 Introduction

Calculation of boundary crossing (or non-crossing) probabilities of Gaussian processes with trend is a long-established and interesting topic of applied probability, see, e.g., [15, 32, 27, 12, 30, 28, 11, 5, 8, 7, 9, 14, 10, 21, 3, 18] and references therein. Numerous applications concerned with the evaluation of boundary non-crossing probabilities relate to mathematical finance, risk theory, queueing theory, statistics, physics, biology among many other fields. In the literature, most of contributions treat the case when the Gaussian process $X(t), t \geq 0$ is a Brownian motion which allows to calculate the boundary non-crossing probability $\mathbb{P}\{X(t) + f(t) < u, t \in [0, T]\}$, for some trend function f and two given constants $T, u > 0$ by various methods (see, e.g., [1, 16]). For particular f including the case of a piecewise constant function, explicit calculations are possible, see, e.g., [20]. Those explicit calculations allow then to approximate the non-crossing probabilities for trend functions of the form $\gamma f(t), t \in [0, T]$ when γ tends to infinity, see [20, 17, 5].

In this paper the centered Gaussian process $X = B^H$ is a fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$ for which no explicit calculations of the boundary non-crossing probability are possible for the most of the trend functions.

Therefore, our interest in this paper is on the derivation of upper and lower bounds for

$$P_f := \mathbb{P}\left\{B^H(t) + f(t) \leq u(t), \forall t \in \mathbb{R}_+\right\} \tag{1}$$

for some admissible trend functions f and measurable functions $u : \mathbb{R}_+ \to \mathbb{R}$ such that $u(0) \geq 0$. In the following we shall consider $f \neq 0$ to belong to the reproducing kernel Hilbert Space (RKHS) of B^H which is denoted by $\mathcal{H}$ defined by the covariance kernel of B^H given as

$$R_H(s,t) := \mathbb{E}\left\{B^H(s)B^H(t)\right\} = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad t, s \geq 0. \tag{2}$$

A precise description of $\mathcal{H}$ is given in Section 2, where also the norm $\|f\|_{\mathcal{H}}$ for $f \in \mathcal{H}$ is defined; for notational simplicity we suppress the Hurst index H and the specification of $\mathbb{R}_+$ avoiding the more common notation $\mathcal{H}_H(\mathbb{R}_+)$.

The lack of explicit formulas (apart from $H = 1/2$ case) for trend functions f poses problems for judging the accuracy of our bounds for P_f. A remedy for that is to consider the asymptotic performance of the bounds

[1]Department of Actuarial Science, University of Lausanne, UNIL-Dorigny, 1015 Lausanne, Switzerland, email:enkelejd.hashorva@unil.ch

[2]Department of Probability, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, 01601 Volodymyrska 64, Kyiv, Ukraine, email: myus@univ.kiev.ua

[3]Department of Mathematics and Mathematical Statistics, Umeå University, SE-901 87 Umeå, Sweden, email:oleg.seleznjev@matstat.umu.se

for trend functions γf with $\gamma \to 0$ and $\gamma \to \infty$. The latter case is more tractable since if for some x_0 we have $f(x_0) > 0$, then (see Corollary 3.1 below)

$$\ln P_{\gamma f} \geq -(1+o(1))\frac{\gamma^2}{2}\|\widehat{f}\|^2_{\mathcal{H}}, \quad \gamma \to \infty, \tag{3}$$

where $\widehat{f} \in \mathcal{H}, \widehat{f} \geq f$ is such that it solves the following minimization problem

$$\text{find the unique } \widehat{f} \in \mathcal{H} \text{ so that } \inf_{g,f\in\mathcal{H},g\geq f}\|g\|_{\mathcal{H}} = \|\widehat{f}\|_{\mathcal{H}}. \tag{4}$$

Clearly, (3) does not show how to find $\widehat{f}$, however it is very helpful for the derivation of upper and lower bounds for P_f since it can be used to check their validity (at least asymptotically).

In this paper, for $f \in \mathcal{H}$ with $f(x_0) > 0$ for some $x_0 > 0$, we find explicitly for $H > 1/2$ the unique solution $\widehat{f} \in \mathcal{H}$ of the minimization problem (4); for $H = 1/2$ this has already been done in [6]. For the case $H \in (0, 1/2)$, we determine again $\widehat{f}$ under the assumption that $\widehat{f} \geq f$. By making use of the Girsanov formula for fBm, we derive upper and lower bounds for P_f in Theorem 3.1 below.

The paper is organized as follows: Section 2 briefly reviews some results from fractional calculus and related Hilbert spaces. We introduce weighted fractional integral operators, fractional kernels and briefly discuss the corresponding reproducing kernel Hilbert spaces. The main result is presented in Section 3. Specific properties of fBm used in our proofs are displayed in Section 4 followed then by two examples for the trend function. All the proofs are relegated to Section 5. A short Appendix concludes the article.

2 Preliminaries

This section reviews basic Riemann-Liouville fractional calculus; a classical reference on this topic is [31]. We use also the notation and results from [29], [2], and [22]. We proceed then with the RKHS of fBm.

Definition 2.1. The (left-sided) Riemann-Liouville fractional integral operator of order α over interval $[0, T]$ (or over $\mathbb{R}_+$) is defined for α and T positive by

$$\left(I^\alpha_{0+}f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-z)^{\alpha-1}f(z)dz, \quad t \in [0,T] \quad (t \in \mathbb{R}_+),$$

where $\Gamma(\cdot)$ is the Euler gamma function. The corresponding right-sided integral operator on $[0, T]$ is defined by

$$\left(I^\alpha_{T-}f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_t^T (z-t)^{\alpha-1}f(z)dz, \quad t \in [0,T],$$

and the right-sided integral operator on $\mathbb{R}_+$ (also known as the Weyl fractional integral operator) is defined by

$$\left(I^\alpha_{\infty-}f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_t^\infty (z-t)^{\alpha-1}f(z)dz, \quad t \in \mathbb{R}_+.$$

Throughout the paper, we suppose that $(I^\alpha_{T-}f)(t) = 0$, for $t > T$. Note that in the case $u^\alpha f(u) \in L_1(\mathbb{R}_+)$, the integral $(I^\alpha_{\infty-}f)$ exists a.a. with respect to Lebesgue measure on $\mathbb{R}_+$ and belongs to $L_1(\mathbb{R}_+)$, see for more details [19]. Next, for $p \geq 1$, denote

$$I^\alpha_+(L_p[0,T]) = \{f : f = I^\alpha_{0+}\varphi \text{ for some } \varphi \in L_p[0,T]\},$$

$$I^\alpha_-(L_p[0,T]) = \{f : f = I^\alpha_{T-}\varphi \text{ for some } \varphi \in L_p[0,T]\},$$

and define similarly $I^\alpha_-(L_p(\mathbb{R}_+))$. If $0 < \alpha < 1$, then the function φ used in the above definitions (it is determined uniquely) coincides for almost all (a.a.) $t \in [0,T]$ $(t \in \mathbb{R})$ with the left- (right-) sided Riemann-Liouville fractional derivative of f of order α. The derivatives are denoted by

$$(I_{0+}^{-\alpha}f)(t) = (\mathcal{D}_{0+}^{\alpha}f)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left(\int_0^t (t-z)^{-\alpha}f(z)dz\right),$$
$$(I_{\infty-}^{-\alpha}f)(t) = (\mathcal{D}_{\infty-}^{\alpha}f)(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left(\int_t^{\infty} (z-t)^{-\alpha}f(z)dz\right),$$

and

$$I_{T-}^{-\alpha}(t) = (\mathcal{D}_{T-}^{\alpha}f)(t) = (\mathcal{D}_{\infty-}^{\alpha}f1_{[0,T]})(t),$$

respectively. Let $f \in I_{-}^{\alpha}(L_p(\mathbb{R}))$ or $I_{\pm}^{\alpha}(L_p[0,T]), p \geq 1, 0 < \alpha < 1$. Then for the corresponding indices $0, T$, and ∞, we have

$$I_{\pm}^{\alpha}\mathcal{D}_{\pm}^{\alpha}f = f.$$

In the case when $f \in L_1(\mathbb{R}_+)$, we have $\mathcal{D}_{\pm}^{\alpha}I_{\pm}^{\alpha}f = f$, see e.g., [31]. In the following we introduce weighted fractional integral operators, fractional kernels and briefly discuss the corresponding RKHS's. Introduce weighted fractional integral operators by

$$(K_{0+}^{H}f)(t) = C_1 t^{H-1/2}(I_{0+}^{H-1/2}u^{1/2-H}f(u))(t),$$
$$(K_{0+}^{H,*}f)(t) = C_1^{-1} t^{H-1/2}(I_{0+}^{1/2-H}u^{1/2-H}f(u))(t),$$
$$(K_{\infty-}^{H}f)(t) = C_1 t^{1/2-H}(I_{\infty-}^{H-1/2}u^{H-1/2}f(u))(t),$$

and

$$(K_{\infty-}^{H,*}f)(t) = C_1^{-1} t^{1/2-H}(I_{\infty-}^{1/2-H}u^{H-1/2}f(u))(t),$$

where $C_1 = \left(\frac{2H\Gamma(H+1/2)\Gamma(3/2-H)}{\Gamma(2-2H)}\right)^{1/2}$. For $H = \frac{1}{2}$ with $\mathbf{I}$ the identity operator we put

$$K_{0+}^{H} = K_{0+}^{H,*} = K_{\infty-}^{1/2} = K_{\infty-}^{1/2,*} = \mathbf{I}.$$

If $H > \frac{1}{2}$ and $u^{H-\frac{1}{2}}f(u) \in L_1(\mathbb{R}_+)$, then $K_{\infty-}^{H,*}K_{\infty-}^{H}f = f$. We can change the order of the operators in the previous equality, i.e., $K_{\infty-}^{H}K_{\infty-}^{H,*}f = f$, provided that $u^{H-\frac{1}{2}}f(u) \in I_{-}^{H-\frac{1}{2}}(L_p(\mathbb{R}_+))$ for some $p \geq 1$ and $H > \frac{1}{2}$, or $u^{\frac{1}{2}-H}f(u) \in L_1(\mathbb{R}_+)$ holds if $H < \frac{1}{2}$. Furthermore, for $f \in L_2(\mathbb{R}_+)$ and $H \in (0,1)$, $K_{0+}^{H}K_{0+}^{H,*}f = f$.
Next, define $K_T^H f = K_{\infty-}^{H}(f1_{[0,T]})$ and $K_T^{H,*}f = K_{\infty-}^{H,*}(f1_{[0,T]})$. For $H \in (0,1)$ and $t > s$, define the fractional kernel

$$K_H(t,s) := \frac{C_1}{\Gamma\Big(H+1/2\Big)}\Big(\Big(\frac{t}{s}\Big)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t (z-s)^{H-\frac{1}{2}}z^{H-\frac{3}{2}}dz\Big).$$

For $H > \frac{1}{2}$, the kernel K_H is simplified to

$$K_H(t,s) = \frac{C_1}{\Gamma\Big(H-\frac{1}{2}\Big)}s^{\frac{1}{2}-H}\int_s^t (z-s)^{H-\frac{3}{2}}z^{H-\frac{1}{2}}dz.$$

In turn, introduce the fractional kernel

$$K_H^*(t,s) = \frac{1}{C_1\Gamma(H+1/2)}\Big(\Big(\frac{t}{s}\Big)^{H-\frac{1}{2}}(t-s)^{\frac{1}{2}-H} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t (z-s)^{\frac{1}{2}-H}z^{H-\frac{3}{2}}dz\Big).$$

For $H < \frac{1}{2}$, the kernel K_H^* is simplified to

$$K_H^*(t,s) = \frac{s^{\frac{1}{2}-H}}{C_1\Gamma(1/2-H)}\int_s^t (z-s)^{-H-\frac{1}{2}}z^{H-\frac{1}{2}}dz.$$

By direct calculations we obtain

$$(K_{\infty-}^{H}1_{[0,t]})(s) = (K_t^H 1_{[0,t]})(s) = K_H(t,s)$$

and

$$(K_{\infty-}^{H,*}1_{[0,t]})(s) = (K_t^{H,*}1_{[0,t]})(s) = K_H^*(t,s).$$

We mention that for $f \in L_p[a,b]$, $g \in L_q[a,b]$ with $0 < \frac{1}{p} + \frac{1}{q} \le 1 + \alpha$ we have the following formula

$$\int_a^b g(x) I_{a+}^{\alpha} f(x)dx = \int_a^b f(x) I_{b-}^{\alpha} g(x)dx.$$

Applying it for $H > \frac{1}{2}$ and $f \in L_p[0,t]$ with $p > 1$ we obtain

$$\int_0^t (K_{\infty-}^{H} 1_{[0,t]})(s) f(s)ds = \int_0^t (K_{0+}^H f)(s)ds.$$

If instead we fix $H < \frac{1}{2}$ and $f \in L_p[0,t]$ with $p > 1$, then we obtain further

$$\int_0^t (K_{\infty-}^{H,*} 1_{[0,t]})(s) f(s)ds = \int_0^t (K_{0+}^{H,*} f)(s)ds.$$

Next, we introduce the RKHS of the fBm, corresponding results for finite interval are described in detail in [13], [29], and [2]. Let $H \in (0,1)$ be fixed and recall that R_H defined in (2) can be defined also as follows

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,u) K_H(s,u) du.$$

Definition 2.2. ([2]) The reproducing kernel Hilbert space (RKHS) of the fractional Brownian motion on $[0,T]$, denoted by $\mathcal{H}[0,T]$ is defined as the closure of the vector space spanned by the set of functions $R_H(t,\cdot), t \in [0,T]$ with respect to the scalar product $\langle R_H(t,\cdot), R_H(s,\cdot)\rangle = R_H(t,s)$, $t,s \in [0,T]$.

In [13] it is shown that $\mathcal{H}[0,T]$ is the set of functions f which can be written as $f(t) = \int_0^t K_H(t,s)\phi(s)ds$ for some $\phi \in L_2([0,T])$. By definition, $\|f\|_{\mathcal{H}[0,T]} = \|\phi\|_{L_2[0,T]}$. We define similarly the RKHS $\mathcal{H} := \mathcal{H}(\mathbb{R}_+)$. Namely, for any $H \in (0,1)$, $\mathcal{H}$ is the set of functions f which can be written as

$$f(t) = \int_0^t K_H(t,s)\phi(s)ds = \int_0^t (K_{\infty-}^H 1_{[0,t]})(s)\phi(s)ds = \int_0^t (K_{0+}^H \phi)(s)ds \tag{5}$$

for some $\phi \in L_2(\mathbb{R}_+)$. Since $f'(t) = (K_{0+}^H \phi)(t)$ and $\phi(t) = (K_{0+}^{H,*} f')(t)$, then we have

$$\|f\|_{\mathcal{H}} = \|\phi\|_{L_2(\mathbb{R}_+)} = \|K_{0+}^{H,*} f'\|_{L_2(\mathbb{R}_+)}.$$

Next, define

$$L_2^H(\mathbb{R}_+) = \{f : K_{\infty-}^H |f| \in L_2(\mathbb{R}_+)\}$$

and if $H \in (0, \frac{1}{2})$, we define further

$$\widetilde{L}_2^H(\mathbb{R}_+) = L_2^H(\mathbb{R}_+) \cap \left\{ f : \mathbb{R}_+ \to \mathbb{R} : \lim_{T\to\infty} \int_0^T t^{1-2H} \left(\int_T^{\infty} u^{H-1/2} f(u)(u-t)^{H-3/2} du \right)^2 dt = 0 \right\}.$$

To this end, for any function g that admits the representation $g(t) = \int_0^t g'(s)ds$ introduce the norm

$$\|g\| = \|g'\|_{L_2(\mathbb{R}_+)}. \tag{6}$$

3 Main result

In this section we study the boundary non-crossing probability P_f defined in (1) for $f \in \mathcal{H}$ and some measurable function $u : \mathbb{R}_+ \to \mathbb{R}$ with $u(0) \ge 0$. Hereafter we assume that $P_0 = \mathbb{P}\left\{B^H(t) \le u(t), \forall t \in \mathbb{R}_+\right\} \in (0,1)$. In applications, see, e.g., [8, 9] it is of interest to calculate the rate of decrease to 0 of $P_{\gamma f}$ as $\gamma \to \infty$ for some $f \in \mathcal{H}$. On the other side, if $\|f\|_{\mathcal{H}}$ is small, we expect that P_f is close to P_0. Set below $\alpha = \Phi^{-1}(P_0)$ where Φ is the distribution function of a $N(0,1)$ random variable. Our first result displays upper and lower bounds for P_f.

Lemma 3.1. *For any $f \in \mathcal{H}$ we have*

$$\Big| P_f - P_0 \Big| \quad \leq \tfrac{1}{\sqrt{2\pi}} \|f\|_{\mathcal{H}}. \tag{7}$$

If further $g \in \mathcal{H}$ is such that $g \geq f$, then

$$\Phi(\alpha - \|g\|_{\mathcal{H}}) \leq P_g \leq P_f \leq \Phi(\alpha + \|f\|_{\mathcal{H}}). \tag{8}$$

Clearly, (7) is useful only if $\|f\|_{\mathcal{H}}$ is small. On the contrary, the lower bound of (8) is important for f such that $\|f\|_{\mathcal{H}}$ is large and $\|g\|_{\mathcal{H}} > 0$. Taking $g = \widehat{f}$, with $\widehat{f}$ being the solution of (3) and noting that for any $\gamma > 0$ we have $\widehat{\gamma f} = \gamma \widehat{f}$ for any $f \in \mathcal{H}$, then the lower bound in (8) implies the following result:

Corollary 3.1. *For any $f \in \mathcal{H}$ such that $f(x_0) > 0$ for some $x_0 \in (0, \infty)$ the claims in (3) and (4) hold.*

Next, let the function f be differentiable with derivative $f' \in L_2(\mathbb{R}_+)$. Then the operator $(K_{0+}^{H,*} f')$ is well-defined. Consider the following assumptions on f:

(i) $(K_{0+}^{H,*} f') \in L_2(\mathbb{R}_+)$, i.e., $f \in \mathcal{H}$.

(ii) Let $h(t) := \int_0^t (K_{0+}^{H,*} f')(s) ds$. We assume that the smallest concave nondecreasing majorant $\widetilde{h}$ of the function h has the right-hand derivative $\widetilde{h}'$ such that $\widetilde{h}' \in L_2(\mathbb{R}_+)$ and moreover the function

$$K(t) := (K_{\infty-}^{H,*} \widetilde{h}')(t)$$

is nonincreasing, $K \in L_2^H(\mathbb{R}_+)$ for $H > \frac{1}{2}$ and $K \in \widetilde{L}_2^H(\mathbb{R}_+)$ for $H < \frac{1}{2}$,

$$K(t) = o(t^{-H}) \text{ as } t \to \infty.$$

(iii) The function $\widetilde{h}'$ can be presented as $\widetilde{h}'(t) = (K_{0+}^{H,*} \hat{f}')(t)$, $t \in \mathbb{R}_+$, for some $\hat{f}' \in L_2(\mathbb{R}_+)$. Evidently, in this case the function $\widetilde{h}$ admits the representation $\widetilde{h}(t) = \int_0^t (K_{0+}^{H,*} \hat{f}')(s) ds$.

We set next for f satisfying $(i) - (iii)$

$$\widehat{f}(t) = \int_0^t \hat{f}'(s) ds = \int_0^t (K_{0+}^H \widetilde{h}')(s) ds.$$

The function $\widehat{f}$ is crucial for our problem as will be shown in our main result below.

Theorem 3.1. *If f satisfies assumptions (i)–(iii), then $\widehat{f} \in \mathcal{H}$ and moreover*

$$P_f \leq P_{f - \widehat{f}} \exp\left(\int_0^\infty u(s) d(-K(s)) - \frac{1}{2} \|\widetilde{h}\|^2 \right). \tag{9}$$

Suppose that $u_- : \mathbb{R}_+ \to \mathbb{R}$ is such that $u_-(t) < u(t), t \in \mathbb{R}_+$. If $H < 1/2$, assume additionally that $\widehat{f} \geq f$. Then for any $H \in (0,1) \setminus \{1/2\}$

$$P_f \geq P_{\widehat{f}} \geq \mathbb{P}\left\{ u_-(t) \leq B^H(t) \leq u(t), t \in \mathbb{R}_+ \right\} \exp\left(I_K - \frac{1}{2} \|\widetilde{h}\|^2 \right) \tag{10}$$

holds, provided that $I_K = \int_0^\infty u_-(s) d(-K(s))$ is finite.

In the next corollary we show that the upper and lower bounds above become (in the log scale) precise when f is large.

Corollary 3.2. *Under the assumptions and notation of Theorem 3.1, if further $f(x_0) > 0$ for some $x_0 \in (0, \infty)$, then*

$$-\ln P_{\gamma f} \sim \frac{\gamma^2}{2} \|\widetilde{h}\|^2, \quad \gamma \to \infty. \tag{11}$$

As a by-product, we solve the minimization problem (4), namely we have:

Corollary 3.3. *Under the assumptions and notation of Theorem 3.1*

$$\inf_{f,g \in \mathcal{H}, g \geq f} \|g\|_{\mathcal{H}} = \|\widehat{f}\|_{\mathcal{H}} = \|\widetilde{h}\|. \tag{12}$$

Remarks: a) If $H \in (1/2, 1)$, then under the conditions (i)–(iii) in Theorem 3.1 we find that $\widehat{f}$ is the explicit solution of the minimization problem (4).

b) The case $H = 1/2$ is discussed in [4], see also [6].

c) It follows from Lemma 7.1 that for $H > \frac{1}{2}$, $\widehat{f} \geq f$ because it immediately follows from that lemma and the inequality $\tilde{h} \geq h$ that $\hat{f}' \geq f'$.

4 Auxiliary results

For the proof of our main result, we need to discuss several properties of fBm's. We shall investigate first the relation between fBm, Molchan martingale, and the underlying Wiener process. Then we present the Girsanov Theorem which is crucial for our analysis.

4.1 Molchan martingale, fBm, and the underlying Wiener process

In what follows we consider a continuous modification of fBm that exists due to well-known Kolmogorov's theorem. Denote by $\mathcal{F}^{B^H} = \{\mathcal{F}_t^{B^H}, t \in \mathbb{R}_+\}$ with $\mathcal{F}_t^{B^H} = \sigma\{B^H(s), 0 \leq s \leq t\}$ the filtration generated by B^H. According to [29], [2], [22], and [26], B^H can be presented as

$$B^H(t) = \int_0^t (K_{\infty-}^H 1_{[0,t]})(s) dW(s) = \int_0^t (K_t^H 1_{[0,t]})(s) dW(s) = \int_0^t K_H(t,s) dW(s), \tag{13}$$

where $W = \{W(t), t \in \mathbb{R}_+\}$ is an "underlying" Wiener process whose filtration coincides with $\mathcal{F}^{B^H}$. Evidently,

$$W(t) = \int_0^t (K_{\infty-}^{H,*} 1_{[0,t]})(s) dB^H(s) = \int_0^t (K_t^{H,*} 1_{[0,t]})(s) dB^H(s) = \int_0^t K_H^*(t,s) dB^H(s). \tag{14}$$

Another form of relations (13) and (14) can be obtained in the following way. According to [26], we can introduce the kernel

$$l_H(t,s) = \left(\frac{\Gamma(3-2H)}{2H\Gamma(3/2-H)^3 \Gamma(H+1/2)} \right)^{1/2} s^{1/2-H} (t-s)^{1/2-H} 1_{[0,t]}(s), \quad s,t \in \mathbb{R}_+ \tag{15}$$

and consider the process

$$M^H(t) = \int_0^t l_H(t,s) dB^H(s), \quad t \in \mathbb{R}_+, \quad H \in (0,1). \tag{16}$$

The process M^H from (16) defines a Gaussian square-integrable martingale with square characteristics $\langle M^H \rangle(t) = t^{2-2H}$, $t \in \mathbb{R}_+$, and with filtration $\mathcal{F}^{M^H} \equiv \mathcal{F}^H$. Then the process $\widetilde{W}(t) = (2-2H)^{-1/2} \int_0^t s^\alpha dM^H(s)$ is a Wiener process with the same filtration. In what follows we consider a continuous modification of any Wiener process. We state next three lemmas which are proved in Section 6.

Lemma 4.1. *The processes $\widetilde{W}$ and W coincide, i.e., are indistinguishable.*

Definition 4.1. ([2], [22], [29]) For any $T \in \mathbb{R}_+$ and $H \in (0,1)$ the Wiener integral w.r.t. fBm is defined as

$$\begin{aligned}\int_0^T f(s)dB^H(s) &= \int_0^T (K^H_{\infty-}(f1_{[0,T]}))(s)dW(s) = \int_0^\infty (K^H_{\infty-}(f1_{[0,T]}))(s)dW(s)\\ &= \int_0^\infty (K^H_T f)(s)dW(s) = \int_0^T (K^H_T f)(s)dW(s)\end{aligned}$$

and the integral $\int_0^T f(s)dB^H(s)$ exists for $f \in L_2^H(\mathbb{R}_+)$.

Now we extend the notion of integration w.r.t. fBm on the $\mathbb{R}_+$ from $[0,T]$ by the following definition.

Definition 4.2. We set

$$\int_0^\infty f(s)dB^H(s) = L_2\text{-}\lim_{T\to\infty} \int_0^T f(s)dB^H(s) \tag{17}$$

whenever this limit exists.

Lemma 4.2. *If $f \in L_2^H(\mathbb{R}_+)$ with $H > \frac{1}{2}$ and $f \in \widetilde{L}_2^H(\mathbb{R}_+)$ for $H < \frac{1}{2}$, then the limit in the right-hand side of (17) exists and*

$$\int_0^\infty f(s)dB^H(s) = \int_0^\infty (K^H_{\infty-}f)(s)dW(s). \tag{18}$$

Lemma 4.3. *Let $h = h(t), t \in R_+$, be a nonrandom measurable function such that*

1. *$h \in L_2^H(\mathbb{R}_+)$ for $H > \frac{1}{2}$ and $h \in \widetilde{L}_2^H(\mathbb{R}_+)$ for $H < \frac{1}{2}$;*
2. *h is nonincreasing;*
3. *$s^H h(s) \to 0$ as $s \to \infty$.*

Then the integral $\int_0^\infty h(s)dB^H(s)$ exists and moreover

$$\int_0^\infty h(s)dB^H(s) = \int_0^\infty B^H(s)d(-h(s)), \tag{19}$$

where the integral in the right-hand side of (19) *is a Riemann-Stieltjes integral with continuous integrand and nondecreasing integrator.*

4.2 Girsanov Theorem for fBm

Let $H \in (0,1)$ and consider a fBm with absolutely continuous drift f that admits the following representation: $B^H(t) + f(t) = B^H(t) + \int_0^t f'(s)ds$. In order to annihilate the drift, there are two equivalent approaches. The first one is to assume that $K_H^*(t,\cdot)f'(\cdot) = (K^{H,*}_{0+}f')(\cdot) \in L_1[0,t]$ for any $t \in R_+$, to equate

$$B^H(t) + f(t) = \widehat{B}^H(t), \tag{20}$$

where $\widehat{B}^H$ is the fBm with respect to the new probability measure, and accordingly to (14), to transform (20) as

$$\int_0^t (K^{H,*}_{\infty-}1_{[0,t]})(s)dB^H(s) + \int_0^t (K^{H,*}_{\infty-}1_{[0,t]})f'(s)ds = \int_0^t (K^{H,*}_{\infty-}1_{[0,t]})(s)d\widehat{B}^H(s),$$

or,

$$\int_0^t K_H^*(t,s)dB^H(s) + \int_0^t K_H^*(t,s)f'(s)ds = \int_0^t K_H^*(t,s)d\widehat{B}^H(s),$$

or, at last,

$$W(t) + \int_0^t (K^{H,*}_{\infty-}1_{[0,t]})(s)f'(s)ds = W(t) + \int_0^t (K^{H,*}_{0+}f')(s)ds = \widehat{W}(t),$$

where $\widehat{W} = \{\widehat{W}_t, t \in \mathbb{R}_+\}$ is a Wiener process with respect to a new probability measure Q, say. The second one is to apply Girsanov's theorem from [25]. We start with (20); suppose that $s^{\frac{1}{2}-H}f'(s) \in L_1[0,t]$ for any $t \in \mathbb{R}_+$ and transform (20) as follows (recall l_H is defined in (15)):

$$M^H(t) + \int_0^t l_H(t,s)f'(s)ds = \widehat{M}^H(t).$$

Further, suppose that the function $q(t) = \int_0^t l_H(t,s)f'(s)ds$ admits the representation

$$q(t) = \int_0^t q'(s)ds. \tag{21}$$

Then

$$(2-2H)^{\frac{1}{2}} \int_0^t s^{\frac{1}{2}-H} dW(s) + \int_0^t q'(s)ds = (2-2H)^{\frac{1}{2}} \int_0^t s^{\frac{1}{2}-H} d\widehat{W}(s),$$

whence $W(t) + (2-2H)^{-1/2} \int_0^t q'(s)s^{H-\frac{1}{2}}ds = \widehat{W}(t)$. Evidently, if the representation (21) holds, then

$$(2-2H)^{-1/2} \int_0^t q'(s)s^{H-\frac{1}{2}}ds = \int_0^t K_H^*(t,s)f'(s)ds = \int_0^t (K_{0+}^{H,*}f')(s)ds. \tag{22}$$

In turn, together with the equivalence, mentioned in the assumption (ii), it means that for $f \in \mathcal{H}$ we have (22). Now we give simple sufficient conditions of existence of q' and $\int_0^t q'(s)s^{H-\frac{1}{2}}ds$. The proof consists in differentiation and integration by parts and therefore it is omitted.

Lemma 4.4. *(i) Let $H < \frac{1}{2}$. Suppose that the drift f is absolutely continuous and for any $t > 0$, the derivative $|f'(s)| \le C(t)s^{H-\frac{3}{2}+\varepsilon}$, $s \le t$, for some $\varepsilon > 0$ and some nondecreasing function $C(t) : \mathbb{R}_+ \to \mathbb{R}_+$. Then for any $t > 0$*

$$q'(t) = \left(\frac{\Gamma(3-2H)}{2H\Gamma(3/2-H)^3\Gamma(H+1/2)}\right)^{1/2} \int_0^t s^{1/2-H}(t-s)^{-1/2-H}f'(s)ds$$

and (22) holds.

(ii) Let $H > \frac{1}{2}$ and suppose that the drift f is absolutely continuous. If further there exists the continuous derivative $(s^{\frac{1}{2}-H}f'(s))'$ such that $\lim_{s\to 0}(s^{\frac{1}{2}-H}f'(s))' = 0$, then for any $t > 0$

$$q'(t) = \left(\frac{\Gamma(3-2H)}{2H\Gamma(3/2-H)^3\Gamma(H+1/2)}\right)^{1/2} \int_0^t (t-s)^{1/2-H}(s^{\frac{1}{2}-H}f'(s))'ds$$

and (22) holds.

So, for a drift $f \in \mathcal{H}$, $B^H(t) + \int_0^t f'(s)ds$ is fBm $\widehat{B}^H(t)$, $t \in \mathbb{R}$, say, under such measure Q that

$$\begin{aligned} \frac{dQ}{dP} &= \exp\Big(-\int_0^\infty (K_{0+}^{H,*}f')(s)dW(s) - \frac{1}{2}\int_0^\infty |(K_{0+}^{H,*}f')(s)|^2 ds\Big) \\ &= \exp\Big(-\int_0^\infty (K_{0+}^{H,*}f')(s)dW(s) - \frac{1}{2}\|f\|_{\mathcal{H}}^2\Big) \end{aligned} \tag{23}$$

if (23) defines a new probability measure leading to the following result.

Theorem 4.1. *Let $f \in \mathcal{H}$, then $B^H(t) + \int_0^t f'(s)ds = \widehat{B}^H(t)$, where $\widehat{B}^H(t)$ is a fBm under a probability measure Q that satisfies relation (23).*

5 Examples of admissible drifts

We present next two examples of drifts satisfying conditions (i)-(iii) imposed in Theorem 3.1.

Example 5.1. In order to construct the drift, we start with h and $\widetilde{h}$. Let $H > \frac{1}{2}$, $h(t) = \widetilde{h}(t) = \int_0^t s^{1/2-H} e^{-s} ds$. Note that $\widetilde{h}' \in L_2(\mathbb{R}_+)$, $\widetilde{h}' > 0$ and decreases on $\mathbb{R}_+$, therefore $\widetilde{h}$ is a concave function as well as h, and evidently, $\widetilde{h}$ is the smallest concave nondecreasing majorant of h. Further we have

$$\begin{aligned}(K_{\infty-}^{H,*}\widetilde{h}')(t) &= -C_1^{-1} t^{1/2-H} \frac{d}{dt}\left(\int_t^\infty (z-t)^{1/2-H} e^{-z} dz\right) \\ &= -C_1^{-1} t^{1/2-H} \frac{d}{dt}\left(\int_0^\infty z^{1/2-H} e^{-z-t} dz\right) \\ &= C_1^{-1}\Gamma\Big(\frac{3}{2}-H\Big) t^{\frac{1}{2}-H} e^{-t} = C_1^{-1}\Gamma\Big(\frac{3}{2}-H\Big)\widetilde{h}'(t).\end{aligned}$$

Consequently, the function $K(t) := (K_{\infty-}^{H,*}\widetilde{h}')(t)$ is nonincreasing. Since further

$$K_{\infty-}^{H}(K_{\infty-}^{H,*}\widetilde{h}')(t) = \widetilde{h}'(t) \in L_2(\mathbb{R}_+),$$

then $K \in L_2^H(\mathbb{R}_+)$ and moreover, $K(t)t^H \to 0$ as $t \to \infty$. It means that condition (ii) holds. Denote f, yet the unknown drift, and let $q(t) = C_2 \int_0^t s^{1/2-H}(t-s)^{1/2-H} f'(s) ds$, with $C_2 := \frac{C_1}{\Gamma\left(H+1/2\right)}$. Then

$$s^{H-1/2} q'(s) = h'(s) = s^{1/2-H} e^{-s}, \quad q'(s) = s^{1-2H} e^{-s}$$

and

$$C_2 \int_0^t (t-s)^{1/2-H} s^{1/2-H} f'(s) ds = \int_0^t s^{1-2H} e^{-s} ds.$$

Hence with $C_3 = C_2 B(\frac{3}{2}-H, H-\frac{1}{2})$ where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, we obtain

$$(H-\frac{1}{2}) C_3 \int_0^t s^{1/2-H} f'(s) ds = \int_0^t (t-s)^{H-1/2} s^{1-2H} e^{-s} ds$$

implying that

$$f(t) = \left(\frac{\Gamma\left(\frac{3}{2}-H\right)}{2H\Gamma(2-2H)\Gamma\left(H+\frac{1}{2}\right)}\right)^{-\frac{1}{2}} \int_0^t s^{H-\frac{1}{2}} \int_0^s (s-z)^{H-\frac{3}{2}} z^{1-2H} e^{-z} dz ds.$$

Since $(K_{0+}^{H,*} f')(t) = C_1 t^{H-1/2} q'(t) = t^{1/2-H} e^{-t} \in L_2(\mathbb{R}_+)$ condition (i) holds. Condition (iii) is clearly satisfied since we can put $\widehat{f} = f$. Note in particular that the assumption $\widehat{f} \geq f$ if $H \in (0, 1/2)$ also holds.

Example 5.2. Let $H < \frac{1}{2}$ and put $h(t) = \widetilde{h}(t) = \int_0^t s^\gamma e^{-s} ds$ with some $0 > \gamma > -\frac{1}{2}$ to have h' and $\widetilde{h}'$ in $L_2(\mathbb{R}_+)$. Then, as before, $\widetilde{h}$ is the smallest nondecreasing concave majorant of h. Further, we may write

$$\begin{aligned}(K_{\infty-}^{H,*}\widetilde{h}')(t) &= C_1^{-1} t^{1/2-H} \int_t^\infty (z-t)^{-H-1/2} z^{H-1/2+\gamma} e^{-z} dz \\ &= -C_1^{-1} t^{1/2-H+\gamma} \int_1^\infty (z-1)^{-H-1/2} z^{H-1/2+\gamma} e^{-zt} dz\end{aligned}$$

and $K(t) := (K_{\infty-}^{H,*}\widetilde{h}')(t)$ is nonincreasing for $\frac{1}{2} - H + \gamma \leq 0$, or $-\frac{1}{2} < \gamma \leq H - \frac{1}{2}$. Moreover, for $\gamma = H - \frac{1}{2}$

$$|(K_{\infty-}^{H,*}\widetilde{h}')(t)| \leq C_1^{-1} t^{1/2-H} e^{-t/2} \int_1^\infty z^{H-3/2} dz,$$

$$K_{\infty-}^{H}(|K_{\infty-}^{H,*}\widetilde{h}'|)(t) = \widetilde{h}'(t) \in L_2(\mathbb{R}_+)$$

implying $K \in L_2^H(\mathbb{R}_+)$ and $\lim_{t\to\infty} K(t)t^H = 0$. Consequently, condition (ii) holds. Similarly to Example 5.1,

$$s^{H-1/2} q'(s) = h'(s) = s^{H-1/2} e^{-s}, \; q'(s) = e^{-s}$$

and $C_2\int_0^t(t-s)^{1/2-H}s^{1/2-H}f'(s)ds=\int_0^t e^{-s}ds$, whence

$$\Big(\frac{1}{2}-H\Big)C_2\int_0^t(t-s)^{-1/2-H}s^{1/2-H}f'(s)ds=1-e^{-t}. \tag{24}$$

It follows from (24) that

$$\Big(\frac{1}{2}-H\Big)C_2B(H+\frac{1}{2},\frac{1}{2}-H)\int_0^t s^{1/2-H}f'(s)ds=\int_0^t(t-s)^{H-1/2}(1-e^{-s})ds.$$

Denote $C_4:=(\frac{1}{2}-H)B(H+\frac{1}{2},\frac{1}{2}-H)$. Then

$$\int_0^t s^{1/2-H}f'(s)ds=\frac{1}{C_4}\int_0^t\frac{(t-s)^{H+1/2}}{H+1/2}e^{-s}ds,$$

and

$$t^{1/2-H}f'(t)=\frac{1}{C_4}\int_0^t(t-s)^{H-1/2}e^{-s}ds.$$

Consequently,

$$f(t)\quad=\quad\int_0^t f'(z)dz=\frac{1}{C_4}\int_0^t e^{-z}\int_z^t s^{H-1/2}(s-z)^{H-1/2}dsdz.$$

Clearly, $(K_{0+}^{H,*}f')(t)=C_1t^{H-1/2}q'(t)=t^{H-1/2}e^{-t}\in L_2(\mathbb{R}_+)$, and condition (i) holds. Condition (iii) is evident.

6 Proofs

6.1 Proofs of auxiliary results

Proof of Lemma 4.1: It was established in [26] that the fBm B^H can be "restored" from $\widetilde{W}$ by the following formula $B^H(t)=\int_0^t K_H(t,s)d\widetilde{W}(s)$, but it means

$$\widetilde{W}(t)=\int_0^t(K_{\infty-}^{H,*}1_{[0,t]})(s)dB^H(s)=W(t)$$

a.s. for any $t\in\mathbb{R}_+$. Since we consider the continuous modifications of all Wiener processes, the proof follows. □

Proof of Lemma 4.2: On one hand, we have that $\int_0^\infty(K_{\infty-}^H f)(s)dW(s)$ exists. On the other hand, we have the equality $\int_0^T f(s)dB^H(s)=\int_0^T(K_{\infty-}^H f1_{[0,T]})(s)dW(s)$. At last,

$$\mathbb{E}\left\{\left(\int_0^\infty(K^Hf)(s)dW(s)-\int_0^T(K_{\infty-}^Hf1_{[0,T]})(s)dW(s)\right)^2\right\}$$

$$=\int_T^\infty((K_{\infty-}^Hf)(s))^2ds+\int_0^T((K_{\infty-}^Hf)(s)-K_T^Hf)(s))^2ds. \tag{25}$$

Since $f\in L_2^H(\mathbb{R}_+)$, we have that $\int_T^\infty((K_{\infty-}^Hf)(s))^2ds\to0$, $T\to\infty$. Further, let $H>\frac{1}{2}$. Then

$$\int_0^T((K_{\infty-}^Hf-K_T^Hf)(s))^2ds=C_1\int_0^T s^{1-2H}\left(\int_T^\infty f(t)t^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}dt\right)^2ds. \tag{26}$$

Since also $|f|\in L_2^H(\mathbb{R}_+)$ then for any $s\leq T$

$$\lim_{T\to\infty}\int_T^\infty|f(t)|t^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}dt=0$$

and the integral is bounded by $\int_s^\infty |f(t)|t^{H-1/2}(t-s)^{H-3/2}dt$. Therefore, the right-hand side of (26) tends to 0 due to the Lebesgue dominated convergence theorem. Next, for $0 < H < \frac{1}{2}$ and by the definition of $\widetilde{L}_2^H(\mathbb{R}_+)$, we have

$$\begin{aligned}\int_0^T ((K_{\infty-}^H f)(s) - (K_T^H f)(s))^2 ds &= C_1^2 \int_0^T s^{1-2H}\Big(\frac{d}{ds}\Big(\int_s^\infty u^{H-\frac{1}{2}} f(u)(u-s)^{H-\frac{1}{2}} du\Big) \\ &\quad -\frac{d}{ds}\Big(\int_s^T u^{H-\frac{1}{2}} f(u)(u-s)^{H-\frac{1}{2}} du\Big)\Big)^2 ds \\ &= C_1^2 \int_0^T s^{1-2H}\Big(\int_T^\infty u^{H-\frac{1}{2}} f(u)(u-s)^{H-\frac{3}{2}} du\Big)^2 ds \\ &= C_1^2 \int_0^T s^{1-2H}\Big(\int_T^\infty u^{H-\frac{1}{2}} f(u)(u-s)^{H-\frac{3}{2}} du\Big)^2 ds \to 0\end{aligned}$$

as $T \to \infty$ implying that the right-hand side of (25) vanishes as $T \to \infty$, hence the claim follows. □

Proof of Lemma 4.3: According to Lemma 4.2, under condition 1) the integral $\int_0^\infty h(s)dB^H(s)$ exists,

$$\int_0^\infty h(s)dB^H(s) = \int_0^\infty (K_{\infty-}^H f)(s)dW(s) = L_2\text{-}\lim_{T\to\infty} \int_0^T f(s)dB^H(s). \tag{27}$$

Further, it was mentioned in [22] that $\int_0^T h(s)dB^H(s)$ is an L_2-limit of the corresponding integrals for the elementary functions:

$$\begin{aligned}\int_0^T h(s)dB^H(s) &= L_2\text{-}\lim_{|\pi|\to 0} \sum_{i=1}^N h(s_{i-1})(B^H(s_i) - B^H(s_{i-1})) \\ &= L_2\text{-}\lim_{|\pi|\to 0} (\sum_{i=1}^N B^H(s_i)(s_{i-1}) - h(s_i)) + B^H(T)h(T)) \\ &= \int_0^T B^H(s)d(-h(s)) + B^H(T)h(T).\end{aligned} \tag{28}$$

In view of (27), the limit in the right-hand side of (28) exists and due to condition 3), it equals $\int_0^\infty B^H(s)d(-h(s))$ establishing the proof. □

6.2 Proofs of the main results

Proof of Lemma 3.1: If $f = 0$, then $\|f\|_{\mathcal{H}} = 0$, hence the first claim follows. Assume therefore that $\|f\|_{\mathcal{H}} > 0$. In view of [24] (see p. 47-48 therein), a standard fBm $B_H(t), t \geq 0$ can be realized in the separable Banach space

$$E = \Big\{\omega : \mathbb{R} \to \mathbb{R},\ \text{continuous}, \quad \omega(0) = 0, \quad \lim_{t\to\infty} \frac{|\omega(t)|}{1+t} = 0\Big\}$$

equipped with the norm $\|\omega\|_E = \sup_{t\geq 0} \frac{|\omega(t)|}{1+t}$. Consequently, Theorem 1' in [23] can be applied, hence

$$\Phi(\alpha - \|f\|_{\mathcal{H}}) \leq P_f \leq \Phi(\alpha + \|f\|_{\mathcal{H}}), \tag{29}$$

where in our notation $\alpha = \Phi^{-1}(P_0)$. Since for any $g \geq f$ we have $P_g \leq P_f$, then the claim in (8) follows. Next, in view of (29), by the mean value theorem (see also Lemma 5 in [21])

$$P_f - P_0 \quad \leq \quad \Phi(\alpha + \|f\|_{\mathcal{H}}) - \Phi(\alpha) = \|f\|_{\mathcal{H}}\Phi'(c) \leq \frac{\|f\|_{\mathcal{H}}}{\sqrt{2\pi}}$$

for some real c and similarly using again (29),

$$P_f - P_0 \quad \geq \quad \Phi(\alpha - \|f\|_{\mathcal{H}}) - \Phi(\alpha) \geq -\frac{\|f\|_{\mathcal{H}}}{\sqrt{2\pi}},$$

hence the proof is complete. □

Proof of Corollary 3.1: In view of (8) for any $\gamma > 0$ and any $g \in \mathcal{H}, g \geq f$ we have

$$P_{\gamma f} \geq P_{\gamma g} \geq \Phi(\alpha - \gamma \|g\|_{\mathcal{H}}).$$

Since $g(x_0) > 0$ follows from $f(x_0) > 0$, then $\|g\|_{\mathcal{H}} > 0$, hence for all γ large

$$\ln P_{\gamma f} \geq -(1 + o(1)) \frac{\gamma^2}{2} \inf_{g \in \mathcal{H}, g \geq f} \|g\|^2_{\mathcal{H}}.$$

Since the norm is a convex function and the set $A_f := \{g \in \mathcal{H}, g \geq f\}$ is convex, then the minimization problem (4) has a unique solution $\widehat{f}$, and thus the proof is complete. □

Proof of Theorem 3.1: Define the function $h(t) = \int_0^t h'(s)ds$ with

$$h'(s) = f_H(s) := (K^{H,*}_{0+} f')(s)$$

and introduce its smallest concave nondecreasing majorant $\widetilde{h}$. As shown in [4] $\widetilde{h}(t) = \int_0^t \widetilde{h}'(s)ds$ and

$$\|h\|^2 := \int_0^\infty (h'(s))^2 ds = \int_0^\infty (f_H(s))^2 ds = \|f\|^2_{\mathcal{H}} = \|\widetilde{h}\|^2 + \|h - \widetilde{h}\|^2.$$

Next, let the probability measure Q be defined by the relation

$$\frac{dQ}{dP} = \exp\left(- \int_0^\infty f_H(s) dW(s) - \frac{1}{2}\|h\|^2 \right) = \exp\left(- \int_0^\infty f_H(s) d\widehat{W}(s) + \frac{1}{2}\|h\|^2 \right), \tag{30}$$

where W is the "underlying" Wiener process, $d\widehat{W} = dW + f_H(s)ds$, $\widehat{W}$ is a Wiener process w.r.t. the measure Q. Note that (30) defines a probability measure since $f_H \in L_2(\mathbb{R}_+)$, due to (i) and Theorem 4.1. Then

$$\begin{aligned} P_f &= \mathbb{E}_Q \left\{ \mathbb{I}\{B^H(t) + f(t) \leq u(t), t \in \mathbb{R}_+\} \frac{dP}{dQ} \right\} \\ &= \mathbb{E}_Q \left\{ \mathbb{I}\{\widehat{B}^H(t) \leq u(t), t \in \mathbb{R}_+\} \exp\left(\int_0^\infty f_H(s) d\widehat{W}(s) - \frac{1}{2}\|h\|^2 \right) \right\} \\ &= \mathbb{E} \left\{ \mathbb{I}\{B^H(t) \leq u(t), t \in \mathbb{R}_+\} \exp\left(\int_0^\infty f_H(s) d\widehat{W}(s) - \frac{1}{2}\|h\|^2 \right) \right\}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \int_0^\infty f_H(s) dW(s) &= \int_0^\infty (f_H(s) - \widetilde{h}'(s)) dW(s) + \int_0^\infty \widetilde{h}'(s) dW(s) \\ &= \int_0^\infty (h'(s) - \widetilde{h}'(s)) dW(s) + \int_0^\infty \widetilde{h}'(s) dW(s). \end{aligned}$$

Next setting $K(t) := (K^{H,*}_{\infty-} \widetilde{h}')(t)$, we have

$$\int_0^\infty \widetilde{h}'(s) dW(s) = \int_0^\infty (K^{H,*}_{\infty-} \widetilde{h}')(s) dB^H(s) = \int_0^\infty K(s) dB^H(s)$$

and both integrals are correctly defined. Indeed, $\widetilde{h}' \in L_2(\mathbb{R}_+)$ implying that $\int_0^\infty \widetilde{h}'(s)dW(s)$ exists. Moreover, in view of (ii) we have $K \in L_2^H(\mathbb{R}_+)$ for $H > \frac{1}{2}$ and $K \in \widetilde{L}_2^H(\mathbb{R}_+)$ for $H < \frac{1}{2}$. Consequently, according to Lemma 4.2 $\int_0^\infty K(s)dB^H(s)$ exists. Furthermore the equality (28) holds. In the light of Lemma 4.3, we get

$$\int_0^\infty K(s) dB^H(s) = \int_0^\infty B^H(s) d(-K(s)).$$

Consequently, condition (iii) implies (set $I_{K,u} := \int_0^\infty u(s)d(-K(s))$)

$$
\begin{aligned}
P_f &= \mathbb{E}\Bigg\{\mathbb{I}\{B^H(t)\le u(t), t\in\mathbb{R}_+\}\exp\Bigg(\int_0^\infty (h'(s)-\widetilde{h}'(s))dW(s)-\frac{1}{2}\|h-\widetilde{h}\|^2+\int_0^\infty B^H(s)d(-K(s))-\frac{1}{2}\|\widetilde{h}\|^2\Bigg)\Bigg\}\\
&\le \mathbb{E}\Bigg\{\mathbb{I}\{B^H(t)\le u(t), t\in\mathbb{R}_+\}\exp\Bigg(\int_0^\infty (h'(s)-\widetilde{h}'(s))dW(s)-\frac{1}{2}\|h-\widetilde{h}\|^2+I_{K,u}-\frac{1}{2}\|\widetilde{h}\|^2\Bigg)\Bigg\}\\
&= \exp\left(I_{K,u}-\frac{1}{2}\|\widetilde{h}\|^2\right)\mathbb{E}\Bigg(\mathbb{I}\{B^H(t)\le u(t), t\in\mathbb{R}_+\}\\
&\quad\times\exp\left(\int_0^\infty\Big((K^{H,*}_{0+}f')(s)-(K^{H,*}_{0+}\widehat{h})(s)\Big)dW(s)-\frac{1}{2}\int_0^\infty\Big((K^{H,*}_{0+}f')(s)-(K^{H,*}_{0+}\widehat{h})(s)\Big)^2ds\right)\\
&= \exp\left(I_{K,u}-\frac{1}{2}\|\widetilde{h}\|)^2\right)P_{f-\widehat{f}}
\end{aligned}
$$

establishing the upper bound (9). In order to prove (10), note that in view of Lemma 7.1 for $H\in(1/2,1)$

$$\widehat{f}\ge f,$$

which is also assumed to hold if $H\in(0,1/2)$. Clearly the above inequality implies that $P_f\ge P_{\widehat{f}}$. As above, we have for some function $u_-(t)<u(t), t\in\mathbb{R}_+$

$$
\begin{aligned}
P_{\widehat{f}} &= \mathbb{E}\Bigg\{\mathbb{I}\{B^H(t)\le u(t), t\in\mathbb{R}_+\}\exp\left(\int_0^\infty B^H(s)d(-K(s))-\frac{1}{2}\|\widetilde{h}\|^2\right)\Bigg\}\\
&\ge \mathbb{E}\Bigg\{\mathbb{I}\{u_-(t)\le B^H(t)\le u(t), t\in\mathbb{R}_+\}\exp\left(\int_0^\infty B^H(s)d(-K(s))-\frac{1}{2}\|\widetilde{h}\|^2\right)\Bigg\}\\
&\ge \mathbb{E}\Bigg\{\mathbb{I}\{u_-(t)\le B^H(t)\le u(t), t\in\mathbb{R}_+\}\exp\left(\int_0^\infty u_-(s)d(-K(s))-\frac{1}{2}\|\widetilde{h}\|^2\right)\Bigg\}\\
&\ge \mathbb{P}\left\{u_-(t)\le B^H(t)\le u(t), t\in\mathbb{R}_+\right\}\exp\left(\int_0^\infty u_-(s)d(-K(s))-\frac{1}{2}\|\widetilde{h}\|^2\right)\Bigg\},
\end{aligned}
$$

hence the proof is complete. □

Proof of Corollary 3.2: Since $\widehat{f}\ge f$ and $f(x_0)>0$, then $\|\widehat{f}\|>0$ and further for any measurable function $u:\mathbb{R}_+\to\mathbb{R}$ with $u(0)\ge 0$

$$
\begin{aligned}
\lim_{\gamma\to\infty}P_{\gamma f-\widehat{\gamma f}} &= \lim_{\gamma\to\infty}P_{\gamma f-\gamma\widehat{f}}\\
&= \lim_{\gamma\to\infty}\mathbb{P}\left\{B^H(t)+\gamma(f(t)-\widehat{f}(t))\le u(t), t\in\mathbb{R}_+\right\}\\
&= \mathbb{P}\left\{B^H(t)\le u(t), t\in\mathbb{R}_+ : f(t)=\widehat{f}(t)\right\}>0.
\end{aligned}
$$

By Theorem 3.1 for all γ large,

$$
\begin{aligned}
P_{\gamma f} &\le P_{\gamma f-\gamma\widehat{f}}\exp\left(-\frac{1}{2}\gamma^2\|\widetilde{h}\|^2+\gamma\int_0^\infty u(s)\,d(-K(s))\right)\\
&= P_{\gamma f-\gamma\widehat{f}}\exp\left(-\frac{1}{2}\gamma^2\|\widetilde{h}\|^2(1+o(1))\right),
\end{aligned}
$$

hence as $\gamma\to\infty$

$$\ln P_{\gamma f}\le -\frac{1}{2}\gamma^2\|\widetilde{h}\|^2(1+o(1))+\ln P_{\gamma f-\gamma\widehat{f}}=-\frac{1}{2}\gamma^2\|\widetilde{h}\|^2(1+o(1)).$$

It is clear that we can find u_- such that $u_-(t) < u(t), t \in (0,\infty)$ such that $\int_0^\infty u_-(t)d(-K(t))$ is finite and $\mathbb{P}\{u_-(t) < B_H(t) \le u(t), t \in R_+\} > 0$. Applying again Theorem 3.1 for such u_- we have as $\gamma \to \infty$

$$\ln P_{\gamma f} \quad \geq \quad -\frac{1}{2}\gamma^2 \|\widetilde{h}\|^2(1+o(1))$$

establishing the proof. □

Proof of Corollary 3.3: In view of (3) and the result of Corollary 3.2, we have

$$\frac{1}{2}\gamma^2 \inf_{g\in\mathcal{H}, g\ge f} \|g\|^2_{\mathcal{H}} \sim \frac{1}{2}\gamma^2 \|\widetilde{h}\|^2$$

as $\gamma \to \infty$. Since further $\|\widehat{f}\|_H = \|\widetilde{h}\|$ and the solution of the minimization problem is unique, then $\widehat{f}$ is its solution, thus the claim follows. □

7 Appendix

Lemma 7.1. *Let $H \in (1/2, 1)$ and suppose that the non-negative function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is such that*

$$g(t) = \int_0^t (K_{0+}^{H,*} f')(s)ds \ge 0$$

for some f such that $(K_{0+}^{H,*} f') \in L_2(\mathbb{R}_+)$ and $f(0) = 0$. Then $f(t) \ge 0, t \in \mathbb{R}_+$, holds.

Proof: Introduce the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(t) = \int_0^t (K_{0+}^{H,*} f')(s)ds$. We have that

$$g(t) = \int_0^t \mathcal{D}_{0+}^{H-\frac{1}{2}}(f'(u)u^{\frac{1}{2}-H})(s)s^{H-\frac{1}{2}}ds, \quad \text{with } f'(u)u^{\frac{1}{2}-H} = I_{0+}^{H-\frac{1}{2}}(g'(t)t^{\frac{1}{2}-H})(u)$$

and

$$\begin{aligned} f(u) &= \int_0^u s^{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}}(g'(t)t^{\frac{1}{2}-H})(s)ds \\ &= \Big(\Gamma\Big(H-\frac{1}{2}\Big)\Big)^{-1} \int_0^u \Big(\int_s^u z^{H-\frac{1}{2}}(z-s)^{H-\frac{3}{2}}dz\Big)g'(s)s^{\frac{1}{2}-H}ds. \end{aligned}$$

Setting $r(s) = s^{\frac{1}{2}-H}\int_s^u z^{H-\frac{1}{2}}(z-s)^{H-\frac{3}{2}}dz$, we may further write for $u > 0$

$$f(u) \quad = \quad -\Big(\Gamma\Big(H-\frac{1}{2}\Big)\Big)^{-1} \int_0^u g(s)r'(s)ds,$$

where

$$\begin{aligned} -r'(s) &= -\Big(s^{\frac{1}{2}-H}\int_s^u z^{H-\frac{1}{2}}(z-s)^{H-\frac{3}{2}}dz\Big)'_s = -(s^{\frac{1}{2}-H}\int_0^{u-s}(z+s)^{H-\frac{1}{2}}z^{H-\frac{3}{2}}dz)'_s \\ &= (H-\frac{1}{2})s^{-\frac{1}{2}-H}\int_0^{u-s}(z+s)^{H-\frac{1}{2}}z^{H-\frac{3}{2}}dz + s^{\frac{1}{2}-H}u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}} \\ &\quad -(H-\frac{1}{2})s^{\frac{1}{2}-H}\int_0^{u-s}(z+s)^{H-\frac{3}{2}}z^{H-\frac{3}{2}}dz \\ &= s^{\frac{1}{2}-H}u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}} + \int_0^{u-s}(H-\frac{1}{2})s^{-\frac{1}{2}-H}(z+s)^{H-\frac{3}{2}}z^{H-\frac{1}{2}}dz > 0, \end{aligned}$$

hence the claim follows. □

Acknowledgments. We are in debt to two referees and an Associate Editor for several suggestions which improved our manuscript. E. Hashorva and Y. Mishura acknowledge support from the SNSF Grants 200021-140633/1, 200020-159246/1 and the project RARE -318984 (an FP7 Marie Curie IRSES Fellowship).

References

[1] L. Alili and P. Patie. Boundary-crossing identities for diffusions having the time-inversion property. *J. Theoret. Probab.*, 23(1):65–84, 2010.

[2] F. Biagini, Y. Hu, B. Øksendal, and T. Zhang. *Stochastic calculus for fractional Brownian motion and applications.* Probability and its Applications. London: Springer, 2008.

[3] W. Bischoff and A. Gegg. The Cameron–Martin theorem for (p-)Slepian processes. *J. Theoret. Probab.*, 2015.

[4] W. Bischoff and E. Hashorva. A lower bound for boundary crossing probabilities of Brownian bridge/motion with trend. *Statist. Probab. Lett.*, 74(3):265–271, 2005.

[5] W. Bischoff, E. Hashorva, J. Hüsler, and F. Miller. Exact asymptotics for boundary crossings of the Brownian bridge with trend with application to the Kolmogorov test. *Ann. Inst. Statist. Math.*, 55(4):849–864, 2003.

[6] W. Bischoff, F. Miller, E. Hashorva, and J. Hüsler. Asymptotics of a boundary crossing probability of a Brownian bridge with general trend. *Methodol. Comput. Appl. Probab.*, 5(3):271–287, 2003.

[7] Bischoff, W. and Hashorva, E. and Hüsler, J. An asymptotic result for non crossing probabilities of Brownian motion with trend. *Commun. Stat., Theory Methods*, 36(13-16):2821–2828, 2007.

[8] Bischoff, W. and Hashorva, E. and Hüsler, J. and Miller, F. On the power of the Kolmogorov test to detect the trend of a Brownian bridge with applications to a change-point problem in regression models. *Stat. Probab. Lett.*, 66(2):105–115, 2004.

[9] Bischoff, W. and Hashorva, E. and Hüsler, J. and Miller, F. Analysis of a change-point regression problem in quality control by partial sums processes and Kolmogorov type tests. *Metrika*, 62(1):85–98, 2005.

[10] K. Borovkov and A.N. Downes. On boundary crossing probabilities for diffusion processes. *Stochastic Process. Appl.*, 120(2):105–129, 2010.

[11] K. Borovkov and A. Novikov. Explicit bounds for approximation rates of boundary crossing probabilities for the Wiener process. *J. Appl. Probab.*, 42(1):82–92, 2005.

[12] E. Csáki, D. Khoshnevisan, and Z. Shi. Boundary crossings and the distribution function of the maximum of Brownian sheet. *Stochastic Process. Appl.*, 90(1):1–18, 2000.

[13] Decreusefond, L. and Üstünel, A.S. Application du calcul des variations stochastiques au mouvement brownien fractionnaire. *Compte-Rendus de l'Academie des Sciences*, 321(2):1605–1608, 1995.

[14] A.N. Downes and K. Borovkov. First passage densities and boundary crossing probabilities for diffusion processes. *Methodol. Comput. Appl. Probab.*, 10(4):621–644, 2008.

[15] J. Durbin. The first-passage density of the Brownian motion process to a curved boundary. *J. Appl. Probab.*, 29(2):291–304, 1992.

[16] J.C. Fu and T-L. Wu. Linear and nonlinear boundary crossing probabilities for Brownian motion and related processes. *J. Appl. Probab.*, 47(4):1058–1071, 2010.

[17] E. Hashorva. Exact asymptotics for boundary crossing probabilities of Brownian motion with piecewise linear trend. *Electron. Comm. Probab.*, 10:207–217 (electronic), 2005.

[18] E. Hashorva and Y. Mishura. Boundary noncrossings of additive Wiener fields. *Lith. Math. J.*, 54(3):277–289, 2014.

[19] E. Hashorva and A.G. Pakes. Tail asymptotics under beta random scaling. *J. Math. Anal. Appl.*, 372(2):496–514, 2010.

[20] A. Janssen and M. Kunz. Brownian type boundary crossing probabilities for piecewise linear boundary functions. *Comm. Statist. Theory Methods*, 33(7):1445–1464, 2004.

[21] A. Janssen and H. Ünlü. Regions of alternatives with high and low power for goodness-of-fit tests. *J. Statist. Plann. Inference*, 138(8):2526–2543, 2008.

[22] C. Jost. Transformation formulas for fractional Brownian motion. *Stochastic Process. Appl.*, 116(10):1341–1357, 2006.

[23] W.V. Li and J. Kuelbs. Some shift inequalities for Gaussian measures. In *High dimensional probability (Oberwolfach, 1996)*, volume 43 of *Progr. Probab.*, pages 233–243. Birkhäuser, Basel, 1998.

[24] M. Mandjes. *Large deviations for Gaussian queues.* John Wiley & Sons Ltd., Chichester, 2007.

[25] Y.S. Mishura. *Stochastic calculus for fractional Brownian motion and related processes*, volume 1929 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2008.

[26] I. Norros, E. Valkeila, and J. Virtamo. An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli*, 5(4):571–587, 1999.

[27] A. Novikov, V. Frishling, and N. Kordzakhia. Approximations of boundary crossing probabilities for a Brownian motion. *J. Appl. Probab.*, 36(4):1019–1030, 1999.

[28] A. Novikov, V. Frishling, and N. Kordzakhia. Time-dependent barrier options and boundary crossing probabilities. *Georgian Math. J.*, 10(2):325–334, 2003.

[29] D. Nualart. Stochastic integration with respect to fractional Brownian motion and applications. In *In: Gonzalez-Barrios, J.M. et al. (eds), Stochastic models (Mexico City, 2002)*, volume 336 of *Contemp. Math.*, pages 3–39. Amer. Math. Soc., Providence, RI, 2003.

[30] K. Pötzelberger and L. Wang. Boundary crossing probability for Brownian motion. *J. Appl. Probab.*, 38(1):152–164, 2001.

[31] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional integrals and derivatives.* Gordon and Breach Science Publishers, Yverdon, 1993. Theory and applications, Edited and with a foreword by S. M. Nikol′skiĭ. Translated from the 1987 Russian original.

[32] L. Wang and K. Pötzelberger. Boundary crossing probability for Brownian motion and general boundaries. *J. Appl. Probab.*, 34(1):54–65, 1997.