TAIL ASYMPTOTICS FOR THE SUM OF TWO HEAVY-TAILED DEPENDENT RISKS

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Abstract

Let X_1, X_2 denote positive heavy-tailed random variables with continuous marginal distribution functions F_1 and F_2, respectively. The asymptotic behavior of the tail of $X_1 + X_2$ is studied in a general copula framework and some bounds and extremal properties are provided. For more specific assumptions on F_1, F_2 and the underlying dependence structure of X_1 and X_2, we survey explicit asymptotic results available in the literature and add several new cases.

Keywords: copula, dependence, mean excess function, regular variation, subexponential distribution, exchangeability, tail dependence

1 Introduction and background

A qualitative and quantitative understanding of the probability of an overshoot of a sum of heavy-tailed risks over a large threshold is of major importance in applied probability and its applications in risk management, such as the determination of risk measures for given portfolios of risks, evaluation of credit risk etc. Under the assumption of independence among the risks, the situation is well understood. In particular, from the very definition of subexponential distributions, given identical marginal distributions, the maximum among the involved risks determines the distribution of the sum and, on the other hand, for non-identical marginals the distribution of the sum is determined by the component with the heaviest tail (see e.g. Asmussen [3, Ch.IX]).

However, for practical purposes the independence assumption is often too restrictive and there is a need for an understanding of the sensitivity of the distribution of sums of risks on the dependence structure between them.

Over the last few years, several results in this direction have been developed. The

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purpose of this paper is to analyze the particular setting of two risk components, both collecting relevant material from the literature under the same umbrella and adding some additional explicit results. Some of the observations are of basic nature, but may help to get a clearer intuitive picture of the matter.
That is, for positive random variables X_1, X_2 with absolutely continuous (heavy-tailed) marginal distribution functions F_1 and F_2, respectively, we are interested in the asymptotic behavior of

$$\mathbb{P}(X_1 + X_2 > x) \tag{1}$$

for large x and given type of dependence among X_1 and X_2. Let us assume that the limit $c = \lim_{x\to\infty} \frac{\overline{F}_2(x)}{\overline{F}_1(x)} \leq 1$ exists (in particular, w.l.o.g. X_1 has the heavier tail in case of non-identically distributed X_1, X_2). At some instances we will focus on the (weighted) sum of exchangeable X_1, X_2, in which case the common marginal distribution function will always be denoted by F. Particularly interesting questions are when (1) is of the same order as for the independent case, and more generally, when the asymptotics of (1) are of the order $d \cdot \mathbb{P}(X_1 > x)$ for some $d \in (0, \infty)$.

For identically distributed X_1 and X_2, if the joint distribution function of X_1 and X_2 can be bounded below by some distribution function $G(x_1, x_2)$ for any $x_1, x_2 \geq 0$, Denuit *et al.* [10] gave the following bounds:

$$\begin{aligned} 1 - \inf_{y\geq 0} \left(F\left(\frac{y}{c_1}\right) + F\left(\frac{x-y}{c_2}\right) - G\left(\frac{y}{c_1}, \frac{x-y}{c_2}\right) \right) \\ \leq \mathbb{P}\left(c_1 X_1 + c_2 X_2 > x\right) \leq 1 - \sup_{y\geq 0} G\left(\frac{y}{c_1}, \frac{x-y}{c_2}\right), \end{aligned} \tag{2}$$

where $c_1, c_2 > 0$ are arbitrary positive constants. For each x, these bounds are best possible, although neither the lower nor the upper bound is the distribution tail of a sum of random variables with marginal distribution F (in particular, the comonotone and counter-monotone copula do in general not provide bounds for the tail of $c_1X_1 + c_2X_2$, contrary to what one might expect at a first glance, see [10] for details).
For positive quadrant dependence (i.e. $\mathbb{P}(X_1 > x, X_2 > x) \geq \mathbb{P}(X_1 > x)\, \mathbb{P}(X_2 > x)$ for all $x \geq 0$) we have $G(x_1, x_2) = F(x_1)F(x_2)$. On the other hand, without any knowledge of the underlying dependence structure, $G(x_1, x_2)$ has to be replaced by the counter-monotone copula $C_W(F(x_1), F(x_2)) = \max\{F(x_1) + F(x_2) - 1, 0\}$, in which form the above result is due to Makarov [21]. For an extension to best-possible bounds on the distribution of general non-decreasing functions of n dependent risks, see for instance Cossette *et al.* [8], Embrechts & Puccetti [12] and Mesfioui & Quessy [23].
However, the above approach is not well-suited for asymptotic considerations and does not make use of the heavy-tail assumption directly. Moreover, one can get more explicit results by specifying classes of dependence structures.

Let $\mathcal{S}$ denote the class of subexponential distributions and $\mathcal{D}$ the class of dominatedly varying distributions (i.e. all distributions on the positive half-line for which $\limsup_{x\to\infty} \overline{F}(x/2)/\overline{F}(x) < \infty$). Furthermore, let $\mathcal{L}$ denote the class of long-tailed distributions (i.e. all distributions on the positive half-line with $\lim_{x\to\infty} \overline{F}(x-y)/\overline{F}(x) = 1$ for all $y > 0$). We will use the notation $F \in \mathcal{R}$ if $\overline{F}(x) = 1 - F(x)$ is regularly varying at infinity with some index $-\alpha < 0$ ($\overline{F} \in \mathcal{R}_{-\alpha}$). Recall that $\mathcal{R} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$ (see e.g. Embrechts *et al.* [11]). The tail probability of weighted sums of independent random variables with regularly varying tails, where the weights are dependent random variables, was studied in Goovaerts et al. [15], see also Tang & Wang [30] for a generalization to the class $D \cap \mathcal{L}$. Asymptotic tail probabilities for negatively associated sums of heavy-tailed random variables were recently investigated in Wang & Tang [31] and Geluk & Ng [14].

For fixed continuous (and especially for identically distributed) marginals, a copula representation of (1) may be considered as a natural tool to analyze the impact of dependence, and we will take up this approach in what follows. For background reading on copulae and their properties, we refer to Joe [17] or Nelsen [24]. Intuitively, there is a trade-off between dependence in the tail and heaviness of F_1: the heavier F_1 is, the stronger the dependence in the tail has to be in order to affect the tail behavior of X_1+X_2. In the paper, this relationship is formalized to some extent.

Recall that the (upper) tail dependence coefficient is defined by

$$\lambda := \lim_{u\to 1} \mathbb{P}(F(X_2) > u | F(X_1) > u)$$

and actually can be interpreted as a property of the underlying copula. If $\lambda = 0$, then X_1 and X_2 are called tail-independent. λ is a frequently used measure of extremal dependence (for estimation procedures, see Frahm *et al.* [13]). For a comparison of various tail dependence measures with a view towards financial time series, see Malevergne and Sornette [22]. For non-identical marginal distributions F_1, F_2, it will turn out more natural to consider the quantity

$$\hat{\lambda} := \lim_{x\to\infty} \mathbb{P}(X_2 > x | X_1 > x).$$

Unlike λ, the quantity $\hat{\lambda} \in [0, 1]$ is not solely a function of the dependence structure, but also of the marginal behavior. The only exception is the case of identically distributed X_1, X_2, where clearly $\lambda = \hat{\lambda}$.
In fact, many available joint tail dependence models have been developed in the framework of bivariate extreme value theory and are based on max-stability (for estimation procedures in this context we refer to Abdous *et al.* [1]). Except for the independent case, all bivariate extreme value distributions have a $\lambda > 0$. On the other hand, as pointed out in Coles *et al.* [7], several classical estimation procedures for λ from a data set might lead to the conclusion $\lambda > 0$ where in fact tail independence is present (see [7] for details and suggestions to overcome this problem). For another model of joint tail dependence in extreme value theory that allows for asymptotic independence, see Ledford & Tawn [20]. An alternative extremal dependence measure feasible for multivariate regularly varying tails is discussed in Resnick [27]. The approach pursued in this paper is related to, but not contained

in the extreme value framework. We are rather interested in the question: Given F_1, F_2, what types of asymptotic behavior of $\mathbb{P}(X_1 + X_2 > x)$ are possible and what assumptions on the marginal distributions and the underlying dependence structure admit an explicit description of that behavior?

Although tail dependence provides a rather restrictive description of the dependence in the tail (for identical marginals, one basically looks at the dependence behavior along the line $X_1 = X_2$ in the tail), in the exchangeable case λ already gives some crude information about the distribution of the sum. Moreover, as will be shown in Section 3.1.2, for $F \in \mathcal{R}$ tail independence is a sufficient condition for insensitivity of tail asymptotics of the sum with respect to dependence, whereas for $F \in \mathcal{S} \cap \mathrm{MDA}(\mathrm{Gumbel})$ (i.e. subexponential distributions in the maximum domain of attraction of the Gumbel distribution) this is not true, as will be shown in Section 2.2.
For certain classes of copulae among X_1 and X_2 (including those of Archimedean type), Juri & Wüthrich [18, 19] established a distributional limit result of conditional dependence in the tail, which in particular refines the description through the coefficient λ. For Archimedean copulae this result could be exploited in Wüthrich [32] and Alink *et al.* [2] to give sharp asymptotics of the tail of $X_1 + X_2$, see Section 3.3. Another related refinement of the coefficient λ based on so-called tail copulae is discussed in Schmidt & Stadtmüller [29].

In Section 2, some general bounds and a copula representation of $\mathbb{P}(X_1 + X_2 > x)$ are discussed. Section 3 then gives explicit results under more specific assumptions on F and the underlying dependence structure. This should be viewed as an outline of several partial answers to the question raised above, setting the stage for further research towards a full understanding of the matter, including the extension to sums of arbitrarily many risks.

2 Some general considerations

Let us first collect some preliminary facts. Recall that $c = \lim_{x\to\infty} \frac{\overline{F}_2(x)}{\overline{F}_1(x)}$.

Lemma 2.1. *$c\,\lambda \le \hat{\lambda} \le \min(c, \lambda)$.*

Proof. We have:

$$\hat{\lambda} = \lim_{x\to\infty} \frac{1 - F_1(x) - F_2(x) + C(F_1(x), F_2(x))}{1 - F_1(x)} \le \lim_{x\to\infty} \frac{1 - F_2(x)}{1 - F_1(x)} = c.$$

Consider $c < 1$ first. Since there exists an $x_0 > 0$ such that $F_2(x) > F_1(x)$ for all $x > x_0$, we have $C(F_1(x), F_2(x)) - C(F_1(x), F_1(x)) \le F_2(x) - F_1(x)$ for all $x > x_0$ (cf. [24]). Thus

$$\hat{\lambda} \le \lim_{x\to\infty} \frac{1 - F_1(x) - F_1(x) + C(F_1(x), F_1(x))}{1 - F_1(x)} = \lambda.$$

In the case $c = 1$, for every $\epsilon > 0$ there exists an $x_0 > 0$ such that $F_1(x) \le F_2(x) + \epsilon\overline{F}_1(x)$ for all $x > x_0$ and hence $C(F_1(x), F_2(x)) - C(F_1(x), F_1(x)) - (F_2(x) - F_1(x)) \le \epsilon\overline{F}_1(x)$. We get $\hat{\lambda} \le \lambda + \epsilon$ and the upper bound follows for $\epsilon > 0$.
For the lower bound, just note that for $F_2(x) > F_1(x)$ one has $C(F_2(x), F_2(x)) - C(F_1(x), F_2(x)) \le F_2(x) - F_1(x)$ and the result follows analogously. □

Lemma 2.2. *(a)* $\mathbb{P}(\max(X_1, X_2) > x) \sim (1 + c - \hat{\lambda})\overline{F}_1(x)$

(b) $\lim_{x\to\infty} \mathbb{P}(X_1 > x | \max(X_1, X_2) > x) = \frac{1}{1+c-\hat{\lambda}}$

Proof. Assertion (a) follows from

$$\begin{aligned}\mathbb{P}(\max(X_1, X_2) > x) &= \mathbb{P}(X_1 > x) + \mathbb{P}(X_2 > x) - \mathbb{P}(X_1 > x, X_2 > x) \\ &= \overline{F}_1(x) + \overline{F}_2(x) - \overline{F}_1(x)\,\mathbb{P}(X_2 > x | X_1 > x)\end{aligned}$$

and (b) is a direct consequence of (a). □

The following trivial bounds can be given:

$$\mathbb{P}(\max(X_1, X_2) > x) \le \mathbb{P}(X_1 + X_2 > x) \le \mathbb{P}(\max(X_1, X_2) > x/2),$$

leading to

$$1+c-\hat{\lambda} \le \liminf_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}_1(x)} \quad \text{and} \quad \limsup_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}_1(x/2)} \le 1+c-\hat{\lambda}. \quad (3)$$

These bounds are determined by the dependence structure through $\hat{\lambda}$. In the absence of any information on the dependence structure, one is left with the "worst case" bounds

$$\overline{F}_1(x) \ll \mathbb{P}(X_1 + X_2 > x) \ll (1 + c)\,\overline{F}_1(x/2).$$

At the same time, the bounds (3) cannot be improved without any further assumptions, since for very heavy tails with $\overline{F}_1(x/2) \sim \overline{F}_1(x)$ we have

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}_1(x)} = 1 + c - \hat{\lambda},$$

so that both bounds are attained (for any value of $\hat{\lambda}$). For such heavy tails, the dependence structure obviously only affects the tail behaviour through $\hat{\lambda}$ and the sum $X_1 + X_2$ is essentially determined by the maximum of the two random variables. But also for distributions with lighter tails than above, the bounds (3) are sharp: The upper bound is attained for comonotone dependence and arbitrary identical marginals (note that in this case $\hat{\lambda} = c = 1$), whereas the lower bound is attained for independence and subexponentiality of X_1.
For identically distributed X_1, X_2 one can use (2) with $G(x, y) = C(F(x), F(y))$ and $y = c_1 x/(c_1 + c_2)$ to get

$$\lambda \le \liminf_{x\to\infty} \frac{\mathbb{P}(c_1X_1 + c_2X_2 > x)}{\mathbb{P}(X_1 > x/(c_1 + c_2))} \quad \text{and} \quad \limsup_{x\to\infty} \frac{\mathbb{P}(c_1X_1 + c_2X_2 > x)}{\mathbb{P}(X_1 > x/(c_1 + c_2))} \le 2-\lambda. \quad (4)$$

Note that both bounds are attained when $\lambda = 1$.

2.1 A copula representation

Proposition 2.3. *Let the random variables X_1 and X_2 be dependent according to an arbitrary absolutely continuous copula function $C(a,b)$ with partial derivative $c_a(a,b) := \frac{\partial C(a,b)}{\partial a}$. Then*

$$\frac{\mathbb{P}(X_1+X_2>x)}{\overline{F}_1(x)} = 1+\int_0^x \frac{1-c_a(F_1(z),F_2(x-z))}{\overline{F}_1(x)}\, F_1(dz). \tag{5}$$

Proof. From the identity

$$\begin{aligned}\frac{\mathbb{P}(X_1+X_2>x)}{\overline{F}_1(x)} &= 1+\frac{F_1(x)-\mathbb{P}(X_1+X_2\le x)}{\overline{F}_1(x)}\\ &= 1+\int_0^x \frac{1-\mathbb{P}(X_2\le x-z|X_1=z)}{\overline{F}_1(x)}\, F_1(dz),\end{aligned}$$

relation (5) follows from the copula representation of the conditional distribution function

$$\mathbb{P}(X_2\le x_2|X_1=x_1)=\mathbb{P}(F_2(X_2)\le F_2(x_2)|F_1(X_1)=F_1(x_1))=c_a(F_1(x_1),F_2(x_2)).$$

□

Formula (5) can also be interpreted geometrically:

$$\mathbb{P}(X_1+X_2>x)=\mathbb{P}(\max(X_1,X_2)>x)+\mathbb{P}(X_1+X_2>x,\max(X_1,X_2)\le x), \tag{6}$$

where the second summand is the integral of the copula density function $c_{ab}(a,b) = \frac{\partial^2 C(a,b)}{\partial a \partial b}$ over the shaded area in Figure 1, so that one obtains

$$\begin{aligned}&\mathbb{P}(X_1+X_2>x)\\ &= 1-C(F_1(x),F_2(x))+\int_{u_1=0}^{F_1(x)}\int_{u_2=F_2(x-F_1^{-1}(u_1))}^{F_2(x)} c_{ab}(u_1,u_2)\,du_2\,du_1,\\ &= 1-\int_{u_1=0}^{F_1(x)}\int_{u_2=0}^{F_2(x-F_1^{-1}(u_1))} c_{ab}(u_1,u_2)\,du_2\,du_1,\end{aligned}$$

which is equivalent to (5).

Since the first summand in (6) is given by Lemma 2.2(a), it suffices to study the contribution of the shaded area in Figure 1 for the tail behavior of the sum. Note also that the lower bound in (3) is sharp whenever the contribution from the shaded area is asymptotically negligible compared to the probability mass in the two stripes to the right and above of it. On the other hand, the upper bound in (3) is sharp whenever the area between the two dashed lines and the lower-bounding curve of the shaded area in Figure 1 is asymptotically negligible to the contribution of the domain above that curve. The latter is in particular fulfilled for comonotonicity and $F_1(x)=F_2(x)$, since then there is only probability mass along the diagonal. Note that the latter does not imply that comonotonicity provides an upper bound for the tail of the sum among all possible dependence structures, see Section 3.1 for a counter-example.

Denote $\hat{c}=\liminf_{x\to\infty}\frac{f_2(x)}{f_1(x)}$ (which coincides with c in case the latter limit exists).

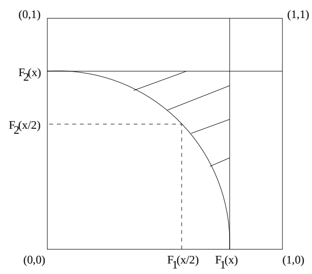

Figure 1: The domain of the copula density function

Proposition 2.4. *Let $F_1 \in \mathcal{L}$ be absolutely continuous with density f_1, F_2 be absolutely continuous with density f_2 and X_1 and X_2 be dependent according to an absolutely continuous copula function $C(a,b)$ where $c_{ab}(a,b)$ is continuous at $b=1$ a.s. (with respect to the Lebesgue measure). Then*

$$\liminf_{x\to\infty} \frac{\mathbb{P}(X_1+X_2>x)}{\overline{F}_1(x)} \geq 1+\hat{c}\int_0^\infty c_{ab}(F_1(z),1)\,F_1(dz). \tag{7}$$

Proof. From (5), Fatou's lemma and de l'Hopital, we obtain

$$\begin{aligned}\liminf_{x\to\infty} \frac{\mathbb{P}(X_1+X_2>x)}{\overline{F}_1(x)} &\geq 1+\int_0^\infty \liminf_{x\to\infty} \frac{1_{\{z\leq x\}}\left(1-c_a\left(F_1(z),F_2(x-z)\right)\right)}{\overline{F}_1(x)}\,F_1(dz)\\ &= 1+\int_0^\infty c_{ab}(F_1(z),1)\;\liminf_{x\to\infty}\frac{f_2(x-z)}{f_1(x-z)}\frac{f_1(x-z)}{f_1(x)}\,F_1(dz).\end{aligned}$$

From the definition of a long-tailed distribution, it immediately follows that $\lim_{x\to\infty}\frac{f_1(x-z)}{f_1(x)}=1$ for all $z>0$, so that we are left with (7). □

Remark 2.1. In the exchangeable case we have for the tail dependence coefficient in terms of (absolutely continuous) copulae

$$\lambda=\lim_{u\to1}\frac{1-2u+C(u,u)}{1-u}=2-2\lim_{u\to1}c_b(u,u)=2-2\lim_{u\to1}\int_0^1 1_{\{z\leq u\}}c_{ab}(z,u)\,dz.$$

If for the integral interchanging integration and limit is justified. then we get

$$\lim_{x\to\infty}\frac{\mathbb{P}(X_1+X_2>x)}{\mathbb{P}(X_1>x)}\geq 1+\int_0^1 c_{ab}(z,1)\,dz\geq 2\int_0^1 c_{ab}(z,1)\,dz=2-\lambda,$$

and the bound (7) is indeed an improvement over the trivial bound (3) whenever $\lambda>0$ (in Section 3.3 we will see cases where interchanging integration and limit is not justified).

If interchanging limits in (5) is justified, then the r.h.s. of (7) gives the correct asymptotic behavior of the limit. This is in particular the case for independence, where $c_{ab}(a,b)=1$. The latter gives rise to a sufficient criterion for interchanging limits in (5):

Proposition 2.5. *Let both $F_1 \in \mathcal{S}$ and F_2 be absolutely continuous and X_1 and X_2 be dependent according to an absolutely continuous copula function $C(a,b)$ with $c_{ab}(a,b) < M$ for all $(a,b) \in [0,1] \times [b_0, 1]$, $b_0 < 1$. Assume that c exists. Then*

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} = 1 + c \int_0^\infty c_{ab}(F_1(z), 1)\, F_2(dz). \tag{8}$$

Proof. Consider representation (5). For the independent case we obviously have

$$\lim_{x\to\infty} \int_0^\infty \frac{1_{\{z\le x\}} \overline{F}_2(x-z)}{\overline{F}_1(x)} F_1(dz) = c. \tag{9}$$

The numerator in (5) can be replaced by

$$1 - c_a(F_1(z), F_2(x-z)) = \int_{x-z}^\infty c_{ab}(F_1(z), F_2(y))\, F_2(dy) \le M\, \overline{F}_2(x-z),$$

where the last inequality holds for $F_2(x-z) > b_0$. If $F_2(x-z) \le b_0$ we have:

$$1 - c_a(F_1(z), F_2(x-z)) \le 1 \le \frac{\overline{F}_2(x-z)}{1-b_0}.$$

Hence (9) for the independent case serves as an upper bound for which interchanging limits is justified. The assertion then follows by virtue of Pratt's Lemma (cf. [25]).
□

Lemma 2.6. *Let X_1 and X_2 be dependent random variables with absolutely continuous marginals F_1 and F_2 according to an absolutely continuous copula $C(a,b)$ such that there exist constants $x_0 < 1$ and $M < \infty$ with $c_{ab}(a,b) < M$ for all $(a,b) \in [x_0, 1]^2$, then $\hat{\lambda} = 0$.*

Proof. Denote with X_1^* and X_2^* independent random variables with the same marginal distributions as X_1 and X_2. For $\min(F_1(x), F_2(x)) > x_0$ we have:

$$\begin{aligned}
\hat{\lambda} &= \lim_{x\to\infty} \mathbb{P}(X_2 > x | X_1 > x) = \lim_{x\to\infty} \frac{\mathbb{P}(X_2 > x, X_1 > x)}{\mathbb{P}(X_1 > x)} \\
&= \lim_{x\to\infty} \frac{1}{\mathbb{P}(X_1 > x)} \int_x^\infty \int_x^\infty c_{ab}(F_1(u_1), F_2(u_2))\, F_1(du_1) F_2(du_2) \\
&\le M \lim_{x\to\infty} \frac{1}{\mathbb{P}(X_1 > x)} \int_x^\infty \int_x^\infty 1\, F_1(du_1) F_2(du_2) \\
&= M \lim_{x\to\infty} \frac{\mathbb{P}(X_2^* > x, X_1^* > x)}{\mathbb{P}(X_1^* > x)} = M \lim_{x\to\infty} \mathbb{P}(X_2^* > x) = 0.
\end{aligned}$$

□

If the copula density function is bounded in some box anchored in $[1,1]$, then we get the following strengthening of Proposition 2.5:

Lemma 2.7. *Let X_1 and X_2 be dependent random variables with absolutely continuous marginals $F_1 \in \mathcal{S}$ and F_2 such that c exists and absolutely continuous copula $C(a,b)$ such that there exist constants $x_0 < 1$ and $M < \infty$ with $c_{ab}(a,b) < M$ for all $(a,b) \in [x_0, 1]^2$, then*

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\mathbb{P}(X_1 > x)} = 1 + c.$$

Proof. Denote with $a(x)$ a function with $\lim_{x\to\infty} a(x) = \infty$ and

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 > x - a(x))}{\mathbb{P}(X_1 > x)} = 1.$$

From (3) we see that we only need to show that

$$\limsup_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\mathbb{P}(X_1 > x)} \leq 1 + c.$$

We have

$$\begin{aligned} \mathbb{P}(X_1 + X_2 > x) \;&\leq\; \mathbb{P}(X_1 > x - a(x) \cup X_2 > x - a(x)) \\ &\quad + \mathbb{P}(X_1 + X_2 > x, \max(X_1, X_2) \leq x - a(x)) \end{aligned} \tag{10}$$

Let X_1^* and X_2^* be independent random variables with the same marginal distributions as X_1 and X_2. For $\inf_{y>x} F_i(a(y)) > x_0$ $(i = 1, 2)$, we have

$$\begin{aligned} &\mathbb{P}(X_1 + X_2 > x, \max(X_1, X_2) \leq x - a(x)) \\ &= \mathbb{P}(X_1 + X_2 > x, \max(X_1, X_2) \leq x - a(x), \min(X_1, X_2) > a(x)) \\ &= \int_{\{X_1+X_2>x, \max(X_1,X_2)\leq x-a(x), \min(X_1,X_2)>a(x)\}} c_{ab}(F_1(u_1), F_2(u_2))\, F_1(du_1) F_2(du_2) \\ &\leq M \int_{\{X_1+X_2>x, \max(X_1,X_2)\leq x-a(x), \min(X_1,X_2)>a(x)\}} 1\, F_1(du_1) F_2(du_2) \\ &= M\, \mathbb{P}(X_1^* + X_2^* > x, \max(X_1^*, X_2^*) \leq x - a(x), \min(X_1^*, X_2^*) > a(x)) \\ &\leq M\, \mathbb{P}(X_1^* + X_2^* > x, \max(X_1^*, X_2^*) \leq x). \end{aligned}$$

Together with $F_1 \in \mathcal{S}$ we get:

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x, \max(X_1, X_2) \leq x - a(x))}{\mathbb{P}(X_1 > x)} = 0.$$

For the first summand in (10) we have

$$\begin{aligned} &\lim_{x\to\infty} \frac{\mathbb{P}(X_1 > x - a(x) \cup X_2 > x - a(x))}{\mathbb{P}(X_1 > x)} \\ &\leq \lim_{x\to\infty} \frac{\mathbb{P}(X_1 > x - a(x)) + \mathbb{P}(X_2 > x - a(x))}{\mathbb{P}(X_1 > x - a(x))} \frac{\mathbb{P}(X_1 > x - a(x))}{\mathbb{P}(X_1 > x)} = 1 + c, \end{aligned}$$

from which the assertion follows. $\square$

2.2 A remark on the role of the mean excess function

In this section we focus on exchangeable copulas and identically distributed random variables X_1 and X_2 with common distribution function F. As already mentioned, the tail dependence coefficient λ is a rather rough measure of the dependence in the tail. The following result uses a somewhat finer criterion of conditional exceedances and can be applied for any type of dependence structure between X_1 and X_2. Recall the definition of the mean excess function $e(x)$ of F given by

$$e(x) = \mathbb{E}(X - x \,|\, X > x) = \int_x^\infty \frac{\overline{F}(u)}{\overline{F}(x)}\, du$$

and note that $e(x) \to \infty$ for $x \to \infty$ for every $F \in \mathcal{S}$ (see e.g. [11]).

Proposition 2.8. *If the mean-excess function $e(x)$ is self-neglecting, i.e.*

$$\lim_{x\to\infty} \frac{e(x + a\, e(x))}{e(x)} = 1 \quad \forall\, a \geq 0, \tag{11}$$

and if

$$\inf_{a>0} \liminf_{x\to\infty} \mathbb{P}(X_2 > a\, e(x) \,|\, X_1 > x) > 0, \tag{12}$$

then

$$\liminf_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} = \infty.$$

Proof. The self-neglecting property (11) implies

$$\lim_{x\to\infty} \frac{\overline{F}(x + a\, e(x))}{\overline{F}(x)} = \mathrm{e}^{-a}$$

(see e.g. [3, p.258]) and we have

$$\begin{aligned} \frac{\overline{F}(x)}{\overline{F}(x - a\, e(x))} &\sim \frac{\overline{F}(x + a\, e(x))}{\overline{F}(x + a\, e(x) - a\, e(x + a\, e(x)))} \\ &\sim \frac{\overline{F}(x + a\, e(x))}{\overline{F}(x)}, \end{aligned}$$

also due to (11). Hence, together with (12),

$$\begin{aligned} \mathbb{P}(X_1 + X_2 > x) &\geq \mathbb{P}(X_1 > x - a\, e(x), X_2 > a\, e(x)) \\ &= \mathbb{P}(X_1 > x - a\, e(x))\, \mathbb{P}(X_2 > a\, e(x) \,|\, X_1 > x - a\, e(x)) \\ &\sim \mathbb{P}(X_1 > x - a\, e(x))\, \mathbb{P}(X_2 > a\, e(x) \,|\, X_1 > x) \\ &\geq \varepsilon\, \mathbb{P}(X_1 > x - a\, e(x)) \\ &\sim \varepsilon\, \mathbb{P}(X_1 > x)\, \mathrm{e}^{a} \end{aligned}$$

for some $\varepsilon > 0$ and any $a > 0$. Hence

$$\liminf_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} \geq \varepsilon\, \mathrm{e}^{a}$$

and the latter is unbounded for $a \to \infty$. □

Remark 2.2. A sufficient condition for (12) to hold is

$$\liminf_{x\to\infty} \mathbb{P}(X_2 > e^*(x) \,|\, X_1 > x) > 0$$

for any $e^*(x)$ with $e^*(x)/e(x) \to \infty$. Condition (11) is satisfied for all subexponential distributions which lie in the domain of attraction of the Gumbel distribution (cf. [6], [11]) (like the lognormal and the Weibull distribution).

In the following we will show that for all $F \in \mathcal{S} \cap$ MDA(Gumbel) there exists a copula such that condition (12) is satisfied. Let $\delta(x) = C(x, x)$ be the diagonal section of a copula C. One can show that

$$C_\delta(a, b) = \min(a, b, 1/2(\delta(a) + \delta(b)) \tag{13}$$

defines another copula with diagonal section $\delta(x)$ (cf. [24]).

Proposition 2.9. *Let X_1 and X_2 be random variables with copula $C_\delta(a, b) = \min(a, b, 1/2(\delta(a) + \delta(b)))$ and marginal distribution $F \in \mathcal{S} \cap$ MDA(Gumbel). If for all $a > 0$ there exists an $x_0 > 0$ such that for all $x > x_0$*

$$\min(F(x), F(ae(x)), 1/2(\delta(F(x)) + \delta(F(ae(x))))) = F(ae(x)), \tag{14}$$

then

$$\lim_{x\to\infty} \mathbb{P}(X_2 > ae(x)|X_1 > x) = 1, \quad \forall a > 0,$$

and hence (12) *is fulfilled.*

Proof. For $x > x_0$ we have:

$$\begin{aligned}
\lim_{x\to\infty} \mathbb{P}(X_2 > ae(x)|X_1 > x) &= \lim_{x\to\infty} \frac{\mathbb{P}(X_2 > ae(x), X_1 > x)}{\mathbb{P}(X_1 > x)} \\
&= \lim_{x\to\infty} \frac{1 - F(x) - F(ae(x)) + C_\delta(F(x), F(ae(x)))}{1 - F(x)} \\
&= \lim_{x\to\infty} \frac{1 - F(x)}{1 - F(x)} = 1.
\end{aligned}$$

□

Hence it remains to find distributions and diagonal sections which fulfill the conditions of Proposition 2.9.

Let $F(x)$ be a distribution with left endpoint $x_L \geq 0$, right endpoint ∞ and inverse distribution function $F^{[-1]}$. Let $g(x)$ be a monotone function with $g(x_L) = 1$ and $g(\infty) = \infty$. Then we can define the functions:

$$\begin{aligned}
h(x) &= \frac{1}{g(F^{[-1]}(x))}, \\
h_1(x) &= 2x + \int_x^1 h(t)\,dt - 1.
\end{aligned}$$

It is easy to see that h is decreasing and $h(0) = 1$, $h(1) = 0$. We can now define the diagonal section:

$$\delta(x) = \max(0, h_1(x)) \tag{15}$$

To see that this indeed defines the diagonal section of a copula we have to prove four properties (cf. [24]):

- $\delta(1) = 1$, which is obviously true.
- $\delta(x) \in [0,1]$: this holds because $h_1'(x) = 2 - h(x) \geq 1$ so that $\delta(x)$ is increasing, $\delta(0) \geq 0$ and $\delta(1) = 1$.
- $0 \leq \delta(x_2) - \delta(x_1) \leq 2(x_2 - x_1)$ for any $x_2 > x_1$. The first inequality is true because δ is increasing and the second follows with:
$$\delta(x_2) - \delta(x_1) \leq h_1(x_2) - h_1(x_1) \leq h_1'(\eta)(x_2 - x_1)$$
for some $\eta \in [0,1]$ and $h_1'(x) \leq 2 \,\forall\, x \in [0,1]$.
- $\delta(x) \leq x$. It suffices to show $h_1(x) - x \leq 0$; since $h_1(1) - 1 = 0$ and $\frac{d}{dx}(h_1(x) - x) = 1 - h(x) \geq 0$, this holds.

For the tail dependence coefficient of this copula we get:

$$\lambda = \lim_{x \to 1} \frac{1 - 2x + \delta(x)}{1 - x} = \lim_{x \to 1} \frac{-2 + 2 - h(1)}{-1} = 0.$$

We want to show that for suitable g condition (14) holds. For sufficiently large x this amounts to:

$$\begin{aligned} F(ae(x)) &\leq \frac{1}{2}\big(\delta(F(x)) + \delta(F(ae(x)))\big) \\ &\leq \frac{1}{2}\left(2F(x) + 2F(ae(x)) + \int_{F(x)}^{1} h(y)\,dy + \int_{F(ae(x))}^{1} h(y)\,dy - 2\right) \end{aligned}$$

or equivalently

$$\overline{F}(x) \leq \frac{1}{2}\left(\int_{F(x)}^{1} h(y)\,dy + \int_{F(ae(x))}^{1} h(y)\,dy\right).$$

It suffices to show

$$\lim_{x \to \infty} \frac{\int_{F(ae(x))}^{1} h(y)\,dy}{\overline{F}(x)} = \infty.$$

Consider

$$\begin{aligned} \lim_{x \to \infty} \frac{\int_{F(ae(x))}^{1} h(y)\,dy}{\overline{F}(x)} &\geq \lim_{x \to \infty} \frac{\int_{1-\overline{F}(ae(x))}^{1-\overline{F}(ae(x))/2} h(y)\,dy}{\overline{F}(x)} \\ &\geq \lim_{x \to \infty} h(1 - \overline{F}(ae(x))/2)\,\frac{\overline{F}(ae(x))}{2\overline{F}(x)} \\ &= \lim_{x \to \infty} \frac{1}{g\left(F^{[-1]}\left(1 - \overline{F}(ae(x))/2\right)\right)}\,\frac{\overline{F}(ae(x))}{2\overline{F}(x)}. \end{aligned} \tag{16}$$

Since $\lim_{x\to\infty} \overline{F}(ae(x))/(2\overline{F}(x)) = \infty$ we can choose g in such a way that (16) tends to ∞. So we have proven:

Theorem 2.10. *Let X_1 and X_2 have marginal distribution function $F \in \mathcal{S} \cap$ MDA(Gumbel). Then there exists a copula for X_1 and X_2 with $\lambda = 0$ such that*

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\mathbb{P}(X_1 > x)} = \infty. \tag{17}$$

3 Some specific cases

3.1 Regularly varying marginal distribution

3.1.1 An upper bound

Proposition 3.1. *Let $\overline{F}_1 \in \mathcal{R}_{-\alpha}$ with $\alpha > 0$. Then*

$$\limsup_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}_1(x)} \le \begin{cases} \left(\hat{\lambda}^{\frac{1}{\alpha+1}} + (1 + c - 2\hat{\lambda})^{\frac{1}{\alpha+1}}\right)^{\alpha+1}, & 0 \le \hat{\lambda} \le \frac{1+c}{3} \\ 2^\alpha(1 + c - \hat{\lambda}), & \frac{1+c}{3} < \hat{\lambda} \le 1. \end{cases} \tag{18}$$

Proof. For any $0 < \delta < 1/2$ we have

$$\begin{aligned}\mathbb{P}(X_1+X_2 > x) &\le \mathbb{P}(\{X_1 > (1-\delta)x\}\cup\{X_2 > (1-\delta)x\}\cup(\{X_1 > \delta x\}\cap\{X_2 > \delta x\})) \\ &\le \overline{F}_1((1-\delta)x)+\overline{F}_2((1-\delta)x)+\mathbb{P}(X_1 > \delta x, X_2 > \delta x)-2\mathbb{P}(X_1 > (1-\delta)x), X_2 > (1-\delta)x)\end{aligned}$$

so that

$$\begin{aligned}&\limsup_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}_1(x)} \\ &\le \limsup_{x\to\infty} \left((1 - 2\hat{\lambda}) \frac{\overline{F}_1((1-\delta)x)}{\overline{F}_1(x)} + \frac{\overline{F}_2((1-\delta)x)}{\overline{F}_1(x)} + \frac{\overline{F}_1(\delta x)}{\overline{F}_1(x)} \mathbb{P}(X_2 > \delta x \mid X_1 > \delta x)\right) \\ &= \frac{1 + c - 2\hat{\lambda}}{(1-\delta)^\alpha} + \frac{\hat{\lambda}}{\delta^\alpha}.\end{aligned}$$

Within the defined range of δ, this upper bound is minimized for

$$\delta^* = \begin{cases} \frac{1}{1+\left(\frac{1+c}{\hat{\lambda}}-2\right)^{\frac{1}{\alpha+1}}}, & 0 \le \hat{\lambda} \le \frac{1+c}{3} \\ \frac{1}{2}, & \frac{1+c}{3} < \hat{\lambda} \le 1, \end{cases}$$

which yields (18). □

Note that this upper bound is sharp for both independence and comonotone dependence when X_1 and X_2 are identically distributed. In particular, together with assertion (a) of Lemma 2.2, we obtain (see also [9])

Corollary 3.2. *If $F_1 \in \mathcal{R}$ and $\hat{\lambda} = 0$, then $\mathbb{P}(X_1 + X_2 > x) \sim (1 + c)\overline{F}_1(x)$.*

Thus for regularly varying tails of the marginals, tail independence suffices to guarantee that the tail of the dependent sum behaves asymptotically as if X_1 and X_2 were independent. From the proof of Proposition 3.1, it becomes clear that this also holds true for any $F \in \mathcal{S}$ with heavier tail than regularly varying. On the other hand, for light-tailed distributions tail independence clearly does not imply such an insensitivity (for instance, consider a bivariate normal distribution, where the dependence is described by the (tail independent) Gaussian copula; in this case, the variance of the sum is a function of the correlation coefficient ρ and the value of ρ does affect the asymptotic behavior of the sum). This gives rise to the question of "how heavy" the marginal tails have to be in order to dominate the "dependence effect" in the tail of the sum, given $\lambda = 0$. Theorem 2.10 of Section 2.2 clarifies this issue by showing that $F \in \mathcal{S} \cap \mathrm{MDA}(\mathrm{Gumbel})$ is not a sufficient condition for that behavior.

For fixed marginals, it was already pointed out by Denuit *et al.* [10] that, unlike the case of stop-loss premiums, the comonotone dependence structure does not always provide an extremal case for the asymptotic behavior of the sum of the tail. The following simple example demonstrates this fact:

Example 3.1. Let $\overline{F}_1 \in \mathcal{R}_{-\alpha}$ with $\alpha > 0$ and $F_1(x) = F_2(x)$. Then for independence between X_1 and X_2, by standard subexponential theory, $\lim_{x\to\infty} \mathbb{P}(X_1 + X_2 > x)/\overline{F}_1(x) = 2$. On the other hand, for comonotone X_1 and X_2 (which due to identical marginals is equivalent to $X_1 = X_2$ a.s.), we have $\lim_{x\to\infty} \mathbb{P}(X_1 + X_2 > x)/\overline{F}_1(x) = 2^\alpha$. Thus, for $\alpha < 1$ the comonotone case does not provide an upper bound.

Intuitively, if the marginal distribution tail is heavy enough, then the two random sources for a possibility of a large sum caused by one of the summands outweighs the effect of summing two large components from one random source.

3.1.2 Multivariate regularly varying tails

A well-known specific way to couple regularly varying marginals is by multivariate regular variation. In our bivariate setting it can be defined as follows: The vector $\mathbf{X} = (X_1, X_2)$ is regularly varying with index $-\alpha < 0$, if there exists a probability measure S on $\mathbb{S}^1$ (the unit sphere in $\mathbb{R}^2$ with respect to the Euclidean norm $|\cdot|$) such that for all $t > 0$

$$\frac{\mathbb{P}(|\mathbf{X}| > tu, \mathbf{X}/|\mathbf{X}| \in \cdot)}{\mathbb{P}(|\mathbf{X}| > u)} \xrightarrow{v} t^{-\alpha}\, S(\cdot) \quad \text{as } u \to \infty,$$

where $\xrightarrow{v}$ stands for vague convergence in $\mathbb{S}^1$ (see for instance Resnick [28]). S is often referred to as the *spectral measure* of $\mathbf{X}$.

With positive random variables X_1, X_2, an equivalent formulation is that there exists a probability measure $S(\cdot)$ on $\mathbb{S}^1_+$ (the restriction of $\mathbb{S}^1$ to the first quadrant) and a function $b(x) \to \infty$ such that

$$b^{-1}(x)\,\mathbb{P}\left(\left(\frac{|\mathbf{X}|}{x}, \frac{\mathbf{X}}{|\mathbf{X}|}\right) \in \cdot\right) \xrightarrow{v} a\, \nu_\alpha \times S \tag{19}$$

in the space of positive Radon measures on $\big((0,\infty] \times \mathbb{S}_+^1\big)$, where $a > 0$ and $\nu_\alpha(t,\infty] = t^{-\alpha}$, $(t > 0, \alpha > 0)$ (cf. Resnick [27]).
The above implies in particular that on every ray from (0,0) into the positive quadrant, we have a regularly varying tail with index $-\alpha$. Moreover, the tail of $c_1X_1+c_2X_2$ for constants $c_1 \geq c_2 > 0$ is also regularly varying with the same index (in fact the relationship between regular variation of $\mathbf{X} = (X_1, X_2)$ and one-dimensional regular variation of linear combinations of its components is much deeper, see Basrak *et al.* [5]).
So for this specific dependence structure among regularly varying marginals, the asymptotic behavior of the sum can be given explicitly. To that end, considering in (19) the events $|\mathbf{X}|/x > t$ for $t = \frac{1}{c_1\cos\varphi + c_2\sin\varphi}$ and $t = \frac{1}{c_1\cos\varphi}$, with $\varphi \in [0,\pi/2]$ denoting the angle corresponding to $\mathbf{X}/|\mathbf{X}|$, we obtain

$$b^{-1}(x)\,\mathbb{P}(c_1X_1 + c_2X_2 > x) \to a \int_0^{\pi/2} (c_1\cos\varphi + c_2\sin\varphi)^\alpha\, S(d\varphi)$$

for some constant $a > 0$ (where in an obvious way we have identified $\mathbb{S}_+^1$ with $[0,\pi/2]$) and

$$b^{-1}(x)\,\mathbb{P}(c_1X_1 > x) \to a \int_0^{\pi/2} c_1^\alpha \cos^\alpha\varphi\, S(d\varphi),$$

so that

$$\mathbb{P}(c_1X_1 + c_2X_2 > x) \sim \overline{F}_1(x/c_1)\,\frac{\int_0^{\pi/2}(c_1\cos\varphi + c_2\sin\varphi)^\alpha\, S(d\varphi)}{\int_0^{\pi/2} c_1^\alpha\cos^\alpha\varphi\, S(d\varphi)}.$$

For exchangeable X_1 and X_2 and $c_1 = c_2 = 1$, we have $S(d\varphi) = S(d(\pi/2 - \varphi))$ and hence

$$\mathbb{P}(X_1 + X_2 > x) \sim 2\,\overline{F}(x)\,\frac{\int_0^{\pi/2}(\cos\varphi + \sin\varphi)^\alpha\, S(d\varphi)}{\int_0^{\pi/2}(\cos^\alpha\varphi + \sin^\alpha\varphi)\, S(d\varphi)}.$$

In particular, the quotient on the right hand side is larger than 1 for $\alpha > 1$, smaller than 1 for $\alpha < 1$ and equal to 1 for $\alpha = 1$ (irrespective of the value of λ). The comonotone case is retrieved when S is concentrated at $\varphi = \pi/4$ which indeed gives $\mathbb{P}(X_1 + X_2 > x) \sim 2^\alpha\overline{F}(x)$. For asymptotic independence, S is concentrated on the two axes, so that $\mathbb{P}(X_1 + X_2 > x) \sim 2\,\overline{F}(x)$. A natural extremal dependence measure in this setting is

$$\rho := 1 - \frac{1}{(\pi/4)^2}\int_0^{\pi/2}\left(\varphi - \frac{\pi}{4}\right)^2 S(d\varphi),$$

see Resnick [27]. Finally, the tail dependence coefficient λ as defined in Section 1 can in this case be obtained by considering the event $|\mathbf{X}|/x > t$ for $t = \frac{1}{\min\{\cos\varphi, \sin\varphi\}}$ in (19), yielding

$$\lambda = \lim_{x\to\infty}\frac{\mathbb{P}(X_1 > x, X_2 > x)}{\mathbb{P}(X_1 > x)} = \frac{2\int_0^{\pi/4}\sin^\alpha\varphi\, S(d\varphi)}{\int_0^{\pi/4}(\sin^\alpha\varphi + \cos^\alpha\varphi)\, S(d\varphi)}.$$

Under further restrictions on the shape of $\mathbf{X}$, the spectral measure S may be explicitly computable (for instance, in case of elliptical distributions with regularly varying tail, see Hult & Lindskog [16]; however, the latter class is not relevant for the present purpose due to our restriction to positive random variables).

Remark 3.1. While in this specific setting, clearly λ is a rougher measure for dependence in the tail than ρ, both measures identify the same distributions as asymptotically independent, i.e. $\rho = \lambda = 0$. In the latter case there are refinements for the study of multivariate regularly varying distributions available, cf. Resnick [26].

3.2 Lognormal marginal distribution

Asmussen & Rojas-Nandayapa [4] considered $X_1 + \cdots + X_n$ where $X_1, \ldots, X_n$ are lognormal with a multivariate Gaussian copula. That is, $X_i = \mathrm{e}^{Y_i}$ where $Y_1, \ldots, Y_n$ are jointly multivariate Gaussian(μ, Σ) for some mean vector μ and some covariance matrix Σ; exchangeability is not required. Their results state that the tail of the sum is asymptotically the same as for the independent case $\Sigma = (\sigma_i^2)_{\mathrm{diag}}$. Note that for lognormal distributed X_1 a multiplication with c_1 is equivalent to changing μ_1 to $\mu_1 + \log c_1$. When specialized to the present setting with $c_1 = c_2 = 1$, this means:

Proposition 3.3. *Let X_1, X_2 be bivariate normal with the same mean μ, the same variance σ^2 and covariance $\rho \in [-1, 1)$. Then*

$$\mathbb{P}(X_1 + X_2 > x) \sim 2\,\mathbb{P}(X_1 > x) \;\sim\; \frac{\sqrt{2/\pi}}{\sigma \log x} \exp\left\{-(\log x - \mu)^2/2\sigma^2\right\}.$$

A short heuristical argument (different from the rigorous, more technical proof of [4]) supporting this result goes as follows. We take $\mu = 0$, $\sigma^2 = 1$, $\rho > 0$ for simplicity. Then we can write

$$Y_1 \;=\; U + V_1\,, \quad Y_2 \;=\; U + V_2\,,$$

where U, V_1, V_2 are independent univariate Gaussian with mean zero and variances a^2, b^2, b^2, respectively, where $a^2 + b^2 = 1$, $a^2 = \rho$. Given $U = u$, X_1 and X_2 are independent lognormals with log-variance b^2, so by subexponential limit theory

$$\begin{aligned}\mathbb{P}\big(X_1 + X_2 > x \,\big|\, U = u\big) \;&=\; \mathbb{P}(\mathrm{e}^{V_1} + \mathrm{e}^{V_2} > x\mathrm{e}^{-u}) \\ &\sim\; \frac{\sqrt{2/\pi}}{b(\log x - u)} \exp\left\{-(\log x - u)^2/2b^2\right\}.\end{aligned}$$

We make the guess

$$\mathbb{P}(X_1 + X_2 > x) \;\approx\; \max_u \frac{1}{a\sqrt{2\pi}} \mathrm{e}^{-u^2/2a^2} \mathbb{P}\big(X_1 + X_2 > x \,\big|\, U = u\big) \tag{20}$$

and ignore everything not in the exponent and constants. Then we have to find the u minimizing

$$\frac{u^2}{2a^2} - \frac{u \log x}{b^2} + \frac{u^2}{2b^2}$$

which (using $a^2+b^2=1$) is easily seen to be $u = a^2 \log x$. Substituting back in (20), we get

$$\begin{aligned} \mathbb{P}(X_1+X_2>x) &\approx \exp\left\{-a^4\log^2 x/2a^2 \;-\; (1-a^2)^2\log^2 x/2b^2\right\} \\ &= \exp\left\{-\log^2 x/2\right\} \end{aligned} \tag{21}$$

in agreement with Proposition 3.3 (we have used $\approx$ to indicate aymptotics at a rough level, that is, rougher than $\sim$ or even logarithmic asymptotics as used in large deviations theory).
Note that the argument contains some information on how X_1+X_2 exceeds x: U must be approximately $u = a^2\log x = \rho \log x$ and either V_1 or V_2 but not both large. Translated back to X_1, X_2, this means that one is larger than x and the other of order $\mathrm{e}^u = x^\rho$.

The above proposition provides an example of lognormal marginals and tail independence (through the Gaussian copula), in which the tail asymptotics of the sum are insensitive to increasing dependence. From Theorem 2.10 we know that lognormality of the marginals is in general not sufficient to guarantee this insensitivity for arbitrary dependence structures. In the following we provide an alternative construction to the one in the proof of Theorem 2.10 of a tail-independent exchangeable random vector (X_1, X_2) with lognormal marginals and

$$\liminf_{x\to\infty} \frac{\mathbb{P}(X_1+X_2>x)}{\overline{F}(x)} = \infty,$$

which is tailored to the lognormal setting:

Lemma 3.4. *There exists a tail-independent exchangeable random vector* (Y_1, Y_2) *with standard normal marginals and* $\left|Y_1 - Y_2\right| = d$ *whenever* $Y_1+Y_2 > y_0$ *for a given* $d>0$ *and* $y_0>0$.

Proof. For $y_1+y_2<0$ simply define the joint distribution as the restriction of the bivariate standard normal distribution with independent marginals to $\{y_1+y_2<0\}$. For $y_1+y_2>0$, let $f(y)$ denote the density of Y_1+Y_2. The problem is to determine f such that

$$\int_0^\infty f(y)\,dy \;=\; \frac{1}{2} \tag{22}$$

and

$$\varphi(y) \;=\; \frac{1}{2}f(y-d) \;+\; \frac{1}{2}f(y+d)\,, \tag{23}$$

where $\varphi(y)$ denotes the density of the standard normal distribution. Let us rewrite

$$f(y) \;=\; \varphi(y)\mathrm{e}^{-dy}g_1(y) \tag{24}$$

for some function $g_1(y)$. Using $\varphi(y+d) \;=\; \varphi(y)\mathrm{e}^{-d^2/2-dy}$, (23) then becomes

$$1 \;=\; \frac{1}{2}\mathrm{e}^{-d^2/2}\left[\mathrm{e}^{dy}\mathrm{e}^{-d(y-d)}g_1(y-d) + \mathrm{e}^{-dy}\mathrm{e}^{-d(y+d)}g_1(y+d)\right],$$

that is

$$2\mathrm{e}^{-d^2/2} \;=\; g_1(y-d) \;+\mathrm{e}^{-2d^2}\,\mathrm{e}^{-2dy} g_1(y+d)\,.$$

Trying the solution $g_1(y) = \sum_{n=0}^{\infty} r_n \mathrm{e}^{-2ndy}$, we obtain

$$\begin{aligned} 2\mathrm{e}^{-d^2/2} &= \sum_{n=0}^{\infty} r_n \mathrm{e}^{-2nd(y-d)} + \mathrm{e}^{-2d^2}\,\mathrm{e}^{-2dy} \sum_{n=0}^{\infty} r_n \mathrm{e}^{-2nd(y+d)} \\ &= \sum_{n=0}^{\infty} r_n \mathrm{e}^{2nd^2}\mathrm{e}^{-2ndy} + \mathrm{e}^{-2d^2} \sum_{n=1}^{\infty} r_{n-1} \mathrm{e}^{-2(n-1)d^2}\mathrm{e}^{-2ndy}. \end{aligned}$$

Identifying coefficients yields $r_0 = 2\mathrm{e}^{-d^2/2}$ and

$$r_n \;=\; -\mathrm{e}^{-4nd^2} r_{n-1}, \quad n \ge 1,$$

leading to

$$r_n = (-1)^n 2\mathrm{e}^{-2d^2(n+1)n-d^2/2}, \quad n \ge 0.$$

Hence

$$g_1(y) = 2\mathrm{e}^{-d^2/2} \sum_{n=0}^{\infty} (-1)^n\,\mathrm{e}^{-2d^2(n+1)n}\,\mathrm{e}^{-2ndy},$$

which is a convergent series for every $y \ge 0$, since it is alternating with coefficients decreasing to zero monotonically. Moreover, $\lim_{y\to\infty} g_1(y) = 2\mathrm{e}^{-d^2/2}$, so that $\int_0^\infty \varphi(y)\mathrm{e}^{-dy} g_1(y)\,dy < \infty$ and the integrand can be normalized in such a way that (22) holds. Finally, from (24) we see that $f(y+d) = o\left(f(y-d)\right)$ as $y \to \infty$ and thus $\lambda = \lim_{y\to\infty} \mathbb{P}(Y_2 > y | Y_1 > y) = 0$. $\square$

Since the copula of a bivariate distribution stays invariant under strictly increasing transformations of the marginals and the tail dependence coefficient is a function of the copula only, Lemma 3.4 can be carried over to the random vector $(X_1, X_2) = (\mathrm{e}^{Y_1}, \mathrm{e}^{Y_2})$ with lognormal marginals. In particular, for large x we then either have $X_1 = X_2\,\mathrm{e}^d$ or $X_2 = X_1\,\mathrm{e}^d$. Hence

$$\begin{aligned} \mathbb{P}(X_1 + X_2 > x) &\sim \mathbb{P}(X_1 + X_2 > x, X_1 = X_2\,\mathrm{e}^d) + \mathbb{P}(X_1 + X_2 > x, X_2 = X_1\,\mathrm{e}^d) \\ &= 2\,\mathbb{P}\left(X_1 > x/(1+e^d)\right). \end{aligned}$$

As for a lognormal random variable $X_1 = e^{Y_1}$ with $Y_1 \sim N(0,1)$, the tail is asymptotically

$$\overline{F}(x) \sim \frac{1}{\sqrt{2\pi}\log x}\,\mathrm{e}^{-\log^2 x},$$

which establishes the alternative counter-example.

3.3 Archimedean copulae

Archimedean copulae are of the form

$$C(a,b) = \phi^{[-1]}(\phi(a) + \phi(b)), \quad 0 \le a, b \le 1, \tag{25}$$

where the generator $\phi(t)$ is a continuous, convex and strictly decreasing function from [0,1] to $[0,\infty]$ such that $\phi(1) = 0$ and $\phi^{[-1]}$ denotes the pseudo-inverse of ϕ defined by

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t), & 0 \le t \le \phi(0), \\ 0, & \phi(0) \le t \le \infty. \end{cases}$$

If $\phi(0) = \infty$, then ϕ is called a *strict* generator.

Lemma 3.5. *Let $F_1 \in \mathcal{S}$ with X_1 and X_2 being dependent according to an Archimedean copula with generator ϕ being twice differentiable with $\phi'(1) = m < 0$ and $\phi''(x) \le M$ and $|\phi'(x)| \le M$ on $[x_0, 1]$ for some $M < \infty$ and $0 < x_0 < 1$, then*

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\mathbb{P}(X_1 > x)} = 1 + c.$$

Proof. Choose $x_1 < 1$ such that $C(x,x) \ge x_0$ for all $x \ge x_1$. For $(a,b) \in [\max(x_0,x_1),1]^2$ we have

$$c_{ab}(a,b) = -\frac{\phi''(C(a,b))\phi'(a)\phi'(b)}{(\phi'(C(a,b)))^3} \le \frac{M^3}{m^3} < \infty.$$

The assertion now follows from Lemma 2.7 □

Proposition 3.6. *Let $C(a,b)$ be an Archimedean copula with twice differentiable generator ϕ and $\lambda > 0$, then $\lim_{b\to1} c_{ab}(a,b) = 0$ and hence the lower bound in* (7) *is* 1.

Proof. Definition (25) implies

$$c_b(a,b) = \frac{\phi'(b)}{\phi'(\phi^{[-1]}(\phi(a) + \phi(b)))}.$$

The tail dependence coefficient λ is given by

$$\lambda = 2 - 2\lim_{u\to1} \frac{\phi'(u)}{\phi'(\phi^{-1}(2\phi(u)))}. \tag{26}$$

Now $\lambda = 0$ unless $\phi'(1) = 0$, in which case we have

$$\lim_{b\to1} c_{ab}(a,b) = \lim_{b\to1} -\frac{\phi''(C(a,b))\phi'(a)\phi'(b)}{(\phi'(C(a,b)))^3} = -\frac{\phi''(a)\phi'(a)\phi'(1)}{(\phi'(a))^3} = 0.$$

□

Example 3.2. Consider the generator $\phi(t) = \log(1 - \theta \log t)$, where $\theta \in (0,1]$ is a dependence parameter (with the limiting case $\theta = 0$ representing independence). The copulae in this family are usually referred to as Gumbel-Barnett copulae, see for instance Nelsen [24, p.97]. It is easily checked that $\phi(t)$ fulfills the conditions of Lemma 3.5 and hence for $F_1 \in \mathcal{S}$ we have

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\mathbb{P}(X_1 > x)} = 1 + c.$$

□

Recall the definition of a survival copula

$$\widehat{C}(a,b) = a + b - 1 + C(1-a, 1-b), \quad 0 \le a, b \le 1,$$

corresponding to the copula $C(a,b)$ (cf. [24]). $\widehat{C}(a,b)$ is itself a copula and exchanges the role of upper and lower tails. Representation (5) can then be rewritten in the form

$$\frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}_1(x)} = 1 + \int_0^x \frac{\widehat{c}_a(\overline{F}_1(z), \overline{F}_2(x-z))}{\overline{F}_1(x)} F_1(dz). \tag{27}$$

For survival copulae of certain Archimedean type, Alink *et al.* [2] recently derived the following remarkable explicit result, which in our setting, for a weighted sum, reads:

Proposition 3.7 (Alink *et al.* 2004)**.** *Let the survival copula be Archimedean with generator $\widehat{\phi}$ regularly varying at 0^+ with index $-\alpha < 0$, let Y_α denote a positive random variable with density $f_\alpha(y) = (1 + y^\alpha)^{-1/\alpha - 1}$ and $c_1 \ge c_2 > 0$.*

(a) If $\overline{F} \in \mathcal{R}_{-\beta}$ with $\beta > 0$, then

$$\mathbb{P}(c_1 X_1 + c_2 X_2 > x) \sim c_1^\beta \left(1 + c_2 \mathbb{E}\left(c_2/c_1 + Y_\alpha^{-1/\beta}\right)^{\beta - 1}\right) \overline{F}(x).$$

(b) If $F \in \mathcal{S}$ and for any $a \in \mathbb{R}$ the relation

$$\lim_{x \to \infty} \overline{F}(x + a\, e(x))/\overline{F}(x) = \mathrm{e}^{-a} \tag{28}$$

holds, where $e(x)$ is again the mean excess function corresponding to F, then

$$\mathbb{P}(c_1 X_1 + c_2 X_2 > x) \sim c_1 \frac{\Gamma\left(1 + \frac{c_1}{(c_1+c_2)\alpha}\right) \Gamma\left(1 + \frac{c_2}{(c_1+c_2)\alpha}\right)}{\Gamma\left(1 + \frac{1}{\alpha}\right)} \overline{F}\left(\frac{x}{c_1 + c_2}\right). \tag{29}$$

Remark 3.2. The assumptions on the generator in the above proposition enforce a strictly positive tail dependence coefficient. More explicitly,

$$\lambda = \lim_{u \to 1} \frac{\widehat{C}(1-u, 1-u)}{1-u} = 2 \lim_{u \to 0} \widehat{c}_a(u,u) = 2 \lim_{u \to 0} \frac{\widehat{\phi}'(u)}{\widehat{\phi}'(\widehat{\phi}^{-1}(2\widehat{\phi}(u)))} = 2^{-1/\alpha}. \tag{30}$$

Remark 3.3. Assumption (28) is equivalent to $F \in$ MDA(Gumbel) (cf. Embrechts *et al.* [11] and also Section 2.2). Hence the conditions in assertion (b) are in particular fulfilled for the lognormal and the Weibull distribution with parameter $\tau < 1$. The constant from (29) (which increases to 1 as $\alpha \to \infty$) can be compared with the trivial upper bound from (4) in view of (30), cf. Figure 2. In particular, it becomes visible that, roughly, for large α (e.g. strong dependence among X_1 and X_2), the dominating contribution for the sum to be large comes from both variables being large.

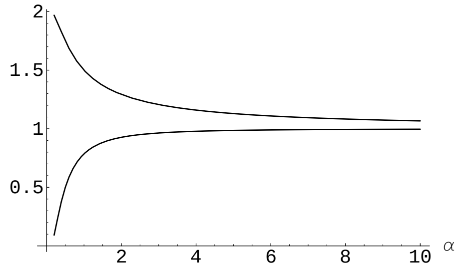

Figure 2: $F \in \mathrm{MDA}(\mathrm{Gumbel})$: exact value from (29) vs. trivial upper bound ($c_1 = c_2 = 1$)

Remark 3.4. Assertion (a) above can be rewritten as

$$\begin{aligned}&\mathbb{P}(c_1X_1 + c_2X_2 > x)\\ &\sim \left(\frac{c_1}{c_1+c_2}\right)^{\beta}\left(1 + c_2\int_0^{\infty}\left(\frac{c_2}{c_1} + y^{-1/\beta}\right)^{\beta-1}(1+y^{\alpha})^{-1-1/\alpha}\,dy\right)\overline{F}\left(\frac{x}{c_1+c_2}\right),\end{aligned}$$

which, in view of

$$\lim_{\beta\to\infty} c_2(c_1/(c_1+c_2))^{\beta}\left(c_2/c_1 + y^{-1/\beta}\right)^{\beta-1} = \frac{c_1c_2}{c_1+c_2}y^{-c_1/(c_1+c_2)},$$

converges to (29) for $\beta \to \infty$. Figure 3 illustrates that already for values of β around 10, the asymptotic behavior of the regularly varying case and the one of the Gumbel case are almost indistinguishable.

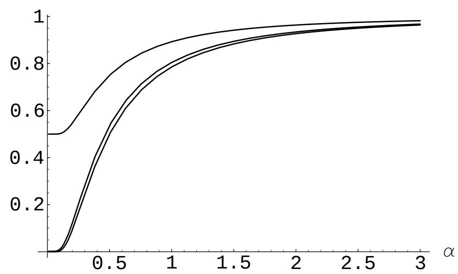

Figure 3: Comparison of constants: $\overline{F} \in \mathcal{R}_{-\beta}$ with $\beta = 2$ and $\beta = 10$ and $F \in \mathrm{MDA}(\mathrm{Gumbel})$ (from top to bottom, $c_1 = c_2 = 1$)

3.4 Farlie-Gumbel-Morgenstern copula

This family of copulae is defined by

$$C(a,b) = a\,b\,(1 + 3\rho_S(1-a)(1-b)), \quad -1/3 \le \rho_S \le 1/3,$$

where ρ_S denotes Spearman's rank correlation coefficient. Here $c_{ab}(a,b) = 1 + 3\rho_S\,(1-2a)\,(1-2b)$ and Lemma 2.7 applies giving

$$\mathbb{P}(X_1 + X_2 > x) \sim (1+c)\overline{F}_1(x).$$

Remark 3.5. Note that the Farlie-Gumbel-Morgenstern copula is tail-independent, providing another example of a dependence structure, for which the first order tail asymptotics of the sum is insensitive to the degree of dependence (measured in terms of ρ_S) irrespective of the heaviness of the marginal tails, as long as $F \in \mathcal{S}$.

3.5 Linear Spearman copula

Finally, we briefly mention the simple case of convex combinations of independence and comonotone dependence, which admits an explicit solution as well. The positive linear Spearman copula is defined by

$$C(a,b) = \lambda \min(a,b) + (1-\lambda)\, a\, b, \quad 0 \le a, b \le 1,$$

where the dependence parameter $\lambda \in [0,1]$ is indeed the tail dependence coefficient. Assuming $F \in \mathcal{S}$ and $\tilde{c} = \lim_{x\to\infty} \overline{F}(x/c_2)/\overline{F}(x/c_1)$ exists, we get

$$\begin{aligned}\mathbb{P}(c_1X_1 + c_2X_2 > x) &= \lambda \mathbb{P}(c_1X_1 + c_2X_2 > x | C_M) + (1-\lambda)\mathbb{P}(c_1X_1 + c_2X_2 > x | C_I) \\ &\sim \lambda \overline{F}(x/(c_1+c_2)) + (1+\tilde{c})(1-\lambda)\overline{F}(x/c_1).\end{aligned}$$

For $\overline{F} \in \mathcal{R}_{-\alpha}$ with $\alpha > 0$, we obtain $\mathbb{P}(c_1X_1 + c_2X_2 > x) \sim ((c_1^\alpha + c_2^\alpha)(1-\lambda) + \lambda\,(c_1+c_2)^\alpha)\overline{F}(x)$. In particular, for $\alpha = 1$ the tail of the sum is asymptotically equivalent to the independent sum for all $\lambda \in [0,1]$ (a comparison with Proposition 3.1 shows that in this example the upper bound (18) is quite rough for larger values of α). On the other hand, for distributions with $\overline{F}(x/c_1) = o(\overline{F}(x/(c_1+c_2)))$, $\mathbb{P}(c_1X_1 + c_2X_2 > x) \sim \lambda \overline{F}(x/(c_1+c_2))$ scales with the tail dependence coefficient λ.

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