Exact Tail Asymptotics of Aggregated Parametrised Risks

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November 15, 2012

Abstract: In this paper we investigate the extremal behaviour of aggregated risk for specific parametrised multivariate dependence framework. Furthermore we discuss conditional limit results and extremal behaviour of both maximum and aggregated log-elliptical risk. Our application establishes the logarithmic efficiency of Rojas-Nandaypa algorithm for rare-event simulation of log-elliptical risks.

Key words and phrases: Risk aggregation; log-elliptical distribution; log-normal distribution; rare-event simulation; logarithmic efficiency.

1 Introduction

Aggregation of risks is a central topic in risk management of insurance and financial institutions. When offering insurance coverage for several lines of business, it is important from a top-down prospective to appropriately quantify the effect of the aggregation of individual risks. In insurance practice, risk models for each separate line of business are constructed, and then all these models are aggregated forming a single model which quantifies the total liability arising from the whole book of business.

In a bivariate setup of two risks Z, W it is of interest to quantify the tail asymptotic behaviour of the aggregated risk $S := Z + W$. Indeed, even for simple cases for which the distribution of (Z, W) is known, the tail asymptotic behaviour of S is in general not readily available.

To illustrate this difficulty suppose that $(\ln Z, \ln W)$ is a bivariate Gaussian random vector with $N(0,1)$ marginals and correlation coefficient $\rho \in (-1,1)$. In view of Asmussen and Rojas-Nandaypa (2008)

$$\boldsymbol{P}\{Z + W > u\} \quad = \quad (1 + o(1))2\boldsymbol{P}\{Z > u\}, \quad u \to \infty, \tag{1.1}$$

which is the first result known for the log-normal risks. Numerous actuarial tasks such as risk management, pricing, or loss reserving make special use of the assumption that the underlying multidimensional distributions are log-normal. This central assumption allows for tractable formulas and thorough understanding of underlying complex relations; the log-normal assumption is due to the role of the Gaussian distribution as a good starting point for more adequate models.

Aggregation of risks in log-normal and some related models has been a central topic in the recent papers Goovaerts et al. (2005), Tang (2006), Embrechts and Puccetti (2008), Ko and Tang (2008), Embrechts et al. (2009), Geluk and Tang (2009), Kortschak and Albrecher (2009), Mitra and Resnick (2009), Degen et al. (2010), Foss and Richards (2010), Hashorva et al. (2010), Hashorva (2010,2012), Li et al. (2010), Asimit et al. (2011), Asmussen et al. (2011), Kortchack (2011).

As shown in Mitra and Resnick (2009), when single risks have distribution functions in the max-domain of attraction (MDA) of the Gumbel distribution, then the tail asymptotics of the aggregated risk is strongly related to conditional limit results and asymptotic independence.

In fact, in diverse applications a parametrised setup can be adopted where for instance

$$\ln Z = B(u), \quad \ln W = B(g(u)), \quad u > 0,$$

with $\{B(u), u > 0\}$ a Brownian motion and $g : \mathbb{R} \to (0, \infty)$ a measurable time transform. The aggregated risk $S(u) := \exp(B(u)) + \exp(B(g(u)))$ depends thus on the parameter u. The determination of the asymptotics of

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$\boldsymbol{P}\{S(u) > u\}$ for this model leads to additional technical difficulties related to the fact that the parameter is controlled by the threshold.

A strong motivation for considering the asymptotics of $\boldsymbol{P}\{S(u) > u\}$ is provided by Asmussen et al. (2011), where the role of u is played by the scale $1/(1-\theta)$ of the covariance matrix Σ of the importance sampling distribution, see Theorem 1 and Theorem 2 therein.

In this paper we first adopt the Mitra-Resnick approach to approximate the probability of the rare event $\{S(u) > u\}$ as u tends to infinity, with $S(u) = X_1(u) + \cdots + X_k(u)$ the aggregated risk. It turns out that for our model, which is governed by asymptotic independence and Gumbel MDA assumption, the maximum risk $M(u) := \max_{1\le i\le k} X_i(u)$ has the same tail asymptotic behaviour ($u \to \infty$) as the aggregated risk.

We consider then in details log-elliptical risks. In order to highlight the relevance of our theoretical findings we present an application concerning the logarithmic efficiency of a conditional Monte Carlo algorithm suggested in Rojas-Nandaypa (2008).

Organisation of the rest of the paper: Section 2 consists of some preliminary results. In Section 3 we extend the Mitra-Resnick model to include the case of parametrised risks. Sections 4 and 5 are concerned with a log-elliptical framework where both conditional limit results and the tail asymptotics of aggregated risks are derived. The application mentioned above is presented in Section 6, while the proofs of all the results are relegated to Section 7. The last Section 8 is a short Appendix.

2 Preliminaries

First we introduce our notation. Vectors in $\mathbb{R}^k, k \ge 2$ are denoted by bold letters such as $\boldsymbol{x} := (x_1, \ldots, x_k)^\top$ and random vectors in $\mathbb{R}^k$ by bold capital letters, say $\boldsymbol{X} := (X_1, \ldots, X_k)^\top$, with $^\top$ the transpose sign. For given $\boldsymbol{\beta} \in [0,\infty)^k, \boldsymbol{x}, \boldsymbol{y}, \in \mathbb{R}^k$ we define

$$\begin{aligned}
\boldsymbol{x} &> \boldsymbol{y}, \text{ if } x_i > y_i, \quad \forall\, i = 1, \ldots, k,\\
\boldsymbol{x} &\ge \boldsymbol{y}, \text{ if } x_i \ge y_i, \quad \forall\, i = 1, \ldots, k,\\
\boldsymbol{x} + \boldsymbol{y} &:= (x_1 + y_1, \ldots, x_k + y_k)^\top,\\
\boldsymbol{x}\boldsymbol{y}^{\boldsymbol{\beta}} &:= (x_1 y_1^{\beta_1}, , \ldots, x_k y_k^{\beta_k})^\top, \quad c\boldsymbol{x} := (cx_1, \ldots, cx_k)^\top, \quad c \in \mathbb{R},\\
\boldsymbol{0} &:= (0, \ldots, 0)^\top \in \mathbb{R}^k, \quad \boldsymbol{1} := (1, \ldots, 1)^\top \in \mathbb{R}^k.
\end{aligned}$$

If a random vector $\boldsymbol{X}$ possesses the distribution function (df) H, we shall indicate this by $\boldsymbol{X} \sim H$. Throughout this paper $\boldsymbol{U}$ is a k-dimensional random vector uniformly distributed on the unit sphere of $\mathbb{R}^k$ being independent of $R > 0$ which has df F with $F(0) = 0$; we write $\overline{F}$ for the survivor function of F (and similarly for other distributions).

In this paper we consider an elliptically symmetric (for short elliptical) random vector $\boldsymbol{Y}$ with stochastic representation

$$\boldsymbol{Y} \overset{d}{=} RA\boldsymbol{U} + \boldsymbol{\mu}, \quad \boldsymbol{\mu} \in \mathbb{R}^k, \tag{2.2}$$

where $A \in \mathbb{R}^{k\times k}$ is a non-singular deterministic matrix. The random variable R is positive and $\boldsymbol{U}$ is uniformly distributed on the unit sphere of $\mathbb{R}^k$ (with respect to L_2 norm). Furthermore R and $\boldsymbol{U}$ are independent. The basic distributional properties of elliptical random vectors are discussed in Cambanis et al. (1981). Instead of (2.2) we write alternatively $\boldsymbol{Y} \sim E_k(\boldsymbol{\mu}, \Sigma, F)$, with $\Sigma := AA^\top$ where F is the distribution function of R. We call $\boldsymbol{X}$ a log-elliptical random vector in $\mathbb{R}^k$ (abbreviate this as $\boldsymbol{X} \sim LE_k(\boldsymbol{\mu}, \Sigma, F)$) if

$$\boldsymbol{X} \overset{d}{=} (\exp(Y_1), \ldots, \exp(Y_k))^\top =: \exp(\boldsymbol{Y}). \tag{2.3}$$

For our investigations we suppose that F with upper endpoint $x_F = \infty$ belongs to the Gumbel MDA with some scaling function w. By definition, we say that the df H belongs to Gumbel MDA if

$$\lim_{u\to\infty} \frac{\overline{H}(u+s/w(u))}{\overline{H}(u)} = \exp(-s), \quad \forall s \in \mathbb{R} \tag{2.4}$$

holds with w some positive scaling function; relation (2.4) is abbreviated hereafter as $H \in GMDA(w)$ or $X \in GMDA(w)$ if $X \sim H$. The asymptotics in (2.4) can be cast into the framework of convergence in distributions. Indeed, (2.4) is equivalent to

$$\lim_{u\to\infty} w(u)\mathcal{Z}_u \overset{d}{\to} \mathcal{E}, \quad u \to \infty, \tag{2.5}$$

where $\mathcal{Z}_u, u > 0$ are defined in the same probability space such that $\mathcal{Z}_u \overset{d}{=} (X-u)|(X > u)$, and $\mathcal{E}$ denotes (throughout this paper) an exponential random variable with mean 1.

Next, let $(X(u), Y(u)), u > 0$ be a bivariate random vector with marginal distributions H_u, G_u, respectively. Given $t_u, u > 0$ define in the same probability space bivariate random vectors $(\mathcal{Z}_u, \mathcal{W}_u), u > 0$ such that

$$(\mathcal{Z}_u, \mathcal{W}_u) \overset{d}{=} \Big((X(u)-t_u), Y(u)\Big)\Big|(X(u) > t_u), \quad u > 0. \tag{2.6}$$

The random variable $\mathcal{Z}_u$ defined in (2.6) differs from that appearing in (2.5) since $X(u)$ depends also on u. Additionally, we use t_u and not u. In the sequel we shall suppose that $t_u, u > 0$

$$\lim_{u\to\infty} t_u = \infty. \tag{2.7}$$

Our proofs are related to convergence stated in (2.5). For $\mathcal{Z}_u$ defined by (2.6) we shall modify (2.5) as follows:

Assumption 1. $[X(u) \in GMDA(w_u)]$ We say that $X(u), u > 0$ satisfies Assumption 1 if there exist positive measurable scaling functions $w_u, u > 0$ such that for any $t_u, u > 0$

$$\lim_{u\to\infty} t_u w_u(t_u) = \infty \tag{2.8}$$

and further

$$\lim_{u\to\infty} w_u(t_u)\mathcal{Z}_u \overset{d}{\to} \mathcal{E}, \quad u \to \infty. \tag{2.9}$$

Note that if $X(u) = X$ and $X \in GMDA(w)$, then Assumption 1 holds with $w_u(x) = w(x)$ since $\lim_{x\to\infty} xw(x) = 0$. We proceed with two other assumptions; the scaling function w_u appearing below relates to Assumption 1.

Assumption 2. $[A_{X(u),Y(u),w_u}]$ For any sequence $t_u, u > 0$ satisfying (2.7), we have the convergence in probability

$$w_u(t_u)\mathcal{W}_u \overset{p}{\to} 0, \quad u \to \infty. \tag{2.10}$$

Assumption 3. $[A_{X(u),Y(u),w_u;L}]$ There exists some $L > 0$ such that for any $t_u, u > 0$ satisfying (2.7)

$$\lim_{u\to\infty} \frac{\boldsymbol{P}\{X(u) > L/w_u(t_u), Y(u) > L/w_u(t_u)\}}{\boldsymbol{P}\{X(u) > t_u\}} = 0. \tag{2.11}$$

We comment briefly on the implication of the above assumptions: if (2.10) holds, then by (2.8)

$$\lim_{u\to\infty} \frac{\boldsymbol{P}\{X(u) > t_u, Y(u) > t_u\}}{\boldsymbol{P}\{X(u) > t_u\}} = \lim_{u\to\infty} \boldsymbol{P}\{\mathcal{W}_u > t_u\} = \lim_{u\to\infty} \boldsymbol{P}\{w_u(t_u)\mathcal{W}_u > t_u w_u(t_u)\} = 0. \tag{2.12}$$

Furthermore, (2.9) and (2.10) yield the joint convergence in distribution

$$\Big(w_u(t_u)\mathcal{Z}_u, w_u(t_u)\mathcal{W}_u\Big) \overset{d}{\to} (\mathcal{E}, 0), \quad u \to \infty,$$

and hence

$$w_u(t_u)\Big(X(u) - t_u + Y(u)\Big)\Big|(X(u) > t_u) \overset{d}{\to} \mathcal{E} + 0 = \mathcal{E}, \quad u \to \infty. \tag{2.13}$$

Indeed, the limit relation in (2.13) is a key result for $S(u) - t_u = X(u) + Y(u)$, which will be used in the proof of Theorem 3.1.

3 Aggregation of Parametrised Risks

In several applications both risks may depend on some deterministic parameters. One example is

$$X(u)=\exp(\overline{\gamma}_u B(\eta_u u)), \quad Y(u)=\exp(\overline{\gamma}_u B(u)), \quad u>0, \tag{3.14}$$

with B a standard Brownian motion and $\eta_u>1, \overline{\gamma}_u>0, u>0$. By the properties of the Brownian motion

$$S(u):=X(u)+Y(u) \quad \stackrel{d}{=} \quad \exp(\overline{\gamma}_u\sqrt{u}[B(1)+\sqrt{\eta_u-1}B^*])+\exp(\overline{\gamma}_u\sqrt{u}B(1)), \quad \forall u>0,$$

with $B^*, B(1)$ independent and $B^* \stackrel{d}{=} B(1)$. If $\eta_u=\eta>1$ does not depend on u, then the tail asymptotics ($u\to\infty$) of $\boldsymbol{P}\{S(u)>u\}$ follows from Theorem 1 in Asmussen et al. (2011). The introduction of the parameter u (through the deflator $1-\theta$ therein) is a novel idea particularly useful for rare-event simulation.

Motivated by Mitra and Resnick (2009) we discuss in this section a general risk aggregation framework by considering the aggregation of a 2-dimensional parametrised random vector $(X(u),Y(u))$; we do not specify the role of the parameter u as in (3.14). The adaption of Mitra-Resnick conditions allows us to derive the asymptotics of $\boldsymbol{P}\{S(u)>u\}$ as $u\to\infty$. Further, under those conditions we show for the maximum risk $M(u):=\max(X(u),Y(u))$ that it is asymptotically tail equivalent to $S(u)$.

Theorem 3.1. *Let $(X(u),Y(u)), u>0$ be a bivariate risk vector. Assume that for any $t_u, u>0$ satisfying* (2.7) *we have*

$$\lim_{u\to\infty}\frac{\boldsymbol{P}\{Y(u)>t_u\}}{\boldsymbol{P}\{X(u)>t_u\}} \quad = \quad c\in[0,\infty). \tag{3.15}$$

Suppose that $X(u)\in GMDA(w_u)$ and Assumptions 2, 3 are satisfied. If further Assumption 2 $[A_{Y(u),X(u),w_u}]$ holds when $c\in(0,\infty)$, then $S(u), M(u)\in GMDA(w_u)$, and moreover

$$\lim_{u\to\infty}\frac{\boldsymbol{P}\{S(u)>t_u\}}{\boldsymbol{P}\{X(u)>t_u\}} \quad = \quad \lim_{u\to\infty}\frac{\boldsymbol{P}\{M(u)>t_u\}}{\boldsymbol{P}\{X(u)>t_u\}}=1+c. \tag{3.16}$$

As in Corollary 2.2 of Mitra and Resnick (2009) we can extend (3.16) under relaxed conditions to a k-dimensional setup focusing on non-negative risks.

Corollary 3.2. *Let $(X_1(u),\ldots,X_k(u))$ be a k-dimensional parameterised random vector with non-negative components such that $X_1(u)\in GMDA(w_u)$, and set $S(u):=\sum_{1\le i\le k}X_i(u), M(u):=\max_{1\le i\le k}X_i(u)$. Assume that for any $t_u, u>0$ satisfying* (2.7)

$$\lim_{u\to\infty}\frac{\boldsymbol{P}\{X_i(u)>t_u\}}{\boldsymbol{P}\{X_1(u)>t_u\}} \quad = \quad c_i\in[0,\infty), \quad 1\le i\le k. \tag{3.17}$$

If further

$$\max_{1\le i\ne j\le k}\lim_{u\to\infty}\frac{\boldsymbol{P}\{w_u(t_u)X_j(u)>z, X_i(u)>t_u\}}{\boldsymbol{P}\{X_1(u)>t_u\}} \quad = \quad 0, \quad \forall z\in(0,\infty), \tag{3.18}$$

and for some positive constants $L_{ij}, 1\le i\ne j\le k$,

$$\max_{1\le i\ne j\le k}\lim_{u\to\infty}\frac{\boldsymbol{P}\{X_i(u)>L_{ij}/w_u(t_u), X_j(u)>L_{ij}/w_u(t_u)\}}{\boldsymbol{P}\{X_1(u)>t_u\}} \quad = \quad 0 \tag{3.19}$$

holds, then $S(u), M(u)\in GMDA(w_u)$, and moreover

$$\lim_{u\to\infty}\frac{\boldsymbol{P}\{S(u)>t_u\}}{\boldsymbol{P}\{X_1(u)>t_u\}} \quad = \quad \lim_{u\to\infty}\frac{\boldsymbol{P}\{M(u)>t_u\}}{\boldsymbol{P}\{X_1(u)>t_u\}}=\sum_{i=1}^k c_i\in[1,\infty). \tag{3.20}$$

Our next result is of interest for the log-elliptical setup. For notational simplicity we assume that both $\boldsymbol{\beta}, \boldsymbol{\lambda}$ appearing in Theorem 3.3 (and in the sequel) are such that

$$\beta_1 \ge \beta_2 \ge \cdots \ge \beta_k, \qquad \text{and } \lambda_1 = \max_{1 \le i \le k} \lambda_i,$$

and the integer m is the multiplicity of β_1. We impose a further condition on parametrised λ's, i.e.,

$$\lim_{u\to\infty} \boldsymbol{\lambda}(u) = \boldsymbol{\lambda} \in [0,\infty)^k, \qquad \text{and } \lambda_i(u) = \lambda_1(u), u > 0, \quad \text{if } \lambda_i = \lambda_1. \tag{3.21}$$

The definitions of $S(u)$ and $M(u)$ remain the same as in Corollary 3.2.

Theorem 3.3. *Let $\boldsymbol{X} = (X_1, \ldots, X_k), k \ge 2$ be a k-dimensional random vector with non-negative components and identical marginal distributions such that $X_1 \in GMDA(w)$, and set $X_i(u) := \lambda_i(u) X_i^{\beta_i \gamma(u)}, i \le k$ with $\boldsymbol{\lambda}(u) \in [0,\infty)^k, \gamma(u) \in (0,\infty), u > 0$ some measurable functions and $\boldsymbol{\beta} \in [0,\infty)^k$. Suppose that both (3.21) and (3.17) hold. Assume that for any $t_u, u > 0$ converging to infinity $\lim_{u\to\infty} \widetilde{w}_{\beta_1\gamma(u)}(t_u) = 0$ with $\widetilde{w}_{1/c}(x) = cx^{c-1}w(x^c), c > 0, x > 0$. If further $\lambda_1(u) X_1^{\beta_1\gamma(u)}, \ldots, \lambda_1(u) X_k^{\beta_1\gamma(u)}$ satisfy (3.18) and (3.19), then we have*

$$\lim_{u\to\infty} \frac{\boldsymbol{P}\{S(u) > t_u\}}{\boldsymbol{P}\{X_1(u) > t_u\}} = \lim_{u\to\infty} \frac{\boldsymbol{P}\{M(u) > t_u\}}{\boldsymbol{P}\{X_1(u) > t_u\}} = \sum_{i \le m: \lambda_i = \lambda_1} c_i. \tag{3.22}$$

Remark 3.4. *(i) If $X(u)$ and $Y(u)$ are independent for all large u, then Assumption 2 [$A_{X(u),Y(u),w_u}$] boils down to*

$$w_u(t_u) Y(u) \xrightarrow{p} 0, \quad u \to \infty,$$

which is satisfied if $\lim_{u\to\infty} w_u(t_u) = 0$ for any $t_u, u > 0$. If $X(u) \stackrel{d}{=} Y(u)$ for all u large, then

$$\lim_{u\to\infty} \frac{\boldsymbol{P}\{S(u) > t_u\}}{\boldsymbol{P}\{X(u) > t_u\}} = 2$$

if additionally for some $L > 0$

$$\lim_{u\to\infty} \frac{(\boldsymbol{P}\{X(u) > L/w_u(t_u)\})^2}{\boldsymbol{P}\{X(u) > t_u\}} = 0. \tag{3.23}$$

If $X(u) = X$ for all large u with X a subexponential random variable, then (3.23) reduces to Mitra-Resnick criterion, see Corollary 2.1 in Mitra and Resnick (2009), and Lemma 2.2 in Hashorva et al. (2010).

(ii) If $X(u) \sim H_u, u > 0$ with $H_u \in GMDA(w_u)$, then for any $\chi \in (1,\infty)$ the limit relation in (2.8) yields

$$\lim_{u\to\infty} \frac{\overline{H_u}(\chi t_u)}{\overline{H_u}(t_u)} \le = 0, \quad r \to \infty. \tag{3.24}$$

(iii) By Assumption 3 and (2.8) for any $C_1, C_2 \in \mathbb{R}$ we have,

$$\lim_{u\to\infty} \frac{\boldsymbol{P}\{X(u) > t_u + C_1/w_u(t_u), Y(u) > t_u + C_2/w_u(t_u)\}}{\boldsymbol{P}\{X(u) > t_u + C_1/w_u(t_u)\}} = 0,$$

which means that $X(u)$ and $Y(u)$ are asymptotically independent.

(iv) Suppose that $\{X(u), Y(u), u > 0\}$ is such that $X(u) \in GMDA(w_u)$ and (3.15) hold, and both Assumption 2 [$A_{X(u),Y(u),w_u}$], Assumption 2 [$A_{Y(u),X(u),w_u}$] are satisfied. If further for any $C \in (0,\infty)$

$$\lim_{u\to\infty} \frac{\boldsymbol{P}\{X(u) > t_u + C/w_u(t_u), Y(u) > t_u + C/w_u(t_u)\}}{\boldsymbol{P}\{X(u) > t_u\}} = \psi_C \in [0,\infty)$$

such that $\lim_{C\downarrow 0} \psi_C = 0$ and

$$\liminf_{u\to\infty} \frac{w_u(t_u + C/w_u(t_u))}{w_u(t_u)} > 0,$$

then we have

$$\liminf_{u\to\infty} \frac{\boldsymbol{P}\{S(u) > t_u\}}{\boldsymbol{P}\{X(u) > t_u\}} \ge 1 + c. \tag{3.25}$$

Example 3.5. *Let (X,Y) be a bivariate random vector with identical non-negative components, and define*

$$X(u) := X^{\beta\gamma(u)}, \quad Y(u) := Y^{\gamma(u)}, \quad \beta \in [1,\infty),$$

with $\gamma(u) \in (0,\infty), u > 0$ such that $\lim_{u\to\infty} \gamma(u) = \gamma \in (0,\infty)$. Suppose that $X \in GMDA(\overline{w}_{\tau,\theta}), \tau, \theta \in (0,\infty)$ with $\overline{w}_{\tau,\theta}(x) = \tau x^{-1}(\ln x)^\theta, x > 0$. If $\ln X$ is a standard Gaussian random variable (with mean 0 and variance 1), then $X \in GMDA(\overline{w}_{1,1})$. It follows easily that $X(u) \in GMDA(\overline{w}_{\tau/\gamma^{1+\theta},\theta})$. Consequently, if (X,Y) are asymptotically independent, which is in particular the case when $(\ln X, \ln Y)$ are jointly Gaussian with correlation $\rho \in [-1,1)$ and $N(0,1)$ components, then also $(X(u),Y(u))$ are asymptotically independent. Hence (3.25) implies

$$\liminf_{u\to\infty} \frac{\boldsymbol{P}\{S(u) > t_u\}}{\boldsymbol{P}\{X(u) > t_u\}} \ge 1 + c, \tag{3.26}$$

with $c = 1$ if $\beta = 1$, and $c = 0$ otherwise.

4 Extremes and Conditional Limit Results for Log-Elliptical Risks

Let $\boldsymbol{X} \sim LE_k(\boldsymbol{\mu}, \Sigma, F)$ be a log-elliptical random vector in $\mathbb{R}^k, k \ge 2$, with stochastic representation (2.2). We assume in the sequel that Σ is non-singular, and it is a correlation matrix with the entries of the main diagonal equal to 1. For any i,j two different indices we have the stochastic representation

$$(X_i, X_j) \stackrel{d}{=} \left(\lambda_i \exp\Big(\rho_{ij} S_1 + \sqrt{1-\rho_{ij}^2} S_2\Big), \lambda_j \exp(S_1)\right), \quad \lambda_l := \exp(\mu_l), \quad 1 \le l \le k, \tag{4.27}$$

where $\boldsymbol{S} \stackrel{d}{=} R_2 \boldsymbol{U}$, with $R_2 \stackrel{d}{=} \sqrt{X_1^2 + X_2^2}$ being independent of $\boldsymbol{U}$ which is uniformly distributed on the unit circle of $\mathbb{R}^2$, and $\rho_{ij} \in (-1,1)$ the ijth entry of Σ. Hence

$$\frac{X_i}{\lambda_i} \stackrel{d}{=} \frac{X_1}{\lambda_1}, \quad i \le k \tag{4.28}$$

and further

$$X_i + X_j \stackrel{d}{=} \lambda_i \exp\Big(\rho_{ij} S_1 + \sqrt{1-\rho_{ij}^2} S_2\Big) + \lambda_j \exp(S_1).$$

Next, we discuss the MDA of a log-elliptical random vector, provided that the defining elliptical random vector has components with df in the Gumbel MDA. Further we show two conditional limit results which are of some interest for the aggregation question treated in the next section.

For a given square matrix Σ we write below Σ_{IJ} for the submatrice of Σ obtained by selecting the rows and the columns of Σ with indices in I and J, respectively. Define similarly Σ_{JI}, Σ_{II}, set $\boldsymbol{x}_I := (x_i, i \in I)^\top, \boldsymbol{x}_J := (x_i, i \in J)^\top$ for any $\boldsymbol{x} \in \mathbb{R}^k$ and write $\boldsymbol{X} \sim LN_k(\Sigma)$ if $\boldsymbol{X} = \exp(\boldsymbol{X})$ with $\boldsymbol{X}$ a k-dimensional Gaussian random vector with mean zero and covariance matrix Σ.

Theorem 4.1. *Let $\overline{\boldsymbol{X}}(u) = \exp(\boldsymbol{Y}(u)) \sim LE_k(\boldsymbol{0}, \Sigma_u, F), u > 0$ be a k-dimensional log-elliptical random vector defined by (2.2) with $R \sim F$ being positive, and let $\Sigma, \Sigma_u, u > 0$ be positive definite correlation matrices with elements $\sigma_{ij}(u), i,j \le k$. Suppose that either $R \in GMDA(w)$ or $Y_1(1) \in GMDA(w)$, and $\lim_{u\to\infty} \Sigma_u = \Sigma$. Let $l \le k$ be a given integer, and let $t_u, u > 0$ be such that $\lim_{u\to\infty} t_u = \infty$. Define*

$$\boldsymbol{Z}_{I,u} := \left(\left(\frac{\overline{X}_i(u)}{t_u^{\sigma_{il}(u)}} \right)^{q(t_u)}, i \in I \right)^\top \Big| (\overline{X}_l(u) > t_u),$$

with $q(x) := \sqrt{w(\ln x)/\ln x}, x \in (0,\infty), I := \{1,\dots,k\} \setminus \{l\}$.
(a) If the scaling function w satisfies $\lim_{u\to\infty} w(u) = \infty$, then for any $t > 0$ $\overline{X}_1(t) \in GMDA(w_*)$, with $w_*(x) =$

$w(\ln x)/x, x>0$. *Furthermore* $\overline{\boldsymbol{X}}(u), u>0$ *has asymptotic independent components.*

(b) For any $\boldsymbol{x} \in (0,\infty)^k$

$$\lim_{u\to\infty} \boldsymbol{P}\{\boldsymbol{Z}_{I,u} \le \boldsymbol{x}_I\} = \boldsymbol{P}\{\boldsymbol{Z}_I \le \boldsymbol{x}_I\} \tag{4.29}$$

holds with $\boldsymbol{Z}_I \sim LN_{k-1}(\Sigma_{II} - \Sigma_{IJ}\Sigma_{JI}), J := \{l\}$.

Example 4.2. *Consider* $(X_1(u), X_2(u))^\top \sim LE_2(\boldsymbol{0}, \Sigma_u, F), u>0$ *a bivariate elliptical random vector. Assume that* $\ln X_1(u) \stackrel{d}{=} \ln X_1(1) \in GMDA(w)$ *where* w *is given by*

$$w(x) = \mathcal{L}(x)x^{\theta-1}, \quad x>0, \tag{4.30}$$

with $\theta>0$ *and* $\mathcal{L}(\cdot)$ *a positive slowly varying function. See Resnick (1987), Bingham et al. (1987) or Embrechts et al. (1997) for details on regular variation. Applying Theorem 4.1 we obtain* $X_1(1) \in GMDA(w_*)$, *where*

$$w_*(x) = \frac{\mathcal{L}(\ln x)(\ln x)^{\theta-1}}{x}, \quad x>0.$$

For $\mathcal{L}(\cdot)$ *constant and* $\theta=2$ *the scaling function* w_* *is proportional to that corresponding to* $(X_1(u), X_2(u))$ *being a bivariate log-normal random vector. If* $\lim_{u\to\infty}\Sigma_u = \Sigma$ *with* Σ *non-singular matrix, then* (3.26) *holds with* $c=1$.

5 Risk Aggregation of Log-Elliptical Risks

The Mitra-Resnick methodology together with the conditional limit results of the previous section enables us to derive the asymptotics of the aggregated risk and the maximum risk in a quite general log-elliptical framework. In view of Example 3.5 the Gumbel MDA assumption for the components of the pertaining risks implies a asymptotic lower bound for the survival function of the sum of log-elliptical risks. Making use of the conditional limiting result obtained above combined with the results of Section 3 we show next the tail equivalence of the sum and the maximum. Our notation below agree with that in Theorem 3.3 and Theorem 4.1.

Theorem 5.1. *Let* $R, \Sigma, \overline{\boldsymbol{X}}(u), \Sigma_u$ *be as in Theorem 4.1, and define* $\boldsymbol{X}(u) := \boldsymbol{\lambda}(u)\overline{\boldsymbol{X}}(u)^{\boldsymbol{\beta}\gamma(u)}, u>0$ *with* $\boldsymbol{\beta}, \boldsymbol{\lambda}, \boldsymbol{\lambda}(u), \gamma(u) \in (0,\infty)^k, u>0$ *satisfying the assumptions of Theorem 3.3. If* $R \in GMDA(w)$, *and further*

$$\lim_{u\to\infty} w(u) = \infty, \qquad \text{and } \lim_{u\to\infty}\frac{\ln w(u)}{u} = \beta_1\gamma\xi, \quad \xi \in \left[0, 1 - \max_{1\le i\ne j\le k}\left(\frac{1+\sigma_{ij}}{2}\right)^{1/2}\right) \tag{5.31}$$

hold, then for any $t_u, u>0$ *with* $\lim_{u\to\infty} t_u = \infty$ *we have (set* $\chi_u := \frac{\ln t_u - \ln \lambda_1(u)}{\beta_1\gamma(u)}, u>0$*)*

$$\begin{aligned}\boldsymbol{P}\{S(u)>t_u\} &= (1+o(1))m_*\boldsymbol{P}\{X_1(u)>\chi_u\} &(5.32)\\ &= (1+o(1))\boldsymbol{P}\{M(u)>t_u\}, &(5.33)\end{aligned}$$

with m_* *being the number of elements of the index set* $\{i\le m : \lambda_i := \lambda_1\}$ *where* m *is the multiplicity of* β_1.

Remark 5.2. *(i) For log-normal distributions with underlying covariance matrix not depending on* u (5.32) *is shown in Theorem 1 of Asmussen et al. (2011), and* (5.33) *in Proposition 1 therein.*

(ii) The assumption $\lim_{u\to\infty}\Sigma_u = \Sigma$ *in Theorem 5.1 can be relaxed to* $\limsup_{u\to\infty}\max_{1\le i\ne j\le k}\sigma_{ij}(u) < b < 1$.

Example 5.3. *Let* $\overline{\boldsymbol{X}}(u) = \exp(\boldsymbol{Y}(u)) \sim LE(\boldsymbol{0}, \Sigma_u, F), u>0$ *be log-elliptical random vectors where* $\Sigma_u, u>0$ *is a correlation matrix. Assume that* $Y_1(1) \in GMDA(w)$, *with* w *as in Example 4.2. Suppose that* $\theta\ge 1$, *and additionally* $\lim_{u\to\infty}\mathcal{L}(u)=\infty$ *if* $\theta=1$. *We may thus write*

$$\lim_{u\to\infty}\frac{\ln w(u)}{u} = \lim_{u\to\infty}\frac{\ln\mathcal{L}(u) + (\theta-1)\ln u}{u} = 0,$$

which implies that (5.31) *holds with* $\xi = 0$. *Consequently, with constants* $\boldsymbol{\beta}, \lambda(u), \gamma(u), u > 0$ *as in Theorem 5.1 equation* (5.32) *can be re-written as*

$$\boldsymbol{P}\{S(u) > t_u\} \quad = \quad (1+o(1))m_* \boldsymbol{P}\Big\{X_1(u) > \frac{\ln t_u - \ln \lambda_1(u)}{\beta_1 \gamma(u)}\Big\}.$$

Note that the special case that $\theta = 2$ *and* $\mathcal{L}(\cdot)$ *is constant includes the case that* $\boldsymbol{Y}(u)$ *is a standard Gaussian random vector (with zero-mean and covariance matrix equal to the correlation matrix).*

Example 5.4. *Consider the bivariate log-normal random vector defined in* (3.14). *We can write*

$$(X(u), Y(u)) \stackrel{d}{=} \Big(\exp(\gamma(u)\sqrt{\eta_u} Z_1(u)), \exp(\gamma(u) Z_2(u))\Big),$$

where

$$Z_1(u) := \frac{B(\eta_u u)}{\sqrt{\eta_u u}} \stackrel{d}{=} B(1), \quad Z_2(u) := \frac{B(u)}{\sqrt{u}} \stackrel{d}{=} B(1), \quad \gamma(u) := \overline{\gamma}_u \sqrt{u} \to \gamma \in (0, \infty).$$

Since $\sqrt{1 - 1/\eta_u} \in (0,1)$ *equals the correlation coefficient of* $Z_1(u)$ *and* $Z_2(u)$ *we need to put some restrictions on* η_u. *Suppose that* $\lim_{u\to\infty} \eta_u = \eta > 1, u > 0$. *Clearly,* $Z_1(u), Z_2(u) \in GMDA(w)$ *with* $w(x) = x, x > 0$. *The condition* (3.15) *for the pair* $\exp(Z_1(u)), \exp(Z_2(u))$ *holds with* $c = 1$, *whereas Assumptions 2 and 3 can be checked to hold along the lines of Example 3.5 in Mitra and Resnick (2009). Applying now Theorem 5.1 with* $\beta_1 = \sqrt{\eta}, \beta_2 = 1, \xi = 0$ *and* $\lambda_1(u) = \lambda_2(u) = 1, u > 0$ *we obtain for any* t_u *such that* $\lim_{u\to\infty} t_u = \infty$

$$\boldsymbol{P}\{S(u) > t_u\} \quad = \quad (1+o(1))\boldsymbol{P}\Big\{B(1) > \frac{\ln t_u}{\sqrt{\eta}\gamma(u)}\Big\}, \quad u \to \infty.$$

6 Application

In this section we establish the logarithmic efficiency of a conditional Monte Carlo algorithm for the simulation of rare events related to aggregated risk. Let $\boldsymbol{X}(u) \sim LE_k(\boldsymbol{\mu}(u), \Sigma_u, F), u > 0$ be a given k-dimensional log-elliptical random vector with $\boldsymbol{\mu}(u) \in \mathbb{R}^k, u > 0$. Further let $\Sigma, \Sigma_u, u > 0$ be positive definite matrices such that $\lim_{u\to\infty} \Sigma_u = \Sigma$, and define $S(u) := X_1(u) + \cdots + X_k(u), u > 0$. Monte Carlo simulation of the rare event probability $\boldsymbol{P}\{S(u) > u\}$ in the log-elliptical setup is suggested in Rojas-Nandayapa (2008). The main idea of the aforementioned contribution rests on the fact that $\boldsymbol{X}(u) \stackrel{d}{=} \exp(RA_u\boldsymbol{U} + \boldsymbol{\mu}(u))$ with $R > 0$ almost surely, $\boldsymbol{U}$ being uniformly distributed on the unit sphere of $\mathbb{R}^k$ and A_u a square matrix satisfying $A_u(A_u)^\top = \Sigma_u, u > 0$. Since R and $\boldsymbol{U}$ are independent we obtain conditioning on $\boldsymbol{U}$

$$\boldsymbol{P}\{S(u) > u\} = \boldsymbol{P}\{h(R, A_u, \boldsymbol{U}) > u\} = \boldsymbol{E}\{\boldsymbol{P}\{h(R, A_u, \boldsymbol{U}) > u)|\boldsymbol{U}\}\}, \quad u > 0,$$

where h is such that $h(R, A_u, \boldsymbol{U}) := S(u)$ (we use the same notation as in the aforementioned paper). Denote in the following by $\boldsymbol{u}$ a simulated value (outcome) of $\boldsymbol{U}$. Since Σ_u is positive definite, for any fixed u, the equation $h(R, A_u, \boldsymbol{u}) = u$ solved for $r > 0$ has at most two solutions denoted by $\psi_L(u, \boldsymbol{u}), \psi_U(u, \boldsymbol{u})$.

For a given outcome $\boldsymbol{u}$ the function h can be $P1$) strictly decreasing, $P2$) both decreasing or increasing, or $P3$) strictly increasing. The properties $P1 - P3$ are examined in Rojas-Nandayapa (2008), p.62. We define $\psi_L(u, \boldsymbol{u}), \psi_U(u, \boldsymbol{u})$ as therein; for instance if $P2$ holds, then there exist at least two different solutions $\psi_L(u, \boldsymbol{u}), \psi_U(u, \boldsymbol{u})$ such that

$$\lim_{u\to\infty} \psi_L(u, \boldsymbol{u}) = -\infty, \text{ and } \lim_{u\to\infty} \psi_U(u, \boldsymbol{u}) = \infty.$$

The estimator of the rare event $\boldsymbol{P}\{S(u) > u\}$ defined in the aforementioned paper is

$$\hat{z}_1(u) \quad = \quad \boldsymbol{P}\{R < \psi_L(u, \boldsymbol{U})\}\mathbf{1}_{\{\psi_L(u,\boldsymbol{U})>0\}} + \boldsymbol{P}\{R > \psi_U(u, \boldsymbol{U})\}, \quad u > 0, \tag{6.34}$$

where the indicator function is needed to make sure that $R > 0$.

The algorithm proposed in Rojas-Nandayapa (2008) consists of the following steps:

1. Simulate the random vector $\boldsymbol{U}$;
2. Calculate $\psi_L(u, \boldsymbol{U}), \psi_U(u, \boldsymbol{U})$;
3. Return $\hat{z}_1(u)$ as in (6.34).

Theorem 6.1. *Let $\boldsymbol{\beta}, \Sigma, \boldsymbol{X}(u), S(u), \Sigma_u, \boldsymbol{\lambda}(u), \gamma(u), u > 0$ be as in Theorem 4.1 where $\lim_{u\to\infty} \Sigma_u = \Sigma$. If Σ is non-singular, then $\hat{z}_1(u)$ is an unbiased estimator of $\boldsymbol{P}\{S(u) > u\}$. If further $R \in GMDA(w)$ where w is given by (4.30) with $\theta \ge 1$, and (5.31) holds, then $\hat{z}_1(u)$ is a logarithmically efficient estimator.*

Theorem 6.1 shows that the algorithm of Rojas-Nandayapa is logarithmically efficient, provided that w has an asymptotic behaviour as a power function. A special case is when $w(x) = x$, which includes the log-normal random vectors, and thus we retrieve the result of Corollary 4.3 in Rojas-Nandayapa (2008).

7 Proofs

Proof of Theorem 3.1 We adopt the proof of Theorem 2.2 in Mitra and Resnick (2009) attempting a shorter alternative way which allows us to omit the analogous results of Proposition 2.1 and Lemma 2.1 therein. Set for $u > 0, L$

$$t_u^* := t_u - L/w_u(t_u), \quad w_u^* := w_u(t_u^*), \quad z_u := t_u^* w_u^*.$$

By Assumption 1 (recall (2.8))

$$\lim_{u\to\infty} t_u^* = \lim_{u\to\infty} t_u[1 - L/(t_u w_u(t_u))] = \infty, \quad \lim_{u\to\infty} z_u = \lim_{u\to\infty} t_u^* w_u^* = \lim_{u\to\infty} t_u w_u(t_u) = \infty.$$

Assumption 1 and Assumption 3 $[A_{X(u),Y(u),w_u;L}]$ imply

$$\begin{aligned}
\lim_{u\to\infty} \frac{\boldsymbol{P}\{S(u) > t_u, X(u) > t_u^*, Y(u) > t_u^*\}}{\boldsymbol{P}\{X(u) > t_u\}} &\le \lim_{u\to\infty} \frac{\boldsymbol{P}\{X(u) > t_u^*\}}{\boldsymbol{P}\{X(u) > t_u\}} \frac{\boldsymbol{P}\{X(u) > t_u^*, Y(u) > t_u^*\}}{\boldsymbol{P}\{X(u) > t_u^*\}} \\
&\le \exp(L) \limsup_{u\to\infty} \frac{\boldsymbol{P}\{X(u) > z_u/w_u^*, Y(u) > z_u/w_u^*\}}{\boldsymbol{P}\{X(u) > t_u^*\}} \\
&= 0.
\end{aligned}$$

Assumption 3 yields further

$$\begin{aligned}
\frac{\boldsymbol{P}\{S(u) > t_u, X(u) \le t_u^*, Y(u) \le t_u^*\}}{\boldsymbol{P}\{X(u) > t_u\}} &\le \frac{\boldsymbol{P}\{X(u) > L/w_u^*, Y(u) > L/w_u^*\}}{\boldsymbol{P}\{X(u) > t_u\}} \\
&\to 0, \quad u \to \infty.
\end{aligned}$$

The joint convergence in (2.13) means simply that (recall $\lim_{u\to\infty} t_u^* = \infty$)

$$w_u^*\Big(S(u) - t_u^*\Big)\Big|(X(u) > t_u^*) \stackrel{d}{\to} \mathcal{E}, \quad u \to \infty, \tag{7.35}$$

with $\mathcal{E}$ a unit exponential random variable, hence together with the Assumption 1

$$\begin{aligned}
\lim_{u\to\infty} \frac{\boldsymbol{P}\{S(u) > t_u, X(u) > t_u^*\}}{\boldsymbol{P}\{X(u) > t_u\}} &= \lim_{u\to\infty} \frac{\boldsymbol{P}\{X(u) > t_u^*\}}{\boldsymbol{P}\{X(u) > t_u\}} \lim_{u\to\infty} \boldsymbol{P}\{w_u^*[X(u) - t_u^* + Y(u)] > L | X(u) > t_u^*\} \\
&\to \exp(L)\exp(-L) = 1, \quad u \to \infty.
\end{aligned}$$

Next, if $c = 0$, then we have

$$\lim_{u\to\infty} \frac{\boldsymbol{P}\{S(u) > t_u, Y(u) > t_u^*\}}{\boldsymbol{P}\{X(u) > t_u\}} \le \lim_{u\to\infty} \frac{\boldsymbol{P}\{Y(u) > t_u^*\}}{\boldsymbol{P}\{X(u) > t_u\}}$$

$$
\begin{aligned}
&= \lim_{u\to\infty} \frac{\boldsymbol{P}\{Y(u)>t_u^*\}}{\boldsymbol{P}\{X(u)>t_u^*\}} \lim_{u\to\infty} \frac{\boldsymbol{P}\{X(u)>t_u^*\}}{\boldsymbol{P}\{X(u)>t_u\}}\\
&= 0,
\end{aligned}
$$

whereas if $c \in (0,\infty)$ by (3.15) and Assumption 1 we conclude that $Y(u) \in GDMA(w_u)$. By Assumption 2 $[A_{Y(u),X(u),w_u}]$, as for (7.35) we get (note we condition on $Y(u) > t_u^*$ below)

$$
w_u^*\Big(S(u)-t_u^*\Big)\Big|(Y(u)>t_u^*) \quad \overset{d}{\to} \quad \mathcal{E}, \quad u\to\infty.
$$

Consequently

$$
\begin{aligned}
\lim_{u\to\infty} \frac{\boldsymbol{P}\{S(u)>t_u, Y(u)>t_u^*\}}{\boldsymbol{P}\{X(u)>t_u\}} &= \lim_{u\to\infty} \frac{\boldsymbol{P}\{Y(u)>t_u^*\}}{\boldsymbol{P}\{Y(u)>t_u\}} \lim_{u\to\infty} \boldsymbol{P}\{w_u^*[S(u)-t_u^*]>L|Y(u)>t_u^*\}\\
&\quad \times \lim_{u\to\infty} \frac{\boldsymbol{P}\{Y(u)>t_u\}}{\boldsymbol{P}\{X(u)>t_u\}}\\
&= \exp(L)\exp(-L)c = c.
\end{aligned}
$$

Hence

$$
\lim_{u\to\infty} \frac{\boldsymbol{P}\{S(u)>t_u\}}{\boldsymbol{P}\{X(u)>t_u\}} = 1+c.
$$

By (2.12) and the fact that for any $u>0$

$$
\begin{aligned}
\boldsymbol{P}\{X(u)>t_u\}+\boldsymbol{P}\{Y(u)>t_u\} &\geq \boldsymbol{P}\{\max(X(u),Y(u))>t_u\}\\
&\geq \boldsymbol{P}\{X(u)>t_u\}+\boldsymbol{P}\{Y(u)>t_u\}-\boldsymbol{P}\{X(u)>t_u, Y(u)>t_u\}
\end{aligned}
$$

we obtain

$$
\lim_{u\to\infty} \frac{\boldsymbol{P}\{M(u)>t_u\}}{\boldsymbol{P}\{X(u)>t_u\}} = 1+c,
$$

and thus the claim follows. □

Proof of Corollary 3.2 With the same arguments as in the proof of Corollary 2.2 of Mitra and Resnick (2009) utilising further Remark 3.4 (iv), we can show for any $t_u, u>0$ such that $\lim_{u\to\infty} t_u = \infty$

$$
\lim_{u\to\infty} \frac{\boldsymbol{P}\{S(u)>t_u\}}{\boldsymbol{P}\{X_1(u)>t_u\}} = \sum_{i\leq k} c_i.
$$

Since further

$$
\begin{aligned}
\sum_{1\leq i\leq k} \boldsymbol{P}\{X_i(u)>t_u\} &\geq \boldsymbol{P}\{\max_{1\leq i\leq k} X_i(u)>u\}\\
&\geq \sum_{1\leq i\leq k} \boldsymbol{P}\{X_i(u)>t_u\} - \sum_{1\leq i\neq j\leq k} \boldsymbol{P}\{X_i(u)>t_u, X_j(u)>t_u\}
\end{aligned}
$$

and by (3.18) (recall (2.8)) for any $1\leq i\neq j\leq k$ we have

$$
\lim_{u\to\infty} \frac{\boldsymbol{P}\{X_i(u)>t_u, X_j(u)>t_u\}}{\boldsymbol{P}\{X_j(u)>t_u\}} = 0.
$$

Next, applying (3.17) we arrive at:

$$
\lim_{u\to\infty} \frac{\boldsymbol{P}\{X_i(u)>t_u, X_j(u)>t_u\}}{\boldsymbol{P}\{X_1(u)>t_u\}} = 0.
$$

Consequently

$$
\lim_{u\to\infty} \frac{\boldsymbol{P}\{M(u)>t_u\}}{\boldsymbol{P}\{X_1(u)>t_u\}} = \sum_{i\leq k} c_i
$$

establishing thus the proof. □

PROOF OF THEOREM 3.3 The proof when $\boldsymbol{\lambda}(u)=\boldsymbol{\lambda}\in(0,\infty)^k, \gamma(u)=\gamma\in(0,\infty)$ do not depend on u is given in Theorem 4.2 of Mitra and Resnick (2009). By the Gumbel MDA assumption

$$X_1^{\beta_1\gamma(u)}\in GMDA(\widetilde{w}_{\beta_1\gamma(u)}),\quad X_1(u)=\lambda_1(u)X_1^{\beta_1\gamma(u)}\in GMDA(w_{\lambda_1(u),\beta_1\gamma(u)}),$$

with

$$w_{a,b}(x)=\widetilde{w}_b(x/a)/a,\qquad \text{where } \widetilde{w}_{1/c}(x)=cx^{c-1}w(x^c),\quad c,x>0.$$

Further, for any $t_u,u>0$ (set $t_{u,i}:=(t_u/\lambda_i(u))^{1/(\beta_i\gamma(u))}, i\le k$)

$$\lim_{u\to\infty}\frac{t_{u,1}}{t_{u,i}}=0<1,\quad \text{if } \beta_1>\beta_j, \text{ and } \lim_{u\to\infty}\frac{t_{u,1}}{t_{u,i}}=\frac{\lambda_i}{\lambda_1},\quad \text{if } \beta_i=\beta_j,\quad i\ne j\le k.$$

Hence for any i such that $\beta_i<\beta_1$ or $\lambda_i<\lambda_1$

$$\lim_{u\to\infty}\frac{\boldsymbol{P}\{\lambda_i(u)X_i^{\beta_i\gamma(u)}>t_u\}}{\boldsymbol{P}\{\lambda_1(u)X_1^{\beta_j\gamma(u)}>t_u\}}\;=\;\lim_{u\to\infty}\frac{\boldsymbol{P}\{X_i>t_{u,i}\}}{\boldsymbol{P}\{X_1>t_{u,i}\}}\lim_{u\to\infty}\frac{\boldsymbol{P}\{X_1>t_{u,i}\}}{\boldsymbol{P}\{X_1>t_{u,1}\}}=:c_ic_i^*=0$$

since if $c_i\in(0,\infty)$, then (3.24) implies $c_i^*=0$. Now, the proof follows along the same lines of the proof of Theorem 4.2 of Mitra and Resnick (2009). □

PROOF OF THEOREM 4.1 (a) First note that the assumption $\Sigma_u, u>0$ is a correlation matrix and (4.28) imply

$$X_i(u)\overset{d}{=}X_j(u)\overset{d}{=}X_1(1),\quad Y_i(u)\overset{d}{=}Y_j(u)\overset{d}{=}Y_1(1)=:Y,\quad 1\le i,j\le k,\quad u>0.$$

By Theorem 4.1 in Hashorva and Pakes (2010) $R\in GMDA(w)$ if and only if $Y\in GMDA(w)$. Hence we may suppose that $Y\in GMDA(w)$. Our assumption $\lim_{u\to\infty}uw_*(u)=\infty$, where $w_*(x):=w(\ln x)/x, x>0$ implies for any $s\in\mathbb{R},u>0$

$$\frac{\boldsymbol{P}\{X_1(u)>x+s/w_*(x)\}}{\boldsymbol{P}\{X_1(u)>x\}}\;=\;\frac{\boldsymbol{P}\{Y>\ln x+\ln(1+s/w(\ln x))\}}{\boldsymbol{P}\{Y>\ln x\}}\to\exp(-s),\quad x\to\infty$$

implying $X_1(1)\in GMDA(w_*)$. In order to show the asymptotic independence of the components of $\boldsymbol{X}(u)$, it suffices to prove that

$$\lim_{u\to\infty}\frac{\boldsymbol{P}\{X_i(u)>\exp(u),X_j(u)>\exp(u)\}}{\boldsymbol{P}\{X_i(u)>\exp(u)\}}=0,\quad 1\le i\ne j\le k.$$

Since $Y\in GMDA(w)$ we have for any $\chi>1$

$$\lim_{u\to\infty}\frac{\boldsymbol{P}\{Y>\chi u\}}{\boldsymbol{P}\{Y>u\}}\;=\;0. \tag{7.36}$$

By (4.27) and applying Lemma 6.1 in Berman (1992) for any $u>0$

$$\begin{aligned}\frac{\boldsymbol{P}\{X_i(u)>\exp(u),X_j(u)>\exp(u)\}}{\boldsymbol{P}\{X_i(u)>\exp(u)\}}\;&\le\;\frac{\boldsymbol{P}\{(1+\sigma_{ij}(u))Y_1(u)+\sqrt{1-(\sigma_{ij}(u))^2}Y_2(u)>2u\}}{\boldsymbol{P}\{Y>u\}}\\&\to\;0,\quad u\to\infty.\end{aligned} \tag{7.37}$$

Consequently

$$\lim_{u\to\infty}\frac{\boldsymbol{P}\{X_i(u)>\exp(u),X_j(u)>\exp(u)\}}{\boldsymbol{P}\{X_i(u)>\exp(u)\}}\;=\;0,$$

and hence the first statement follows.

(b) Define for $u>0$ and some $l\le k$

$$\boldsymbol{Y}_{I,u}:=\boldsymbol{Y}_{I,u}|(Y_l(u)>u).$$

If $\Sigma_u, u>0$ does not depend on u in view of Balakrishnan and Hashorva (2013) (see also Hashorva and Jaworski (2012)) for any $\boldsymbol{x} \in \mathbb{R}^k$

$$\lim_{u\to\infty} \boldsymbol{P}\Big\{ \sqrt{w(u)/u}[\boldsymbol{Y}_{I,u} - u(\Sigma_u)_{IJ}] \le \boldsymbol{x}_I \Big\} = \boldsymbol{P}\{ \boldsymbol{Z}_I \le \boldsymbol{x}_I\},$$

with $\boldsymbol{Z}_I \in \mathbb{R}^{k-1}$ a Gaussian random vector with mean zero and covariance matrix $\Sigma_{II} - \Sigma_{IJ}\Sigma_{JI}, J := \{l\}$. If Σ_u depends on u, the proof of the above convergence can be established along the lines of the proof of the aforementioned theorems, and thus the result follows. □

Proof of Theorem 5.1 Since $\lim_{u\to\infty} w(u) = \infty$ applying Theorem 4.1 we have $\overline{X}_1(1) \in GMDA(w_*), w_*(x) = w(\ln x)/x$. Set next $Y := Y_1(1)$ and define

$$\tau_u := \frac{1}{\beta_1 \gamma(u)}, \quad s_u := \tau_u \ln t_u, \quad u>0.$$

If follows that $X_1(u) \in GMDA(w_u), u>0$ with w_u defined by

$$w_u(x) = \tau_u x^{\tau_u -1} w_*(x^{\tau_u}), \qquad x \in (0,\infty).$$

In view of (4.27) we can establish the proof by applying Theorem 3.3 restricting further to the 2-dimensional setup, i.e., $k=2$ and $\beta_1 = \beta_2$. By the assumption on the scaling function

$$w(x) \le \exp(\beta_1 \gamma(\xi+\varepsilon)x) \tag{7.38}$$

for some $\varepsilon>0$ such that $\xi+\varepsilon<1$. Hence

$$\lim_{u\to\infty} w_u(t_u) = \lim_{u\to\infty} \tau_u t_u^{\tau_u -1}\frac{w(\tau_u \ln t_u)}{t_u^{\tau}} \le \lim_{u\to\infty} \tau_u \frac{\exp\Big(\beta_1\gamma(\xi+\varepsilon)\tau_u \ln t_u\Big)}{t_u} = 0.$$

Next, we show further that both Assumption 2 and 3 are satisfied. By (4.29) for any $z \in (0,\infty)$

$$\lim_{u\to\infty} \boldsymbol{P}\Big\{ \frac{\overline{X}_1(u)}{t_u^{\sigma_{12}(u)}} > z^{1/q(t_u)} \Big| \overline{X}_2(u) > t_u \Big\} \quad = \quad \boldsymbol{P}\Big\{ \sqrt{1-\sigma_{12}^2} Z > \ln z \Big\}, \tag{7.39}$$

with $q(x) = \sqrt{w(\ln x)/\ln x}, x>0$ and Z a standard Gaussian random variable. The assumption $\lim_{u\to\infty}\Sigma_u = \Sigma$, with Σ being non-singular and the fact that $\lim_{u\to\infty} uw(u) = \infty$ imply

$$\lim_{u\to\infty} \sigma_{12}(u) = \sigma_{12} \in (-1,1), \quad \lim_{u\to\infty} w_u(s_u)s_u = \infty.$$

For any $a,u>0$ we obtain (set $\widetilde{t}_u := t_u^{\tau_u}, u>0$)

$$\begin{aligned}
\boldsymbol{P}\Big\{ w_u(t_u)X_1(u) > a \Big| X_2(u) > t_u \Big\} &= \boldsymbol{P}\Big\{ w_u(t_u)(\overline{X}_1(u))^{1/\tau_u} > a \Big| \overline{X}_2(u) > t_u^{\tau_u}\Big\} \\
&= \boldsymbol{P}\Big\{ \Big(\frac{\overline{X}_1(u)}{\widetilde{t}_u^{\sigma_{12}(u)}}\Big)^{1/\tau_u} > (\widetilde{t}_u)^{[1-\sigma_{12}(u)]/\tau_u} \frac{a}{\tau_u w(s_u)} \Big| \overline{X}_2(u) > \widetilde{t}_u \Big\} \\
&= \boldsymbol{P}\Big\{ \Big(\frac{\overline{X}_1(u)}{\widetilde{t}_u^{\sigma_{12}(u)}}\Big)^{q(\widetilde{t}_u)} > \widetilde{t}_u^{q(\widetilde{t}_u)[1-\sigma_{12}(u)]} \Big(\frac{a}{\tau_u w(s_u)}\Big)^{q(s_u)\tau_u} \Big| \overline{X}_2(u) > \widetilde{t}_u \Big\} \\
&= \boldsymbol{P}\Big\{ \Big(\frac{\overline{X}_1(u)}{\widetilde{t}_u^{\sigma_{12}(u)}}\Big)^{q(\widetilde{t}_u)} > \chi_{a,u} \Big| \overline{X}_2(u) > \widetilde{t}_u \Big\},
\end{aligned}$$

with

$$\begin{aligned}
\ln(\chi_{a,u}) &:= q(\widetilde{t}_u)\ s_u \Big[1 - \sigma_{12}(u) + \tau_u \frac{\ln a - \ln \tau_u}{s_u} - \tau_u \frac{\ln w(s_u)}{s_u} \Big] \\
&= \sqrt{s_u w(s_u)} \Big[1 - \sigma_{12} + o(1) - \xi \Big].
\end{aligned}$$

Since

$$\sqrt{(1+\sigma_{12})/2} \geq \sigma_{12}, \quad \forall \sigma_{12} \in (-1,1)$$

by (5.31) we have $1-\sigma_{12}-\xi>0$. Cnsequently,

$$\lim_{u\to\infty} \ln(\chi_{a,u}) = \infty$$

and thus Assumption 2 follows by (7.39). Condition (5.31) implies that there exists some $b \in (1,\infty)$ such that as $u \to \infty$

$$-\frac{\ln w_u(t_u)}{\ln t_u} = 1 - \frac{\ln \tau_u}{\ln t_u} - \tau_u \frac{\ln w(s_u)}{s_u}(1+o(1))b\sqrt{(1+\sigma_{12})/2}.$$

Hence as in (7.37) for all u large we have

$$\begin{aligned}\frac{\boldsymbol{P}\{X_1(u) > \frac{1}{w_u(t_u)}, X_2(u) > \frac{1}{w_u(t_u)}\}}{\boldsymbol{P}\{X_1(u) > t_u\}} &\leq \frac{\boldsymbol{P}\{\sqrt{2(1+\sigma_{12}(u))}Y > -2\tau_u \ln w_u(t_u)\}}{\boldsymbol{P}\{Y > \tau_u \ln t_u\}}\\ &= \frac{\boldsymbol{P}\{Y > (1+o(1))bs_u\}}{\boldsymbol{P}\{Y > s_u\}}\\ &\to 0, \quad u \to \infty\end{aligned}$$

and thus Assumption 3 holds with $L=1$, hence the proof is complete. □

Proof of Theorem 6.1 By the assumptions on $\Sigma_u, u>0$ it follows as in Theorem 4.2 of Rojas-Nandayapa (2008) (therein Σ_u does not depend on u) that $\hat{z}_1(u)$ is an unbiased estimator of $\boldsymbol{P}\{S(u)>u\}$. We show next that $\hat{z}_1(u)$ is a logarithmically efficient estimator, which is implied (see e.g., Rojas-Nandayapa (2008)) if for any $\varepsilon > 0$

$$\limsup_{u\to\infty} \frac{\boldsymbol{E}\{\hat{z}_1^2(u)\}}{(\boldsymbol{P}\{S(u)>u\})^{2-\varepsilon}} < \infty. \tag{7.40}$$

As in the proof of Theorem 4.2 of Rojas-Nandayapa (2008) we obtain for all large u

$$\boldsymbol{E}\{\hat{z}_1^2(u)\} \leq \boldsymbol{P}\Big\{R > \frac{\ln u - \mu_1(u) + \sqrt{(\ln u - \mu(u))^2 - 4\sigma(u)}}{2\sigma(u)}\Big\},$$

where

$$\mu_1(u) = \ln \lambda_1(u), \quad \mu(u) = \max_{1\leq j\leq k} \ln \lambda_i(u), \quad \sigma(u) = \beta_1\gamma(u), \quad u>0.$$

By the assumptions and Corollary 12.3.1 in Berman (1992)

$$\begin{aligned}\boldsymbol{P}\{X_1(u)>u\} &= \frac{1}{2}\boldsymbol{P}\{|X_1(u)|>u\}\\ &= (1+o(1))\frac{2^{(k-1)/2-1}\Gamma((k-1)/2)}{\Gamma(1/2)}(uw(u))^{-(k-1)/2}\boldsymbol{P}\{R>u\}, \quad u\to\infty.\end{aligned}$$

Since further the scaling function w given by (4.30) satisfies condition (5.31) with $\xi=0$ (see Example 4.2), then (5.32) holds. Next, applying Lemma 8.1 we establish (7.40), and thus the claimed result follows. □

8 Appendix

Lemma 8.1. *Let F be a univariate df such that $F \in GMDA(w)$. For any $a, \delta \in (0,\infty)$ if w is given by (4.30) with $\theta \in (0,\infty)$, then*

$$\lim_{u\to\infty} \frac{\overline{F}^{1+\delta}(u-a)}{\overline{F}(u)} = 0. \tag{8.41}$$

Proof of Lemma 8.1 By the assumption on F we have for some $z_0 > 0$ and all u large

$$\overline{F}(u) = c(x) \exp\Big(-\int_{z_0}^{u} \widetilde{w}(s)\, ds\Big),$$

where $c(x)$ is a positive measurable function satisfying $\lim_{u\to\infty} c(u) = c \in (0,\infty)$ and $\widetilde{w}$ is a positive measurable function such that $\lim_{u\to\infty} w(u)/\widetilde{w}(u) = 1$. See Resnick (1987) or Embrechts et al. (1997) for alternative representations of F. We may thus write for ε, a two positive constants (set for notational simplicity $z_0 = 0$)

$$\begin{aligned}\frac{\overline{F}^{1+\delta}(u-a)}{\overline{F}(u)} &= \frac{(c(u))^{1+\delta}}{c(u)} \exp\Big(-(1+\delta)\int_0^{u-a} \widetilde{w}(s)\, ds + \int_0^{u} \widetilde{w}(s)\, ds\Big)\\ &= (1+o(1)) \exp\Big(-\delta \int_0^{u-a} \widetilde{w}(s)\, ds + \int_{u-a}^{u} \widetilde{w}(s)\, ds\Big), \quad u \to \infty.\end{aligned}$$

If $\theta \in (0,1)$, since $\widetilde{w}(x) \sim w(x) = \mathcal{L}(x)x^{\theta-1}$, then $\lim_{u\to\infty} w(u) = \lim_{u\to\infty} \widetilde{w}(u) = 0$. Further, $F \in GMDA(w)$ implies

$$\lim_{u\to\infty} \frac{\overline{F}(u-a)}{\overline{F}(u)} = \lim_{u\to\infty} \frac{\overline{F}(u - [aw(u)]/w(u))}{\overline{F}(u)} = \exp(\lim_{u\to\infty} aw(u)) = 1,$$

hence (8.41) follows. If $\theta \ge 1$, then for any $\tau \in (0,1)$ and u large

$$\int_0^{u-a} \widetilde{w}(s)\, ds = u\int_0^{1-a/u} \widetilde{w}(us)\, ds > u \int_0^{\tau} \widetilde{w}(us)\, ds = (1+o(1))\frac{\tau^\theta}{\theta} u\widetilde{w}(u),$$

which follows by Theorem B.1.12 in De Haan and Ferreira (2006). With a similar argument we find that

$$\lim_{u\to\infty} \frac{\int_0^{u-a} \widetilde{w}(s)\, ds}{\int_{u-a}^{u} \widetilde{w}(s)\, ds} = \infty,$$

and thus again (8.41) follows. □

Acknowledgments. I am in debt to the referee for numerous suggestions and corrections which improved the manuscript. Partial support by the Swiss National Science Foundation Grant 200021-134785 is kindly acknowledged.

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