Calculation of Bayes Premium for Conditional Elliptical Risks

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Abstract: In this paper we discuss the calculation of the Bayes premium for conditionally elliptical multivariate risks. In our framework the prior distribution is allowed to be very general requiring only that its probability density function satisfies some smoothness conditions. Based on previous results of Landsman and Nešlehová (2008) and Hamada and Valdez (2008) we show in this paper that for conditionally multivariate elliptical risks the calculation of the Bayes premium is closely related to Brown identity and the celebrated Stein's Lemma.

Key words and phrases: Bayes premium; Credibility premium; Elliptically symmetric distribution; Stein's Lemma; Brown identity; Gaussian multivariate model.

1 Introduction

In the setup of classical credibility theory (see e.g., Bühlmann and Giesler (2006), Mikosch (2006), or Kaas et al. (2008)) calculation of the Bayes premium is a central task. Considering the L_2 loss function, that task reduces to the calculation of the conditional expectation $\boldsymbol{E}\{\Theta|X\}$ with Θ being some random parameter with some probability density function h and X a random loss measure, say for instance the net loss amount. When the conditional random variable $X|\Theta=\theta$ has the Normal distribution function with mean θ and variance σ^2 (write $X \sim \mathcal{N}(\theta,\sigma^2)$), then for $\Theta \sim \mathcal{N}(\mu,\tau^2), \tau>0$ we have almost surely (see Bühlmann and Giesler (2006))

$$\boldsymbol{E}\{\Theta|X\} \quad = \quad X + \frac{\sigma^2}{\sigma^2+\tau^2}(\mu - X). \tag{1.1}$$

The identity (1.1) is a direct consequence of the fact that the conditional distributions of Gaussian (or Normal) random vectors are again Gaussian. In fact, (1.1) can be stated for general Θ with some probability density function h, in the form known in the literature as Brown identity (see e.g., DasGupta (2010)), i.e,

$$\boldsymbol{E}\{\Theta - X|X=x\} \quad = \quad \sigma^2 \frac{\boldsymbol{E}\{h'(x+Y)\}}{\boldsymbol{E}\{h(x+Y)\}}, \quad x\in\mathbb{R}, \quad Y \sim \mathcal{N}(0,\sigma^2), \tag{1.2}$$

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provided that the derivative h' of h exits and the ratio in the right-hand side above is finite.
Since the Gaussian distribution is a canonical example of elliptically symmetric (for short elliptical) distributions, in this paper our main interest is to show Brown identity for $\boldsymbol{X}$ a d-dimensional elliptical random vector, and thus deriving an explicit expression of Bayes premium for conditional elliptical models. The parameter Θ, which in our framework below is a d-dimensional random vector, is assumed to possess a probability density function h satisfying some regularity conditions. It turns out that Brown identity is closely related to Stein's lemma, which is recently discussed for elliptical random vectors in Landsman (2006), Landsman and Nešlehová (2008), and Hamada and Valdez (2008).

Several influential papers such as Landsman and Valdez (2003), Goovaerts et al. (2005), Vanduffel et al. (2008), Valdez et al. (2009) and among many others in the actuarial literature have derived tractable properties of elliptical random vectors which allow for important applications in insurance and risk management. Our results show that this class of random vectors is also tractable in the Bayesian paradigm which is a key pillar of actuarial science and practice.

Outline of the rest of the paper: In Section 2 we give some preliminary results and definitions. The main result is presented in Section 3. Proofs and some additional results are relegated to Section 4.

2 Preliminaries

We shall discuss first some distributional properties of elliptical random vectors and then we shall present an extension of (1.1) to univariate elliptical risks. Let $A \in \mathbb{R}^{d \times d}, d \geq 1$ be a non-singular square matrix, and consider an elliptical random vector $\boldsymbol{X} = (X_1, \ldots, X_d)^\top$ with stochastic representation

$$\boldsymbol{X} \stackrel{d}{=} RA\boldsymbol{U} + \boldsymbol{\mu}, \quad \boldsymbol{\mu} \in \mathbb{R}^d, \tag{2.1}$$

with $R > 0$ a random radius being independent of $\boldsymbol{U}$ which is uniformly distributed on the unit sphere of $\mathbb{R}^d$ (with respect to L_2-norm). Throughout in the following $\Sigma = AA^\top$, with A a $d \times d$ non-singular matrix, h denotes the probability density function of the random parameter Θ (in the d-dimensional setup we shall write instead $\boldsymbol{\Theta}$), and R will be referred to as the radial component of $\boldsymbol{X}$.

Here $\stackrel{d}{=}$ and $^\top$ stand for the equality of the distribution functions and the transpose sign, respectively. For the basic distributional properties of elliptical random vectors see e.g., Cambanis et al. (1981), Valdez and Chernih (2003) or Denuit et al. (2006).

It is well-known that the elliptically distributed random vector $\boldsymbol{X}$ as in (2.1) possesses a probability density function f if and only if its radial component R possesses a probability density function. Moreover, f is given by

$$f(\boldsymbol{x}) \quad = \quad \frac{1}{c\sqrt{det(\Sigma)}} g\Big(\frac{(\boldsymbol{x}-\boldsymbol{\mu})^\top \Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}{2}\Big), \quad \forall \boldsymbol{x} \in \mathbb{R}^d, \tag{2.2}$$

where the positive measurable function g is the so-called *density generator* satisfying

$$c \quad = \quad \frac{(2\pi)^{d/2}}{\Gamma(d/2)} \int_0^\infty u^{d/2-1} g(u)\, du \in (0, \infty). \tag{2.3}$$

When the distribution function of R has a finite upper endpoint $\omega := \sup\{x \in \mathbb{R} : \boldsymbol{P}\{R \le x\} < 1\} \in (0,\infty)$, then we take $g(x) = 0$ for all $x > \omega$. For any $x > 0$ set further

$$\widetilde{g}(x) = \int_x^\infty g(s)\,ds,$$

which is well-defined if $\boldsymbol{E}\{|X_1|\} < \infty$ or equivalently $\boldsymbol{E}\{R\} < \infty$. We note in passing that $\widetilde{g}$ is also a density generator if $\boldsymbol{E}\{R^2\} < \infty$, see Lemma 4.1 below.

If $\boldsymbol{X}$ has stochastic representation (2.1) with generator g, then we write

$$\boldsymbol{X} \sim E_d(\boldsymbol{\mu}, \Sigma, g).$$

A canonical example of elliptically symmetric random vectors is an $\boldsymbol{X}$ being Gaussian with covariance matrix Σ and density generator

$$g(x) = \exp(-x), \quad x > 0. \tag{2.4}$$

Consequently, we have

$$\widetilde{g}(x) = g(x), \quad x > 0. \tag{2.5}$$

We now present the extension of (1.2) to the elliptical framework for the 1-dimensional setup.

Theorem 2.1. *Let $Y \sim E_1(0,1,g)$, and let Θ be a random parameter with differentiable probability density function h. Assume that $\boldsymbol{E}\{Y^2\} \in (0,\infty)$ and let $\widetilde{Y} \sim E_1(0,1,\widetilde{g})$. If $X|\Theta = \theta \sim E_1(\theta,1,g)$ such that for some $x \in \mathbb{R}$*

$$\mathcal{M}_h(x) := \boldsymbol{E}\{h(x+Y)\} \in (0,\infty), \quad \boldsymbol{E}\{|h'(x+\widetilde{Y})|\} < \infty, \tag{2.6}$$

then the Bayes premium is

$$\boldsymbol{E}\{\Theta|X = x\} = x + c_1 \frac{\mathcal{L}_h(x)}{\mathcal{M}_h(x)}, \quad c_1 := \frac{\int_0^\infty r^{-1/2}\widetilde{g}(r)dr}{\int_0^\infty r^{-1/2} g(r)dr}, \tag{2.7}$$

with $\mathcal{L}_h(x) := \boldsymbol{E}\{h'(x+\widetilde{Y})\}$.

Remarks: *a) For simplicity, in the above theorem we consider only $\sigma = 1$. The general case with $\sigma \in (0,\infty)$ can be easily derived by multiplying the right hand side of (2.7) by σ^2 since $\boldsymbol{E}\{(\Theta - X)|X\} = \sigma^2 c_1 \frac{\mathcal{L}_h(X)}{\mathcal{M}_h(X)}$, see the main result in the next section.*

b) When the probability density function h of Θ satisfies $h(x) \le a|x|^{\alpha-1}\exp(-b|x|^c), x \in \mathbb{R}$ with a, α, b, c some positive constants, then condition (2.6) is satisfied. A particular instance is the Gaussian case which we discuss below in some more details.

Example 1. (*Gaussian risks*) Under the setup of Theorem 2.1 consider the special case of generator $g(x) = \exp(-x), x \in \mathbb{R}$ and $Y \sim \mathcal{N}(0,\sigma^2), \sigma \in (0,\infty)$. Clearly, Y is an elliptical random variable, i.e., $Y \sim E_1(0,\sigma^2,g)$. Let further $\Theta \sim \mathcal{N}(\mu,\tau^2), \mu \in \mathbb{R}, \tau \in (0,\infty)$ be also a Gaussian random variable and denote by $h(x;\mu,\tau^2)$ its probability density function. By (2.5) we have $c_1 = 1$. In view of Theorem 2.1, since

$$h'(s+t;\mu,\tau^2) = \frac{\mu - s - t}{\tau^2} h(s+t;\mu,\tau^2), \quad s,t \in \mathbb{R}$$

and by the fact that $\tilde{Y}$ and Y have the same distribution, we obtain

$$\begin{aligned}\boldsymbol{E}\{(\Theta - X)|X = x\} &= \sigma^2 \frac{\int_{s\in\mathbb{R}} h'(x+s;\mu,\tau^2)h(s;0,\sigma^2)\,ds}{\int_{s\in\mathbb{R}} h(x+s;\mu,\tau^2)h(s;0,\sigma^2)\,ds} \\ &= \frac{1}{\tau^2}\int_{s\in\mathbb{R}} (\mu - x - s)h\Big(s;(\mu-x)\frac{\sigma^2}{\sigma^2+\tau^2}, (\frac{1}{\sigma^2}+\frac{1}{\tau^2})^{-1}\Big)\,ds \\ &= \frac{\sigma^2}{\tau^2}(\mu-x)(1-\frac{\sigma^2}{\sigma^2+\tau^2}) = \frac{\sigma^2}{\sigma^2+\tau^2}(\mu-x)\end{aligned}$$

for any $x \in \mathbb{R}$. Consequently, the previous claim in (1.1) follows immediately.

3 Main Result

In this section we focus on multivariate d-dimensional conditional elliptical models. Let therefore $\boldsymbol{X}, \boldsymbol{\Theta}$ be two d-dimensional random vectors such that

$$(\boldsymbol{X}|\boldsymbol{\Theta} = \boldsymbol{\theta}) \quad \sim \quad E_d(\boldsymbol{\theta}, \Sigma, g),$$

with $\boldsymbol{\Theta}$ a d-dimensional random parameter with probability density function h. Again, as in the univariate setup, the credibility premium is calculated (under L_2 loss function) by the conditional expectation $\boldsymbol{E}\{\boldsymbol{\Theta}|\boldsymbol{X}\}$, which for the Gaussian framework is closely related to Brown identity, see DasGupta (2010).

In the sequel, we consider $\boldsymbol{\Theta}$ such that its probability density function h is almost differentiable, adopting the following definition from Stein (1981).

Definition 3.1. *A function $q : \mathbb{R}^d \to \mathbb{R}$ is almost differentiable if there exists $\nabla q : \mathbb{R}^d \to \mathbb{R}^d$ such that*

$$q(\boldsymbol{x}+\boldsymbol{z}) - q(\boldsymbol{x}) \quad = \quad \int_0^1 \boldsymbol{z}^\top \nabla q(\boldsymbol{x}+t\boldsymbol{z})dt, \quad \forall \boldsymbol{x}, \boldsymbol{z} \in \mathbb{R}^d.$$

Note that ∇q is the vector function of component-wise partial derivatives, and it is almost surely unique.

We derive below the expression for the Bayes premium, which boils down to the multivariate Brown identity for elliptical risks. The importance of our result is that it also shows the direct connection between Brown identity and Stein's Lemma for elliptically symmetric risks.

Theorem 3.2. *Let $\boldsymbol{Y} \sim E_d(\boldsymbol{0}, \Sigma, g)$ with $\boldsymbol{0} = (0, \dots, 0)^\top \in \mathbb{R}^d$ be a d-dimensional elliptical random vector with radial component $R > 0$. Assume that $(\boldsymbol{X}|\boldsymbol{\Theta} = \boldsymbol{\theta}) \sim E_d(\boldsymbol{\theta}, \Sigma, g)$, where the random parameter $\boldsymbol{\Theta}$ has probability density function h which is almost differentiable. If $\boldsymbol{E}\{R^2\} \in (0, \infty)$ and for some $\boldsymbol{x} \in \mathbb{R}^d$ we have*

$$\mathcal{M}_h(\boldsymbol{x}) := \boldsymbol{E}\{h(\boldsymbol{x}+\boldsymbol{Y})\} \in (0,\infty), \quad \boldsymbol{E}\{|\nabla h(\boldsymbol{x}+\widetilde{\boldsymbol{Y}})|\} < \infty, \tag{3.1}$$

with $\widetilde{\boldsymbol{Y}} \sim E_d(\boldsymbol{0}, \Sigma, \widetilde{g})$, then the Bayes premium is

$$\boldsymbol{E}\{\boldsymbol{\Theta}|\boldsymbol{X} = \boldsymbol{x}\} = \boldsymbol{x} + c_d \Sigma \frac{\mathcal{L}_h(\boldsymbol{x})}{\mathcal{M}_h(\boldsymbol{x})}, \quad c_d := \frac{\boldsymbol{E}\{R^2\}}{d}, \tag{3.2}$$

with $\mathcal{L}_h(\boldsymbol{x}) := \boldsymbol{E}\{\nabla h(\boldsymbol{x}+\widetilde{\boldsymbol{Y}})\}$. Moreover we have

$$\boldsymbol{E}\{\boldsymbol{Y} h(\boldsymbol{x}+\boldsymbol{Y})\} \quad = \quad c_d \Sigma \boldsymbol{E}\{\nabla h(\boldsymbol{x}+\widetilde{\boldsymbol{Y}})\}. \tag{3.3}$$

Remarks: a) In order to retrieve the expression of the constant c_d appearing in Theorem 2.1 we need to write it in terms of g and $\widetilde{g}$ as

$$c_d = \frac{\int_0^\infty r^{d/2-1}\widetilde{g}(r)\,dr}{\int_0^\infty r^{d/2-1}g(r)\,dr}.$$

When $g(t) = \exp(-t)$ by (2.5) we immediately get that $c_d = 1$. Further in view of (2.4) both $\widetilde{\boldsymbol{Y}}, \boldsymbol{Y}$ have the Gaussian distribution with mean zero and covariance matrix Σ. Consequently, by (3.3)

$$\boldsymbol{E}\{\boldsymbol{Y}h(\boldsymbol{x}+\boldsymbol{Y})\} \quad = \quad \Sigma\boldsymbol{E}\{\nabla h(\boldsymbol{x}+\boldsymbol{Y})\}, \tag{3.4}$$

where $\boldsymbol{Y}$ is a mean-zero Gaussian random vector with covariance matrix Σ. For Σ the identity matrix, (3.4) appears in Lemma 2 of Stein (1981). In the case that $\boldsymbol{Y}$ is elliptically symmetric (3.3) is established in Landsman (2006); see also Landsman and Nešlehová (2008) and Hamada and Valdez (2008).
b) Clearly, for non-Gaussian risks the Bayes premium is given by (3.2) and is in general not a credibility premium.
c) In several tractable cases such as Gaussian scale mixture distributions the distribution of $\widetilde{\boldsymbol{Y}}$ can be explicitly calculated, see Landsman and Nešlehová (2008).
d) The referee of the paper suggested the validity of our result when Σ is semi-positive definite. If we go through our definitions and the results of Theorem 3.2, we see that Σ appears without its inverse matrix, whose existence is not possible for Σ semi-positive definite. This observation suggests that the assumption that Σ is positive-definite in our main result is redundant. In the bivariate setup, this condition has been removed in Hamada and Valdez (2008). With some extra technical efforts (but with a different proof that we give here), it is possible to drop that assumption. In the context of Bayes premium there is no particular motivation to allow for singular matrices Σ. Therefore in order to avoid some extra technical details, we shall postpone the proof to a forthcoming technical manuscript.

Example 2. (*Multivariate Gaussian model*) We denote the multivariate Gaussian distribution with mean $\boldsymbol{\mu}$ and covariance matrix Σ as $\mathcal{N}_d(\boldsymbol{\mu}, \Sigma)$. Its probability density function is denoted by $f(\boldsymbol{x}; \boldsymbol{\mu}, \Sigma), \boldsymbol{\mu} \in \mathbb{R}^d$. Next, assume that $\boldsymbol{X}|\boldsymbol{\Theta} \sim \mathcal{N}_d(\boldsymbol{\Theta}, \Sigma)$ where $\boldsymbol{\Theta} \sim \mathcal{N}_d(\boldsymbol{\mu}, \Sigma_0)$. If both Σ and Σ_0 are non-singular covariance matrices, using further the fact that for the multivariate Gaussian density functions we have that (set $B := (\Sigma_0^{-1} + \Sigma^{-1})^{-1}$)

$$f(\boldsymbol{y}; \boldsymbol{0}, \Sigma) f(\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{\mu}, \Sigma_0) \propto f(\boldsymbol{y}, \Sigma_0^{-1}B(\boldsymbol{\mu}-\boldsymbol{x}), B),$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ (with $\propto$ meaning proportionality), then with the proportionality constants canceling out in the ratios of Eq. (3.2) we obtain

$$\boldsymbol{E}\{(\boldsymbol{\Theta}-\boldsymbol{X})|\boldsymbol{X}\} = \Sigma\boldsymbol{E}\{\Sigma_0^{-1}(\boldsymbol{\mu}-\boldsymbol{x}-\boldsymbol{Y})\},$$

where

$$\boldsymbol{Y} \sim \mathcal{N}_d(\Sigma_0^{-1}B(\boldsymbol{\mu}-\boldsymbol{x}), B).$$

Consequently,

$$\begin{aligned}
\boldsymbol{E}\{(\boldsymbol{\Theta}-\boldsymbol{X})|\boldsymbol{X}=\boldsymbol{x}\} &= \Sigma\Sigma_0^{-1}\Big(\boldsymbol{\mu}-\boldsymbol{x}-\Sigma_0^{-1}B(\boldsymbol{\mu}-\boldsymbol{x})\Big)\\
&= \Sigma\Sigma_0^{-1}\Big(I_d-(I_d+\Sigma_0\Sigma^{-1})^{-1}\Big)(\boldsymbol{\mu}-\boldsymbol{x})\\
&= \Sigma\Sigma_0^{-1}\Big(I_d+\Sigma\Sigma_0^{-1}\Big)^{-1}(\boldsymbol{\mu}-\boldsymbol{x})
\end{aligned}$$

$$= \left(\Sigma_0\Sigma^{-1} + I_d\right)^{-1}(\boldsymbol{\mu} - \boldsymbol{x}),$$

where I_d denotes the $d \times d$ identity matrix. An interesting special case is when $\Sigma = a\Sigma_0$ with a some positive constant. Clearly, Σ is positive definite if and only if Σ_0 is positive definite matrix. Applying the formula above, we obtain

$$\boldsymbol{E}\{(\boldsymbol{\Theta} - \boldsymbol{X})|\boldsymbol{X}\} = \frac{a}{1+a}(\boldsymbol{\mu} - \boldsymbol{X}).$$

In particular, when $\Sigma = \sigma^2 I_d$ and $\Sigma = \tau^2 I_d$ with σ, τ positive constant we obtain the result presented in Example 1.

4 Proofs

Next we provide a lemma which clarifies the properties of $\widetilde{g}$ needed to proceed with the proofs of the main result.

Lemma 4.1. *Let $\boldsymbol{Y} \sim E_d(\boldsymbol{0}, I_d, g)$ be a given elliptical random vector. If $\boldsymbol{E}\{R^2\} \in (0,\infty)$, then $\widetilde{g}$ is a density generator of a d-dimensional elliptical random vector, and moreover*

$$\int_0^\infty r^{d/2-1}\widetilde{g}(r)\,dr = \frac{\boldsymbol{E}\{R^2\}}{d}\int_0^\infty r^{d/2-1}g(r)\,dr \in (0,\infty). \tag{4.1}$$

Proof: The random vector $\boldsymbol{Y}$ has radial decomposition with positive radial component R which has probability density function f_R given by (set $K := \int_0^\infty r^{d/2-1}g(r)\,dr$)

$$f_R(r) = \frac{r^{d-1}g(r^2/2)}{2^{d/2-1}K}, \quad r > 0. \tag{4.2}$$

Since g is a density generator, partial integration implies

$$\begin{aligned}
\int_0^\infty r^{d/2-1}\widetilde{g}(r)\,dr &= \int_0^\infty r^{d/2-1}\left(\int_r^\infty g(s)\,ds\right)dr \\
&= \int_0^\infty g(s)\left(\int_0^s r^{d/2-1}\,dr\right)ds \\
&= \frac{2}{d}\int_0^\infty s^{d/2}g(s)\,ds \\
&= \frac{K}{d}\int_0^\infty t^{d+1}\frac{g(t^2/2)}{2^{d/2-1}K}\,dt \\
&= \frac{K}{d}\int_0^\infty t^2 f_R(t)\,dt \\
&= \frac{K}{d}\boldsymbol{E}\{R^2\} \in (0,\infty),
\end{aligned}$$

hence the claim follows. □

PROOF OF THEOREM 2.1 By the assumption $(X|\Theta = \theta) \sim E_1(\theta, 1, g)$ it follows that the conditional random variable $\Theta|X = x$ has probability density function $q(\cdot|x)$ given by

$$q(\theta|x) = \frac{h(\theta)g((x-\theta)^2/2)}{\int_{\mathbb{R}} h(\theta)g((x-\theta)^2/2)d\theta}$$

$$= \frac{h(\theta)g((x-\theta)^2/2)}{\mathcal{M}_h(x)},$$

with

$$\mathcal{M}_h(x) := \boldsymbol{E}\{h(x-Y)\} = \boldsymbol{E}\{h(x+Y)\}$$

for x such that $\boldsymbol{P}\{X \le x\} \in (0,1)$. Consequently,

$$\mathcal{M}_h(x)\boldsymbol{E}\{(\Theta - X)|X = x\} = \boldsymbol{E}\{Yh(x-Y)\}.$$

Since Y is symmetric about 0, i.e., $Y \stackrel{d}{=} -Y$ we have further

$$\mathcal{M}_h(x)\boldsymbol{E}\{(\Theta - X)|X = x\} = \boldsymbol{E}\{Yh(x+Y)\},$$

with $Y \sim E_1(0,1,g)$. By the assumptions and Lemma 4.1 $\widetilde{g}$ is a density generator with some normalising constant $\widetilde{c} \in (0,\infty)$ as in (2.3). Since we assume that $\boldsymbol{E}\{Y^2\} \in (0,\infty)$, then

$$c_1 = \frac{\int_0^\infty s^{-1/2}\widetilde{g}(s)ds}{\int_0^\infty s^{-1/2}g(s)ds} = \frac{\widetilde{c}}{c}\boldsymbol{E}\{Y^2\}$$

is finite with $c \in (0,\infty)$ the normalising constant of g. By Lemma 2 of Hamada and Valdez (2008) (see also Theorem 1 of Landsmann (2006)) for any $x \in \mathbb{R}$ we obtain

$$\boldsymbol{E}\{Yh(x+Y)\} \quad = \quad c_1\boldsymbol{E}\{h'(x+\widetilde{Y})\},$$

with $\widetilde{Y} \sim E_1(0,1,\widetilde{g})$, and thus the claim follows. □

Proof of Theorem 3.2 Let $\boldsymbol{Y} \sim E_d(\boldsymbol{0},\Sigma,g)$ with $\boldsymbol{0} = (0,\ldots,0)^\top \in \mathbb{R}^d$ and let A be a square matrix such that $AA^\top = \Sigma$. Note first that as in the univariate case we have $\boldsymbol{Y}$ is symmetric about origin, i.e.,

$$\boldsymbol{Y} \stackrel{d}{=} -\boldsymbol{Y}.$$

As in the proof of Theorem 2.1 for any $\boldsymbol{x} \in \mathbb{R}^d$ in the support of $\boldsymbol{X}$ we have

$$\boldsymbol{E}\{\boldsymbol{\Theta}|\boldsymbol{X} = \boldsymbol{x}\} = \boldsymbol{x} + \frac{\boldsymbol{E}\{\boldsymbol{Y}h(\boldsymbol{x}+\boldsymbol{Y})\}}{\mathcal{M}_h(\boldsymbol{x})}, \tag{4.3}$$

provided that $\mathcal{M}_h(\boldsymbol{x}) := \boldsymbol{E}\{h(\boldsymbol{x}+\boldsymbol{Y})\}$ is finite and non-zero. Applying Lemma 3 of Landsman and Nešlehová (2008) we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \boldsymbol{v}h(\boldsymbol{v})g(\boldsymbol{v}^\top\boldsymbol{v}/2)\,d\boldsymbol{v} &= \int_{\mathbb{R}^d} \nabla h(\boldsymbol{v})\left(\int_{\boldsymbol{v}^\top\boldsymbol{v}/2}^\infty g(u)\,du\right)d\boldsymbol{v} \\ &= \int_{\mathbb{R}^d} \nabla h(\boldsymbol{v})\widetilde{g}(\boldsymbol{v}^\top\boldsymbol{v}/2)\,d\boldsymbol{v}.\end{aligned}$$

Hence with c as in (2.3) and $\widetilde{\boldsymbol{Y}} \stackrel{d}{=} A\boldsymbol{V} \sim E_d(\boldsymbol{0},\Sigma,\widetilde{g})$ we may further write

$$\begin{aligned}\boldsymbol{E}\{\boldsymbol{Y}h(\boldsymbol{x}+\boldsymbol{Y})\} &= \frac{1}{c}\Sigma\int_{\mathbb{R}^d} \nabla h(\boldsymbol{x}+A\boldsymbol{v})\widetilde{g}(\boldsymbol{v}^\top\boldsymbol{v}/2)\,d\boldsymbol{v} \\ &= \frac{1}{c}\Sigma\int_{\mathbb{R}^d} \frac{1}{\sqrt{det(\Sigma)}}\nabla h(\boldsymbol{x}+\boldsymbol{y})\widetilde{g}(\boldsymbol{y}^\top\Sigma^{-1}\boldsymbol{y}/2)\,d\boldsymbol{y} \\ &= \frac{\widetilde{c}}{c}\Sigma\boldsymbol{E}\{\nabla h(\boldsymbol{x}+\widetilde{\boldsymbol{Y}})\}.\end{aligned}$$

In view of Lemma 4.1

$$\frac{\widetilde{c}}{c} = \frac{\int_0^\infty u^{d/2-1}\widetilde{g}(u)du}{\int_0^\infty u^{d/2-1}g(u)du} = \frac{\boldsymbol{E}\{R^2\}}{d}$$

and thus the rest of the proof proceeds as in the univariate case, hence the claim follows. □

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