Second-order Tail Asymptotics of Deflated Risks

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April 11, 2014

Abstract: Random deflation of risk models is an interesting topic for both theoretical and practical actuarial problems. In this paper, we investigate second-order tail asymptotics of the deflated risk $X = RS$ under the assumptions of second-order regular variation on the survival functions of the risk R and the deflator S. Our findings are applied to derive second-order expansions of Value-at-Risk. Further we investigate the estimation of small tail probability for deflated risks and then discuss the asymptotics of the aggregated deflated risk.

Key words and phrases: Random deflation; Value-at-Risk; Risk aggregation; Second-order regular variation; Estimation of tail probability.

JEL classification: G22

MSC: 60G70, 62G32, 62G20, 91B30

1 Introduction

Let R be a non-negative random variable (rv) with distribution function (df) F being independent of the rv $S \in (0, 1)$ with df G. If R models the loss amount of a financial risk, and S models a random deflator for a particular time-period, then the product $X = RS$ represents the deflated value of R at the end of the time-period under consideration. Random deflation is a natural phenomena in various actuarial applications attributed to the time-value of money. When large values or extremes are of interest, for instance for reinsurance pricing and risk management purposes, it is important to link the behaviors of the risk R and the random deflator S. Intuitively, we expect that large values observed for R are not significantly influenced by the random deflation. However, this is not always the case; a precise analysis driven by some extreme value theory models is given in Tang and Tsitsiashvili (2004), Tang (2006, 2008), Hashorva et al. (2010), Arendarczyk and Dębicki (2011), Tang and Yang (2012), Zhu and Li (2012), Hashorva (2013), Yang and Hashorva (2013), Yang and Wang (2013), and the references therein. The results of the aforementioned papers are obtained mainly under a first-order asymptotic condition for the survival function or the quantile function in extreme value theory, i.e., the df F under consideration belongs to the max-domain of attraction (MDA) of a univariate extreme value distribution $Q_\gamma, \gamma \in \mathbb{R}$, abbreviated as $F \in D(Q_\gamma)$, which means that

$$F^n(a_n x + b_n) \to \exp\left(-(1+\gamma x)^{-1/\gamma}\right) =: Q_\gamma(x), \quad 1+\gamma x > 0, \quad n \to \infty \tag{1.1}$$

holds for some constants $a_n > 0$ and $b_n \in \mathbb{R}, n \geq 1$, see Resnick (1987). The parameter γ is called the extreme value index; according to $\gamma > 0, \gamma = 0$ and $\gamma < 0$, the df F belongs to the MDA of the Fréchet distribution, the Gumbel distribution and the Weibull distribution, respectively.

In order to derive some more informative asymptotic results, second-order regular variation (2RV) conditions are widely used in extreme value theory. Here we only mention de Haan and Resnick (1996) for the uniform convergence rate of F^n to its ultimate extreme value distribution Q_γ under 2RV, and Beirlant et al. (2009, 2011), Ling et al. (2012) and the references therein for the asymptotic distributions of the extreme value index estimators under consideration.

Indeed, almost all the common loss distributions including log-gamma, absolute t, log-normal, Weibull, Benktander II, Beta (cf. Table 2 in the Appendix) possess 2RV properties; actuarial applications based on those properties are developed in the recent contributions Hua and Joe (2011), Mao and Hu (2012, 2013) and Yang (2013).

The main contributions of this paper concern the second-order expansions of the tail probability of the deflated risk $X = RS$ which are then illustrated by several examples. Our main findings are utilized for the formulations of three applications, namely approximation of Value-at-Risk, estimation of small tail probability of the deflated risk, and the derivation of the tail asymptotics of aggregated risk under deflation.

The rest of this paper is organized as follows. In Section 2 we present our main results under second-order regular variation conditions. Section 3 shows the efficiency of our second-order asymptotics through some illustrating examples. Section 4 is dedicated to three applications. The proofs of all results are relegated to Section 5. We conclude the paper with a short Appendix.

2 Main results

We start with the definitions and some properties of regular variation followed by our principal findings. A measurable function $f : [0, \infty) \to \mathbb{R}$ with constant sign near infinity is said to be of second-order regular variation with parameters $\alpha \in \mathbb{R}$ and $\rho \leq 0$, denoted by $f \in 2\mathrm{RV}_{\alpha,\rho}$, if there exists some function A with constant sign near infinity satisfying

$\lim_{t\to\infty} A(t) = 0$ such that for all $x > 0$ (cf. Bingham et al. (1987) and Resnick (2007))

$$\lim_{t\to\infty} \frac{f(tx)/f(t) - x^{\alpha}}{A(t)} = x^{\alpha} \int_1^x u^{\rho-1}\, du =: H_{\alpha,\rho}(x). \tag{2.1}$$

Here, A is referred to as the auxiliary function of f. Note that (2.1) implies $\lim_{t\to\infty} f(tx)/f(t) = x^{\alpha}$, i.e., f is regularly varying at infinity with index $\alpha \in \mathbb{R}$, denoted by $f \in \mathrm{RV}_{\alpha}$. RV_0 is the class of slowly varying functions. For f eventually positive, it is of second-order Π-variation with the second-order parameter $\rho \le 0$, denoted by $f \in 2\mathrm{ERV}_{0,\rho}$, if there exist some functions a and A with constant signs near infinity and $\lim_{t\to\infty} A(t) = 0$ such that for all x positive

$$\lim_{t\to\infty} \frac{\frac{f(tx)-f(t)}{a(t)} - \ln x}{A(t)} = \psi(x) := \begin{cases} \frac{x^{\rho}-1}{\rho}, & \rho < 0, \\ \frac{\ln^2 x}{2}, & \rho = 0 \end{cases} \tag{2.2}$$

(cf. Resnick (2007)), where the functions a and A are referred to as the first-order and the second-order auxiliary functions of f, respectively. From Theorem B.3.1 in de Haan and Ferreira (2006) we see that $a \in 2\mathrm{RV}_{0,\rho}$ with auxiliary function A, and that $|A| \in \mathrm{RV}_{\rho}$. In fact, (2.2) implies $\lim_{t\to\infty}(f(tx) - f(t))/a(t) = \ln x$ for all $x > 0$, which means f is Π-varying with auxiliary function a, denoted by $f \in \Pi(a)$.

We shall keep the notation of the Introduction for R and $S \in (0, 1)$, denoting their df's by F and G, respectively, whereas the df of $X = RS$ will be denoted by H. Throughout this paper, let $\bar{F}_0 = 1 - F_0$ denote the survival function of a given df F_0.

Next, we present our main results. Theorem 2.1 gives a second-order counterpart of Breiman's Lemma (see Breiman (1965)) while Theorem 2.3 and Theorem 2.6 include refinements of the tail asymptotics of products derived in Hashorva et al. (2010).

Theorem 2.1. *If $F \in D(Q_{1/\alpha_1})$ satisfies $\bar{F} \in 2\mathrm{RV}_{-\alpha_1,\tau_1}$ with auxiliary function $\tilde{A}$ for some $\alpha_1 > 0$ and $\tau_1 \le 0$, then*

$$\frac{\bar{H}(x)}{\bar{F}(x)} = \mathbb{E}\left\{S^{\alpha_1}\right\}\left[1 + \mathcal{E}(x)\right], \tag{2.3}$$

where $\mathcal{E}(x) = \left(\mathbb{E}\left\{S^{\alpha_1-\tau_1}\right\}/\mathbb{E}\left\{S^{\alpha_1}\right\} - 1\right)\tilde{A}(x)/\tau_1(1+o(1))$ as $x \to \infty$, and thus $\bar{H} \in 2\mathrm{RV}_{-\alpha_1,\tau_1}$ with auxiliary function

$$A^*(x) = \frac{\mathbb{E}\left\{S^{\alpha_1-\tau_1}\right\}}{\mathbb{E}\left\{S^{\alpha_1}\right\}}\tilde{A}(x).$$

Remark 2.2. *a) The expression for $\tau_1 = 0$ is understood throughout this paper as its limit as $\tau_1 \to 0$.*

b) Under the assumptions of Theorem 2.1, Breiman's Lemma only implies

$$\frac{\bar{H}(x)}{\bar{F}(x)} = \mathbb{E}\left\{S^{\alpha_1}\right\}\left[1 + \mathcal{E}^*(x)\right]$$

with $\lim_{x\to\infty} \mathcal{E}^*(x) = 0$, while the error term $\mathcal{E}(x)$ in (2.3) not only converges to 0 as $x \to \infty$, but it shows also the speed of convergence being determined by $\tilde{A}(x)$.

Next, we shall consider the cases that F belongs to the MDA of the Gumbel distribution and the Weibull distribution, respectively. Compared to the heavy-tail case above, we need to impose some assumptions on the tail of S; see Hashorva et al. (2010). In our setting, we strengthen L (see (2.4) below for an accurate definition) to be of second-order regular variation.

We shall write $Y \sim Q$ for some rv Y with df Q, whereas $Q^{\leftarrow}$ denotes the generalized left-continuous inverse of Q (also for Q which are not dfs). Since both H and F have the same upper endpoint $x_H = x_F := \sup\{y : F(y) < 1\}$, then all the limit relations below are for $x \uparrow x_F$ unless otherwise specified. Further, for some $\alpha_2 > 0$ we set

$$L(x) = x^{\alpha_2} \bar{G}\left(1 - \frac{1}{x}\right), \quad K(\alpha_2, \rho) = \begin{cases} \frac{(1-\rho)^{-\alpha_2}-1}{\rho}\Gamma(\alpha_2+1), & \rho < 0, \\ \frac{\alpha_2 \Gamma(\alpha_2+2)}{2}, & \rho = 0, \end{cases} \tag{2.4}$$

where $\Gamma(\cdot)$ is the Euler Gamma function, and define

$$w(x) = \frac{1}{\mathbb{E}\{R - x | R > x\}}, \quad \eta(x) = x w(x). \tag{2.5}$$

Hereafter the generalized left-continuous inverses of $1/\bar{F}$ and $1/\bar{H}$ are denoted respectively by

$$U{=}U_R = (1/\bar{F})^{\leftarrow} \quad \text{and} \quad U_X = (1/\bar{H})^{\leftarrow}.$$

Theorem 2.3. *Let F be strictly increasing and continuous in the left neighborhood of x_F and let $U \in 2\mathrm{ERV}_{0,\rho}, \rho \le 0$ with auxiliary functions $1/w(U)$ and $\tilde{A}$. If $L \in 2\mathrm{RV}_{0,\tau_2}, \tau_2 < 0$ with auxiliary function A, then*

$$\frac{\bar{H}(x)}{\bar{F}(x)\bar{G}\left(1 - 1/\eta(x)\right)} = \Gamma(\alpha_2 + 1) + \mathcal{E}(x), \tag{2.6}$$

where $K(\alpha_2, \rho), \eta(x)$ are defined in (2.4), (2.5), and

$$\mathcal{E}(x) = \left[\frac{\Gamma(\alpha_2 - \tau_2 + 1) - \Gamma(\alpha_2 + 1)}{\tau_2} A(\eta(x)) - \frac{\alpha_2 \Gamma(\alpha_2 + 2)}{\eta(x)} + K(\alpha_2, \rho)\tilde{A}\left(\frac{1}{\bar{F}(x)}\right)\right](1 + o(1)).$$

In view of our second result above, the error term $\mathcal{E}(x)$ converges to 0 as $x \uparrow x_F$ with a speed which is determined by $A(\eta(x)), 1/\eta(x)$ and $\tilde{A}(1/\bar{F}(x))$. In general, it is not clear which of these terms is asymptotically relevant for the definition of the error term $\mathcal{E}(x)$. For instance in Example 3.3 below $\tilde{A}(1/\bar{F}(x))$ determines $\mathcal{E}(x)$. However, Example 3.4 shows the opposite, namely $\tilde{A}(1/\bar{F}(x))$ does not appear in our second-order approximation.

Corollary 2.4. *Under the conditions of Theorem 2.3, with ψ and w given by (2.2) and (2.5), respectively, then for $z \in \mathbb{R}$*

$$\frac{\bar{H}(x + z/w(x))}{\exp(-z)\bar{H}(x)} = 1 + \mathcal{E}(x), \quad \mathcal{E}(x) = \left[\left(\psi(e^{-z}) + \alpha_2 \frac{e^{\rho z} - 1}{\rho}\right)\tilde{A}\left(\frac{1}{\bar{F}(x)}\right) - \frac{\alpha_2 z}{\eta(x)}\right](1 + o(1)), \tag{2.7}$$

where $(e^{\rho z} - 1)/\rho$ is interpreted as z for $\rho = 0$. Thus $U_X \in 2\mathrm{ERV}_{0,0}$ with auxiliary functions $\breve{a}$ and $\breve{A}$ given by

$$\breve{a}(x) = \tilde{a}(x)\left(1 - \frac{\alpha_2 \tilde{a}(x)}{U_X(x)} + \alpha_2 \tilde{A}\left(\frac{1}{\bar{F}(U_X(x))}\right)\right), \quad \breve{A}(x) = -\frac{\alpha_2^2 \tilde{a}^2(x)}{U_X^2(x)} + \tilde{A}\left(\frac{1}{\bar{F}(U_X(x))}\right), \tag{2.8}$$

where $\tilde{a} = 1/w(U_X)$.

Numerous dfs in the MDA of the Gumbel distribution have Weibull tails (see Embrechts et al. (1997) and Table 1 in the Appendix); specifically such a distribution function F has the representation

$$\bar{F}(x) = \exp(-V(x)), \quad V^{\leftarrow}(x) = x^{\theta}\ell(x), \ \theta > 0, \tag{2.9}$$

where ℓ denote a positive slowly varying function at infinity, and θ is called the Weibull tail coefficient of F.

Corollary 2.5. *Under the conditions of Theorem 2.3, if instead we assume that F is given by* (2.9) *and* $\ell \in 2\mathrm{RV}_{0,\rho'}, \rho' \le 0$ *with auxiliary function b, then*

$$\bar{H}(x) = \exp(-V(x))\bar{G}\left(1 - \frac{1}{V(x)}\right)\Gamma(\alpha_2+1)\theta^{\alpha_2}\left[1 + \mathcal{E}(x)\right], \tag{2.10}$$

with

$$\mathcal{E}(x) = \left(\frac{\alpha_2}{\theta}b(V(x)) + \frac{\dfrac{\Gamma(\alpha_2-\tau_2+1)}{\theta^{\tau_2}\Gamma(\alpha_2+1)} - 1}{\tau_2}A(V(x)) - \frac{\alpha_2(\alpha_2+1)(\theta+1)}{2V(x)}\right)(1+o(1)),$$

and thus

$$\bar{H}(x) = \exp(-V^*(x)), \quad (V^*)^{\leftarrow}(x) = x^{\theta}\ell^*(x),$$

where $\ell^* \in 2\mathrm{RV}_{0,\rho'^*}$ *with auxiliary function* $b^*(x) = b(x) + \theta\alpha_2(\ln x)/x, \rho'^* = \max(\rho', -1)$.

Theorem 2.1 and Corollary 2.4 illustrate that the tail asymptotics of the product $X = RS$ mainly depends on the heavier factor R. Corollary 2.5 shows that for the Weibull tail distributions, the Weibull tail properties of X are inherited from the factor R in the presence of random deflation. The result of Corollary 2.5 is of particular interest for the estimation of tail probabilities, see Section 4.2.

Our last theorem shows that for both R and S belonging to the MDA of the Weibull distribution, the tail of the product $X = RS$ is heavier than those of the factors R and S.

Theorem 2.6. *Let F be strictly increasing and continuous in the left neighborhood of $x_F = 1$. Assume that for some* $\alpha_1 > 0, \tau_1 \le 0, 1 - U \in 2\mathrm{RV}_{-1/\alpha_1,\tau_1/\alpha_1}$ *with auxiliary function $\tilde{A}$. If further* $L \in 2\mathrm{RV}_{0,\tau_2}, \tau_2 \le 0$ *with auxiliary function A, then*

$$\frac{\bar{H}(x)}{\bar{F}(x)\bar{G}(x)} = \alpha_1 B\left(\alpha_1, \alpha_2+1\right) + \mathcal{E}(x), \tag{2.11}$$

where

$$\begin{aligned}\mathcal{E}(x) = \Bigg[&-\frac{\alpha_1^2\alpha_2}{\tau_1}\big(B\left(\alpha_2, \alpha_1-\tau_1+1\right) - B\left(\alpha_2, \alpha_1+1\right)\big)\tilde{A}\left(\frac{1}{\bar{F}(x)}\right) + \alpha_1\alpha_2 B\left(\alpha_1+1, \alpha_2+1\right)(1-x)\\ &+ \frac{\alpha_1}{\tau_2}\big(B\left(\alpha_1, \alpha_2-\tau_2+1\right) - B\left(\alpha_1, \alpha_2+1\right)\big)A\left(\frac{1}{1-x}\right)\Bigg](1+o(1)),\end{aligned}$$

with $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b), a, b > 0$.

Remark 2.7. *Recall that for a df F with a finite upper endpoint x_F belonging to MDA of the Weibull distribution, then for some $\alpha_1 > 0, \tau_1 \le 0, x_F - U \in 2\mathrm{RV}_{-1/\alpha_1, \tau_1/\alpha_1}$ with auxiliary function $\tilde{A}$, is equivalent that $\bar{F}(x_F - 1/x) \in 2\mathrm{RV}_{-\alpha_1,\tau_1}$ with auxiliary function $\tilde{A}^*(x) = -\alpha_1^2 \tilde{A}\left(1/\bar{F}(x_F - 1/x)\right)$ and $|\tilde{A}^*| \in \mathrm{RV}_{\tau_1}$ (cf. Theorem 2.3.8 in de Haan and Ferreira (2006)). Thus* (2.11) *holds with*

$$\begin{aligned}\mathcal{E}(x) = &\left[\frac{\alpha_2}{\tau_1}\left(B\left(\alpha_2, \alpha_1 - \tau_1 + 1\right) - B\left(\alpha_2, \alpha_1 + 1\right)\right)\tilde{A}^*\left(\frac{1}{1-x}\right) + \alpha_1\alpha_2 B\left(\alpha_1 + 1, \alpha_2 + 1\right)(1-x)\right.\\ &\left.+ \frac{\alpha_1}{\tau_2}\left(B\left(\alpha_1, \alpha_2 - \tau_2 + 1\right) - B\left(\alpha_1, \alpha_2 + 1\right)\right)A\left(\frac{1}{1-x}\right)\right](1 + o(1)).\end{aligned}$$

Remark 2.8. *Under the assumptions of Theorem 2.6, $\bar{H}(1 - 1/x) \in 2\mathrm{RV}_{-\alpha,\tau}$ with $\alpha = \alpha_1 + \alpha_2$ and $\tau = \max(-1, \tau_1, \tau_2)$.*

3 Examples

In this section, six examples are presented to illustrate estimation errors of the second-order expansions given by Section 2 and the first-order asymptotics by Breiman (1965) and Hashorva et al. (2010). We use the R-Project to calculate the exact value of $\bar{H}(x)$. Fig. 1$\sim$ Fig. 5 illustrate the advantage of our second-order tail approximations.

Example 3.1. *(Fréchet case with Pareto distribution)* Let R be a random variable with a Pareto df F given by

$$\bar{F}(x) = \left(\frac{\theta}{x+\theta}\right)^{\alpha}, \quad x > 0,\ \alpha, \theta > 0$$

denoted in the sequel by $R \sim Pareto(\alpha, \theta)$. Suppose that $S \sim beta(a, b)$ where $beta(a, b)$ stands for the Beta distribution with positive parameters a and b and probability density function (pdf)

$$g(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1,\ a, b > 0. \tag{3.1}$$

We have that $\bar{F} \in 2\mathrm{RV}_{-\alpha,-1}$ with auxiliary function $\tilde{A}(x) = \alpha\theta/x$ and $\mathbb{E}\left\{S^\kappa\right\} = B(a+\kappa, b)/B(a, b)$ for all $\kappa > 0$. By Theorem 2.1 with $\alpha_1 = \alpha$ and $\tau_1 = -1$

$$\bar{H}(x) = \bar{F}(x)\mathbb{E}\left\{S^\alpha\right\}[1 + \mathcal{E}(x)] = \left(\frac{\theta}{x+\theta}\right)^{\alpha} \frac{B(a+\alpha, b)}{B(a,b)}[1 + \mathcal{E}(x)],$$

with

$$\mathcal{E}(x) = \left(1 - \frac{\mathbb{E}\left\{S^{\alpha+1}\right\}}{\mathbb{E}\left\{S^{\alpha}\right\}}\right)\tilde{A}(x)(1 + o(1)) = \frac{\alpha\theta b}{(\alpha + a + b)x}(1 + o(1)).$$

Fig. 1 compares the first-order and the second-order asymptotic expansions with the exact true value $\bar{H}(x)$ when $R \sim Pareto(\alpha, \theta), S \sim beta(a, b)$ with $(\alpha, \theta, a, b) = (1, 1, 1, 2)$ (left) and $(\alpha, \theta, a, b) = (2, 1, 1, 2)$ (right). As expected, we find that the second-order tail asymptotics is more accurate than the first-order one.

Example 3.2. *(Fréchet case with Beta distribution of second kind)* Let R be a random variable with Beta distribution of second kind with positive parameters a, b, i.e., $R \stackrel{d}{=} 1/R_0 - 1, R_0 \sim beta(b, a)$, denoted by $R \sim beta_2(a, b)$ (here $\stackrel{d}{=}$ stands for equality of distribution function). It follows from (3.1) that

$$\mathbb{P}(R_0 < x) = \frac{x^b}{bB(b,a)} \left[1 - \frac{(a-1)b}{1+b} x(1+o(1))\right], \quad x \downarrow 0,$$

and thus

$$\bar{F}(x) = \mathbb{P}(R > x) = \mathbb{P}\left(R_0 < \frac{1}{1+x}\right) = \frac{x^{-b}}{bB(b,a)} \left[1 - \frac{(a+b)b}{(1+b)x}(1+o(1))\right], \quad x \to \infty, \tag{3.2}$$

i.e., $\bar{F} \in 2\mathrm{RV}_{-b,-1}$ with auxiliary function $\tilde{A}(x) = (a+b)b/((1+b)x)$. Let $S \sim beta(c, d)$, and then $\mathbb{E}\{S^\kappa\} = B(c+\kappa, d)/B(c, d)$ for all $\kappa > 0$. In view of Theorem 2.1 with $\alpha_1 = b$ and $\tau_1 = -1$

$$\bar{H}(x) = \bar{F}(x)\mathbb{E}\left\{S^b\right\}[1+\mathcal{E}(x)] = \frac{x^{-b}}{bB(b,a)} \left[1 - \frac{(a+b)b}{(1+b)x}(1+o(1))\right] \frac{B(b+c,d)}{B(c,d)}[1+\mathcal{E}(x)],$$

with

$$\mathcal{E}(x) = \left(1 - \frac{\mathbb{E}\left\{S^{b+1}\right\}}{\mathbb{E}\left\{S^b\right\}}\right) \tilde{A}(x)(1+o(1)) = \frac{d}{b+c+d} \frac{(a+b)b}{(1+b)x}(1+o(1)).$$

In particular, for $a = c + d$,

$$\bar{H}(x) = \frac{x^{-b}}{bB(b,c)} \left[1 - \frac{(b+c)b}{(1+b)x}(1+o(1))\right],$$

which is the second-order expansion of survival function of $beta_2(c, b)$ (cf. (3.2)), and coincides with the fact that $X \sim beta_2(c, b)$, see Lemma 5 in Balakrishnan and Hashorva (2011). Fig. 2 compares the first-order and the second-order expansions with the exact true value $\bar{H}(x)$ when $R \sim beta_2(a, b), S \sim beta(c, d)$ with $(a, b, c, d) = (3, 2, 1, 2)$ (left) and $(a, b, c, d) = (2, 2, 1, 2)$ (right). As expected, we find that the second-order tail asymptotics is more accurate than the first-order one.

Example 3.3. *(Gumbel case with $\rho = 0$)* Let R be a random variable with df F given by

$$\bar{F}(x) = \exp\left(-\frac{cx}{1-x}\right), \quad 0 < x < 1, c > 0, \tag{3.3}$$

denoted in the sequel by $R \sim E(1, c)$. If follows that $F \in D(Q_0)$ with $w(x) = c/(1-x)^2$, and $U \in 2\mathrm{ERV}_{0,0}$ with auxiliary functions

$$a(x) = \frac{1}{w(U(x))}, \qquad \tilde{A}(x) = -\frac{2}{c + \ln x}.$$

If $S \sim beta(a, b)$, then the df G of S satisfies

$$\bar{G}\left(1 - \frac{1}{x}\right) = \frac{x^{-b}}{bB(a,b)} \left(1 - \frac{b(a-1)}{(b+1)x}(1+o(1))\right), \quad x \to \infty, \tag{3.4}$$

i.e., $\bar{G}(1 - 1/x) = x^{-b}L(x), L \in 2\mathrm{RV}_{0,-1}$ with auxiliary function

$$A(x) = \frac{b(a-1)}{(b+1)x}.$$

Consequently,

$$\frac{1}{\eta(x)} = \frac{(1-x)^2}{cx}, \quad \tilde{A}\left(\frac{1}{\bar{F}(x)}\right) = -\frac{2(1-x)}{c}, \quad A(\eta(x)) = \frac{b(a-1)}{(b+1)}\frac{(1-x)^2}{cx}.$$

By Theorem 2.3 with $\alpha_2 = b, \tau_2 = -1$ and $\rho = 0$

$$\bar{H}(x) = \bar{F}(x)\bar{G}\left(1 - \frac{(1-x)^2}{cx}\right)\Gamma(b+1)[1+\mathcal{E}(x)],$$

with

$$\mathcal{E}(x) = K(b,0)\tilde{A}\left(\frac{1}{\bar{F}(x)}\right)(1+o(1)) = \frac{b(b+1)}{c}(1-x)(1+o(1)).$$

Example 3.4. *(Gumbel case with $\rho < 0$)* Let $R \sim F$ with

$$\bar{F}(x) = \frac{1-\exp(-\exp(-x))}{p}, \quad x > 0,\, p = 1 - e^{-1}. \tag{3.5}$$

It follows that $F \in D(Q_0)$ with constant scaling function $w(x) = 1$ and its tail quantile function is

$$U(x) = \ln\frac{x}{p} - \frac{p}{2x}(1+o(1)).$$

Furthermore, $U \in 2\mathrm{ERV}_{0,-1}$ with auxiliary functions

$$a(x) = 1, \quad \tilde{A}(x) = \frac{p}{2x}.$$

Next, suppose that $S \sim beta(a,b)$. Thus (see (3.4))

$$\frac{1}{\eta(x)} = \frac{1}{x}, \quad \tilde{A}\left(\frac{1}{\bar{F}(x)}\right) = \frac{1}{2}e^{-x}, \quad A(\eta(x)) = \frac{b(a-1)}{(b+1)x}.$$

By Theorem 2.3 with $\alpha_2 = b, \tau_2 = -1$ and $\rho = -1$

$$\bar{H}(x) = \bar{F}(x)\bar{G}\left(1 - \frac{1}{x}\right)\Gamma(b+1)[1+\mathcal{E}(x)],$$

with

$$\mathcal{E}(x) = -\left[\frac{b^2(a-1)}{(b+1)x} + \frac{b(b+1)}{x}\right](1+o(1)).$$

Fig. 3 shows the efficiency of the second-order asymptotics of $\bar{H}$ when $R \sim E(1,c)$ with $c = 1$ and $S \sim beta(1,1/2)$ (left); and when R follows the left-truncated Gumbel distribution (3.5) and $S \sim beta(1,1)$ (right).

Example 3.5. *(Gumbel case with Weibull tail)* Let $R \sim \Gamma(\alpha,\lambda)$ with pdf

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}, \quad x > 0,\ \lambda,\alpha > 0.$$

The tail quantile function of F is

$$U(x) = \frac{1}{\lambda}(\ln x - \ln\Gamma(\alpha))\left[1 + \frac{(\alpha-1)\ln\ln x}{\ln x - \ln\Gamma(\alpha)}(1+o(1))\right].$$

Thus $F \in D(Q_0)$ with $w(x) = \lambda$ and $U \in 2\mathrm{RV}_{0,0}$ with second-order auxiliary function

$$\tilde{A}(x)=\frac{1-\alpha}{\ln^2 x}$$

(cf. Table 1 in the Appendix). Next, let $S \sim beta(a, b)$, note that the survival function satisfies (3.4). Consequently,

$$\frac{1}{\eta(x)} = \frac{1}{\lambda x}, \quad \tilde{A}\left(\frac{1}{\bar{F}(x)}\right) = \frac{1-\alpha}{(\lambda x)^2}, \quad A(\eta(x)) = \frac{b(a-1)}{(b+1)\lambda x}.$$

By Theorem 2.3 with $\alpha_2 = b, \tau_2 = -1$ and $\rho = 0$

$$\bar{H}(x) = \bar{F}(x)\bar{G}\left(1 - \frac{1}{\lambda x}\right)\Gamma(b+1)[1 + \mathcal{E}(x)],$$

where

$$\mathcal{E}(x) = -\frac{b}{\lambda x}\left[\frac{b(a-1)}{b+1} + b + 1\right](1 + o(1)).$$

Thus

$$\begin{aligned}
\bar{H}(x) &= \frac{(\lambda x)^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)}\left[1 + \frac{\alpha-1}{\lambda x}(1+o(1))\right]\frac{(\lambda x)^{-b}\Gamma(b+1)}{bB(a,b)}\left(1 - \frac{b(a-1)}{\lambda(b+1)x}(1+o(1))\right) \\
&\quad \times \left[1 - \frac{b}{\lambda x}\left(\frac{b(a-1)}{b+1} + b + 1\right)(1+o(1))\right] \\
&= \frac{(\lambda x)^{\alpha-b-1}e^{-\lambda x}}{\Gamma(a)\Gamma(\alpha)/\Gamma(a+b)}\left[1 + \frac{\alpha - b(a+b) - 1}{\lambda x}(1+o(1))\right]. \qquad (3.6)
\end{aligned}$$

On the other hand, in view of Corollary 2.5, both R and X are in the MDA of the Weibull distribution with (cf. Table 1 in the Appendix)

$$\theta = 1,\ \rho' = -1,\ b(x) = \frac{(1-\alpha)\ln x}{x} \quad \text{and} \quad \rho'^* = -1,\ b^*(x) = b(x) + \frac{\theta\alpha_2 \ln x}{x} = \frac{(1-\alpha+b)\ln x}{x}, \qquad (3.7)$$

which is consistent with (3.6). In particular, if $\alpha = a + b$, then (3.6) and (3.7) are consistent with the well-known result $X \sim \Gamma(a, \lambda)$ (cf. Hashorva (2013)).

In Fig. 4, we choose $(\alpha, \lambda, a, b) = (1, 1, 1/2, 1/2)$ (left) and $(\alpha, \lambda, a, b) = (1, 2, 1/2, 1/2)$(right). We observe that the second-order expansion of the tail probability is much closer to the true values.

Example 3.6. *(Weibull case)* Let $R \sim beta(a_1, b_1)$ and $S \sim beta(a_2, b_2)$. By (3.4), $1 - U \in 2\mathrm{RV}_{-1/b_1,-1/b_1}$ with auxiliary function

$$\tilde{A}(x) = -\frac{a_1 - 1}{b_1(b_1+1)}\left(\frac{x}{b_1 B(a_1, b_1)}\right)^{-1/b_1}$$

and $\bar{G}(1 - 1/x) = x^{-b_2}L(x), L \in 2\mathrm{RV}_{0,-1}$ with auxiliary function

$$A(x) = \frac{b_2(a_2-1)}{(b_2+1)x}.$$

Hence

$$\tilde{A}\left(\frac{1}{\bar{F}(x)}\right) = -\frac{a_1-1}{b_1(b_1+1)}(1-x), \quad A\left(\frac{1}{1-x}\right) = \frac{b_2(a_2-1)}{b_2+1}(1-x).$$

By Theorem 2.6 with $\alpha_1 = b_1, \alpha_2 = b_2, \tau_1 = \tau_2 = -1$ and

$$\bar{H}(x) = \bar{F}(x)\bar{G}(x)\left[b_1 B\left(b_1, b_2+1\right) + \mathcal{E}(x)\right],$$

with

$$\mathcal{E}(x) = b_1 b_2 B(b_1+1, b_2+1)\left(1 + \frac{a_1-1}{b_1+1} + \frac{a_2-1}{b_2+1}\right)(1-x)(1+o(1)).$$

In particular, for $a_2 + b_2 = a_1$

$$\begin{aligned}\bar{H}(x) = {} & \frac{(1-x)^{b_1+b_2} B(b_1, b_2+1)}{b_2 B(a_1, b_1) B(a_2, b_2)}\left[1 + \left(\frac{b_1+b_2}{b_1+b_2+1}\left(1 + \frac{a_1-1}{b_1+1} + \frac{a_2-1}{b_2+1}\right)\right.\right. \\ & \left.\left. - \left(\frac{b_1(a_1-1)}{b_1+1} + \frac{b_2(a_2-1)}{b_2+1}\right)\right)(1-x)(1+o(1))\right] \\ = {} & \frac{(1-x)^{b_1+b_2}}{(b_1+b_2)B(a_2, b_1+b_2)}\left[1 - \frac{(b_1+b_2)(a_2-1)}{b_1+b_2+1}(1-x)(1+o(1))\right],\end{aligned}$$

which is the second-order expansion of survival function of $beta(a_2, b_1+b_2)$ (cf. (3.4)), and coincides with the fact that $X \sim beta(a_2, b_1+b_2)$ (cf. Hashorva (2013)).

In Fig. 5, we simulate the cases with $(a_1, b_1, a_2, b_2) = (4, 2, 2, 2)$ (left) and $(a_1, b_1, a_2, b_2) = (4, 2, 2, 3)$ (right). We observe that the second-order expansion of the tail probability is much closer to the true values.

4 Applications

4.1 Approximation of Value-at-Risk

In insurance and risk management applications, Value-at-Risk (denoted by VaR) is an important risk measure; see e.g., Denuit et al. (2006). In the following we shall analyse the asymptotics of $\mathrm{VaR}_p(X)$ in case that R has a heavy tail and a Weibull tail, respectively. Recall that VaR at probability level p for R is defined by

$$\mathrm{VaR}_p(R) = \inf\{y : F(y) \ge p\} = U(1/(1-p)). \tag{4.1}$$

With the same notation introduced as before, if $\bar{F} \in RV_{-\alpha}, \alpha > 0$, then by Breiman's Lemma

$$\bar{H}(x) = \mathbb{E}\left\{S^\alpha\right\} \bar{F}(x)(1+o(1)) = \bar{F}((\mathbb{E}\left\{S^\alpha\right\})^{-1/\alpha} x)(1+o(1)),$$

implying the following first-order asymptotics

$$\mathrm{VaR}_p(X) = (\mathbb{E}\left\{S^\alpha\right\})^{1/\alpha} \mathrm{VaR}_p(R)(1+o(1)), \quad p \uparrow 1. \tag{4.2}$$

Refining the above, we derive the following second-order asymptotics

$$\mathrm{VaR}_p(X) = (\mathbb{E}\left\{S^\alpha\right\})^{1/\alpha} \mathrm{VaR}_p(R)[1 + \mathcal{E}(p)], \quad \mathcal{E}(p) = \left(\frac{\mathbb{E}\left\{S^{\alpha-\tau}\right\}}{(\mathbb{E}\left\{S^\alpha\right\})^{1-\tau/\alpha}} - 1\right)\frac{\tilde{A}(\mathrm{VaR}_p(R))}{\alpha\tau}(1+o(1)) \tag{4.3}$$

provided that $\bar{F} \in 2\mathrm{RV}_{-\alpha,\tau}, \alpha > 0, \tau < 0$ with auxiliary function $\tilde{A}$.

Indeed, there exists some positive constant c such that (cf. Hua and Joe (2011))

$$\bar{F}(x) = cx^{-\alpha}\left[1 + \frac{\tilde{A}(x)}{\tau}(1 + o(1))\right]$$

for sufficiently large x. Thus, by Theorem 2.1

$$\bar{H}(x) = cx^{-\alpha}\mathbb{E}\{S^\alpha\}\left[1 + \frac{\mathbb{E}\{S^{\alpha-\tau}\}}{\mathbb{E}\{S^\alpha\}}\frac{\tilde{A}(x)}{\tau}(1 + o(1))\right].$$

Therefore, in view of Theorem 1.5.12 in Bingham et al. (1987)

$$\mathrm{VaR}_p(R) = \left(\frac{c}{1-p}\right)^{1/\alpha}\left[1 + \frac{\tilde{A}(\mathrm{VaR}_p(R))}{\alpha\tau}(1 + o(1))\right], \quad p \uparrow 1$$

and

$$\mathrm{VaR}_p(X) = \left(\frac{c\mathbb{E}\{S^\alpha\}}{1-p}\right)^{1/\alpha}\left[1 + \frac{\mathbb{E}\{S^{\alpha-\tau}\}}{\mathbb{E}\{S^\alpha\}}\frac{\tilde{A}(\mathrm{VaR}_p(X))}{\alpha\tau}(1 + o(1))\right], \quad p \uparrow 1.$$

Consequently, by $|\tilde{A}| \in \mathrm{RV}_\tau$ and (4.2) we obtain the second-order asymptotics (4.3).

In what follows, we will consider the case that F is in the MDA of the Gumbel distribution. Since most of such distributions are Weibull tail distributions (cf. Table 1 and Table 2 in the Appendix), we focus on the asymptotics of $\mathrm{VaR}_p(X)$ in terms of $\mathrm{VaR}_p(R)$ (see (4.4) below) under the conditions of Corollary 2.5. Note that $\bar{F}$ has a Weibull tail satisfying the second-order condition (cf. (2.9))

$$\bar{F}(x) = \exp(-V(x)), \quad \text{with } V^{\leftarrow}(x) = x^\theta \ell(x),\ \theta > 0$$

and $\ell \in 2\mathrm{RV}_{0,\rho'}, \rho' \le 0$ with auxiliary function b. By (4.1)

$$\mathrm{VaR}_p(R) = V^{\leftarrow}(-\ln(1-p)) = (-\ln(1-p))^\theta \ell(-\ln(1-p)).$$

In view of Corollary 2.5 (see (2.10))

$$\bar{H}(x) = \exp\left(-V(x) - \alpha_2 \ln V(x) + \ln L^*(V(x))\right),$$

where L^* denotes a slowly varying function. Recalling that $\ln L^*(V(x)) = o(\ln V(x))$ (see Bingham et al. (1987)), we have as $p \uparrow 1$

$$\begin{aligned}
\mathrm{VaR}_p(X) &= V^{\leftarrow}\left(-\ln(1-p)\left[1 - \alpha_2\frac{\ln(-\ln(1-p))}{-\ln(1-p)}(1 + o(1))\right]\right)\\
&= \left(\ln\frac{1}{1-p}\right)^\theta [1 - \theta\alpha_2\varpi(p)(1 + o(1))]\,\ell\left(\ln\frac{1}{1-p}\right)\left[1 + \frac{(1-\alpha_2\varpi(p))^{\rho'} - 1}{\rho'}b\left(\ln\frac{1}{1-p}\right)(1 + o(1))\right]\\
&= \mathrm{VaR}_p(R)\left[1 - \theta\alpha_2\varpi(p)(1 + o(1))\right], \quad \text{with } \varpi(p) = \frac{\ln(-\ln(1-p))}{-\ln(1-p)}.
\end{aligned} \tag{4.4}$$

4.2 Estimations of tail probability

In many insurance applications it is important to estimate the tail probability of the extreme risks. In what follows, we investigate this problem under the random scaling framework. Let $\{(R_i, S_i), i = 1, \ldots, n\}$ be a random sample from (R, S), and thus $X_i := R_i S_i, i \le n$ is a sample of size n from $X \stackrel{d}{=} RS$. Our goal is to estimate $p = \mathbb{P}(X > x)$ with sufficiently large x. One possible estimation is via the empirical df if x is in the region of the sample $X_i, i \le n$ with $X_i = R_i S_i, i = 1, \ldots, n$. In general, we consider how to estimate $p_n := \mathbb{P}(X > x_n)$ as $x_n \to \infty$. Hereafter, we write $R_{n-k+1,n}, S_{n-k+1,n}$ and $X_{n-k+1,n}, k \le n$ as the associated increasing order statistics, and assume that $R \sim F$ and $S \in (0, 1)$ are independent.

First we consider the case that $\bar{F} \in 2\mathrm{RV}_{-\alpha,\tau}, \alpha > 0, \tau < 0$ with the second-order auxiliary function $\tilde{A}$. By Hua and Joe (2011), there exists a positive constant c such that

$$\bar{F}(x)=cx^{-\alpha}(1 + \tilde{A}(x)/\tau(1 + o(1))) =: cx^{-\alpha}(1 + \alpha\delta(x)),$$

i.e., $F \in \mathcal{F}_{1/\alpha,\tau}$ with $\delta(x) = \tilde{A}(x)/(\alpha\tau)$ in the terminology of Beirlant et al. (2009). By Theorem 2.1

$$\bar{H}(x)=\bar{F}(x) \left(\mathbb{E}\left\{S^{\alpha}\right\} + \mathbb{E}\left\{S^{\alpha}(S^{-\tau} - 1)\right\} \alpha\delta(x)(1 + o(1))\right). \tag{4.5}$$

In order to estimate $\bar{H}(x)$ with $x = x_n$ given, we use the estimators of α, δ, τ and $\bar{F}$ proposed by Beirlant et al. (2009). Let $y_{k,n} = x/R_{n-k,n}, \widehat{\tau}_{k,n} = \widehat{\rho}_n/H_{k,n}$ with $\widehat{\rho}_n$ some weakly consistent estimator of $\rho = \tau/\alpha$ based on samples from the parent R, denote

$$H_{k,n} = \frac{1}{k}\sum_{i=1}^{k} \ln \frac{R_{n-i+1,n}}{R_{n-k,n}}, \quad E_{k,n}(s) = \frac{1}{k}\sum_{i=1}^{k} (\frac{R_{n-i+1,n}}{R_{n-k,n}})^{s}, s \le 0$$

and

$$\widehat{\alpha}_{k,n} = \left(H_{k,n} - \widehat{\delta}_{k,n}\frac{\widehat{\rho}_n}{1 - \widehat{\rho}_n}\right)^{-1}, \quad \widehat{\delta}_{k,n} = H_{k,n}(1 - 2\widehat{\rho}_n)(1 - \widehat{\rho}_n)^3\widehat{\rho}_n^{-4} \left(E_{k,n}(\frac{\widehat{\rho}_n}{H_{k,n}}) - \frac{1}{1 - \widehat{\rho}_n}\right). \tag{4.6}$$

Thus, by (4.5), the tail probability p_n can be estimated as (denoted by $\widehat{p}_{k,n}(R, S)$)

$$\widehat{p}_{k,n}(R, S) = \widehat{\bar{F}}(x) \left(\widehat{\mathbb{E}\left\{S^{\alpha}\right\}} + (\widehat{\mathbb{E}\left\{S^{\alpha-\tau}\right\}} - \widehat{\mathbb{E}\left\{S^{\alpha}\right\}})\frac{\widehat{\delta}_{k,n}}{H_{k,n}}\right), \tag{4.7}$$

with

$$\widehat{\bar{F}}(x) = \frac{k}{n}\left(y_{k,n}\left(1 + \widehat{\delta}_{k,n}(1 - y_{k,n}^{\widehat{\tau}_{k,n}})\right)\right)^{-\widehat{\alpha}_{k,n}}, \quad \widehat{\mathbb{E}\left\{S^{\alpha}\right\}} = \frac{1}{n}\sum_{i=1}^{n} S_i^{\widehat{\alpha}_{k,n}}, \quad \widehat{\mathbb{E}\left\{S^{\alpha-\tau}\right\}} = \frac{1}{n}\sum_{i=1}^{n} S_i^{\widehat{\alpha}_{k,n}-\widehat{\tau}_{k,n}}. \tag{4.8}$$

On the other hand, by Theorem 2.1, X has the same second-order tail behavior as that of R. Consequently, p_n can be directly estimated by using samples from X. We denote that estimator by $\widehat{p}_{k,n}(X)$, given as (in contrast to (4.7), (4.8))

$$\widehat{p}_{k,n}(X) = \frac{k}{n}\left(y^{*}_{k,n}\left(1 + \widehat{\delta}^{*}_{k,n}(1 - (y^{*}_{k,n})^{\widehat{\tau}^{*}_{k,n}})\right)\right)^{-\widehat{\alpha}^{*}_{k,n}}, \tag{4.9}$$

with $y^*_{k,n} = x/X_{n-k,n}$ and $\widehat{\delta}^*_{k,n}, \widehat{\tau}^*_{k,n}, \widehat{\alpha}^*_{k,n}$ are $\widehat{\delta}_{k,n}, \widehat{\tau}_{k,n}, \widehat{\alpha}_{k,n}$ with the order statistics replaced by $X_{n-k+1,n}, k \le n-1$. Relying on (4.7) and (4.9), we shall perform some simulations to compare the finite sample behaviors of $\widehat{\alpha}_{k,n}, \widehat{p}_{k,n}(R,S)$ and $\widehat{\alpha}^*_{k,n}, \widehat{p}_{k,n}(X)$. Since $\tau = -1$ holds in most applications, we take $\widehat{\tau}_{k,n} = -1$ and $\widehat{\rho}_n = -H_{k,n}$ in the simulations. Here we simulate 100 samples of size $n = 1000$ from $R \sim Pareto(2,1)$ and $S \sim beta(1,2)$, and estimate $1/\alpha = 0.5$ and $p = \mathbb{P}(X > 3) = 0.01298$. It turns out that the bias as well as the mean squared errors based on the information of R and S is much smaller than that on the reduced information of RS, see Fig. 6.

Next, we investigate the case of $F \in D(Q_0)$. For convenience, we consider only the estimation comparisons for F having Weibull tails. Since by Corollary 2.5, both R and X have Weibull tails with the same Weibull tail coefficient θ and further the second-order parameter ρ'^* is greater than -1, we consider the bias-reduced Weibull tail coefficient estimators $\widehat{\theta}$ by Diebolt et al. (2008)

$$\widehat{\theta} = \widehat{\theta}(k,R) = \bar{Z}_k - \widehat{b}(\ln(n/k))\bar{x}_k, \tag{4.10}$$

with

$$\widehat{b}(\ln(n/k)) = \frac{\sum_{i=1}^k (x_i - \bar{x}_k) Z_i}{\sum_{i=1}^k (x_i - \bar{x}_k)^2}$$

and

$$x_j = \frac{\ln(n/k)}{\ln(n/j)}, \quad Z_j = j \ln \frac{n}{j} \ln \frac{R_{n-j+1,n}}{R_{n-j,n}}, \quad \bar{x}_k = \frac{\sum_{j=1}^k x_j}{k}, \quad \bar{Z}_k = \frac{\sum_{j=1}^k Z_j}{k}.$$

Based on the bias-reduced tail quantile estimators provided by Diebolt et al. (2008), given by

$$\widehat{x}_{p_n} = R_{n-k,n} \left(\frac{\ln(1/p_n)}{\ln(n/k)} \right)^{\widehat{\theta}} \exp\left(\widehat{b}(\ln(n/k)) \frac{(\ln(1/p_n)/\ln(n/k))^{\widehat{\rho}'} - 1}{\widehat{\rho}'} \right)$$

with p_n known, we can solve the dual problem and estimate the tail probability $\bar{F}(x)$ for given x as follows

$$\widehat{\bar{F}}(x) = \exp\left(-\ln(n/k) \left(\frac{x}{R_{n-k,n}} \right)^{1/\widehat{\theta}} \exp\left(-\widehat{b}(\ln(n/k)) \frac{(x/R_{n-k,n})^{\widehat{\rho}'/\widehat{\theta}} - 1}{\widehat{\theta}\widehat{\rho}'} \right) \right), \tag{4.11}$$

where $\widehat{\rho}'$ is a consistent estimator of ρ'. Since $\bar{F}(x) = \exp(-V(x))$, we have

$$\widehat{V}(x) = -\ln \widehat{\bar{F}}(x), \quad \widehat{b}(V(x)) = \widehat{b}(\ln(n/k)) \left(\frac{\widehat{V}(x)}{\ln(n/k)} \right)^{\widehat{\rho}'}. \tag{4.12}$$

Further, we remark that $S \sim G$ with $\bar{G}(1 - 1/x) \in 2\mathrm{RV}_{-\alpha_2,\tau_2}$ is equivalent to $S^* := 1/(1-S) \sim G^*$ with $\bar{G}^* \in 2\mathrm{RV}_{-\alpha_2,\tau_2}$. Hence, using the estimations of tail probability by Beirlant et al. (2009), we have

$$\widehat{\bar{G}}\left(1 - \frac{1}{V(x)}\right) = \frac{k}{n}\left(y_{k,n}(1 + \widehat{\delta}_{k,n}(1 - y_{k,n}^{\widehat{\tau}_2(k)})) \right)^{\widehat{\alpha}_2(k)}, \quad \widehat{A}(V(x)) = \widehat{\alpha}_2(k)\widehat{\tau}_2(k)\widehat{\delta}_{k,n} y_{k,n}^{\widehat{\tau}_2(k)}, \tag{4.13}$$

where $y_{k,n} = \widehat{V}(x)/S^*_{n-k,n}$ and $\widehat{\delta}_{k,n}, \widehat{\tau}_2(k), \widehat{\alpha}_2(k)$ are estimated with the order statistics replaced by $S^*_{n-k,n} := 1/(1 - S_{n-k,n})$ in (4.8). Therefore, combining (4.10)–(4.13), the estimator of $p = \bar{H}(x)$, denoted by $p_k(R,S)$, is then in view of Corollary 2.5 given by

$$\widehat{p}_{k,n}(R,S) = \widehat{\bar{F}}(x)\widehat{\bar{G}}(1 - 1/V(x))\Gamma(\widehat{\alpha}_2(k) + 1)(\widehat{\theta})^{\widehat{\alpha}_2(k)}$$

$$\times \left[1 + \frac{\widehat{\alpha}_2(k)}{\widehat{\theta}} \widehat{b}(V(x)) + \frac{\frac{\Gamma(\widehat{\alpha}_2(k) - \widehat{\tau}_2(k) + 1)}{(\widehat{\theta})^{\widehat{\tau}_2(k)} \Gamma(\widehat{\alpha}_2(k) + 1)} - 1}{\widehat{\tau}_2(k)} \widehat{A}(V(x)) - \frac{\widehat{\alpha}_2(k)(\widehat{\alpha}_2(k) + 1)(\widehat{\theta} + 1)}{2\widehat{V}(x)} \right]. \tag{4.14}$$

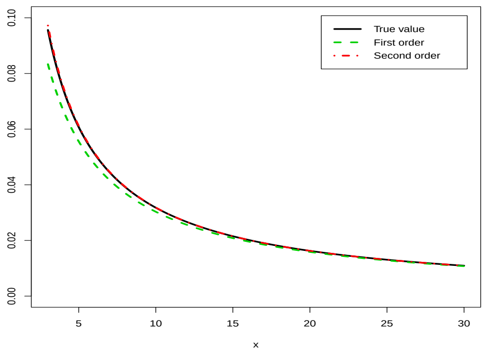
On the other hand, by Corollary 2.5, we can estimate $p = \bar{H}(x)$ directly based on samples from X as

$$\widehat{p}_{k,n}(X) = \exp\left(-\ln(n/k) \left(\frac{x}{X_{n-k,n}} \right)^{1/\widehat{\theta}} \exp\left(-\widehat{b}^*(\ln(n/k)) \frac{(x/X_{n-k,n})^{\widehat{\rho}'^*/\widehat{\theta}} - 1}{\widehat{\theta}\widehat{\rho}'^*} \right) \right), \tag{4.15}$$

where $\widehat{\rho}'^*$ is a consistent estimator of ρ'^* and $\widehat{\theta}, \widehat{b}^*$ are computed by (4.10) with samples $R_i, i \le n$ replaced by $X_i = R_i S_i, i \le n$.

Now, we generate 100 samples of size $n = 1000$ from $R \sim W(2,1)$ and $S \sim beta(2,3)$ to compare the finite sample behaviors of estimators of $\theta = 1/2$ and $p = \mathbb{P}(X > 3) = 2.1186 \times 10^{-7}$ given by (4.10), (4.14) and (4.15). In the simulation we take $\widehat{\tau}_2(k) = -1, \widehat{\rho}' = \widehat{\rho}'^* = -1$ and plot mean values and mean squared errors of $\widehat{\theta}$ and $\ln(\widehat{p}_k/p), k = 50, \ldots, 4500$, with $\widehat{p}_k = \widehat{p}_{k,n}(R,S), \widehat{p}_{k,n}(X)$, respectively (cf. (4.14) and (4.15)).

Fig. 7 shows that our estimators of θ and tail probability based on the original data (indicated by the red dotted line $(\cdot - \cdot)$) have much wider stable regions with less bias even the true value of ρ' is $-\infty$, see Table 1.



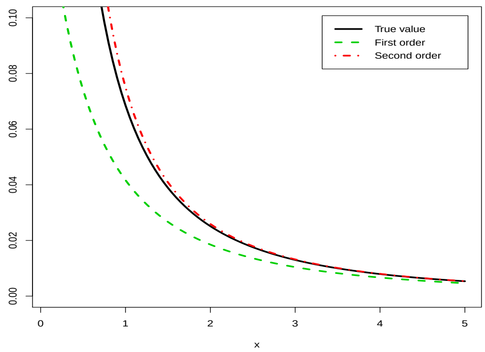


Figure 1: Tail $\bar{H}$ when $R \sim Pareto(1,1), S \sim beta(1,2)$ (left) and $R \sim Pareto(2,1), S \sim beta(1,2)$ (right).

4.3 Linear combinations of random contractions

Motivated by the dependence structure of elliptical random vectors, Hashorva et al. (2010) discussed the first-order tail asymptotics of the aggregated risks of certain bivariate random vectors which we shall introduce next. Let therefore (V_1, V_2) be a bivariate scale mixture random vector with stochastic representation

$$(V_1, V_2) \stackrel{d}{=} R(I_1 S, I_2 \sqrt{1 - S^2}), \tag{4.16}$$

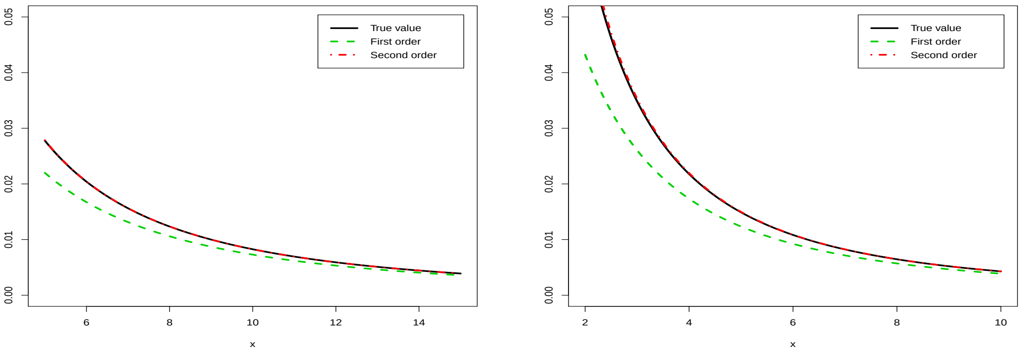


Figure 2: Tail $\bar{H}$ when $R \sim beta_2(3,2), S \sim beta(1,2)$ (left) and $R \sim beta_2(2,2), S \sim beta(1,2)$ (right).

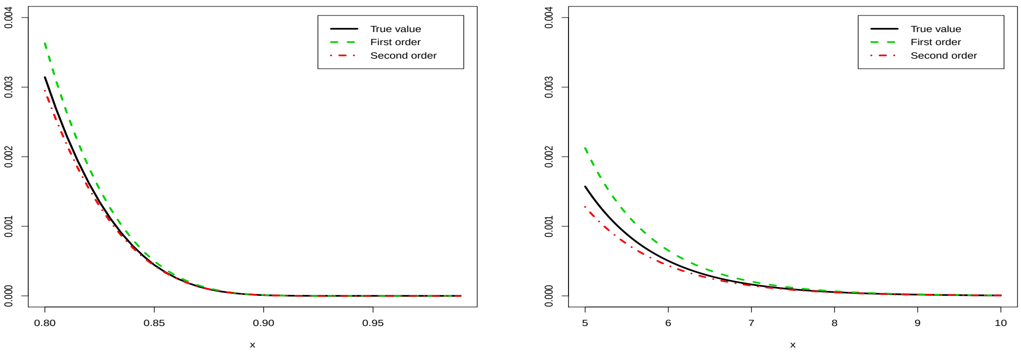


Figure 3: Tail $\bar{H}$ when $R \sim E(1,c)$ with $c = 1$ and $S \sim beta(1,1/2)$ (left) and R is left-truncated Gumbel distributed and $S \sim beta(1,1)$ (right).

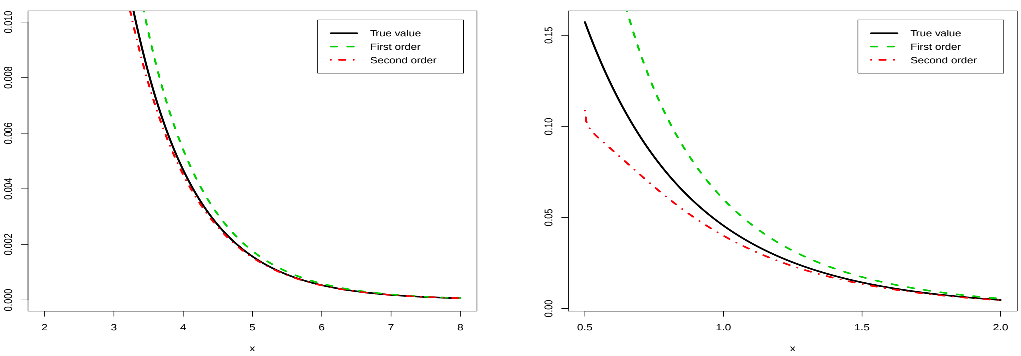


Figure 4: Tail $\bar{H} \sim \Gamma(a,\lambda)$ when $R \sim \Gamma(\alpha,\lambda)$ and $S \sim beta(a,b)$ for $(\alpha,\lambda,a,b) = (1,1,1/2,1/2)$ (left) and $(\alpha,\lambda,a,b) = (1,2,1/2,1/2)$ (right).

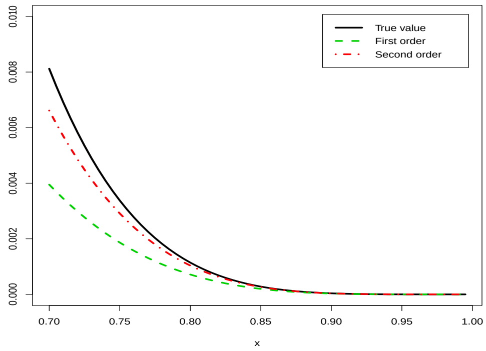


Figure 5: Tail $\bar{H}$ when $R \sim beta(4,2), S \sim beta(2,2)$ (left) and $R \sim beta(4,2), S \sim beta(2,3)$ (right).

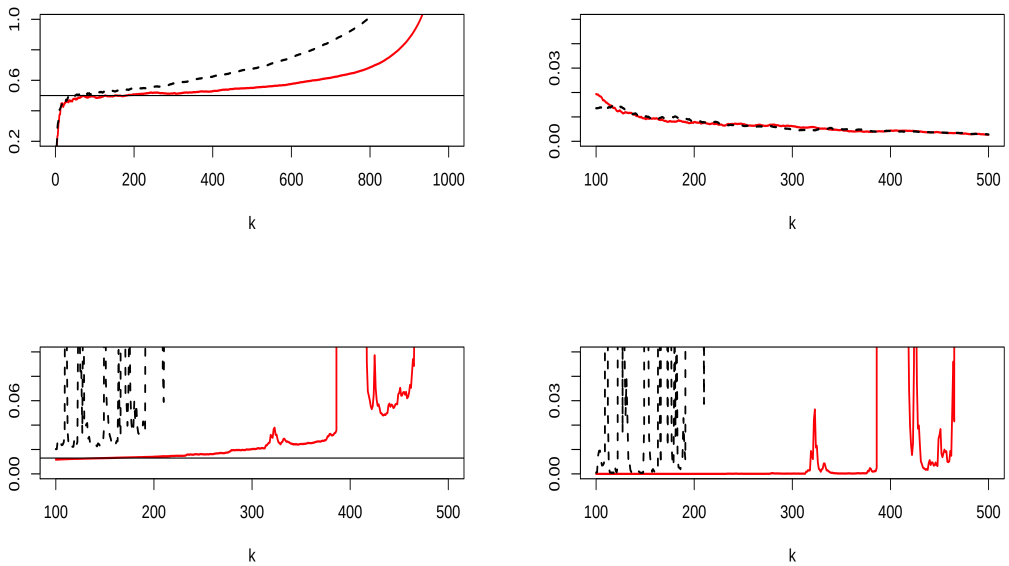


Figure 6: Finite behaviors of mean values (left) and mean squared errors (right) of $1/\widehat{\alpha}_{k,n}$ and $\widehat{p}_k = \widehat{p}_{k,n}(R,S), \widehat{p}_{k,n}(X)$ respectively give by (4.6), (4.7) and (4.9), where $1/\alpha = 1/2$ and $p = \mathbb{P}(X > 3) = 0.01298$, which are indicated by the horizontal lines. The line and the dotted line stand for the estimators based on the original samples from RS and RS with $R \sim Pareto(2,1), S \sim beta(1,2)$, respectively.

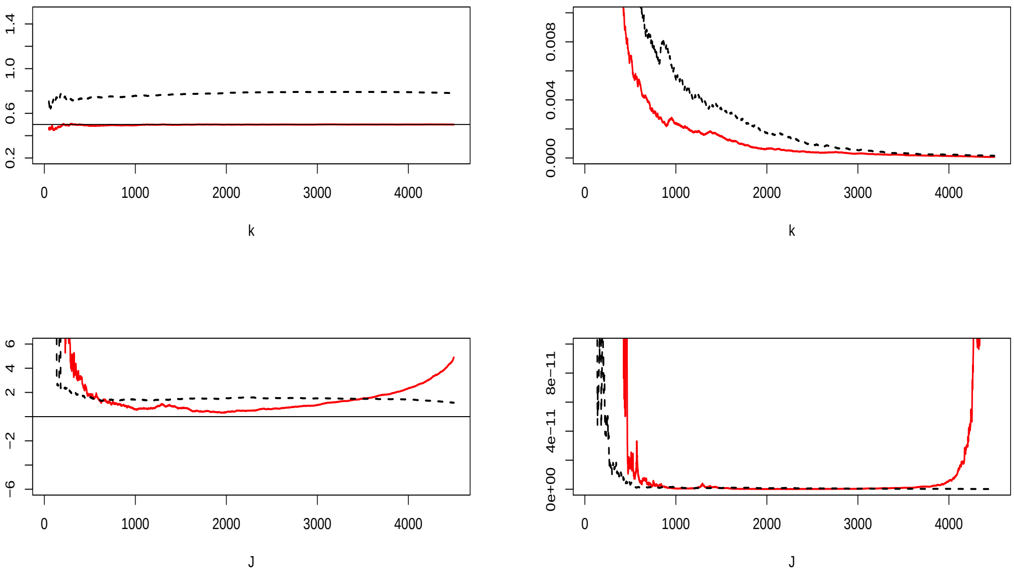


Figure 7: Finite behaviors of mean values (left) and mean squared errors (right) of $\widehat{\theta}$ (above) and $\ln(\widehat{p}_k/p)$ (bottom) with $\widehat{\theta}$, $\widehat{p}_k = \widehat{p}_{k,n}(R,S)$, $\widehat{p}_{k,n}(X)$ respectively given by (4.10), (4.14) and (4.15), where $\theta = 1/2$ and $p = \mathbb{P}(X > 3) = 2.1186 \times 10^{-7}$, which are indicated by the horizontal lines. The line and the dotted line stand for the estimators based on the original samples from R, S and RS with $R \sim Weibull(2,1), S \sim beta(2,3)$, respectively.

where $R \sim F$, is almost surely positive, $S \sim G$ is a scaling random variable taking values in $(0,1)$, while I_1, I_2 assume values in $\{1,-1\}$. Hashorva et al. (2010) studied the tail asymptotics of the aggregated risk

$$V(\lambda) = \lambda V_1 + \sqrt{1-\lambda^2} V_2 = R(\lambda I_1 S + \sqrt{1-\lambda^2} I_2 \sqrt{1-S^2}) =: RS^*(\lambda) \tag{4.17}$$

for $\lambda \in (0,1)$. In what follows, we derive the second-order tail asymptotics of $V(\lambda)$ given by (4.17). Specifically, we suppose that for small $x > 0$

$$\mathbb{P}(|S-\lambda| \le x) = c_\lambda x^{\alpha_\lambda}(1 + L_\lambda(x) x^{\tau_\lambda}), \quad \alpha_\lambda, \tau_\lambda \in (0,\infty) \quad \text{and} \quad \lambda \in [0,1], \tag{4.18}$$

where c_λ is a positive constant and $|L_\lambda|$ is slowly varying at 0. Set

$$q_\lambda = \mathbb{P}(I_1 = I_2 = 1)\mathbb{I}\{\lambda \in (0,1)\} + \mathbb{P}(I_2 = 1)\mathbb{I}\{\lambda = 0\} + \mathbb{P}(I_1 = 1)\mathbb{I}\{\lambda = 1\}, \tag{4.19}$$

with $\mathbb{I}\{\cdot\}$ the indicator function.

Lemma 4.1. *Let I_1, I_2 be two random variables taking values $-1, 1$ with probability $q_\lambda \in (0,1]$ defined by (4.19) and being independent of the scaling random variable $S \sim G$. For given $\lambda \in [0,1]$, suppose further that the df G satisfies (4.18) for small $x > 0$. Then for $S^*(\lambda)$ defined in (4.17) we have as $x \downarrow 0$*

a) If $\lambda \in (0,1)$, then

$$\mathbb{P}(S^*(\lambda) > 1 - x) = q_\lambda c_\lambda (2x(1-\lambda^2))^{\alpha_\lambda/2} \left[1 + \mathcal{A}_\lambda(x)\right],$$

with

$$\mathcal{A}_\lambda(x) = \left(L_\lambda(\sqrt{x})(2x(1-\lambda^2))^{\tau_\lambda/2} - \frac{\lambda \alpha_\lambda}{\sqrt{2(1-\lambda^2)}} x^{1/2} \right)(1 + o(1)).$$

b) If $\lambda = 0$, then

$$\mathbb{P}(S^*(\lambda) > 1 - x) = q_\lambda c_\lambda (2x)^{\alpha_\lambda/2} \left[1 + \mathcal{A}_\lambda(x)\right], \quad \mathcal{A}_\lambda(x) = \left(L_\lambda(\sqrt{x})(2x)^{\tau_\lambda/2} - \frac{\alpha_\lambda x}{4} \right)(1 + o(1)).$$

c) If $\lambda = 1$, then

$$\mathbb{P}(S^*(\lambda) > 1 - x) = q_\lambda c_\lambda x^{\alpha_\lambda} \left[1 + \mathcal{A}_\lambda(x)\right], \quad \mathcal{A}_\lambda(x) = L_\lambda(x) x^{\tau_\lambda}.$$

In view of Lemma 4.1, we have $\mathbb{P}(S^*(\lambda) > 1 - 1/x) \in 2\mathrm{RV}_{-\alpha,\tau}$ with α, τ and auxiliary function A defined by

$$\alpha = \begin{cases} \alpha_\lambda/2, & \lambda \in [0,1), \\ \alpha_\lambda, & \lambda = 1; \end{cases} \qquad \tau = \begin{cases} -\min(\tau_\lambda, 1)/2, & \lambda \in (0,1), \\ -\min(\tau_\lambda, 2)/2, & \lambda = 0, \\ -\tau_\lambda, & \lambda = 1; \end{cases} \qquad A(x) = \tau \mathcal{A}_\lambda(1/x). \tag{4.20}$$

Next, utilizing Theorem 2.3, Theorem 2.6 and Lemma 4.1, we give the second-order tail approximation of $V(\lambda)$.

Theorem 4.2. *Let $V(\lambda)$ be defined in* (4.17) *for $\lambda \in [0,1]$ and suppose that the conditions of Lemma 4.1 hold.*

a) If $F \in D(Q_0)$ and its tail quantile function $U \in 2\mathrm{ERV}_{0,\rho}, \rho \le 0$ with auxiliary functions $1/w(U)$ and $\tilde{A}$, then for $x \uparrow x_F$ (recall $\eta(x) = xw(x)$)

$$\mathbb{P}(V(\lambda) > x) = \bar{F}(x)\mathbb{P}\left(S^*(\lambda) > 1 - \frac{1}{\eta(x)}\right)$$
$$\times \left[\Gamma(\alpha+1) + \left(\frac{\Gamma(\alpha-\tau+1) - \Gamma(\alpha+1)}{\tau} A(\eta(x)) + K(\alpha,\rho)\tilde{A}\left(\frac{1}{\bar{F}(x)}\right)\right)(1+o(1))\right].$$

b) If $F \in D(Q_{-1/\alpha_1}), \alpha_1 > 0$ and $x_F = 1$. Furthermore, we assume that its tail quantile function U satisfies $1 - U \in 2\mathrm{RV}_{-1/\alpha_1, \tau_1/\alpha_1}$ with auxiliary function $\tilde{A}$, then for $x \downarrow 0$

$$\mathbb{P}(V(\lambda) > 1-x) = \bar{F}(1-x)\mathbb{P}(S^*(\lambda) > 1-x)$$
$$\times \left[\alpha_1 B(\alpha_1, \alpha+1) + \left(\frac{\alpha\alpha_1^2}{\tau_1}\left[B(\alpha,\alpha_1+1) - B(\alpha,\alpha_1-\tau_1+1)\right]\tilde{A}\left(\frac{1}{\bar{F}(1-x)}\right)\right.\right.$$
$$\left.\left.+\frac{\alpha_1}{\tau}\left[B(\alpha_1,\alpha-\tau+1) - B(\alpha_1,\alpha+1)\right]A\left(\frac{1}{x}\right)\right)(1+o(1))\right].$$

Here α, τ and A are those defined in (4.20)*, and $\mathbb{P}(S^*(\lambda) > 1-x)$ is given by Lemma 4.1.*

Remark 4.3. *a) If S has Beta distribution with positive parameters a and b, then* (4.18) *holds for $\lambda = 0, 1$ and $\alpha_0 = a, \alpha_1 = b, \tau_0 = \tau_1 = 1$,*

$$c_0 = \frac{1}{aB(a,b)}, \quad L_0(x) = -\frac{(b-1)a}{a+1}(1+o(1)), \quad c_1 = \frac{1}{bB(a,b)}, \quad L_1(x) = -\frac{(a-1)b}{b+1}(1+o(1)).$$

b) If G has pdf g which has a continuous third derivative g''', then condition (4.18) *holds for any $\lambda \in (0,1)$ and*

$$\alpha_\lambda = 1, \quad c_\lambda = 2g(\lambda), \quad L_\lambda(x) = \frac{g'''(\lambda)}{6g'(\lambda)}(1+o(1)), \quad \tau_\lambda = 2.$$

c) If S has Beta distribution with parameters $1/2, 1/2$ and I_1, I_2 are independent with mean 0 being further independent of S, then (V_1, V_2) is spherically distributed, and

$$V(\lambda) \stackrel{d}{=} I_1 RS \stackrel{d}{=} I_2 R\sqrt{1-S^2}$$

for all $\lambda \in [0,1]$. Thus the tail asymptotics of $V(\lambda)$ can be directly obtained by Theorem 2.3 and Theorem 2.6 in Section 2.

5 Proofs

Proof of Theorem 2.1 It follows from Breiman's Lemma that

$$\lim_{x\to\infty} \frac{\bar{H}(x)}{\bar{F}(x)} = \mathbb{E}\left\{S^{\alpha_1}\right\}.$$

We consider two cases $\tau_1 < 0$ and $\tau_1 = 0$ separately. For $\tau_1 < 0$, by Lemma 5.2 of Draisma et al. (1999), for every $\epsilon > 0$, there exists $x_0 = x_0(\epsilon) > 0$ such that for all $x > x_0$ and all $s \in (0,1)$

$$\left| \frac{\bar{F}(x/s)/\bar{F}(x) - s^{\alpha_1}}{\tilde{A}(x)} - s^{\alpha_1} \frac{s^{-\tau_1} - 1}{\tau_1} \right| \le \epsilon (C_1 + C_2 s^{\alpha_1} + C_3 s^{\alpha_1 - \tau_1 - \epsilon}),$$

with some positive constants C_1, C_2 and C_3 not depending on x and s. Therefore, by the dominated convergence theorem

$$\lim_{x\to\infty} \frac{1}{\tilde{A}(x)} \left(\frac{\bar{H}(x)}{\bar{F}(x)} - \mathbb{E}\left\{ S^{\alpha_1} \right\} \right) = \int_0^1 \lim_{x\to\infty} \frac{\bar{F}(x/s)/\bar{F}(x) - s^{\alpha_1}}{\tilde{A}(x)} \, dG(s) = \mathbb{E}\left\{ S^{\alpha_1} \frac{S^{-\tau_1} - 1}{\tau_1} \right\}.$$

For $\tau_1 = 0$, note that for all $\alpha_1 > 0$, the function $f(s) = s^{\alpha_1} \ln(1/s)$ is continuous in $(0,1]$ and $\lim_{s\downarrow 0} f(s) = 0$. We have that $f(s)$ is bounded on $[0,1]$ and $\mathbb{E}\{f(S)\}$ exists. Similarly as above for $\tau_1 < 0$, we have if $\tau_1 = 0$ that

$$\lim_{x\to\infty} \frac{1}{\tilde{A}(x)} \left(\frac{\bar{H}(x)}{\bar{F}(x)} - \mathbb{E}\left\{ S^{\alpha_1} \right\} \right) = \mathbb{E}\left\{ S^{\alpha_1} \ln S^{-1} \right\}$$

establishing the proof. $\square$

Proof of Theorem 2.3 Letting $t = 1/\bar{F}(x)$, note that $x \uparrow x_F$ if and only if $t \to \infty$, and

$$\bar{H}(x) = \int_x^{x_F} \bar{G}\left(\frac{x}{y}\right) dF(y) = \int_t^\infty \bar{G}\left(\frac{U(t)}{U(s)}\right) d\left(1 - \frac{1}{s}\right) = t^{-1} \int_0^1 \bar{G}\left(1 - \frac{U(t/s) - U(t)}{U(t/s)}\right) ds.$$

We rewrite the left-hand side of (2.6) as (recall $\bar{G}(1 - 1/x) = x^{-\alpha_2} L(x)$)

$$\begin{aligned}
\frac{\bar{H}(x)}{\bar{F}(x)\bar{G}(1 - 1/\eta(x))} &= \int_0^1 \frac{\bar{G}\left(1 - (U(t/s) - U(t))/U(t/s)\right)}{\bar{G}\left(1 - a(t)/U(t)\right)} ds \\
&= \int_0^1 \left(\frac{U(t/s) - U(t)}{a(t)} \frac{U(t)}{U(t/s)} \right)^{\alpha_2} \frac{L\left(\frac{U(t)}{a(t)} \Big/ \left(\frac{U(t/s) - U(t)}{a(t)} \frac{U(t)}{U(t/s)} \right) \right)}{L\left(\frac{U(t)}{a(t)} \right)} ds \\
&= \int_0^1 (\Theta_t(s))^{\alpha_2} \frac{L(\Xi_t(s))}{L(\varphi_t)} ds, \qquad (5.1)
\end{aligned}$$

where

$$\Theta_t(s) = q_t(s)\phi_t(s), \quad \Xi_t(s) = \frac{\varphi_t}{\Theta_t(s)}, \quad \varphi_t = \frac{U(t)}{a(t)}$$

and

$$q_t(s) = \frac{U(t/s) - U(t)}{a(t)}, \quad a(t) = \frac{1}{w(U(t))}, \quad \phi_t(s) = \frac{U(t)}{U(t/s)}.$$

Further we decompose (5.1) as

$$\begin{aligned}
\frac{\bar{H}(x)}{\bar{F}(x)\bar{G}(1 - 1/\eta(x))} - \Gamma(\alpha_2 + 1) &= \int_0^1 \left((q_t(s))^{\alpha_2} - \ln^{\alpha_2}(1/s) \right) ds - \int_0^1 (q_t(s))^{\alpha_2} \left(1 - (\phi_t(s))^{\alpha_2}\right) ds \\
&\quad + \int_0^1 (\Theta_t(s))^{\alpha_2} \left(\frac{L(\Xi_t(s))}{L(\varphi_t)} - 1 \right) ds =: I_t - II_t + III_t. \qquad (5.2)
\end{aligned}$$

Since (5.1) tends to $\Gamma(\alpha_2 + 1)$ by Theorem 3.1 in Hashorva et al. (2010), the rest of the proof is concerned with the derivation of the convergence rates of the three terms on the right-hand side of (5.2).

By Lemma 5.2 in Draisma et al. (1999), for every $\epsilon > 0$, there exists $t_0 = t_0(\epsilon) > 0$ such that for all $t > t_0$ and all $s \in (0,1)$

$$\left| \frac{q_t(s) - \ln(1/s)}{\tilde{A}(t)} - \psi(1/s) \right| \le \epsilon (C_1 + C_3 s^{-\rho-\epsilon}),$$

with some positive constants C_1 and C_3 not depending on s and t. Therefore, by Taylor's expansion and the dominated convergence theorem

$$\lim_{t\to\infty} \frac{I_t}{\tilde{A}(t)} = \int_0^1 \alpha_2 \ln^{\alpha_2-1}(1/s)\psi(1/s)\, ds = K(\alpha_2, \rho), \tag{5.3}$$

with $\psi(\cdot)$ and $K(\alpha_2, \rho)$ defined in (2.2) and (2.4), respectively.

For the second term II_t, recall that $U \in \Pi(a)$ implies that $U \in \mathrm{RV}_0$ and $\varphi_t \to \infty$ as $t \to \infty$. By Corollary B.2.10 of de Haan and Ferreira (2006), for all $s \in (0,1)$ and sufficiently large t

$$0 \le q_t(s) \le cs^{-\epsilon}, \quad 0 \le \phi_t(s) = \left(1 + \frac{q_t(s)}{\varphi_t}\right)^{-1} \le 1 \tag{5.4}$$

for some $c > 1$ and any $\epsilon > 0$ implying

$$\frac{1 - \phi_t(s)}{1/\varphi_t} \le q_t(s) \le cs^{-\epsilon}.$$

Therefore, again by Taylor's expansion and the dominated convergence theorem

$$\begin{aligned} \lim_{t\to\infty} \frac{II_t}{1/\varphi_t} &= \alpha_2 \int_0^1 \ln^{\alpha_2+1}(1/s)\, ds \\ &= \alpha_2 \Gamma(\alpha_2 + 2). \end{aligned} \tag{5.5}$$

Finally, we show below that (5.6) holds for the third term III_t

$$\begin{aligned} &\lim_{t\to\infty} \frac{III_t}{A(\varphi_t)} - \frac{\Gamma(\alpha_2 - \tau_2 + 1) - \Gamma(\alpha_2 + 1)}{\tau_2} \\ &= \lim_{t\to\infty} \int_0^1 (\Theta_t(s))^{\alpha_2} \left(\frac{L(\Xi_t(s))/L(\varphi_t) - 1}{A(\varphi_t)} - \frac{(\Theta_t(s))^{-\tau_2} - 1}{\tau_2} \right) ds = 0. \end{aligned} \tag{5.6}$$

Recall that $L \in 2\mathrm{RV}_{0,\tau_2}$ with auxiliary function A. Again by Lemma 5.2 in Draisma et al. (1999), for every $\epsilon > 0$, there exists $t_0 = t_0(\epsilon) > 0$ such that for all $\varphi_t > t_0$, the integral of the right-hand side of (5.6) is dominated by

$$\begin{aligned} &\int_{\{s: s\in(0,1), \Xi_t(s) > t_0\}} \epsilon (\Theta_t(s))^{\alpha_2} (C_1 + C_3 (\Theta_t(s))^{-\tau_2} \exp(\epsilon |\ln(\Theta_t(s))|)\, ds \\ &+ \int_{\{s: s\in(0,1), \Xi_t(s) < t_0\}} (\Theta_t(s))^{\alpha_2} \left| \frac{L(\Xi_t(s))/L(\varphi_t) - 1}{A(\varphi_t)} \right| ds \\ &+ \int_{\{s: s\in(0,1), \Xi_t(s) < t_0\}} (\Theta_t(s))^{\alpha_2} \left| \frac{(\Theta_t(s))^{-\tau_2} - 1}{\tau_2} \right| ds =: J_{1t} + J_{2t} + J_{3t}. \end{aligned} \tag{5.7}$$

Recall that (5.4) implies that $f_t(s) = (\Theta_t(s))^{\alpha}$, $s \in (0,1)$ is integrable for all $\alpha > 0$ and sufficiently large t. Thus, J_{1t} tends to 0 since ϵ is arbitrarily small, whereas J_{3t} tends to 0 due to $\varphi_t / t_0 \to \infty$.

It suffices to prove that $\lim_{t\to\infty} J_{2t} = 0$. To this end, we need the two statements as in (5.8) and (5.9) below. Next, note that $L \in 2\mathrm{RV}_{0,\tau_2}, \tau_2 < 0$ implies that L is ultimately bounded away from 0 and

$$L(t) = t^{\alpha_2}\bar{G}(1-1/t) \le t^{\alpha_2}, \quad L(t) > 1/M$$

hold for some given $M > 0$ and sufficiently large t. By Potter bounds (cf. Proposition B.1.9 in de Haan and Ferreira (2006)), for any $\epsilon > 0$, there exists $t_0 = t_0(\epsilon) > 0$ such that $\min(\varphi_t, \Xi_t(s)) > t_0$

$$\frac{L(\Xi_t(s))}{L(\varphi_t)} \le c\max((\Theta_t(s))^{\epsilon}, (\Theta_t(s))^{-\epsilon}),$$

otherwise for $\varphi_t > t_0, \Xi_t(s) \le t_0$ such that

$$\frac{L(\Xi_t(s))}{L(\varphi_t)} \le \frac{(\Xi_t(s))^{\alpha_2}}{1/M} \le M t_0^{\alpha_2}. \tag{5.8}$$

Note that $|A|$ is ultimately decreasing and $|A| \in \mathrm{RV}_{\tau_2}$. By the Karamata Representation (cf. Resnick (1987), p.17), for any given $\delta > 0$ and $t_0 < \varphi_t < \Theta_t(s)t_0$

$$|A(\varphi_t)| \ge |A(\Theta_t(s)t_0)| \ge K_2(\Theta_t(s))^{\tau_2-\delta}|A(t_0)|, \tag{5.9}$$

with $K_2 \in (0,1)$ a constant. Therefore, the integrand of J_{2t} is dominated by

$$\frac{Mt_0^{\alpha_2}+1}{K_2|A(t_0)|}(\Theta_t(s))^{\alpha_2-\tau_2+\delta} \le \frac{Mt_0^{\alpha_2}+1}{K_2|A(t_0)|}(cs^{-\epsilon})^{\alpha_2-\tau_2+\delta}.$$

Hence, by the dominated convergence theorem, J_{2t} tends to 0 as $t \to \infty$. Consequently, we have that (5.7) tends to 0 as $t \to \infty$, and thus (5.6) follows establishing the proof. □

Proof of Corollary 2.4 For $a = 1/w(U)$ the first-order auxiliary function of U, note that, by Theorem B.3.1 in de Haan and Ferreira (2006), we have $a \in 2\mathrm{RV}_{0,\rho}, \rho \le 0$ with auxiliary function $\tilde{A}$. Thus, for sufficiently large x

$$\frac{w\left(x+z/w(x)\right)}{w(x)} = 1 - \frac{e^{\rho z}-1}{\rho}\tilde{A}\left(\frac{1}{\bar{F}(x)}\right)(1+o(1)) \tag{5.10}$$

holds for all $z \in \mathbb{R}$ (here $(e^{\rho z}-1)/\rho$ is interpreted as z for $\rho = 0$). Since further $\bar{G}(1-1/x) \in 2\mathrm{RV}_{-\alpha_2,\tau_2}$ and $|A| \in RV_{\tau_2}$, we have

$$\begin{aligned}
\frac{\bar{G}\left(1-\frac{1}{\eta(x+z/w(x))}\right)}{\bar{G}(1-1/\eta(x))} &= \left(\frac{\eta(x+z/w(x))}{\eta(x)}\right)^{-\alpha_2}\left(1+\frac{\left(\frac{\eta(x+z/w(x))}{\eta(x)}\right)^{\tau_2}-1}{\tau_2}A(\eta(x))(1+o(1))\right)\\
&= \left(\frac{x+z/w(x)}{x}\frac{w(x+z/w(x))}{w(x)}\right)^{-\alpha_2}\left[1+o\left(\frac{1}{\eta(x)}\right)+o\left(\tilde{A}\left(\frac{1}{\bar{F}(x)}\right)\right)\right]\\
&= 1-\left[\frac{\alpha_2 z}{\eta(x)}-\alpha_2\frac{e^{\rho z}-1}{\rho}\tilde{A}\left(\frac{1}{\bar{F}(x)}\right)\right](1+o(1)).
\end{aligned} \tag{5.11}$$

Recall that $U \in 2\mathrm{ERV}_{0,\rho}$ with auxiliary function $\tilde{A}$, and

$$\frac{\bar{F}\left(x+z/w(x)\right)}{\bar{F}(x)} = e^{-z}\left(1+\psi(e^{-z})\tilde{A}\left(\frac{1}{\bar{F}(x)}\right)\right). \tag{5.12}$$

The claim (2.7) follows from (2.6), (5.10)–(5.12) and the fact that

$$\lim_{x\to x_F}\eta(x)\tilde{A}\left(\frac{1}{\bar{F}(x)}\right)=\lim_{t\to\infty}\frac{\tilde{A}(t)}{a(t)/U(t)}=0 \tag{5.13}$$

for $\rho<0$ (cf. Lemma B.3.16 in de Haan and Ferreira (2006)).

Using (5.13) and the relation $h(h^{\leftarrow}(t))=t(1+o(1))$ as $t\to\infty$ with $h=1/\bar{H}$ in (2.7), we have that $U_X\in 2\mathrm{ERV}_{0,0}$ with auxiliary functions $\breve{a}$ and $\breve{A}$ stated by (2.8). □

Proof of Corollary 2.5 First, note that $U(t)=V^{\leftarrow}(\ln t)=(\ln t)^{\theta}\ell(\ln t)$ with $\ell\in 2\mathrm{RV}_{0,\rho'}$ with auxiliary function b. We have

$$\begin{aligned} U(tx)=V^{\leftarrow}(\ln tx)&=(\ln t)^{\theta}\ell(\ln t)\left(1+\frac{\ln x}{\ln t}\right)^{\theta}\frac{\ell(\ln t(1+\ln x/\ln t))}{\ell(\ln t)}\\ =&U(t)\left(1+\theta\frac{\ln x}{\ln t}+\frac{\theta(\theta-1)}{2}\frac{\ln^2 x}{\ln^2 t}(1+o(1))\right)\left(1+b(\ln t)\frac{(1+\ln x/\ln t)^{\rho'}-1}{\rho'}(1+o(1))\right). \end{aligned}$$

Therefore, $U\in 2\mathrm{ERV}_{0,0}$ with auxiliary functions a and $\tilde{A}$ as

$$a(t)=\frac{\theta+b(\ln t)}{\ln t}U(t),\quad \tilde{A}(t)=\frac{\theta-1+(\rho'-1)b(\ln t)/\theta}{\ln t}.$$

This implies that

$$\eta(x)=\frac{x}{a(1/\bar{F}(x))}=\frac{V(x)}{\theta+b(V(x))},\quad \tilde{A}\left(\frac{1}{\bar{F}(x)}\right)=\frac{\theta-1+(\rho'-1)b(V(x))/\theta}{V(x)}. \tag{5.14}$$

By Theorem 2.3,

$$\begin{aligned} \bar{H}(x)=&\bar{F}(x)\bar{G}\left(1-\frac{1}{V(x)}\right)\left(\frac{\eta(x)}{V(x)}\right)^{-\alpha_2}\left[1+\frac{\left(\frac{\eta(x)}{V(x)}\right)^{\tau_2}-1}{\tau_2}A(V(x))(1+o(1))\right]\Gamma(\alpha_2+1)\\ &\times\left[1+\left(\frac{\frac{\Gamma(\alpha_2-\tau_2+1)}{\Gamma(\alpha_2+1)}-1}{\tau_2}\left(\frac{\eta(x)}{V(x)}\right)^{\tau_2}A(V(x))\right.\right.\\ &\left.\left.-\left(\theta+b(V(x))-\frac{\theta-1+(\rho'-1)b(V(x))/\theta}{2}\right)\frac{\alpha_2(\alpha_2+1)}{V(x)}\right)(1+o(1))\right]\\ =&\exp(-V(x))\bar{G}\left(1-\frac{1}{V(x)}\right)\Gamma(\alpha_2+1)\theta^{\alpha_2}\\ &\times\left[1+\left(\frac{\alpha_2}{\theta}b(V(x))+\frac{\frac{\Gamma(\alpha_2-\tau_2+1)}{\theta^{\tau_2}\Gamma(\alpha_2+1)}-1}{\tau_2}A(V(x))-\frac{(\theta+1)\alpha_2(\alpha_2+1)}{2V(x)}\right)(1+o(1))\right] \qquad (5.15)\\ =:&\exp(-V(x))(V(x))^{-\alpha_2}L^*(V(x)), \qquad (5.16) \end{aligned}$$

where (5.15) is due to (5.14) and $\bar{G}(1-1/x)\in 2\mathrm{RV}_{-\alpha_2,\tau_2}$ with auxiliary function A. Clearly, L^* is a slowly varying function. Therefore, letting the right-hand side of (5.16) equal to $1/s$, and solving the equation of x, we have $V(x)=\ln s(1+o(1))$ and

$$U_X(s)=V^{\leftarrow}\left(\ln\frac{sL^*(V(x))}{(V(x))^{\alpha_2}}\right)$$

$$= \left(\ln s - \alpha_2 \ln V(x)\left(1 - \frac{\ln L^*(V(x))}{\alpha_2 \ln V(x)}\right)\right)^{\theta} \ell\left(\ln s - \alpha_2 \ln V(x)\left(1 - \frac{\ln L^*(V(x))}{\alpha_2 \ln V(x)}\right)\right)$$
$$= (\ln s - \alpha_2 \ln\ln s(1+o(1)))^{\theta}\,\ell(\ln s)(1 + o(\ln\ln s/\ln s)).$$

The last step is due to $\ell \in 2\mathrm{RV}_{0,\rho'}$ and the property of slowly varying function: $\ln L^*(V(x))/\ln V(x) \to 0$ (see Bingham et al. (1987)). Hence

$$\bar{H}(x) = \exp(-V^*(x)), \quad (V^*)^{\leftarrow}(x) = x^{\theta}\left(1 - \alpha_2 \frac{\ln x}{x}\right)^{\theta} \ell^*(x).$$

Thus the claim in Corollary 2.5 follows from $\ell^* \in 2\mathrm{RV}_{0,\rho'^*}$ with $\rho'^* = \max(\rho', -1)$ and auxiliary function

$$b^*(x) = b(x) + \frac{\theta\alpha_2 \ln x}{x}.$$

We complete the proof. □

Proof of Theorem 2.6 First, by arguments similar to (5.1) for the case that $F \in D(Q_0)$, we have

$$\frac{\bar{H}(x)}{\bar{F}(x)\bar{G}(x)} = \int_0^1 (\Theta_t(s))^{\alpha_2} \frac{L(\varphi_t/\Theta_t(s))}{L(\varphi_t)}\,ds,$$

where $t = 1/\bar{F}(x), x = U(t)$ and

$$\Theta_t(s) = q_t(s)\phi_t(s), \quad \varphi_t = \frac{1}{1-U(t)}, \quad \text{with} \quad q_t(s) = \frac{U(t/s) - U(t)}{1-U(t)},\ \phi_t(s) = \frac{1}{U(t/s)}.$$

Next,

$$\frac{\bar{H}(x)}{\bar{F}(x)\bar{G}(x)} - \alpha_1 B(\alpha_1, \alpha_2+1) = \int_0^1 \left(q_t(s)\right)^{\alpha_2} - (1 - s^{1/\alpha_1})^{\alpha_2}\,ds$$
$$+ \int_0^1 (q_t(s))^{\alpha_2}((\phi_t(s))^{\alpha_2} - 1)\,ds + \int_0^1 (\Theta_t(s))^{\alpha_2}\left(\frac{L(\varphi_t/\Theta_t(s))}{L(\varphi_t)} - 1\right) ds$$
$$=: I_t + II_t + III_t. \tag{5.17}$$

It remains thus to derive the convergence rate of each term above. By Lemma 5.2 in Draisma et al. (1999), for every $\epsilon > 0$, there exists $t_0 = t_0(\epsilon) > 0$ such that for all $t > t_0$ and all $s \in (0,1)$

$$\left|\frac{q_t(s) - (1 - s^{1/\alpha_1})}{\tilde{A}(t)} + s^{1/\alpha_1}\frac{s^{-\tau_1/\alpha_1} - 1}{\tau_1/\alpha_1}\right| \le \epsilon(C_1 + C_2 s^{1/\alpha_1} + C_3 s^{(1-\tau_1)/\alpha_1 - \epsilon}),$$

with some positive constants C_1, C_2 and C_3 not depending on s and t. Therefore, by Taylor's expansion and the dominated convergence theorem

$$\begin{aligned}\lim_{t\to\infty}\frac{I_t}{\tilde{A}(t)} &= -\alpha_2 \int_0^1 (1 - s^{1/\alpha_1})^{\alpha_2 - 1} s^{1/\alpha_1}\frac{s^{-\tau_1/\alpha_1} - 1}{\tau_1/\alpha_1}\,ds \\ &= -\frac{\alpha_2\alpha_1^2}{\tau_1}(B(\alpha_2, \alpha_1 - \tau_1 + 1) - B(\alpha_2, \alpha_1 + 1)).\end{aligned}$$

Here, (5.18) for $\tau_1 = 0$ is understood as

$$-\alpha_2 \int_0^1 (1 - s^{1/\alpha_1})^{\alpha_2 - 1} s^{1/\alpha_1} \lim_{\tau_1 \to 0} \frac{s^{-\tau_1/\alpha_1} - 1}{\tau_1/\alpha_1}\,ds$$

$$= \lim_{\tau_1 \to 0} -\frac{\alpha_2 \alpha_1^2}{\tau_1} \big(B(\alpha_2, \alpha_1 - \tau_1 + 1) - B(\alpha_2, \alpha_1 + 1) \big)$$

(cf. Corollary 4.4 in Mao and Hu (2012)). For II_t, note that $q_t(s) \in (0,1), \varphi_t \to \infty$ and thus for all $s \in (0,1)$

$$0 \le \frac{\phi_t(s) - 1}{1/\varphi_t} = \frac{(1 - (1 - q_t(s))/\varphi_t)^{-1} - 1}{1/\varphi_t} = \frac{1 - q_t(s)}{1 - (1 - q_t(s))/\varphi_t} \le \frac{1}{1 - 1/\varphi_t} \to 1$$

as $t \to \infty$. Therefore, by Taylor's expansion and the dominated convergence theorem

$$\begin{aligned} \lim_{t\to\infty} \frac{II_t}{1/\varphi_t} &= \int_0^1 \lim_{t\to\infty} (q_t(s))^{\alpha_2} \frac{(1 + (\phi_t(s) - 1))^{\alpha_2} - 1}{1/\varphi_t} \, ds \\ &= \alpha_2 \int_0^1 (1 - s^{1/\alpha_1})^{\alpha_2} s^{1/\alpha_1} \, ds = \alpha_1 \alpha_2 B(\alpha_1 + 1, \alpha_2 + 1). \end{aligned} \tag{5.18}$$

Finally, we consider the third term III_t. By Lemma 5.2 in Draisma et al. (1999), for every $\epsilon > 0$, there exists $t_0 = t_0(\epsilon) > 0$ such that for all $\varphi_t > t_0$ and all $s \in (0,1)$

$$\begin{aligned} &\left| (\Theta_t(s))^{\alpha_2} \left(\frac{L\big(\frac{\varphi_t}{\Theta_t(s)}\big)/L(\varphi_t) - 1}{A(\varphi_t)} - \frac{(\Theta_t(s))^{-\tau_2} - 1}{\tau_2} \right) \right| \\ &\le \epsilon (C_1 + C_2 (\Theta_t(s))^{\alpha_2} + C_3 (\Theta_t(s))^{\alpha_2 - \tau_2 - \epsilon}) \le \epsilon (C_1 + C_2 + C_3). \end{aligned}$$

The last step is due to $\Theta_t(s) \le 1$ for all $s \in (0,1)$ and $t > 0$. Hence, by the dominated convergence theorem

$$\begin{aligned} \lim_{t\to\infty} \frac{III_t}{A(t)} &= \int_0^1 \lim_{t\to\infty} (\Theta_t(s))^{\alpha_2} \frac{(\Theta_t(s))^{-\tau_2} - 1}{\tau_2} \, ds \\ &= \int_0^1 (1 - s^{1/\alpha_1})^{\alpha_2} \frac{(1 - s^{1/\alpha_1})^{-\tau_2} - 1}{\tau_2} \, ds = \frac{\alpha_1}{\tau_2} \big(B(\alpha_1, \alpha_2 - \tau_2 + 1) - B(\alpha_1, \alpha_2 + 1) \big). \end{aligned} \tag{5.19}$$

Consequently, the claim follows from (5.18), (5.18) and (5.19). □

Proof of Lemma 4.1 We only give the proof of the case $\lambda \in (0,1)$. The other cases can be verified by similar arguments. Clearly, for $\lambda \in (0,1), S^*(\lambda) \le 1$ and it is bounded away from unit unless $I_1 = I_2 = 1$, and when the event $\{I_1 = I_2 = 1\}$ occurs, $S^*(\lambda) \uparrow 1$ if and only if $|S - \lambda| \downarrow 0$. For small $x > 0$, the event

$$\{S^*(\lambda) > 1 - x\} = \{(S - \lambda)^2 + 2\lambda x S < 2x - x^2\}$$

occurs is equivalent that

$$(S - \lambda)^2 < 2x\big((1 - \lambda^2) - \lambda\sqrt{2x(1 - \lambda^2)}(1 + o_p(1))\big).$$

Consequently, the claim follows from (4.18). □

6 Appendix

This appendix includes two tables. Table 1 contains Weibull tail distributions satisfying the second-order regular varying conditions and Table 2 shows several distributions in the maximum domain of attraction of the Fréchet distribution, the Gumbel distribution and the Weibull distribution in the second-order framework.

Table 1: **Weibull tail distributions**

Weibull tail distributions	Tail $\bar{F}$ or pdf f	θ	ρ	$b(x)$
Gamma ($\Gamma(\alpha,\lambda)$)	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x},\ \lambda,\alpha>0,\alpha\neq 1$	1	-1	$\frac{(1-\alpha)\ln x}{x}$
Absolute Normal ($\lvert N(0,1)\rvert$)	$f(x)=\frac{2}{\sqrt{2\pi}}e^{-x^2/2}$	$\frac{1}{2}$	-1	$\frac{\ln x}{4x}$
Weibull ($W(\beta,c)$)	$\bar{F}(x)=\exp(-cx^\beta),\ c,\beta>0$	$\frac{1}{\beta}$	$-\infty$	0
Perturbed Weibull ($PW(\beta,\alpha)$)	$\bar{F}(x)=e^{-x^\beta(C+Dx^{-\alpha})},\ \alpha,\beta,C>0,D\in\mathbb{R}$	$\frac{1}{\beta}$	$-\frac{\alpha}{\beta}$	$\frac{\alpha D}{\beta^2}C^{\alpha/\beta-1}x^{-\alpha/\beta}$
Modified Weibull ($MW(\beta,c)$)	$Y\ln Y\sim F, Y\sim W(\beta,c)$	$\frac{1}{\beta}$	0	$\frac{1}{\ln x}$
Benktander II ($\mathcal{B}II(\beta,\lambda)$)	$\bar{F}(x)=x^{-(1-\beta)}\exp(-\frac{\lambda}{\beta}(x^\beta-1)),\ \lambda>0,0<\beta<1$	$\frac{1}{\beta}$	-1	$\frac{(1-\beta)\ln x}{\beta^2 x}$
Extended Weibull ($\mathcal{E}W(\beta,\alpha)$)	$\bar{F}(x)=r(x)\exp(-x^\beta),\ \beta\in(0,1),r\in\mathrm{RV}_{-\alpha},\alpha\in\mathbb{R}$	$\frac{1}{\beta}$	-1	$\frac{\alpha\ln x}{\beta^2 x}$
Logistic	$\bar{F}(x)=\frac{2}{1+e^x}$	1	-1	$-\frac{\ln 2}{x}$
Gumbel ($G(\mu)$)	$\bar{F}(x)=1-\exp(-\exp(\mu-x)),\ \mu\neq 0$	1	-1	$-\frac{\mu}{x}$

Weibull tail distributions: $\bar{F}(x)=\exp(-V(x)), V^{\leftarrow}(x)=x^\theta\ell(x)$ and $\ell\in 2\mathrm{RV}_{0,\rho}$ with auxiliary function b.

Table 2: **Risks satisfying the second-order regular variation conditions**

Fréchet MDA	Tail $\bar{F}$ or pdf f	α	τ	$A(x)$
Pareto	$\bar{F}(x) = \left(\frac{\theta}{\theta+x}\right)^{\alpha}, \quad \theta, \alpha > 0$	α	-1	$\frac{\alpha\theta}{x}$
Fréchet	$\bar{F}(x) = 1 - \exp(-x^{-\alpha})$	α	$-\alpha$	$\frac{\alpha x^{-\alpha}}{2}$
Burr	$\bar{F}(x) = (1 + x^{b})^{-a}$	ab	$-b$	abx^{-b}
Hall-Weiss	$\bar{F}(x) = \frac{1}{2}x^{-\alpha}(1 + x^{\tau}), \quad \alpha > 0, \tau < 0$	α	τ	τx^{τ}
$F(m,n)$	$f(x) = \frac{(m/n)^{m/2}}{B(m/2,n/2)} x^{m/2-1}\left(1 + \frac{mx}{n}\right)^{-(m+n)/2}$	$\frac{n}{2}$	-1	$\frac{(m+n)n^2}{2m(n+2)x}$
Log-gamma	$f(x) = \frac{\alpha^{\beta}}{\Gamma(\beta)}(\ln x)^{\beta-1}x^{-\alpha-1}, \quad \alpha, \beta > 0$	α	0	$\frac{\beta-1}{\ln x}$
Inv-gamma	$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{-\alpha-1}e^{-\beta/x}, \quad \alpha, \beta > 0$	α	-1	$\frac{\alpha\beta}{(\alpha+1)x}$
Absolute t	$f(x) = \frac{2\Gamma((v+1)/2)}{\sqrt{v\pi}\Gamma(v/2)}(1 + x^2/v)^{-(v+1)/2}, \quad v \in \mathbb{N}$	v	-2	$\frac{v^2(v+1)}{(v+2)x^2}$
Weibull MDA	**Tail $\bar{F}(x_F - 1/x)$ or pdf f**	α	τ	$A(x)$
Beta	$f(x) = \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}, \quad a, b > 0$	b	$-1 \ (a \neq 1)$	$\frac{b(a-1)}{(b+1)x}$
Reverse-Burr	$\bar{F}(x_F - 1/x) = (1 + x^{b})^{-a}$	ab	$-b$	abx^{-b}
Extreme value Weibull	$\bar{F}(x_F - 1/x) = 1 - \exp(-x^{-\alpha})$	α	$-\alpha$	$\frac{\alpha x^{-\alpha}}{2}$
Gumbel MDA	**Tail $\bar{F}$ or pdf f**	ρ	$a(x)$	$A(x)$
Gamma	$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}, \quad \lambda, \alpha > 0$	0	$\left(1 + \frac{\alpha-1}{\ln x}\right)/\lambda$	$\frac{1-\alpha}{\ln^2 x}$
Absolute Normal	$f(x) = \frac{2}{\sqrt{2\pi}}e^{-x^2/2}$	0	$\frac{U_1(2x)}{2\ln(2x)}$	$-\frac{1}{2\ln x}$
Log-normal	$f(x) = \frac{1}{\sqrt{2\pi}x}\exp(-\frac{\ln^2 x}{2})$	0	$\frac{\exp(U_1(x))}{\sqrt{2\ln x}}$	$\frac{1}{\sqrt{2\ln x}}$
Logistic	$\bar{F}(x) = \frac{2}{1+e^{x}}$	-1	1	$\frac{1}{2x}$
Truncated Gumbel	$\bar{F}(x) = \frac{1-\exp(-e^{-x})}{1-e^{-1}}$	-1	1	$\frac{1-e^{-1}}{2x}$
Exponential with finite x_F	$\bar{F}(x) = \exp(-\frac{c}{x_F - x} + \frac{c}{x_F}), \quad c > 0, x_F > 0$	0	$\frac{c}{(\ln x + c/x_F)^2}$	$-\frac{2}{\ln x}$
Weibull	$\bar{F}(x) = \exp(-cx^{\beta}), \quad c > 0, \beta \in (0,1)$	0	$\frac{(\ln x)^{1/\beta-1}}{\beta c^{1/\beta}}$	$\frac{1/\beta-1}{\ln x}$
Benktander I	$\bar{F}(x) = \left(1 + \frac{2\beta}{\alpha}\ln x\right)\exp(-\beta\ln^2 x - (\alpha+1)\ln x)$	0	$\frac{U_2(x)}{2\sqrt{\beta\ln x}}$	$\frac{1}{2\sqrt{\beta\ln x}}$
Benktander II	$\bar{F}(x) = x^{-(1-\beta)}\exp(-\frac{\alpha}{\beta}(x^{\beta} - 1)), \alpha > 0, 0 < \beta < 1$	0	$a^*(x)$	$\frac{1/\beta-1}{\ln x}$

$$a^*(x) = \frac{1-(1-\beta)/(\beta(\alpha/\beta+\ln x))}{\beta(\alpha/\beta+\ln x)}U(x), \qquad U(x) = \left(\frac{\beta}{\alpha}((\alpha/\beta + \ln x) - (1-\beta)\ln U(x))\right)^{1/\beta}$$

$$U_1(x) = \sqrt{2\ln x} - \frac{\ln(4\pi\ln x)}{2\sqrt{2\ln x}}, \qquad U_2(x) = \exp\left(-\frac{\alpha+1}{2\beta} + \sqrt{\frac{\ln x}{\beta}} + \frac{\ln\ln x + \ln(4\beta/\alpha^2) + (\alpha+1)^2/(2\beta)}{4\sqrt{\beta\ln x}}\right)$$

For the Fréchet MDA $\bar{F} \in 2\mathrm{RV}_{-\alpha,\tau}$ with auxiliary function A. Further for the Weibull MDA $\bar{F}(x_F - 1/x) \in 2\mathrm{RV}_{-\alpha,\tau}$ with auxiliary function A and a finite upper endpoint x_F. Finally, note that for the Gumbel MDA the tail quantile function $U \in 2\mathrm{ERV}_{0,\rho}$ with the first-order auxiliary function a and the second-order auxiliary function A.

Acknowledgements. We are in debt to the referees for suggestions which improved the manuscript significantly. The authors acknowledge partial support by the Swiss National Science Foundation grants 200021-134785, 200021-140633/1 and RARE -318984 (an FP7 Marie Curie IRSES Fellowship). The third author was also supported by the National Natural Science Foundation of China grant o.11171275 and the Natural Science Foundation Project of CQ no. cstc2012jjA00029.

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