MAXIMA OF A TRIANGULAR ARRAY OF MULTIVARIATE GAUSSIAN SEQUENCE

ENKELEJD HASHORVA, LIANG PENG, AND ZHICHAO WENG

Abstract: It is known that the normalized maxima of a sequence of independent and identically distributed bivariate normal random vectors with correlation coefficient $\rho \in (-1,1)$ is asymptotically independent, which implies that using bivariate normal distribution will seriously underestimate extreme co-movement in practice. By letting ρ depend on the sample size and go to one with certain rate, Hüsler and Reiss (1989) showed that the normalized maxima of Gaussian random vectors can become asymptotically dependent so as to well predict the co-movement observed in the market. In this paper, we extend such a study to a triangular array of a multivariate Gaussian sequence, which further generalizes the results in Hsing, Hüsler and Reiss (1996) and Hashorva and Weng (2013).

Key Words: Correlation coefficient; maxima; stationary Gaussian triangular array

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1. Introduction

Let $(X_1^{(1)}, X_1^{(2)}), \cdots, (X_n^{(1)}, X_n^{(2)})$ be independent and identically distributed bivariate normal random vectors with zero means, unit variances and correlation coefficient $\rho \in [-1,1]$. For each $x \in \mathbb{R}$, put

$$u_n(x) = x/a_n + b_n \quad \text{with} \quad a_n = \sqrt{2\ln n} \quad \text{and} \quad b_n = \sqrt{2\ln n} - \frac{\ln\ln n + \ln(4\pi)}{2\sqrt{2\ln n}}. \tag{1.1}$$

When $|\rho| < 1$, it is known that for any $x, y \in \mathbb{R}$

$$\Psi_\rho(u_n(x), u_n(y)) := \mathbb{P}\left(\max_{1\le i\le n} X_i^{(1)} \le u_n(x), \max_{1\le i\le n} X_i^{(2)} \le u_n(y)\right) \to e^{-e^{-x}-e^{-y}} \quad \text{as} \quad n\to\infty,$$

where the limit becomes the joint distribution of two independent Gumbel random variables, and the choices of a_n and b_n in (1.1) can be found in Resnick (1987). In this case, $X_1^{(1)}$ and $X_1^{(2)}$ are called asymptotically independent (see Sibuya (1960)). Although normal distributions have many good properties and receive much attention in risk management (see McNeil, Frey and Embrechts (2005) for some overviews), this asymptotic independence property does seriously underestimate extreme co-movement observed in practice. To overcome this drawback, Hüsler and Reiss (1989) proposed to let $\rho = \rho(n)$ depend on the sample size n such that

$$(1-\rho(n))\ln n \to \lambda \in [0,\infty] \quad \text{as} \quad n\to\infty, \tag{1.2}$$

and then showed that

$$\lim_{n\to\infty} \Psi_{\rho(n)}(u_n(x), u_n(y)) = e^{-\Phi(\sqrt{\lambda}+\frac{x-y}{2\sqrt{\lambda}})e^{-y} - \Phi(\sqrt{\lambda}+\frac{y-x}{2\sqrt{\lambda}})e^{-x}} =: H_\lambda(x,y) \tag{1.3}$$

for $x, y \in \mathbb{R}$, where Φ denotes the standard normal distribution function. It follows from (1.3) that the limit distribution H_λ (referred to as the Hüsler-Reiss distribution) is not a product distribution when $\lambda \in [0,\infty)$, i.e., $X_1^{(1)}$ and $X_1^{(2)}$ are asymptotically dependent in this case. Using (1.2), Frick and Reiss (2013) extended the above limit to the maxima of

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normal copulas. Some other extensions of Hüsler and Reiss (1989) to more general triangular arrays have been made in the literature too as reviewed below.

Consider a triangular array of normal random variables $X_{n,i}, i = 1, 2, \cdots, n = 1, 2, \cdots$ such that for each n, $\{X_{n,i}, i \geq 1\}$ is a stationary normal sequence with mean zero, variance one and covariance $\rho_{n,j} = \mathbb{E}\{X_{n,1}X_{n,j+1}\}$. Motivated by condition (1.2), by assuming that

$$(1 - \rho_{n,j}) \ln n \to \delta_j \in (0, \infty] \quad \text{for all} \quad j \geq 1 \tag{1.4}$$

as $n \to \infty$, and some other conditions on $\rho_{n,j}$, Hsing, Hüsler and Reiss (1996) showed that

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{1\leq j\leq n} X_{n,j} \leq u_n(x)\right) = e^{-\theta e^{-x}} \tag{1.5}$$

holds for all $x \in \mathbb{R}$, where

$$\theta = \mathbb{P}\left(A/2 + \sqrt{\delta_k} W_k \leq \delta_k \quad \text{for all} \quad k \geq 1 \quad \text{such that} \quad \delta_k < \infty\right),$$

with A being a standard exponential random variable independent of W_k and $\{W_k : \delta_k < \infty, k \geq 1\}$ being jointly normal with zero means and

$$\mathbb{E}\{W_i W_j\} = \frac{\delta_i + \delta_j - \delta_{|i-j|}}{2\sqrt{\delta_i \delta_j}}.$$

Here θ is set to be 1 if all δ_k's are infinite. Recently French and Davis (2013) generalized this study to a Gaussian random field on a lattice.

Another extension of Hüsler and Reiss (1989) made by Hashorva and Weng (2013) is to study a triangular array of 2-dimensional stationary Gaussian sequence as follows.

Consider a triangular array of bivariate normal random vectors $X_{n,j} = (X_{n,j}^{(1)}, X_{n,j}^{(2)}), j = 1, 2, \cdots, n = 1, 2, \cdots$ such that for each n, $\{X_{n,j}, j \geq 1\}$ is a Gaussian sequence with mean zero, variance one and covariance

$$\mathbb{E}\left\{X_{n,k}^{(i)} X_{n,l}^{(j)}\right\} = \rho_{ij}(|k - l|, n) \quad \text{for} \quad i, j = 1, 2.$$

By assuming that

$$\lim_{n\to\infty} (1 - \rho_{12}(0, n)) \ln n = \lambda \in [0, \infty] \tag{1.6}$$

and

$$\sigma := \max_{1\leq k<n, n\geq 1, 1\leq i,j\leq 2} |\rho_{ij}(k, n)| < 1, \quad \lim_{n\to\infty} \max_{l_n \leq k<n, 1\leq i,j\leq 2} \rho_{ij}(k, n) \ln n = 0, \tag{1.7}$$

where $l_n = [n^\alpha]$ for some $\alpha \in (0, \frac{1-\sigma}{1+\sigma})$, Hashorva and Weng (2013) proved that

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{1\leq k\leq n} X_{n,k}^{(1)} \leq u_n(x), \max_{1\leq k\leq n} X_{n,k}^{(2)} \leq u_n(y)\right) = H_\lambda(x, y) \tag{1.8}$$

for all $x, y \in \mathbb{R}$. Taking $y = \infty$ in (1.8), we have

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{1\leq k\leq n} X_{n,k}^{(1)} \leq u_n(x)\right) = e^{-e^{-x}} \quad \text{for} \quad x \in \mathbb{R}.$$

That is, convergence in (1.8) excludes the possibility that (1.4) holds for $X_{n,k}^{(1)}$ and $X_{n,k}^{(2)}$. This motivates us to investigate the limit of $\mathbb{P}\left(\max_{1\leq k\leq n} X_{n,k}^{(1)} \leq u_n(x), \max_{1\leq k\leq n} X_{n,k}^{(2)} \leq u_n(y)\right)$ when (1.6) holds and (1.4) holds for both $X_{n,k}^{(1)}$ and $X_{n,k}^{(2)}$. Such a study will generalize the results in both Hsing, Hüsler and Reiss (1996) and Hashorva and Weng (2013).

Some other recent extensions of Hüsler and Reiss (1989) consist in to drop the Gaussian assumption. For example, Hashorva (2013) studied the maxima of some spherical processes; Hashorva, Kabluchko and Wübker (2012) investigated the maxima of χ^2-random vectors; Manjunath, Frick and Reiss (2012) discussed the maxima in the setup of extremal discriminant analysis; Engelke, Kabluchko and Schlather (2014) analyzed the maxima for some type of conditional Gaussian models.

We organize this paper as follows. Section 2 derives the limit for the normalized componentwise maxima of a triangular array of d-dimensional normal random vectors when (1.4) and (1.6) hold. All proofs are put in Section 3.

2. Main results

Throughout we consider a triangular array $\boldsymbol{X}_{n,k} = (X_{n,k}^{(1)}, \cdots, X_{n,k}^{(d)}), k = 1, 2, \cdots, n = 1, 2, \cdots$ such that for each n, $\{\boldsymbol{X}_{n,k}, k \geq 1\}$ is a d-dimensional stationary Gaussian sequence with mean zero, variance one and correlations given by $\mathbb{E}\left\{X_{n,k}^{(i)} X_{n,l}^{(j)}\right\} = \rho_{ij}(|k-l|, n)$ for $k, l = 1, 2, \cdots$ and $i, j = 1, 2, \cdots, d$.

Hereafter A stands for a unit exponential random variable being independent of all other random elements involved and $\boldsymbol{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$.

Theorem 2.1. *Let $\{\boldsymbol{X}_{n,k}, k, n \geq 1\}$ be a d-dimensional centered stationary Gaussian triangular array satisfying* $(\delta_{ii}(0) = 0)$

$$\left\{\begin{array}{ll} \lim_{n\to\infty}(1-\rho_{ij}(k,n))\ln n = \delta_{ij}(k) \in (0, \infty] & \text{for} \quad i, j = 1, \cdots, d; k = 1, 2, \cdots \\ \lim_{n\to\infty}(1-\rho_{ij}(0,n))\ln n = \delta_{ij}(0) \in (0, \infty] & \text{for} \quad i, j = 1, \cdots, d, i \neq j. \end{array}\right. \tag{2.1}$$

Suppose that there exist positive integers l_n, r_n satisfying

$$\lim_{n\to\infty} \frac{l_n}{r_n} = 0, \qquad \lim_{n\to\infty} \frac{r_n}{n} = 0, \tag{2.2}$$

$$\lim_{n\to\infty} \frac{n^2}{r_n} \sum_{i,j=1}^{d} \sum_{s=l_n}^{n} |\rho_{ij}(s,n)| \exp\left(-\frac{2\ln n - \ln\ln n}{1+|\rho_{ij}(s,n)|}\right) = 0 \tag{2.3}$$

and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{i,j=1}^{d} \sum_{s=m}^{r_n} n^{-\frac{1-\rho_{ij}(s,n)}{1+\rho_{ij}(s,n)}} \frac{(\ln n)^{-\rho_{ij}(s,n)/(1+\rho_{ij}(s,n))}}{\sqrt{1-\rho_{ij}^2(s,n)}} = 0. \tag{2.4}$$

Then

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{1\leq k\leq n} X_{n,k}^{(1)} \leq u_n(x_1), \cdots, \max_{1\leq k\leq n} X_{n,k}^{(d)} \leq u_n(x_d)\right) = \exp\left(-\sum_{i=1}^{d} \vartheta_i(\boldsymbol{x}) e^{-x_i}\right), \quad \forall \boldsymbol{x} \in \mathbb{R}^d, \tag{2.5}$$

where

$$\begin{aligned} \vartheta_1(\boldsymbol{x}) = \quad & \mathbb{P}\Big(\tfrac{A}{2} + \sqrt{\delta_{t1}(k-1)} W_{k,1}^{(t)} \leq \delta_{t1}(k-1) + \tfrac{x_t - x_1}{2}, 1 \leq t \leq d, \\ & \qquad \textit{for all} \quad k \geq 2 \quad \textit{such that} \quad \delta_{t1}(k-1) < \infty\Big) \end{aligned} \tag{2.6}$$

and for $i = 2, \cdots, d$

$$\begin{aligned} \vartheta_i(\boldsymbol{x}) = \quad & \mathbb{P}\Big(\tfrac{A}{2} + \sqrt{\delta_{si}(0)} W_{1,i}^{(s)} \leq \delta_{si}(0) + \tfrac{x_s - x_i}{2}, 1 \leq s < i, \delta_{si}(0) < \infty, \\ & \tfrac{A}{2} + \sqrt{\delta_{ti}(k-1)} W_{k,i}^{(t)} \leq \delta_{ti}(k-1) + \tfrac{x_t - x_i}{2}, 1 \leq t \leq d, \\ & \textit{for all} \quad k \geq 2 \quad \textit{such that} \quad \delta_{ti}(k-1) < \infty\Big), \end{aligned} \tag{2.7}$$

where $\{W_{k,i}^{(t)}, 1 \leq t \leq d, \delta_{ti}(k-1) < \infty, k \geq 1\}$ are jointly normal with zero means and for each $i = 1, \cdots, d$

$$Cov(W_{k,i}^{(j)}, W_{l,i}^{(t)}) \quad = \quad \frac{\delta_{ji}(k-1) + \delta_{ti}(l-1) - \delta_{jt}(|k-l|)}{2\sqrt{\delta_{ji}(k-1)\delta_{ti}(l-1)}}, \tag{2.8}$$

where $j,t=1,\cdots,d$, and $k,l\ge 1$ if $i\ne j$ and $i\ne t$, and $k,l\ge 2$ if $i=j$ or $i=t$.

Remark 2.1. *i) The ϑ's above should be set to 1 if all δ's involved are equal to infinity. If only a finite number of δ's is not equal to infinity, then ϑ's are all positive and thus the limit in (2.5) is a max-stable distribution function. As mentioned in Remark 2 of French and Davis (2013), for some tractable correlation functions it is possible to show that ϑ's are positive.*
ii) If condition (2.1) holds with $\delta_{ij}(k)=\infty$ for any index $i,j\le d$ and $k\ge 1$, then we can write

$$\vartheta_i(\boldsymbol{x})=\vartheta_i(x_1,\cdots,x_i),\quad \boldsymbol{x}\in\mathbb{R}^d, i\le d,$$

and the limiting distribution becomes $G(\boldsymbol{x})=e^{-\sum_{i=1}^d \vartheta_i(x_1,\cdots,x_i)e^{-x_i}}$, which coincides with the d-dimensional max-stable Hüsler-Reiss distribution.
iii) As in Theorem 2.2 of Hsing, Hüsler and Reiss (1996), conditions (2.2), (2.3) and (2.4) can be replaced by

$$\lim_{n\to\infty}\sum_{1\le i,j\le d}\max_{l_n\le k\le n}|\rho_{ij}(k,n)|\ln n=0\quad \textit{for some}\quad l_n=o(n)$$

and

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{i,j=1}^{d}\sum_{s=m}^{l_n} n^{-\frac{1-\rho_{ij}(s,n)}{1+\rho_{ij}(s,n)}}\frac{(\ln n)^{-\rho_{ij}(s,n)/(1+\rho_{ij}(s,n))}}{\sqrt{1-\rho_{ij}^2(s,n)}}=0.$$

These last two conditions are easier to check than those in Theorem 2.1.

3. Proofs

For notational simplicity we shall define

$$M_{k,l}^{(i)}=\max_{k<s\le l}X_{n,s}^{(i)},\quad M_l^{(i)}=M_{0,l}^{(i)}=\max_{1\le s\le l}X_{n,l}^{(i)},\quad M_{l,l}^{(i)}=-\infty$$

for $i=1,2,\cdots,d$, $k=1,\cdots,l$ and $l=1,\cdots,n$. Before proving the theorem, we need some lemmas.

Lemma 3.1. *For any $n\times d$ random matrix $\{X_{n,k}^{(i)},1\le k\le n,1\le i\le d\}$ and any vector of constants $(u^{(1)},\cdots,u^{(d)})$ we have*

$$\begin{aligned}(3.1)\qquad \mathbb{P}\left(\bigcup_{i=1}^{d}\{M_n^{(i)}>u^{(i)}\}\right) &= \sum_{k=1}^{n}\mathbb{P}\left(X_{n,k}^{(1)}>u^{(1)},\bigcap_{t=1}^{d}\{M_{k,n}^{(t)}\le u^{(t)}\}\right)\\ &\quad+\sum_{i=2}^{d}\sum_{k=1}^{n}\mathbb{P}\left(X_{n,k}^{(i)}>u^{(i)},\bigcap_{s=1}^{i-1}\{M_{k-1,n}^{(s)}\le u^{(s)}\},\bigcap_{t=i}^{d}\{M_{k,n}^{(t)}\le u^{(t)}\}\right).\end{aligned}$$

Proof. The case of $d=1$ directly follows from O'Brien (1987). We shall prove the case of $d=2$ and then use the induction method to conclude that the lemma holds for any $d\ge 2$.

It is straightforward to check that for any $s\ge 0$ and $i=1,\cdots,d$,

$$\begin{aligned}\mathbb{P}\left(M_{s,n}^{(i)}>u^{(i)}\right) &= \mathbb{P}\left(X_{n,n}^{(i)}>u^{(i)}\right)+\mathbb{P}\left(M_{s,n-1}^{(i)}>u^{(i)},X_{n,n}^{(i)}\le u^{(i)}\right)\\ &= \mathbb{P}\left(X_{n,n}^{(i)}>u^{(i)},M_{n,n}^{(i)}\le u^{(i)}\right)+\mathbb{P}\left(X_{n,n-1}^{(i)}>u^{(i)},M_{n-1,n}^{(i)}\le u^{(i)}\right)\\ &\quad+\mathbb{P}\left(M_{s,n-2}^{(i)}>u^{(i)},X_{n,n-1}^{(i)}\le u^{(i)},X_{n,n}^{(i)}\le u^{(i)}\right).\end{aligned}$$

Continuing the above decomposition, we have

$$\mathbb{P}\left(M_{s,n}^{(i)} > u^{(i)}\right) = \sum_{k=s+1}^{n} \mathbb{P}\left(X_{n,k}^{(i)} > u^{(i)}, M_{k,n}^{(i)} \le u^{(i)}\right) \tag{3.2}$$

for any $s \ge 0$. For proving that (3.1) holds for the case of $d = 2$, we first note that

$$\begin{aligned}
&\mathbb{P}\left(M_n^{(1)} \le u^{(1)}, M_n^{(2)} > u^{(2)}\right)\\
&= \sum_{k=1}^{n} \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \le u^{(2)}, M_n^{(1)} \le u^{(1)}\right)\\
&= \sum_{k=1}^{n} \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \le u^{(2)}, M_{k-1,n}^{(1)} \le u^{(1)}\right)\\
&\quad - \sum_{k=1}^{n} \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \le u^{(2)}, M_{k-1,n}^{(1)} \le u^{(1)}, M_{k-1}^{(1)} > u^{(1)}\right)\\
&= \sum_{k=1}^{n} \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \le u^{(2)}, M_{k-1,n}^{(1)} \le u^{(1)}\right)\\
&\quad - \sum_{k=1}^{n} \sum_{l=1}^{k-1} \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \le u^{(2)}, X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \le u^{(1)}\right)\\
&= \sum_{k=1}^{n} \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \le u^{(2)}, M_{k-1,n}^{(1)} \le u^{(1)}\right)\\
&\quad - \sum_{l=1}^{n-1} \sum_{k=l+1}^{n} \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \le u^{(2)}, X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \le u^{(1)}\right)\\
&= \sum_{k=1}^{n} \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \le u^{(2)}, M_{k-1,n}^{(1)} \le u^{(1)}\right) - \sum_{l=1}^{n-1} \mathbb{P}\left(M_{l,n}^{(2)} > u^{(2)}, X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \le u^{(1)}\right), \qquad (3.3)
\end{aligned}$$

which can be used to show that

$$\begin{aligned}
&\mathbb{P}\left(\{M_n^{(1)} > u^{(1)}\} \cup \{M_n^{(2)} > u^{(2)}\}\right)\\
&= \mathbb{P}\left(M_n^{(1)} > u^{(1)}\right) + \mathbb{P}\left(M_n^{(1)} \le u^{(1)}, M_n^{(2)} > u^{(2)}\right)\\
&= \sum_{l=1}^{n} \mathbb{P}\left(X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \le u^{(1)}\right) + \sum_{k=1}^{n} \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \le u^{(2)}, M_{k-1,n}^{(1)} \le u^{(1)}\right)\\
&\quad - \sum_{l=1}^{n-1} \mathbb{P}\left(M_{l,n}^{(2)} > u^{(2)}, X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \le u^{(1)}\right)\\
&= \mathbb{P}\left(X_{n,n}^{(1)} > u^{(1)}\right) + \sum_{l=1}^{n-1} \mathbb{P}\left(X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \le u^{(1)}, M_{l,n}^{(2)} \le u^{(2)}\right)\\
&\quad + \sum_{k=1}^{n} \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \le u^{(2)}, M_{k-1,n}^{(1)} \le u^{(1)}\right)\\
&= \sum_{l=1}^{n} \mathbb{P}\left(X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \le u^{(1)}, M_{l,n}^{(2)} \le u^{(2)}\right) + \sum_{k=1}^{n} \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \le u^{(2)}, M_{k-1,n}^{(1)} \le u^{(1)}\right),
\end{aligned}$$

i.e., (3.1) holds for $d = 2$.

Next, suppose that (3.1) holds for $d = m-1 > 2$, i.e.,

$$\begin{aligned}\mathbb{P}\left(\bigcup_{i=1}^{m-1}\{M_n^{(i)} > u^{(i)}\}\right) &= \sum_{k=1}^{n}\mathbb{P}\left(X_{n,k}^{(1)} > u^{(1)}, \bigcap_{t=1}^{m-1}\{M_{k,n}^{(t)} \le u^{(t)}\}\right)\\ &\quad + \sum_{i=2}^{m-1}\sum_{k=1}^{n}\mathbb{P}\left(X_{n,k}^{(i)} > u^{(i)}, \bigcap_{s=1}^{i-1}\{M_{k-1,n}^{(s)} \le u^{(s)}\}, \bigcap_{t=i}^{m-1}\{M_{k,n}^{(t)} \le u^{(t)}\}\right).\end{aligned} \tag{3.4}$$

In view of (3.2) and (3.4), we have

$$\begin{aligned}&\mathbb{P}\left(\bigcap_{i=1}^{m-1}\{M_n^{(i)} \le u^{(i)}\}, M_n^{(m)} > u^{(m)}\right)\\ &= \sum_{k=1}^{n}\mathbb{P}\left(X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \le u^{(m)}, \bigcap_{i=1}^{m-1}\{M_{k-1,n}^{(i)} \le u^{(i)}\}\right)\\ &\quad -\sum_{k=1}^{n}\mathbb{P}\left(X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \le u^{(m)}, \bigcap_{i=1}^{m-1}\{M_{k-1,n}^{(i)} \le u^{(i)}\}, \bigcup_{j=1}^{m-1}\{M_{k-1}^{(j)} > u^{(j)}\}\right)\\ &= \sum_{k=1}^{n}\mathbb{P}\left(X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \le u^{(m)}, \bigcap_{i=1}^{m-1}\{M_{k-1,n}^{(i)} \le u^{(i)}\}\right)\\ &\quad -\sum_{l=1}^{n-1}\mathbb{P}\left(M_{l,n}^{(m)} > u^{(m)}, \bigcap_{i=1}^{m-1}\{M_{l,n}^{(i)} \le u^{(i)}\}, X_{n,l}^{(1)} > u^{(1)}\right)\\ &\quad -\sum_{j=2}^{m-1}\sum_{l=1}^{n-1}\mathbb{P}\left(X_{n,l}^{(j)} > u^{(j)}, \bigcap_{s=1}^{j-1}\{M_{l-1,n}^{(s)} \le u^{(s)}\}, \bigcap_{t=j}^{m-1}\{M_{l,n}^{(t)} \le u^{(t)}\}, M_{l,n}^{(m)} > u^{(m)}\right).\end{aligned} \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}&\mathbb{P}\left(\bigcup_{i=1}^{m}\{M_n^{(i)} > u^{(i)}\}\right)\\ &= \mathbb{P}\left(\bigcup_{i=1}^{m-1}\{M_n^{(i)} > u^{(i)}\}\right) + \mathbb{P}\left(\bigcap_{i=1}^{m-1}\{M_n^{(i)} \le u^{(i)}\}, M_n^{(m)} > u^{(m)}\right)\\ &= \sum_{l=1}^{n}\mathbb{P}\left(X_{n,l}^{(1)} > u^{(1)}, \bigcap_{i=1}^{m}\{M_{l,n}^{(i)} \le u^{(i)}\}\right)\\ &\quad + \sum_{j=2}^{m}\sum_{l=1}^{n}\mathbb{P}\left(X_{n,l}^{(j)} > u^{(j)}, \bigcap_{s=1}^{j-1}\{M_{l-1,n}^{(s)} \le u^{(s)}\}, \bigcap_{t=j}^{m}\{M_{l,n}^{(t)} \le u^{(t)}\}\right),\end{aligned}$$

i.e., (3.1) holds for $d = m$. Hence the lemma follows from the induction method. □

Lemma 3.2. *Let $\{\boldsymbol{X}_{n,k}, k, n \ge 1\}$ be a d-dimensional centered stationary Gaussian triangular array. If there exist positive integers l_n and r_n such that (2.2) and (2.3) hold, then we have for any $x_i \in \mathbb{R}, i \le d$*

$$\lim_{n\to\infty}\left(\mathbb{P}\left(\bigcap_{i=1}^{d}\{M_n^{(i)} \le u_n(x_i)\}\right) - \left(\mathbb{P}\left(\bigcap_{i=1}^{d}\{M_{r_n}^{(i)} \le u_n(x_i)\}\right)\right)^{q_n}\right) = 0, \tag{3.6}$$

where $q_n = [n/r_n]$.

Proof. Define $N_n = \{1, 2, \cdots, n\}$ for any positive integer n and set

$$N_{r_n q_n} = (I_1 \cup J_1) \cup (I_2 \cup J_2) \cup \cdots \cup (I_{q_n} \cup J_{q_n}),$$

with $I_s = \{(s-1)r_n+1, \cdots, sr_n - l_n\}$ and $J_s = \{sr_n - l_n + 1, \cdots, sr_n\}$ for $s = 1, 2, \cdots, q_n$. Since $r_n q_n \le n < (r_n+1)q_n < r_n q_n + l_n$, we get $|N_n \backslash N_{r_n q_n}| < q_n < l_n$, where $|K|$ means the length of the interval $K \subset \mathbb{R}$. Further, define sets I_{q_n+1} and J_{q_n+1} by

$$\begin{aligned} I_{q_n+1} &= \{r_n q_n - r_n + l_n + 1, \cdots, r_n q_n - 1, r_n q_n\}, \\ J_{q_n+1} &= \{r_n q_n + 1, \cdots, r_n q_n + l_n - 1, r_n q_n + l_n\}. \end{aligned}$$

Clearly, $|I_{q_n+1}| = r_n - l_n$, $|J_{q_n+1}| = l_n$ and $I_{q_n+1} \subset N_{r_n q_n}$ and $N_n \backslash N_{r_n q_n} \subset J_{q_n+1}$. Using the fact that

$$l_n = o(r_n), \quad l_n = o(n), \quad \lim_{n\to\infty} n(1-\Phi(u_n(x_i))) = e^{-x_i}$$

we obtain

$$\begin{aligned} 0 &\le \mathbb{P}\left(\bigcap_{s=1}^{q_n}\bigcap_{i=1}^{d}\{M^{(i)}(I_s) \le u_n(x_i)\}\right) - \mathbb{P}\left(\bigcap_{i=1}^{d}\{M_n^{(i)} \le u_n(x_i)\}\right) \\ &\le \sum_{s=1}^{q_n+1}\sum_{i=1}^{d}\mathbb{P}\left(M^{(i)}(I_s) \le u_n(x_i) < M^{(i)}(J_s)\right) \\ &\le \sum_{s=1}^{q_n+1}\sum_{i=1}^{d}\mathbb{P}\left(u_n(x_i) < M^{(i)}(J_s)\right) \\ &\le (q_n+1)l_n\sum_{i=1}^{d}(1-\Phi(u_n(x_i))) \\ &\to 0 \end{aligned}$$

as $n \to \infty$, where $M^{(i)}(I_s) = \max_{j\in I_s} X_{n,j}^{(i)}$. Using Berman's inequality given in Li and Shao (2001) (see also Piterbarg (1996)) and (2.3)

$$\begin{aligned} &\left|\mathbb{P}\left(\bigcap_{s=1}^{q_n}\bigcap_{i=1}^{d}\{M^{(i)}(I_s) \le u_n(x_i)\}\right) - \prod_{s=1}^{q_n}\mathbb{P}\left(\bigcap_{i=1}^{d}\{M^{(i)}(I_s) \le u_n(x_i)\}\right)\right| \\ \le\ & (q_n-1)\frac{1}{2\pi}\sum_{i,j=1}^{d}\sum_{1\le s<t\le n, t-s>l_n}\left|\arcsin(\rho_{ij}(t-s,n))\right|\exp\left(-\frac{u_n^2(x_i)+u_n^2(x_j)}{2(1+|\rho_{ij}(t-s,n)|)}\right) \\ \le\ & C\frac{n^2}{r_n}\sum_{i,j=1}^{d}\sum_{s=l_n}^{n}|\rho_{ij}(s,n)|\exp\left(-\frac{2\ln n - \ln\ln n}{1+|\rho_{ij}(s,n)|}\right) \\ \to\ & 0 \quad \text{as} \quad n\to\infty, \end{aligned}$$

where C is some positive constant. Since further

$$\begin{aligned} 0 &\le \prod_{s=1}^{q_n}\mathbb{P}\left(\bigcap_{i=1}^{d}\{M^{(i)}(I_s) \le u_n(x_i)\}\right) - \prod_{s=1}^{q_n}\mathbb{P}\left(\bigcap_{i=1}^{d}\{M^{(i)}(I_s\cup J_s) \le u_n(x_i)\}\right) \\ &\le \sum_{s=1}^{q_n}\sum_{i=1}^{d}\mathbb{P}\left(u_n(x_i) < M^{(i)}(J_s)\right) \to 0 \end{aligned}$$

as $n \to \infty$, the lemma follows. □

Remark 3.1. *If $\{s_n, n \ge 1\}$ is a sequence of positive integers such that $s_n = o(n)$ and $r_n = o(s_n)$, then clearly both (2.2) and (2.3) hold with r_n replaced by s_n. From the proof above we see that these two conditions are the only assumptions of Lemma 3.2. Hence (3.6) still holds if we substitute q_n by $t_n = [n/s_n]$.*

Lemma 3.3. *Under the conditions of Theorem 2.1, for any bounded index set $K \subset \{2, 3, \cdots\}$ and each $c \in \{2, \cdots, d\}$ we have*

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\left(X_{n,1}^{(s)} \le u_n(x_s), 1 \le s < c, X_{n,k}^{(t)} \le u_n(x_t), 1 \le t \le d, k \in K | X_{n,1}^{(c)} > u_n(x_c)\right)\\
= \;& \mathbb{P}(\tfrac{A}{2} + \sqrt{\delta_{sc}(0)} W_{1,c}^{(s)} \le \delta_{sc}(0) + \tfrac{x_s - x_c}{2}, 1 \le s < c, \delta_{sc}(0) < \infty,\\
& \tfrac{A}{2} + \sqrt{\delta_{tc}(k-1)} W_{k,c}^{(t)} \le \delta_{tc}(k-1) + \tfrac{x_t - x_c}{2}, 1 \le t \le d, \textit{for all } \; k \in K \textit{ such that } \; \delta_{tc}(k-1) < \infty).
\end{aligned} \tag{3.7}$$

Further

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\left(X_{n,k}^{(t)} \le u_n(x_t), 1 \le t \le d, k \in K | X_{n,1}^{(1)} > u_n(x_1)\right)\\
= \;& \mathbb{P}(\tfrac{A}{2} + \sqrt{\delta_{t1}(k-1)} W_{k,1}^{(t)} \le \delta_{t1}(k-1) + \tfrac{x_t - x_1}{2}, 1 \le t \le d, \textit{for all } \; k \in K \; \textit{ such that } \; \delta_{t1}(k-1) < \infty),
\end{aligned} \tag{3.8}$$

and $\{W_{k,i}^{(t)}, 1 \le t \le d, \delta_{ti}(k-1) < \infty, k \in \{1\} \cup K\}$ are jointly normal with zero means and

$$Cov(W_{k,i}^{(j)}, W_{l,i}^{(t)}) \;=\; \frac{\delta_{ji}(k-1) + \delta_{ti}(l-1) - \delta_{jt}(|k-l|)}{2\sqrt{\delta_{ji}(k-1)\delta_{ti}(l-1)}},$$

where $i, j, t = 1, \cdots, d,$ and $k, l \in \{1\} \cup K$ if $i \ne j$ and $i \ne t,$ and $k, l \in K$ if $i = j$ or $i = t$.

Proof. We follow the arguments in the proof of Lemma 4.1 in Hsing, Hüsler and Reiss (1996). First like (4.1) therein we have for each $c \in \{2, \cdots, d\}$,

$$\begin{aligned}
&\mathbb{P}\left(X_{n,1}^{(s)} \le u_n(x_s), 1 \le s < c, X_{n,k}^{(t)} \le u_n(x_t), 1 \le t \le d, k \in K | X_{n,1}^{(c)} > u_n(x_c)\right)\\
\sim\; & \int_0^\infty \mathbb{P}\left(X_{n,1}^{(s)} \le u_n(x_s), 1 \le s < c, X_{n,k}^{(t)} \le u_n(x_t), 1 \le t \le d, k \in K \big| X_{n,1}^{(c)} = T_n(x_c, z)\right)\\
& \times \exp\left(-z - \frac{z^2}{2u_n^2(x_c)}\right) dz,
\end{aligned} \tag{3.9}$$

where $T_n(x_c, z) = u_n(x_c) + z/u_n(x_c)$. Let $\{Y_{n,k,c}^{(i)}, 1 \le i \le d, k \in \{1\} \cup K\}$ have the same distribution as the conditional distribution of $\{X_{n,k}^{(i)}, 1 \le i \le d, k \in \{1\} \cup K\}$ given $X_{n,1}^{(c)} = T_n(x_c, z)$. Then

$$\mathbb{E}\left\{Y_{n,k,c}^{(i)}\right\} = \rho_{ic}(k-1, n) T_n(x_c, z)$$

and

$$Cov(Y_{n,k,c}^{(i)}, Y_{n,k,c}^{(j)}) = \rho_{ij}(|k-l|, n) - \rho_{ic}(k-1, n)\rho_{jc}(l-1, n)$$

for $i, j \in \{1, \cdots, d\}$ and $k, l \in \{1\} \cup K$. Further define

$$Z_{n,k,c}^{(i)} = \frac{Y_{n,k,c}^{(i)} - \rho_{ic}(k-1, n) T_n(x_c, z)}{\sqrt{1 - \rho_{ic}^2(k-1, n)}},$$

where $1 \le i \le d$, and $k \in \{1\} \cup K$ if $i \ne c$, and $k \in K$ if $i = c$. Then we have

$$Cov(Z_{n,k,c}^{(i)}, Z_{n,l,c}^{(j)}) = \frac{\rho_{ij}(|k-l|, n) - \rho_{ic}(k-1, n)\rho_{jc}(l-1, n)}{\sqrt{(1 - \rho_{ic}^2(k-1, n))(1 - \rho_{jc}^2(l-1, n))}} \to \frac{\delta_{ic}(k-1) + \delta_{jc}(l-1) - \delta_{ij}(|k-l|)}{2\sqrt{\delta_{ic}(k-1)\delta_{jc}(l-1)}},$$

where $i,j\in\{1,\cdots,d\}$, and $k,l\in\{1\}\cup K$ if $c\neq i$ and $c\neq j$, and $k,l\in K$ if $c=i$ or $c=j$. Thus, using $u_n^2(x)\sim 2\ln n$ for $x\in\mathbb{R}$ we have

$$\begin{aligned}
&\mathbb{P}\left(Y_{n,1,c}^{(s)}\le u_n(x_s), 1\le s<c, Y_{n,k,c}^{(t)}\le u_n(x_t), 1\le t\le d, k\in K\right)\\
=\ &\mathbb{P}\Big(\tfrac{1}{2}\rho_{sc}(0,n)z+\sqrt{\tfrac{1+\rho_{sc}(0,n)}{2}}\sqrt{\tfrac{u_n^2(x_c)(1-\rho_{sc}(0,n))}{2}}Z_{n,1,c}^{(s)}\le\tfrac{1}{2}(u_n(x_s)u_n(x_c)-\rho_{sc}(0,n)u_n^2(x_c)),\\
&\tfrac{1}{2}\rho_{tc}(k-1,n)z+\sqrt{\tfrac{1+\rho_{tc}(k-1,n)}{2}}\sqrt{\tfrac{u_n^2(x_c)(1-\rho_{tc}(k-1,n))}{2}}Z_{n,k,c}^{(t)}\\
&\le\tfrac{1}{2}(u_n(x_t)u_n(x_c)-\rho_{tc}(k-1,n)u_n^2(x_c)),\quad\text{for}\quad 1\le s<c, 1\le t\le d, k\in K\Big)\\
\to\ &\mathbb{P}\Big(\tfrac{z}{2}+\sqrt{\delta_{sc}(0)}W_{1,c}^{(s)}\le\delta_{sc}(0)+\tfrac{x_s-x_c}{2}, 1\le s<c, \delta_{sc}(0)<\infty,\\
&\tfrac{z}{2}+\sqrt{\delta_{tc}(k-1)}W_{k,c}^{(t)}\le\delta_{tc}(k-1)+\tfrac{x_t-x_c}{2}, 1\le t\le d,\\
&\text{for all}\ \ k\in K\ \ \text{such that}\ \ \delta_{tc}(k-1)<\infty\Big).
\end{aligned}\tag{3.10}$$

Therefore, (3.7) follows by (3.10) and (3.9). The proof of (3.8) can be established with similar arguments. Hence the claim follows. □

Lemma 3.4. *Under the conditions of Theorem 2.1, for $c\in\{1,\cdots,d\}$ we have*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\bigcup_{i=1}^{d}\bigcup_{j=m}^{r_n}\{X_{n,j}^{(i)}>u_n(x_i)\}\Big|X_{n,1}^{(c)}>u_n(x_c)\right)=0.$$

Proof. It suffices to show that for each fixed $i\in\{1,\cdots,d\}$

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\bigcup_{j=m}^{r_n}\{X_{n,j}^{(i)}>u_n(x_i)\}\Big|X_{n,1}^{(c)}>u_n(x_c)\right)=0.$$

As in the proof of Lemma 3.3, write with $a_{nj}(z)=\rho_{ic}(j-1,n)(u_n(x_c)+z/u_n(x_c))$ and $b_{nj}:=\sqrt{1-\rho_{ic}^2(j-1,n)}$

$$\mathbb{P}\left(\bigcup_{j=m}^{r_n}\{X_{n,j}^{(i)}>u_n(x_i)\}|X_{n,1}^{(c)}>u_n(x_c)\right)\ \sim\ \int_0^\infty\mathbb{P}\left(\bigcup_{j=m}^{r_n}\{a_{nj}(z)+Z_{n,j,c}^{(i)}b_{nj}>u_n(x_i)\}\right)\exp\left(-z-\frac{z^2}{2u_n^2(x_c)}\right)dz.$$

Hence, we only need to show that for each fixed $z_0>0$

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_0^{z_0}\mathbb{P}\left(\bigcup_{j=m}^{r_n}\{a_{nj}(z)+Z_{n,j,c}^{(i)}b_{nj}>u_n(x_i)\}\right)\exp\left(-z-\tfrac{z^2}{2u_n^2(x_c)}\right)dz=0,$$

which follows if we show

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sup_{0\le z\le z_0}\sum_{j=m}^{r_n}\mathbb{P}\left(a_{nj}(z)+Z_{n,j,c}^{(i)}b_{nj}>u_n(x_i)\right)=0.\tag{3.11}$$

In view of the derivation of (4.4) in Hsing, Hüsler and Reiss (1996), condition (2.4) implies

$$\lim_{m\to\infty}\limsup_{n\to\infty}\max_{m\le j\le r_n}((1-\rho_{ic}(j-1,n))\ln n)^{-1}=0.$$

Thus, for large n and $j\in[m,r_n]$ we have

$$\theta_{nj}:=\frac{u_n(x_i)-u_n(x_c)\rho_{ic}(j-1,n)}{\sqrt{1-\rho_{ic}^2(j-1,n)}}-\frac{z\rho_{ic}(j-1,n)}{u_n(x_c)\sqrt{1-\rho_{ic}^2(j-1,n)}}>0.$$

By the fact that $1-\Phi(x)\le x^{-1}\varphi(x)$ for $x>0$, we obtain

$$\mathbb{P}\left(Z_{n,j,c}^{(i)}>\theta_{nj}\right)\ \le\ \frac{1}{\theta_{nj}\sqrt{2\pi}}\exp\left(-\frac{1}{2}\theta_{nj}^2\right).$$

Next, for some positive constant C depending only on x_i,x_c and z_0 we have

$$\theta_{nj}^2\ \le\ C+\frac{1-\rho_{ic}(j-1,n)}{1+\rho_{ic}(j-1,n)}b_n^2$$

$$\leq \quad C+\frac{1-\rho_{ic}(j-1,n)}{1+\rho_{ic}(j-1,n)}(2\ln n-\ln\ln n),$$

which implies that

$$\mathbb{P}\left(Z^{(i)}_{n,j,c}>\theta_{nj}\right)\leq C^{*}b_{nj}^{-1}n^{-\frac{1-\rho_{ic}(j-1,n)}{1+\rho_{ic}(j-1,n)}}(\ln n)^{-\frac{\rho_{ic}(j-1,n)}{1+\rho_{ic}(j-1,n)}} \tag{3.12}$$

for some C^* depending on x_i, x_c and z_0. Hence (3.11) follows from (3.12), i.e., the lemma holds. □

Proof of Theorem 2.1. In view of Lemma 3.3

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{P}\left(\bigcap_{t=1}^{d}\{M^{(t)}_{1,m}\leq u_n(x_t)\}\middle|X^{(1)}_{n,1}>u_n(x_1)\right)=\vartheta_1(\boldsymbol{x})$$

and for $i=2,\cdots,d$

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{P}\left(\bigcap_{s=1}^{i-1}\{M^{(s)}_{m}\leq u_n(x_s)\},\bigcap_{t=i}^{d}\{M^{(t)}_{1,m}\leq u_n(x_t)\}\middle|X^{(i)}_{n,1}>u_n(x_i)\right)=\vartheta_i(\boldsymbol{x}),$$

with $\vartheta_1(\boldsymbol{x})$ and $\vartheta_i(\boldsymbol{x})$ defined in (2.6) and (2.7) respectively, and by making use of Lemma 3.4

$$\lim_{n\to\infty}\mathbb{P}\left(\bigcap_{t=1}^{d}\{M^{(t)}_{1,r_n}\leq u_n(x_t)\}\middle|X^{(1)}_{n,1}>u_n(x_1)\right)=\vartheta_1(\boldsymbol{x})$$

and for $i=2,\cdots,d$

$$\lim_{n\to\infty}\mathbb{P}\left(\bigcap_{s=1}^{i-1}\{M^{(s)}_{r_n}\leq u_n(x_s)\},\bigcap_{t=i}^{d}\{M^{(t)}_{1,r_n}\leq u_n(x_t)\}\middle|X^{(i)}_{n,1}>u_n(x_i)\right)=\vartheta_i(\boldsymbol{x}).$$

Hence, by $n(1-\Phi(u_n(x)))\to e^{-x}$ as $n\to\infty$, the theorem follows if further

$$\begin{aligned}&\mathbb{P}\left(\bigcap_{i=1}^{d}\{M^{(i)}_{n}\leq u_n(x_i)\}\right)\\&-\exp\left(-n\mathbb{P}\left(X^{(1)}_{n,1}>u_n(x_1),\bigcap_{t=1}^{d}\{M^{(t)}_{1,r_n}\leq u_n(x_t)\}\right)\right.\\&\qquad\left.-n\sum_{i=2}^{d}\mathbb{P}\left(X^{(i)}_{n,1}>u_n(x_i),\bigcap_{s=1}^{i-1}\{M^{(s)}_{r_n}\leq u_n(x_s)\},\bigcap_{t=i}^{d}\{M^{(t)}_{1,r_n}\leq u_n(x_t)\}\right)\right)\\&\to 0\quad\text{as}\quad n\to\infty.\end{aligned}$$

Following the arguments in the proof of Theorem 2.1 in O'Brien (1987), we first derive an asymptotic upper bound for $p_{n,d}:=\mathbb{P}\left(\bigcap_{i=1}^{d}\{M^{(i)}_{n}\leq u_n(x_i)\}\right)$. Utilising (3.6) and Lemma 3.1 for all large n we obtain

$$\begin{aligned}p_{n,d}&=\left(\mathbb{P}\left(\bigcap_{i=1}^{d}\{M^{(i)}_{r_n}\leq u_n(x_i)\}\right)\right)^{q_n}+o(1)\\&=\left(1-\mathbb{P}\left(\bigcup_{i=1}^{d}\{M^{(i)}_{r_n}>u_n(x_i)\}\right)\right)^{q_n}+o(1)\\&\leq\left(1-r_n\mathbb{P}\left(X^{(1)}_{n,1}>u_n(x_1),\bigcap_{t=1}^{d}\{M^{(t)}_{1,r_n}\leq u_n(x_t)\}\right)\right.\\&\qquad\left.-\sum_{i=2}^{d}r_n\mathbb{P}\left(X^{(i)}_{n,1}>u_n(x_i),\bigcap_{s=1}^{i-1}\{M^{(s)}_{r_n}\leq u_n(x_s)\},\bigcap_{t=i}^{d}\{M^{(t)}_{1,r_n}\leq u_n(x_t)\}\right)\right)^{q_n}+o(1)\\&\leq\exp\left(-n\mathbb{P}\left(X^{(1)}_{n,1}>u_n(x_1),\bigcap_{t=1}^{d}\{M^{(t)}_{1,r_n}\leq u_n(x_t)\}\right)\right.\end{aligned}$$

$$-n\sum_{i=2}^{d}\mathbb{P}\left(X_{n,1}^{(i)}>u_n(x_i),\bigcap_{s=1}^{i-1}\{M_{r_n}^{(s)}\le u_n(x_s)\},\bigcap_{t=i}^{d}\{M_{1,r_n}^{(t)}\le u_n(x_t)\}\right)\right)+o(1).$$

The rest of the proof is dedicated to the derivation of an asymptotic lower bound for $p_{n,d}$. Choose a sequence of positive integers $\{s_n, n\ge 1\}$ such that $r_n=o(s_n), s_n=o(n)$, and (3.6) holds with r_n replaced by s_n and q_n replaced by $t_n=[n/s_n]$. In view of the assumptions (see Remark 3.1) this is possible. Since $r_n=o(s_n)$, we have

$$\mathbb{P}\left(M_{r_n}^{(i)}>u_n(x_i)\right)=o\left(\mathbb{P}\left(M_{s_n}^{(i)}>u_n(x_i)\right)\right),\quad 1\le i\le d. \tag{3.13}$$

We proceed by induction showing that as $n\to\infty$

$$\begin{aligned}
&\mathbb{P}\left(\bigcup_{i=1}^{d}\{M_{s_n}^{(i)}>u_n(x_i)\}\right)\\
=\;&\left(\sum_{k=1}^{s_n-r_n}\mathbb{P}\left(X_{n,k}^{(1)}>u_n(x_1),\bigcap_{t=1}^{d}\{M_{k,s_n}^{(t)}\le u_n(x_t)\}\right)\right.\\
&\left.+\sum_{i=2}^{d}\sum_{k=1}^{s_n-r_n}\mathbb{P}\left(X_{n,k}^{(i)}>u_n(x_i),\bigcap_{s=1}^{i-1}\{M_{k-1,s_n}^{(s)}\le u_n(x_s)\},\bigcap_{t=i}^{d}\{M_{k,s_n}^{(t)}\le u_n(x_t)\}\right)\right)(1+o(1)).
\end{aligned}\tag{3.14}$$

If $d=1$, as in O'Brien (1987), we have

$$\begin{aligned}
\mathbb{P}\left(M_{s_n}^{(1)}>u_n(x_1)\right)&=\mathbb{P}\left(M_{s_n-r_n}^{(1)}>u_n(x_1),M_{s_n-r_n,s_n}^{(1)}\le u_n(x_1)\right)+\mathbb{P}\left(M_{r_n}^{(1)}>u_n(x_1)\right)\\
&=\left(\sum_{k=1}^{s_n-r_n}\mathbb{P}\left(X_{n,k}^{(1)}>u_n(x_1),M_{k,s_n}^{(1)}\le u_n(x_1)\right)\right)(1+o(1))\quad\text{as}\quad n\to\infty.
\end{aligned}$$

For $d=2$, by (3.2), (3.3) and stationarity we have

$$\begin{aligned}
&\mathbb{P}\left(\{M_{s_n}^{(1)}>u_n(x_1)\}\cup\{M_{s_n}^{(2)}>u_n(x_2)\}\right)\\
=\;&\mathbb{P}\left(M_{s_n}^{(1)}>u_n(x_1)\right)+\mathbb{P}\left(M_{s_n}^{(2)}>u_n(x_2),M_{s_n}^{(1)}\le u_n(x_1)\right)\\
=\;&\mathbb{P}\left(M_{s_n}^{(1)}>u_n(x_1)\right)+\mathbb{P}\left(M_{s_n}^{(2)}>u_n(x_2),M_{s_n}^{(1)}\le u_n(x_1)\right)\\
&-\mathbb{P}\left(M_{s_n-r_n,s_n}^{(2)}>u_n(x_2),M_{s_n-r_n,s_n}^{(1)}\le u_n(x_1)\right)+\mathbb{P}\left(M_{r_n}^{(2)}>u_n(x_2),M_{r_n}^{(1)}\le u_n(x_1)\right)\\
=\;&\left(\sum_{k=1}^{s_n-r_n}\mathbb{P}\left(X_{n,k}^{(1)}>u_n(x_1),M_{k,s_n}^{(1)}\le u_n(x_1)\right)+\sum_{k=1}^{s_n}\mathbb{P}\left(X_{n,k}^{(2)}>u_n(x_2),M_{k,s_n}^{(2)}\le u_n(x_2),M_{k-1,s_n}^{(1)}\le u_n(x_1)\right)\right.\\
&-\sum_{k=1}^{s_n-1}\mathbb{P}\left(M_{k,s_n}^{(2)}>u_n(x_2),X_{n,k}^{(1)}>u_n(x_1),M_{k,s_n}^{(1)}\le u_n(x_1)\right)\\
&-\sum_{k=s_n-r_n+1}^{s_n}\mathbb{P}\left(X_{n,k}^{(2)}>u_n(x_2),M_{k,s_n}^{(2)}\le u_n(x_2),M_{k-1,s_n}^{(1)}\le u_n(x_1)\right)\\
&\left.+\sum_{k=s_n-r_n+1}^{s_n-1}\mathbb{P}\left(M_{k,s_n}^{(2)}>u_n(x_2),X_{n,k}^{(1)}>u_n(x_1),M_{k,s_n}^{(1)}\le u_n(x_1)\right)\right)(1+o(1))\\
=\;&\left(\sum_{k=1}^{s_n-r_n}\mathbb{P}\left(X_{n,k}^{(1)}>u_n(x_1),M_{k,s_n}^{(1)}\le u_n(x_1),M_{k,s_n}^{(2)}\le u_n(x_2)\right)\right.\\
&\left.+\sum_{k=1}^{s_n-r_n}\mathbb{P}\left(X_{n,k}^{(2)}>u_n(x_2),M_{k,s_n}^{(2)}\le u_n(x_2),M_{k-1,s_n}^{(1)}\le u_n(x_1)\right)\right)(1+o(1)),
\end{aligned}$$

i.e., (3.14) holds for $d=2$. Assume next that (3.14) holds for $d=m-1>2$. By (3.13)

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{i=1}^{m}\{M_{s_n}^{(i)}>u_n(x_i)\}\right) &= \mathbb{P}\left(\bigcup_{i=1}^{m-1}\{M_{s_n}^{(i)}>u_n(x_i)\}\right)+\mathbb{P}\left(\bigcap_{i=1}^{m-1}\{M_{s_n}^{(i)}\le u_n(x_i)\},M_{s_n}^{(m)}>u_n(x_m)\right)\\
&= \left(\mathbb{P}\left(\bigcup_{i=1}^{m-1}\{M_{s_n}^{(i)}>u_n(x_i)\}\right)+\mathbb{P}\left(\bigcap_{i=1}^{m-1}\{M_{s_n}^{(i)}\le u_n(x_i)\},M_{s_n}^{(m)}>u_n(x_m)\right)\right.\\
&\quad \left.-\mathbb{P}\left(\bigcap_{i=1}^{m-1}\{M_{s_n-r_n,s_n}^{(i)}\le u_n(x_i)\},M_{s_n-r_n,s_n}^{(m)}>u_n(x_m)\right)\right)(1+o(1)).
\end{aligned}$$

Consequently (3.5) implies that (3.14) holds for $d=m$. According to (3.14), by stationarity we have

$$\begin{aligned}
&\mathbb{P}\left(\bigcup_{i=1}^{d}\{M_{s_n}^{(i)}>u_n(x_i)\}\right)\\
\le\ & \left(\sum_{k=1}^{s_n-r_n}\mathbb{P}\left(X_{n,k}^{(1)}>u_n(x_1),\bigcap_{t=1}^{d}\{M_{k,r_n+k-1}^{(t)}\le u_n(x_t)\}\right)\right.\\
&\left.+\sum_{i=2}^{d}\sum_{k=1}^{s_n-r_n}\mathbb{P}\left(X_{n,k}^{(i)}>u_n(x_i),\bigcap_{s=1}^{i-1}\{M_{k-1,r_n+k-1}^{(s)}\le u_n(x_s)\},\bigcap_{t=i}^{d}\{M_{k,r_n+k-1}^{(t)}\le u_n(x_t)\}\right)\right)(1+o(1))\\
\le\ & \left(s_n\mathbb{P}\left(X_{n,1}^{(1)}>u_n(x_1),\bigcap_{t=1}^{d}\{M_{1,r_n}^{(t)}\le u_n(x_t)\}\right)\right.\\
&\left.+\sum_{i=2}^{d}s_n\mathbb{P}\left(X_{n,1}^{(i)}>u_n(x_i),\bigcap_{s=1}^{i-1}\{M_{r_n}^{(s)}\le u_n(x_s)\},\bigcap_{t=i}^{d}\{M_{1,r_n}^{(t)}\le u_n(x_t)\}\right)\right)(1+o(1)).
\end{aligned}$$

Since by our choice of the sequence $\{s_n,n\ge1\}$

$$p_{n,d}=\left(\mathbb{P}\left(\bigcap_{i=1}^{d}\{M_{s_n}^{(i)}\le u_n(x_i)\}\right)\right)^{t_n}+o(1)\quad\text{as}\quad n\to\infty$$

we have

$$\begin{aligned}
&\mathbb{P}\left(\bigcap_{i=1}^{d}\{M_{n}^{(i)}\le u_n(x_i)\}\right)\\
\ge\ & \exp\left(-n\mathbb{P}\left(X_{n,1}^{(1)}>u_n(x_1),\bigcap_{t=1}^{d}\{M_{1,r_n}^{(t)}\le u_n(x_t)\}\right)\right.\\
&\left.-n\sum_{i=2}^{d}\mathbb{P}\left(X_{n,1}^{(i)}>u_n(x_i),\bigcap_{s=1}^{i-1}\{M_{r_n}^{(s)}\le u_n(x_s)\},\bigcap_{t=i}^{d}\{M_{1,r_n}^{(t)}\le u_n(x_t)\}\right)\right)+o(1).
\end{aligned}$$

Hence the theorem holds. □

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Enkelejd Hashorva, Department of Actuarial Science, University of Lausanne,, UNIL-Dorigny, 1015 Lausanne, Switzerland

E-mail address: `Enkelejd.Hashorva@unil.ch`

Liang Peng, Department of Risk Management and Insurance, Georgia State University, Atlanta, GA 30303,

E-mail address: `lpeng@gsu.edu`

Zhichao Weng, Department of Actuarial Science, University of Lausanne,, UNIL-Dorigny, 1015 Lausanne, Switzerland

E-mail address: `wengzhichao166@126.com`