Multivariate Approximation Methods for the Pricing of Catastrophe-linked Bonds*

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Abstract

In this paper we develop Quasi-Monte Carlo techniques for the evaluation of high-dimensional integrals that occur in financial applications, namely in the pricing of default-risky catastrophe-linked bonds in a model including stochastic interest rates, basis risk and default risk. It is shown that these techniques clearly outperform classical Monte Carlo integration in terms of efficiency. The methods are based on number-theoretic low-discrepancy sequences such as Halton, Sobol and Faure sequences.

1 Introduction

Quasi-Monte Carlo integration has turned out to be a powerful tool for integrals occurring in various branches of applied mathematics (see e.g. [1,2,24]). In this paper we investigate the efficiency of Quasi-Monte Carlo techniques for pricing default-risky catastrophe-linked bonds in a model including stochastic interest rates, basis risk and default risk. In Section 2 we describe the Quasi-Monte Carlo (QMC) method in general. In Section 3 we develop a suitable contingent claim model for these bonds. Section 4 discusses implementation issues and gives numerical illustrations of the superiority of QMC methods over classical Monte Carlo. Furthermore the sensitivity of CAT bond prices with respect to various model parameters is investigated.

2 Multivariate Integration using QMC Sequences

After suitable transformation of the integration domain, an s-dimensional integral can be written in the form

$$I(f) = \int_{U^s} f(\mathbf{x})\, d\mathbf{x}, \tag{2.1}$$

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where f is a function defined on the s-dimensional unit cube $U^s = [0,1]^s$. The basic idea of Monte Carlo integration is now to choose N integration points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ randomly in U^s and to approximate the integral $I(f)$ by the arithmetic mean

$$I_N(f) = \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n).$$

Chebyshev's inequality gives a probabilistic bound on the integration error of Monte Carlo integration

$$P\left(\left| \int_{U^s} f(\mathbf{x})\, d\mathbf{x} - \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n) \right| \geq \sqrt{\frac{\mathrm{Var}(f(\mathbf{x}))}{\epsilon N}} \right) \leq \epsilon, \qquad (2.2)$$

where $\mathrm{Var}(f(\mathbf{x}))$ is the variance of $f(\xi)$, and the probable error thus turns out to be proportional to $1/\sqrt{N}$ and is independent of the dimension s (see e.g. [15]). A crucial aspect, however, is the availability of an efficient pseudo-random number generator.

The use of deterministic uniformly distributed point sequences (instead of pseudo-random sequences) has proven to be an efficient alternative to classical Monte Carlo, especially for dimensions $s < 30$. Such nonrandom sequences are called Quasi-Monte Carlo sequences. A well-known measure for the uniformness of the distribution of a sequence $\{\mathbf{y}_n\}_{1 \leq n \leq N}$ in U^s is the star-discrepancy

$$D_N^*(\mathbf{y}_n) = \sup_{I \in J_0^s} \left| \frac{A(\mathbf{y}_n; I)}{N} - \lambda_s(I) \right| ,$$

where J_0^s is the set of all intervals of the form $[0, \vec{\eta}) = [0, \eta_1) \times [0, \eta_2) \times \ldots \times [0, \eta_s)$ with $0 \leq \eta_i < 1$, $i = 1, \ldots, s$ and $A(\mathbf{y}_n; I)$ is the number of points of the sequence $\{\mathbf{y}_n\}_{1 \leq n \leq N}$ that lie in I. $\lambda_s(I)$ denotes the s-dimensional Lebesgue-measure of I.

The notion of discrepancy can be used for obtaining an upper bound for the error of Quasi-Monte Carlo integration, since by the Koksma-Hlawka inequality

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{y}_n) - \int_{[0,1)^s} f(u) du \right| \leq V(f) D_N^*(\mathbf{y}_1, \ldots, \mathbf{y}_N), \qquad (2.3)$$

where $V(f)$ is the total variation of f in the sense of Hardy and Krause (see e.g. Drmota and Tichy [10]). This error bound is deterministic (in contrast to the probabilistic error bound (2.2)).

The Koksma-Hlawka inequality can be viewed as an error bound in the sense of worst case analysis. In order to establish an error bound in the sense of average case analysis we have to introduce the L^p-discrepancy for $p \geq 1$ defined by

$$D_n^{(p)}(\mathbf{y}_n) = \left(\int_{U^s} \left| \frac{A(\mathbf{y}_n; [0, \overrightarrow{\eta}))}{N} - \eta_1 \cdots \eta_s \right|^p d\eta_1 \cdots d\eta_s \right)^{1/p} .$$

The star-discrepancy corresponds to $p = \infty$. Furthermore, the L^p-discrepancy and the star-discrepancy are related by the following inequality ($C(s,p)$ denoting an explicitly known constant depending only on s and p):

$$D_N^{(p)} \leq D_N^* \leq C(s,p) \left(D_N^{(p)} \right)^{\frac{p}{s+p}} ,$$

which is due to Niederreiter, Tichy and Turnwald [23]. The central problem of average case analysis of numerical integration is to determine the minimal cost for approximation of $I(f)$ with error $\leq \epsilon$ for a given class $\mathcal{F}$ of functions, where $\mathcal{F}$ is equipped with a probability measure. A very convenient choice for $\mathcal{F}$ is the class of continuous functions equipped with the Wiener measure. The average complexity of numerical integration can be expressed as the expectation with respect to Wiener measure μ_w of the number of function evaluations and the number of operations necessary for the computation of $I_N(f)$. Woźniakowski [27] computed the average case error in terms of the L^2-discrepancy:

$$D_N^{(2)}(\mathbf{1} - \mathbf{y}_n) = \left(\int_{\mathcal{F}} \left(I(f) - I_N(f) \right)^2 d\mu_w \right)^{1/2} , \tag{2.4}$$

where $\mathbf{1} = (1, \ldots, 1)$.
In particular for s not too large, certain Quasi-Monte Carlo sequences have turned out to be superior to crude probabilistic Monte Carlo techniques in many applications. This is the case for so-called low discrepancy sequences, i.e. sequences for which

$$D_N^*(\mathbf{y}_n) \leq C_s \frac{(\log N)^s}{N} \tag{2.5}$$

holds with an explicitly computable constant C_s. The best known low discrepancy sequences satisfy a discrepancy bound of type (2.5). However, for the L^2-discrepancy bounds of better asymptoptic order in N are known. Recently Chen and Skriganov [5] explicitly constructed a sequence $(\mathbf{z}_n)$ such that

$$D_N^{(2)}(\mathbf{z}_n) \leq C'_s \frac{(\log N)^{\frac{1}{2}(s-1)}}{N},$$

which is the best possible asymptotic order by the following general lower bound due to Roth [25]:

$$D_N^{(2)} \geq c_s \frac{(\log N)^{\frac{s-1}{2}}}{N}.$$

From Roth's bound and (2.4) it can immediately be deduced that the average case complexity of numerical integration (on the space of continuous functions equipped with Wiener measure) satisfies a bound of the order $\mathcal{O}(\epsilon^{-1}(\log \epsilon - 1)^{\frac{s-1}{2}})$, where ϵ is the error in the approximation of $I(f)$ by $I_N(f)$ (see [27]).

For the star-discrepancy only slight improvements of Roth's bound are known. The most recent one is due to Baker [3]:

$$D_N^* \geq c_s' \frac{(\log N)^{\frac{s-1}{2}}}{N} \left(\frac{\log \log N}{\log \log \log N}\right)^{\frac{1}{2s-2}}$$

for $N \geq N_0$.

Bounds for the constants C_s are usually pessimistic and often the actual error made by Quasi-Monte Carlo integration is much lower than the bound implied by C_s (see e.g. [4]).

However, for applications in mathematical finance the dimensions s may be quite big (several hundreds). Thus discrepancy bounds in s and N are of interest. Recently this problem was successfully attacked by Heinrich, Novak, Wasilkowski and Woźniakowski [14]. They considered the quantity

$$disc(N, s) = \inf_{(y_1, \ldots, y_N) \in [0,1]^{Ns}} D_N^*(\mathbf{y}_n)$$

and obtained the estimate

$$disc(N, s) \leq C s^{\frac{1}{2}} N^{-\frac{1}{2}}$$

with an absolute constant C. The proof depends on probabilistic methods and can be generalized to L^p-discrepancies. As a consequence the minimum number $K(s, \epsilon)$ of points $\mathbf{y}_n$ $(1 \leq n \leq N)$ satisfying $disc(N, s) \leq \epsilon$ fulfills a bound of the type $K(s, \epsilon) = \mathcal{O}(s\epsilon^{-2})$. Thus the problem of finding points with $disc(N, s) \leq \epsilon$ can be solved in polynomial time.

The following low discrepancy sequences will be used in the sequel:

- The Halton sequence [13] is defined as a sequence of vectors in U^s based on the digit representation of n in base p_i

$$\xi_n = (b_{p_1}(n), b_{p_2}(n), \ldots, b_{p_s}(n)), \tag{2.6}$$

  where p_i is the ith prime number and $b_p(n)$ is the digit reversal function for base p given by

$$b_p(n) = \sum_{k=0}^{\infty} n_k p^{-k-1}, \qquad n = \sum_{k=0}^{\infty} n_k p^k,$$

  where the n_k are integers. One could also use pairwise coprime base numbers, but the error estimate turns out to be best possible for prime bases p_n. For Halton sequences C_s tends to infinity super-exponentially for $s \to \infty$.

Better error bounds can be obtained for low-discrepancy sequences based on (t, m, s)-nets. These nets are based on the b-adic representation of vectors in U^s. Instead of optimizing the discrepancy itself, only the discrepancy with respect to elementary intervals J in base b is considered, i.e. $J = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1) b^{-d_i})$ with integers $d_i \geq 0$ and integers $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$, and point sequences in U^s are constructed such that the discrepancy with respect to these intervals J is optimal for subsequences of length $N = b^m$. Let $\#(J, N)$ denote the number of points of a sequence $\{\mathbf{y}_n\}_{1 \leq n \leq N}$ that lie in J. A point set $\mathcal{P}$ with $\mathrm{card}(\mathcal{P}) = b^m$ is now called a (t, m, s)-net, if $\#(J, b^m) = b^t$ for every elementary interval J with $\lambda_s(J) = b^{t-m}$. The parameter t is a quality parameter. For $t = 0$ we have the minimal discrepancy of the point set $\mathcal{P}$ with respect to the family of elementary intervals.

For an integer $t \geq 0$, a sequence $\xi_1, \xi_2, \ldots$ of points in U^s is called a (t, s)-sequence in base b, if for all integers $k \geq 0$ and $m > t$, the point set consisting of the ξ_n with $kb^m < n \leq (k+1)b^m$ is a (t, m, s)-net in base b.

- The Sobol sequence is a (t, s)-sequence in base 2 with values t that depend on s. For a construction of this sequence we refer to [26]. The corresponding constant C_s is much lower than for the Halton sequence.

- The Faure sequence [12] is a $(0, s)$-sequence with an even lower constant C_s.

Since Quasi-Monte Carlo methods are especially competitive for low dimensions ($s < 30$), the best numerical performance can be achieved by implementing hybrid Monte Carlo techniques, i.e. Quasi-Monte Carlo sequences for the initial dimensions and the remaining dimensions are then simulated by crude Monte Carlo.

3 A Model for Pricing Catastrophe-Linked Bonds

We now apply the QMC technique to a particular problem of mathematical finance, namely the pricing of a catastrophe-linked bond (CAT bond for short), which is a liability hedge instrument for insurance companies. As an alternative to traditional reinsurance, an insurer can issue (i.e. sell) these bonds under the condition of debt-forgiveness triggers in case the aggregate claims to be paid by the insurer (or a certain composite index of losses among various companies) exceeds a specified threshold K. In that way the issuer of these bonds avoids credit risk that might arise with traditional reinsurance concepts. Since during the last decades the insurance losses due to natural catastrophes have increased dramatically, CAT bonds have become a popular hedging tool and one of the key issues is to correctly price these financial instruments (see e.g. [7,8,18]).
A crucial step in determining a fair price of the CAT bond is to model the dynamics of the assets and the losses of the insurer as well as the dynamics of the (stochastic) interest rate. In this paper we adopt the approach of Lee and Yu [17], which also includes the consideration of default risk (cf. Section 3.2) and basis risk (cf. Section 3.3). For convenience, we briefly repeat the relevant specifications of the model from [17]:
The asset dynamics of the insurer are modeled by

$$\frac{dA_t}{A_t} = \mu_A\, dt + \phi\, dr_t + \sigma_A\, dW_{A,t}, \tag{3.1}$$

where A_t is the value of the insurer's total assets at time t, r_t is the instantaneous interest rate at time t (with ϕ denoting the interest rate elasiticity), $W_{A,t}$ is the Wiener process representing the risk orthogonal to interest rate risk (frequently referred to as credit risk), and μ_A and σ_A are the drift and volatility due to this credit risk of the insurer. The stochastic interest rate r_t is assumed to follow the Cox-Ingersoll-Ross process [6]

$$dr_t = \kappa(m - r_t)\, dt + \nu\sqrt{r_t}\, dZ_t,$$

where κ, m and ν are given parameters and Z_t is a Wiener process independent of $W_{A,t}$. Under the risk-neutral pricing measure Q, the process can thus be written as

$$\frac{dA_t}{A_t} = r_t\,dt + \phi\nu\sqrt{r_t}\,dZ_t^* + \sigma_A\,dW_{A,t}^*, \tag{3.2}$$

where $W_{A,t}^*$ and Z_t^* are independent Wiener processes under Q and

$$dr_t = \kappa^*(m^* - r_t)\,dt + \nu\sqrt{r_t}\,dZ_t^* \tag{3.3}$$

represent the dynamics of the interest rate process under Q with

$$\kappa^* = \kappa^* + \lambda_r, \qquad m^* = \frac{\kappa m}{\kappa + \lambda_r} \quad \text{and} \quad dZ_t^* = dZ_t + \frac{\lambda_r\sqrt{r_t}}{\nu}\,dt,$$

where λ_r denotes the market price of interest rate risk (cf. [17]).
The loss process is assumed to follow a compound Poisson process

$$C_t = \sum_{j=1}^{N(t)} X_j, \tag{3.4}$$

where X_j denotes the amount of losses caused by the jth catastrophe for the issuing insurance company during the specific period and $N(t)$ is a homogeneous Poisson process with intensity λ. Thus C_t represents the aggregate loss at time t for the issuing firm. It is shown in [17] that the above aggregate loss process retains the original distribution characteristics when changing from the physical to the risk-neutral probability measure Q.
Under the risk-neutral pricing measure, the value of the CAT bond price is given by the discounted expectation of its various payoffs in the risk-neutral world. In the following subsections we will specify the payoffs of the CAT bonds under three different scenarios:

3.1 Default-free CAT Bonds

First we consider the simple case where there is no default risk. The payoff $P_{CAT}(T)$ at maturity time T is then given by

$$P_{CAT}(T) = \begin{cases} 1 & \text{if} \quad C_T \le K \\ p & \text{if} \quad C_T > K, \end{cases} \tag{3.5}$$

where K is the trigger level in the CAT bond provisions and p is the proportion of the principal that has to be paid to bondholders when the amount of losses exceeds K. The face amount of any CAT bond considered in this paper is assumed to be 1. Under the assumptions of the model, the CAT bond price at time 0 is then given by

$$P_{CAT}(0) = A(0,T)e^{-B(0,T)r_0}\left[\sum_{j=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^j}{j!}F^j(K)+p\left(1-\sum_{j=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^j}{j!}F^j(K)\right)\right], \tag{3.6}$$

where $F^j(K) = Pr(X_1 + \ldots + X_j \leq K)$ denotes the jth convolution of the claim size distribution F and

$$A(0,T) = \left(\frac{2\gamma e^{(\kappa^*+\gamma)T/2}}{(\kappa^*+\gamma)(e^{\gamma T}-1)+2\gamma}\right)^{2\kappa^* m^*/\nu^2}, \; B(0,T) = \frac{2(e^{\gamma T}-1)}{(\kappa^*+\gamma)(e^{\gamma T}-1)+2\gamma}$$

with $\gamma = \sqrt{\kappa^{*2}+2\nu^2}$.

3.2 Default-Risky CAT Bonds Without Basis Risk

We now include the possibility of insolvency of the insurer in our model. Typically, CAT bondholders have priority for salvage over other debtholders. However, in this case bondholders still might only be repaid part of the principal and thus the default risk will lower the bond price. The payoff of the CAT bond in a model including default risk is thus given by

$$P_{def}(T) = \begin{cases} 1 & \text{if } \; C_T \leq K \text{ and } C_T \leq A_T - a\,L \\ p & \text{if } \; K < C_T < A_T - p\,a\,L, \\ \frac{\max\{A_T - C_T, 0\}}{a\,L} & \text{otherwise,} \end{cases} \tag{3.7}$$

where K and p are defined as in Section 3.1, L is the face amount of the issuing firm's total debts including the face amount of the CAT bonds and a is the ratio of the total CAT bonds' face amount over total outstanding debts.
The CAT bond price at issuing time 0 is then given by

$$P_{def}(0) = E_Q[e^{-\int_0^T r(t)dt}\, P_{def}(T)],$$

where E_Q denotes the expectation with respect to the risk-neutral pricing measure Q.

3.3 Default-Risky CAT Bonds With Basis Risk

In the preceding model the issuing firm obviously has the incentive to settle claims more generously when the incurred loss is close to the trigger level (an effect known as moral hazard). To circumvent this problem, an alternative design of a CAT bond is to define the forgiveness triggers as a function of a composite index of losses (among various firms exposed to the same catastrophic events)

$$C_{index,t} = \sum_{j=1}^{N(t)} X_{index,j}, \tag{3.8}$$

rather than on the individual aggregate loss process C_t of the issuing firm. Here $X_{index,j}$ denotes the amount of losses relevant for the index caused by the jth catastrophe. X_j and $X_{index,j}$, respectively, for $j = 1, \dots, N(t)$ are assumed to be iid random variables with (marginal) distribution function F, whereas X_j and $X_{index,j}$ are dependent according to some given copula structure (in [17] the dependence structure is modeled by correlation coefficients for the logarithm of lognormal marginals; our copula approach allows to generalize this approach to arbitrary marginal distributions; moreover the well-known deficiency of the correlation coefficient being a measure of linear dependence only is avoided (see e.g. [11])). In this paper we choose Frank's copula and Spearman's rank correlation coefficient ρ as the measure of positive dependence ($0 \le \rho \le 1$), see e.g. [21]. The gap between the insurer's actual loss and the composite index of losses (called the basis risk) may cause the insurer to default on his debt. Thus basis risk affects the bond price. The amount of basis risk is represented by ρ (with larger basis risk for smaller values of ρ).
The payoff of a default-risky CAT bond including basis risk is given by

$$P_{index}(T) = \begin{cases} 1 & \text{if } C_{index,T} \le K \text{ and } C_T \le A_T - a\,L \\ p & \text{if } K < C_{index,T} < A_T - p\,a\,L, \\ \frac{\max\{A_T - C_T, 0\}}{a\,L} & \text{otherwise.} \end{cases} \tag{3.9}$$

Accordingly, the corresponding CAT bond price at issuing time 0 is given by

$$P_{index}(0) = E_Q[e^{-\int_0^T r(t)dt} P_{index}(T)].$$

4 Numerical Analysis

In this section we first briefly discuss some implementation techniques needed to exploit and optimize the numerical advantages of Quasi-Monte Carlo integration for our pricing model of CAT bonds. We then present numerical results and compare the various solution methods in terms of efficiency.

The following refined version of the Koksma-Hlawka inequality (2.3) shows that one can improve the performance of QMC algorithms, if the first dimensions of the QMC sequences (which in most constructions have a lower discrepancy in the lower-dimensional faces, see e.g. [22]) are assigned to those variables which are responsible for most of the integrand's variation:

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{y}_n) - \int_{[0,1)^s} f(u)du \right| \leq \sum_{l=0}^{s-1} \sum_{F_l} D_N^* \left(\mathbf{y}_n^{(F_l)} \right) V^{(s-l)}(f^{(F_l)}), \tag{4.1}$$

where the second sum is extended over all $(s-l)$-dimensional faces F_l of the form $y_{i_1} = \cdots = y_{i_l} = 1$, the discrepancy $D_N^*(\mathbf{y}_n^{(F_l)})$ is computed in the face of $[0,1]^s$ containing $\left(\mathbf{y}_n^{(F_l)}\right)$, and $V^{(s-l)}(f^{(F_l)})$ is the variation (in the sense of Vitali) of the restriction $f^{(F_l)}$ of f to F_l (see e.g. [10]).
For this reason, in the algorithms designed in this paper the first QMC dimension is always used to determine the number $N(T)$ of claims occurring until maturity. Our required distribution is calculated from the uniform distribution by standard inversion techniques (cf. [9]).

Let us first consider the case of default-free CAT bonds. Equations (3.5) and (3.6) show that determining the corresponding price reduces to simulating a compound Poisson process (i.e. the number and size of claims in the given time interval). Since catastrophic events are rare, this problem therefore has rather low dimension from the perspective of numerical integration; for the parameter values chosen in this paper, the dimension typically is between 3 to 5 (note that calculating the expected value of an s-dimensional random variable is equivalent to evaluating an s-dimensional integral).

The dimension of the problem increases considerably, when default risk of the bonds is included in the model. Due to the dependence between the asset process A_t and the stochastic interest rate process r_t one has to simulate the interest rates explicitly (opposed to the case of default-free bonds, where a suitable choice of numeraire led to (3.6)). In accordance with [17], we discretize the stochastic differential equation (3.3) to

$$r_{i+1} = r_i + \kappa^*(m^* - r_i)(t_{i+1} - t_i) + \nu\sqrt{r_i(t_{i+1} - t_i)} Z_{i+1}, \tag{4.2}$$

where the time span $t_{i+1} - t_i$ is a week, which for $T = 1$ year introduces 52 dimensions. Here Z_i (i=1..52) are iid random variables with standard normal distribution. A well-known problem in this context is that due to (4.2) negative interest rates are possible. However, this only happens for a negligible fraction of sample paths and thus it seems safe to ignore these paths (i.e. not to use them in the calculations). For the asset process, we solve the stochastic differential equation (3.2) to obtain

$$A_t = A_s \exp\left[(1 - \frac{\phi^2\nu^2}{2}) \int_s^t r_u \, du - \frac{\sigma^2}{2}(t-s) + \phi\nu \int_s^t \sqrt{r_u} dZ_u^* + \sigma W_{t-s}^*\right],$$

where $t > s$, and discretize the solution in the following way

$$A_{i+1} = A_i \exp\Bigg[(1 - \tfrac{\phi^2\nu^2}{2})(t_{i+1} - t_i)\tfrac{r_{i+1}+r_i}{2} - \tfrac{\sigma^2}{2}(t_{i+1} - t_i) + \\ + \phi\nu\tfrac{\sqrt{r_{i+1}}+\sqrt{r_i}}{2}\sqrt{(t_{i+1} - t_i)}Z_{i+1} + \sigma\sqrt{(t_{i+1} - t_i)}W_{i+1}\Bigg],$$

where W_i are iid standard normal random variables ($i = 1, \ldots, 52$), mutually independent of Z_i ($i = 1, \ldots, 52$). Therefore we face a problem with dimension larger than 100.
In view of (2.3) and (2.5), theoretically QMC integration should lead to good results for small dimensions only. However, empirically it turns out that QMC techniques can be very effective also for high dimensional numerical integration. One explanation is the low effective dimension of many integrands. This concept (introduced in [4]) formalizes the idea that only a few variables are responsible for most of the variation of the integrand. One well-known technique for reducing the effective dimension of the integrand when simulating a Wiener process is to use Brownian bridges (cf. [20]), which is based on a reordering of the discretization points and using conditional distributions to calculate the corresponding sample paths (see also [16]). We will implement and compare this technique in the sequel.

As indicated in Section 2, in many practical situations QMC methods turn out to be particularly effective, if QMC sequences are used for the "effective" dimensions only and the remaining dimensions are then simulated using pseudo-random numbers. This hybrid Monte Carlo approach increases the simulation speed of pure QMC, while the convergence speed of the relative error in terms of N is nearly unafffected. Throughout this paper, all QMC algorithms are hybrid algortihms with 30 QMC dimensions, and we use the *Mersenne Twister*

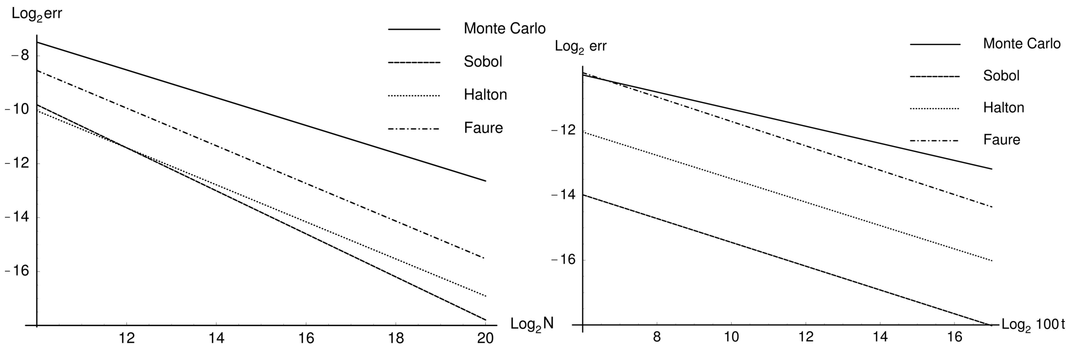

Figure 1: Approximation of default-free CAT bond prices (Lognormal claims): Relative error as a function of the sample size N (left) and calculation time t (right)

as our pseudorandom number generator, which basically is an optimized version of a multiplicative congruential algorithm (see [19]). In our calculations, the first QMC dimension is always used to generate the number of losses $N(T)$, and the following initial dimensions are used to generate the catastrophic loss amounts X_j (resp. $X_{index,j}$). For consistency of the convergence analysis it is important to fix the number of QMC dimensions reserved for the loss amounts. The remaining QMC dimensions are then assigned to the most important variates of the asset process and the interest rate process, respectively.

The introduction of basis risk makes it necessary not only to generate claim amounts (of the index) $X_{index,j}$, but also random variables X_j dependent on $X_{index,j}$ according to the specified copula structure (see Section 3.3). This can be achieved by implementing a modified inversion technique involving the partial derivatives of the copula function (see e.g. [21], Chapter 2). The dimension of this model is slightly bigger than without basis risk.

Table 1 now gives the parameter values used in the calculations below. For the catastrophic claim size distribution, the heavy-tailed Lognormal and Weibull distributions are chosen with typical parameters in practical situations.

Figure 1 depicts least-squares linear regression fits of the relative error of QMC and Monte Carlo techniques as a function of the sample size N and the calculation time t (measured in seconds), respectively. The "exact" value, in lack of an analytic solution, is obtained by a Monte Carlo simulation over 50 million

T	term of the contract (in years)	1
σ_A	volatility of credit risk	0.05
ϕ	interest elasticity of asset	-3
A_0/L	asset/liability ratio	1.1
r_0	initial instantaneous interest rate	0.05
k	magnitude of mean reverting force	0.2
m	long run mean of interest rates	0.05
ν	volatility of interest rates	0.1
λ_r	market price of interest risk	-0.01
K	trigger level	100
p	ratio of principal paid back in case the trigger has been pulled	0.5
a	ratio of amount of CAT bonds to total debts	0.1
L	insurer's total amount of debts	100
Claim size distribution	density	
Lognormal	$f(x) = \frac{1}{\sqrt{2\pi}\cdot 2x} e^{-\frac{(\log x - 2)^2}{8}}$	
Weibull	$f(x) = 0.32x^{0.96}e^{-0.16\,x^{1.96}}$	

Table 1: Parameter values and catastrophic claim size distributions

sample paths. For every N, the estimate of the Monte Carlo error was obtained by averaging the error of 25 runs with N sample paths each. The plot shows that the QMC algorithms outperform crude Monte Carlo significantly, both for a given number of sample paths N and given calculation time t. The latter plot also contains the following information: for a given calculation time t, what is the relative error of the estimate that can be achieved by each of these methods? In the case of default-free CAT bonds, the Sobol sequence seems to be best suited for the corresponding integrand.

Figure 2 shows a comparison of MC and QMC algorithms for the estimation of the price of default-risky CAT bonds without basis risk. The subscript $_{BB}$ refers to the Brownian bridge algorithm of the respective QMC technique. Again the QMC algorithms turn out to be much more effective, both in terms of N and t. The Brownian bridge approach discussed above leads to a slight improvement over the respective plain QMC algorithm. Note that the dimension of the problem in Figure 2 is much higher than that of Figure 1. Especially for large values of N and t, the Sobol and Halton sequences seem to be particularly competitive.

Default-free CAT bonds					
CD	N	MC	Sobol	Halton	Faure
Lognormal	2^{10}	4.7E-03	1.2E-03	7.4E-04	4.6E-03
$\lambda = 2$	2^{15}	9.8E-04	5.3E-04	2.2E-04	1.1E-04
	2^{20}	1.5E-04	4.9E-06	1.6E-05	2.8E-06

Default-risky CAT bonds without basis risk								
CD	N	MC	Sobol	Halton	Faure	S_{BB}	H_{BB}	F_{BB}
LN	2^{10}	1.0E-02	4.9E-03	1.1E-03	1.0E-02	3.8E-03	7.5E-05	9.9E-03
$\lambda = 2$	2^{15}	2.3E-03	5.5E-05	2.2E-04	2.2E-04	1.0E-04	1.8E-04	5.1E-04
	2^{20}	3.9E-04	1.6E-04	1.2E-04	1.0E-04	8.8E-05	8.4E-05	5.5E-06
Weibull	2^{10}	8.6E-03	1.1E-03	7.1E-03	1.8E-02	2.1E-03	5.7E-03	2.0E-02
$\lambda = 4$	2^{15}	1.8E-03	1.5E-03	1.3E-03	8.2E-04	1.8E-03	7.1E-04	6.8E-04
	2^{20}	1.9E-04	2.6E-05	8.9E-05	1-2E-04	8.4E-05	8.4E-05	8.0E-05

Default-risky CAT bonds with basis risk								
CD	N	MC	Sobol	Halton	Faure	S_{BB}	H_{BB}	F_{BB}
LN	2^{11}	1.1E-02	4.4E-03	6.5E-03	9.5E-04	1.3E-03	2.2E-03	2.4E-04
ρ=0.5	2^{15}	2.5E-03	8.8E-05	4.7E-04	1.4E-03	2.9E-04	4.6E-04	7.8E-04
$\lambda = 2$	2^{20}	4.3E-04	3.6E-05	6.3E-05	3.3E-05	5.8E-05	7.0E-05	1.4E-04
Weibull	2^{11}	6.2E-03	4.6E-03	1.9E-03	1.8E-02	3.0E-03	1.4E-04	1.7E-02
ρ=0.8	2^{15}	1.4E-03	4.4E-04	9.0E-04	1.5E-05	1.1E-04	1.5E-03	3.9E-04
$\lambda = 4$	2^{20}	3.2E-04	1.5E-04	1.0E-04	2.7E-05	7.4E-05	2.7E-05	7.9E-05
Weibull	2^{11}	5.5E-03	4.1E-04	2.0E-03	1.6E-02	6.9E-04	6.7E-04	1.5E-03
$\rho = 0.3$	2^{15}	1.6E-03	8.7E-04	1.3E-03	3.2E-03	1.1E-03	1.8E-03	1.4E-04
$\lambda = 4$	2^{20}	3.3E-04	2.7E-05	9.6E-05	6.6E-05	9.9E-05	3.3E-05	1.1E-04

Table 2: Relative errors of MC and QMC algorithms for given claim distributions (CD) and values of ρ and λ.

Figure 3 captures the situation in the case of default-risky CAT bonds including basis risk for various dependence levels ρ. Again, Monte Carlo methods are always beaten by their deterministic alternatives. Some of the QMC sequences outperform the Monte Carlo approach by several orders of magnitude. Note that the size of the relative error in all these plots is very small, which further emphasizes the competitiveness of QMC techniques (e.g. for Weibull distributed claims and $\rho = 0.8$, using the Sobol Brownian bridge technique, a relative error of $5 \cdot 10^{-4}$ can be achieved in about 0.1 seconds!).
As a numerical illustration, Table 2 gives relative errors of our MC and QMC algorithms in all three models for various parameter choices and increasing N.

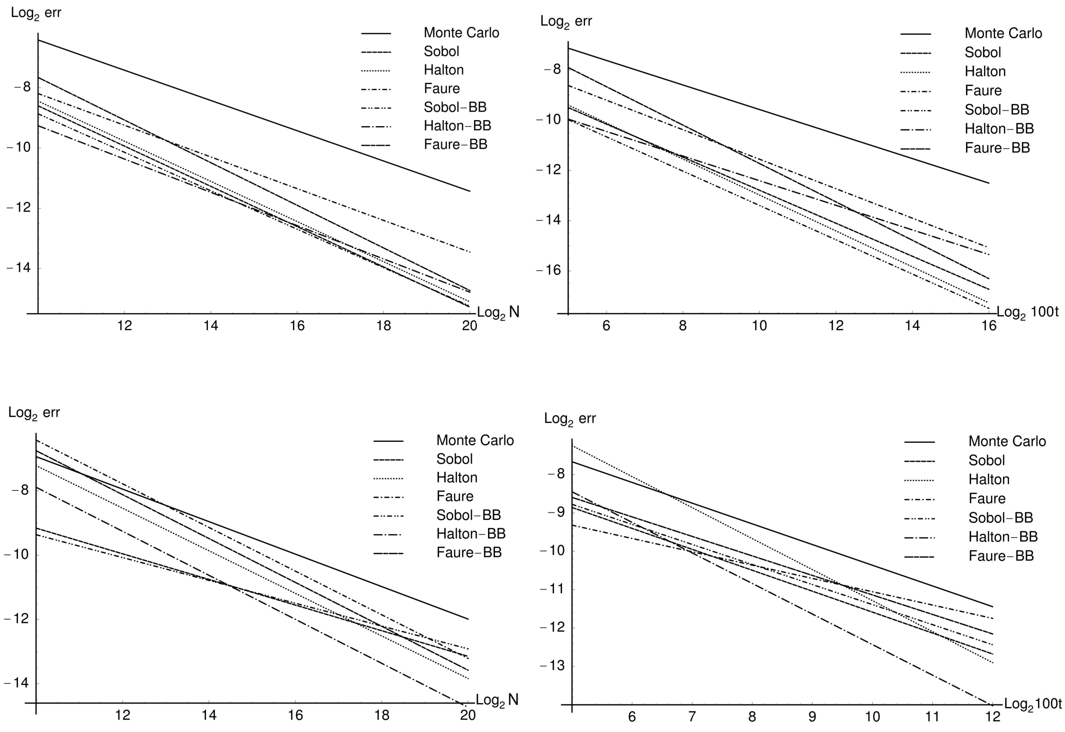

Figure 2: Approximation of default-risky CAT bond prices without basis risk (Lognormal claims (above) and Weibull claims (below)): Relative error as a function of N (left) and calculation time t (right)

Note that it can happen that QMC estimates are closer to the exact value for lower values of N. This may happen since the actual estimate for a given N may be above or below the average behavior represented by the regression fit. This effect does not occur for the Monte Carlo approach, since we already average over 25 simulation runs.

Finally, Figure 4 depicts the CAT bond prices in the presence of basis risk as function of the dependence parameter ρ (representing the amount of basis risk) and the ratio a of the total CAT bonds' face value over total outstanding debts. Moreover the CAT bond prices as a function of ρ and trigger level K are given. Numerical illustrations are given in Table 3. An increase in basis risk leads to considerably lower CAT bond prices, which shows that neglecting basis risk can be very dangerous for holders of these bonds.

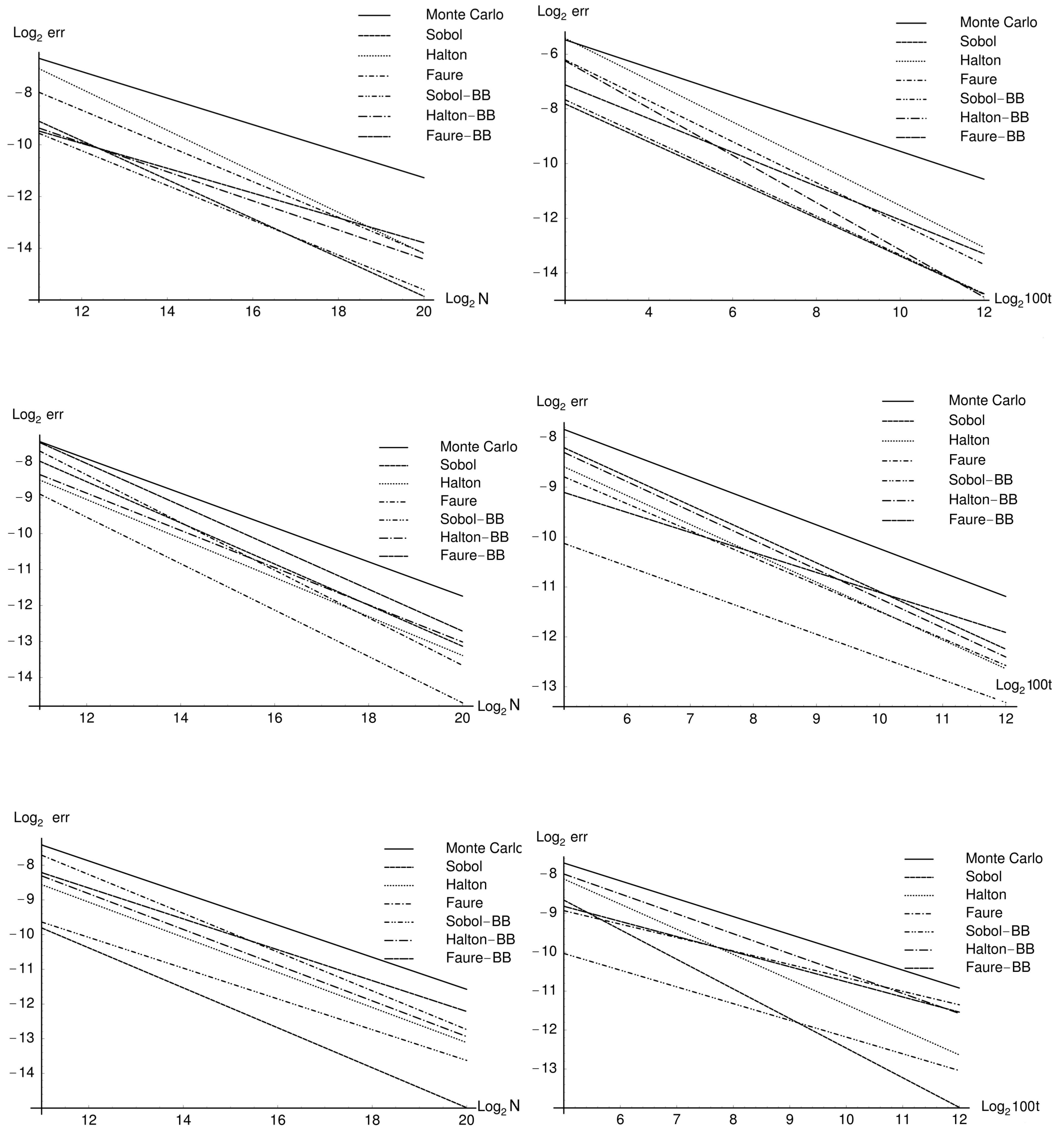

Figure 3: Approximation of default-risky CAT bond prices with different levels of basis risk (Lognormal claims, $\rho = 0.5$ (top); Weibull claims, $\rho = 0.8$ (middle) and Weibull claims, $\rho = 0.3$ (bottom)): Relative error as a function of N (left) and calculation time t (right).

a	**Spearman** ρ					K	**Spearman** ρ				
	0	0.2	0.5	0.8	1		0	0.2	0.5	0.8	1
0.1	0.695	0.700	0.709	0.722	0.758	80	0.682	0.687	0.697	0.711	0.740
0.3	0.683	0.688	0.698	0.712	0.747	90	0.689	0.694	0.703	0.717	0.750
0.5	0.668	0.674	0.684	0.699	0.729	100	0.695	0.699	0.708	0.722	0.757
0.7	0.65	0.656	0.666	0.681	0.705	110	0.700	0.704	0.713	0.726	0.762
0.9	0.624	0.631	0.641	0.655	0.672	120	0.704	0.709	0.717	0.730	0.764

Table 3: CAT bond prices for various levels of basis risk (ρ), ratio a and trigger levels K.

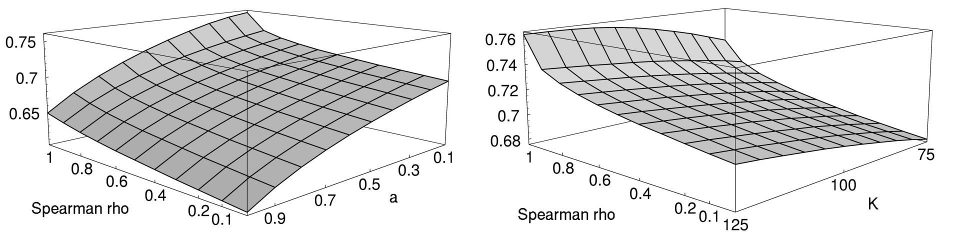

Figure 4: Default-risky CAT bond prices as a function of basis risk (ρ) and ratio a of CAT bonds in total debt (left), respectively trigger level K (right).

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