EXTREMES OF VECTOR-VALUED GAUSSIAN PROCESSES

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Abstract. The seminal papers of Pickands [1, 2] paved the way for a systematic study of high exceedance probabilities of both stationary and non-stationary Gaussian processes. Yet, in the vector-valued setting, due to the lack of key tools including Slepian's Lemma, there has not been any methodological development in the literature for the study of extremes of vector-valued Gaussian processes. In this contribution we develop the uniform double-sum method for the vector-valued setting, obtaining the exact asymptotics of the high exceedance probabilities for both stationary and n on-stationary Gaussian processes. We apply our findings to the operator fractional Brownian motion and Ornstein-Uhlenbeck process.

Key Words: High exceedance probability, Vector-valued Gaussian process, Operator fractional Ornstein-Uhlenbeck processes, Operator fractional Brownian motion, Uniform double-sum method, Vector-valued Borell-TIS inequality, Vector-valued Piterbarg inequality.

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1. Introduction

The asymptotic analysis of probabilities of rare events has been the topic of numerous past contributions and is still an active area of research. In this article the rare events of interests are the high exceedances of vector-valued Gaussian processes, i.e., we shall investigate the exact approximation of

$$p_{\boldsymbol{b}}(T,u)=\mathbb{P}\left\{\exists t\in[0,T]:X_j(t)>ub_j, j\le d\right\}$$

as $u\to\infty$ with $\boldsymbol{X}(t)=(X_1(t),\dots,X_d(t))^\top$, $t\in[0,T]$ a given centered $\mathbb{R}^d$-valued Gaussian process with a.s. continuous sample paths and given constants b_i's. In order to avoid trivialities, hereafter we shall assume that at least one of the b_i's is positive.

The approximation of $p_{\boldsymbol{b}}(T,u)$ is of interest in various applications including statistics, ruin theory, queueing theory, see e.g., [3–7]. Large deviation type results related to the vector-valued setting of this contribution are obtained in [8, 9], see [10–12] for various interesting findings for non-Gaussian $\boldsymbol{X}$.
Even the seemingly trivial case that $\boldsymbol{X}$ has independent components is quite challenging, see the recent contributions by Azais and Pham [3, 13]. We refer also to important work by Piterbarg and Zhdanov [14], where the exact asymptotics of the tail distribution of extremes of $g(\boldsymbol{X}(t))$ for the class of centered stationary $\mathbb{R}^d$-valued Gaussian processes and family of smooth homogeneous real-valued functions g was considered. We note that in our setting function $\tilde{g}$ defined as $\tilde{g}(\boldsymbol{X}(t)):=\min_{j\le d}X_j(t)/b_j$ for $b_j>0, j=1,...,d$ plays an analogous role to g in [14]. Unfortunately, the methodology developed in [14] is not applicable for $\tilde{g}$. We note tha the smooth case has been also discussed in [15]. The available results in the literature that concern $p_{\boldsymbol{b}}(T,u)$ for Gaussian processes with dependent components cover only linear transformations of an $\mathbb{R}^d$-valued Brownian motion, see [16, 17]. The independence of increments and the self-similarity property of Brownian motion are essential properties used in the aforementioned contributions, which also determine limits of applicability of methods used in [16, 17].

In the one-dimensional case three different methods are often utilised when dealing with the asymptotics of extremes of Gaussian processes: Pickands method which is based on the discretisation of supremum and the negligibility of double-sum term (see also [18] for further refinements), Piterbarg's approach which makes particular use of continuous mapping theorem (see [19]), and Berman's method that capitalises on the relation between supremum and sojourn times (see [20, 21]). All the above techniques are heavily based on the following fundamental results

i) Slepian lemma, see e.g., [22, Thm 2.2.1];
ii) Borell-TIS and Piterbarg inequalities, see e.g., [22, Thm 2.1.1] and [23, Thm 8.1];
iii) uniform version of the classical Pickands-Piterbarg lemma, see [24, Lem 2.1].

Roughly speaking, in the one-dimensional setting, Slepian lemma, Borell-TIS and Piterbarg inequalities are essential for the non-stationary case. The first one is utilised to approximate rare events by switching to stationary Gaussian processes, whereas the both inequalities show that only a small neighbourhood around the point of the maximum of the variance (assumed to be unique) is responsible for the rare-event approximation; see, e.g., the seminal monograph by Piterbarg [25].

One of the reasons for the lack of methodological approach for studying extremes of vector-valued Gaussian process is that the three key tools mentioned above are not available in the general vector-valued setting. In fact, the existing extensions of Slepian lemma in the form of Gordon inequality, are not generally applicable in higher dimensions (apart from very special cases like processes with independent components, see e.g., [26, Lem 5.1]), whereas an extension of Borell-TIS and Piterbarg inequalities requires a deep understanding of the problem at hand, which has been addressed in this paper.
In this contribution exact asymptotics of $p_{\boldsymbol{b}}(T,u)$ as $u \to \infty$ for both stationary and non-stationary $\boldsymbol{X}$ are derived by levering the uniform double-sum method to the vector-valued setting. The key to the methodology developed in this contribution is what we refer to as the uniform Pickands-Piterbarg lemma, see Lemma 4.7 below.
We briefly explain the main ideas underlying the approach taken in this paper pointing out some subtle issues related to uniform approximations that appear to have been overlooked in the literature; [27] takes particular care of those issues in the one-dimensional setting.
The main attempts of the double-sum method consist in proving that

$$p_{\boldsymbol{b}}(T,u) \sim \sum_{i=1}^{N_u} \mathbb{P}\left\{\exists t \in T_i(u):\ X_j(t) > ub_j,\ j \le d\right\} =: \Sigma(u) \tag{1.1}$$

as $u \to \infty$, where $T_k(u)$, $k \le N_u$ are disjoint compact intervals covering $[0,T]$.

Commonly, $\Sigma(u)$ is referred to as the *single-sum term*. Each term of the single-sum, say the jth one, is approximated by some function $\theta_j(u)$ as $u \to \infty$. However, for non-stationary processes, such approximation does not imply $\Sigma(u) \sim \sum_{j=1}^{N_u} \theta_j(u)$ as $u \to \infty$, since typically N_u tends to infinity as $u \to \infty$. This holds true if the aforementioned approximation of $\theta_j(u)$ is uniform for all positive integers $j \le N_u$.
In the literature this fact has been not taken care of systematically; a notable exception is [28]. As a result numerous proofs in the literature have certain gaps. We take special care of this key uniformity issue by deriving a uniform version of the Pickands-Piterbarg lemma, see Lemma 4.7. Recently, for the one-dimensional setting, an alternative approach that solves previous gaps in the literature concerning uniformity issues has been suggested in [29, 30]. However, due to lack of Slepian lemma for general vector-valued Gaussian processes that approach is not applicable for the studies of this contribution.

In view of Bonferroni inequality, $\Sigma(u)$ is an upper bound for $p_{\boldsymbol{b}}(T,u)$ and a lower bound is given by $\Sigma(u)-\Sigma\Sigma(u)$ where the so-called *double-sum term* is given by

$$\Sigma\Sigma(u) = \sum_{i=1}^{N_u} \sum_{i<j\le N_u} \mathbb{P}\left\{\exists (s,t) \in T_i(u)\times T_j(u):\ X_k(s) > ub_k,\ X_l(t)) > ub_l,\ k,\ l \le d\right\}.$$

Showing the asymptotic negligibility of $\Sigma\Sigma(u)$ as $u \to \infty$ is typically a hard and technical problem, since the asymptotic bounds derived for its summands need also be uniform for all positive integers $i,\ j \le N_u$.
A subtle novelty of our approach for non-stationary $\boldsymbol{X}$ is that we do not use the common approach to standardise the process and then substitute it by a stationary process (utilising Slepian lemma). The reason is that, as previously mentioned, Slepian lemma does not hold in general for vector-valued Gaussian processes.

An application of our findings concerns the study of the behaviour of asymptotics of supremum tail distribution of operator fractional Brownian motion (fBm) discussed briefly below (see for details Section 3.2). Let H be a $d \times d$ real-valued matrix with eigenvalues $h_i \in (0,1]$, $i \le d$. A centered, sample continuous $\mathbb{R}^d$-valued Gaussian process $\boldsymbol{X}(t)$, $t \in \mathbb{R}$ is said to be an *operator fBm* with index H, if it has stationary increments and is operator self-similar in the sense that

$$\{\boldsymbol{X}(\lambda t),\ t \in \mathbb{R}\} \stackrel{d}{=} \left\{\sum_{k=0}^{\infty} (\log \lambda)^k \frac{H^k}{k!} \boldsymbol{X}(t),\ t \in \mathbb{R}\right\} \tag{1.2}$$

for any $\lambda > 0$, where $\stackrel{d}{=}$ stands for the equality of finite-dimensional distributions. Let $h_* = \min_{1\le i\le d} h_i$. However, if further $\boldsymbol{X}$ is time-reversible, i.e., $\mathbb{E}\left\{\boldsymbol{X}(t)\boldsymbol{X}(s)^\top\right\} = \mathbb{E}\left\{\boldsymbol{X}(s)\boldsymbol{X}(t)^\top\right\}$ for all t and s, in view of Proposition 3.3 in Section 3, as $u \to \infty$, for positive b_i's

$$p_{\boldsymbol{b}}(T,u) \sim C u^{\max(0,\frac{1-2h_*}{h_*})} \mathbb{P}\left\{X_j(T) > ub_j, j \le d\right\}.$$

Throughout this paper $\sim$ means the asymptotic equivalence as $u \to \infty$. If $h_* < 1/2$, then C is given in the form of Pickands-type constant and for $h_* = 1/2$ it corresponds to the so-called Piterbarg-type constant.
Other applications illustrating findings of this contribution are concerned with stationary $\boldsymbol{X}$ being the Lamperti transform of some operator fBm or $\boldsymbol{X}$ being an operator fractional Ornstein-Uhlenbeck (fO-U) process.
We note that in the special case when the coordinates of $\boldsymbol{X}$ are mutually independent, our main results given in Theorems 2.1 and 2.4 in Section 2 recover findings of [26].

Brief organisation of the paper. Main results of this paper are presented in Section 2 with proofs relegated to Section 5. We dedicate Section 3 some important examples and then present in Section 4 several auxiliary results; their proofs are relegated to Appendix. We conclude this section by introducing some standard notation.

Notation. All vectors in $\mathbb{R}^d$ are written in bold letters, for instance $\boldsymbol{b} = (b_1,\dots,b_d)^\top$, $\boldsymbol{0} = (0,\dots,0)^\top$ and $\boldsymbol{1} = (1,\dots,1)^\top$. For two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we write $\boldsymbol{x} > \boldsymbol{y}$ if $x_i > y_i$ for all $1 \le i \le d$. Given a real-valued matrix $A = (a_{ij})$ we shall write A_{IJ} for the submatrix of A determined by keeping the rows and columns of A with row indices in the non-emtpy set I and column indices in the non-empty set J, respectively. If A is a $d\times d$ matrix, then $\|A\|_{\mathrm{F}} = \sqrt{\sum_{1\le i,j\le d} a_{ij}^2}$ denotes its Frobenius norm. In our notation $\mathcal{I}_d$ is the $d\times d$ identity matrix and $\mathrm{diag}(\boldsymbol{x}) = \mathrm{diag}(x_1,\dots,x_d)$ stands for the diagonal matrix with entries x_i, $i = 1,\dots,d$ on the main diagonal, respectively.
Let in the sequel $\Sigma \in \mathbb{R}^{d\times d}$ be a positive definite matrix with inverse Σ^{-1}. If $\boldsymbol{b} \in \mathbb{R}^d \setminus (-\infty,0]^d$, then the quadratic programming problem $\Pi_\Sigma(\boldsymbol{b})$

$$\Pi_\Sigma(\boldsymbol{b}) = \text{minimise } \boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x} \text{ under the linear constraint } \boldsymbol{x} \ge \boldsymbol{b} \tag{1.3}$$

has a unique solution $\widetilde{\boldsymbol{b}} \ge \boldsymbol{b}$ and there exists a unique non-empty index set $I \subset \{1,\dots,d\}$ such that

$$\widetilde{\boldsymbol{b}}_I = \boldsymbol{b}_I, \quad \widetilde{\boldsymbol{b}}_J = \Sigma_{IJ}(\Sigma_{II})^{-1}\boldsymbol{b}_I \ge \boldsymbol{b}_J, \quad \boldsymbol{w}_I = (\Sigma_{II})^{-1}\boldsymbol{b}_I > \boldsymbol{0}_I, \quad \boldsymbol{w}_J = \boldsymbol{0}_J \tag{1.4}$$

and $\boldsymbol{w} = \Sigma^{-1}\widetilde{\boldsymbol{b}}$, where coordinates $J = \{1,\dots,d\}\setminus I$ (which can be empty) are responsible for dimension-reduction phenomena, while coordinates belonging to I play essential role in the exact asymptotics. We refer to Lemma 4.1 below for more details.

2. Main Results

As mentioned in the Introduction, we are interested in the exact asymptotics of

$$p_{\boldsymbol{b}}(T,u) = \mathbb{P}\left\{\exists t \in [0,T] : \ \boldsymbol{X}(t) > u\boldsymbol{b}\right\}, \quad u \to \infty \tag{2.1}$$

for a centered $\mathbb{R}^d$-valued Gaussian process $\boldsymbol{X}(t)$, $t \in [0,T]$ and any $\boldsymbol{b} \in \mathbb{R}^d$ with at least one positive component.
We shall state our main results for $\boldsymbol{X}$ stationary and non-stationary separately.
Hereafter for Gaussian processes defined on some compact parameter set $E \subset \mathbb{R}^k$ we shall assume its a.s. sample continuity. Let $\boldsymbol{X}(t)$, $t \in E$ be a centered d-dimensional vector-valued Gaussian process i.e., $\boldsymbol{X}(t) = (X_1(t),\dots,X_d(t))^\top$, $t \in E$ is a column vector Gaussian process. Let for any t, $s \in E$

$$R(t,s) = \mathbb{E}\left\{\boldsymbol{X}(t)\boldsymbol{X}(s)^\top\right\}$$

be the covariance matrix function (cmf) of $\boldsymbol{X}$, which is a matrix-valued non-negative definite function in the sense that

$$\sum_{i,j=1}^n \boldsymbol{v}_i^\top R(t_i,t_j)\boldsymbol{v}_j \ge 0$$

for any $t_i \in E, \boldsymbol{v}_i \in \mathbb{R}^d, i \le n$. Conversely, if $R\colon E\times E \mapsto \mathbb{R}^{d\times d}$ is a matrix-valued non-negative definite function such that $R(t,s) = R(s,t)^\top$ for any t, $s \in E$, then there exists an $\mathbb{R}^d$-valued Gaussian process $\boldsymbol{X}(t)$, $t \in E$ with cmf R. Note that in the definition of positive definite or non-negative definite matrices we do not require the matrices to be symmetric.

Let V be a $d\times d$ real-valued matrix and let $\alpha\in(0,2]$ be given. An interesting example of a cmf determined by V is

$$R_{\alpha,V}(t,s)=S_\alpha(t,V)+S_\alpha(-s,V)-S_\alpha(t-s,V),\quad t,\,s\in\mathbb{R}, \tag{2.2}$$

with

$$S_\alpha(t,V)=|t|^\alpha\left(V\mathbf{1}_{\{t\ge 0\}}+V^\top\mathbf{1}_{\{t<0\}}\right)=|t|^\alpha\left(V^+ + V^-\,\mathrm{sgn}(t)\right)$$

and

$$V^+=\frac{1}{2}\left(V+V^\top\right),\quad V^-=\frac{1}{2}\left(V-V^\top\right).$$

Note that we use the standard notation $\mathrm{sgn}(t)=1$ for $t\ge 0$ and $\mathrm{sgn}(t)=-1$ for $t<0$.
By [31, Prop 9], the matrix-valued function $R_{\alpha,V}$ defined by (2.2) is a non-negative definite function if and only if the Hermitian matrix

$$V^\star=\sin\left(\frac{\pi\alpha}{2}\right)V^+-\sqrt{-1}\cos\left(\frac{\pi\alpha}{2}\right)V^- \tag{2.3}$$

is non-negative definite. Furthermore, under the above conditions, one can define a multivariate fBm $\boldsymbol{Y}(t),t\in\mathbb{R}$ with $R_{\alpha,V}$ as its cmf. The classical Pickands constants (see [1]) are defined in terms of a standard fBM. A multidimensional analog of Pickands constants can be defined utilising $\boldsymbol{Y}$. Specifically, for compact $E\subset\mathbb{R}$ set

$$\mathcal{H}_{\alpha,V}(E)=\int_{\mathbb{R}^d}e^{\mathbf{1}^\top\boldsymbol{x}}\mathbb{P}\left\{\exists t\in E:\ \boldsymbol{Y}(t)-S_\alpha(t,V)\mathbf{1}>\boldsymbol{x}\right\}d\boldsymbol{x}. \tag{2.4}$$

As shown in the proof of Theorem 2.1 the function $t\mapsto\mathcal{H}_{\alpha,V}([0,t])$, $t>0$ is sub-additive, which implies that

$$\mathcal{H}_{\alpha,V}=\lim_{T\to\infty}\frac{\mathcal{H}_{\alpha,V}([0,T])}{T}=\inf_{T>0}\frac{\mathcal{H}_{\alpha,V}([0,T])}{T}\in[0,\,\infty). \tag{2.5}$$

We shall call $\mathcal{H}_{\alpha,V}$ the multidimensional Pickands constant.

2.1. **Stationary case.**

Let $\boldsymbol{X}(t)$, $t\in\mathbb{R}$ be a centered, $\mathbb{R}^d$-valued stationary Gaussian process with cmf $R(t,s)$. The stationarity of $\boldsymbol{X}$ means that $R(s+t,s)=R(t,0)=:\mathcal{R}(t)$ for any $s,\,t$. Letting $\Sigma=\mathcal{R}(0)$ we have that for each fixed t, the matrix $\Sigma-\mathcal{R}(t)$ is non-negative definite but not necessarily symmetric, which is reflected in the formula

$$\mathbb{E}\left\{\left[\boldsymbol{X}(t)-\boldsymbol{X}(0)\right]\left[\boldsymbol{X}(t)-\boldsymbol{X}(0)\right]^\top\right\}\quad=\quad\Sigma-\mathcal{R}(t)+\Sigma-\mathcal{R}(t)^\top.$$

Hereafter I stands for the unique non-empty subset of $\{1,\ldots,d\}$ that determines the solution $\widetilde{\boldsymbol{b}}$ of the quadratic programming problem $\Pi_\Sigma(\boldsymbol{b})$ (defined in the Introduction) and $\boldsymbol{w}=\Sigma^{-1}\widetilde{\boldsymbol{b}}$ has non-negative components.

In this section we shall impose the following assumptions:
(B1) $\Sigma_{II}-\mathcal{R}_{II}(t)$ is positive definite for every t;
(B2) There exists a $d\times d$ real matrix V such that $\boldsymbol{w}^\top V\boldsymbol{w}>0$ with $\boldsymbol{w}=\Sigma^{-1}\widetilde{\boldsymbol{b}}$ and further

$$\Sigma-\mathcal{R}(t)\sim t^\alpha V\quad\text{as } t\downarrow 0 \tag{2.6}$$

holds for some $\alpha\in(0,2]$.

Here for two matrix-valued functions $A(t)=(a_{ij}(t))$ and $C(t)=(c_{ij}(t))$, we write $A(t)\sim C(t)$ if $a_{ij}(t)\sim c_{ij}(t)$ for all $(i,\,j)$ as $t\to 0$. Note in passing that since $\mathcal{R}(-t)=\mathcal{R}(t)^\top$, then by (2.6) we have that

$$\Sigma-\mathcal{R}(t)\sim|t|^\alpha V^\top,\quad t\uparrow 0.$$

Moreover, sufficient and necessary condition for V to satisfy (2.6) by a stationary Gaussian process $\boldsymbol{X}(t)$, $t\in\mathbb{R}$, is that $V^\star$ given by (2.3) is non-negative definite.

Theorem 2.1. *If both (B1) and (B2) hold, then as $u\to\infty$*

$$\lim_{u\to\infty}\frac{p_{\boldsymbol{b}}(T,u)}{Tu^{2/\alpha}\mathbb{P}\left\{\boldsymbol{X}(0)>u\boldsymbol{b}\right\}}=\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}, \tag{2.7}$$

where $V_{\boldsymbol{w}}=\mathrm{diag}(\boldsymbol{w})V\mathrm{diag}(\boldsymbol{w})$ and $\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}\in(0,\infty)$. Moreover, (2.7) holds with T replaced by T_u, provided that $\lim_{u\to\infty}T_u u^{2/\alpha}\mathbb{P}\left\{\boldsymbol{X}(0)>u\boldsymbol{b}\right\}=0$.

If $\boldsymbol{w}^\top V \boldsymbol{w} = 0$ in Assumption (B2), the Pickands constant $\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}$ is not necessarily positive. For example, when $V_{II} = 0$, we have that $V_{\boldsymbol{w}}$ is the zero matrix. In this case, $\mathcal{H}_{\alpha,V_{\boldsymbol{w}}} = 0$ and (2.7) does not provide correct order of the asymptotics for $p_{\boldsymbol{b}}(T,u)$. It is not a simple task to give a complete characterization of the asymptotics for $p_{\boldsymbol{b}}(T,u)$, however in the following special case, we can obtain the approximation of $p_{\boldsymbol{b}}(T,u)$ and give further an explicit formula for the corresponding Pickands constant.

Theorem 2.2. *Suppose that $\alpha = 1$ and Assumption (B1) is satisfied. If there exists a $d \times d$ anti-symmetric matrix V such that*

$$\Sigma - \mathcal{R}(t) \sim tV, \quad t \to 0 \tag{2.8}$$

and $(V\boldsymbol{w})_I \neq \boldsymbol{0}_I$, then the statements of Theorem 2.1 hold with $\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}$ replaced by $\frac{1}{2}\sum_{1\le i\le d} w_i |(V\boldsymbol{w})_i| > 0$.

2.2. **Non-stationary case.** We discuss next the case of non-stationary $\boldsymbol{X}$. Let for $t_0,\, t \in [0,T]$

$$\Sigma(t) = R(t,t), \quad \Sigma = \Sigma(t_0)$$

and assume that Σ is non-singular. As in the stationary case, for $\boldsymbol{b} \in \mathbb{R}^d \setminus (-\infty,0]^d$ we set $\boldsymbol{w} = \Sigma^{-1}\widetilde{\boldsymbol{b}}$. Recall that $\widetilde{\boldsymbol{b}}$ is the unique solution of (1.3) and $\boldsymbol{w}_I = (\Sigma_{II})^{-1}\boldsymbol{b}_I > \boldsymbol{0}_I$, $\boldsymbol{w}_J = \boldsymbol{0}_J$ as mentioned in (1.4), where I, J are defined with respect to $\Pi_\Sigma(\boldsymbol{b})$; see Lemma 4.1. Next, for any $t \in [0,T]$ define

$$\sigma_{\boldsymbol{b}}^2(t) = \min_{\boldsymbol{z}\in[0,\infty)^d : \boldsymbol{z}^\top \boldsymbol{b} > 0} \frac{\boldsymbol{z}^\top \Sigma(t) \boldsymbol{z}}{(\boldsymbol{z}^\top \boldsymbol{b})^2} = \frac{1}{\min_{\boldsymbol{x}\ge \boldsymbol{b}} \boldsymbol{x}^\top \Sigma^{-1}(t) \boldsymbol{x}}, \tag{2.9}$$

where the second equality holds under the assumption that $\Sigma(t)$ is non-singular, see Lemma 4.1. We shall refer to $\sigma_{\boldsymbol{b}}^2(t)$, $t \in [0,T]$ as the generalized variance function of $\boldsymbol{X}$.

For the 1-dimensional case ($d = 1$) it is known from several works of V.I. Piterbarg that the local behaviour of the variance function around a unique maximizing point is crucial for the exact asymptotics of supremum tail distribution of non-stationary Gaussian processes.

In the vector-valued setting, the situation is more complex since the local structure of the generalized variance function in the neighbourhood of its maximizer is crucial. Therefore, the following set of assumptions relates to both the covariance function and the generalized variance function of $\boldsymbol{X}$. Namely, we shall assume that:

(D1) $\sigma_{\boldsymbol{b}}^2(t)$, $t \in [0,T]$ is continuous and attains its unique maximum at $t_0 \in [0,T]$;

(D2) For all t in $[0,T]$, there exists a continuous $d \times d$ real matrix function $A(t)$, $t \in [0,T]$ such that

$$\Sigma(t) = A(t)A(t)^\top, \quad t \in [0,T] \tag{2.10}$$

and there exist a $d \times d$ real matrix Ξ and some $\beta > 0$ such that as $t \to t_0$

$$A(t) = A(t_0) - |t-t_0|^\beta \Xi + o(|t-t_0|^\beta), \tag{2.11}$$

with

$$\tau_{\boldsymbol{w}} := \boldsymbol{w}^\top \Xi A(t_0)^\top \boldsymbol{w} > 0; \tag{2.12}$$

(D3) There exist $\alpha \in (0,2]$ and a $d \times d$ real matrix D such that for $t > s$

$$R(t,s) = A(t)\left(\mathcal{I}_d - (t-s)^\alpha D + o(|t-s|^\alpha)\right) A(s)^\top \tag{2.13}$$

as $t \to t_0,\ s \to t_0$;

(D4) There exist $\gamma \in (0,2]$, $C \in (0,\infty)$ such that for all s, t in an open neighbourhood of t_0

$$\mathbb{E}\left\{ |\boldsymbol{X}(t) - \boldsymbol{X}(s)|^2 \right\} \le C\,|t-s|^\gamma. \tag{2.14}$$

Remark 2.3. *i) As shown in Appendix, (2.11) implies that*

$$\sigma_{\boldsymbol{b}}^2(t_0) - \sigma_{\boldsymbol{b}}^2(t) \sim \frac{2\tau_{\boldsymbol{w}}}{(\widetilde{\boldsymbol{b}}^\top \Sigma^{-1} \widetilde{\boldsymbol{b}})^2} |t-t_0|^\beta \quad \text{as } t \to t_0. \tag{2.15}$$

Thus if $\tau_{\boldsymbol{w}} > 0$, then $\sigma_{\boldsymbol{b}}^2(t)$ has a local maximum point at $t = t_0$. Conversely, if $\sigma_{\boldsymbol{b}}^2(t)$ attains its maximum at $t = t_0$, then we have $\tau_{\boldsymbol{w}} \ge 0$.

ii) By (D2), Assumption (D4) follows if for some $\gamma > 0$

$$\|A(t) - A(s)\|_{\mathrm{F}} \le C\,|t-s|^\gamma$$

for all s,t in an open neighbourhood of t_0.

iii) *Assumption (D4) is used to control the behavior of the process $\boldsymbol{X}(t)$ for $u^{-\beta/2}\log^{2/\beta} u \le |t-t_0| \le \theta$, where θ is a sufficiently small number (see application of Lemma 4.5 to Equation (5.43) in the proof of Theorem 2.4). In the case $d=1$, (D4) is not needed, since we can use Slepian inequality and then consider the standardised process, see [32]. For $d>1$, Slepian-type inequalities do not hold. Instead we suppose that (D4) holds.*

For a multivariate fBm $\boldsymbol{Y}(t)$, $t \in \mathbb{R}$ with cmf $R_{\alpha,V}$ given by (2.2) and a matrix W, we introduce the multivariate Piterbarg constant

$$\mathcal{P}_{\alpha,V,W} = \lim_{\Lambda\to\infty} \int_{\mathbb{R}^d} e^{\mathbf{1}^\top \boldsymbol{x}} \mathbb{P}\left\{\exists t \in [0,\Lambda] : \boldsymbol{Y}_t - \left[S_\alpha(t,V) + |t|^\alpha W\right]\mathbf{1} > \boldsymbol{x}\right\} d\boldsymbol{x}, \tag{2.16}$$

provided the limit exists. The following theorem constitutes the main result of this section. For compactness of the presentation we suppose that $t_0 = 0$ in (D1)-(D3); the other cases are commented in Remark 2.5.

Theorem 2.4. *Let $\boldsymbol{X}(t)$, $t \in [0,T]$ be a centered $\mathbb{R}^d$-valued Gaussian process satisfying (D1)–(D4) with $t_0 = 0$ and set $A = A(t_0)$, $V = ADA^\top$, $V_{\boldsymbol{w}} = \mathrm{diag}(\boldsymbol{w})V\mathrm{diag}(\boldsymbol{w})$ and $W_{\boldsymbol{w}} = \mathrm{diag}(\boldsymbol{w})\Xi A^\top \mathrm{diag}(\boldsymbol{w})$.*

(1) If $\beta > \alpha$ and $\boldsymbol{w}^\top V \boldsymbol{w} > 0$, then as $u \to \infty$

$$p_{\boldsymbol{b}}(T,u) \sim \mathcal{H}_{\alpha,V_{\boldsymbol{w}}} \Gamma(1/\beta+1)\tau_{\boldsymbol{w}}^{-\beta} u^{\frac{2}{\alpha}-\frac{2}{\beta}} \mathbb{P}\{\boldsymbol{X}(t_0) > u\boldsymbol{b}\}, \tag{2.17}$$

where $\mathcal{H}_{\alpha,V_{\boldsymbol{w}}} \in (0,\infty)$.

(2) If $\beta = \alpha$, then as $u \to \infty$

$$p_{\boldsymbol{b}}(T,u) \sim \mathcal{P}_{\alpha,V_{\boldsymbol{w}},W_{\boldsymbol{w}}} \mathbb{P}\{\boldsymbol{X}(t_0) > u\boldsymbol{b}\}, \tag{2.18}$$

where $\mathcal{P}_{\alpha,V_{\boldsymbol{w}},W_{\boldsymbol{w}}} \in (0,\infty)$.

(3) If $\beta < \alpha$, then as $u \to \infty$

$$p_{\boldsymbol{b}}(T,u) \sim C_{\boldsymbol{w}} \mathbb{P}\{\boldsymbol{X}(t_0) > u\boldsymbol{b}\}, \tag{2.19}$$

where $C_{\boldsymbol{w}} = 1 + \tau_{\boldsymbol{w}}^{-1}\sum_{i\in I} w_i \max(0, -(\Xi A^\top \boldsymbol{w})_i)$.

Remark 2.5. *i) The constant $C_{\boldsymbol{w}}$ in (2.19) equals 1 if and only if $(\Xi A^\top \boldsymbol{w})_I \ge \mathbf{0}_I$. Moreover, (2.19) holds with the same constant if $t_0 \in (0,T]$.*

ii) If $t_0 \in (0,T)$ or $t_0 = T$, then Pickands constant $\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}$ in (2.17) has to be replaced by $\mathcal{H}_{\alpha,V_{\boldsymbol{w}}} + \mathcal{H}_{\alpha,V_{\boldsymbol{w}}^\top}$ or $\mathcal{H}_{\alpha,V_{\boldsymbol{w}}^\top}$, respectively. Analogously, Piterbarg constant is defined by (2.16) with $[0,\Lambda]$ replaced by $[-\Lambda,\Lambda]$ or $[-\Lambda,0]$ if $t_0 \in (0,T)$ or $t_0 = T$, respectively.

3. Examples

In this section we apply the findings of Section 2 to three important classes of vector-valued Gaussian processes, namely operator fO-U processes, operator fBm's and their Lamperti transforms.

3.1. **Operator fO-U process.** Let H be a symmetric matrix with all eigenvalues $h_1, \ldots, h_d$ belonging to $(0,1]$ and consider a centered stationary a.s. continuous $\mathbb{R}^d$-valued Gaussian processes $\boldsymbol{X}(t)$, $t \ge 0$ with cmf

$$R(t,s) = e^{-|t-s|^{2H}},$$

where $t^H = \exp(H \log t)$ for $t > 0$. We call $\boldsymbol{X}$ *an operator fO-U process.*

The existence of an fO-U process follows from the fact that, by the symmetry of H, we can write $H = Q\mathrm{diag}(h_1,\ldots,h_d)Q^\top$ for some orthogonal matrix Q. Hence $\boldsymbol{X}(t) \overset{d}{=} Q\boldsymbol{Z}(t)$, $t \ge 0$, where $Z_i(t)$, $t \ge 0$, $i = 1, \ldots, d$, are mutually independent stationary Gaussian processes with covariance function $r_i(t) = e^{-|t|^{2h_i}}$, respectively. Consequently, setting $\mathcal{R}(t) = R(t,0)$ we have

$$\mathcal{R}(t) = \mathcal{I}_d - |t|^{2h_\star} Q\widetilde{\mathcal{I}}Q^\top + o(|t|^{2h_\star})$$

as $t \to 0$, where $h_* = \min_{1\le i\le d} h_i$ and

$$\widetilde{\mathcal{I}} = \mathrm{diag}(e_1, \ldots, e_d), \text{ with } e_i = \begin{cases} 0 & \text{if } h_i > h_\star \\ 1 & \text{if } h_i = h_\star. \end{cases} \tag{3.1}$$

Then (B2) holds with $\alpha = 2h_*$, $\Sigma = \mathcal{I}_d$ and $V = Q\widetilde{\mathcal{I}}Q^\top$. Let $\widetilde{\boldsymbol{b}}$ be the solution to the quadratic programming problem (1.3), that is, $\widetilde{\boldsymbol{b}}_i = b_i \vee 0$ for $1 \le i \le d$.
In view of Theorem 2.1 we arrive at the following result.

Proposition 3.1. *Let $\boldsymbol{X}(t)$, $t \in [0,T]$ be an $\mathbb{R}^d$-valued operator fO-U process with a symmetric matrix H with all eigenvalues belonging to $(0,1]$. If $\widetilde{\boldsymbol{b}}^\top Q\widetilde{\mathcal{I}}Q^\top \widetilde{\boldsymbol{b}} > 0$, then as $u \to \infty$*

$$p_{\boldsymbol{b}}(T,u) \sim T\mathcal{H}_{2h_*, V_{\widetilde{\boldsymbol{b}}}} u^{1/h_*} \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\},$$

where $V_{\widetilde{\boldsymbol{b}}} = \mathrm{diag}(\widetilde{\boldsymbol{b}}) Q\widetilde{\mathcal{I}}Q^\top \mathrm{diag}(\widetilde{\boldsymbol{b}})$.

3.2. **Operator fBm.** Let H be a $d\times d$ matrix and let $\boldsymbol{X}(t)$, $t \in \mathbb{R}$ be a centered, a.s. continuous $\mathbb{R}^d$-valued *operator fBm* with index H. We shall assume that the following conditions hold:

(O1) There exists an invertible matrix Q such that $H = QUQ^{-1}$ with $U = \mathrm{diag}(h_1, \ldots, h_d)$ and $h_1, \ldots, h_d \in (0,1]$;

(O2) $\Sigma = \mathbb{E}\left\{\boldsymbol{X}(1)\boldsymbol{X}(1)^\top\right\}$ is non-singular and $\boldsymbol{X}$ is time-reversible, i.e., $\mathbb{E}\left\{\boldsymbol{X}(t)\boldsymbol{X}(s)^\top\right\} = \mathbb{E}\left\{\boldsymbol{X}(s)\boldsymbol{X}(t)^\top\right\}$ for all t and s.

Since $\boldsymbol{X}$ is time-reversible, we have that the cmf of $\boldsymbol{X}$ is given by

$$R(t,\, s) = \frac{1}{2}\left(|t|^H \Sigma |t|^{H^\top} + |s|^H \Sigma |s|^{H^\top} - |t-s|^H \Sigma |t-s|^{H^\top}\right);$$

see for example [33].

For notational simplicity we shall suppose that $T = 1$. Write $\Sigma = A^2$ for some $d \times d$ symmetric real-valued matrix A. Then $\Sigma(t) = \mathbb{E}\left\{\boldsymbol{X}(t)\boldsymbol{X}(t)^\top\right\} = A(t)A(t)^\top$, where

$$A(t) = t^H A = A(s) - s^{-1}(s-t)HA(s) + o(|s-t|). \tag{3.2}$$

Let $\sigma_{\boldsymbol{b}}^2(t)$ be the generalized variance function of $\boldsymbol{X}$ defined by (2.9). Since in general the behaviour of $\sigma_{\boldsymbol{b}}^2(t)$ may be quite complex, we focus on a tractable class, supposing that:

(O3) The function $\sigma_{\boldsymbol{b}}^2(t)$, $t \in [0,1]$ attains its unique maximum at $t = 1$.

Remark 3.2. *Assumption (O3) holds if $H\Sigma$ is positive definite in the sense that $\boldsymbol{y}^\top H\Sigma \boldsymbol{y} > 0$ for all $\boldsymbol{y} \neq \boldsymbol{0}$. Indeed, we have from (2.15) that*

$$\sigma_{\boldsymbol{b}}^2(s) - \sigma_{\boldsymbol{b}}^2(t) = \frac{2\tau_{\boldsymbol{w}}(s)}{\left(\widetilde{\boldsymbol{b}}(s)^\top \Sigma^{-1}(s)\widetilde{\boldsymbol{b}}(s)\right)^2}(s-t) + o(|s-t|),$$

where $\widetilde{\boldsymbol{b}}(t)$ is the solution to the quadratic programming problem $\min_{\boldsymbol{x}\geq \boldsymbol{b}} \boldsymbol{x}^\top \Sigma^{-1}(t)\boldsymbol{x}$, $\boldsymbol{w}(s) = \Sigma^{-1}(s)\widetilde{\boldsymbol{b}}(s)$ and

$$\tau_{\boldsymbol{w}}(s) = s^{-1}\boldsymbol{w}(s)^\top HA(s)A(s)^\top \boldsymbol{w}(s) = s^{-1}\boldsymbol{w}(s)^\top s^H H\Sigma\, s^{H^\top}\boldsymbol{w}(s).$$

Since $H\Sigma$ is positive definite, we have that $\tau_{\boldsymbol{w}}(s) > 0$ for all $s > 0$ and hence $\sigma_{\boldsymbol{b}}^2(s)$ is strictly increasing in s. Another condition to ensure Assumption (O3) is that $\boldsymbol{b} > \boldsymbol{0}$ and $H = \mathrm{diag}(h_1, \ldots, h_d)$ with $h_i \in (0,1)$, $1 \leq i \leq d$. Under this setting, the cone $t^{-H}S_{\boldsymbol{b}}$ with $S_{\boldsymbol{b}} = \{\boldsymbol{x} :\ \boldsymbol{x} \geq \boldsymbol{b}\}$, is strictly increasing in $t \in (0,1]$ and therefore

$$\min_{\boldsymbol{x}\geq \boldsymbol{b}} \boldsymbol{x}^\top \Sigma^{-1}(t)\boldsymbol{x} = \min_{\boldsymbol{y}\in t^{-H}S_{\boldsymbol{b}}} \boldsymbol{y}^\top \Sigma^{-1}\boldsymbol{y}$$

is strictly decreasing.

By (3.2) we have that (2.11) holds with $\beta = 1$ and $\Xi = HA$. Setting further $h_* = \min_{1\leq i\leq d} h_i$, we have

$$\begin{aligned} & A(t)^{-1}R(t,\, s)\left(A(s)^{-1}\right)^\top \\ = \ & \frac{1}{2}\left[A\left(\frac{t}{s}\right)^{H^\top} A^{-1} + A^{-1}\left(\frac{s}{t}\right)^H A - A^{-1}\left(\frac{|t-s|}{t}\right)^H \Sigma \left(\frac{|t-s|}{s}\right)^{H^\top} A^{-1}\right] \\ = \ & \mathcal{I}_d - (t-s)D_1 + O(|t-s|^2) - |t-s|^{2h_*}D_2 + o(|t-s|^{2h_*}) \end{aligned}$$

as $t \uparrow 1$, $s \uparrow 1$ and $|t-s| \to 0$, where

$$D_1 = \frac{1}{2}\left(AH^\top A^{-1} - A^{-1}HA\right), \quad D_2 = \frac{1}{2}A^{-1}Q\widetilde{\mathcal{I}}Q^{-1}\Sigma(Q\widetilde{\mathcal{I}}Q^{-1})^\top A^{-1},$$

with $\widetilde{\mathcal{I}}$ given by (3.1). If $D_1 = 0$, or equivalently $\Sigma H^\top = H\Sigma$, then Assumption (D3) holds with $\alpha = 2h_*$ and $D = D_2$. If $H^\top\Sigma \neq \Sigma H$, then Assumption (D3) also holds for $h_* < 1/2$ with $\alpha = 2h_*$ and $D = D_2$, for $h_* = 1/2$ with $\alpha = 1$ and $D = D_1 + D_2$, whereas for $h_* > 1/2$ with $\alpha = 1$ and $D = D_1$. Note that D_1 is anti-symmetric and hence $\boldsymbol{w}^\top AD_1A^\top \boldsymbol{w} = 0$.

Applying Theorem 2.4, we have the following asymptotics for operator fBm $\boldsymbol{X}$.

Proposition 3.3. *Let $\boldsymbol{X}(t)$, $t \in [0,1]$ be an operator fBm with index H. Suppose that (O1)-(O3) hold and $\tau_{\boldsymbol{w}} = \tau_{\boldsymbol{w}}(1) = \boldsymbol{w}^\top H\Sigma \boldsymbol{w} > 0$.*

i) If $h_* < 1/2$ and $\boldsymbol{w}^\top A D_2 A\boldsymbol{w} > 0$, then

$$p_{\boldsymbol{b}}(1,u) \sim \mathcal{H}_{2h_*, V_{\boldsymbol{w}}} \tau_{\boldsymbol{w}}^{-1} u^{\frac{1-2h_*}{h_*}} \mathbb{P}\left\{\boldsymbol{X}(1) > u\boldsymbol{b}\right\},$$

with $V_{\boldsymbol{w}} = \mathrm{diag}(\boldsymbol{w}) A D_2 A\mathrm{diag}(\boldsymbol{w})$.

ii) If $h_* = 1/2$, then

$$p_{\boldsymbol{b}}(1,u) \sim \mathcal{P}_{1, V_{\boldsymbol{w}}, W_{\boldsymbol{w}}} \mathbb{P}\left\{\boldsymbol{X}(1) > u\boldsymbol{b}\right\},$$

with $V_{\boldsymbol{w}} = \mathrm{diag}(\boldsymbol{w}) A\left(D_1 + D_2\right) A\mathrm{diag}(\boldsymbol{w})$ and $W_{\boldsymbol{w}} = \mathrm{diag}(\boldsymbol{w}) H\Sigma \mathrm{diag}(\boldsymbol{w})$.

iii) If $h_* > 1/2$, then

$$p_{\boldsymbol{b}}(1,u) \sim C_{\boldsymbol{w}} \mathbb{P}\left\{\boldsymbol{X}(1) > u\boldsymbol{b}\right\},$$

with $C_{\boldsymbol{w}} = 1 + \tau_{\boldsymbol{w}}^{-1} \sum_{i \in I} w_i \max\left(0, -\left(H\widetilde{\boldsymbol{b}}\right)_i\right)$.

3.3. **Lamperti transform of operator fBm's.** Let $\boldsymbol{Y}(t)$, $t \ge 0$ be an operator fBm with index H. Suppose that (O1)-(O2) hold and let $\boldsymbol{X}(t) = (e^{-t})^H \boldsymbol{Y}(e^t)$, $t \ge 0$ be the Lamperti transform of $\boldsymbol{Y}$. We follow the notation introduced in Section 3.2. Clearly, $\mathbf{X}$ is stationary with

$$\begin{aligned}\mathbb{E}\left\{\boldsymbol{X}(t)\boldsymbol{X}(s)^\top\right\} =& \frac{1}{2}\left(\Sigma e^{(t-s)H^\top} + e^{-(t-s)H}\Sigma - |1-e^{-(t-s)}|^H \Sigma |1-e^{t-s}|^{H^\top}\right)\\ =& \Sigma - (t-s)\frac{1}{2}\left(H\Sigma - \Sigma H^\top\right) - |t-s|^{2h_*} Q\widetilde{\mathcal{I}}Q^{-1}\Sigma\left(Q\widetilde{\mathcal{I}}Q^{-1}\right)^\top + o\left(|t-s|^{2h_*}\right)\end{aligned}$$

for $t \ge s$, as $t - s \to 0$. Set $\tilde{V} = Q\widetilde{\mathcal{I}}Q^{-1}\Sigma(Q\widetilde{\mathcal{I}}Q^{-1})^\top$. Recall that $\widetilde{\boldsymbol{b}}$ solves the quadratic programming problem (1.3) and we set $\boldsymbol{w} = \Sigma^{-1}\widetilde{\boldsymbol{b}}$. Applying Theorem 2.1 and 2.2, we have the following proposition.

Proposition 3.4. *Let $\boldsymbol{X}$ be the Lamperti transform of an operator fBm with index H satisfying (O1)-(O2).*

i) Assume $H\Sigma = \Sigma H^\top$ or $H\Sigma \ne \Sigma H^\top$ but $h_* < 1/2$. If $\boldsymbol{w}^\top \widetilde{V}\boldsymbol{w} > 0$, then as $u \to \infty$

$$p_{\boldsymbol{b}}(T,u) \sim T\,\mathcal{H}_{\alpha, V_{\boldsymbol{w}}} u^{2/\alpha} \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}, \tag{3.3}$$

with $\alpha = 2h_*$ and $V = \widetilde{V}$.

ii) Assume $H\Sigma \ne \Sigma H^\top$ and $h_* = 1/2$. If $\boldsymbol{w}^\top \widetilde{V}\boldsymbol{w} > 0$, then (3.3) holds with $\alpha = 1$ and $V = H\Sigma - \Sigma H^\top + \widetilde{V}$.

iii) Assume $H\Sigma \ne \Sigma H^\top$ and $h_* > 1/2$. Set $\alpha = 1$ and $V = H\Sigma - \Sigma H^\top$. If $(V\boldsymbol{w})_I \ne \boldsymbol{0}_I$, then (3.3) holds with $\mathcal{H}_{\alpha, V_{\boldsymbol{w}}}$ replaced by $\frac{1}{2}\sum_{1 \le i \le d} w_i \left|(V\boldsymbol{w})_i\right|$.

4. Auxiliary Results

In this section we include some key tools for vector-valued Gaussian processes, which will be used in the proofs of the main results and are of some interest on their own right. We postpone all the proofs of lemmas presented in this section to Appendix. We explain first the properties of the solution of $\Pi_\Sigma(\boldsymbol{b})$ followed by a section on uniform approximation of tails of functionals of Gaussian processes.

4.1. **Quadratic programming problem.** For a given non-singular $d \times d$ real matrix Σ we consider the quadratic programming problem

$$\Pi_\Sigma(\boldsymbol{b}) : \text{minimise } \boldsymbol{x}^\top \Sigma^{-1}\boldsymbol{x} \text{ under the linear constraint } \boldsymbol{x} \ge \boldsymbol{b}. \tag{4.1}$$

Below $J = \{1, \ldots, d\} \setminus I$ can be empty; the claim in (4.3) is formulated under the assumption that J is non-empty.

Lemma 4.1. *Let $d \ge 2$ and Σ a $d \times d$ symmetric positive definite matrix with inverse Σ^{-1}. If $\boldsymbol{b} \in \mathbb{R}^d \setminus (-\infty, 0]^d$, then $\Pi_\Sigma(\boldsymbol{b})$ has a unique solution $\widetilde{\boldsymbol{b}}$ and there exists a unique non-empty index set $I \subset \{1, \ldots, d\}$ with $m \le d$ elements such that*

$$\widetilde{\boldsymbol{b}}_I = \boldsymbol{b}_I \ne \boldsymbol{0}_I, \tag{4.2}$$

$$\widetilde{\boldsymbol{b}}_J = \Sigma_{JI}(\Sigma_{II})^{-1}\boldsymbol{b}_I \ge \boldsymbol{b}_J, \quad (\Sigma_{II})^{-1}\boldsymbol{b}_I > \boldsymbol{0}_I, \tag{4.3}$$

$$\min_{\boldsymbol{x} \ge \boldsymbol{b}} \boldsymbol{x}^\top \Sigma^{-1}\boldsymbol{x} = \widetilde{\boldsymbol{b}}^\top \Sigma^{-1}\widetilde{\boldsymbol{b}} = \boldsymbol{b}_I^\top (\Sigma_{II})^{-1}\boldsymbol{b}_I > 0, \tag{4.4}$$

$$\max_{\boldsymbol{z} \in [0,\infty)^d : \boldsymbol{z}^\top \boldsymbol{b} > 0} \frac{(\boldsymbol{z}^\top \boldsymbol{b})^2}{\boldsymbol{z}^\top \Sigma \boldsymbol{z}} = \frac{(\boldsymbol{w}^\top \boldsymbol{b})^2}{\boldsymbol{w}^\top \Sigma \boldsymbol{w}} = \min_{\boldsymbol{x} \ge \boldsymbol{b}} \boldsymbol{x}^\top \Sigma^{-1}\boldsymbol{x}, \tag{4.5}$$

with $\boldsymbol{w} = \Sigma^{-1}\widetilde{\boldsymbol{b}}$ satisfying $\boldsymbol{w}_I = (\Sigma_{II})^{-1}\boldsymbol{b}_I > \boldsymbol{0}_I, \boldsymbol{w}_J = \boldsymbol{0}_J$.

Define the solution map of the quadratic programming problem (1.3) by $\mathcal{P} : \Sigma^{-1} \mapsto \widetilde{\boldsymbol{b}}$ with $\widetilde{\boldsymbol{b}}$ the unique solution to $\Pi_\Sigma(\boldsymbol{b})$. The next result is a special case of [34, Thm 3.1].

Lemma 4.2. *$\mathcal{P}$ is Lipschitz continuous on compact subset of the space of real $d \times d$ symmetric positive definite matrices.*

4.2. **Uniform tail appropximation for functionals of families of Gaussian processes.** A key role in the analysis of extremes of Gaussian processes is played by the continuous mapping theorem, the idea appeared first in [35, 36] and it is used extensively in the monographs [19, 25].
Our main tool that shall compensate for the lack of Slepian lemma is the uniform approximation of the supremum tail distribution of threshold-dependent Gaussian processes. We present below a general result where the tail distributions for a continuous functional of a family of Gaussian processes are uniformly approximated.
Let $\{\boldsymbol{X}_{u,\tau}(t),\ t \in E\}$, $u > 0$, $\tau \in Q_u \subset \mathbb{R}$ be a family of centered, d-dimensional Gaussian processes with a.s. continuous sample paths and parameter space E which is assumed to be a compact subset of $\mathbb{R}^k$. For notational simplicity we discuss below only case $k = 1$. Denote its cmf by $R_{u,\tau}(t,s) = \mathbb{E}\left\{\boldsymbol{X}_{u,\tau}(t)\boldsymbol{X}_{u,\tau}(s)^\top\right\}$ and let $C(E)$ be the separable Banach space of all $\mathbb{R}^d$-valued continuous functions on E equipped with the sup-norm and assume for simplicity that the origin 0 of $\mathbb{R}^k$ belongs to E.

Lemma 4.3. *Suppose that $\boldsymbol{X}_{u,\tau}(0) = \boldsymbol{0}$ almost surely and $\boldsymbol{Y}(t)$, $t \in E$ is a Gaussian process with a.s. continuous sample paths. Let $\boldsymbol{f}_{u,\tau}(t)$, $\boldsymbol{f}(t)$, $t \in E$ be deterministic $\mathbb{R}^d$-valued continuous functions. Assume that*

$$\lim_{u\to\infty} \sup_{t\in E,\tau\in Q_u} \left|\boldsymbol{f}_{u,\tau}(t) - \boldsymbol{f}(t)\right| = 0 \tag{4.6}$$

and

$$\lim_{u\to\infty} \sup_{t,s\in E,\tau\in Q_u} \left\|R_{u,\tau}(t,s) - \mathbb{E}\left\{\boldsymbol{Y}(t)\boldsymbol{Y}(s)^\top\right\}\right\|_{\mathrm{F}} = 0. \tag{4.7}$$

If further for some $C \in (0,\infty)$, $\gamma \in (0,2]$ and any $s,\ t \in E$

$$\limsup_{u\to\infty} \max_{1\le i\le d} \sup_{\tau\in Q_u} \mathbb{E}\left\{[X_{i,u,\tau}(t) - X_{i,u,\tau}(s)]^2\right\} \le C\,|t-s|^\gamma\,, \tag{4.8}$$

then for any continuous functional $\Gamma : C(E) \to \mathbb{R}$,

$$\lim_{u\to\infty} \sup_{\tau\in Q_u} \left|\mathbb{P}\left\{\Gamma(\boldsymbol{X}_{u,\tau}(\cdot) - \boldsymbol{f}_{u,\tau}(\cdot)) \le s\right\} - \mathbb{P}\left\{\Gamma(\boldsymbol{Y}(\cdot) - \boldsymbol{f}(\cdot)) \le s\right\}\right| = 0 \tag{4.9}$$

is valid for all continuity point s of the distribution function of $\Gamma(\boldsymbol{Y}(\cdot) - \boldsymbol{f}(\cdot))$.

Application of Lemma 4.7 requires the determination of the continuity points of the functional $\Gamma(\boldsymbol{Y}(\cdot) - \boldsymbol{f}(\cdot))$. The next result is useful in that context.

Lemma 4.4. *If $\boldsymbol{Y}(t)$, $t \in E$ is a Gaussian process with a.s. bounded sample paths, then $\mathbb{P}\{\exists t \in E : \ \boldsymbol{Y}(t) > \boldsymbol{x}\}$ is continuous on $\mathbb{R}^d$, except at most at points of Lebesgue measure 0 in $\mathbb{R}^d$ belonging to $\bigcup_{i=1}^d \mathbb{R}^i \times \{s_i\} \times \mathbb{R}^{d-i-1}$, where $s_i = \inf\{s : \mathbb{P}\{\sup_{t\in E} Y_i(t) \le s\} > 0\}$ for $i = 1, ..., d$.*

4.3. **Borell-TiS & Piterbarg inequalities.** The Borell-TIS inequality, see e.g. [22], is very useful and crucial in numerous theoretical problems. Under some weak assumptions, its refinement i.e., Piterbarg inequality ([23, Thm 8.1]) gives a more precise upper bound for supremum tail distribution of Gaussian random fields. In the following lemma we present an extension of these tools to vector-valued setting.

Lemma 4.5. *Let $\boldsymbol{Z}(t)$, $t \in E$ be a separable centered d-dimensional vector-valued Gaussian process having components with a.s. continuous trajectories. Assume that $\Sigma(t) = \mathbb{E}\left\{\boldsymbol{Z}(t)\boldsymbol{Z}(t)^\top\right\}$ is non-singular for all $t \in E$. Let $\boldsymbol{b} \in \mathbb{R}^d \setminus (-\infty, 0]^d$ and define $\sigma_{\boldsymbol{b}}^2(t)$ as in (2.9). If $\sigma_{\boldsymbol{b}}^2 = \sup_{t\in E} \sigma_{\boldsymbol{b}}^2(t) \in (0,\infty)$, then there exists some positive constant μ such that for all $u > \mu$*

$$\mathbb{P}\{\exists t \in E : \ \boldsymbol{Z}(t) > u\boldsymbol{b}\} \le \exp\left(-\frac{(u-\mu)^2}{2\sigma_{\boldsymbol{b}}^2}\right). \tag{4.10}$$

If further for some $C \in (0,\infty)$ and $\gamma \in (0,2]$

$$\sum_{1\le i\le d} \mathbb{E}\left\{(Z_i(t) - Z_i(s))^2\right\} \le C\,|t-s|^\gamma \tag{4.11}$$

and

$$\left\|\Sigma^{-1}(t) - \Sigma^{-1}(s)\right\|_{\mathrm{F}} \le C\,|t-s|^\gamma \tag{4.12}$$

hold for all $t,\ s \in E$, then for all u positive

$$\mathbb{P}\left\{\exists t \in E: \ \boldsymbol{Z}(t) > u\boldsymbol{b}\right\} \le C_* \mathrm{mes}(E)\ u^{\frac{2d}{\gamma}-1} \exp\left(-\frac{u^2}{2\sigma_{\boldsymbol{b}}^2}\right), \tag{4.13}$$

where C_* is some positive constant not depending on u.
In particular, if $\sigma_{\boldsymbol{b}}(t),\ t \in E$ is continuous and achieves its unique maximum at some fixed point $t_0 \in E$, then (4.13) is still valid if (4.11) and (4.12) are assumed to hold only for all $s,\ t \in E$ in an open neighbourhood of t_0.

4.4. **Uniform approximation on short intervals (Pickands-Piterbarg Lemma).** Following notation introduced in Section 4.2 denote by $R_{u,\tau}(\cdot,\cdot)$ the cmf of $\boldsymbol{X}_{u,\tau}$. We shall impose the following assumptions:

(A1) For all large u and all $\tau \in Q_u$ the matrix $\Sigma_{u,\tau} = R_{u,\tau}(0,0)$ is positive definite and

$$\lim_{u\to\infty} u \sup_{\tau\in Q_u} \left\|\Sigma - \Sigma_{u,\tau}\right\|_{\mathrm{F}} = 0 \tag{4.14}$$

holds for some positive definite matrix Σ;

(A2) There exist a continuous $\mathbb{R}^d$-valued function $\boldsymbol{d}(t), t \in E$ and a continuous matrix-valued function $K(t,s)$, $(t,s) \in E \times E$ such that

$$\lim_{u\to\infty} \sup_{\tau\in Q_u, t\in E} u \left\|\Sigma_{u,\tau} - R_{u,\tau}(t,0)\right\|_{\mathrm{F}} = 0, \tag{4.15}$$

$$\lim_{u\to\infty} \sup_{\tau\in Q_u, t\in E} \left| u^2 \left[\Sigma_{u,\tau} - R_{u,\tau}(t,0)\right] \Sigma^{-1} \widetilde{\boldsymbol{b}} - \boldsymbol{d}(t) \right| = 0 \tag{4.16}$$

and

$$\lim_{u\to\infty} \sup_{\tau\in Q_u, s,t\in E} \left\| u^2 \left[R_{u,\tau}(t,s) - R_{u,\tau}(t,0)\Sigma_{u,\tau}^{-1} R_{u,\tau}(0,s)\right] - K(t,s) \right\|_{\mathrm{F}} = 0; \tag{4.17}$$

(A3) There exist positive constants C and $\gamma \in (0,2]$ such that for any $s,\ t \in E$

$$\sup_{\tau\in Q_u} u^2 \mathbb{E}\left\{\left|\boldsymbol{X}_{u,\tau}(t) - \boldsymbol{X}_{u,\tau}(s)\right|^2\right\} \le C\left|t-s\right|^{\gamma}. \tag{4.18}$$

Remark 4.6. *(i) The existence of $\Sigma_{u,\tau}^{-1}$ follows from the positive definiteness of Σ and condition (4.14). Further since*

$$\begin{aligned} &R_{u,\tau}(t,s) - R_{u,\tau}(t,0)\Sigma_{u,\tau}^{-1} R_{u,\tau}(0,s) \\ =&\mathbb{E}\left\{\left[\boldsymbol{X}_{u,\tau}(t) - R_{u,\tau}(t,0)\Sigma_{u,\tau}^{-1}\boldsymbol{X}_{u,\tau}(0)\right]\left[\boldsymbol{X}_{u,\tau}(s) - R_{u,\tau}(s,0)\Sigma_{u,\tau}^{-1}\boldsymbol{X}_{u,\tau}(0)\right]^{\top}\right\}, \end{aligned}$$

then $K(t,s)$ is a matrix-valued non-negative definite function on $E \times E$ with $K(t,s) = K(s,t)^{\top}$. Consequently, for $K_{\boldsymbol{w}}(t,s) = \mathrm{diag}(\boldsymbol{w})K(t,s)\mathrm{diag}(\boldsymbol{w})$ with $\boldsymbol{w}$ some vector in $\mathbb{R}^d\backslash\{\boldsymbol{0}\}$ there exists a centered, $\mathbb{R}^d$-valued Gaussian random field $\boldsymbol{Y}(t),\ t \in E$ with $\boldsymbol{Y}(0) = 0$ and cmf $K_{\boldsymbol{w}}$.

(ii) If for some continuous matrix-valued function $V(t,s) \in \mathbb{R}^{d\times d}$, $(t,s) \in E \times E$

$$\lim_{u\to\infty} \sup_{\tau\in Q_u, s,t\in E} \left| u^2 \Big[\Sigma_{u,\tau} - R_{u,\tau}(t,s)\Big] - V(t,s) \right| = 0, \tag{4.19}$$

then (A2) holds with $\boldsymbol{d}(t) = V(t,0)\boldsymbol{w}$ and $K(t,s) = V(t,0) + V(0,s) - V(t,s)$.

(iii) Let $A_{u,\tau}(t)$ (resp. $A_{u,\tau}$) be the square roots of the positive definite matrices $\Sigma_{u,\tau}(t)$ (resp. $\Sigma_{u,\tau}$). Note that

$$R_{u,\tau}(t,s) = A_{u,\tau}(t)C_{u,\tau}(t,s)A_{u,\tau}(s)^{\top},$$

with $C_{u,\tau}(t,s)$ the cmf of $A_{u,\tau}(t)^{-1}\boldsymbol{X}_{u,\tau}(t)$. Under the condition that $\lim_{u\to\infty} A_{u,\tau}(t) = A$ uniformly in $t \in E$ and $\tau \in Q_u$, the convergence in (4.17) only depends on the correlation structure of $\boldsymbol{X}_{u,\tau}(t)$, that is, if we suppose that

$$\lim_{u\to\infty} \sup_{\tau\in Q_u, s,t\in E} \left| u^2 \left[C_{u,\tau}(t,s) - C_{u,\tau}(t,0)C_{u,\tau}(0,s)\right] - \widetilde{K}(t,s) \right| = 0, \tag{4.20}$$

then (4.17) holds with $K(t,s) = A\widetilde{K}(t,s)A^{\top}$.

For $\boldsymbol{Y}(t),\ t \in E$ a centered $\mathbb{R}^d$-valued Gaussian process with a.s. continuous sample paths with cmf $K(s,t)$, s, $t \in E$ and an $\mathbb{R}^d$-valued function $\boldsymbol{d}$ define below

$$H_{\boldsymbol{Y},\boldsymbol{d}}(E) = \int_{\mathbb{R}^d} e^{\boldsymbol{1}^{\top}\boldsymbol{x}} \mathbb{P}\left\{\exists t \in E: \ \boldsymbol{Y}(t) - \boldsymbol{d}(t) > \boldsymbol{x}\right\} d\boldsymbol{x}. \tag{4.21}$$

Lemma 4.7. *Suppose that $\boldsymbol{X}_{u,\tau}(t)$, $t \in E$, $u > 0$, $\tau \in Q_u$ satisfy (A1)-(A3). Let $\boldsymbol{w} = \Sigma^{-1}\widetilde{\boldsymbol{b}}$ where $\widetilde{\boldsymbol{b}}$ is the unique solution of $\Pi_\Sigma(\boldsymbol{b})$. If $\boldsymbol{Y}(t)$, $t \in E$ has cmf $R(t,s) = \mathrm{diag}(\boldsymbol{w})K(t,s)\mathrm{diag}(\boldsymbol{w})$ and $\boldsymbol{d}_{\boldsymbol{w}}(t) = \mathrm{diag}(\boldsymbol{w})\boldsymbol{d}(t)$, then we have*

$$\lim_{u\to\infty} \sup_{\tau \in Q_u} \left| \frac{\mathbb{P}\left\{\exists t \in E: \ \boldsymbol{X}_{u,\tau}(t) > u\boldsymbol{b}\right\}}{\mathbb{P}\left\{\boldsymbol{X}_{u,\tau}(0) > u\boldsymbol{b}\right\}} - H_{\boldsymbol{Y},\boldsymbol{d}_{\boldsymbol{w}}}(E) \right| = 0. \tag{4.22}$$

Remark 4.8. *If we suppose stronger assumptions on $\Sigma_{u,\tau}$, for instance*

$$\lim_{u\to\infty} \sup_{t \in Q_u} \left\| u^2 \left[\Sigma - \Sigma_{u,\tau}\right] - \Xi \right\|_{\mathrm{F}} = 0,$$

then as $u \to \infty$

$$\mathbb{P}\left\{\boldsymbol{X}_{u,\tau}(0) > u\boldsymbol{b}\right\} \sim e^{-\boldsymbol{w}^\top \Xi \boldsymbol{w}/2}\mathbb{P}\left\{\boldsymbol{\mathcal{N}} > u\boldsymbol{b}\right\},$$

where $\boldsymbol{\mathcal{N}}$ is a centered Gaussian vector with covariance matrix Σ.

5. Proofs of Main Results

Recall that I and J with $|I| = m$ are the index sets related to $\Pi_\Sigma(\boldsymbol{b})$ with unique solution $\widetilde{\boldsymbol{b}}$, where $\boldsymbol{b} \in \mathbb{R}^d$ is assumed to have at least one positive component. As before we set $\boldsymbol{w} = \Sigma^{-1}\widetilde{\boldsymbol{b}}$. By (1.4) and Assumption (B2) we have that $\boldsymbol{w}_J = \boldsymbol{0}_J$ and

$$\xi_{\boldsymbol{w}} = \boldsymbol{w}^\top V \boldsymbol{w} = \boldsymbol{w}_I^\top V_{II} \boldsymbol{w}_I > 0. \tag{5.1}$$

Otherwise specified, in the following $\boldsymbol{Y}(t)$, $t \in \mathbb{R}$ is a centered $\mathbb{R}^d$-valued Gaussian process with cmf

$$\mathrm{diag}(\boldsymbol{w})R_{\alpha,V}(t,s)\mathrm{diag}(\boldsymbol{w})$$

where $R_{\alpha,V}$ is defined in (2.2).
Hereafter, throughout this paper we use the lower case constants $c_1, c_2, \ldots$ to denote generic constants used in the proofs, whose exact values are not important and can be changed line to line. The labelling of the constants starts anew in every proof.

5.1. Proof of Theorem 2.1.

Let $\boldsymbol{X}(t)$, $t \in [0,T]$ be a stationary centered $\mathbb{R}^d$-valued Gaussian process. Before proceeding to the proof of Theorem 2.1, we shall derive some useful asymptotic bounds. The first lemma is crucial for the negligibility of the double-sum. Below we set $\Delta(\lambda,\Lambda) = [0,\Lambda] \times [\lambda, \lambda+\Lambda]$ and

$$P_{\boldsymbol{b}}(\lambda,\Lambda,u) = \mathbb{P}\left\{\exists (t,s) \in u^{-2/\alpha}\Delta(\lambda,\Lambda): \ \boldsymbol{X}(t) > u\boldsymbol{b}, \ \boldsymbol{X}(s) > u\boldsymbol{b}\right\}. \tag{5.2}$$

Lemma 5.1. *If Assumption (B2) holds, then there exist positive constants C, ε and n_0 such that for every $\lambda \geq n_0\Lambda > 0$ with $\lambda + \Lambda < \varepsilon u^{2/\alpha}$*

$$P_{\boldsymbol{b}}(\lambda,\Lambda,u) \leq C\,\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\} e^{-\frac{\lambda^\alpha}{16}\xi_{\boldsymbol{w}}}. \tag{5.3}$$

Proof of Lemma 5.1: Since

$$P_{\boldsymbol{b}}(\lambda,\Lambda,u) \leq \mathbb{P}\left\{\exists (t,s) \in u^{-2/\alpha}\Delta(\lambda,\Lambda): \ \boldsymbol{X}_I(t) > u\boldsymbol{b}, \boldsymbol{X}_I(s) > u\boldsymbol{b}\right\}$$

and $\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\} \sim c_1 \mathbb{P}\left\{\boldsymbol{X}_I(0) > u\boldsymbol{b}_I\right\}$ as $u \to \infty$, it suffices to prove that

$$\mathbb{P}\left\{\exists (t,s) \in u^{-2/\alpha}\Delta(\lambda,\Lambda): \ \boldsymbol{X}_I(t) > u\boldsymbol{b}_I, \ \boldsymbol{X}_I(s) > u\boldsymbol{b}_I\right\} \leq C\,\mathbb{P}\left\{\boldsymbol{X}_I(0) > u\boldsymbol{b}_I\right\} e^{-\frac{\lambda^\alpha}{16}\xi_{\boldsymbol{w}}}.$$

Without loss of generality, in the rest of the proof we assume that $I = \{1,\ldots,d\}$ and write $\boldsymbol{X}$ instead of $\boldsymbol{X}_I$, $\boldsymbol{w}$ instead of $\boldsymbol{w}_I$ and so on.
Set below

$$V^+ = \frac{1}{2}(V + V^\top), \quad V(t) = S_\alpha(t,V) = |t|^\alpha \left(V\boldsymbol{1}_{\{t\geq 0\}} + V^\top \boldsymbol{1}_{\{t<0\}}\right).$$

By Assumption (B2), for every $\delta > 0$ there exists $\varepsilon > 0$ such that for every $|t| < \varepsilon$ we have

$$\left\|\Sigma - \mathcal{R}(t) - V(t)\right\|_{\mathrm{F}} \leq \delta |t|^\alpha. \tag{5.4}$$

Set $R_u(t) = \mathcal{R}(u^{-2/\alpha}t)$ and define $\boldsymbol{X}_u(t,s) = \frac{1}{2}\left(\boldsymbol{X}(u^{-2/\alpha}t) + \boldsymbol{X}(u^{-2/\alpha}s)\right)$ which has cmf

$$\begin{aligned} R_u(t,s;t_1,s_1) =& \mathbb{E}\left\{\boldsymbol{X}_u(t,s)\boldsymbol{X}_u(t_1,s_1)^\top\right\} \\ =& \frac{1}{4}\left(R_u(t-t_1) + R_u(s-s_1) + R_u(t-s_1) + R_u(s-t_1)\right). \end{aligned}$$

Further set

$$\Sigma_{u,\lambda} = \mathbb{E}\left\{\boldsymbol{X}_u(0,\lambda)\boldsymbol{X}_u(0,\lambda)^\top\right\} = \frac{1}{4}\left[2\Sigma + R_u(\lambda) + R_u(-\lambda)\right]$$

and

$$V(t,s;t_1,s_1)=V(t-t_1)+V(s-s_1)+V(t-s_1)+V(s-t_1)-V(\lambda)-V(-\lambda).$$

In view of (5.4) we have for $\lambda\in(0,\varepsilon u^{2/\alpha})$ that

$$\left\|u^2(\Sigma-\Sigma_{u,\lambda})-\frac{\lambda^\alpha}{2}V^+\right\|_{\rm F}\le\frac{1}{2}\delta\lambda^\alpha \tag{5.5}$$

and

$$\left\|u^2\left(\Sigma_{u,\lambda}-R_u(t,s;t_1,s_1)\right)-\frac{1}{4}V(t,s;t_1,s_1)\right\|_{\rm F}\le\frac{3}{2}\delta\lambda^\alpha. \tag{5.6}$$

Since $\Sigma_{u,\lambda}^{-1}-\Sigma^{-1}=\Sigma_{u,\lambda}^{-1}\left(\Sigma-\Sigma_{u,\lambda}\right)\Sigma^{-1}$, we have from (5.5) that

$$\left\|\Sigma_{u,\lambda}^{-1}-\Sigma^{-1}\right\|_{\rm F}\quad=\quad O(\lambda^\alpha u^{-2}) \tag{5.7}$$

and

$$\begin{aligned}&\left\|u^2\left(\Sigma_{u,\lambda}^{-1}-\Sigma^{-1}\right)-\frac{\lambda^\alpha}{2}\Sigma^{-1}V^+\Sigma^{-1}\right\|_{\rm F}\\ \le&\left\|\Sigma^{-1}\left[u^2\left(\Sigma-\Sigma_{u,\lambda}\right)-\frac{\lambda^\alpha}{2}V^+\right]\Sigma^{-1}\right\|_{\rm F}+\left\|u^2\left(\Sigma_{u,\lambda}^{-1}-\Sigma^{-1}\right)\left(\Sigma-\Sigma_{u,\lambda}\right)\Sigma^{-1}\right\|_{\rm F}\\ \le&\delta\lambda^\alpha\left\|\Sigma^{-1}\right\|_{\rm F}^2.\end{aligned} \tag{5.8}$$

Therefore, for $\delta>0$ sufficiently small

$$u^2\boldsymbol{b}^\top\Sigma_{u,\lambda}^{-1}\boldsymbol{b}\ge u^2\boldsymbol{b}^\top\Sigma^{-1}\boldsymbol{b}+\frac{\lambda^\alpha}{4}\xi_{\boldsymbol{w}}. \tag{5.9}$$

Conditioning on $\boldsymbol{X}_u(0,\lambda)=u\boldsymbol{b}-u^{-1}\boldsymbol{x}=:a_u(\boldsymbol{x})$ we obtain further

$$\begin{aligned}P_{\boldsymbol{b}}(\lambda,\Lambda,u)\le&\mathbb{P}\left\{\exists(t,s)\in\Delta(\lambda,\Lambda):\ \boldsymbol{X}_u(t,s)>u\boldsymbol{b}\right\}\\ =&u^{-d}\int_{\mathbb{R}^d}\mathbb{P}\left\{\exists(t,s)\in\Delta(\lambda,\Lambda):\ \boldsymbol{X}_u(t,s)>u\boldsymbol{b}\,\middle|\,\boldsymbol{X}_u(0,\lambda)=a_u(\boldsymbol{x})\right\}\varphi_{\Sigma_{u,\lambda}}(a_u(\boldsymbol{x}))d\boldsymbol{x}\\ =&u^{-d}\int_{\mathbb{R}^d}J_u(\boldsymbol{x})\varphi_{\Sigma_{u,\lambda}}(a_u(\boldsymbol{x}))d\boldsymbol{x},\end{aligned}$$

where

$$J_u(\boldsymbol{x})=\mathbb{P}\left\{\exists(t,s)\in\Delta(\lambda,\Lambda):\ \boldsymbol{\chi}_u(t,s)>\boldsymbol{x}\right\}$$

and $\boldsymbol{\chi}_u(t,s)$ is the conditioned process $u(\boldsymbol{X}_u(t,s)-u\boldsymbol{b})+\boldsymbol{x}$ given $\boldsymbol{X}_u(0,\lambda)=a_u(\boldsymbol{x})$. By (5.7) and (5.9)

$$\begin{aligned}\varphi_{\Sigma_{u,\lambda}}(a_u(\boldsymbol{x}))\le&\varphi_{\Sigma_{u,\lambda}}(u\boldsymbol{b})\exp\left(\boldsymbol{b}^\top\left(\Sigma_{u,\lambda}\right)^{-1}\boldsymbol{x}\right)\\ \le&\varphi_\Sigma(u\boldsymbol{b})\exp\left(-\frac{\lambda^\alpha\xi_{\boldsymbol{w}}}{8}\right)e^{\left(\boldsymbol{w}+O(\lambda^\alpha u^{-2})\right)^\top\boldsymbol{x}}.\end{aligned} \tag{5.10}$$

Consequently, for all u large enough

$$P_{\boldsymbol{b}}(\lambda,\Lambda,u)\le 2u^{-d}\varphi_\Sigma(u\boldsymbol{b})\exp\left(-\frac{\lambda^\alpha}{8}\xi_{\boldsymbol{w}}\right)\int_{\mathbb{R}^d}e^{\left(\boldsymbol{w}+O(\lambda^\alpha u^{-2})\right)^\top\boldsymbol{x}}J_u(\boldsymbol{x})d\boldsymbol{x}. \tag{5.11}$$

Given $F\subset\{1,\dots,d\}$ let $\Omega_F=\left\{\boldsymbol{x}\in\mathbb{R}^d:\ x_i>0,\,x_j<0,\,i\in F,\,j\notin F\right\}$. If F is empty i.e., $F=\emptyset$, then

$$\int_{\Omega_\emptyset}e^{\left(\boldsymbol{w}+O(\lambda^\alpha u^{-2})\right)^\top\boldsymbol{x}}J_u(\boldsymbol{x})d\boldsymbol{x}\le\int_{\Omega_\emptyset}e^{\left(\boldsymbol{w}+O(\lambda^\alpha u^{-2})\right)^\top\boldsymbol{x}}d\boldsymbol{x}\le\frac{2}{\prod_{1\le i\le d}w_i}. \tag{5.12}$$

Assume next that $F\ne\emptyset$ and define for $u>0$

$$d_u(t,s)=-\mathbb{E}\left\{\boldsymbol{\chi}_u(t,s)\right\},\quad \eta_u(t,s)=\boldsymbol{w}_F^\top\left(\boldsymbol{\chi}_{u,F}(t,s)+\boldsymbol{d}_{u,F}(t,s)\right).$$

For all $\boldsymbol{x}\in\Omega_F$

$$J_u(\boldsymbol{x})\le\mathbb{P}\left\{\exists(t,s)\in\Delta(\lambda,\Lambda):\ \eta_u(t,s)>\boldsymbol{w}_F^\top\boldsymbol{x}_F-\boldsymbol{w}_F^\top\boldsymbol{d}_{u,F}(t,s)\right\}. \tag{5.13}$$

Since $\boldsymbol{X}_u(0,\lambda)$ is independent of $\boldsymbol{X}_u(t,s)-R_u(t,s;0,\lambda)\Sigma_{u,\lambda}^{-1}\boldsymbol{X}_u(0,\lambda)$ we obtain

$$d_u(t,s)\quad=\quad u^2\left[\Sigma_{u,\lambda}-R_u(t,s;0,\lambda)\right]\Sigma_{u,\lambda}^{-1}\boldsymbol{b}-\left[\Sigma_{u,\lambda}-R_u(t,s;0,\lambda)\right]\Sigma_{u,\lambda}^{-1}\boldsymbol{x}$$

and the cmf of $\boldsymbol{\chi}_u$ is

$$K_u(t,s;t_1,s_1) = u^2\left[R_u(t,s;t_1,s_1) - R_u(t,s;0,\lambda)\Sigma_{u,\lambda}^{-1}R_u(0,\lambda;t_1,s_1)\right].$$

Set

$$\boldsymbol{d}(t,s) = \frac{1}{4}\left[t^\alpha + (s-\lambda)^\alpha + s^\alpha - \lambda^\alpha\right]V\boldsymbol{w} - \frac{1}{4}\left(\lambda^\alpha - (\lambda-t)^\alpha\right)V^\top\boldsymbol{w}$$

and

$$\begin{aligned}K(t,s;t_1,s_1) =& \frac{1}{4}\left(V(t,s;0,\lambda) + V(0,\lambda;t_1,s_1) - V(t,s;t_1,s_1)\right)\\ =& \frac{1}{4}\left[\psi(t,s;t_1,s_1)V + \psi(t_1,s_1;t,s)V^\top - V(t-t_1) - V(s-s_1)\right],\end{aligned}$$

where

$$\psi(t,s;t_1,s_1) = t^\alpha + (s-\lambda)^\alpha + s^\alpha - \lambda^\alpha + (\lambda-t_1)^\alpha - (s-t_1)^\alpha.$$

By (5.6) and (5.8), there exists $c_2 > 0$ such that for $\lambda + \Lambda < \varepsilon u^{2/\alpha}$ and (t,s), $(t_1,s_1) \in \Delta(\lambda,\Lambda)$

$$\left|\boldsymbol{d}_u(t,s) - \boldsymbol{d}(t,s) - \left[\Sigma_{u,\lambda} - R_u(t,s;0,\lambda)\right]\Sigma_{u,\lambda}^{-1}\boldsymbol{x}\right| \le c_2\delta\lambda^\alpha \tag{5.14}$$

holds and further

$$\|K_u(t,s;t_1,s_1) - K(t,s;t_1,s_1)\|_{\mathrm{F}} \le c_2\delta\lambda^\alpha. \tag{5.15}$$

Using the inequality

$$\left|p^h - q^h\right| \le h\max(p^{h-1},q^{h-1})\,|p-q| \tag{5.16}$$

valid for all p,q and h positive, we obtain that

$$|\boldsymbol{d}(t,s)| \le c_3\lambda^{\alpha-1}\Lambda \quad \text{and} \quad \|K(t,s;t_1,s_1)\|_{\mathrm{F}} \le c_3\lambda^{\alpha-1}\Lambda$$

holds for some positive constant c_3. Therefore, we can choose n_0 large enough so that, for $\lambda \ge n_0\Lambda$ and $\lambda + \Lambda < \varepsilon u^{2/\alpha}$

$$\inf_{(t,s)\in\Delta(\lambda,\Lambda)}\left[-\boldsymbol{w}_F^\top\boldsymbol{d}_{u,F}(t,s)\right] \ge -\frac{1}{2}\boldsymbol{w}_F^\top\boldsymbol{x}_F - c_4\delta\lambda^\alpha \tag{5.17}$$

and

$$\sigma_F^2 = \sup_{(t,s)\in\Delta(\lambda,\Lambda)}\mathrm{Var}\left(\eta_u(t,s)\right) \le c_4\delta\lambda^\alpha. \tag{5.18}$$

Since the conditional variance is always less than or equal to the unconditional one, we have that

$$\begin{aligned}&\mathrm{Var}\left(\eta_u(t,s) - \eta_u(t_1,s_1)\right)\\ \le& u^2\mathrm{Var}\left(\boldsymbol{w}_F^\top\boldsymbol{X}_{u,F}(t,s) - \boldsymbol{w}_F^\top\boldsymbol{X}_{u,F}(t_1,s_1)\right)\\ \le& \frac{u^2}{2}\left[\mathrm{Var}\left(\boldsymbol{w}_F^\top\boldsymbol{X}_F(u^{-2/\alpha}t) - \boldsymbol{w}_F^\top\boldsymbol{X}_F(u^{-2/\alpha}t_1)\right) + \mathrm{Var}\left(\boldsymbol{w}_F^\top\boldsymbol{X}_F(u^{-2/\alpha}s) - \boldsymbol{w}_F^\top\boldsymbol{X}_F(u^{-2/\alpha}s_1)\right)\right]\\ =& u^2\boldsymbol{w}_F^\top\left(\Sigma_{FF} - \mathcal{R}_{FF}(u^{-2/\alpha}(t-t_1))\right)\boldsymbol{w}_F + u^2\boldsymbol{w}_F^\top\left(\Sigma_{FF} - \mathcal{R}_{FF}(u^{-2/\alpha}(s-s_1))\right)\boldsymbol{w}_F\\ \le& c_5\left(|t-t_1|^\alpha + |s-s_1|^\alpha\right).\end{aligned}$$

Now Piterbarg inequality (c.f. [25, Theorem 8.1]) and (5.13), (5.17) imply that

$$\begin{aligned}J_u(\boldsymbol{x}) \le& c_6\left(\frac{\boldsymbol{w}_F^\top\boldsymbol{x}_F - 2c_4\delta\lambda^\alpha}{\sigma_F}\right)^{4/\alpha}\exp\left(-\frac{\left(\boldsymbol{w}_F^\top\boldsymbol{x}_F - 2c_4\delta\lambda^\alpha\right)^2}{8\sigma_F^2}\right)\\ \le& c_7\exp\left(-\frac{\left(\boldsymbol{w}_F^\top\boldsymbol{x}_F - 2c_4\delta\lambda^\alpha\right)^2}{16c_4\delta\lambda^\alpha}\right)\end{aligned}$$

for $\boldsymbol{w}_F^\top\boldsymbol{x}_F > 2c_4\delta\lambda^\alpha$. It follows that

$$\begin{aligned}&\int_{\Omega_F} e^{\left(\boldsymbol{w}+O(\lambda^\alpha u^{-2})\right)^\top\boldsymbol{x}}J_u(\boldsymbol{x})d\boldsymbol{x}\\ \le& c_8\left(\delta\lambda^\alpha\right)^{|F|}e^{2c_4\delta\lambda^\alpha} + c_8\int_{2c_4\delta\lambda^\alpha}^\infty y^{|F|-1}\exp\left(2y - \frac{(y-2c_4\delta\lambda^\alpha)^2}{16c_4\delta\lambda^\alpha}\right)dy\\ \le& c_9\exp\left(c_{10}\delta\lambda^\alpha\right),\end{aligned} \tag{5.19}$$

where $|F|$ is the cardinality of the set F. Together with (5.11) and (5.12) we obtain further

$$P_{\boldsymbol{b}}(\lambda,\Lambda,u) \le c_{11}u^{-d}\varphi_{\Sigma}(u\boldsymbol{b})\exp\left(-\frac{\lambda^{\alpha}}{8}\xi_{\boldsymbol{w}}+c_{10}\delta\lambda^{\alpha}\right).$$

Choosing $\delta>0$ small enough so that $c_{10}\delta\le\xi_{\boldsymbol{w}}/16$, we have

$$P_{\boldsymbol{b}}(\lambda,\Lambda,u)\le c_{12}u^{-m}\varphi_{\Sigma}(u\boldsymbol{b})\exp\left(-\frac{\lambda^{\alpha}}{16}\xi_{\boldsymbol{w}}\right)\sim c_{13}\mathbb{P}\left\{\boldsymbol{X}(0)>u\boldsymbol{b}\right\}\exp\left(-\frac{\lambda^{\alpha}}{16}\xi_{\boldsymbol{w}}\right)$$

as $u\to\infty$, which establishes the proof. □

Corollary 5.2. *Under Assumption (B2), there are positive constants C and ε such that for every $\lambda>\Lambda>0$ with $\lambda+\Lambda<\varepsilon u^{2/\alpha}$ we have*

$$\frac{P_{\boldsymbol{b}}(\lambda,\Lambda,u)}{\mathbb{P}\left\{\boldsymbol{X}(0)>u\boldsymbol{b}\right\}}\le C\Lambda^2(\lambda-\Lambda)^{-2}e^{-\frac{(\lambda-\Lambda)^{\alpha}}{16}\xi_{\boldsymbol{w}}}. \tag{5.20}$$

Proof of Corollary 5.2. Hereafter $\lfloor x\rfloor$ stands for the integer part of $x\in\mathbb{R}$. Let n_0 be the constant specified in Lemma 5.1. By Lemma 5.1, it suffices to consider the case $\lambda<n_0\Lambda$. Let $k_0=\lfloor\frac{n_0\Lambda}{\lambda-\Lambda}\rfloor+1$ and $\Lambda_0=\Lambda/k_0$. For $0\le k,\, l\le k_0-1$ we define

$$A_{kl}=\left\{\exists(t,s)\in u^{-2/\alpha}\left([k\Lambda_0,\,(k+1)\Lambda_0]\times[\lambda+l\Lambda_0,\,\lambda+(l+1)\Lambda_0]\right):\ \boldsymbol{X}(t)>u\boldsymbol{b},\,\boldsymbol{X}(s)>u\boldsymbol{b}\right\}$$

and thus in this notation

$$P_{\boldsymbol{b}}(\lambda,\Lambda,u)\le\sum_{k,l=0}^{k_0-1}\mathbb{P}\left\{A_{kl}\right\}.$$

Since $\lambda+(l-k-1)\Lambda_0\ge\lambda-\Lambda\ge n_0\Lambda_0$ Lemma 5.1 implies

$$\mathbb{P}\left\{A_{kl}\right\}=P_{\boldsymbol{b}}(\lambda+(l-k)\Lambda_0,\Lambda_0,u)\le C\mathbb{P}\left\{\boldsymbol{X}(0)>u\boldsymbol{b}\right\}e^{-\frac{(\lambda-\Lambda)^{\alpha}}{16}\xi_{\boldsymbol{w}}}.$$

The claim follows from the fact that $k_0^2\le 4n_0^2\Lambda^2(\lambda-\Lambda)^{-2}$. □

Proof of Theorem 2.1. Let in the following for $\Lambda>0$, $u>0$

$$\Delta_k=[k\Lambda u^{-2/\alpha},(k+1)\Lambda u^{-2/\alpha}],\quad 0\le k\le N_T=\lfloor\frac{T}{\Lambda u^{-2/\alpha}}\rfloor.$$

Since $\boldsymbol{X}$ is stationary, we have

$$\begin{aligned}(N_T+1)&\mathbb{P}\left\{\exists t\in\Delta_0:\ \boldsymbol{X}(t)>u\boldsymbol{b}\right\}\\ \ge\ &\mathbb{P}\left\{\exists t\in[0,T]:\ \boldsymbol{X}(t)>u\boldsymbol{b}\right\}\\ \ge\ &N_T\mathbb{P}\left\{\exists t\in\Delta_0:\ \boldsymbol{X}(t)>u\boldsymbol{b}\right\}-2\sum_{k=1}^{N_T}(N_T-k)P_{\boldsymbol{b}}(k\Lambda,\Lambda,u),\end{aligned} \tag{5.21}$$

where $P_{\boldsymbol{b}}(k\Lambda,\Lambda,u)$ is defined by (5.2). By Lemma 4.7 and the stationarity of $\boldsymbol{X}$, as $u\to\infty$

$$\mathbb{P}\left\{\exists t\in\Delta_k:\ \boldsymbol{X}(t)>u\boldsymbol{b}\right\}\sim H_{\boldsymbol{Y},\boldsymbol{d_w}}([0,\Lambda])\mathbb{P}\left\{\boldsymbol{X}(0)>u\boldsymbol{b}\right\},$$

with $\boldsymbol{Y}$ as defined at the beginning of this section and

$$\boldsymbol{d_w}(t)=S_{\alpha}(t,V_{\boldsymbol{w}})\mathbf{1},\quad V_{\boldsymbol{w}}=\mathrm{diag}(\boldsymbol{w})V\mathrm{diag}(\boldsymbol{w}). \tag{5.22}$$

Note that in our notation $H_{\boldsymbol{Y},\boldsymbol{d_w}}([0,\Lambda])=\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}([0,\Lambda])$, consequently

$$\limsup_{u\to\infty}\frac{\mathbb{P}\left\{\exists t\in[0,T]:\ \boldsymbol{X}(t)>u\boldsymbol{b}\right\}}{Tu^{2/\alpha}\mathbb{P}\left\{\boldsymbol{X}(0)>u\boldsymbol{b}\right\}}\le\frac{1}{\Lambda}\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}([0,\Lambda]). \tag{5.23}$$

The stationarity of $\boldsymbol{X}$ implies that the function $\Lambda\mapsto\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}([0,\Lambda])$ is sub-additive. Therefore, the limit

$$\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}=\lim_{\Lambda\to\infty}\Lambda^{-1}\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}([0,\Lambda])$$

exists and is finite. The sum in (5.21) is bounded by

$$A_1+A_2+A_3:=N_TP_{\boldsymbol{b}}(\Lambda,\Lambda,u)+N_T\sum_{k=2}^{N_{\varepsilon}}P_{\boldsymbol{b}}(k\Lambda,\Lambda,u)+N_T\sum_{k=N_{\varepsilon}+1}^{N_T}P_{\boldsymbol{b}}(k\Lambda,\Lambda,u).$$

In view of Lemma 4.7, Corollary 5.2 and the Piterbarg inequality stated in Lemma 4.5 the negligibility of the double-sum follows with the same arguments as in the 1-dimensional case. Here we spell out the details for readers' convenience.

We first estimate A_3. For $k \ge N_\varepsilon + 1$, the distance between Δ_0 and Δ_k is at least $\varepsilon/2$. Note that the variance matrix of $\boldsymbol{X}(t) + \boldsymbol{X}(s)$ is

$$\Sigma(t,s) = 2\Sigma + \mathcal{R}(t-s) + \mathcal{R}(s-t).$$

In view of Assumption (B1) $(\Sigma_{II}(t,s))^{-1} - (\Sigma_{II})^{-1}$ is strictly positive definite for $t \neq s$, which implies that

$$\tau = \inf_{(t,s)\in\Delta_0\times\Delta_k} \inf_{\boldsymbol{v}_I \ge \boldsymbol{b}_I} \boldsymbol{v}_I^\top (\Sigma(t,s)_{II})^{-1} \boldsymbol{v}_I > \inf_{\boldsymbol{v}_I \ge \boldsymbol{b}_I} \boldsymbol{v}_I^\top (\Sigma_{II})^{-1} \boldsymbol{v}_I = \boldsymbol{b}_I^\top (\Sigma_{II})^{-1} \boldsymbol{b}_I > 0. \tag{5.24}$$

By the Piterbarg inequality stated in Lemma 4.5

$$\begin{aligned} P_{\boldsymbol{b}}(k\Lambda, \Lambda, u) \le & \mathbb{P}\left\{\exists (t,s) \in \Delta_0 \times \Delta_k : \ \boldsymbol{X}_I(t) + \boldsymbol{X}_I(s) > 2u\boldsymbol{b}_I\right\} \\ \le & c_1 \Lambda^2 u^{-2/\alpha} u^{\frac{4}{\gamma}-1} \exp\left(-\frac{u^2}{2}\tau\right). \end{aligned}$$

It follows that

$$A_3 = O\left(\exp\left(-\frac{u^2}{2}(\tau - \delta)\right)\right) \tag{5.25}$$

for some $0 < \delta < \tau - \boldsymbol{b}_I^\top (\Sigma_{II})^{-1} \boldsymbol{b}_I$. For $2 \le k \le N_\varepsilon$, we have from Corollary 5.2 that

$$\begin{aligned} \limsup_{u\to\infty} \frac{A_2}{Tu^{2/\alpha}\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \quad & \le \quad c_2 \Lambda \sum_{k=1}^{N_\varepsilon} (k\Lambda)^{-2} \exp\left(-\frac{k^\alpha \Lambda^\alpha}{16}\xi_{\boldsymbol{w}}\right) \\ & \le \quad c_3 \Lambda^{-1} \exp\left(-\frac{\Lambda^\alpha}{16}\xi_{\boldsymbol{w}}\right). \end{aligned} \tag{5.26}$$

Now we consider A_1. Note that

$$P_{\boldsymbol{b}}(\Lambda, \Lambda, u) \le P_{\boldsymbol{b}}(\Lambda + \sqrt{\Lambda}, \Lambda, u) + \mathbb{P}\left\{\exists t \in u^{-2/\alpha}[0, \sqrt{\Lambda}] : \ \boldsymbol{X}(t) > u\boldsymbol{b}\right\}.$$

Applying Corollary 5.2, Lemma 4.7 and the subadditivity of $\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}([0,\Lambda])$ we obtain

$$\limsup_{u\to\infty} \frac{A_1}{Tu^{2/\alpha}\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \le c_5 \left[\exp\left(-\frac{\Lambda^{\alpha/2}}{16}\xi_{\boldsymbol{w}}\right) + \mathcal{H}_{\alpha,V_{\boldsymbol{w}}}([0,1])\Lambda^{-1/2}\right]. \tag{5.27}$$

Putting all the bounds (5.21)–(5.27) together, for any Λ_1 and $\Lambda_2 > 0$ we obtain that

$$\begin{aligned} \frac{\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}([0,\Lambda_1])}{\Lambda_1} \quad & \ge \quad \limsup_{u\to\infty} \frac{\mathbb{P}\left\{\exists t \in [0,T] : \ \boldsymbol{X}(t) > u\boldsymbol{b}\right\}}{Tu^{2/\alpha}\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \\ & \ge \quad \liminf_{u\to\infty} \frac{\mathbb{P}\left\{\exists t \in [0,T] : \ \boldsymbol{X}(t) > u\boldsymbol{b}\right\}}{Tu^{2/\alpha}\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \\ & \ge \quad \frac{\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}([0,\Lambda_2])}{\Lambda_2} - c_6 \Lambda_2^{-1} \exp\left(-\frac{\Lambda_2^\alpha}{16}\xi_{\boldsymbol{w}}\right) \\ & \quad\ -c_7 \exp\left(-\frac{\Lambda_2^{\alpha/2}}{16}\xi_{\boldsymbol{w}}\right) - c_8 \mathcal{H}_{\alpha,V_{\boldsymbol{w}}}([0,1])\Lambda_2^{-1/2}. \end{aligned} \tag{5.28}$$

Consequently, the constant $\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}$ is positive. This and (5.28) establish the proof when T does not depend on u. The case T dependent on u follows with analogous calculations. □

5.2. Proof of Theorem 2.2.

Lemma 5.3. *For every $\boldsymbol{v} \in \mathbb{R}^d$ such that $a = \mathbf{1}^\top \boldsymbol{v} \ge 0$ and $\Lambda \ge 0$ we have*

$$\int_{\mathbb{R}^d} e^{\mathbf{1}^\top \boldsymbol{x}} \mathbf{1}_{\{\exists t \in [0,\Lambda]: \ \boldsymbol{x} < -t\boldsymbol{v}\}} d\boldsymbol{x} = \begin{cases} 1 + \Lambda \sum_{i=1}^d v_i^-, & \text{if } a = 0, \\ 1 + \frac{1-e^{-a\Lambda}}{a} \sum_{i=1}^d v_i^- & \text{if } a > 0, \end{cases}$$

where $x^- = \max(0, -x)$, $x \in \mathbb{R}$.

Proof of Lemma 5.3. Set $b = \sum_{i=1}^d v_i^-$ and

$$F = \{1 \le i \le d : \ v_i > 0\}, \quad \bar{F} = \{1, \ldots, d\} \setminus F.$$

Define $\Omega_t = \{\boldsymbol{x} < -t\boldsymbol{v}\}$ and let $\Omega = \bigcup_{0 \le t \le \Lambda} \Omega_t$. For a Borel set $A \subset \mathbb{R}^d$, let $\mathcal{I}(A) = \int_A e^{\mathbf{1}^\top \boldsymbol{x}} d\boldsymbol{x}$. Then for every $t \ge 0$ we have $\mathcal{I}(\Omega_t) = e^{-at}$. Note that for $t < s$, $\Omega_s \cap \Omega_t = \{\boldsymbol{x}_F < -s\boldsymbol{v}_F, \ \boldsymbol{x}_{\bar{F}} < -t\boldsymbol{v}_{\bar{F}}\}$. Since $\mathcal{I}(\Omega_s \cap \Omega_t) = e^{-(a+b)s+bt}$,

$$\mathcal{I}(\Omega_s \setminus \Omega_t) = e^{-at-(a+b)(s-t)}\left(e^{bt} - 1\right).$$

On the other hand, $\bigcup_{t<u<s} \Omega_u \subset \{\boldsymbol{x}_F < -t\boldsymbol{v}_F,\ \boldsymbol{x}_{\bar{F}} < -s\boldsymbol{v}_{\bar{F}}\}$ and hence

$$\mathcal{I}\left(\bigcup_{t<u<s} \Omega_u \setminus \Omega_t\right) \le e^{-at}\left(e^{b(s-t)}-1\right).$$

Let $t_k = k\Lambda/n$. Since

$$\bigcup_{0\le k\le n-1} (\Omega_{t_{k+1}} \setminus \Omega_{t_k}) \subset \Omega \setminus \Omega_0 \subset \bigcup_{0\le k\le n-1}\left[\left(\bigcup_{t_{k+1}\le u\le t_k} \Omega_u\right)\setminus \Omega_{t_k}\right]$$

we have

$$\sum_{k=0}^{n-1} e^{-at_k - \frac{\Lambda(a+b)}{n}}\left(e^{\frac{\Lambda b}{n}}-1\right) \le \mathcal{I}(\Omega) - 1 \le \sum_{k=0}^{n-1} e^{-at_k}\left(e^{\frac{\Lambda b}{n}}-1\right).$$

Letting $n\to\infty$ we complete the proof. □

Proof of Theorem 2.2. We follow the same idea as in the proof of Theorem 2.1 and only spell out the necessary changes.

For $\Lambda > 0$, from Lemma 4.7 and the fact that $V^\top = -V$ we obtain

$$\lim_{u\to\infty} \frac{\mathbb{P}\left\{\exists t\in[0,\Lambda u^{-2}]:\ \boldsymbol{X}(t) > u\boldsymbol{b}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} = H_{0,\boldsymbol{d_w}}([0,\Lambda]),$$

where $\boldsymbol{d_w}(t) = t\mathrm{diag}(\boldsymbol{w})V\boldsymbol{w}$ and the constant $H_{0,\boldsymbol{d_w}}([0,\Lambda])$ is given by (4.21). By Lemma 5.3, we have that

$$\lim_{\Lambda\to\infty} \frac{H_{0,\boldsymbol{d_w}}([0,\Lambda])}{\Lambda} = \frac{\sum_{i\in I} w_i\,|(V\boldsymbol{w})_i|}{2} > 0. \tag{5.29}$$

Let below

$$M = \{i\in I:\ (V\boldsymbol{w})_i > 0\}, \quad \overline{M} = I\setminus M.$$

By the assumption $(V\boldsymbol{w})_I \ne \boldsymbol{0}_I$, both M and $\overline{M}$ are non-empty. Using that (5.21) also holds under the conditions of Theorem 2.2, now we analyze the sum in (5.21). Without loss of generality, we assume that $I = \{1,\ldots,d\}$. Define

$$\Delta_k = [0,\ \Lambda]\times[k\Lambda,\ (k+1)\Lambda], \quad 0 < k \le N_T = \frac{Tu^2}{\Lambda}.$$

We first estimate the sum in (5.21) for large k. By (2.8), for any $\delta > 0$, there exists $\varepsilon > 0$ such that $\|\Sigma - R(t) - tV\|_{\mathrm{F}} \le \delta\lambda$ for $|t| < \varepsilon$. Assume that $(k+1)\Lambda \le \varepsilon u^2$. Let $\boldsymbol{\chi}_u(t)$ be the conditioned process $u(\boldsymbol{X}(u^{-2}t) - u\boldsymbol{b}) + \boldsymbol{x}$ given $\boldsymbol{X}(0) = u\boldsymbol{b} - u^{-2}\boldsymbol{x}$. Then

$$\begin{aligned}&\mathbb{P}\left\{\exists(t,s)\in u^{-2}\Delta_k:\ \boldsymbol{X}(t) > u\boldsymbol{b},\ \boldsymbol{X}(s) > u\boldsymbol{b}\right\}\\ \le&\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}\int_{\mathbb{R}^d} e^{\boldsymbol{w}^\top\boldsymbol{x}}\mathbb{P}\left\{\exists(t,s)\in\Delta_k:\ \boldsymbol{\chi}_u(t) > \boldsymbol{x},\ \boldsymbol{\chi}_u(s) > \boldsymbol{x}\right\}d\boldsymbol{x}.\end{aligned}$$

Setting $\boldsymbol{d}_u(t) = -\mathbb{E}\left\{\boldsymbol{\chi}_u(t)\right\}$ and $\boldsymbol{d}(t) = tV\boldsymbol{w}$ we have

$$\left|\boldsymbol{d}_u(t) - \boldsymbol{d}(t) - O(\lambda u^{-2})\boldsymbol{x}\right| \le c_1\delta\lambda$$

and

$$\left\|\mathbb{E}\left\{(\boldsymbol{\chi}_u(t) + \boldsymbol{d}_u(t))(\boldsymbol{\chi}_u(s) + \boldsymbol{d}_u(s))^\top\right\}\right\|_{\mathrm{F}} \le c_1\delta\lambda$$

for some $c_1 > 0$. Consequently, for all $(t,s)\in\Delta_k$

$$\begin{aligned}\boldsymbol{d}_{u,M}(s) &\ge \lambda(V\boldsymbol{w})_M - c_1\delta\lambda - O(\lambda u^{-2})\boldsymbol{x}_M,\\ \boldsymbol{d}_{u,\overline{M}}(t) &\ge \Lambda(V\boldsymbol{w})_{\overline{M}} - c_1\delta\lambda - O(\lambda u^{-2})\boldsymbol{x}_{\overline{M}}.\end{aligned}$$

By the change of variables $\boldsymbol{y}_M = \boldsymbol{x}_M - \lambda(V\boldsymbol{w})_M + c_1\delta\lambda$ and $\boldsymbol{y}_{\overline{M}} = \boldsymbol{x}_{\overline{M}} - \Lambda(V\boldsymbol{w})_{\overline{M}} + c_1\delta\lambda$ and the assumption $\boldsymbol{w}^\top V\boldsymbol{w} = 0$, we have that

$$\begin{aligned}&\int_{\mathbb{R}^d} e^{\boldsymbol{w}^\top\boldsymbol{x}}\mathbb{P}\left\{\exists(t,s)\in\Delta_k:\ \boldsymbol{\chi}_u(t) > \boldsymbol{x},\ \boldsymbol{\chi}_u(s) > \boldsymbol{x}\right\}d\boldsymbol{x}\\ \le&\int_{\mathbb{R}^d} d\boldsymbol{x}\, e^{\boldsymbol{w}^\top\boldsymbol{x}}\,\mathbb{P}\Big\{\exists(t,s)\in\Delta_k:\ \boldsymbol{\chi}_{u,F}(s) > \boldsymbol{x}_M - \lambda(V\boldsymbol{w})_M + c_1\delta\lambda + O(\lambda u^{-2})\boldsymbol{x}_M,\\ &\qquad\qquad \boldsymbol{\chi}_{u,\overline{M}}(t) > \boldsymbol{x}_{\overline{M}} - \Lambda(V\boldsymbol{w})_{\overline{M}} + c_1\delta\lambda + O(\lambda u^{-2})\boldsymbol{x}_{\overline{M}}\Big\}\\ \le&e^{-\boldsymbol{w}_M^\top(V\boldsymbol{w})_M(\lambda-\Lambda)+c_2\delta\lambda}\int_{\mathbb{R}^d} e^{(\boldsymbol{w}+O(\lambda u^{-2}))^\top\boldsymbol{y}}g_u(\boldsymbol{y})d\boldsymbol{y},\end{aligned}$$

where

$$g_u(\boldsymbol{y}) = \mathbb{P}\left\{\exists (t,s) \in [0,\Lambda]\times[0,\Lambda] : \ \boldsymbol{Z}_{u,M}(s) > \boldsymbol{y}_F, \ \boldsymbol{Z}_{u,\overline{M}}(t) > \boldsymbol{y}_{\overline{M}}\right\}$$

and

$$\boldsymbol{Z}_{u,M}(t) = \boldsymbol{\chi}_{u,M}(t+k\Lambda) - \boldsymbol{d}_{u,M}(t+k\Lambda), \quad \boldsymbol{Z}_{u,\overline{M}}(t) = \boldsymbol{\chi}_{u,\overline{M}}(t) - \boldsymbol{d}_{u,\overline{M}}(t).$$

For $F \subset \{1,\ldots,d\}$ with $m = |F|$ let

$$\Omega_F = \{\boldsymbol{y} \in \mathbb{R}^d : \ y_i > 0, \ y_j < 0, \ i \in F, \ j \notin F\}$$

and set

$$\eta(t_1,\ldots,t_m) = \sum_{i\in F} \boldsymbol{Z}_{u,i}(t_i).$$

Note that

$$\sup_{t_i\in[0,\Lambda], i\le m} \mathbb{E}\left\{\eta(t_1,\ldots,t_m)^2\right\} \le c_3\delta\lambda.$$

Applying Borell-TIS inequality we obtain

$$\int_{\mathbb{R}^d} e^{(\boldsymbol{w}+O(\lambda u^{-2}))^\top \boldsymbol{y}} g_u(\boldsymbol{y}) d\boldsymbol{y} \le e^{c_4\delta\lambda}.$$

Therefore, there exists $n_0 > 0$ such that for all $k \ge n_0$ and $(k+1)\Lambda \le \varepsilon u^2$

$$\mathbb{P}\left\{\exists (t,s) \in u^{-2}\Delta_k : \ \boldsymbol{X}(t) > u\boldsymbol{b}, \ \boldsymbol{X}(s) > u\boldsymbol{b}\right\} \le \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\} e^{-\frac{\boldsymbol{w}_M^\top (V\boldsymbol{w})_M}{2}\lambda}.$$

It remains to estimate the sum in (5.21) for $1 \le k \le n_0$. We have from Lemma 4.7 that

$$\begin{aligned}
&\limsup_{u\to\infty} \frac{\mathbb{P}\left\{\exists (t,s) \in \Delta_k : \ \boldsymbol{X}(t) > u\boldsymbol{b}, \ \boldsymbol{X}(s) > u\boldsymbol{b}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \\
&\le \int_{\mathbb{R}^d} e^{\boldsymbol{w}^\top \boldsymbol{x}} \mathbb{P}\left\{\exists (t,s) \in \Delta_k : \ -\boldsymbol{d}(t) > \boldsymbol{x}, \ -\boldsymbol{d}(s) > \boldsymbol{x}\right\} d\boldsymbol{x} \\
&\le \int_{\mathbb{R}^d} e^{\boldsymbol{w}^\top \boldsymbol{x}} \mathbf{1}_{\{\boldsymbol{x}_M < -\lambda(V\boldsymbol{w})_M, \ \boldsymbol{x}_{\overline{M}} < -\Lambda(V\boldsymbol{w})_{\overline{M}}\}} d\boldsymbol{x} \\
&\le c_5 e^{-\boldsymbol{w}_M^\top (V\boldsymbol{w})_M(\lambda-\Lambda)},
\end{aligned}$$

where $\boldsymbol{d}(t) = tV\boldsymbol{w}$.

From the above we conclude the negligibility of the sum in (5.21). The rest of the proof follows by the same arguments as given in the proof of Theorem 2.1. □

5.3. **Proof of Theorem 2.4.** We present first several supporting lemmas and then continue with the proof of Theorem 2.4.

For the next two lemmas we impose the assumptions of Theorem 2.4. Set below $\delta_u = u^{-2/\beta}\log^{2/\beta} u$ and recall that in view of (2.12)

$$\tau_{\boldsymbol{w}} = \boldsymbol{w}^\top \Xi A^\top \boldsymbol{w} = \boldsymbol{w}_I^\top (\Xi A^\top)_{II} \boldsymbol{w}_I > 0.$$

Lemma 5.4. *There exist positive constants C, u_0 and Λ_0 such that for $\Lambda \ge \Lambda_0$ and $u \ge u_0$*

$$\mathbb{P}\left\{\exists t \in [\Lambda u^{-\frac{2}{\beta}}, \ \delta_u] : \ \boldsymbol{X}(t) > u\boldsymbol{b}\right\} \le C \exp\left(-\tau_{\boldsymbol{w}}\Lambda^\beta\right) \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}. \tag{5.30}$$

Proof of Lemma 5.4: Similarly as in the proof of Lemma 5.1, without loss of generality, we may assume that $I = \{1,\ldots,d\}$. Letting $\nu = \min(\alpha,\beta)$ and $\theta_u = \Lambda u^{-2/\nu}$. For $1 \le k \le N_u = \lceil \delta_u/\theta_u \rceil$ we define $\boldsymbol{X}_{u,k}(t) = \boldsymbol{X}(k\theta_u + tu^{-2/\nu})$. Then

$$R_{u,k}(t,s) = \mathbb{E}\left\{\boldsymbol{X}_{u,k}(t)\boldsymbol{X}_{u,k}(s)^\top\right\} = R(k\theta_u + tu^{-2/\nu}, k\theta_u + su^{-2/\nu}),$$

where $R(t,s)$ is the cmf of $\boldsymbol{X}$. Setting next $\Sigma_{u,k} = \mathbb{E}\left\{\boldsymbol{X}_{u,k}(0)\boldsymbol{X}_{u,k}(0)^\top\right\} = \Sigma(k\theta_u)$ we have (see also the proof of Lemma 4.7)

$$\begin{aligned}
&\mathbb{P}\left\{\exists t \in [k\theta_u, (k+1)\theta_u] : \ \boldsymbol{X}(t) > u\boldsymbol{b}\right\} \\
&\le u^{-d}\varphi_{\Sigma_{u,k}}(u\boldsymbol{b}) \int_{\mathbb{R}^d} e^{\boldsymbol{b}^\top \Sigma_{u,k}^{-1}\boldsymbol{x}} \mathbb{P}\left\{\exists t \in [0,\Lambda] : \ \boldsymbol{\chi}_{u,k}(t) > \boldsymbol{x}\right\} d\boldsymbol{x},
\end{aligned}$$

where $\boldsymbol{\chi}_{u,k}(t)$ is the conditional process $u\left(\boldsymbol{X}_{u,k}(t) - u\boldsymbol{b}\right) + \boldsymbol{x}$ given $\boldsymbol{X}_{u,k}(0) = u\boldsymbol{b} - u^{-1}\boldsymbol{x}$. Note that

$$\boldsymbol{d}_{u,k}(t) = -\mathbb{E}\left\{\boldsymbol{\chi}_{u,k}(t)\right\} = u^2\left[\Sigma_{u,k} - R_{u,k}(t,0)\right]\Sigma_{u,k}^{-1}\boldsymbol{b} + \left[\Sigma_{u,k} - R_{u,k}(t,0)\right]\Sigma_{u,k}^{-1}\boldsymbol{x}$$

and

$$\begin{aligned}K_{u,k}(t,s)=&\mathbb{E}\left\{\left[\boldsymbol{\chi}_{u,k}(t)-\boldsymbol{d}_{u,k}(t)\right]\left[\boldsymbol{\chi}_{u,k}(s)-\boldsymbol{d}_{u,k}(t)\right]^\top\right\}\\ =&u^2\left[R_{u,k}(t,s)-R_{u,k}(t,0)\Sigma_{u,k}^{-1}R_{u,k}(0,s)\right].\end{aligned}$$

By Assumptions (D2) and (D3) with $R_{\alpha,V}$ defined in (2.2)

$$K_{u,k}(t,s)\to\begin{cases}R_{\alpha,V}(t,s), & \text{if } \beta\ge\alpha\\ 0, & \text{if } \beta<\alpha\end{cases}\tag{5.31}$$

as $u\to\infty$, where the convergence is uniform in $(t,s)\in[0,\Lambda]\times[0,\Lambda]$ and $1\le k\le N_u$. By Assumptions (D2) and (D3) again, for every $\varepsilon>0$, there is u_0 such that for $u\ge u_0$

$$\left|\boldsymbol{d}_{u,k}(t)-\boldsymbol{d}(t)-\left[\Sigma_{u,k}-R_{u,k}(t,0)\right]\Sigma_{u,k}^{-1}\boldsymbol{x}\right|\le\varepsilon u_* k^\beta\Lambda^\beta,\tag{5.32}$$

with $u_*=\min(1,u^{2-\frac{2\beta}{\alpha}})$ and

$$\boldsymbol{d}(t)=\begin{cases}|t|^\alpha V\boldsymbol{w}, & \text{if } \beta>\alpha,\\ \left[(k\Lambda+t)^\alpha-(k\Lambda)^\alpha\right]\Xi A^\top\boldsymbol{w}+|t|^\alpha V\boldsymbol{w}, & \text{if } \beta=\alpha,\\ \left[(k\Lambda+t)^\beta-(k\Lambda)^\beta\right]\Xi A^\top\boldsymbol{w}, & \text{if } \beta<\alpha.\end{cases}$$

In the above derivation we used (5.16) with $h=\beta$ for the case $\beta>\alpha$. Consequently,

$$\lim_{u\to\infty}\max_{1\le k\le N_u}\left|u^{2-2\beta/\alpha}\left((k\Lambda+t)^\beta-(k\Lambda)^\beta\right)\right|=0.$$

As in the proof of Lemma 5.1, we define for $F\subset\{1,\dots,d\}$

$$\Omega_F=\{\boldsymbol{x}\in\mathbb{R}^d:\ x_i>0,\ x_j<0,\ i\in F\text{ and } j\notin F\}.$$

Applying (5.16) we have that

$$\sup_{t\in[0,\Lambda]}\left|\boldsymbol{w}_F^\top\boldsymbol{d}_F(t)\right|\le\begin{cases}c_1\Lambda^\alpha, & \text{if } \beta>\alpha,\\ c_1\left(1+k^{\beta-1}\right)\Lambda^\beta, & \text{if } \beta\le\alpha.\end{cases}$$

It follows from (5.32) that for every $1\le k\le N_u$ and all u large enough

$$\sup_{t\in[0,\Lambda]}\left|\boldsymbol{w}_F^\top\boldsymbol{d}_{u,k,F}(t)\right|\le\frac{1}{2}\boldsymbol{w}_F^\top\boldsymbol{x}_F+c(k,\Lambda),\tag{5.33}$$

with

$$c(k,\Lambda)=\begin{cases}c_1\Lambda^\alpha+\varepsilon u_* k^\beta\Lambda^\beta, & \text{if } \beta>\alpha,\\ c_1\left(1+k^{\beta-1}\right)\Lambda^\beta+\varepsilon k^\beta\Lambda^\beta, & \text{if } \beta\le\alpha.\end{cases}$$

Setting $\eta_{u,k}(t)=\boldsymbol{w}_F^\top\left(\boldsymbol{\chi}_{u,k}(t)+\boldsymbol{d}_{u,k}(t)\right)$ for every $\boldsymbol{x}\in\Omega_F$ we have

$$\mathbb{P}\left\{\exists t\in[0,\Lambda]:\ \boldsymbol{\chi}_{u,k}(t)>\boldsymbol{x}\right\}\le\mathbb{P}\left\{\sup_{t\in[0,\Lambda]}\eta_{u,k}(t)>\frac{1}{2}\boldsymbol{w}_F^\top\boldsymbol{x}_F-c(k,\Lambda)\right\}.$$

By (5.31), the variance of $\eta_{u,k}(t)$, $0\le t\le\Lambda$ is bounded uniformly (with respect to $k\le N_u$) by $\sigma^2=c_2\Lambda^\alpha$ for $\beta\ge\alpha$ and $\sigma^2=c_2$ for $\beta<\alpha$. Consequently, Piterbarg inequality implies

$$\mathbb{P}\left\{\exists t\in[0,\Lambda]:\ \boldsymbol{\chi}_{u,k}(t)>\boldsymbol{x}\right\}\le c_3\left(\frac{\boldsymbol{w}_F^\top\boldsymbol{x}_F-2c(k,\Lambda)}{\sigma}\right)^{2/\gamma}\exp\left(-\frac{\left(\boldsymbol{w}_F^\top\boldsymbol{x}_F-2c(k,\Lambda)\right)^2}{8\sigma^2}\right).$$

Similarly to the derivation of (5.19) in the proof of Lemma 5.1 it follows that

$$\int_{\mathbb{R}^d}e^{\boldsymbol{b}^\top\Sigma_{u,k}^{-1}\boldsymbol{x}}\mathbb{P}\left\{\exists t\in[0,\Lambda]:\ \boldsymbol{\chi}_{u,k}(t)>\boldsymbol{x}\right\}d\boldsymbol{x}\le c_4e^{c_5\left(\sigma^2+c(k,\Lambda)\right)}\le c_6e^{c_5c(k,\Lambda)}.$$

Assumption (D2) implies further

$$u^2\boldsymbol{b}^\top\left(\Sigma_{u,k}^{-1}-\Sigma^{-1}\right)\boldsymbol{b}=u^2\boldsymbol{b}^\top\Sigma_{u,k}^{-1}\left(\Sigma-\Sigma_{u,k}\right)\Sigma^{-1}\boldsymbol{b}=2\tau_{\boldsymbol{w}}u_*k^\beta\Lambda^\beta+o\left(u_*k^\beta\Lambda^\beta\right).\tag{5.34}$$

It follows that

$$\frac{\mathbb{P}\left\{\exists t\in[k\theta_u,(k+1)\theta_u]:\ \boldsymbol{X}(t)>u\boldsymbol{b}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(0)>u\boldsymbol{b}\right\}}\le c_7\exp\left(-\frac{3}{2}\tau_{\boldsymbol{w}}u_*k^\beta\Lambda^\beta+c(k,\Lambda)\right)\tag{5.35}$$

for u large enough. If $\beta > \alpha$, then the left-hand side of (5.30) is at most

$$\sum_{k=\lfloor u^{\frac{2}{\alpha}-\frac{2}{\beta}}\rfloor}^{N_u} \mathbb{P}\left\{\exists t \in [k\theta_u,(k+1)\theta_u]:\ \boldsymbol{X}(t) > u\boldsymbol{b}\right\} \le c_8 \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\} \exp\left(-\frac{3}{2}\tau_{\boldsymbol{w}}\Lambda^{\beta} + c_1\Lambda^{\alpha}\right),$$

hence the thesis of the lemma follows by taking $\Lambda \ge \Lambda_0$ with $\Lambda_0^{\beta-\alpha}\tau_{\boldsymbol{w}} > 2c_1$.
Now assume that $\beta \le \alpha$. Choose k_0 so that $c_1(k_0^{-\beta} + k_0^{-1}) < \tau_{\boldsymbol{w}}/2$. By (5.35), we have for $k > k_0$

$$\mathbb{P}\left\{\exists t \in [k\theta_u,(k+1)\theta_u]: \boldsymbol{X}(t) > u\boldsymbol{b}\right\} \le c_8 \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\} \exp\left(-\tau_{\boldsymbol{w}} k^{\beta}\Lambda^{\beta}\right). \tag{5.36}$$

It remains to consider the case $1 \le k \le k_0$. Set $\widetilde{\Lambda} = \Lambda/k_0$ and note that our choice of k_0 is independent of Λ. By (5.36) we have that

$$\begin{aligned}\mathbb{P}\left\{\exists t \in [k\theta_u,(k+1)\theta_u]:\ \boldsymbol{X}(t) > u\boldsymbol{b}\right\} &\le \sum_{j=k_0 k}^{k_0(k+1)-1} \mathbb{P}\left\{\exists t \in [j\widetilde{\Lambda}u^{-2/\nu},(j+1)\widetilde{\Lambda}u^{-2/\nu}]:\ \boldsymbol{X}(t) > u\boldsymbol{b}\right\}\\ &\le \sum_{j=k_0 k}^{k_0(k+1)-1} c_8 \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\} \exp\left(-\frac{1}{2}\tau_{\boldsymbol{w}} u_* j^{\beta}\widetilde{\Lambda}^{\beta}\right)\\ &\le c_9 \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\} \exp\left(-\frac{1}{2}\tau_{\boldsymbol{w}} u_* k^{\beta}\Lambda^{\beta}\right),\end{aligned}$$

which together with (5.36) completes the proof. □

Corollary 5.5. *If $\alpha, V, \boldsymbol{w}$ and $W = \Xi A^{\top}$ are as in Theorem 2.4, then $\mathcal{P}_{\alpha,V_{\boldsymbol{w}},W_{\boldsymbol{w}}} \in (0,\infty)$.*

Proof of Corollary 5.5: First, we note that with $\boldsymbol{d}_{\boldsymbol{w}}(t) = \mathrm{diag}(\boldsymbol{w})V(t)\boldsymbol{w}$ and $\boldsymbol{Y}$ as in Theorem 2.4, we have $\mathcal{P}_{\alpha,V_{\boldsymbol{w}},W_{\boldsymbol{w}}} = \lim_{\Lambda\to\infty} H_{\boldsymbol{Y},\boldsymbol{d}_{\boldsymbol{w}}+\boldsymbol{f}_{\boldsymbol{w}}}([0,\Lambda])$, where

$$V_{\boldsymbol{w}} = \mathrm{diag}(\boldsymbol{w})V\mathrm{diag}(\boldsymbol{w}), \quad W_{\boldsymbol{w}} = \mathrm{diag}(\boldsymbol{w})\Xi A^{\top}\mathrm{diag}(\boldsymbol{w}), \boldsymbol{f}_{\boldsymbol{w}}(t) = |t|^{\alpha} W_{\boldsymbol{w}}\mathbf{1}. \tag{5.37}$$

Since $H_{\boldsymbol{Y},\boldsymbol{d}_{\boldsymbol{w}}+\boldsymbol{f}_{\boldsymbol{w}}}([0,\Lambda])$ is increasing in Λ, it suffices to prove that it is uniformly bounded.
Fix $0 < \Lambda_0 < \Lambda$. By Lemma 5.4 for two positive constants c, C

$$\limsup_{u\to\infty} \frac{\mathbb{P}\left\{\exists t \in [\Lambda_0 u^{-2/\alpha}, \Lambda u^{-2/\alpha}]:\ \boldsymbol{X}(t) > u\boldsymbol{b}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \le Ce^{-c\Lambda_0^{\alpha}}.$$

It follows that

$$\begin{aligned}H_{\boldsymbol{Y},\boldsymbol{d}_{\boldsymbol{w}}+\boldsymbol{f}_{\boldsymbol{w}}}([0,\Lambda]) &= \lim_{u\to\infty} \frac{\mathbb{P}\left\{\exists t \in [0,\Lambda u^{-2/\alpha}]:\ \boldsymbol{X}(t) > u\boldsymbol{b}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}}\\ &\le \lim_{u\to\infty} \frac{\mathbb{P}\left\{\exists t \in [0,\Lambda_0 u^{-2/\alpha}]:\ \boldsymbol{X}(t) > u\boldsymbol{b}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} + Ce^{-c\Lambda_0^{\alpha}}\\ &\le H_{\boldsymbol{Y},\boldsymbol{d}_{\boldsymbol{w}}+\boldsymbol{f}_{\boldsymbol{w}}}([0,\Lambda_0]) + Ce^{-c\Lambda_0^{\alpha}},\end{aligned}$$

which completes the proof. □
Set in the following $\Delta(\tau,\lambda,\Lambda) = [\tau,\tau+\Lambda]\times[\lambda,\lambda+\Lambda]$ and

$$P_{\boldsymbol{b}}(\tau,\lambda,\Lambda,u) = \mathbb{P}\left\{\exists (t,s) \in u^{-2/\alpha}\Delta(\tau,\lambda,\Lambda):\ \boldsymbol{X}(t) > u\boldsymbol{b},\ \boldsymbol{X}(s) > u\boldsymbol{b}\right\}.$$

Lemma 5.6. *If $\beta > \alpha$ and $\xi_{\boldsymbol{w}} = \boldsymbol{w}^{\top}V\boldsymbol{w} > 0$, then for every $0 < \tau + \Lambda < \lambda \le N_u$ with u large enough*

$$P_{\boldsymbol{b}}(\tau,\lambda,\Lambda,u) \le C_1\Lambda^2 \exp\left(-C_2\left(\lambda-\tau-\Lambda\right)^{\alpha}\right) u^{-d}\varphi_{\Sigma_{u,\tau,\lambda}}(u\boldsymbol{b}),$$

where C_1, C_2 are two positive constants and

$$\Sigma_{u,\tau,\lambda} = \frac{1}{4}\mathbb{E}\left\{(\boldsymbol{X}(u^{-\alpha/2}\tau) + \boldsymbol{X}(u^{-\alpha/2}\lambda))(\boldsymbol{X}(u^{-\alpha/2}\tau) + \boldsymbol{X}(u^{-\alpha/2}\lambda))^{\top}\right\}. \tag{5.38}$$

Proof of Lemma 5.6: The proof is similar to that of Lemma 5.1 and we only sketch the main ideas. We shall assume for simplicity that $|I| = d$. Set $\boldsymbol{X}_u(t,s) = \frac{1}{2}\left(\boldsymbol{X}(u^{-\alpha/2}t) + \boldsymbol{X}(u^{-\alpha/2}s)\right)$ and define

$$P(\tau,\lambda,\Lambda) = \mathbb{P}\left\{\exists (t,s) \in u^{-\alpha/2}\Delta(\tau,\lambda,\Lambda):\ \boldsymbol{X}(t) + \boldsymbol{X}(s) > 2u\boldsymbol{b}\right\}.$$

For any $u > 0$

$$\frac{u^d P(\tau,\lambda,\Lambda)}{\varphi_{\Sigma_{u,\tau,\lambda}}(u\boldsymbol{b})} \le \int_{\mathbb{R}^d} e^{\boldsymbol{b}^{\top}\Sigma_{u,\tau,\lambda}^{-1}\boldsymbol{x}} \mathbb{P}\left\{\exists (t,s) \in \Delta(\tau,\lambda,\Lambda):\ \boldsymbol{\chi}_{u,\tau,\lambda}(t,s) > \boldsymbol{x}\right\} d\boldsymbol{x},$$

where $\boldsymbol{\chi}_{u,\tau,\lambda}(t,s)$ is the conditioned process $u\left(\boldsymbol{X}_u(t,s)-u\boldsymbol{b}\right)+\boldsymbol{x}$ given $\boldsymbol{X}_u(\tau,\lambda)=u\boldsymbol{b}-u^{-1}\boldsymbol{x}$. Define $\boldsymbol{d}_{u,\tau,\lambda}(t)=-\mathbb{E}\left\{\boldsymbol{\chi}_{u,\tau,\lambda}(t)\right\}$ and let $R_u(t,s;t_1,s_1)$ be the cmf of $\boldsymbol{\chi}_{u,\tau,\lambda}$. Set further

$$d(t,s)=\frac{1}{4}\left[(t-\tau)^\alpha+(s-\tau)^\alpha+(\lambda-t)^\alpha+(s-\lambda)^\alpha-2(\lambda-\tau)^\alpha\right]$$

and

$$\begin{aligned}r(t,s;t_1,s_1)=&(t-\tau)^\alpha+(s-\tau)^\alpha+(t_1-\tau)^\alpha\\&+(s_1-\tau)^\alpha+(\lambda-t)^\alpha+(\lambda-t_1)^\alpha+(s-\lambda)^\alpha+(s_1-\lambda)^\alpha\\&-|t-t_1|^\alpha-|s-s_1|^\alpha-(s_1-t)^\alpha-(s-t_1)^\alpha-2(\lambda-\tau)^\alpha.\end{aligned}$$

By Assumptions (D2) and (D3), we have that for every $\varepsilon>0$ and $(t,s)\in\Delta(\tau,\lambda,\Lambda)$

$$\left|\boldsymbol{d}_{u,\tau,\lambda}(t,s)-d(t,s)V\boldsymbol{w}-\left[\Sigma_{u,\tau,\lambda}-R_u(t,s;\tau,\lambda)\right]\Sigma_{u,\tau,\lambda}^{-1}\boldsymbol{x}\right|\le\varepsilon\,(\lambda-\tau)^\alpha \tag{5.39}$$

and

$$\left\|R_u(t,s;t_1,s_1)-\frac{1}{4}r(t,s;t_1,s_1)V\right\|_{\mathrm{F}}\le\varepsilon\,(\lambda-\tau)^\alpha\,, \tag{5.40}$$

provided u is large enough. Now, by the same argument as in the proof of Lemma 5.1, there exist positive constants c_1 and n_0 such that for every $\lambda-\tau\ge n_0\Lambda>0$ with $\lambda\le N_u$ and u large enough

$$P(\tau,\lambda,\Lambda)\le C_1u^{-d}\varphi_{\Sigma_{u,\tau,\lambda}}(u\boldsymbol{b})\exp\left(-\xi_{\boldsymbol{w}}^2\frac{(\lambda-\tau)^\alpha}{16}\right). \tag{5.41}$$

It remains to consider the case $\lambda-\tau\le n_0\Lambda$. Set $\boldsymbol{X}_{u,\tau}(t)=\boldsymbol{X}(u^{-\alpha/2}(\tau+t))$ with cmf $R_{u,\tau}(t,s)=R(u^{-\alpha/2}(\tau+t),u^{-\alpha/2}(\tau+s))$ and define $\Sigma_{u,\tau}=R_{u,\tau}(0,0)$. Note that

$$\lim_{u\to\infty}u^2\left[\Sigma_{u,\tau}-R_{u,\tau}(t,0)\right]=|t|^\alpha V \tag{5.42}$$

and

$$\lim_{u\to\infty}u^2\left[R_{u,\tau}(t,s)-R_{u,\tau}(t,0)\Sigma_{u,\tau}^{-1}R_{u,\tau}(0,s)\right]=R_{\alpha,V}(t,s)$$

uniformly in $\lambda-\tau\le n_0\Lambda$. Recall that we defined $\boldsymbol{Y}$ as a centered $\mathbb{R}^d$-valued Gaussian process with cmf $\mathrm{diag}(\boldsymbol{w})R_{\alpha,V}\mathrm{diag}(\boldsymbol{w})$ and

$$\boldsymbol{d}_{\boldsymbol{w}}(t)=|t|^\alpha\mathrm{diag}(\boldsymbol{w})V\boldsymbol{w}=V_{\boldsymbol{w}}\boldsymbol{1},\quad V_{\boldsymbol{w}}=\mathrm{diag}(\boldsymbol{w})V\mathrm{diag}(\boldsymbol{w}).$$

Analogous to Lemma 4.7

$$P_{\boldsymbol{b}}(\tau,\lambda,\Lambda,u)\quad\sim\quad H(\lambda-\tau,\Lambda)u^{-d}\varphi_{\Sigma_{u,\tau,\lambda}}(u\widetilde{\boldsymbol{b}})$$

as $u\to\infty$, where (set G=$[0,\Lambda]\times[\lambda-\tau,\lambda-\tau+\Lambda]$)

$$H(\lambda-\tau,\Lambda)=\int_{\mathbb{R}^d}e^{\boldsymbol{1}^\top\boldsymbol{x}}\mathbb{P}\left\{\exists(t,s)\in G:\ \boldsymbol{Y}(t)-\boldsymbol{d}_{\boldsymbol{w}}(t)>\boldsymbol{x},\boldsymbol{Y}(s)-\boldsymbol{d}_{\boldsymbol{w}}(s)>\boldsymbol{x}\right\}d\boldsymbol{x},$$

with $\boldsymbol{d}_{\boldsymbol{w}}(t)=S_\alpha(t,V_{\boldsymbol{w}})\boldsymbol{1}=|t|^\alpha\mathrm{diag}(\boldsymbol{w})V\boldsymbol{w}$. By Corollary 5.2 we have the following upper bound

$$H(\lambda-\tau,\Lambda)\le c_2\Lambda^2\exp\left(-c_3(\lambda-\tau-\Lambda)^\alpha\right),$$

which together with (5.41) establishes the proof. □

Proof of Theorem 2.4. For $\delta_u=u^{-2/\beta}\log^{2/\beta}u$ and some $\theta>0$ sufficiently small by Assumption (D2)

$$\max_{t\in[\delta_u,\theta]}\sigma_{\boldsymbol{b}}(t)\le\sigma_{\boldsymbol{b}}^2(0)+c\delta_u^\beta=\sigma_{\boldsymbol{b}}^2(0)+cu^{-2}\log^2u,$$

which together with the assumption that $\sigma_{\boldsymbol{b}}(t)$ attains its unique maximum at $t_0=0$ and the vector version of Piterbarg inequality derived in Lemma 4.5 yield

$$\mathbb{P}\left\{\exists t\in[\delta_u,T]:\ \boldsymbol{X}(t)>u\boldsymbol{b}\right\}\le CTu^{2/\gamma-1}\exp\left(-\frac{u^2}{2}\left(\sigma_{\boldsymbol{b}}^2(0)+c_2u^{-2}\log^2u\right)\right) \tag{5.43}$$

and thus it suffices to consider the asymptotics of $\mathbb{P}\left\{\exists[0,\delta_u]:\ \boldsymbol{X}(t)>u\boldsymbol{b}\right\}$ as $u\to\infty$.
Recall that in our notation

$$\boldsymbol{w}=\Sigma^{-1}\widetilde{\boldsymbol{b}},\quad V=ADA^\top,\quad W=\Xi A^\top,\quad V_{\boldsymbol{w}}=\mathrm{diag}(\boldsymbol{w})V\mathrm{diag}(\boldsymbol{w})$$

and $H_{\boldsymbol{Y},\boldsymbol{d}_{\boldsymbol{w}}}([0,\Lambda])=\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}([0,\Lambda])$ with $\boldsymbol{d}_{\boldsymbol{w}}=S_\alpha(t,V_{\boldsymbol{w}})\boldsymbol{1}$ is the constant defined in (2.4).
We divide the rest of the proof into three separate cases.

1. Case $\beta > \alpha$**.** For given Λ, S positive we define $N_u = \lfloor S u^{\frac{2}{\alpha}-\frac{2}{\beta}}/\Lambda \rfloor$. Consider a Gaussian process $\boldsymbol{X}_{u,k}(t) = \boldsymbol{X}(u^{-2/\alpha}(k\Lambda + t))$ with cmf $R_{u,k}(t,s) = R(u^{-2/\alpha}(k\Lambda+t), u^{-2/\alpha}(k\Lambda+s))$. By Assumption (D2) (set $\Sigma_{u,k} = R_{u,k}(0,0)$)

$$u^2\left(\Sigma - \Sigma_{u,k}\right) \sim \left(u^{\frac{2}{\beta}-\frac{2}{\alpha}} k\Lambda\right)^{\beta}\left(A\Xi^\top + \Xi A^\top\right), \quad u \to \infty.$$

Hence, $\tau_{\boldsymbol{w}} = \boldsymbol{w}^\top A \Xi^\top \boldsymbol{w} > 0$ implies further

$$\mathbb{P}\left\{\boldsymbol{X}_{u,k}(0) > u\boldsymbol{b}\right\} = \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\} \exp\left(-\tau_{\boldsymbol{w}}\left(u^{\frac{2}{\beta}-\frac{2}{\alpha}} k\Lambda\right)^{\beta}(1+o(1))\right). \tag{5.44}$$

In view of both Assumptions (D2) and (D3)

$$\lim_{u\to\infty} u^2\left[\Sigma_{u,k} - R_{u,k}(t,0)\right] = |t|^{\alpha} V \tag{5.45}$$

and

$$\lim_{u\to\infty} u^2\left[R_{u,k}(t,s) - R_{u,k}(t,0)\Sigma_{u,k}^{-1}R_{u,k}(0,s)\right] = R_{\alpha,V}(t,s) \tag{5.46}$$

uniformly for all non-negative integers $k \le N_u$ and $t,\ s \in [0,\Lambda]$. Define for $k \in \mathbb{N}, u > 0$

$$A_k = \{\exists t \in [k\Lambda u^{-2/\alpha}, (k+1)\Lambda u^{-2/\alpha}] :\ \boldsymbol{X}(t) > u\boldsymbol{b}\}.$$

In view of (5.45) and (5.46), we have from Lemma 4.7 that

$$\mathbb{P}\left\{A_k\right\} \sim H_{\boldsymbol{Y},\boldsymbol{d_w}}([0,\Lambda])\mathbb{P}\left\{\boldsymbol{X}_{u,k}(0) > u\boldsymbol{b}\right\}, \quad u \to \infty \tag{5.47}$$

uniformly for all non-negative integers $k \le N_u$. This together with (5.44) implies that

$$\begin{aligned}\frac{\sum_{k=0}^{N_u}\mathbb{P}\left\{A_k\right\}}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} &\sim \frac{\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}([0,\Lambda])}{\Lambda}\, u^{\frac{2}{\alpha}-\frac{2}{\beta}} \sum_{k=0}^{N_u} \exp\left(-\tau_{\boldsymbol{w}}\left(u^{\frac{2}{\beta}-\frac{2}{\alpha}}k\Lambda\right)^{\beta}(1+o(1))\right) u^{\frac{2}{\beta}-\frac{2}{\alpha}}\Lambda \\ &\sim \frac{\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}([0,\Lambda])}{(\tau_{\boldsymbol{w}})^{\beta}\Lambda} u^{\frac{2}{\alpha}-\frac{2}{\beta}} \int_0^S e^{-x^\beta}dx\end{aligned}$$

as $u \to \infty$. In view of Lemma 5.4 for some $c_1 > 0$ we have

$$\mathbb{P}\left\{\exists t \in [Su^{-2/\beta}, \delta_u] :\ \boldsymbol{X}(t) > u\boldsymbol{b}\right\} \le c_1 \exp\left(-\tau_{\boldsymbol{w}}\frac{S^\beta}{2}\right)\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}.$$

Letting $S \to \infty$ and then $\Lambda \to \infty$ yields further

$$\limsup_{u\to\infty} \frac{\mathbb{P}\left\{\exists t \in [0,T] :\ \boldsymbol{X}(t) > u\boldsymbol{b}\right\}}{u^{2/\alpha - 2/\beta}\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \le \mathcal{H}_{\alpha,V_{\boldsymbol{w}}}\Gamma(1/\beta+1)\tau_{\boldsymbol{w}}^{-\beta}, \tag{5.48}$$

where

$$\mathcal{H}_{\alpha,V_{\boldsymbol{w}}} = \lim_{\Lambda\to\infty}\frac{\mathcal{H}_{\alpha,V_{\boldsymbol{w}}}([0,\Lambda])}{\Lambda} \in (0,\infty)$$

is defined in (2.5). Next, we show the negligibility of the double-sum term. Note first that

$$\lim_{u\to\infty} u^2\left[\Sigma_{u,k} - \Sigma_{u,k\Lambda,j\Lambda}\right] = \frac{1}{2}(j-k)^{\beta}\Lambda^{\beta}$$

uniformly for all non-negative integers $k < j \le N_u$, where $\Sigma_{u,\tau,\lambda}$ is defined in (5.38). By Lemma 5.6, for some $c_2 > 0$

$$\begin{aligned}\sum_{j=k+1}^{N_u}\mathbb{P}\left\{A_kA_j\right\} &\le c_2\Lambda^2 u^{-|I|}\sum_{j=k+1}^{N_u}\exp\left(-\theta\left(j-k-1\right)^{\alpha}\Lambda^{\alpha}\right)\varphi_{\Sigma_{u,k\Lambda,j\Lambda}}(u\widetilde{\boldsymbol{b}}) \\ &\le c_2\Lambda^2\exp\left(-\theta\Lambda^{\beta}\right)u^{-|I|}\varphi_{\Sigma_{u,k}}(u\widetilde{\boldsymbol{b}}),\end{aligned} \tag{5.49}$$

which implies that the double-sum $\sum_{0\le k<j\le N_u}\mathbb{P}\left\{A_kA_j\right\}$ is negligible compared to the single-sum if we let $\Lambda \to \infty$. Therefore we complete the proof of (2.17).

2. Case $\beta = \alpha$**.** Let in the following $V_{\boldsymbol{w}}$, $W_{\boldsymbol{w}}$, $\boldsymbol{f_w}$ be as in (5.37). It is straightforward to see from Lemma 4.7 that

$$\mathbb{P}\left\{\exists t \in [0, \Lambda u^{-2/\alpha}] :\ \boldsymbol{X}(t) > u\boldsymbol{b}\right\} \sim H_{\boldsymbol{Y},\boldsymbol{d_w}+\boldsymbol{f_w}}([0,\Lambda])\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}. \tag{5.50}$$

Applying Lemma 5.4 we obtain

$$\mathbb{P}\left\{\exists t \in [\Lambda u^{-2/\alpha}, \delta_u] :\ \boldsymbol{X}(t) > u\boldsymbol{b}\right\} \le c_3 e^{-c_4\Lambda^\alpha}. \tag{5.51}$$

Combining (5.43), (5.50) and (5.51) and letting $\Lambda \to \infty$ the claim in (2.18) follows utilising Corollary 5.5.

3. Case $\beta < \alpha$. The proof is similar to the case $\beta = \alpha$. Define $\boldsymbol{X}_u(t) = \boldsymbol{X}(u^{-2/\beta}t)$, $u > 0$ with cmf $R_u(t,s) = R(u^{-2/\beta}t, u^{-2/\beta}s)$. By Assumptions (D2) and (D3)

$$\lim_{u\to\infty} u^2\left[\Sigma - R_u(t,0)\right] = |t|^\beta \Xi A^\top$$

and

$$\lim_{u\to\infty} u^2\left[R_u(t,s) - R_u(t,0)\Sigma^{-1}R_u(0,s)\right] = \boldsymbol{0}.$$

By Lemma 4.7, as $u \to \infty$

$$\mathbb{P}\left\{\exists t \in [0, \Lambda u^{-2/\beta}]: \ \boldsymbol{X}(t) > u\boldsymbol{b}\right\} \sim H_{\boldsymbol{0},\boldsymbol{f_w}}([0,\Lambda])\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\},$$

where

$$H_{\boldsymbol{0},\boldsymbol{f_w}}([0,\Lambda]) = \int_{\mathbb{R}^d} e^{\boldsymbol{1}^\top \boldsymbol{x}} \boldsymbol{1}_{\{\exists t\in[0,\Lambda]:\ -\boldsymbol{f_w}|t|^\beta > \boldsymbol{x}\}} d\boldsymbol{x}.$$

Further Lemma 5.3 implies

$$\lim_{\Lambda\to\infty} H_{\boldsymbol{0},\boldsymbol{f_w}}([0,\Lambda]) = 1 + \frac{\sum_{i\in I} w_i \max(0, -(\Xi A^\top \boldsymbol{w})_i)}{\boldsymbol{w}^\top \Xi A^\top \boldsymbol{w}},$$

which together with Lemma 5.4 yields (2.19). □

6. Appendix

We present the proofs of (2.15), Lemmas 4.1, 4.3, 4.5 and 4.7.

Proof of (2.15). In view of Assumption (D2) we have as $t \to t_0$

$$\Sigma^{-1}(t) - \Sigma^{-1} = \Sigma^{-1}\left(\Sigma - \Sigma(t)\right)\Sigma^{-1}(t) = |t-t_0|^\beta\, \Sigma^{-1}\left(A\Xi^\top + \Xi A^\top\right)\Sigma^{-1} + o\left(|t-t_0|^\beta\right),$$

where $A = A(t_0)$. Let $\widetilde{\boldsymbol{b}}(t)$ be the unique solution to the quadratic programming problem (1.3) with Σ replaced by $\Sigma(t)$. Then we have from the fact that $\sigma_{\boldsymbol{b}}^{-2}(t) = \min_{\boldsymbol{x}\ge\boldsymbol{b}} \boldsymbol{x}^\top\Sigma^{-1}(t)\boldsymbol{x}$,

$$\widetilde{\boldsymbol{b}}(t)^\top\left(\Sigma^{-1}(t) - \Sigma^{-1}\right)\widetilde{\boldsymbol{b}}(t) \le \sigma_{\boldsymbol{b}}^{-2}(t) - \sigma_{\boldsymbol{b}}^{-2}(t_0) \le \widetilde{b}^\top\left(\Sigma^{-1}(t) - \Sigma^{-1}\right)\widetilde{b}.$$

This and the fact that $\widetilde{b}(t)$ is Lipschitz continuous (c.f. Lemma 4.2) complete the proof. □

Proof of Lemma 4.1: All the claims apart from (4.5) are known, see e.g., [16, Lem 2.1]. Repeating the arguments of [9, Lem 1] (therein $\boldsymbol{b} > \boldsymbol{0}$ is assumed) we have for any $\boldsymbol{z} \ge \boldsymbol{0}$ with $\boldsymbol{z}^\top\boldsymbol{b} \ge 0$ and B a square matrix such that $BB^\top = \Sigma$

$$0 \le \boldsymbol{z}^\top\boldsymbol{b} = \inf_{\boldsymbol{x}\ge\boldsymbol{b}} \boldsymbol{z}^\top\boldsymbol{x} \le \left|B^\top\boldsymbol{w}\right| \inf_{\boldsymbol{x}\ge\boldsymbol{b}}\left|B^{-1}\boldsymbol{x}\right| = \sqrt{\boldsymbol{z}^\top\Sigma\boldsymbol{z}} \inf_{\boldsymbol{x}\ge\boldsymbol{b}} \boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x}$$

implying thus

$$\max_{\boldsymbol{z}\in[0,\infty)^d:\boldsymbol{z}^\top\boldsymbol{b}>0} \frac{(\boldsymbol{z}^\top\boldsymbol{b})^2}{\boldsymbol{z}^\top\Sigma\boldsymbol{z}} \le \min_{\boldsymbol{x}\ge\boldsymbol{b}} \boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x}.$$

By the properties of the unique solution $\widetilde{\boldsymbol{b}}$ of $\Pi_\Sigma(\boldsymbol{b})$ we have that the unique solution $\boldsymbol{w}$ of the dual programming problem of $\Pi_\Sigma(\boldsymbol{b})$ is given by $\boldsymbol{w} = \Sigma^{-1}\widetilde{\boldsymbol{b}}$. We have $\widetilde{\boldsymbol{b}}_I = \boldsymbol{b}_I$ and if J is non-empty, then $\widetilde{\boldsymbol{b}}_J = \Sigma_{JI}\Sigma_{II}^{-1}\boldsymbol{b}_I$. Since Σ is non-singular, then $(\Sigma^{-1})_{JJ}\Sigma_{JI} = -(\Sigma^{-1})_{JI}\Sigma_{II}$. Consequently, we obtain

$$\boldsymbol{w}_I = (\Sigma^{-1})_{II}\boldsymbol{b}_I + (\Sigma^{-1})_{IJ}\Sigma_{JI}(\Sigma_{II})^{-1}\boldsymbol{b}_I = [(\Sigma^{-1})_{II} + (\Sigma^{-1})_{IJ}\Sigma_{JI}(\Sigma_{II})^{-1}]\boldsymbol{b}_I = (\Sigma_{II})^{-1}\boldsymbol{b}_I$$

and

$$\boldsymbol{w}_J = (\Sigma^{-1})_{JI}\boldsymbol{b}_I + (\Sigma^{-1})_{JJ}\Sigma_{JI}(\Sigma_{II})^{-1}\boldsymbol{b}_I = [(\Sigma^{-1})_{JI} + (\Sigma^{-1})_{JJ}\Sigma_{JI}()\Sigma_{II})^{-1}]\boldsymbol{b}_I = \boldsymbol{0}_J$$

hold implying that

$$\boldsymbol{w}^\top\boldsymbol{b} = \boldsymbol{w}_I^\top\boldsymbol{b}_I = \boldsymbol{b}_I^\top(\Sigma_{II})^{-1}\boldsymbol{b}_I = \inf_{\boldsymbol{x}\ge\boldsymbol{b}} \boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x} > 0,$$

which yields further

$$\frac{(\boldsymbol{w}^\top\boldsymbol{b})^2}{\boldsymbol{w}^\top\Sigma\boldsymbol{w}} = \boldsymbol{b}_I^\top(\Sigma_{II})^{-1}\boldsymbol{b}_I = \min_{\boldsymbol{x}\ge\boldsymbol{b}} \boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x}.$$

Hence (4.5) follows and the proof is complete. □

PROOF OF LEMMA 4.3: First note that a family of vector-valued processes is tight if its components are tight, see e.g., [37, Cor 1.3]. Let $\tau_u \in Q_u, u>0$ be given and set $\boldsymbol{Z}_u(t)=\boldsymbol{X}_{u,\tau_u}(t)-\boldsymbol{f}_{u,\tau_u}(t)$. For given $s_i \in \mathbb{R},, t_i\in E, i\le n$ by Berman's comparison lemma (see e.g., [21]) and (4.7) for some $c>0$

$$\begin{aligned}&\left|\mathbb{P}\left\{X_{i,u,\tau_u}(t_k)-f_{i,u,\tau_u}(t_k)\le s_i, i\le d, k\le n\right\}-\mathbb{P}\left\{Y_i(t_k)-f_{i,u,\tau_u}(t_k)\le s_i, i\le d,k\le n\right\}\right|\\ &\quad\le\ c\sum_{1\le l\le k\le n, 1\le i\le j\le n}\left|\mathrm{cov}(X_{i,u,\tau_u}(t_l),X_{j,u,\tau_u}(t_k))-\mathrm{cov}(Y_i(t_l),Y_j(t_k))\right|\\ &\quad\to\ 0,\quad u\to\infty.\end{aligned}$$

Since $Y_i(t_k), i\le d, k\le n$ has a continuous distribution, then (4.8) yields

$$\lim_{u\to\infty}\left|\mathbb{P}\left\{Y_i(t_k)-f_{i,u,\tau_u}(t_k)\le s_i, i\le d,k\le n\right\}-\mathbb{P}\left\{Y_i(t_k)-f_i(t_k)\le s_i, i\le d, k\le n\right\}\right| \;=\; 0.$$

Consequently, the finite-dimensional distributions of $\boldsymbol{Z}_u$ converge in distribution to those of $\boldsymbol{Y}-\boldsymbol{f}$ as $u\to\infty$. Moreover, condition (4.8) implies that each component of $\boldsymbol{X}_{u,\tau}$, $u>0$ is tight, see [19, Prop 9.7]. By (4.8) each component of $\boldsymbol{Z}_u, u>0$ is also tight, and thus $\boldsymbol{Z}_u, u>0$ is tight. Since by assumption Γ is continuous, the continuous mapping theorem implies that for any continuity point s of $\Gamma(\boldsymbol{Y}(\cdot)-\boldsymbol{f}(\cdot))$ we have

$$\lim_{u\to\infty}\mathbb{P}\left\{\Gamma(\boldsymbol{Z}_u(\cdot))>s\right\}=\mathbb{P}\left\{\Gamma(\boldsymbol{Y}(\cdot)-\boldsymbol{f}(\cdot))>s\right\}, \tag{6.1}$$

hence

$$\lim_{u\to\infty}\sup_{\tau\in Q_u}\left|\mathbb{P}\left\{\Gamma(\boldsymbol{X}_{u,\tau}(\cdot)-\boldsymbol{f}_{u,\tau}(\cdot))>s\right\}-\mathbb{P}\left\{\Gamma(\boldsymbol{Y}(\cdot)-\boldsymbol{f}(\cdot))>s\right\}\right|=0.$$

Indeed, if the above is not satisfied, then for a given $\varepsilon>0$ and all u large we can find τ_u such that $\left|\mathbb{P}\left\{\Gamma(\boldsymbol{X}_{u,\tau_u}(\cdot)-\boldsymbol{f}_{u,\tau_u}(\cdot))>s\right\}-\mathbb{P}\left\{\Gamma(\boldsymbol{Y}(\cdot)-\boldsymbol{f}(\cdot))>s\right\}\right|>\varepsilon$ which is a contradiction in view of (6.1), hence the proof is complete. □

PROOF OF LEMMA 4.4: Let $\boldsymbol{x}\in\mathbb{R}^d$ and $\boldsymbol{\delta}=(\delta_1,...,\delta_d)^\top\ge\boldsymbol{0}$ be given. The proof follows from combination of the fact that

$$\begin{aligned}&\mathbb{P}\left\{\exists t\in E:\ \boldsymbol{Y}(t)>\boldsymbol{x}-\delta\right\}-\mathbb{P}\left\{\exists t\in E:\ \boldsymbol{Y}(t)>\boldsymbol{x}+\delta\right\}\\ &\quad\le\ \sum_{i=1}^d\mathbb{P}\left\{\sup_{t\in E}Y_i(t)\in(x_i-\delta_i,x_i+\delta_i]\right\}\end{aligned}$$

and Tsirelson Theorem (see, e.g., [38, Thm 7.1] and [39]) which implies that $\mathbb{P}\left\{\sup_{t\in E}Y_i(t)\le x\right\}$ is continuous $i=1,\dots,d$ except at most at one point $s_i=\inf\{s:\mathbb{P}\left\{\sup_{t\in E}Y_i(t)\le x\right\}>0\}$. □

PROOF OF LEMMA 4.5: For $\boldsymbol{z}\in[0,\infty)^d$ such that $\boldsymbol{z}^\top\boldsymbol{b}>0$, we define $Y_{\boldsymbol{z}}(t)=\widetilde{\boldsymbol{z}}^\top\boldsymbol{Z}(t)$ with $\widetilde{\boldsymbol{z}}=\boldsymbol{z}/(\boldsymbol{z}^\top\boldsymbol{b})$. Clearly, $\{\boldsymbol{Z}(t)>u\boldsymbol{b}\}\subset\{Y_{\boldsymbol{z}}(t)>u\}$ for such $\boldsymbol{z}$ implying

$$\mathbb{P}\left\{\exists t\in E:\ \boldsymbol{Z}(t)>u\boldsymbol{b}\right\}\ \le\ \inf_{\boldsymbol{z}\in[0,\infty)^d:\boldsymbol{z}^\top\boldsymbol{b}>0}\mathbb{P}\left\{\sup_{t\in E}Y_{\boldsymbol{z}}(t)>u\right\}. \tag{6.2}$$

The proof of (4.10) follows by a direct application of Borell-TIS inequality to $Y_{\boldsymbol{z}}(t)$ (see e.g., [22]) with

$$\mu=\mathbb{E}\left\{\sup_{t\in E}Y_{\boldsymbol{z}}(t)\right\}<\infty.$$

Next, if $\Sigma(t)$ is non-singular for $t\in E$, choose $\boldsymbol{v}(t)\ge\boldsymbol{b}$ such that it minimises $\boldsymbol{v}^\top(t)\Sigma^{-1}(t)\boldsymbol{v}(t)$ and set $\boldsymbol{z}(t)=\Sigma^{-1}(t)\boldsymbol{v}(t)$. By Lemma 4.1 and (4.11)

$$\sup_{\boldsymbol{z}\ge[0,\infty)^d:\boldsymbol{z}^\top\boldsymbol{b}>0}\mathrm{Var}(Y_{\boldsymbol{z}}(t))=\mathrm{Var}(Y_{\boldsymbol{z}(t)}(t))=\frac{1}{\boldsymbol{v}(t)^\top\Sigma^{-1}(t)\boldsymbol{v}(t)}>0.$$

In view of Lemma 4.2, (4.12) and the compactness of E for $\boldsymbol{f}=\boldsymbol{v}$ or $\boldsymbol{f}=\boldsymbol{w}$ we obtain

$$|\boldsymbol{f}(s)-\boldsymbol{f}(t)|\le C_1\,|t-s|^\gamma \tag{6.3}$$

for some positive constant C_1. Note that since E is compact

$$\inf_{t\in E}\boldsymbol{v}(t)^\top\boldsymbol{b}>0. \tag{6.4}$$

It follows that $\boldsymbol{f}(t)=\widetilde{\boldsymbol{z}}(t)=\boldsymbol{z}(t)/[\boldsymbol{z}(t)^\top\boldsymbol{b}]$ also satisfies (6.3) (for some other constant C_1). This and (4.11) imply the γ-Hölder continuity of $Y_{\widetilde{\boldsymbol{z}}}(t)=\widetilde{\boldsymbol{z}}(t)^\top\boldsymbol{Z}(t)$. Therefore (4.13) follows by applying [25, Thm 8.1] to $Y_{\widetilde{\boldsymbol{z}}}$. If $\sigma_{\boldsymbol{b}}^2(t), t\in E$ has a unique maximum at $t_0\in E$ and is continuous, the claim follows by using first Borell-TIS inequality and then applying Piterbarg inequality for the neighbourhood of t_0, see the derivation of (32) and (33) in [26]. □

Proof of Lemma 4.7: For notational simplicity we shall assume that the index set J is not empty. Define $\overline{\boldsymbol{u}} \in \mathbb{R}^d$ such that $\overline{\boldsymbol{u}}_I$ has all components equal u and $\overline{\boldsymbol{u}}_J$ has all components equal 1. Set next

$$\boldsymbol{Z}_{u,\tau}(t) = \overline{\boldsymbol{u}} \cdot \left[\boldsymbol{X}_{u,\tau}(t) - u\widetilde{\boldsymbol{b}}\right] + \boldsymbol{x}, \quad \boldsymbol{\chi}_{u,\tau}(t) = \left(\boldsymbol{Z}_{u,\tau}(t) | \boldsymbol{Z}_{u,\tau}(0) = \boldsymbol{0}\right),$$

where in our notation $\boldsymbol{x} \cdot \boldsymbol{y} = (x_1 y_1, \ldots, x_d y_d)^\top$ for $\boldsymbol{x}$ and $\boldsymbol{y}$ two vectors in $\mathbb{R}^d$. For any $u > 0$ we obtain

$$\begin{aligned}
&\mathbb{P}\left\{\exists t \in E: \ \boldsymbol{X}_{u,\tau}(t) > u\boldsymbol{b}\right\} \\
=&\mathbb{P}\left\{\exists t \in E: \ \left(\boldsymbol{X}_{u,\tau}(t) - u\widetilde{\boldsymbol{b}}\right) > u(\boldsymbol{b} - \widetilde{\boldsymbol{b}})\right\} \\
=&u^{-m} \int_{\mathbb{R}^d} \mathbb{P}\left\{\exists t \in E: \ \left(\boldsymbol{X}_{u,\tau}(t) - u\widetilde{\boldsymbol{b}}\right) > u(\boldsymbol{b} - \widetilde{\boldsymbol{b}}) \big| \boldsymbol{X}_{u,\tau}(0) = u\widetilde{\boldsymbol{b}} - \boldsymbol{x}/\overline{\boldsymbol{u}}\right\} \varphi_{\Sigma_{u,\tau}}(u\widetilde{\boldsymbol{b}} - \boldsymbol{x}/\overline{\boldsymbol{u}})\, d\boldsymbol{x} \\
=&u^{-m} \int_{\mathbb{R}^d} \mathbb{P}\left\{\exists t \in E: \ \boldsymbol{Z}_{u,\tau}(t) > \boldsymbol{x} + \overline{\boldsymbol{u}}u(\boldsymbol{b} - \widetilde{\boldsymbol{b}}) \big| \boldsymbol{Z}_{u,\tau}(0) = \boldsymbol{0}\right\} \varphi_{\Sigma_{u,\tau}}(u\widetilde{\boldsymbol{b}} - \boldsymbol{x}/\overline{\boldsymbol{u}})\, d\boldsymbol{x} \\
=&u^{-m} \int_{\mathbb{R}^d} \mathbb{P}\left\{\exists t \in E: \ \left(\boldsymbol{\chi}_{u,\tau}(t) - u\overline{\boldsymbol{u}}(\boldsymbol{b} - \widetilde{\boldsymbol{b}})\right) > \boldsymbol{x}\right\} \varphi_{\Sigma_{u,\tau}}(u\widetilde{\boldsymbol{b}} - \boldsymbol{x}/\overline{\boldsymbol{u}})\, d\boldsymbol{x}.
\end{aligned}$$

Assumption (A2) implies

$$\begin{aligned}
\mathbb{E}\left\{\boldsymbol{\chi}_{u,\tau}(t)\right\} =&\overline{\boldsymbol{u}} \cdot \left\{R_{u,\tau}(t,0)\Sigma_{u,\tau}^{-1}(u\widetilde{\boldsymbol{b}} - \boldsymbol{x}/\overline{\boldsymbol{u}}) - u\widetilde{\boldsymbol{b}}\right\} + \boldsymbol{x} \\
=&\begin{pmatrix} u^2\left\{(R_{u,\tau}(t,0) - \Sigma_{u,\tau})\Sigma_{u,\tau}^{-1}\widetilde{\boldsymbol{b}}\right\}_I \\ u\left\{(R_{u,\tau}(t,0) - \Sigma_{u,\tau})\Sigma_{u,\tau}^{-1}\widetilde{\boldsymbol{b}}\right\}_J \end{pmatrix} + \overline{\boldsymbol{u}} \cdot \left\{(R_{u,\tau}(t,0) - \Sigma_{u,\tau})\Sigma_{u,\tau}^{-1}(\boldsymbol{x}/\overline{\boldsymbol{u}})\right\} \\
\to&\begin{pmatrix} -\boldsymbol{d}_I(t) \\ \boldsymbol{0}_J \end{pmatrix}
\end{aligned} \tag{6.5}$$

and

$$\begin{aligned}
&\mathbb{E}\left\{\left[\boldsymbol{\chi}_{u,\tau}(t) - \mathbb{E}\left\{\boldsymbol{\chi}_{u,\tau}(t)\right\}\right]\left[\boldsymbol{\chi}_{u,\tau}(s) - \mathbb{E}\left\{\boldsymbol{\chi}_{u,\tau}(s)\right\}\right]^\top\right\} \\
=&\mathrm{diag}(\overline{\boldsymbol{u}})\left[R_{u,\tau}(t,s) - R_{u,\tau}(t,0)\Sigma_{u,\tau}^{-1}R_{u,\tau}(0,s)\right]\mathrm{diag}(\overline{\boldsymbol{u}}) \\
\to&\begin{pmatrix} K_{II}(t,s) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}
\end{aligned} \tag{6.6}$$

uniformly in t, $s \in E$ and $\tau \in Q_u$ as $u \to \infty$.
Define the index set L as the maximal subset of J such that $\tilde{b}_i = b_i$ for any $i \in L$. Hence $\tilde{b}_i = b_i$ for all $i \in I \cup L$ and $\tilde{b}_i > b_i$ for $i \in J \setminus L$. For simplicity we shall assume that L is non-empty. We write next

$$a_u(\boldsymbol{x}) := \mathbb{P}\left\{\exists t \in E: \ \boldsymbol{\chi}_{u,\tau}(t) - u\overline{\boldsymbol{u}}(\boldsymbol{b} - \widetilde{\boldsymbol{b}}) > \boldsymbol{x}\right\} = \mathbb{P}\left\{\Gamma\left(\widetilde{\boldsymbol{\chi}}_{u,\tau}(\cdot)\right) > 0\right\},$$

where

$$\widetilde{\boldsymbol{\chi}}_{u,\tau,i}(t) := \boldsymbol{\chi}_{u,\tau,i}(t) - x_i, \ i \in I \cup L; \quad \widetilde{\boldsymbol{\chi}}_{u,\tau,j}(t) := u^{-1}\boldsymbol{\chi}_{u,\tau,j}(t) - (b_j - \widetilde{b}_j) - u^{-1}x_j, \ j \in J \setminus L,$$

and Γ is the continuous functional on the Banach space $C(E)$ (of all $\mathbb{R}^d$-valued continuous functions on E) given by

$$\Gamma(\boldsymbol{\omega}(\cdot)) = \sup_{t \in E} \min_{1 \le i \le d} \omega_i(t), \quad \boldsymbol{\omega} \in C(E).$$

Let $\boldsymbol{W}_I(t)$, $t \ge 0$ denotes a centered Gaussian process with K_{II} as its cmf and let $\widetilde{\boldsymbol{W}}(t) = \left(\boldsymbol{W}_I(t), \boldsymbol{0}_J\right)^\top$. It follows from (6.5) and (6.6) that the assumptions of Lemma 4.3 hold true with $\boldsymbol{X}_{u,\tau}(t)$, $\boldsymbol{Y}(t)$, $\boldsymbol{f}_{u,\tau}(t)$ and $\boldsymbol{f}(t)$ replaced respectively by $\widetilde{\boldsymbol{\chi}}_{u,\tau}(t) - \mathbb{E}\left\{\widetilde{\boldsymbol{\chi}}_{u,\tau}(t)\right\}$, $\widetilde{\boldsymbol{W}}(t)$, $-\mathbb{E}\left\{\widetilde{\boldsymbol{\chi}}_{u,\tau}(t)\right\}$ and $\widetilde{\boldsymbol{d}}(t) = \left(\boldsymbol{d}_I(t) + \boldsymbol{x}_I, \boldsymbol{x}_L, \boldsymbol{b}_{J\setminus L} - \widetilde{\boldsymbol{b}}_{J\setminus L}\right)^\top$. Since

$$\mathbb{P}\left\{\Gamma\left(\widetilde{\boldsymbol{W}}(\cdot) - \widetilde{\boldsymbol{d}}(\cdot)\right) > 0\right\} = \boldsymbol{1}_{\{\boldsymbol{x}_L < \boldsymbol{0}_L\}}\mathbb{P}\left\{\exists t \in E: \ \boldsymbol{W}_I(t) - \boldsymbol{d}_I(t) > \boldsymbol{x}_I\right\},$$

we have from Lemma 4.4 that 0 is a continuity point of the distribution function of $\Gamma\left(\widetilde{\boldsymbol{W}}(\cdot) - \widetilde{\boldsymbol{d}}(\cdot)\right)$ for almost all $x \in \mathbb{R}^d$. Applying Lemma 4.3, we obtain that

$$\lim_{u \to \infty} \sup_{\tau \in Q_u} \left| a_u(\boldsymbol{x}) - \boldsymbol{1}_{\{\boldsymbol{x}_L < \boldsymbol{0}_L\}}\mathbb{P}\left\{\exists t \in E: \ \boldsymbol{W}_I(t) - \boldsymbol{d}_I(t) > \boldsymbol{x}_I\right\}\right| = 0 \tag{6.7}$$

holds for almost all $\boldsymbol{x} \in \mathbb{R}^d$. Note that, by Lemma 4.1 there exists a positive constant λ such that for all u large and all $\tau \in Q_u$

$$\frac{1}{2}(\boldsymbol{x}/\overline{\boldsymbol{u}})^\top \Sigma_{u,\tau}^{-1}(\boldsymbol{x}/\overline{\boldsymbol{u}}) \geq \lambda\left(\boldsymbol{x}/\overline{\boldsymbol{u}}\right)^\top\left(\boldsymbol{x}/\overline{\boldsymbol{u}}\right) \geq \lambda \boldsymbol{x}_J^\top \boldsymbol{x}_J.$$

Hence (set $\boldsymbol{w} = \Sigma^{-1}\widetilde{\boldsymbol{b}}$ and recall that $\boldsymbol{w}_I = (\Sigma_{II})^{-1}\boldsymbol{b}_I > \boldsymbol{0}_I$, $\boldsymbol{w}_J = \boldsymbol{0}_J$ and $\boldsymbol{w}^\top \boldsymbol{x} = \boldsymbol{w}_I^\top \boldsymbol{x}_I$ for any $\boldsymbol{x} \in \mathbb{R}^d$)

$$\begin{aligned}
\varphi_{\Sigma_{u,\tau}}(u\widetilde{\boldsymbol{b}} - \boldsymbol{x}/\overline{\boldsymbol{u}}) =& \varphi_{\Sigma_{u,\tau}}(u\widetilde{\boldsymbol{b}}) \exp\left(u\left(\boldsymbol{x}/\overline{\boldsymbol{u}}\right)^\top \Sigma_{u,\tau}^{-1}\widetilde{\boldsymbol{b}} - \frac{1}{2}\left(\boldsymbol{x}/\overline{\boldsymbol{u}}\right)^\top \Sigma_{u,\tau}^{-1}\left(\boldsymbol{x}/\overline{\boldsymbol{u}}\right)\right)\\
=& \varphi_{\Sigma_{u,\tau}}(u\widetilde{\boldsymbol{b}}) \exp\left(\boldsymbol{w}_I^\top \boldsymbol{x}_I - \frac{1}{2}(\boldsymbol{x}/\overline{\boldsymbol{u}})^\top \Sigma_{u,\tau}^{-1}(\boldsymbol{x}/\overline{\boldsymbol{u}})) + u\widetilde{\boldsymbol{b}}^\top\left(\Sigma_{u,\tau}^{-1} - \Sigma^{-1}\right)(\boldsymbol{x}/\overline{\boldsymbol{u}})^\top\right)\\
\leq& \varphi_{\Sigma_{u,\tau}}(u\widetilde{\boldsymbol{b}}) \exp\left(\boldsymbol{w}_I^\top \boldsymbol{x}_I + u\widetilde{\boldsymbol{b}}^\top\left(\Sigma_{u,\tau}^{-1} - \Sigma^{-1}\right)(\boldsymbol{x}/\overline{\boldsymbol{u}}) - \lambda \boldsymbol{x}_J^\top \boldsymbol{x}_J\right).
\end{aligned}$$

In view of (6.7) and Assumption (A1) as $u \to \infty$

$$\begin{aligned}
&\mathbb{P}\left\{\exists t \in E:\ \boldsymbol{X}_{u,\tau}(t) > u\boldsymbol{b}\right\}\\
&\sim u^{-m}\varphi_{\Sigma_{u,\tau}}(u\widetilde{\boldsymbol{b}}) \int_{\mathbb{R}^d} e^{\boldsymbol{x}_I^\top \boldsymbol{w}_I - \frac{1}{2}\boldsymbol{x}_J^\top (\Sigma^{-1})_{JJ}\boldsymbol{x}_J} \mathbf{1}_{\{\boldsymbol{x}_L < \boldsymbol{0}_L\}} \mathbb{P}\left\{\exists t \in E:\ \boldsymbol{W}_I(t) - \boldsymbol{d}_I(t) > \boldsymbol{x}_I\right\} d\boldsymbol{x}.
\end{aligned} \tag{6.8}$$

The asymptotic equivalence above follows by the dominated convergence theorem, which is justified by the following arguments. First note that

$$w_i = ((\Sigma_{II})^{-1}\boldsymbol{b}_I)_i > 0, \quad \forall i \in I.$$

Split the region of the integration on sets where the ith coordinates of $\boldsymbol{x}_I$ are either positive or negative. If $x_i < 0$, then the domination of the integrand for this coordinate is clear since $w_i > 0$. Suppose that we deal therefore with a region Ω where the first l components are negative and the $m - l$ components in the index set F are positive. Then (recall the definition of $a_u(\boldsymbol{x})$ above)

$$a_u(\boldsymbol{x}) \leq \mathbb{P}\left\{\exists t \in E:\ \boldsymbol{\chi}_{u,\tau,F}(t) > \boldsymbol{x}_F\right\} \leq \mathbb{P}\left\{\exists t \in E:\ \mathbf{1}_F^\top \boldsymbol{\chi}_{u,\tau,F}(t) > \mathbf{1}_F^\top \boldsymbol{x}_F\right\}.$$

It follows from (6.5), (6.6) and Assumptions (A1), (A2) that for any $t \in E$

$$\mathbb{E}\left\{\mathbf{1}_F^\top \boldsymbol{\chi}_{u,\tau,F}(t)\right\} \leq c + \delta \mathbf{1}_F^\top \boldsymbol{x}_F \quad \text{and } \mathbb{E}\left\{(\mathbf{1}_F^\top \boldsymbol{\chi}_{u,\tau,F}(t))^2\right\} \leq \sigma^2$$

for some $\delta > 0$, $\sigma > 0$ and some constant c. By the Borell-TIS inequality (c.f. [25, Theorem D.1])

$$a_u(\boldsymbol{x}) \quad \leq \quad \mathbb{P}\left\{\exists t \in E:\ \mathbf{1}_F^\top \boldsymbol{\chi}_{u,\tau,F}(t) > \mathbf{1}_F^\top \boldsymbol{x}_F\right\} \leq e^{-\varepsilon\left(\boldsymbol{x}_F^\top \boldsymbol{x}_F\right)}$$

for some $\varepsilon > 0$ small enough, hence the domination of the integrand follows.
Taking in particular $E = \{0\}$ in (6.8) implies as $u \to \infty$

$$\mathbb{P}\left\{\boldsymbol{X}_{u,\tau}(0) > u\boldsymbol{b}\right\} \sim u^{-m}\varphi_{\Sigma_{u,\tau}}(u\widetilde{\boldsymbol{b}}) \left(\prod_{i \in I} w_i\right)^{-1} \int_{\mathbb{R}^{|J|}} e^{-\frac{1}{2}\boldsymbol{x}_J^\top (\Sigma^{-1})_{JJ}\boldsymbol{x}_J} \mathbf{1}_{\{\boldsymbol{x}_L < \boldsymbol{0}_L\}} d\boldsymbol{x}_J.$$

Therefore, as $u \to \infty$

$$\begin{aligned}
&\mathbb{P}\left\{\exists t \in E:\ \boldsymbol{X}_{u,\tau}(t) > u\boldsymbol{b}\right\}\\
&\sim \mathbb{P}\left\{\boldsymbol{X}_{u,\tau}(0) > u\boldsymbol{b}\right\} \left(\prod_{i \in I} w_i\right) \int_{\mathbb{R}^m} e^{\boldsymbol{w}_I^\top \boldsymbol{x}_I} \mathbb{P}\left\{\exists t \in E:\ \boldsymbol{W}_I(t) - \boldsymbol{d}_I(t) > \boldsymbol{x}_I\right\} d\boldsymbol{x}_I,
\end{aligned}$$

which complete the proof. □

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