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ESSAYS ON FUNDING MECHANISMS, ASSET ALLOCATION AND CALIBRATION OF AN NUITEES IN SWISS PENSION FUNDS

Müller Philipp

UNIL | Université de Lausanne

FACULTÉ DES HAUTES ÉTUDES COMMERCIALES

DÉPARTEMENT DE SCIENCES ACTUARIELLES

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THÈSE DE DOCTORAT

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pour l'obtention du grade de
Docteur ès Sciences Actuarielles

par

Philipp Müller

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Jury

Prof. Olivier Cadot, Président
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Sans se prononcer sur les opinions de l'auteur, la Faculté des Hautes Etudes Commerciales de l'Université de Lausanne autorise l'impression de la thèse de Monsieur Philipp Müller, titulaire d'un bachelor et d'un master en mathématiques économiques de l'Université d'Ulm, en vue de l'obtention du grade de docteur ès sciences actuarielles.

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Le doyen

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Faculty of Business and Economics

Doctorate in Actuarial Science

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and have found it to meet the requirements for a doctoral thesis.
All revisions that I or committee members
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Declaration of Authorship

I, Philipp MÜLLER, declare that this thesis titled, ESSAYS ON FUNDING MECHANISMS, ASSET ALLOCATION AND CALIBRATION OF ANNUITIES IN SWISS PENSION FUNDS and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this university.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this university or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Parts of this thesis appeared in the publications that are indicated in the relevant chapters.
- Where the thesis is based on work done by myself jointly with my supervisor, I have made clear exactly what was done by others and what I have contributed myself.

Signature

14.06.18

Date

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ESSAYS ON FUNDING MECHANISMS, ASSET ALLOCATION AND CALIBRATION OF AN NUITEES IN SWISS PENSION FUNDS

Müller Philipp

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Summary

This thesis focuses on the pension fund system in Switzerland. As the conditions on the financial markets have changed, the funds are facing a plethora of challenges. These involve, among others, the increasing lifetime of individuals, the conditions on the capital markets, the political system as well as governance and regulation. There is consequently a need for reforming the pension fund legislation. In this thesis, three selected topics that are connected to this, are studied more closely. The first topic addresses the stability of a fund. By raising additional contributions and distributing surpluses, it can maintain a desired safety level. It is studied how different choices for the governance affect the stability. It is found that raising additional contributions is beneficial for safety, while risk averse members dislike the instability originating from bonus payments. The second topic deals with the investment strategy of a pension fund. It is analyzed how the asset allocation changes when historical return characteristics are taken into account in a more detailed way. Different distributions assumptions are compared with respect to how they are able to replicate the characteristics of the investment classes. It is found that there is a considerable change in the portfolio and in the investment returns when the third and fourth moments and different distribution assumptions. The final topic revolves around the right calibration of the pension payment during retirement. It depends on the lifetime of the retiree and the investment strategy of the pension fund whether the savings suffice to cover the payments. It is compared how these factors influence the choice of the pension amount. The results indicate that a higher lifetime and a lower investment return require a decrease of the pension. Therein, the impact of the investment is found to be higher than the one of the lifetime.

Cette thèse est axée sur le système des caisses de pension en Suisse. Avec le changement des conditions sur les marchés financiers, les caisses font face à une pléthore de défis. Ceux-ci incluent, parmi d'autres, l'allongement de la durée de vie des individus, les conditions sur les marchés financiers, le système politique, ainsi que la gouvernance et la réglementation. Par conséquent, il y a une nécessité de réformer la législation concernant les caisses de pension. Dans cette thèse, trois sujets sélectionnés, liés à ce thème, sont étudiés plus en détail. Le premier sujet aborde la stabilité d'une caisse de pension. En exigeant des contributions supplémentaires ou en distribuant les excédents, le niveau de sûreté est maintenu. L'impact sur la stabilité d'une caisse selon les choix de gouvernance est étudié. Il est constaté que la levée de contributions supplémentaires est bénéfique pour la sûreté, tandis que les membres averses aux risques n'apprécient pas l'instabilité provenant des prestations supplémentaires. Le deuxième sujet traite de la stratégie d'investissement d'une caisse de pension. On analyse comment l'allocation des actifs change quand les caractéristiques historiques sont prises en compte d'une façon plus détaillée. Différentes hypothèses sur les distributions du rendement sont comparées par rapport à leur capacité de reproduire les caractéristiques des classes d'actifs. Un changement considérable dans le portefeuille et les rendements est constaté si les troisième et quatrième moments, ainsi que les différentes hypothèses sur les distributions sont utilisés. Le dernier sujet porte sur la calibration appropriée de la rente pendant la retraite. Elle dépend de la durée de vie du retraité et de la stratégie d'investissement de la caisse de pension si l'épargne est suffisante pour couvrir les paiements. On compare comment ces facteurs influencent le niveau de la rente. Les résultats indiquent qu'une durée de vie supérieure et un rendement inférieur des investissements impliquent des rentes plus faibles. De plus, l'impact de l'investissement s'avère plus important que celui de la durée de vie.

Chapter 1

Introduction

The pension fund systems in many countries are under pressure these days. As many of them were originally set a long time ago, they are facing problems with respect to the changed conditions that are present today. Consequently, there is a need to reform the systems and adjust them to the new circumstances. The challenges that pension funds are facing are manifold. It is therefore of great importance to address the prevailing problems in a timely manner. For the pension fund system in Switzerland, the challenges correspond to the ones that can also be found in many other countries. In addition to this, there are certain topics that originate from the specific characteristics of the Swiss legislation. This thesis takes a close look at the Swiss pension fund system and the problems that it is facing. This encompasses analyzing the origin and the scope as well as consequential difficulties. Going further, selected topics are closely examined and analyzed in detail.

In the second Chapter of this thesis, the challenges that the Swiss pension fund system is facing, are presented and discussed in detail. This involves factors that influence the system from the outside as well as internal ones that originate from the need for adjustments. One of the trends that have been discussed most in connection to pension funds is longevity. With constant improvements in healthcare and other fields, the lifetime of individuals has increased significantly. At the same time, the fertility rate in developed countries has been decreasing over the last decades. These two developments represent parameter risks that need to be accounted for in the modeling of the fund. Along with individuals living longer, the age structure of the members is changing. Adding to this, the returns from investing on the capital market are under pressure, as the historic returns exhibit high volatility and the risk-free interest rate is on a historical low. In response to this, the investment strategies of the funds need to be adapted. One of the biggest influences on the pension fund system lies with the legislator. It is the government who chooses the regulatory aspects and sets the parameters of the social security system. In addition to this, it is the responsibility of the pension funds to set up their organizations in accordance with safety requirements while taking the structure of their members into consideration. In this process, the societal changes in today's society need to be accounted for. Individuals change their employer more frequently and demand more flexibility with respect to the moment of their retirement. A reform of the pension fund system also needs to be carried out in such a way that it preserves the financial stability. With respect to this, there have been attempts to improving the standards for the risk reporting.

The financial stability of a pension fund is studied more closely in the third Chapter. There, the impact of funding mechanisms on the stability of pension funds and the utility of their members is analyzed. To this end, the Swiss second pillar of old age provisions and the different risk dimensions that it inherits, are examined. This involves a comparison of the savings account at retirement to the overall payments of an individual member. With the help of stochastic simulations, sensitivity analyses and capital return scenarios, the effects of extra contributions and surplus distributions with respect to the funding ratio are analyzed over time. For the fund, this involves finding a choice of parameters that leads to a better financial stability. With respect to the insureds, the goal is to reach a higher utility. The results show that the payment of remediation measures is an effective tool in stabilizing the fund. For the distribution of surpluses the outcomes reveal that they lead to a higher overall volatility of the savings account. This is not favored by risk averse individuals. In addition, the sensitivity analysis shows that a proper choice for the model parameters is important.

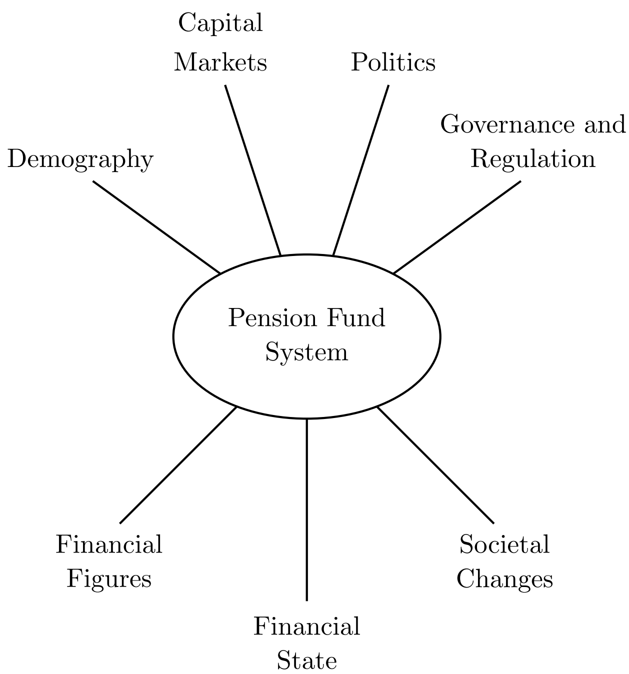
With respect to the investment returns, Chapter 4 looks at the optimal investment strategy of a pension fund. As the risk-free interest rate does not suffice to meet the required investment goals, an optimal mix of the available assets is of importance. In this context, the optimal asset allocation according to minimum variance theory is compared to one that makes use of an extended utility function. This is achieved by optimizing a utility function that takes the third and fourth moments of the returns into account as well. This way, the skewness and the kurtosis of the asset returns are used for determining the optimal allocation. For the simulation of the different asset classes, several return distributions are compared. The goal of this is to find the best fit for the historic data and consequently improve the simulation results. Using the optimal asset allocation and the distributions that provide the best fit, the assets and the liabilities of a pension fund are simulated in a one-period model. The results are then analyzed by looking at different key measures such as the funding ratio, the underfunding probability and selected quantiles of the funding ratio. Besides the simulation done for a target return, the optimization process is also carried out with target values for the funding ratio and the underfunding probability. For the extended utility function, a sensitivity analysis with respect to risk aversion towards the third and fourth moments is performed. This way, the impacts from the preferences for the different moments of the portfolio return are compared.

In the fifth Chapter, the retirement phase of a defined contribution pension fund is studied. In Switzerland, the annuity payments to the pensioners are calculated based on the available savings at retirement by using a conversion rate. The adequate choice of the used conversion rate is therefore of great importance for the financial stability of the pension fund. A value that is too high would lead to a lack of savings. Conversely, choosing it too low would not be favored by the pensioners. The conversion rate, within legal limits, depends on the capital market returns from investing the savings, the mortality of the members, and the technical interest rate. It is studied what impact these financial and biometric risk parameters have on the choice of the conversion rate. To this end, a sensitivity analysis regarding the three factors is carried out using as far as possible analytic expressions. Going further into detail, selected scenarios for the asset returns are considered. In order to reflect the dispersion in the historic investment returns, the impact of the capital market is studied for different values of the volatility. All of the three factors are found to have an important influence on the conversion rate. When looking at the results, it can be concluded that the impact of the investment returns exceeds the one of the mortality rate. Further, a decrease of the technical interest rate needs to meet with a decrease of the conversion rate as well. Overall, the conversion rate is found to be very sensitive to changes in the different factors. Its appropriate choice by the regulator is therefore of high importance for the stability of the pension funds in the retirement phase.

Chapter 2

Challenges in the Swiss Pension Fund System

The pension fund system is facing great challenges in today's environment. While the risk-free returns from the capital market have hit bottom, the lifetimes of the members are ever-increasing. Along with that, the lifestyle of individuals has changed. These altered conditions make an adjustment of the pension system necessary. In this regard, there is disagreement in the political system in Switzerland about the type and scope of reforms. The changed conditions on the market are consequently putting the financial state of the pension funds under pressure. As the conditions on the market are becoming more difficult, there are further efforts to change the governance and regulation with respect to reporting more risk-oriented financial figures. In this Chapter, we give an overview of the challenges that are present in the Swiss pension fund market. This encompasses factors that influence the system externally such as the demography, the capital markets, the politics and the governance and regulation as well as ones that follow from it such as the societal changes, the financial state and the financial figures of funds.



2.1 Demography

Over the past years and decades, there has been a strong worldwide trend of the population growing older. Thanks to better medical treatments as well as life- and workstyle improvements, the life expectancy of individuals has been increasing constantly. For Switzerland, the life expectancy of newborns has increased from 77.9 years in 1990 to 83.6 years in 2018 (United Nations, 2018). Projections further suggest that this trend in increasing longevity will continue. Estimations suggest that by 2050 individuals will be living for 87.4 years and by 2100 they are expected to reach 93.3 years. Figure 2.1 shows the historic and predicted future development of the curtate expected lifetime for individuals at age 65 in Switzerland based on own modeling using data from the Human Mortality Database and by fitting a Lee-Carter model. The increase in the curtate expected lifetime for the male, the female and the overall population is relatively constant. With respect to the future, we further notice that the lifetimes of the different groups can be expected to continue increasing onwards with a linear slope. It can further be seen, that the difference in expected lifetime for men and women has been decreasing over the last decades.

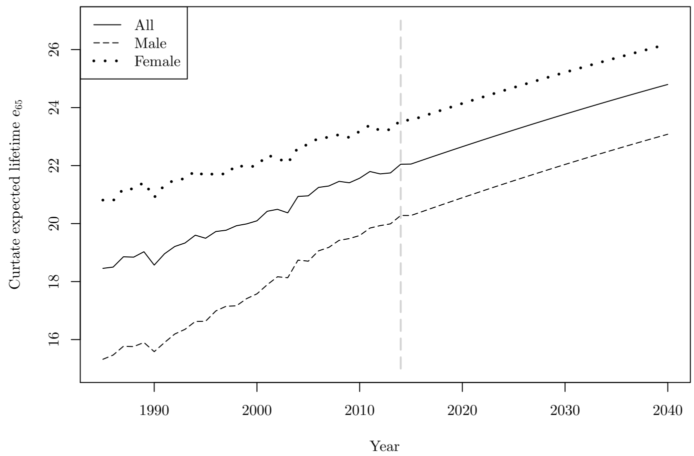


Figure 2.1: Plot of e_{65}, the curtate life expectancy of 65-year old individuals in Switzerland.

This demographic development is posing a great challenge for the social security system and pension funds in particular. As individuals live longer, they are requiring a higher total of pension payments. This, in turn, is posing problems for the pension funds, as the old-age savings of the members may not suffice in order to make those additional payments. The consequence would be a redistribution of the savings from the younger generation towards the older one. This solidarity between young and old fund members is under pressure from the development of the birth rate. The birth rate for European countries has decreased from 1.98 in 1980 down to 1.62 in 2018. For the Swiss population, the values are more stable, albeit lower, amounting to 1.54 in 1980 and 1.55 in 2018. For the future, only a small increase is expected, with an estimated birth rate of 1.7 in the year 2100. Together, the developments of increasing lifetime and low birth rates lead to a shift in the distribution of age groups. For 2018, the share of individuals under the age of 25 from the Swiss population amounts to 26%. At the same time, elderly people over the age of 60 already make up 24% of the population. For the future, these values are expected to change in favor of the elder. Consequently, it can be expected that there are fewer individuals who will be paying contributions while there will be more pensioners who will receive payments. The financial state of pension funds could therefore be under pressure, as the contribution might not suffice anymore in order to make the annuity payments. While in defined contribution pension schemes this evolution is critical to an individual level account (insufficient funds compared to the longevity), in

defined benefit schemes the inter-generational equilibrium is even more at risk.
It has been discussed widely in the literature what the consequences for the social system will be. Some authors argue that countries that already have a high share of older people are well prepared for the future changes (Herrmann, 2011). Others argue that the impacts of longevity can be compensated for (Bloom et al., 2011). Necessary changes that are suggested include higher savings contributions, a higher retirement age and lower pension payments. This way, the misbalance between the accumulated savings at retirement and the pension payments can be reduced. In addition, a higher labor participation of the elderly and a reduction of early retirement appear necessary (Dorn and Sousa-Poza, 2005). The lower birth rate is further assumed to lead to a higher work participation of women (Bloom et al., 2010). The necessary changes to the pension system depend on the action-taking of the regulator (cf. Section 2.3). With the demographical changes already solidifying today, the stability of the pension funds depends on the political system making changes and proposing reforms in response to the altered circumstances in order to make the pension system sustainable.

2.2 Capital Markets

The members of a defined contribution pension fund are entitled to an interest on their savings. For the Swiss pension fund system, the legislator sets a minimum interest rate that active members should receive on the obligatory part of the accumulated capital (see BVV2, Art. 12). For the fund, this consequently poses the challenge of investing its assets in a way such that the demanded return is achieved while satisfying a certain safety level. As the risk-free interest rate has been decreasing over the last years, this is becoming more difficult. As an illustration for the (quasi) risk-free interest rate, Figure 2.2 shows the return of 10-year Swiss government bonds over the last 20 years. We observe that, while the return was

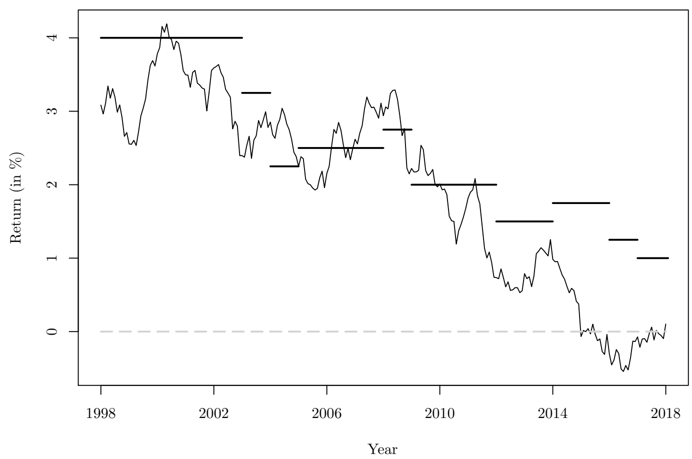


Figure 2.2: Plot of the return of 10-year Swiss government bonds and the BVV2 minimum guaranteed interest rate from 1998 to 2018.

higher than 4% in 2000, it has since dropped and even taken negative values. In response to this, the guaranteed interest rate has been adjusted over the last years. For 2017, it has been fixed to 1%. The historic course of the minimum interest rate is given in Table 2.1. While the minimum guaranteed interest rate used to be below or close to the risk-free interest rate (represented by 10-year Swiss government bonds), it is now higher than it (Swisscanto, 2018). The consequence of this is that pension funds need to engage in a riskier investment strategy in order to reach higher returns. This involves investing more in asset classes such as real estate, stocks and hedge funds and other alternative classes. For the year 2017

Year	Interest rate	Year	Interest rate
1985	4.00%	2009	2.00%
2003	3.25%	2012	1.50%
2004	2.25%	2014	1.75%
2005	2.50%	2016	1.25%
2008	2.75%	2017	1.00%

Table 2.1: Evolution of the BVV2 minimum guaranteed interest rate.

for example, pension funds in Switzerland have been investing 22.5% of their assets in real estate, 30.7% in stocks and merely 6.3% in alternative investments (Swisscanto, 2018). As the share of bonds has been decreasing over the past years, many pension funds state that one of their main goals is to reach a higher real estate share. In addition to this, more than half of the funds support the elimination of the investment limits proposed by the legislator (BVV2, Art. 55). While this change of the investment strategy makes higher investment returns possible, it comes at the price of a higher volatility and risk. Figure 2.3 shows the annual return of the Pictet BVG 40 pension fund index, an index that is often taken as reference.[1] It can be seen, that the average annual return fluctuates strongly, reaching values

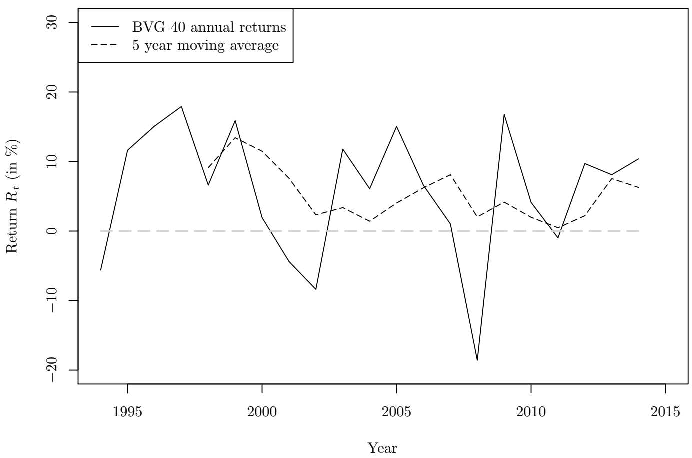


Figure 2.3: Historic annual returns r_t of the Pictet BVG 40 pension fund index together with the five-year moving average.

between 20% and −20% over the last 20 years. This consequently increases the risk of being underfunded in given years and requiring remediation measures. A survey among Swiss pension funds showed that the target investment return for 2018 amounts to 3.1% on average (Swisscanto, 2018).

The higher share of the real estate in the portfolios of pension funds comes at a risk. As the risk-free interest rate has been at a low value over the past few years, it has led to a strong increase of building activities (Credit Suisse, 2018). This has led to an increase of unused housing especially in rural areas. Due to this, the risk of a bubble on the real estate market in Switzerland has been increasing. For the cities of Zurich and Geneva, for example, housing prices are moderately overvalued (UBS, 2017). The situation remains sensitive to interest rate changes as an increase of the risk-free interest rate has a greater impact when yields are lower.

The development of the capital market returns also plays an important role for the technical interest rate. The technical interest rate represents the discounting factor on the liabilities that is used in order

[1] See `https://www.am.pictet/en/switzerland/articles/lpp-indices` for further information.

to calculate the technical reserves of a pension fund. The pension fund has to choose the technical interest rate in a way such that it takes the composition and the characteristics of the institution into account. At the same time, it is supposed to remain below the capital market return that is to be expected (BVG, Art. 52). Within these guidelines, the board of the pension fund is free to choose a value for the technical interest rate. In order to give guidance, a formula for calculating a reference value for the technical interest rate has been defined by the Swiss chamber of pension fund experts (SKPE, 2015). It states that the technical interest rate should be calculated as a weighted average of 2/3 of the average performance over the last 20 years and 1/3 of the return of ten-year Swiss government bonds. In order to remain below the expected capital market return, this value is then reduced by 0.5% and rounded down. An overview of the historic values of the technical reference interest rate is given in Table 2.2. As the liabilities of the pension funds in Switzerland are very high, a change of the technical

Year	Technical Interest Rate	Year	Technical Interest Rate
2005	4.50%	2012	3.50%
2006	4.50%	2013	3.00%
2007	4.50%	2014	3.00%
2008	4.00%	2015	2.75%
2009	3.75%	2016	2.25%
2010	4.25%	2017	2.00%
2011	3.50%		

Table 2.2: Evolution of the technical reference interest rate from 2005 to 2017.

interest rate can lead to strong changes in the required capital. In fact, in 2016, the pensions that were paid out, reached a total of CHF 42.5 bn (BSV, 2017a). It is consequently of high importance to set the technical interest in a reasonable and appropriate way. To this end, the regulatory authority seeks to revise the regulations for calculating and setting the technical interest rate (OAK BV, 2017b). The challenges and difficulties that lie around this process are described more in detail in the following Section.

2.3 Politics

The legislator plays a key role in the pension funds system as the laws and regulations are set by politics. This is especially the case with respect to reforms of the laws that are in force. It is therefore important for the regulator to monitor the pension system, check that it works properly and correct possible shortcomings. The pension system that is in force in Switzerland today has been introduced in 1985. As there have been changes with respect to the lifetime of the members and the birth rate (cf. Section 2.1) however, the contributions and the payments are not in balance anymore. For the first pillar of the system, which is based on redistributing the savings of the actives to the pensioners, there has been a deficit of CHF 767 Mio. in 2016 (BSV, 2017a). For that year, the missing funds were compensated for by high investment returns. A small capital reserve is still available in the first pillar funds as well (OAK BV, 2018). In the second pillar, a redistribution from the insureds to the retirees also takes place As noted before (cf. Section 2.2), the conditions on the capital market have changed considerably over the last years, leading to lower and more volatile returns. It has consequently become necessary to adapt the pension system to these new circumstances. In this regard, several attempts have been made in order to reform the social security system in Switzerland (BSV, 2017b). In 2004, the first major revision of the social security system was made (BSV, 2016). The changes included, among others, an increase of the retirement age for women from 62 up to 64 years. In addition, the conversion rate was decreased from 7.2% to 6.8%, with both adjustments taking place gradually over the course of ten years. The annual values are given in Table 2.3.

Year	Conversion Rate	Year	Conversion Rate
1985 – 2004	7.20%	2010	7.00%
2005	7.15%	2011	6.95%
2006	7.10%	2012	6.90%
2007	7.10%	2013	6.85%
2008	7.05%	2014 – 2018	6.80%
2009	7.05%		

Table 2.3: Evolution of the conversion rate.

The conversion rate is of importance with respect to the pension payments as the annual annuity that pensioners receive is calculated by multiplying the savings at retirement with the conversion rate. While this reform was accepted by the electorate, further changes need to be made. In 2010, an attempt to decrease the conversion rate further and to increase the retirement age for women to 65 years, was declined. Another reform proposal called "Reform 2020", was made in 2017 (BSV, 2014). It included, among others:

- A uniform retirement age of 65 years for men and women.
- More flexibility with respect to retiring earlier or later, allowing individuals to retire from the age of 62 up to the age of 70.
- A decrease of the conversion rate from 6.8% down to 6%.

This reform was rejected by the electorate as well. The consequence is that the funding gap of the pension system persists and grows further over time.
The recent reform proposals were mainly targeted at changing the retirement age and the conversion rate. However, the possibilities are more extensive. While a decrease of the conversion rate would lead to a lowering of the pension payments, the goal of increasing the retirement age is to increase the savings and reduce the annuity duration. Instead of increasing the retirement age, it would also be possible to increase the contributions from working individuals. This way, the actives would accumulate more funds throughout their work live and could still retire at the same age. As the regulations for calculating the savings contribution are extensive, there are many ways to adjust the legislation. By changing the regulations, the regulator is able to restore the stability of the pension system in general and the equilibrium of the payment streams in particular. In order to assess the state of the pension funds, the legislator can examine different parameters. One key figure that describes the ratio of the assets and the liabilities of a fund, is the funding ratio (cf. Section 2.6).

2.4 Governance and Regulation

The regulation of pension funds in Switzerland is taking place on three levels. The base is set by the regulator who establishes the laws that the pension fund system is based on. In addition to this, the Supervisory Commission for occupational pensions (OAK BV) sets up a regulation for the supervisory practice for all pension funds. At the level of the pension funds is the Swiss Chamber of Pension

Actuaries (SKPE). Its function is to act as an advisor for funds. To this end, it gives recommendations with respect to the practical application of the regulations, legal aspects and the investment. The board of the pension fund is taking final decisions. Its duties lie in setting up the regulations of the fund in accordance with the existing legislation and statutory orders. This involves setting up the policies with respect to the savings plans of the actives as well as the pension schemes for the members. Going more into detail, they are responsible for choosing an investment strategy for the assets and setting the technical interest rate for the valuation of its liabilities (typically following the expert's advice).
Especially for the supplementary part that exceeds the legal minimum, the board has in comparison a lot of freedom with respect to choosing the framework of the fund. The consequence of this is that there is a high complexity involved in leading a pension fund. As the regulations are giving them great freedom, the members of the foundation of the fund need to take all the existing risks and difficulties into account as well as possible. This has led to a consolidation, coming with a strong decrease of the number of funds in Switzerland over the past years. At the same time, the number of actives has increased. Figure 2.4 shows the development of the number of Swiss pension funds and the active members from 2004 to 2016.

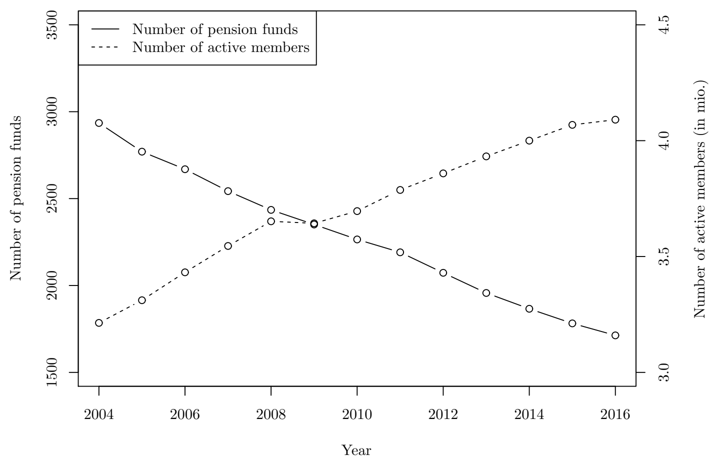


Figure 2.4: Number of pension funds and active members in Switzerland from 2004 to 2016.

As we can see, the number of funds has been decreasing almost linearly from close to 3 000 in 2004 down to about 1 700 in 2016. The number of funds has consequently almost halved over the course of 12 years. Meanwhile, the number of active members increased from 3.2 Mio. in 2004 up to more than 4 Mio. in 2016, corresponding to an increase of almost 30%. A strong trend of consolidation is observed in the market. As mentioned before, this is linked to a complex environment from a capital, demographic, and regulatory point of view. Together with the steady increase in the number of members, this has a concentration effect on the market. As there a fewer, yet larger funds, the cluster risk also increases. Along with this, the level of freedom for funds to choose many aspects on their own makes their supervision challenging.
An advantage of this setup is that the board of the fund is able to structure the fund more closely in accordance with the age and member structure. However, as a consequence, the demands of corporate governance are increasing. This is especially posing challenges for institutions that are still of small and medium size. Overall, the current system is characterized by great flexibility but also a high responsibility that comes together with it. Going forward, it needs to be asked whether this is an advantage or if it would be preferable to have a more restrictive system (e.g. principle-based or rule-based regulation). While this would take some of the flexibility away, it could in turn improve the ability of pension funds

to remain on the market and consequently lead to a higher safety level. In Section 2.7, we further discuss current plans and discussions with respect to reporting financial risk figures of pension funds to the supervisory authorities.

2.5 Societal Changes

Together with the increased lifetime of individuals (cf. Section 2.1), changes are taking place in the structure of the society. As people live longer, they also spend more time in good health. Being consequently more active at a higher age, their spending behavior changes as well. The legislator in Switzerland has set itself as a goal that retired members should receive a pension that corresponds to about 60% of their last salary (BBL, 1976). With respect to the change in lifestyle of the elderly, it is to be questioned whether this value is still appropriate as of today or if it needs to be adjusted. Furthermore, working biographies are more heterogeneous with people changing their employers more often than in the past. This entails also changes of the pension fund affiliation with ever-changing rules and uncertainties in pension planning.

In order to relieve the retirement system, it is necessary to either increase the capital at retirement, reduce the pension payments or apply a mixture of the two. To this end, increasing the retirement age would help with achieving both these goals. The life expectancy in Switzerland is among the highest worldwide with men living 81.5 years on average and women 85.3 years (BFS, 2017). Yet still, the retirement age remains at 65 years for men and 64 years for women (BVG, Art. 13), while there is currently a worldwide trend of increasing it as more than half of the OECD countries have decided on increases of the retirement age (OECD, 2017). There is great potential for a relieve of the social security system in increasing the retirement age. To this end, 61% of the companies in Switzerland already offer possibilities to work past the legal retirement age today (Cosandey, 2015). However, political initiatives to protect old age employees may aggravate their chances on the job market. Measures such as a prohibition of releases from a certain age and generous pension plans in the case of early retirement would serve to protect employed individuals. At the same time, this would lead to difficulties to find a new job and consequently increase the risk of long-time unemployment. Other measures that would lead to an increase of the labor costs, would have a comparable effect. While employers should therefore be encouraged to employ old age individuals, protective measures for older individuals could have a contrary effect.

In addition to an increase of old age occupation, a more flexible transition into retirement should also be strived for. In this fashion, plans that involve part-time work at the end of the work life should be offered by companies. The reduced income that is connected to this, could be compensated for by already obtaining a part of the pension benefits.

2.6 Financial State

It is the task of the pension fund to ensure that there is a balance between the assets and the liabilities that it has (BVG, Art. 65). In this, the assets contain, among others, the contributions of the members, a contribution reserve from the employers and the technical reserves (SKPE, 2014). The liabilities are mainly made up of the savings accounts of the actives and the retired members as well as additional reserves. The ratio of the assets over the liabilities denotes the funding ratio of the pension fund.[2] If the funding ratio is below 100%, the institution is said to be underfunded. Conversely, a ratio of more than 100% is denoted as overfunding. A plot of the funding ratios from 2004 to 2017 is presented in Figure 2.5 (Swisscanto, 2009, 2018). As we can see, the values have been above 100% in almost every

[2] An extensive overview of the management of assets and liabilities in pension funds can be found in Solari (2002).

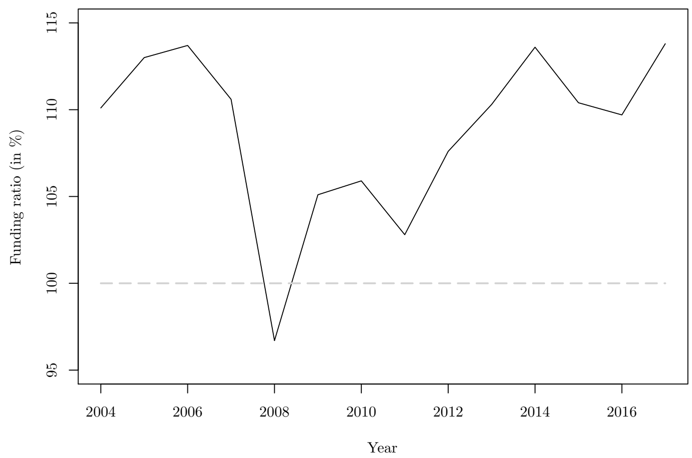


Figure 2.5: Average funding ratio of Swiss pension funds from 2004 to 2017.

year. Pension institutions were only underfunded on average in 2008 with a value of 96.7%. The reason for this lies in negative investment returns due to the financial crisis in that year. On average, pension funds had a capital market return of −12.7% in 2008 (Swisscanto, 2018). In the following years, the gap between assets and liabilities was closed. At the end of 2017, the average funding ratio amounted to 113.8%.

Once a pension fund has become underfunded, it is obliged to inform the authorities and take measures in order to increase the funding ratio to 100% again (BVG, Art. 65d). To this end, it is supposed to work out measures in order to correct the underfunding. The legislation lists several ways how to restore the financial state of a fund. Those include, among others (SKPE, 2014):

- A temporary reduction of the interest rate on the savings below the legal minimum.
- Additional contributions by the employer.
- Remediation measures from the actives and their employers.
- Contributions from the retired individuals.

The effectiveness of the remediation measures depends on the age structure and the ratio of active and retired members of the pension fund. For example, additional contributions can pose considerable additional expenditures for young members. Consequently, they would tend to prefer a reduction of the interest rate. Conversely, older members would most likely prefer to avoid a reduction of the interest rate. As their savings accumulate to much higher amounts, the impact on their accounts would be more significant. It is the task of the pension fund to choose the measures in a way such that the stabilization of the fund is achieved without causing too much stress to its members.

The recovery plan has to be reported to and approved by the supervisory authorities. In addition, the regulator is surveilling the proper execution of the measures. The pension funds are supposed to aim for the duration of the recovery measures to be as short as possible. The reasoning behind this is that the institution should be exposed to a further deterioration of the financial state for as little time as possible. Consequently, the remediation measures are not supposed to last for longer than five to seven years (OAK BV, 2017c).

In the case of overfunding, i.e. a funding ratio above 100%, the pension fund has to establish a reserve for future return fluctuations. This way, it is able to mitigate the impact of years with low investment returns (BVV2, Art. 48e). Once the reserves are high enough, the fund is able to distribute additional

funds as a bonus to its members. This bonus can be conducted by increasing the interest rate as well as by paying monetary amounts.
In Chapter 3, we analyze the impact of pension funding mechanisms with respect to the stability and payoff of a pension funds. To this end, we analyze the accumulation phase of a Swiss pension fund. We assume that depending on the state of the fund, additional contributions from the members may be required or that surpluses can be distributed as a bonus. Among others, we find that while remediation measures are helpful with respect to stabilization, they also lead to an increased volatility. Surplus distributions, in turn, lower the relative payoff utility of the members and lead to an increased frequency of additional payments. We therefore conclude that pension funds can profit from a cautious funding policy that targets an increased stability and achieves a lower volatility.

2.7 Financial Figures

Following the current risks and challenges on the Swiss pension fund market, a more thorough and timely surveillance of the risk profile and financial status of funds appears necessary. The legislator is therefore trying to establish a regular report of the financial risk figures of every pension fund (OAK BV, 2016). This report is supposed to encompass the three risk dimensions of financial safety, the ability to be restructured and the current financing situation. To this end, a number of parameters have been proposed that should help in assessing the risk status of a pension fund (OAK BV, 2017a). These include, among others, the following parameters:

- Pension capital
- Technical reserves
- Sum of the savings of the members
- Expected investment return
- Technical interest rate
- Target value for the fluctuation reserve
- Target value for the funding ratio
- Biometric assumptions (e.g. mortality tables)

In addition, it is to be stated what impact the following scenarios have:

- Reduction of the funding ratio when reducing the technical interest rate by 0.5%.
- Impact on the funding ratio when reducing the return on the savings on the actives by 1%.
- Impact on the funding ratio when raising 1% of the contributions of the actives as remediation measures.
- Effect of a reduction the interest rate with respect to remediation.

These risk figures are meant to help the board of the pension funds as well as the supervisory authorities in assessing the risk situation of the fund. By looking at the various parameters and sensitivities, the structure of the members of the funds and the long-term horizon of the business are to be taken into consideration.
The reactions from pension fund representatives to this proposal were mostly negative in 2017. The main criticism was that the majority of the figures are already included in the respective annual reports. The

additional value of a risk report would therefore be comparatively small. In addition, it has been argued that it is not clear yet whether certain figures are supposed to be based on estimates or if they should be computed with all technical reserves taken into consideration. The latter case would be connected to additional costs for the funds. In a similar fashion, it has been asked if the risk report would be legally binding or not. As the added value of the proposed report was perceived as rather small and open questions about certain parameters remained, the introduction of the report has been postponed in order to review the criticism from practitioners. While the opinions on the proposed report of the risk figures diverge, there is a broad agreement that a more risk oriented reporting and surveillance is necessary. While this is still work in progress, an extensive risk reporting from the proposed report up to some kind of solvency test for pension funds could be possible in the future.

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Chapter 3

The Impact of Pension Funding Mechanisms on the Stability and the Payoff from Swiss DC Pension Schemes

Adequately funding occupational pension funds is a major concern for society in general and individual contributors in particular. The low returns accompanied with high volatility in capital markets have put many funds in distress. While the basic contributions are mostly defined by the state, the fund's situation may require additional contributions from the insureds or may allow the distribution of surpluses. In this Chapter, we focus on the accumulation phase of a defined contribution plan in Switzerland with minimum returns and annual solvency targets in terms of an assets-to-liabilities funding ratio. From the viewpoint of the pension fund, we evaluate the outcome of selected funding mechanisms on the solvency situation. Taking the perspective of the contributors, we analyze the payoff and the utility. Combining both prospects, we discuss the boundary values that trigger the various participation mechanisms and their impact. We find that remediation measures, while stabilizing the fund, yield a higher volatility in the insured's contributions. Further, surplus distributions lower the relative payoff utility of the fund's members and increase the frequency of remediation measures. Overall, insureds and pension funds will profit from a cautious surplus distribution policy that focuses on keeping the stability high and lowers the volatility of the result.

Note: This is joint work with J. Wagner and has been published in the The Geneva Papers on Risk and Insurance - Issues and Practice, volume 42, issue 3, pp. 423–452. The authors are thankful for the comments on earlier versions of this manuscript by participants of the *Western Risk and Insurance Annual Meeting* 2016, the International Conference *Mathematical and Statistical Methods for Actuarial Sciences and Finance* 2016, the *Lyon-Lausanne Seminar* 2016, the *European Actuarial Journal Conference* 2016, the *European Group of Risk and Insurance Economists Seminar* 2016 and the *Annual Meeting of the German Association for Actuarial Science* 2017.

3.1 Introduction

Since the introduction of the Swiss pension system and occupational pension funds (second pillar) in particular, the demographic and capital market framework conditions have changed. Life expectancy is increasing, while birthrate is decreasing, causing the ratio of the number of active workers to the number of retirees to decline over the years (see OECD, 2015b). In the financial markets, many asset classes have delivered historically low returns and at the same time exhibited increased volatility in the last two decades (see OECD, 2015a).

The demographic issues and changing capital markets from the last financial crisis and ongoing turbulences are highlighted by most practitioners (see Credit Suisse, 2014). There are also other factors that change the environment. For example, at the society level, family and living structures along with work conditions have changed (see, e.g., Maas et al., 2015). Flexibility in work time management, statutory and effective retirement age and new disability and old-age dependency challenges need to be considered. Many technical parameters in Swiss pension funds (e.g., the minimum interest rate, the conversion rate) and their adaptations depend on political decisions. Reforms have been strongly rejected in the last few years, and the definitions of technical and actuarial parameters have undergone lengthy political processes. The currently planned reform in Switzerland called "Altersvorsorge 2020"[1] only yields partial answers. In the 2008 financial crisis and its aftermath, the solvency of many pension funds has been stressed. The funding ratio has dropped below 100% in many cases, which has put pressure on funds to move towards consolidation and sustainability considerations (see Swisscanto, 2015). This pressure comes along with operational risks with regard to compliance, higher transparency and governance requirements. Furthermore, one has to consider increasing wealth transfers between younger and older contributor groups (Avanzi and Purcal, 2014; Eling, 2012) or potentially unfair mechanisms regarding employees who change their employer and pension fund.[2] These changes pose challenges to the Swiss system and the ones in other countries. Many aspects of these problems have been discussed by practitioners and politicians, but they have been given much less attention in academic research. For example, pension funds in good shape have started paying bonuses to their members while it is unclear whether this is optimal and increases their utility (see, e.g. Jacquemart, 2014; Lisse, 2014). Independent research and a solid academic foundation are important in an area where the stakeholders, contributors, pension funds, actors from the industry and politicians have diverging interests and opinions. This is why we focus on the following research question: What is the impact of funding mechanisms on the pension fund stability and the utility of the insured?

While our research holds true for the accumulation of funds in pension funds in many countries (with certain adaptations), we apply our modeling to the Swiss second pillar pension system and study different dimensions of risk that affect pension schemes and their members. We study the impact that remediation measures and surplus distribution have on the stability and the payoff of a fund and the utility of its members. For this, we put the available funds at the term of the savings phase in relation to the total payments, i.e. regular contributions and remediation costs. Our research involves, among others, the adequate choice of parameters, the model's sensitivity and the impact of capital market scenarios. While this is adequate when analyzing the pension fund for the active insureds during the accumulation phase, limitations for drawing conclusions on the overall state of the fund exist (e.g., exclusion of the bonuses legally due to pensioners, credits and debits from mortality).

The academic literature analyzing different types of pension schemes is abundant. Sharpe (1976) is

[1] See `http://www.bsv.admin.ch/altersvorsorge_2020`, September 2016.

[2] Contributors changing their employer must change to the pension linked to the new company. Thereby, the assets are transferred whereas, e.g. potential remediation measures to improve the overall state of the pension fund, remain with the previous institution. Exceptions may apply in the case of partial liquidation of the fund (see BVG, Art. 53).

one of the first to rigorously analyze pension insurance provided by a sponsor. Black (1976) discusses both the optimal pattern of contributions and the optimal investment policy for the assets of a pension fund. Typically, stochastic pension fund modeling is used, as can be found in, e.g., O'Brien (1986), Bacinello (1988) and Dufresne (1989). The topic of asset allocation is studied from different perspectives in the literature as well. By using a simple stochastic model, Cairns et al. (2006) incorporate asset, salary and interest rate risk in the derivation of optimal investment strategies. While many actuarial papers analyze risk from demographic changes, financial risk in pension funds is less extensively considered in the existing literature: Most recently, by integrating assets and liabilities as well as solvency requirements, Berdin and Gründl (2015) consider a representative German life insurer and its asset allocation and outstanding liabilities. Generating a stochastic term structure of interest rates and stock market returns, the authors simulate investment returns in a multi-period setting. Based on empirically calibrated parameters, the evolution of the balance sheet is observed with a special focus on the solvency situation. Looking at participating life insurance contracts, Schmeiser and Wagner (2014) try to find a suitable value for the guaranteed interest rate. Their results show that as the risk-free interest rate approaches the guaranteed one, the equity falls to zero, as there is no longer any benefit from risky investments. This is relevant for pension funds too, as a minimum interest rate must be credited annually to the accounts of the contributors (see Broeders et al., 2011; Mirza and Wagner, 2016). A study of the impact of product features and contributor types on lapse in life insurance contracts can be found in Eling and Kiesenbauer (2013). Using a data set from a German life insurer, they conclude that the contract age and the premium type have the most important impact on the lapse rate. An analysis of the relationship between the liability structure and the asset allocation of defined benefit pension funds is performed by Alestalo and Puttonen (2006). Examining data from Finnish pension funds, the authors find that the liability structure does indeed influence the asset allocation, with the age structure of the members being one source of correlation. Examining data from different countries, Ghilarducci (2010) finds that there is a positive correlation between the spending for pensions and education. By combining a stochastic pension fund model with a traffic light approach, Braun et al. (2011) measure the shortfall probability of Swiss occupational pension funds in order to assist stakeholders in making decisions. Examining Dutch pension funds, Broeders et al. (2016) find empirical evidence for herding behavior in the asset allocation of institutions. By analyzing the optimality of supervisory rules, Chen and Clever (2015) show that both the security mechanisms and risk measures used by regulators influence the optimality of the regulations. The optimality of the boundaries used for the objective funding ratio and the optimal dividends are also discussed by Gerber and Shiu (2003). Finally, in a recent working paper, and closest to our undertaking, Avanzi et al. (2016) formally analyze the iteration of surplus dividend payments and the funding ratio of pension funds.
Recent statistical and industrial publications in Switzerland consider the current state of pension funds from a practical point of view. Often, they focus on the ongoing reforms, underline challenges that the system is facing, and discuss relevant funding ratios or intergenerational wealth transfers. An overview of the situation can be found in Albrecher et al. (2016). Some authors analyze the financial situation of funds, discuss possible reforms and ways to go forward (Bischofberger and Walser, 2011). Eling (2012) considers the current wealth distribution and transfer mechanisms among young and old generations in Switzerland. The aging population and the long-term (financial) perspectives are also in the focus of UBS (2014). In his recent book, Cosandey (2014) discusses reforms for fair intergenerational mechanisms and justice.

Our research aims at building on and extending the current state of knowledge by considering the framework of private Swiss pension funds, accounting for the currently changing environmental conditions, and including both the asset and liability perspectives. Using stochastic simulations and considering an

individual contributor's account, we construct a model to assess the extra contribution and the surplus distribution mechanisms of defined contribution pension plans.[3] From the institutional perspective, the funding ratio and the stability of the fund are taken into consideration. In particular, we look at the changes in the funding ratio over time for different funding mechanisms. This includes limits for the distribution of bonuses and methods for determining the additional contributions. We analyze what leads to greater stability of the fund and to higher final contributor utility. We study several scenarios for the capital market returns in order to examine the fund's ability to cope with periods of low and high market returns. We obtain the distribution of the final payoff and its sensitivity to the different mechanisms.
Our results indicate that the distribution of bonuses is connected to a higher volatility of the account value at retirement. For risk averse individuals, this leads to a decrease of their relative certainty equivalent. Thus, from an insured's perspective, it is typically favorable not to get surpluses credited to the account during the contract period. With respect to pension fund policies, we believe that funds should consider measures that help reducing the volatility in the outcomes (e.g. by distributing less, less often, or accumulating larger reserves before doing so). From this increased stability, the insured as well as the fund would be able to profit. More specifically, our main findings are as follows. First, it is deduced that charging remediation measures helps secure the stability of the fund in years following low market returns. Funds in good health can distribute bonuses to their clients while still maintaining their good state. However, this may be detrimental to the utility of the insured since remediation measures may be required afterwards. For these methods to be fully effective, the right choice of parameters is crucial, as our sensitivity analysis shows. Long-lasting periods of low returns have a strong impact on the fund because remediation and bonus measures influence the contributor's payoff.

The remainder of the Chapter is structured as follows. The second section introduces the model framework and explains the processes that take place. The implementation and choice of parameters are given in the third part. Section 4 covers the numerical analysis. This includes several funding mechanisms and a sensitivity analysis of the results. Additionally, the accounts of the insured at interim time points and capital market scenarios are studied. The final section discusses the results and concludes.

3.2 Model Framework

To properly control for actuarial gains and losses over time, a scenario-based stochastic approach seems natural. By performing numerical simulations, we examine how the accounts of members evolve over time. For this, we look at an individual model contributor in a multi-period setting and take the simplified balance sheet approach comparable to that in Eling and Holder (2013) or Broeders et al. (2011), which is depicted in Figure 3.1.

Assets	Liabilities
Assets A_t	Accumulated Contributions C_t
Remediation measures K_t	Bonuses B_t

Figure 3.1: Simplified balance sheet in time t.

The annual contributions of an individual insured in time periods $t = 0, \ldots, T$ increase the pension assets (A_t) and lead to a liability changing over time (cf. Figure 3.2) linked directly to the contribu-

[3] The work in this thesis mostly applies to defined contribution pension funds. Other schemes, in particular defined benefit ones, exist in Switzerland as well.

tions (C_t). Additional contributions (cf. remediation measures K_t introduced below) are accounted to the asset side while bonus payments (B_t) account for the liabilities. We model a pension fund by simulating the assets of the fund limited to an individual contributor. The assets follow a stochastic process for the rates of return at time t. The liabilities evolve according to the legally fixed minimum interest rate r_{PL}.

For every year that has passed, the fund compares how assets and liabilities relate to each other. This involves looking at the funding ratio $F_t = (A_t + K_t)\,/\,(C_t + B_t)$ (Equation (3.8)), which is the key regulatory measure for Swiss pension funds (BVV2, Art. 44). Depending on F_t being higher or lower than some predefined threshold, actions are modeled along predefined mechanisms (cf. Sections 3.2.2 and 3.3.3).

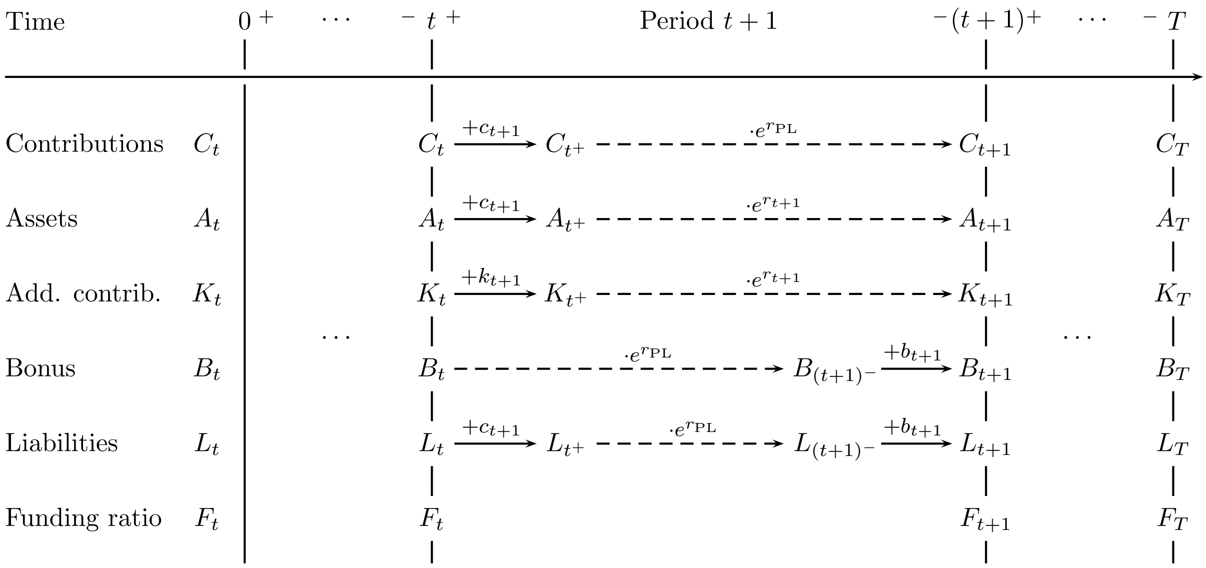


Figure 3.2: Illustration of the contract variables and the cash flows during the saving period along the contract timeline from time $t = 0$ until T. In our model t^+ denotes the beginning of period $t + 1$ and $(t + 1)^-$ the end of it. Contributions are credited at the beginning of the period (t^+) just after the calculation of the funding ratio in t. Bonus payments are granted in $(t + 1)^-$ and thereafter the end-of-period funding ratio is evaluated (time $t + 1$). For the description of the variables see Sections 3.2.1 and 3.2.2.

3.2.1 Contributions, Asset Evolution and Funding Ratio

We consider an active insured at age x contributing during its working time. The time horizon covers the T periods, i.e., times $t = 0, \ldots, T$. Adjustments of the key variables occur at the beginning and end of a period, i.e., in t^+ and $(t + 1)^-$.

Basic Contributions The annual contributions c_t depend on the salary and conversion factors linked to the age of the insured. From the salary S_{t+1} in period $t + 1$, the coordinated period salary $\hat{S}_{t+1}$ is calculated by subtracting the coordination deduction S_t^{cd}. Only the part between a minimum value $\hat{S}_t^{\min}$ and a maximum $\hat{S}_t^{\max}$ falls under the legal minimum rates (BVG, Art. 8). The coordinated salary in period $t + 1$, $t = 0, \ldots, T - 1$, is

$$\hat{S}_{t+1} = \min\left\{\max\left\{S_{t+1} - S_t^{\text{cd}}; \hat{S}_t^{\min}\right\}; \hat{S}_t^{\max}\right\}. \tag{3.1}$$

The contribution c_{t+1} is determined by multiplication of the coordinated salary $\hat{S}_{t+1}$ with a contribution rate $f^{\mathrm{S}}_{x,t}$, depending on the age x and the time $t = 0, \ldots, T-1$ (BVG, Art. 16), i.e.,

$$c_{t+1} = f^{\mathrm{S}}_{x,t} \cdot \hat{S}_{t+1}. \tag{3.2}$$

Contributions are assumed to be paid at the beginning of each period, i.e., for period $t+1$, the contribution c_{t+1} is paid at time t^{+}. While in practice the payment of c_{t+1} is split up between employees and employers, we do not distinguish the origin of the contributions in our model. The member is assumed to be concerned by the total account value (see also Section 3.2.3). During every period, the member receives a minimum interest rate r_{PL} on the sum of its contributions C_t. The value at time $t+1$ is

$$C_{t+1} = (C_t + c_{t+1}) \cdot e^{r_{\mathrm{PL}}}, \tag{3.3}$$

with $C_0 = 0$.

Asset Evolution The assets A_t represent the funds that are available for paying the liabilities towards the members. They consist of the paid contributions and the returns from investing them. As for C_t, c_{t+1} is added to the assets at the beginning of each period, i.e.,

$$A_{t^+} = A_t + c_{t+1}, \tag{3.4}$$

with $A_0 = 0$. The fund must be self-financing with the contributions and capital market earnings. To simulate the return from the capital market, a basic stock model is applied. We use a geometric Brownian motion with drift μ_{B} and volatility σ_{B}, i.e.,

$$dA_t = \mu_{\mathrm{B}} A_t dt + \sigma_{\mathrm{B}} A_t dW_t, \tag{3.5}$$

where $W = (W_t)_{t \geq 0}$ is a standard Brownian motion (Björk, 2004). The return in period $t+1$, then, is

$$r_{t+1} = \ln\left[\frac{A_{t+1}}{A_t^{+}}\right] = \mu_{\mathrm{B}} - \frac{\sigma_{\mathrm{B}}^2}{2} + \sigma_{\mathrm{B}} \cdot N_{0,1}, \tag{3.6}$$

with $N_{0,1}$ a standard normally distributed random variable. At the end of the period, the assets A_{t+1} are

$$A_{t+1} = A_{t^+} \cdot e^{r_{t+1}} = (A_t + c_{t+1}) \cdot e^{r_{t+1}}. \tag{3.7}$$

Funding Ratio The funding ratio is determined by dividing the total assets, i.e., equity and additional contributions over the liabilities. Additional contributions may be due in case of periods with underfunding (cf. Section 3.2.2). The funding ratio therefore is

$$F_{t+1} = \frac{A_{t+1} + K_{t+1}}{L_{t+1}} = \frac{A_{t+1} + K_{t+1}}{C_{t+1} + B_{t+1}}, \tag{3.8}$$

where the liabilities L_{t+1} equal the contributions C_{t+1} and surpluses B_{t+1}. Bonuses are distributed when the fund is in good health (cf. Section 3.2.2).

The funding ratio is calculated at the end of period $t+1$. Depending on its value, it is decided whether bonuses can be distributed or whether additional contributions need to be charged. A value below 100% corresponds to underfunding, while $F_{t+1} > 100\%$ means overfunding.

3.2.2 Monitoring of the Funding Ratio and Funding Mechanisms

When the funding ratio exceeds a certain threshold, the surpluses can be distributed to the members. In the case of underfunding, remediation measures may be necessary. The concept of using boundaries for the funding ratio, that we apply therein, has been discussed earlier by, e.g., Gerber and Shiu (2003). In the following, the corresponding mechanisms, that are used in our analysis, are explained.[4]

Situation of Underfunding and Additional Contributions

We consider a procedure that automatically determines recovery contributions. In practice this is not an automated process. The board of the fund evaluates the underfunding with consideration of the fund's overall situation (e.g., market environment, investment portfolio, characteristics of the members). If recovery measures have been decided upon, the employers of the insured may also be involved in covering deficits (BVG, Art. 65d).

Once F_t drops below 100%, the assets do not suffice to meet the obligations. Therefore, the insured may be requested to pay additional contributions. Additional contributions are paid at the beginning of the following year. We present two methods for calculating the additional contributions k_{t+1} in period $t+1$. Our first method (UF1) is based on a share z of the funding gap $L_t - (A_t + K_t)$. This comes into action once the funding ratio drops below a limit $F_{\min}$. Additional contributions at time t^+ then are

$$k_{t+1} = z \cdot (L_t - (A_t + K_t)), \tag{3.9}$$

where z represents the share of $L_t - (A_t + K_t)$ to be paid.

The second method (UF2) is based on a Value-at-Risk approach.[5] Here, k_{t+1} is set such that at the end of the next period, the funding ratio falls below 100% only with probability q. The probability q is typically a small number. In the Solvency II regulation for private insurances, for example, q is set to 0.5%. In our sensitivity analyses, we use a 1% to 10% one-year underfunding probability threshold (see Table 3.3). In our model, realizations of the funding ratio are evaluated through

$$\hat{F}_{t+1}(k_{t+1}) = \frac{(A_t + c_{t+1}) \cdot e^{r_{t+1}} + (K_t + k_{t+1}) \cdot e^{r_{t+1}}}{(L_t + c_{t+1}) \cdot e^{r_{\mathrm{PL}}}}, \tag{3.10}$$

where r_{t+1} is a realization of the asset return in the following period. Thus, additional contributions k_{t+1} must satisfy the equation

$$VaR_{1-q}\left(1 - \hat{F}_{t+1}\right) = \inf\left\{x \,\middle|\, \mathbb{P}\left(\hat{F}_{t+1}(k_{t+1}) < 100\% - x\right) \le q\right\} \stackrel{!}{=} 0, \tag{3.11}$$

with $\stackrel{!}{=}$ denoting that the Value-at-Risk has to be equal to zero. Thus, the equation

$$\inf\left\{x \,\middle|\, \mathbb{P}\left(\frac{(A_t + c_{t+1}) \cdot e^{r_{t+1}} + (K_t + k_{t+1}) \cdot e^{r_{t+1}}}{(L_t + c_{t+1}) \cdot e^{r_{\mathrm{PL}}}} < 100\% - x\right) \le q\right\} = 0 \tag{3.12}$$

[4] In contrast to life insurance companies, regulations such as Solvency II and the Swiss Solvency Test (SST) do not apply to Swiss pension funds. The reason why a transfer of these regulations has not been performed yet can be found in the differences between funds and insurers. In contrast to insurance companies, gains and losses are distributed among the members. Additionally, the contractual relationship between the member and the pension fund is quite rigid. For example, employees are automatically affiliated in the pension plan connected to the employer. Due to these characteristics, a temporary phase of underfunding can be dealt with. Pension funds stay in business and pursue their investment strategies even when they are underfunded. Also, it is the decision of the board of the fund if, and to what extent, remediation measures and surplus payments are to be made. This stands in strong contrast to life insurance companies regulated by market authorities that require strict solvency calculations and adequate capitalization on a year-to-year basis. While there have been efforts to suggest regulations comparable to Solvency II and the SST for pension funds (see, e.g. Schweizerische Kammer der Pensionskassen-Experten, 2012; Braun et al., 2011), there are currently no regulations with respect to this.

[5] Note that in practice, the use of method (UF1) is more common among Swiss pension funds. Furthermore, it is the board of the fund that ultimately decides on when charging remediation measures as well as on their amount.

needs to be solved numerically for k_{t+1}.[6] The intuition behind this is that higher additional contributions k_{t+1} increase the funding ratio and decrease the probability of financial distress, i.e., the probability that the funding ratio falls below 100% in the next period. We therefore aim to find the smallest value for k_{t+1} for which the probability that the funding ratio falls below 100% ($x \stackrel{!}{=} 0$) is smaller or equal to a small value q. The contributions k_{t+1} are added to the assets, but not to the account of the insured. Because K_t is invested on the capital market, the return in period $t+1$ is r_{t+1}. We have

$$K_{t+1} = K_{t^+} \cdot e^{r_{t+1}} = (K_t + k_{t+1}) \cdot e^{r_{t+1}}. \tag{3.13}$$

Situation of Overfunding and Surplus Distribution

In years where market returns exceed r_{PL}, the assets of the pension fund grow. Part of the surplus can be distributed to the members (BVG, Art. 68a). Additionally, it is required that the fund holds a reserve in case of fluctuation of the assets (BVV2, Art. 48e). Therefore, we assume for our base scenario (see, e.g., the cases shaded in gray in our sensitivity analysis, Table 3.3) that surpluses can only be distributed once a certain reserve on the liabilities has been accumulated (see the definition of the parameters in Table 3.1 and Footnote 14 in Section 3.3.3). We assume that a bonus b_{t+1} is paid out at the end of a period if $F_{(t+1)^-}$ exceeds a limit F^{L}_{t+1}.[7] Because it represents an obligation, the sum of surpluses B_{t+1} is part of the liabilities. Such payments cause a drop of the funding ratio. We assume that b_{t+1} is chosen such that from the threshold F^{L}_{t+1}, the decrease equals ΔF_{t+1}. Subsequently, b_{t+1} is derived from

$$F_{t+1} = \frac{A_{t+1} + K_{t+1}}{C_{t+1} + B_{(t+1)^-} + b_{t+1}} \stackrel{!}{=} F^{\mathrm{L}}_{t+1} - \Delta F_{t+1}. \tag{3.14}$$

The bonuses B_t grow with the rate r_{PL} and their value at time $t+1$ is

$$B_{t+1} = B_{(t+1)^-} + b_{t+1} = B_t \cdot e^{r_{\mathrm{PL}}} + b_{t+1}. \tag{3.15}$$

While in our model B_t increases with a rate of r_{PL}, this is not required in practice, as bonuses do not fall under the legal minimum (see, e.g., Avanzi and Purcal (2014)). In the above, we assume that surpluses are paid out as a lump sum. In practice, it is more common to assign bonuses as increased interest rates on the insured's account. The methods can be converted into each other, i.e.,

$$L_{t+1} = (L_t + c_{t+1}) \cdot e^{r_{\mathrm{PL}}} + b_{t+1} \equiv (L_t + c_{t+1}) \cdot e^{r_{\mathrm{PL}} + r_{\mathrm{b}}} = (L_t + c_{t+1}) \cdot e^{r_{\mathrm{eff}}}, \tag{3.16}$$

with

$$r_{\mathrm{eff}} = r_{\mathrm{PL}} + r_{\mathrm{b}} \geq r_{\mathrm{PL}}. \tag{3.17}$$

In our discussion, we focus on the lump sum payments.

3.2.3 Contributor Valuation

To evaluate the payoff and utility of the members, we use several indicators. To assess the contributor's return on its contributions, we use the internal rate of return r_{c+b+k} defined as follows: If the insured were only to receive the return r_{c+b+k} on its regular contributions, the account value would equal the

[6] For solving Equation (3.12), we use a numerical root-finding algorithm. A reliable and quick method is, e.g., the method proposed by Brent (1974).

[7] This can be compared with the dividend distribution analyzed in Avanzi et al. (2016).

value of contributions and bonuses minus remediation measures at time T, i.e.,

$$\sum_{t=1}^{T} c_t \cdot e^{r_{c+b+k}\cdot(T-t+1)} \overset{!}{=} C_T + B_T - K_T.^{8} \tag{3.18}$$

To measure the utility, we use the certainty equivalent $u^{-1}\left(\mathbb{E}\left[u\left(L_T\right)\right]\right)$. For this, we use a constant relative risk aversion utility as introduced, e.g., in Broeders et al. (2011),

$$u(x) = \frac{x^{1-\rho}}{1-\rho}, \text{ with } \rho > 0,\ \rho \neq 1. \tag{3.19}$$

In order to take the amounts paid in the various cases (regular contributions and irregular remediation measures) into account in our analysis, we focus on the relative certainty equivalent given by

$$\frac{u^{-1}\left(\mathbb{E}\left[u\left(L_T\right)\right]\right)}{C_T + \mathbb{E}\left[K_T\right]}. \tag{3.20}$$

3.3 Implementation and Parameterization

In order to simulate the course of the assets we use Monte Carlo simulations. The results are obtained using $N = 100\,000$ realizations in every period. We first introduce a reference case setting with the starting values for the various model parameters. The parameters are summarized in Table 3.1 and described in the following.

Parameter	Variable	Value
Number of periods	T	40
Legislation		
Coordination deduction (at time $t = 0$)	S_0^{cd}	CHF 24 675
Minimum coordinated salary ($t = 0$)	$\hat{S}_0^{\min}$	CHF 3 525
Maximum coordinated salary ($t = 0$)	$\hat{S}_0^{\max}$	CHF 59 925
Total contribution rate of age class 25 − 34	$f_{x,t}^{\text{S}}$	7%
Total contribution rate of age class 35 − 44	$f_{x,t}^{\text{S}}$	10%
Total contribution rate of age class 45 − 54	$f_{x,t}^{\text{S}}$	15%
Total contribution rate of age class 55 − 65	$f_{x,t}^{\text{S}}$	18%
Salary growth rate	r_{S}	1%
Minimum interest rate	r_{PL}	1.25%
Risk-free interest rate	r_{f}	1%
Capital market		
Drift of the geometric Brownian motion process	μ_B	3.0%
Volatility of the geometric Brownian motion process	σ_B	6.0%
Pension fund governance		
Minimum funding ratio	$F_{\min}$	100%
Proportion of missing assets to be paid	z	90%
Quantile for additional contributions	q	1%
Upper bound for distributing surpluses	$F_{t+1}^{\text{L}} \equiv F^{\text{L}}$	110%
Difference of bonus bounds	$\Delta F_{t+1} \equiv \Delta F$	2%
Utility of the member		
Risk aversion	ρ	30

Table 3.1: Input parameters for the reference case.

[8] While the internal rate of return is commonly used in practice, it has certain drawbacks. Those include, among others, assumptions on the reinvestment. For further details, see e.g. Lorie and Savage (1955).

3.3.1 Legislation

We consider one type of insured working from age 25 until retirement at 65, corresponding to $T = 40$ working years. Their salary starts at CHF 55 000 (first period) and grows linearly to CHF 82 300 (present value), corresponding to the Swiss average at time zero. Additionally, the salary grows with a rate of $r_{\mathrm{S}} = 1\%$ per year in order to reflect the increase of salaries related to the increase of prices.[9] The pension fund contributions are set by the legislator. The coordination deduction S_0^{cd} for 2016 is CHF 24 675. The minimum and maximum coordinated salaries $\hat{S}_0^{\min}$ and $\hat{S}_0^{\max}$ are CHF 3 525 and CHF 59 925 (BVV2, Art. 5). They are adapted over time with the rate r_{S}. The contribution factor $f_{x,t}^{\mathrm{S}}$ changes with the age of the contributor. We consider total contributions, i.e. the factors correspond to the contributions by employers and employees.[10] Since the salary S_t and the contribution rates $f_{x,t}^{\mathrm{S}}$ grow with age and time, c_t is higher in later years. For 2016, the minimum interest rate r_{PL} is 1.25% (BVV2, Art. 12).[11] We first use this as a constant value as we do for the distribution of the capital market returns, since the general interest environment remains unchanged. When studying the capital market scenarios in Section 3.4.4, we allow for variations in r_{PL}, however (see also Footnote 15). For the risk-free interest rate r_{f}, a value of 1% is chosen. This is related to r_{S}, the increase in prices. It follows that $\sum_{t=1}^{40} c_t / \sum_{t=1}^{40} \hat{S}_t$ is constant and equals 13.71%. When no bonuses are paid, a fixed value for r_{PL} leads to constant liabilities at retirement, i.e. $L_{40} = 361\,212$.

3.3.2 Capital Market

To calibrate the parameters of the asset process, the LPP-40 sub-index of the Pictet LPP 2000 index is utilized. With an equity portion of 40%, this index is close to the average investments of larger pension funds in Switzerland. It also contains approximately 40% of foreign currency investments.[12] We parametrize a geometric Brownian motion based on the annualized average monthly performance from January 2000 to December 2015, i.e., we calculate the index performance using monthly data and choose the (rounded) annualized values of $\mu_{\mathrm{B}} = 3\%$ for the drift and $\sigma_{\mathrm{B}} = 6\%$ for the volatility.[13]

3.3.3 Pension Fund Governance and Utility of the Members

If F_t falls below $F_{\min}$, the fund can ask for remediation measures. In the reference case, we use a lower limit of $F_{\min} = 100\%$ (legal minimum). For our first method, a proportion $z = 90\%$ of $L_t - A_t$ is used. The purpose of this is to reduce the one-time capital outlay for the members and spread the remediation expenses over a longer time. The second method is based on a Value-at-Risk approach. The additional payment k_t is set such that in the following period, the fund is underfunded with a probability of $q = 1\%$. Once the funding ratio exceeds F_{t+1}^{L}, a bonus can be distributed. We use a constant upper limit of $F_{t+1}^{\mathrm{L}} \equiv F^{\mathrm{L}} = 110\%$[14] and assume that surpluses are distributed until the funding ratio has

[9] This corresponds to the historical salary changes also found in the adaptations of the BVV2 salary boundaries.

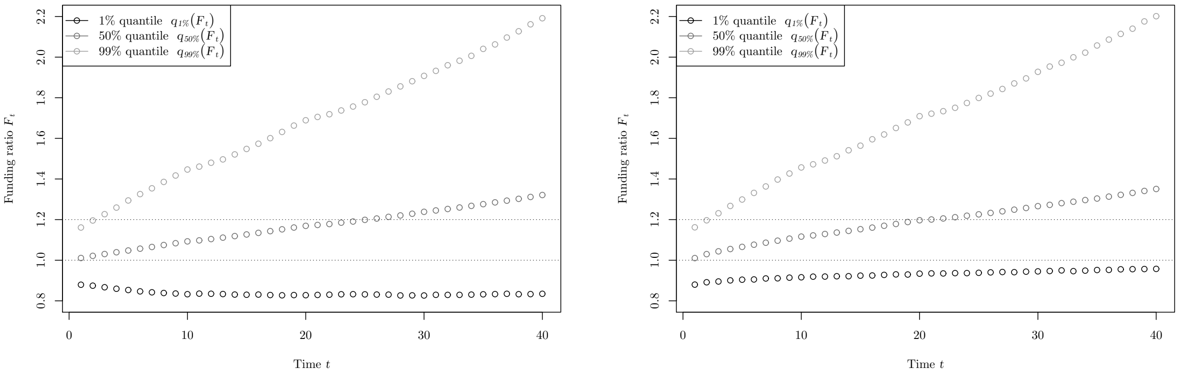
[10] In our analysis, we do not differentiate between the sources of the contributions, but we focus on the total payoff at time T.

[11] In our work we use the term $e^{r_{\mathrm{PL}}}$ for crediting the minimum guaranteed interest rate. This represents an approximation in continuous time of the value $1 + r_{\mathrm{PL}}$ in discrete time that is required by law. The minimum interest rate is adjusted periodically and equals 1% for the year 2018.

[12] The composition of the index is 60% bonds and 40% equities, with about 40% of the investments made in foreign currencies. For further information, see `https://www.group.pictet/corporate/en/home/institutional_investors/lpp_indices/lpp2000.html`, September 2016.

[13] We chose to use annualized values based on the monthly observations to have a larger statistical basis (192 observations). The annualized expected return is calculated from the monthly expected return by multiplying by 12. The corresponding volatility is obtained from multiplication by $\sqrt{12}$. For comparison, when calculating the performance on the base of the only 16 annual data points, we find that the expected return remains unchanged and yields 3% while the volatility is about 2% higher in the considered period.

[14] In our base case, bonuses can only be distributed if the funding ratio exceeds 110%, i.e. when reserves of 10% on top of the value of the liabilities are accumulated. This reference scenario corresponds to the target values mostly observed in practice (5% to 10%). In our sensitivity analysis, we vary $F^{\mathrm{L}} = 110\%$ through very low and high values ranging from 102%



(a) Quantiles of the funding ratio F_t in case (A)   (b) Quantiles of the funding ratio F_t in case (B)

Figure 3.3: Illustration of the funding ratio F_t in the cases (A) (no additional contributions, no bonus payments) and (B) (additional contributions, no bonus payments). The graphs depict the 1%, 50% and 99% quantiles of F_t. The parameters are as in the reference case given in Table 3.1.

decreased to $F^{\mathrm{L}} - \Delta F = 108\%$, corresponding to $\Delta F_{t+1} \equiv \Delta F = 2\%$. In the insured's utility, we use a risk aversion factor of $\rho = 30$.

3.4 Numerical Analysis and Discussion

3.4.1 Funding Mechanisms: Impact over Time

We assess the impact of remediation measures and surplus distributions by analyzing F_t, k_t and b_t for

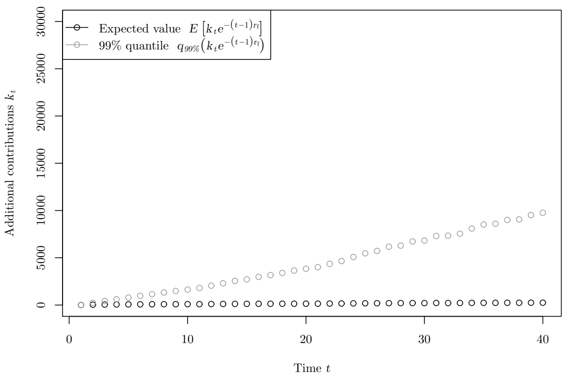
- case (A), with neither additional contributions nor surplus distribution,
- case (B), with only additional contributions, and
- case (C), with both remediation measures and surplus distribution.

The additional contributions are calculated along method (UF1). The parameters are as in Table 3.1.

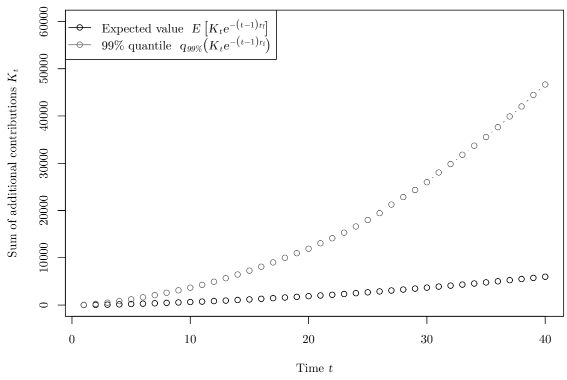
Funding Ratios and Remediation Measures In Figure 3.3, the 1%, 50% and 99% quantiles of F_t are given for the cases (A) and (B). In Figure 3.3(a) (case A), the 1% quantile of F_t always stays below 100%. It starts at approximately 90% and subsequently drops to about 80%. The 50% quantile $q_{50\%}(F_t)$ starts at almost 100% and grows to about 130% in $t = 40$. This drift is related to the difference between μ_B and r_{PL}. In case (B), in Figure 3.3(b), the additional contributions only affect the 1% quantile, now reaching almost 100% in T. Remediation measures lead to an improvement of underfunding situations, while the other quantiles remain unchanged. The 99% quantile grows from nearly 120% to 220% making the distribution of surpluses possible.

Expected Remediation Measures For case (B), Figure 3.4 shows the development of k_t and K_t. Figure 3.4(a) shows the expected present value of k_t and the 99% quantile. The 99% quantile of the additional contributions grows and reach more than CHF 10 000 at time T. The present expected value stays close to zero. As in Figure 3.3, the fund is always overfunded in the upper 50% of all cases.
For the present value of K_t (see Figure 3.4(b)), the 99% quantile reaches about CHF 50 000. The shape of the $q_{99\%}\left(K_t \cdot e^{-(t-1)r_{\mathrm{f}}}\right)$ curve follows from the additional contributions being credited with the returns r_{t+1}. The expected value exceeds zero (see line 7 in Table 3.3 from the sensitivity analysis

to 118% corresponding to reserves of 2% to 18% of the liabilities (see Table 3.3).



(a) Present value and 99% quantile of the additional contributions k_t in case (B)



(b) Present value and 99% quantile of the sum of additional contributions K_t in case (B)

Figure 3.4: Illustration of the discounted remediation measures k_t and their sum K_t in case (B) (additional contributions, no bonus payments). The graphs depict the expected present value and the 99% quantile of k_t and K_t. The parameters are as in the reference case in Table 3.1.

in Section 3.4.2). The expected remediation measures discounted to time zero $\mathbb{E}\left[k_t \cdot e^{-(t-1)r_f} | k_t > 0\right]$ amount to CHF 1 520. They are levied four times on average (cf. Table 3.3). The overall expected payments are approximately CHF 6 080.

Surplus Distribution For case (C), Figure 3.5 depicts the 1%, 50% and 99% quantiles of F_t from periods 1 to 40. Distributing surpluses leads to the 1% quantile $q_{1\%}(F_t)$ being around 90%. The 99% quantile $q_{99\%}(F_t)$ lies at 110%, which equals F^L. The 50% quantile $q_{50\%}(F_t)$ starts at approximately 100% and converges to about 107% (below $F^L - \Delta F = 110\% - 2\% = 108\%$).

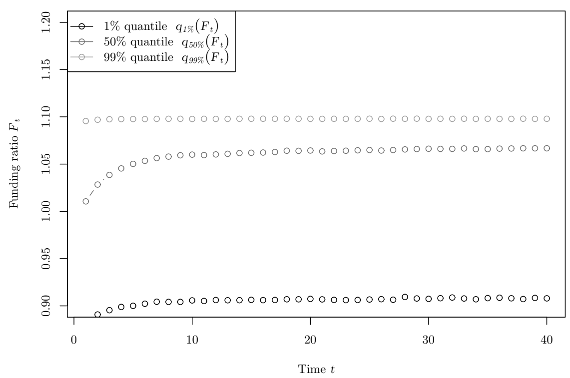
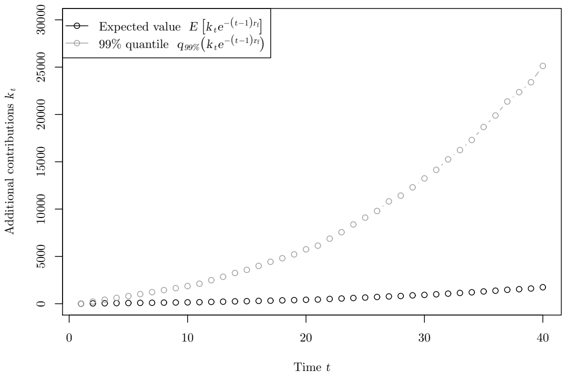
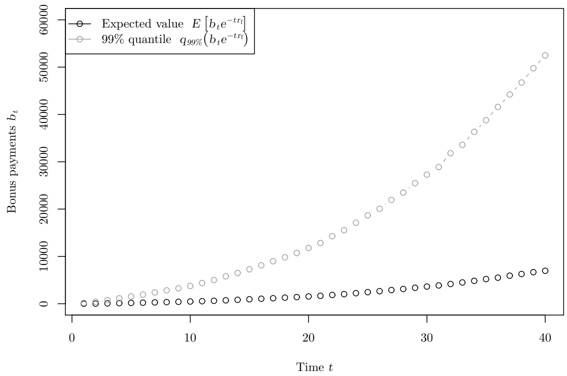


Figure 3.5: Illustration of the funding ratio F_t in case (C) (additional contributions, bonus payments). The graph depicts the 1%, 50% and 99% quantiles of F_t. The parameters are as in the reference case given in Table 3.1.

Additional Contributions and Bonus Payments Figure 3.6 depicts the expected present value and the 99% quantile of k_t and b_t. The expected present value of k_t exceeds CHF 3 000 at time T, whereas b_t reaches approximately CHF 8 000. The surpluses that are paid out on average are thus approximately double of what needs to be paid in remediation cases. A similar conclusion can be drawn for the quantile values where the surpluses of best cases are twice the value of remediation measures. Due to the drift of the investment process, they increase exponentially. In T, $\mathbb{E}\left[k_t \cdot e^{-(t-1)r_f}\right]$ reaches almost CHF 250 while $\mathbb{E}\left[b_t \cdot e^{-t\, r_f}\right]$ is about CHF 10 000. This matches the observations made regarding the funding ratio where $q_{50\%}(F_t)$ is around 107%. With the given bonus bounds, $\mu_B =$ 3% and $r_{PL} = 1.25\%$, the fund is likely to be overfunded, i.e. b_t grows faster than k_t. This causes higher volatility in the payout stream, i.e., despite bonuses in some years, additional contributions need to be made in others (cf. discussion in Section 3.4.2).



(a) Present value and 99% quantile of the additional contributions k_t in case (C)



(b) Present value and 99% quantile of the bonus payments b_t in case (C)

Figure 3.6: Illustration of the discounted extra contributions k_t and the bonus payments b_t in case C (additional contributions, bonus payments). The graphs depict the expected present value as well as the 99% quantile of k_t and b_t. The parameters are as in the reference case given in Table 3.1.

3.4.2 Sensitivity Analysis

We study key indicators at time $T = 40$ from the contributor's and the fund's perspectives and analyze how sensitive the results are to parameter changes. In Table 3.3, the columns labeled 1 to 7 contain the input values, columns 8 to 13 the insured's perspective, 14 to 19 the funding levels and 20 to 25 the surplus distribution and the remediation measures. A detailed explanation of the columns is given in Table 3.2.

In the first line of Table 3.3, case (A) with neither remediation measures nor surplus distributions is analyzed. Rows 2 to 11, show case (B) with additional contributions. This includes changing $F_{\min}$ for method (UF1) and q for (UF2). Lines 12 to 32 cover case (C) with remediation measures and bonus payments. There, F^{L} is altered first. Next, we vary the difference of bonus bounds ΔF. Subsequently, the parameters $F_{\min}$ and q are changed as for case (B). When changing one variable, the others are kept constant (cf. Table 3.1).

	Item	Description
1	Case	(A) no additional contributions, no bonus payments, (B) only additional contributions, (C) additional contributions and bonus payments (Section 3.4.1)
2	UF (indicator)	Calculation method for remediation measures: 1 = percentage of funding ratio, 2 = Value-at-Risk approach (Section 3.2.2)
3	$F_{\min}$	Minimum funding ratio targeted (Equation (3.8))
4	q	Probability for the funding ratio to fall below 100% within one year (Equation (3.11))
5	Bonus (indicator)	Use of surplus distribution: 0 = no bonus payments, 1 = bonus is distributed (Section 3.2.2)

6	F^{L}	Upper bound for distributing bonus (Equation (3.14))
7	ΔF	Difference of bonus bounds (Equation (3.14))
8	$\mathbb{E}\left[L_{40}\right]$	Expected liabilities in $t=40$
9	$\frac{\sigma[L_{40}]}{\mathbb{E}[L_{40}]}$	Relative volatility of L_{40}
10	$\gamma\left[L_{40}\right]$	Skewness of L_{40}
11	$u^{-1}\left(\mathbb{E}\left[u\left(L_{40}\right)\right]\right)$	Certainty equivalent in $t=40$ (Equation (3.19))
12	$\frac{u^{-1}(\mathbb{E}[u(L_{40})])}{C_{40}+\mathbb{E}[K_{40}]}$	Relative certainty equivalent in $t=40$
13	$\mathbb{E}\left[r_{c+b+k}\right]$	Internal rate of return (Equation (3.18))
14	$\mathbb{E}\left[F_t\right]$	Expected funding ratio
15	$\mathbb{E}\left[q_{1\%}\left(F_t\right)\right]$	Expected 1% quantile of the funding ratio
16	$\mathbb{E}\left[q_{50\%}\left(F_t\right)\right]$	Expected 50% quantile of the funding ratio
17	$\mathbb{E}\left[q_{99\%}\left(F_t\right)\right]$	Expected 99% quantile of the funding ratio
18	$q_{1\%}\left(\sum_{t=1}^{40} \mathbb{1}_{\{F_t<1\}}\right)$	1% quantile of the number of years in underfunding
19	$q_{50\%}\left(\sum_{t=1}^{40} \mathbb{1}_{\{F_t>F^L\}}\right)$	50% quantile of the number of years with F_t exceeding F^{L}
20	$\frac{\mathbb{E}\left[\sum_{t=1}^{40} k_t \cdot e^{-(t-1)r_{\mathrm{f}}}\right]}{\sum_{t=1}^{40} c_t \cdot e^{-(t-1)r_{\mathrm{f}}}}$	Ratio of the expected present sum of additional contributions over the sum of present regular contributions
21	$\mathbb{E}\left[\sum_{t=1}^{40} \mathbb{1}_{\{k_t>0\}}\right]$	Expected number of years with remediation measures required
22	$\mathbb{E}\left[k_t \cdot e^{-(t-1)r_{\mathrm{f}}} \mid k_t>0\right]$	Expected present value of remediation measures when required
23	$\frac{\mathbb{E}\left[\sum_{t=1}^{40} b_t \cdot e^{-t\,r_{\mathrm{f}}}\right]}{\sum_{t=1}^{40} c_t \cdot e^{-(t-1)r_{\mathrm{f}}}}$	Ratio of the expected present sum of distributed surpluses over the sum of present regular contributions
24	$\mathbb{E}\left[\sum_{t=1}^{40} \mathbb{1}_{\{b_t>0\}}\right]$	Expected number of years with surpluses required
25	$\mathbb{E}\left[b_t \cdot e^{-t\,r_{\mathrm{f}}} \mid b_t>0\right]$	Expected present value of distributed surpluses when required

Table 3.2: Description of the items in Table 3.3.

Impact of Minimum Funding Ratio $F_{\min}$ For case (A), the mean of the effective return $\mathbb{E}\left[r_{c+b+k}\right]$ in column 13 (Table 3.3) equals the minimum interest rate $r_{\mathrm{PL}} = 1.25\%$. Introducing remediation measures according to (UF1) in case (B) leads to higher overall payments. Since k_t is not credited to the liabilities, $\mathbb{E}\left[r_{c+b+k}\right]$ decreases from 1.16% for a lower boundary of $F_{\min}$ equal to 90% down to 1.10% for $F_{\min} = 100\%$. For the additional contributions, a lower value for $F_{\min}$ leads to fewer remediation payments (column 21). The average amount k_t paid is highest for a low value of $F_{\min}$ and decreases as the boundary is raised (column 22). Column 20 contains the ratio $\mathbb{E}[\sum_{t=1}^{40} k_t \cdot e^{-(t-1)r_{\mathrm{f}}}]/\sum_{t=1}^{40} c_t \cdot e^{-(t-1)r_{\mathrm{f}}}$: lower values of $F_{\min}$ lead to lower additional contributions, ranging from 2.6% for $F_{\min} = 100\%$ to 1.5% for $F_{\min} = 90\%$. While there are considerable changes in the amount and frequency of additional contributions, $\mathbb{E}[F_t]$ is nearly constant at approximately 1.20.

Value-At-Risk Approach For the Value-at-Risk approach (UF2) in case (B), the sensitivity is more important. The remediation measures resulting from $1-q$ increasing from 90% up to 99% are higher than for (UF1). Columns 21 and 22 show that additional contributions are more frequent. The payments remain at a similar level as for high values of $F_{\min}$. The $\mathbb{E}\left[\sum_{t=1}^{40} k_t \cdot e^{-(t-1)r_f}\right] / \sum_{t=1}^{40} c_t \cdot e^{-(t-1)r_f}$ ratio goes up to 8%. Due to the increased contributions, F_t increases too. With q decreasing from 10% to 1%, the mean of F_t grows from 1.24 to 1.29.

Impact of the Upper Distribution Limit F^{L} In case (C), significantly higher effective returns are observed. Changing F^{L} from 102% to 118% leads to fewer bonus payments (column 24). Since bonuses on average decrease by more than CHF 1 000, the ratio $\mathbb{E}\left[\sum_{t=1}^{40} b_t \cdot e^{-t\, r_f}\right] / \sum_{t=1}^{40} c_t \cdot e^{-(t-1)r_f}$ decreases too. Meanwhile, the amount and the frequency of additional contributions decrease. For $\mathbb{E}[r_{c+b+k}]$, a higher threshold for surplus distributions leads to a decrease. Overall, $\mathbb{E}[F_t]$ rises together with F^{L}.

Bonus Bounds ΔF As seen in columns 23 to 25, increasing ΔF from 0.01 to 0.06 leads to a strong increase in average bonuses. While the expected number of payments decreases, a growth of about 25% take place in $\mathbb{E}\left[\sum_{t=1}^{40} b_t \cdot e^{-t\, r_f}\right] / \sum_{t=1}^{40} c_t \cdot e^{-(t-1)r_f}$. Due to larger amounts being distributed, remediation measures become higher and more frequent. The expected return $\mathbb{E}[r_{c+b+k}]$ and the certainty equivalent $u^{-1}\left(\mathbb{E}\left[u\left(L_{40}\right)\right]\right)$ increase. While $\Delta F = 0.01$ leads to a certainty equivalent that equals approximately CHF 420 000, a change to $\Delta F = 0.06$ leads to an increase in $u^{-1}\left(\mathbb{E}\left[u\left(L_{40}\right)\right]\right)$ of about CHF 14 000.

Insured Perspective When raising the upper bound for distributing bonuses F^{L}, the certainty equivalent $u^{-1}\left(\mathbb{E}\left[u\left(L_{40}\right)\right]\right)$ decreases together with $\mathbb{E}[r_{c+b+k}]$ by more than 20% (Table 3.3, lines 12 – 16). Insureds should thus prefer lower values of F^{L}, coming along with regular bonus payments. This is supported by the relative certainty equivalent being the highest for $F^{\mathrm{L}} = 102\%$. Contrary movements follow from varying ΔF. While $u^{-1}\left(\mathbb{E}\left[u\left(L_{40}\right)\right]\right)$ grows together with it, the relative certainty equivalent (column 12) decreases by approximately 5%. This is due to higher bonus payments causing an increased need for remediation measures in years with lower market returns.

In fact, members should prefer that no bonuses are distributed, as can be seen in Figure 3.7. In this, the relative certainty equivalent is depicted depending on the difference of bonus bounds ΔF and the upper boundary for distributing bonuses F^{L} in case C with $F_{\min} = 100\%$. The different colors correspond to the values of the relative certainty equivalent $u^{-1}\left(\mathbb{E}\left[u\left(L_{40}\right)\right]\right)$. In this, darker colors represent a higher relative certainty equivalent and lighter ones a lower one. As the graph shows, the influence of F^{L} is bigger than that of ΔF as the changes when moving horizontally on the graph are much larger than the ones when moving vertically. This is consistent with the results of the sensitivity analysis in Table 3.3 (lines 12 – 22). Overall, it can be seen that the relative certainty equivalent is larger for smaller values of the upper bound F^{L}. Therefore, members should prefer that no bonuses are distributed, as this gives them the highest relative certainty equivalent. It can thus be said, that the common belief of clients profiting from surplus distributions is a fallacy. As we can see, the benefit from keeping additional funds as protection against times with lower capital market returns, is higher than the one from distributing them to the insureds. Pension funds should therefore put more stress on accumulating reserves rather than distributing funds to their clients.

Impact of Remediation Measures on Surpluses Varying $F_{\min}$, the changes for the additional contributions k_t correspond to case (B). With an increasing value for $F_{\min}$, additional payments become lower but more frequent. Consequently, the ratio $\mathbb{E}\left[\sum_{t=1}^{40} k_t \cdot e^{-(t-1)r_f}\right] / \sum_{t=1}^{40} c_t \cdot e^{-(t-1)r_f}$ of additional over regular contributions increases with a higher threshold $F_{\min}$ inducing an increase in $\mathbb{E}\left[L_{40}\right]$. Bonuses are, on average, distributed at approximately 1.3 more points in time. Together with an increase of $\mathbb{E}\left[b_t \cdot e^{-t\, r_f} | b_t > 0\right]$ by approximately 2%, $\mathbb{E}\left[\sum_{t=1}^{40} b_t \cdot e^{-t\, r_f}\right] / \sum_{t=1}^{40} c_t \cdot e^{-(t-1)r_f}$ grows by nearly 15% from one extreme to the other. For the Value-at-Risk approach, we see that for low values of q, both k_t

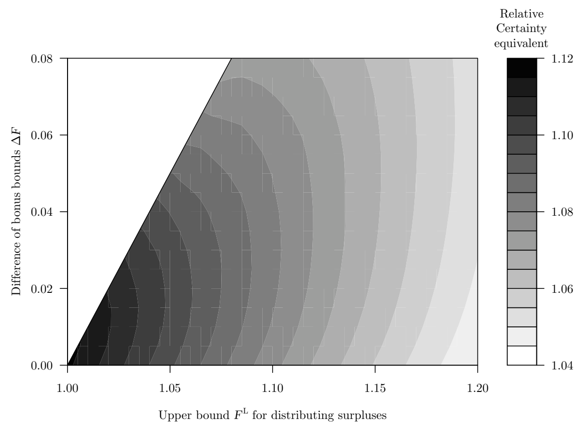


Figure 3.7: Illustration of the relative certainty equivalent depending on the difference of bonus bounds ΔF and the upper bound F^{L} for distributing bonuses.

and b_t grow considerably. For $q = 2.5\%$, $\mathbb{E}\left[\sum_{t=1}^{40} b_t \cdot e^{-t\, r_{\mathrm{f}}}\right] / \sum_{t=1}^{40} c_t \cdot e^{-(t-1) r_{\mathrm{f}}}$ exceeds 150%. The remediation measures grow similarly, reaching a ratio of approximately 249.1% for $q = 1\%$. This follows from the additional contributions causing a rise of the funding ratio. Subsequently, F_t often exceeds F^{L} and the additional contributions are redistributed as bonuses, leading to an increase in the certainty equivalent of approximately 80%. As the relative certainty equivalent drops down to almost 70%, this is unfavorable for insureds.

3.4.3 Interim Valuation

In today's working environment, employees change their jobs more frequently than before (see e.g. Cosandey, 2014). Often this is linked to a change of the pension fund associated with the employer. For the fund, remediation is most important. If contributors leave, they are entitled by law to receive their regular contributions and the minimum interest rate that has been paid. Remediation measures are only credited to the assets A_t but not to the accounts and remain with the fund. For our model, bonuses are credited to the accounts of the members and therefore remain with the insured when leaving. Exceptions may apply in the case of a partial liquidation (see, e.g., FZG, Art. 23), where bonuses could be canceled by required remediation measures. We examine the results after 10 years (insured aged 35) and 20 years (age 45) and compare them with the ones at retirement for the scenarios (A), (B) and (C). Numerical results are reported in Table 3.4.

Valuation at Time $t = 10$ For cases (B) and (C) in $t = 10$, $\mathbb{E}[K_t]$ stays below CHF 1 000. Since C_t is no more than CHF 25 000 at that time, this can be considered a low amount. Compared to $t = 40$, less than 10% of $\mathbb{E}[K_t]$ is paid in the first ten years in case (B). Introducing surpluses in (C) leads to a small increase of the remediation measures. Simultaneously, the ratio of the expected sums of additional contributions in $t = 10$ and 40 decreases to less than 3% (0.85 vs. 32.10). The bonuses lead to a growth in the expected liabilities of more than CHF 2 000, more than twice the amount of the remediation measures. The ratio $\mathbb{E}[L_t] / (C_t + \mathbb{E}[K_t])$ grows and is larger than one. The same holds for $u^{-1}(\mathbb{E}[u(L_t)]) / (C_t + \mathbb{E}[K_t])$. The distribution of surpluses also leads to higher remediation measures and to a higher volatility causing the relative certainty equivalent to be almost 5% lower than $\mathbb{E}[L_t] / (C_t + \mathbb{E}[K_t])$ (1.008 vs. 1.054).

Valuation at Time $t = 20$ In case (B), $\mathbb{E}[K_t]$ exceeds CHF 2 000, corresponding to an increase by a factor of three (against 670 in $t = 10$). They reach approximately 30% (2.11 vs. 7.69) of the value

Cases and parameters								Valuation of final payoff in T and effective returns					
	1	2	3	4	5	6	7	8	9	10	11	12	13
	Case	UF (indicator)	$F_{\min}$	q (in %)	Bonus (indicator)	F^{L}	ΔF	$\mathbb{E}[L_{40}]$ (in thousands)	$\frac{\sigma[L_{40}]}{\mathbb{E}[L_{40}]}$ (in %)	$\gamma[L_{40}]$	$u^{-1}(\mathbb{E}[u(L_{40})])$ (in thousands)	$\frac{u^{-1}(\mathbb{E}[u(L_{40})])}{C_{40}+\mathbb{E}[K_{40}]}$	$\mathbb{E}[r_{c+b+k}]$ (in %)
1	A							361.2	0		361.2	1.000	1.25
2	B	1	**0.90**		0			361.2	0		361.2	0.988	1.16
3	B	1	**0.92**		0			361.2	0		361.2	0.986	1.15
4	B	1	**0.94**		0			361.2	0		361.2	0.983	1.13
5	B	1	**0.96**		0			361.2	0		361.2	0.981	1.12
6	B	1	**0.98**		0			361.2	0		361.2	0.980	1.11
7	B	1	**1.00**		0			361.2	0		361.2	0.979	1.10
8	B	2		**10.0**	0			361.2	0		361.2	0.961	0.96
9	B	2		**5.0**	0			361.2	0		361.2	0.954	0.90
10	B	2		**2.5**	0			361.2	0		361.2	0.948	0.85
11	B	2		**1.0**	0			361.2	0		361.2	0.938	0.76
12	C	1	1.00		1	**1.02**	0.02	623.9	15.0	0.80	504.6	1.100	3.37
13	C	1	1.00		1	**1.06**	0.02	544.3	15.1	0.93	450.7	1.089	2.99
14	C	1	1.00		1	**1.10**	0.02	501.7	15.2	1.04	423.2	1.076	2.74
15	C	1	1.00		1	**1.14**	0.02	475.0	15.1	1.13	407.0	1.063	2.54
16	C	1	1.00		1	**1.18**	0.02	455.7	14.9	1.22	396.4	1.050	2.36
17	C	1	1.00		1	1.10	**0.01**	495.2	15.1	1.06	419.8	1.075	2.69
18	C	1	1.00		1	1.10	**0.02**	501.7	15.2	1.04	423.2	1.076	2.74
19	C	1	1.00		1	1.10	**0.03**	508.3	15.3	1.01	426.2	1.076	2.78
20	C	1	1.00		1	1.10	**0.04**	514.8	15.4	0.99	429.0	1.074	2.82
21	C	1	1.00		1	1.10	**0.05**	521.6	15.5	0.96	431.6	1.072	2.86
22	C	1	1.00		1	1.10	**0.06**	528.6	15.7	0.93	434.0	1.068	2.90
23	C	1	**0.90**		1	1.10	0.02	483.7	15.6	1.10	409.9	1.088	2.74
24	C	1	**0.92**		1	1.10	0.02	487.4	15.5	1.09	412.9	1.086	2.73
25	C	1	**0.94**		1	1.10	0.02	491.5	15.4	1.08	415.9	1.083	2.73
26	C	1	**0.96**		1	1.10	0.02	496.0	15.3	1.06	419.2	1.080	2.73
27	C	1	**0.98**		1	1.10	0.02	499.9	15.2	1.05	421.9	1.077	2.74
28	C	1	**1.00**		1	1.10	0.02	501.7	15.2	1.04	423.2	1.076	2.74
29	C	2		**10.0**	1	1.10	0.02	581.9	14.7	0.88	479.2	1.044	2.87
30	C	2		**5.0**	1	1.10	0.02	685.0	16.2	0.83	534.7	0.991	3.09
31	C	2		**2.5**	1	1.10	0.02	903.2	19.3	0.86	635.8	0.888	3.43
32	C	2		**1.0**	1	1.10	0.02	1 430.6	23.2	0.90	854.3	0.742	4.04

Table 3.3: Valuation of final payoff and effective returns in cases (A), (B) and (C) (see Section 3.4.1). The parameter values are as in Table 3.1.

	Funding levels, bonuses and additional contributions											
	14	15	16	17	18	19	20	21	22	23	24	25
	$\mathbb{E}\left[F_t\right]$	$\mathbb{E}\left[q_{1\%}\left(F_t\right)\right]$	$\mathbb{E}\left[q_{50\%}\left(F_t\right)\right]$	$\mathbb{E}\left[q_{99\%}\left(F_t\right)\right]$	$q_{1\%}\left(\sum_{t=1}^{40} \mathbb{1}_{\{F_t<1\}}\right)$	$q_{50\%}\left(\sum_{t=1}^{40} \mathbb{1}_{\{F_t>F^L\}}\right)$	$\mathbb{E}\left[\frac{\sum_{t=1}^{40} k_t \cdot e^{-(t-1)r_f}}{\sum_{t=1}^{40} c_t \cdot e^{-(t-1)r_f}}\right]$ (in %)	$\mathbb{E}\left[\sum_{t=1}^{40} \mathbb{1}_{\{k_t>0\}}\right]$	$\mathbb{E}\left[k_t \cdot e^{-(t-1)r_f} \mid k_t > 0\right]$ (in thousands)	$\mathbb{E}\left[\frac{\sum_{t=1}^{40} b_t \cdot e^{-t\,r_f}}{\sum_{t=1}^{40} c_t \cdot e^{-(t-1)r_f}}\right]$ (in %)	$\mathbb{E}\left[\sum_{t=1}^{40} \mathbb{1}_{\{b_t>0\}}\right]$	$\mathbb{E}\left[b_t \cdot e^{-t\,r_f} \mid b_t > 0\right]$ (in thousands)
1	1.17	0.84	1.14	1.57	0	25						
2	1.19	0.90	1.15	1.58	0	26	1.5	0.5	6.41			
3	1.19	0.90	1.15	1.58	0	26	1.8	0.8	5.01			
4	1.20	0.91	1.16	1.58	0	26	2.0	1.2	3.87			
5	1.20	0.92	1.16	1.58	0	27	2.3	1.8	2.93			
6	1.20	0.93	1.16	1.58	0	27	2.5	2.7	2.16			
7	1.20	0.93	1.16	1.58	0	27	2.6	4.0	1.52			
8	1.24	0.98	1.19	1.61	0	31	4.9	6.6	1.74			
9	1.26	1.00	1.21	1.62	0	33	5.8	7.4	1.84			
10	1.27	1.01	1.22	1.63	0	35	6.7	8.1	1.94			
11	1.29	1.03	1.24	1.65	0	36	8.0	8.9	2.11			
12	0.98	0.88	1.00	1.02	10	19	31.0	15.6	4.64	73.7	18.8	9.14
13	1.01	0.90	1.03	1.05	4	15	16.8	10.9	3.60	51.4	14.7	8.14
14	1.04	0.91	1.05	1.09	1	11	10.3	7.9	3.04	39.4	11.6	7.91
15	1.07	0.91	1.07	1.12	0	9	7.0	6.2	2.64	31.9	9.4	7.92
16	1.09	0.92	1.09	1.15	0	7	5.2	5.2	2.33	26.6	7.7	8.03
17	1.05	0.91	1.06	1.09	1	12	9.4	7.4	2.94	37.6	12.6	6.96
18	1.04	0.91	1.05	1.09	1	11	10.3	7.9	3.04	39.4	11.6	7.91
19	1.04	0.90	1.05	1.09	2	11	11.2	8.3	3.13	41.3	10.8	8.94
20	1.03	0.90	1.05	1.09	2	10	12.2	8.8	3.23	43.1	10.0	10.04
21	1.03	0.90	1.04	1.09	2	9	13.2	9.3	3.33	45.0	9.4	11.20
22	1.03	0.90	1.03	1.09	3	9	14.4	9.7	3.45	47.0	8.8	12.43
23	1.03	0.87	1.04	1.09	1	10	5.0	1.0	11.21	34.4	10.3	7.78
24	1.03	0.88	1.05	1.09	1	10	6.1	1.5	9.17	35.4	10.6	7.81
25	1.04	0.89	1.05	1.09	1	11	7.3	2.3	7.37	36.6	10.9	7.84
26	1.04	0.90	1.05	1.09	1	11	8.6	3.5	5.77	37.8	11.2	7.87
27	1.04	0.90	1.05	1.09	1	11	9.8	5.2	4.33	38.9	11.5	7.90
28	1.04	0.91	1.05	1.09	1	11	10.3	7.9	3.04	39.4	11.6	7.91
29	1.06	0.94	1.07	1.09	0	17	31.2	16.8	4.34	61.9	16.7	8.67
30	1.06	0.96	1.09	1.09	0	21	57.0	37.5	3.54	90.9	20.7	10.24
31	1.07	0.98	1.11	1.11	0	26	112.7	39.0	6.74	152.3	25.4	13.96
32	1.07	1.00	1.13	1.13	0	31	249.1	39.0	14.89	301.0	30.7	22.89

Table 3.3: Valuation of final payoff and effective returns in cases (A), (B) and (C) (see Section 3.4.1). The parameter values are as in Table 3.1 (continued).

	Case	$C_t \cdot e^{-t\,r_f}$ (in thousands)	$\mathbb{E}\left[K_t \cdot e^{-t\,r_f}\right]$ (in thousands)	$\mathbb{E}\left[L_t \cdot e^{-t\,r_f}\right]$ (in thousands)	C_t (in thousands)	$\mathbb{E}\left[K_t\right]$ (in thousands)	$\mathbb{E}\left[L_t\right]$ (in thousands)	$\frac{\mathbb{E}[L_t]}{C_t+\mathbb{E}[K_t]}$	$\frac{u^{-1}(\mathbb{E}[u(L_t)])}{C_t+\mathbb{E}[K_t]}$
$t = 10$	A	23.8	0	23.8	26.3	0	26.3	1	1
	B	23.8	0.61	23.8	26.3	0.67	26.3	0.975	0.975
	C	23.8	0.77	25.9	26.3	0.85	28.6	1.054	1.008
$t = 20$	A	65.4	0	65.4	79.9	0	79.9	1	1
	B	65.4	1.73	65.4	79.9	2.11	79.9	0.974	0.974
	C	65.4	3.43	77.7	79.9	4.18	94.9	1.129	1.032
$t = 40$	A	242.1	0	242.1	361.2	0	361.2	1	1
	B	242.1	5.16	242.1	361.2	7.69	361.2	0.979	0.979
	C	242.1	21.52	336.3	361.2	32.10	501.7	1.276	1.076

Table 3.4: Simulation results for cases (A), (B) and (C) after 10, 20 and 40 years. The parameter values are as in Table 3.1.

in $t = 40$. Distributing surpluses increases $\mathbb{E}[K_t]$. While it doubles, the ratio of $\mathbb{E}[K_t]$ for $t = 20$ and 40 decreases to less than 15% (4.18 vs. 32.10). The distribution of bonuses leads to the expected liabilities gaining approximately CHF 15 000. This is more than six times larger than in time $t = 10$ (15 vs. 2.3). Thus, while the remediation measures increase, the distributed bonuses increase even more. This also holds for $\mathbb{E}[L_t] / (C_t + \mathbb{E}[K_t])$ which grows by approximately 7% (1.129 vs. 1.054). In case (B), it stays at 0.974. The relative certainty equivalent experiences a gain of more than 2% (1.032 vs. 1.008) in case (C) and is smaller than $\mathbb{E}[L_t] / (C_t + \mathbb{E}[K_t])$.

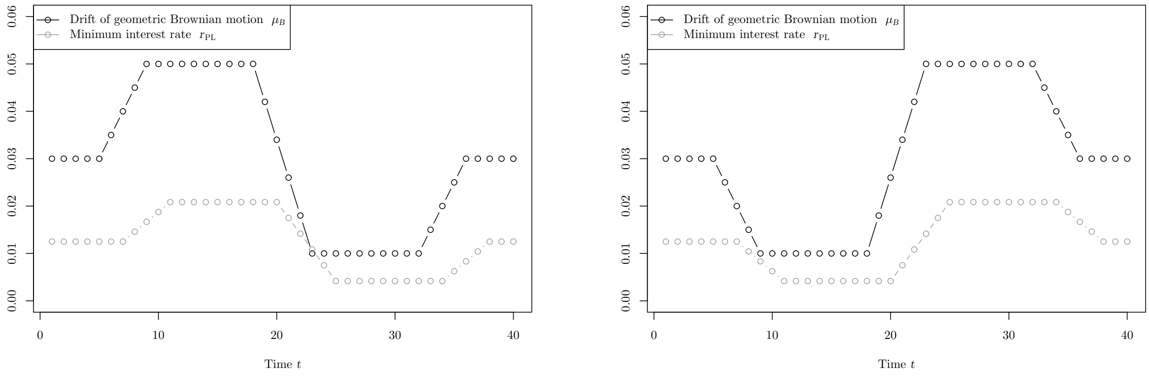
Discussion In early years, the paid amounts remain rather low. As the salary and the conversion factor grow over time, most of the contributions are paid towards the end of the time frame. The required amounts in the case of underfunding remain low in early years. Consequently, when changing pension funds in early years, the effect of contributors not taking remediation measures with themselves remains fairly low. Distributing surpluses increases the relative certainty equivalent.

3.4.4 Impact of Capital Market Scenarios

Focusing on case (C), we analyze capital market scenarios. We consider the reference case and let the drift μ_B follow a predefined path using two scenarios. In the first one, μ_B equals the reference value in the first five periods. This is followed by ten periods with high returns of 5% (i.e. 2% increase). Subsequently, it drops to 1% for another ten periods (mimicking a crisis and a post-crisis environment, see e.g. Europe after the 2008 financial crisis). For the last five time points it returns to 3%. For scenario two, the course of μ_B is mirrored. The minimum interest rate r_{PL} follows μ_B at a ratio of $r_{\mathrm{PL}}/\mu_B = 1.25/3 = 41.67\%$ with a delay of two years. This simulates a delayed adaptation of r_{PL}, reflecting practice, e.g., in Switzerland, where the minimum interest rate is adapted through a political process (BVG, Art. 15).[15] The paths of μ_B and r_{PL} for both scenarios are depicted in Figure 3.8.

Simulation Results Table 3.5 reports the results. Figures 3.9 to 3.11 illustrate the development of the means of F_t, k_t and b_t. The periods of increased and decreased drift μ_{B} are shown as light and dark

[15] In the Swiss system, a commission regularly decides about changes of r_{PL}. For this, they take the market conditions into account by using a rolling average of government bonds as a benchmark. We mirror this process in our analysis by adjusting the guaranteed interest rate r_{PL} with a delay of two years at a fixed ratio of r_{PL}/μ_B.



(a) μ_B and r_{PL} for Scenario 1.

(b) μ_B and r_{PL} for Scenario 2.

Figure 3.8: Illustration of the drift μ_B of the geometric Brownian motion and the minimum interest rate r_{PL} for scenarios 1 and 2.

gray areas, respectively.

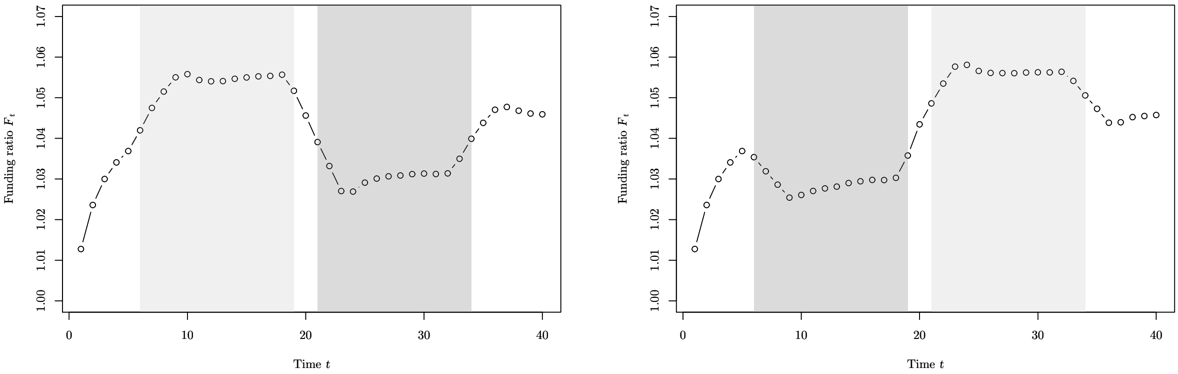
	$\mathbb{E}[F_t]$	$\mathbb{E}\left[\sum_{t=1}^{40} \mathbb{1}_{\{k_t>0\}}\right]$	$\mathbb{E}\left[k_t \cdot e^{-(t-1)r_f} \mid k_t > 0\right]$ (in thousands)	$\mathbb{E}\left[\sum_{t=1}^{40} \mathbb{1}_{\{b_t>0\}}\right]$	$\mathbb{E}\left[b_t \cdot e^{-t\,r_f} \mid b_t > 0\right]$ (in thousands)	$\mathbb{E}[L_{40}]$ (in thousands)	$\mathbb{E}[r_{c+b+k}]$ (in %)	$\frac{\sigma[L_{40}]}{\mathbb{E}[L_{40}]}$ (in %)	$u^{-1}(\mathbb{E}[u(L_{40})])$ (in thousands)	$\frac{u^{-1}(\mathbb{E}[u(L_{40})])}{C_{40}+\mathbb{E}[K_{40}]}$
Reference case	1.04	7.9	3.04	11.6	7.91	501.7	2.74	15.2	423.2	1.076
Scenario 1	1.04	8.1	3.71	11.8	6.94	468.1	2.17	14.2	403.1	1.041
Scenario 2	1.04	8.2	2.67	11.8	9.26	549.5	3.33	16.1	450.8	1.111

Table 3.5: Simulation results for the reference case (C) and scenarios 1 and 2 (cf. Table 3.1).

Funding Ratio F_t From Figure 3.9(a) it can be seen that during the times of increased market returns, the funding ratio increases sharply. From approximately 101%, it rises to more than 105%. The decrease of μ_{B} has an immediate impact on F_t which falls below 103%. As in the case of high returns, F_t converges quickly to this value and subsequently changes only little. The recovery of μ_{B} to 3% at the end of the time frame also leads to the funding ratio returning to 104%. It can be seen in Table 3.5 that, as in the reference case, $\mathbb{E}[F_t]$ equals 104%.

For the second scenario, the development of F_t is analogous. After an increase during the first periods, the lower drift causes the mean funding ratio to drop to approximately 102.5%. In the subsequent periods with higher market returns, F_t rises to nearly 106%. With μ_{B} returning to 3%, the funding ratio decreases to 104%. As in the first scenario, F_t reacts quickly to changes in μ_{B} and stays nearly constant once the drift stabilizes. The expected value of F_t over all 40 periods is again equal to 104%. It can thus be concluded that two periods of high and low capital market returns of similar severity and length do not influence the expected funding ratio, regardless of how they are ordered.

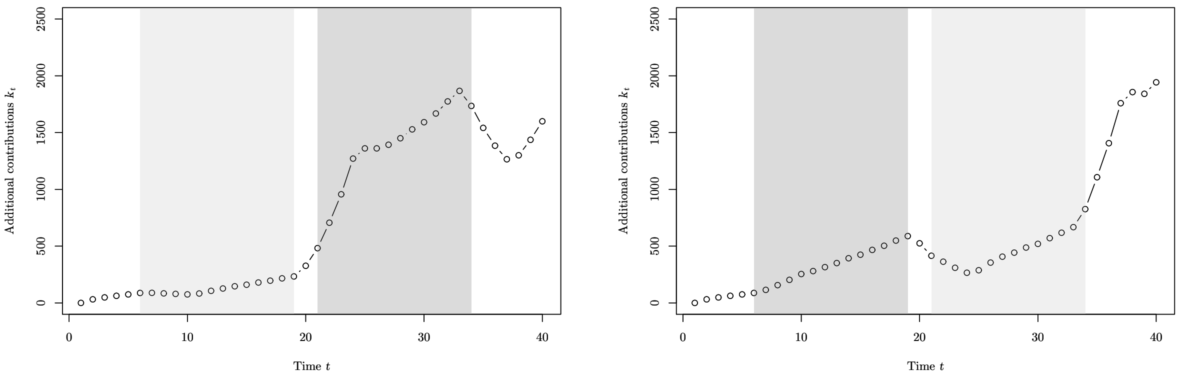
Remediation Measures k_t Figure 3.10 depicts the present values of the remediation measures payed. In the first case, k_t initially stays very low, hardly exceeding CHF 200. This is because the high returns lead to overfunding. With the subsequent drop of μ_{B}, the additional payments escalate quickly, reaching



(a) Mean of the funding ratio F_t in scenario 1. (b) Mean of the funding ratio F_t in scenario 2.

Figure 3.9: Illustration of the means of the funding ratio F_t in scenarios 1 and 2.

almost CHF 2 000. Towards the end of the time frame, the curve first decreases and then rejoins the course of the reference case, settling at approximately CHF 1 500. The remediation measures for the

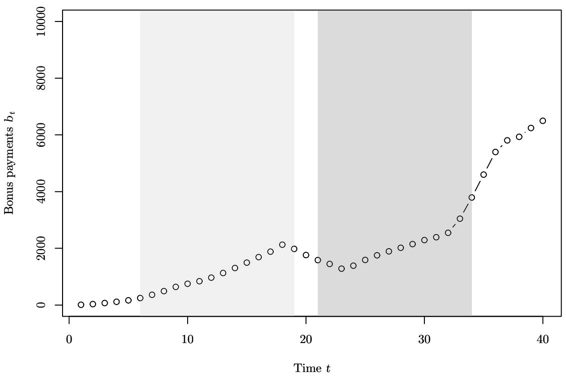


(a) Present value of the remediation measures k_t in scenario 1. (b) Present value of the remediation measures k_t in scenario 2.

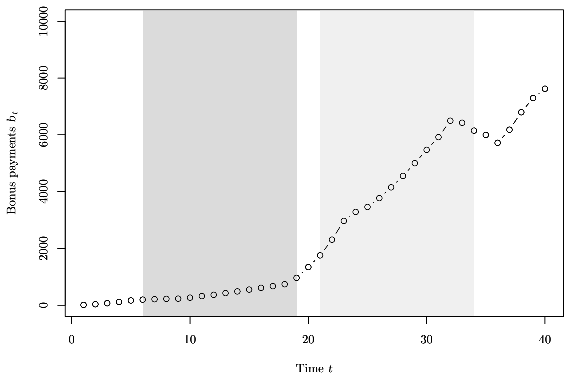
Figure 3.10: Illustration of the present values of the remediation measures k_t in scenarios 1 and 2.

second scenario grow regularly until the end of period 20, exceeding CHF 500. The successive higher market returns then cause them to halve. While μ_{B} equals 5%, the remediation measures again grow only slowly. With the drift returning to 3%, k_t increases strongly. In time T, its mean is above CHF 2 000. Comparing the two cases through Table 3.5, the expected number of additional payments is almost the same, exceeding the reference case. The expected amount that is paid is higher for the first scenario, reaching more than CHF 3 710. The second scenario is approximately 30% lower than that with the reference case located almost in the middle between the two cases. The reason for the differences in $\mathbb{E}\left[k_t \cdot e^{-(t-1)r_{\mathrm{f}}} | k_t > 0\right]$ can be found in the development of the contributor accounts over time. When capital market returns are low at early time points, the amounts needed to compensate are still relatively low. At later times, the required amounts are much larger as assets A_t and contributions C_t are much higher.

Distributed Surpluses b_t For the first scenario, the present value of the bonus payments increases very little at the beginning, reaching approximately CHF 2 000. The low capital market returns then cause a slight decline, which is followed by an increase similar to that at the beginning. The recovery of μ_{B} to 3% then leads to a strong increase to about CHF 6 000. For the second case, the payments during the first 20 periods only reach approximately CHF 1 000. The high returns in later times lead to



(a) Present value of the distributed surpluses b_t in scenario 1.



(b) Present value of the distributed surpluses b_t in scenario 2.

Figure 3.11: Illustration of the present values of the distributed surpluses b_t in scenarios 1 and 2.

very high mean surpluses being distributed reaching a maximum of about CHF 7 000. The return of μ_B to 3% only leads to a small decrease before the curve proceeds to grow as in the reference case.
While the expected number of bonus payments in Table 3.5 amounts to nearly 12 for both scenarios, the expected distributed surpluses are a lot higher in the second scenario (CHF 9 260). The reference case has fewer time points where bonuses are being paid and an expected value for b_t that lies between the two cases.

Liabilities and Certainty Equivalent For the second scenario the expected value of the liabilities $\mathbb{E}[L_{40}]$ is higher than for the first, amounting to a difference of more than CHF 80 000. For $\mathbb{E}\left[r_{c+b+k}\right]$, there is an increase. The lower remediation measures lead to a growth of the effective return. Both impacts together cause a strong increase of $\mathbb{E}\left[r_{c+b+k}\right]$. In the second scenario, the effective return is 3.33%. The first case only reaches 2.17%. For the certainty equivalent, the results are similar to the expected liabilities. While the second scenario reaches about CHF 450 000, the first one only exceeds CHF 400 000. This also holds for $u^{-1}\left(\mathbb{E}\left[u\left(L_{40}\right)\right]\right)/\left(\mathbb{E}\left[C_{40}\right]+\mathbb{E}\left[K_{40}\right]\right)$. Here, the reference case has a value of 1.076. The first scenario is close to this, reaching 1.041. The second one reaches 1.111. Analyzing the relative volatility $\sigma\left[L_{40}\right]/\mathbb{E}\left[L_{40}\right]$, the first scenario has the lowest fluctuation, with a value of 14.2%. While obtaining the highest effective returns, the second scenario is also coupled to a high volatility of 16.1%.

3.5 Discussion and Conclusion

Remediation Measures Considering the cases without (A) and with additional contributions (B), we observe an improvement of funding levels connected with their charging. While the higher quantiles are in good funding, the 1% quantile remains below 100% if no remediation measures are charged. In case (B), we see an improvement of this subgroup. Utilizing remediation measures thus leads to a stabilization of the fund.

Surplus Distribution When only charging remediation measures, the funding ratio rises above 100% in the considered reference setting. Therefore, excessive funds can be distributed, leading to an increase in the insured's absolute certainty equivalent. Additionally, bonuses exceed remediation measures. However, with increasing bonuses remediation costs increase as well, causing higher volatility in the annual payments and thus lower utility relative to the total costs for the insured.

Calibration Remediation measures lead to an improved stability of pension funds. For this to be fully effective, an adequate assignment of all model parameters is essential. Small changes in variables can already lead to a strong impact on the outcomes. For example, a decrease in $F_{\min}$ of only 2% in case (C) leads to an increase of k_t of more than 40%.

Interim Valuation When members leave a fund, additional contributions remain with the fund while bonuses leave with the insured. Our results show that the insured's account is still fairly low after 10 and 20 years. It can therefore be concluded that a change of pension funds can be made without a large impact on the savings. This circumstance, however, changes dramatically in later years.

Capital Market Scenarios Letting the capital market returns follow a predefined path, we imitate periods of both very low and very high returns. The results show that, especially for later years, additional contributions can rise substantially if a funding gap occurs. The same is true for the distributed surpluses. The amounts that are charged or distributed can make up a great percentage of the overall cash flows. Capital market scenarios therefore need to be taken into account with close attention.

Risk Bearing With the worldwide trend from defined benefit (DB) plans towards defined contribution (DC) plans, it is interesting to put our findings into this overall context. In fact, while in DC plans the members can often choose the investment strategy, they bear a large part of the capital market risks since mostly only minimum benefits are guaranteed upfront. Looking at changing interest rate assumptions, Godwin et al. (1996) find that funds are likely to change their interest rate assumptions to increase their latitude concerning contractual relationships. Poterba et al. (2007) find that on average the retirement wealth from DC plans exceeds the one from DB plans. However, DC schemes are more likely to generate very low outcomes. In a similar spirit, Vigna and Haberman (2001) analyze the financial risk in a DC pension plan to derive an optimal investment strategy. They conclude that there is a large variability in the level of pension achieved at retirement. Our results are in line with these findings. As the distribution of bonuses increases the volatility of the payoff, the relative certainty equivalent of risk averse individuals decreases.

Policy Recommendations This work analyzes the value of the accounts of the insured at retirement. We observe that the utilized funding mechanisms and their specific calibration can have an important impact on the stability of the fund and the utility of its contributors. In our model, the charging of remediation measures and the distribution of surpluses takes place automatically. In practice, decisions concerning these actions would typically be made by the board of the fund. From this we conclude that the role of the board and the governance is crucial for the management of a fund. Although our model is fitted to the Swiss pension fund system, an interpretation of our findings in the light of the rules in place in other countries and an extension of the results to other types of DC pension plans should be straightforward. Also, in practice, the administration costs, the mortality of the members and the decumulation phase where pensions are paid out should be accounted for to get more realistic results. Nevertheless, we observe that while considering solvency constraints, distributing bonuses may increase the risks borne by members and the common belief of the insured profiting from surplus distribution is a fallacy. Given the current trends of the low interest rate environment and volatility in many markets, such mechanisms should be used with caution.

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Chapter 4

Optimal Asset Allocation in Pension Funds Under Consideration of Higher Moments

The low interest rates that prevail on many capital markets impose great challenges for the asset management of financial organizations. They try to achieve target returns for their clients, a solid one-period funding ratio and a low one-period underfunding probability. Optimal investment shares obtained from minimum variance theory only take the first two moments of the asset return distribution into account and leave important properties disregarded. In our work, we aim to study the impact of capital allocation strategies for pension funds in Switzerland. Thereby, we compare classic Markowitz theory with an extended Taylor series approach for the utility function. It is further analyzed how the assumption of normally distributed returns drives the asset allocation when compared with using the distributions corresponding to the best fit of the historical data. Taking the extended utility function including the first four central moments and the alternative return distributions, we simulate the assets of a pension fund in a one-period model with the Monte Carlo method. A comparison of these results with those obtained from the classic minimum variance theory concludes that a considerable change of the portfolio weights takes place. Additionally, we find that multi-dimensional risk factors corresponding to preferences with respect to the different moments of the portfolio return distribution significantly affect the asset allocation. Our research is relevant for theory and practice alike. Financial institutions can profit from making use of higher dimensional utility functions in their asset allocation strategies.

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4.1 Introduction

In the current low interest rate environment, a proper asset allocation strategy is crucial for financial institutions like life insurers and pension funds. Often being obliged to grant their clients a guaranteed interest rate on their savings, they face the issue of having to find a suitable combination of asset classes to invest in. With respect to Swiss pension funds, this topic is of great importance. As funds face large obligations towards their members, it is necessary for them to have an investment strategy that balances expected return and volatility. While the expected return must suffice to earn the promised interest for the clients, the volatility should not be too high. Otherwise, years with very low capital market returns could lead to a strong decrease of the funding ratio and thus put the fund's solvency at risk. It is therefore crucial to choose a combination of assets that meets the needed returns while maintaining a certain safety level. In our work, we seek to examine this subject by looking at specific asset allocation techniques under selected return distribution assumptions. To this respect, we study the impact that higher moments have when included in the decision taker's utility function. We analyze how using different distributions for simulating the assets leads to a better fit of the historic data and thus to improved simulation results. In order to compare the resulting allocations, we simulate the assets of a fund and analyze the results after one period.

Classical portfolio selection based on the minimum variance approach dates back to Markowitz (1952). By taking only the first two moments into consideration, it is possible to derive a closed-form solution for the minimum variance portfolio and make statements about the future development of the asset returns (Samuelson, 1970). For modeling return distributions, it is well known that the normal distribution does not suffice in order to model the characteristics of asset returns (Mandelbrot, 1963; Maringer, 2008). Since the normal distribution is symmetric and puts only little weight on the tails, it misses important aspects. Among others, it has been found that skewness is present in asset returns (Harvey and Siddique, 1999; Jondeau and Rockinger, 2003) and therefore has an impact on pricing (Harvey and Siddique, 2000; Hwang and Satchell, 1999; Chang et al., 2013). Consequently, alternative models have been developed that extend utility theory to further moments and to incorporate three or more parameters (Jean, 1971, 1973; Ingersoll, 1975). This makes it possible to consider, among others, the skewness of return distributions (Athayde and Flôres, 2004). With respect to pension funds, the funding ratio and the optimal portfolio selection have been studied from different perspectives (O'Brien, 1986). Those include, among others, determining the optimal asset allocation with respect to longevity, income and inflation (Yang and Huang, 2009; Battocchio and Menoncin, 2004; Haberman and Vigna, 2002). For the Swiss pension fund system, research has been conducted considering the solvency and overall stability of a pension fund (Braun et al., 2011; Eling, 2013). Drawing from this, recommendations for regulatory application are given in order to ensure the stability of the system. Being aimed at the Swiss pension system, our research focuses on the correct choice of asset classes as well as their simulation. To this end, we look at five different types of assets. In order to find the optimal allocation strategy, classic Markowitz theory uses the first two moments of expected returns and covariance in order to find the efficient portfolio. In our work, we look at extensions of this by comparing how the use of higher moments, such as the skewness and the kurtosis, changes the results. With research having shown both, an impact from higher moments on some financial products (Prakash et al., 2003), as well as none for several others (Jondeau and Rockinger, 2006; Peiró, 1999), we aim to analyze their implications in the case of typical asset allocations of pension funds in Switzerland. Our research question is: How does the use of higher moments change the optimal asset allocation and do alternative distributions lead to a better fit of the historical data?

To start our study, we consider a pension fund with a given asset-liability situation and which is regulated by prevalent rules of the Swiss pension fund system. Our aim is to find the asset allocation that allows the fund to reach a given expected target return, a given funding ratio or a given low underfunding probability. Having established an optimal allocation strategy, we simulate the assets of the fund. For this, we first make use of a multivariate normal distribution. However, asset returns exhibit characteristics such as skewness and heavy tails, that cannot be fully reflected by using a normal distribution. Therefore, we perform the simulation also by using alternative distributions taking their specific correlation structure into account. This way, the historic data can be fitted in a more flexible way that should lead to improved simulation results. With the optimal allocation corresponding to the objective return and a distribution that fits the historic data, we simulate the assets and liabilities of the pension fund in a one-period model. While the assets evolve according to the simulated returns, the liabilities are assumed to be credited with the guaranteed interest rate. Based on the results, it is analyzed in what state the fund is at the end of the period. Among others, this involves examining key figures such as the expected funding ratio, selected quantiles of that ratio and the underfunding probability. In this context, we compare what implications the use of an asset-liability approach has compared to a classical analysis. In this, the asset allocation can be set up with regard to a desired funding ratio or underfunding probability rather than a target return.
The remainder of the Chapter is organized as follows. Section two introduces the framework that we use for modeling the pension fund. Additionally, the optimization problem for determining the optimal asset allocation is presented. The third section presents the asset classes together with their descriptive statistics and the fitting of return distributions. Section four presents the optimization and simulation results. This comprises both, the results for the minimum variance portfolio with normally distributed returns as well as the extended utility along with the best-fit return distributions. Following this, we compare the results and conduct a sensitivity analysis regarding the preference of risk with respect to the different moments. The final section concludes.

4.2 Model Framework

In the following, we introduce a simple asset allocation framework and solvency indicators for pension funds. This involves describing the processes that take place within the fund as well as in the assets that it invests in. We formally describe the use of higher moments of the asset return distributions and develop on optimal portfolio theory in which we use different utility functions that incorporate higher moments.

4.2.1 Pension Funds and Key Funding Indicators in a One-Period Model

For the pension fund, we focus on a simplified representation of the accumulation phase of a defined contribution fund in Switzerland. We examine how the assets A_0 and the liabilities L_0 evolve in a one-period model from time $t = 0$ to $t = 1$, given an asset allocation and legal minimum increases of the liabilities. Thereby, we disregard fluctuations, annuitization and deaths. At the end of the period, the state of the fund is analyzed by considering the funding ratio and the probability of underfunding.

The assets A_0 represent the capital that is available to the fund for investing on the capital market at time zero. This way, it is able to earn the amount that needs to be awarded to its clients as annual interest returns. In our model, we assume that the fund invests in n different asset classes $i = 1, \ldots, n$. The shares α_i, that are invested in the respective classes i, are summarized in the vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)'$. These shares are in the focus of the present study. In Switzerland, the regulator imposes limits on the asset shares that can be invested in the various asset types (see BVV2, 2017, Art. 55). Therefore, we introduce an upper limit $\alpha_i^{\max}$ for each asset class i. We assume that the entire assets are invested and suppose that no short sales are made. Consequently, it holds for the asset shares α_i that $0 \leq \alpha_i \leq \alpha_i^{\max}$, $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i = 1$.

The stochastic asset returns in the period are denoted by $\boldsymbol{r} = (r_1, \ldots, r_n)'$. They are the only source of risk in our study. With the returns r_i denoting the different assets and α_i the investment shares, the overall portfolio return is

$$r_{\mathrm{A}} = \boldsymbol{\alpha}' \cdot \boldsymbol{r} = \sum_{i=1}^{n} \alpha_i \cdot r_i. \tag{4.1}$$

Starting with a value of A_0 at time $t = 0$ and considering continuous compounding of the interest return, the value of the assets at time $t = 1$ is given by

$$A_1 = A_0 \cdot e^{r_{\mathrm{A}}} = A_0 \cdot e^{\sum_{i=1}^{n} \alpha_i \cdot r_i}. \tag{4.2}$$

The liabilities L_t represent the obligations that the fund has towards its members. This includes the regular contributions that have been paid by the actives as well as surpluses that the fund can distribute when being in good health. In the Swiss system, funds are required to credit their members at least a minimum interest rate r_{L} on the compulsory part of their second pillar pension savings. The value of r_{L} is set by the legislator at the end of every year according to the prevailing conditions on the financial market (see BVV2, 2017, Art. 12). In our model, we assume that the liabilities start with a value of L_0 at time zero and are compounded with r_{L} over the course of the period. Their value at time one therefore is

$$L_1 = L_0 \cdot e^{r_{\mathrm{L}}}. \tag{4.3}$$

While in our model L_1 is deterministic (no fluctuations, no mortality, no payouts, no surpluses credited), A_1 is a stochastic outcome and depends on the asset allocation and the market returns. Having obtained the values of the assets and the liabilities at time $t = 1$, we analyze the distribution of the state of the fund. For this, we first consider the funding ratio F_t calculated by dividing the assets by the liabilities, i.e.

$$F_t = \frac{A_t}{L_t}, \ t = 0, 1. \tag{4.4}$$

We analyze the mean $\mathbb{E}[F_1]$ as well as selected quantiles q_x of F_1 in order to examine the range of the distribution of the funding ratio at time one in our simulations. We consider the 1%, 50% and 99% quantiles, denoted by $q_{1\%}(F_1)$, $q_{50\%}(F_1)$ and $q_{99\%}(F_1)$.

Further, we compute the probability of underfunding, i.e. the probability of the funding ratio falling below 100% at time one. This way, it is measured how exposed to insolvency the fund is. The underfunding probability at time one is defined as

$$\mathbb{P}[F_1 < 100\%]. \tag{4.5}$$

The asset returns $\boldsymbol{r}$ are of particular importance, as they influence the allocation $\boldsymbol{\alpha}$ and the distribution of A_1. For this, it is important to have information about the distribution and the dependency structure of the asset portfolio.

4.2.2 Analyzing Higher Moments of the Return Distribution

For analyzing the returns of the asset portfolio, and subsequently the impact of utility preferences, we examine the first four moments introduced in the following. These first four moments are the expected return μ, the volatility σ, the skewness $\bar{\gamma}$ and the kurtosis $\bar{\kappa}$ (see, e.g., Bhandari and Das, 2009) of the investment portfolio return. For the expected return of the portfolio, it holds that

$$\mu = \mathbb{E}[r_{\mathrm{A}}] = \mathbb{E}\left[\sum_{i=1}^{n} \alpha_i \cdot r_i\right] = \sum_{i=1}^{n} \alpha_i \cdot \mu_i = \boldsymbol{\alpha}' \cdot \boldsymbol{\mu}, \tag{4.6}$$

with $\boldsymbol{\mu} = \{\mu_i\}$ the vector of mean returns corresponding to the first moment of the asset classes i.
The volatility σ of the portfolio return is calculated as the square root of the portfolio variance σ^2 which in turn is defined as

$$\sigma^2 = \mathrm{Var}[r_{\mathrm{A}}] = \mathrm{Var}\left[\sum_{i=1}^{n} \alpha_i \cdot r_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \cdot \alpha_j \cdot \sigma_{ij}^2 = \boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha}, \tag{4.7}$$

with σ_{ij}^2, the covariance of assets i and j, being the elements of the covariance matrix $\Sigma = \{\sigma_{ij}\}$, for which it holds that

$$\Sigma = \mathbb{E}[(\boldsymbol{r} - \boldsymbol{\mu})(\boldsymbol{r} - \boldsymbol{\mu})']. \tag{4.8}$$

The variance σ^2 corresponds to the second central moment.
In order to calculate skewness and variance of the portfolio return, it is necessary to determine the third and fourth central moments γ and κ. They are defined as

$$\gamma = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \alpha_i \cdot \alpha_j \cdot \alpha_k \cdot \gamma_{ijk} = \boldsymbol{\alpha}' \cdot \Gamma \cdot (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha}), \tag{4.9}$$

and

$$\kappa = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \alpha_i \cdot \alpha_j \cdot \alpha_k \cdot \alpha_l \cdot \kappa_{ijkl} = \boldsymbol{\alpha}' \cdot K \cdot (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} \otimes \boldsymbol{\alpha}), \tag{4.10}$$

with $\otimes$ representing the Kronecker product. In this, γ_{ijk} represents the co-skewness between assets i, j and k and κ_{ijkl} the co-kurtosis of asset classes i, j, k and l. They are the elements of the co-skewness matrix $\Gamma = \{\gamma_{ijk}\}$ and the co-kurtosis matrix $K = \{\kappa_{ijkl}\}$, which are defined as

$$\Gamma = \mathbb{E}[(\boldsymbol{r} - \boldsymbol{\mu}) \cdot (\boldsymbol{r} - \boldsymbol{\mu})' \otimes (\boldsymbol{r} - \boldsymbol{\mu})'], \tag{4.11}$$

and

$$K = \mathbb{E}[(\boldsymbol{r} - \boldsymbol{\mu}) \cdot (\boldsymbol{r} - \boldsymbol{\mu})' \otimes (\boldsymbol{r} - \boldsymbol{\mu})' \otimes (\boldsymbol{r} - \boldsymbol{\mu})']. \tag{4.12}$$

The co-skewness matrix is of dimension $n \times n^2$ and the co-kurtosis matrix of dimension $n \times n^3$.
When analyzing the results of our simulations, we use the standardized moments to facilitate the comparison of different distributions. Using the central moments, the portfolio skewness, which describes the "asymmetry" of the return distribution, is then calculated as the *standardized* third central moment, i.e.

$$\bar{\gamma} = \frac{\gamma}{\sigma^3}. \tag{4.13}$$

The portfolio kurtosis, which measures the "heaviness" of the tails of the return distribution, is defined as the fourth *standardized* central moment (see, e.g., Boudt et al., 2008), i.e.

$$\bar{\kappa} = \frac{\kappa}{\sigma^4}. \tag{4.14}$$

4.2.3 Optimal Portfolio Theory

In this section, we first derive a formulation of a utility function $U(\boldsymbol{\alpha}, \boldsymbol{r})$ based on a limited number of moments of the distribution of $\boldsymbol{r}$. We then consider two particular cases for optimizing the choice of the

asset allocation, the minimum variance approach by Markowitz (1952) as well as an alternative one that includes the third and fourth moments of skewness and kurtosis.
When deciding on how to invest in the various asset classes, a rational investor would aim to choose the portfolio that yields the highest utility $U(\boldsymbol{\alpha}, \boldsymbol{r})$. The utility, in return, depends both, on the (uncertain) return vector $\boldsymbol{r} = (r_1, \ldots, r_n)'$ from the available asset classes as well as on the (selected) portfolio weights $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)'$. Looking at a Taylor series expansion with respect to the variable $\boldsymbol{r}$ of the utility function, we get

$$U(\boldsymbol{\alpha}, \boldsymbol{r}) = U(\boldsymbol{\alpha}, \boldsymbol{r}) + \frac{U'(\boldsymbol{\alpha}, \boldsymbol{r})}{1!} \cdot (\boldsymbol{r} - \boldsymbol{\mu}) + \frac{U''(\boldsymbol{\alpha}, \boldsymbol{r})}{2!} \cdot (\boldsymbol{r} - \boldsymbol{\mu})^2 + \frac{U'''(\boldsymbol{\alpha}, \boldsymbol{r})}{3!} \cdot (\boldsymbol{r} - \boldsymbol{\mu})^3 + \frac{U^{(4)}(\boldsymbol{\alpha}, \boldsymbol{r})}{4!} \cdot (\boldsymbol{r} - \boldsymbol{\mu})^4 + \ldots \tag{4.15}$$

Taking the expected value of $U(\boldsymbol{\alpha}, \boldsymbol{r})$, we get the expected utility $\mathbb{E}[U(\boldsymbol{\alpha}, \boldsymbol{r})]$, defined as

$$\begin{aligned}\mathbb{E}[U(\boldsymbol{\alpha}, \boldsymbol{r})] = U(\boldsymbol{\alpha}, \boldsymbol{r}) + \frac{U'(\boldsymbol{\alpha}, \boldsymbol{r})}{1!} \cdot \mathbb{E}[(\boldsymbol{r} - \boldsymbol{\mu})] + \frac{U''(\boldsymbol{\alpha}, \boldsymbol{r})}{2!} \cdot \mathbb{E}\left[(\boldsymbol{r} - \boldsymbol{\mu})^2\right] + \frac{U'''(\boldsymbol{\alpha}, \boldsymbol{r})}{3!} \cdot \mathbb{E}\left[(\boldsymbol{r} - \boldsymbol{\mu})^3\right] \\ + \frac{U^{(4)}(\boldsymbol{\alpha}, \boldsymbol{r})}{4!} \cdot \mathbb{E}\left[(\boldsymbol{r} - \boldsymbol{\mu})^4\right] + \ldots \end{aligned} \tag{4.16}$$

As it holds that

$$\begin{aligned} &\mathbb{E}[(\boldsymbol{r} - \boldsymbol{\mu})] = 0, \\ &\mathbb{E}\left[(\boldsymbol{r} - \boldsymbol{\mu})^2\right] = \boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha}, \\ &\mathbb{E}\left[(\boldsymbol{r} - \boldsymbol{\mu})^3\right] = \boldsymbol{\alpha}' \cdot \Gamma \cdot (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha}), \text{ and} \\ &\mathbb{E}\left[(\boldsymbol{r} - \boldsymbol{\mu})^4\right] = \boldsymbol{\alpha}' \cdot K \cdot (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} \otimes \boldsymbol{\alpha}), \end{aligned} \tag{4.17}$$

we can simplify the expected utility function above and get the following approximation of order four in $\boldsymbol{\alpha}$ (see, e.g., Beardsley et al., 2012)

$$\mathbb{E}[U(\boldsymbol{\alpha}, \boldsymbol{r})] \approx U(\boldsymbol{\alpha}, \boldsymbol{r}) + \frac{U''(\boldsymbol{\alpha}, \boldsymbol{r})}{2!} \cdot \boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha} + \frac{U'''(\boldsymbol{\alpha}, \boldsymbol{r})}{3!} \cdot \boldsymbol{\alpha}' \cdot \Gamma \cdot (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha}) + \frac{U^{(4)}(\boldsymbol{\alpha}, \boldsymbol{r})}{4!} \cdot \boldsymbol{\alpha}' \cdot K \cdot (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} \otimes \boldsymbol{\alpha}). \tag{4.18}$$

Normally Distributed Returns

In the case of multivariate normally distributed asset returns, the third and fourth moment equal zero. Consequently, the expected utility function in Equation (4.18) reduces to

$$\mathbb{E}[U(\boldsymbol{\alpha}, \boldsymbol{r})] = U(\boldsymbol{\alpha}, \boldsymbol{r}) + \frac{1}{2} \cdot U''(\boldsymbol{\alpha}, \boldsymbol{r}) \cdot \boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha}. \tag{4.19}$$

As the utility function is invariant with respect to positive and monotone transformations (see, e.g., Levy and Markowitz, 1979), we are able to define a function of equivalent utility $V_1(\boldsymbol{\alpha}, \boldsymbol{r})$, for which it holds that (see, e.g., Braun et al., 2017)

$$V_1(\boldsymbol{\alpha}, \boldsymbol{r}) = \mu - \lambda_1 \cdot \boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha}. \tag{4.20}$$

In this, $\lambda_1 > 0$ implies $U''(\boldsymbol{\alpha}, \boldsymbol{r}) < 0$ and thus serves as a measure for risk aversion. Conversely, $\lambda_1 < 0$ corresponds to a risk taking behavior while $\lambda_1 = 0$ stands for a risk-neutral decision making. Consequently, as $V_1(\boldsymbol{\alpha}, \boldsymbol{r})$ is of equivalent utility to $\mathbb{E}[U(\boldsymbol{\alpha}, \boldsymbol{r})]$, the combination of parameters (shares α) that maximize $V_1(\boldsymbol{\alpha}, \boldsymbol{r})$ will also turn out to maximize the expected utility $\mathbb{E}[U(\boldsymbol{\alpha}, \boldsymbol{r})]$.
To find the portfolio that maximizes the utility function, we search for the classic Markowitz (1952) minimum variance portfolio. In this, the aim is to find an optimal set of portfolio weights $\boldsymbol{\alpha}^* = (\alpha_1^*, \ldots, \alpha_n^*)'$

for the different asset classes that minimizes the variance of the resulting portfolio while achieving a certain target return μ^*. We therefore have the constrained optimization problem

$$\boldsymbol{\alpha}^* = \arg\min_{\boldsymbol{\alpha}} \left[\lambda_1 \cdot \boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha}\right], \tag{4.21}$$

with the objective

$$\boldsymbol{\alpha}' \cdot \boldsymbol{\mu} = \mu^*, \tag{4.22}$$

where

$$0 \leq \alpha_i \leq \alpha_i^{\max}\ ,\ i = 1, \ldots, n, \tag{4.23}$$

and

$$\boldsymbol{\alpha}' \cdot \mathbb{1} = 1. \tag{4.24}$$

The investment restrictions on α_i require that no short sales are made, the share α_i, that is invested in each asset class, remains below the respective regulatory upper limit $\alpha_i^{\max}$ (Equation 4.23) and the entire assets are invested on the capital market (Equation 4.24).
Depending on the perspective taken, the objective in Equation (4.22) is interchangeable with other target conditions considering the funding ratio or the underfunding probability, leading to conditions of the type

$$\mathbb{E}\left[F_1\right] = \bar{F}, \tag{4.25}$$

where $\bar{F}$ is a given funding ratio target, and

$$\mathbb{P}\left[F_1 < 100\%\right] = \epsilon. \tag{4.26}$$

where ϵ is a predetermined one-year probability for underfunding (i.e., $F_1 < 100\%$).[1]

General Case

While the optimization problem introduced in Equations (4.21 ff.) can be solved relatively easily and yields the minimum variance portfolio, it relies on the assumption that the asset returns follow a multivariate normal distribution. Research shows that in practice, capital market returns do not show Gaussian properties (see, e.g., Jondeau et al., 2007; Cont, 2001). As the historical values exhibit clear signs of asymmetry and heavy tails, it is in many situations not reasonable to assume that the third and fourth moment are equal to zero (cf. Section 4.3.1, Table 4.1). Therefore, the expected utility function can in general not be simplified to only include the variance of the asset returns. Instead, we aim to find the optimal investment weights $\boldsymbol{\alpha}^*$ with respect to the (still approximate) expected utility function given in Equation (4.18). If we again follow the above procedure of applying positive and monotone transformations, we obtain the utility function $V_2(\boldsymbol{\alpha}, \boldsymbol{r})$, which is defined as

$$V_2(\boldsymbol{\alpha}, \boldsymbol{r}) = \mu - \lambda_1 \cdot \boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha} + \lambda_2 \cdot \boldsymbol{\alpha}' \cdot \Gamma \cdot (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha}) - \lambda_3 \cdot \boldsymbol{\alpha}' \cdot K \cdot (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} \otimes \boldsymbol{\alpha}). \tag{4.27}$$

Again, λ_1 serves as a measure for risk aversion, with $\lambda_1 > 0$ being equivalent to $U''(\boldsymbol{\alpha}, \boldsymbol{r}) < 0$. Analogously, the new parameters, λ_2 and λ_3 represent risk preferences (for $\lambda_2, \lambda_3 > 0$) regarding the third and fourth moments of skewness and kurtosis. As individuals prefer odd moments and try to avoid even ones (see, e.g. Chiu, 2010; Scott and Horvath, 1980), the signs of the terms in $V_2(\boldsymbol{\alpha}, \boldsymbol{r})$ alternate. The reasoning behind this is, that the second moment represents the dispersion of the asset returns, which

[1] When the investment restrictions on α_i from Equation (4.23) are used, a closed-form solution of the optimization problem can not be derived. We consequently solve the constrained optimization problem using numerical approximations. For further details, see, e.g. Samuelson (1970) and Harvey et al. (2010).

a risk averse investor would aim to keep as low as possible. For the skewness, it holds that a negative skewness corresponds to the mass of the distribution being concentrated on the right with a longer left tail. Correspondingly, a positive third moment has the mass of the distribution shifted towards the left while the right tail is longer. Due to the characteristics of the tails, a risk-averse investor would prefer a positive skewness of the asset returns, as it reduces the risk of extreme losses (low returns). The kurtosis serves as a measure for the tails of the distribution. For this, a large value corresponds to distinctive peaks with little weight on tails, whereas a small value signifies lower peaks and heavy tails. Risk-averse individuals would therefore prefer a smaller kurtosis.
Taking the equivalent utility function $V_2(\boldsymbol{\alpha}, \boldsymbol{r})$ into account, Equation (4.21) of the optimization problem becomes

$$\boldsymbol{\alpha}^* = \arg\min_{\boldsymbol{\alpha}} \left[\lambda_1 \cdot \boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha} - \lambda_2 \cdot \boldsymbol{\alpha}' \cdot \Gamma \cdot (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha}) + \lambda_3 \cdot \boldsymbol{\alpha}' \cdot K \cdot (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} \otimes \boldsymbol{\alpha})\right], \tag{4.28}$$

with the same target conditions laid out above in Equations (4.22), (4.25) or (4.26), and under the investment restrictions of Equations (4.23) and (4.24) on $\boldsymbol{\alpha}$.
In the following, we aim to compare the two optimization problems and their outcomes with each other. This involves fitting distributions to the historical asset return data presented in the following section, as well as determining the covariance, co-skewness and co-kurtosis. Based on those, we aim to compute the optimal solutions of the two optimization problems and consequently analyze their simulation results.

4.2.4 Outline of the Methodology

To conclude the model presentation, we present the steps that are performed in our model. This involves solving the optimization problem, simulating the asset process and analyzing the corresponding results. A visualization of the model is given in Figure 4.1.

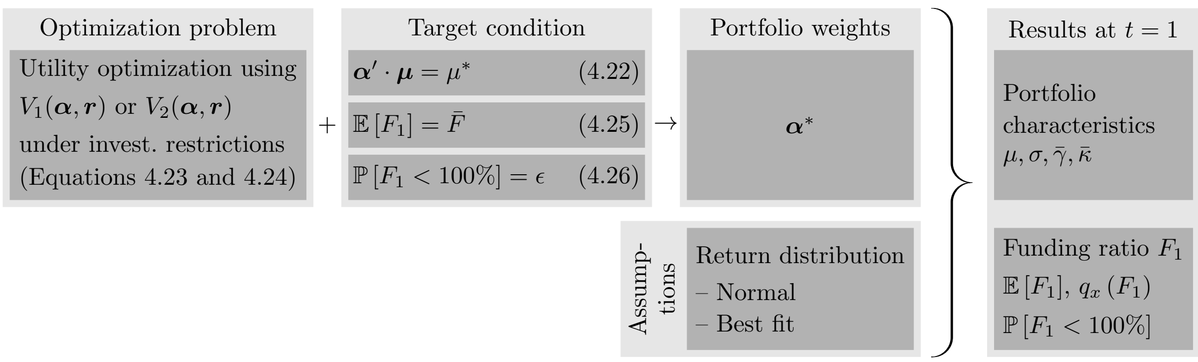


Figure 4.1: Synopsis of the steps involved in the model.

Starting with the optimization problem, we choose whether to use the utility function $V_1(\boldsymbol{\alpha}, \boldsymbol{r})$ or $V_2(\boldsymbol{\alpha}, \boldsymbol{r})$. This way, the optimal portfolio weights are calculated either based on the second or on the second, third and fourth moment of the asset classes. Additionally, we select one of the three target conditions from Equations (4.22), (4.25) and (4.26), which correspond to target values for the return μ^*, the funding ratio $\bar{F}$ and the underfunding probability ϵ. All optimizations are subject to the investment constraints in Equations (4.23) and (4.24), i.e. the assets are invested entirely on the capital market with no short sales being made and the respective shares α_i having to remain below a legal upper limit of $\alpha_i^{\max}$.
By solving the optimization problem (either using V_1 or V_2) with a target condition and the investment restrictions (4.23) and (4.24), we obtain the optimal portfolio weights $\boldsymbol{\alpha}^* = (\alpha_1^*, \ldots, \alpha_n^*)'$.[2] In the next

[2] For solving the optimization problem, we use a numerical algorithm. A reliable and quick method is, e.g., the augmented

step, we then simulate the asset returns at time $t = 1$. In order to do this, we select the distribution of the asset classes by choosing between the normal distribution and the one obtained from fitting the historic data. Using the asset weights, we simulate the assets A_1 at time $t = 1$ and analyze the resulting portfolio return r_{A} as well as the funding ratio F_1. Among others, this involves looking at key figures such as the mean drift μ, the volatility σ, the standardized third central moment $\bar{\gamma}$ and the standardized fourth central moment $\bar{\kappa}$. For the funding ratio F_1, we analyze the expected value $\mathbb{E}[F_1]$, selected quantiles $q_x(F_1)$ and the underfunding probability $\mathbb{P}[F_1 < 100\%]$.
For the optimization problem with a target return μ^* from Equation (4.22), the optimization process can be performed directly. For the target conditions in Equations (4.25) and (4.26) this cannot be done in one step, as they depend on the funding ratio F_1 at time one. Therefore, the optimization and simulation need to be performed iteratively in order to obtain the optimal portfolio weights yielding the expected funding ratio $\bar{F}$ and respectively the underfunding probability ϵ.

4.3 Asset Return Statistics and Distribution

In order to model the assets that the pension fund can invest in, we use five different asset classes that represent the most common investment types. Those involve the money market, government bonds, real estate, stocks and hedge funds (representing riskier investments). As our research focuses on the Swiss pension fund system, we make use of financial products that are connected to the Swiss market. For the money market, we use the 3-month CHF Libor interest rate (Bloomberg: SZC0TR03). The government bonds are represented by the SBI government total return index (Bloomberg: SBIDGT). It mimics the market for Swiss obligations and, this way, gives information about the level of interest rates in the Swiss market. For the real estate data, we use the Swiss Exchange real estate funds (Bloomberg: SWIIT). They include a multitude of real estate stocks and trusts based in Switzerland. For an asset that represents stock returns, we choose the Swiss Performance Index SPI (Bloomberg: SPI:IND). It encompasses the stocks of all companies that are traded on the Swiss exchange market. For hedge funds, there is a lack of funds that have been operating over the whole time span. We therefore choose to utilize an equally weighted basket of 14 different hedge funds.[3] We base our study on historical data from the beginning of 1996 till the end of 2015, i.e. a time horizon of 20 years. As for the returns, we look at monthly values (due to daily data not being available for hedge funds).

4.3.1 Descriptive Statistics of Asset Returns

An overview of the characteristics of the asset classes introduced above is given in Table 4.1. For the

Asset class	Mean return μ_i	Volatility σ_i	Skewness $\bar{\gamma}_i$	Kurtosis $\bar{\kappa}_i$
Money Market (MM)	1.14%	0.34%	16.54%	19.81%
Government Bonds (GB)	3.95%	3.72%	3.74%	28.28%
Real Estate (RE)	5.28%	7.07%	−12.70%	33.34%
Stocks (ST)	7.27%	15.57%	−27.13%	41.76%
Hedge Funds (HF)	8.72%	7.58%	0.45%	24.04%

Table 4.1: Overview of asset classes with their respective annualized mean return μ_i, volatility σ_i, skewness $\bar{\gamma}_i$ and kurtosis $\bar{\kappa}_i$ in the time period 1996–2015.[4]

lagrangian adaptive barrier minimization algorithm provided by the package Alabama in R.

[3]The hedge funds that are comprised in our basket have the Bloomberg codes ORBOPEF BH, SCHFODH SW, GLOINVA KY, GABINTL VI, EDFAGFI BH, ROTNEMI GU, GAMMUTI VI, CGUTUSD BA, EDFGCBE NA, EDFGCFI NA, SLRGLII KY, TALWNDI VI, EDFCURI BH and PHFUNDI BH.

annualized mean returns μ_i, it can be seen that the money market achieves about one percent, whereas the government bonds reach almost four percent in the given period. The real estate and stocks have an average return that is about one, respectively three percent higher than that. The highest value is obtained by the hedge funds with a value of 8.72%, about 1.5% higher than the one of the stocks. We observe that money market investments show low volatility with a value of 0.34%. For the government bonds and the real estate, higher values of 3.72% and 7.07% can be observed. The highest volatility can be found in the stock market, which reaches more than 15%. The variation in the hedge funds reaches about half of that with a value of 7.58%. For the skewness, we observe that the real estate and the stocks are strongly negatively skewed with values of −12.70% and −27.13%. This corresponds to the mass of the returns being shifted to the right (above the mean) and the left tail being longer (corresponding to lower returns). Therefore, while positive returns are overall more likely, extreme losses can occur as well. For the money market, we calculate a value of 16.54%. Thus, while the mass of the distribution is to the left, the left tail is very thin as well. For the government bonds and the hedge funds, the skewness takes comparatively small positives values. For the kurtosis, we obtain only positive values, ranging from about 20% for the money market to roughly 40% for the stocks. All assets therefore exhibit fatter tails.
The correlation matrix of the five asset classes is given in Table 4.2. The money market (MM) returns show only little to almost no correlation with the returns from other classes. For the government bonds (GB), an increase in the correlations can be noticed. At the same time, all the values stay below 0.2. The returns from the real estate (RE) show a higher correlation with those from stocks (ST) and hedge funds (HF). The highest correlation can be observed between the stocks and the hedge funds with a value of about 0.3.

Asset class	MM	GB	RE	ST	HF
MM	1	0.134	−0.094	−0.167	0.109
GB	0.134	1	0.191	−0.183	0.052
RE	−0.094	0.191	1	0.229	0.210
ST	−0.167	−0.183	0.229	1	0.304
HF	0.109	0.052	0.210	0.304	1

Table 4.2: Correlation matrix of the five asset classes.

4.3.2 Fitting of Return Distributions

For simulating the asset returns, the multivariate normal distribution is not able to take the skewness and kurtosis of the returns into account. Furthermore, it is not suited for the modeling of fat tails. We therefore fit the historic data with alternative distributions in order to better describe the empirical distribution. This includes, among others, a better fit of the asymmetry as well as the tails of the returns. In this section, we present the distributions that we consider in our study, show the results that we obtain when fitting them to the historical return data and demonstrate how we simulate the multivariate asset returns. The distributions we consider are the Cauchy, Logistic and Normal-Inverse-Gaussian (NIG) distribution.
Serving as a reference case, we first look at the normal distribution. This way, the fits of the alternative distributions cannot only be compared to each other, but also to the one from the normal distribution.

[4] As the empirical moments of the third and fourth order are calculated by taking the third and fourth power of the returns, they are particularly sensitive to changes in the data. Different historic values can consequently result in strong changes in the empirical moments. This needs to be taken into account when calculating the skewness and kurtosis.

The Cauchy distribution is symmetric and has two degrees of freedom, the location and the scale parameter. The logistic distribution has two parameters as well, a location and a scale parameter. While also being symmetric, it puts more weight on the tails than the normal distribution. The NIG distribution has four degrees of freedom, making it very flexible and able to replicate skewed and heavy-tailed return distributions. We use the Akaike information criterion (AIC, see Akaike, 1973) values to evaluate the goodness-of-fit. The results are reported in Table 4.3.

	Money Market	Gvt. Bonds	Real Estate	Stocks	Hedge Funds
Normal	−2 628.35	−1 485.09	−1 178.72	−800.79	−1 144.47
Cauchy	−2 531.36	−1 410.13	−1 126.66	−775.29	−1 051.68
Logistic	−2 616.45	−1 485.15	−1 186.16	−816.72	−1 138.91
NIG	−2 651.30	−1 482.60	−1 183.97	−833.06	−1 140.47

Table 4.3: AIC values from fitting distributions to the historic monthly returns using 20 years of data.

When comparing the results, we conclude that the NIG distribution provides the best fit on our return data for the money market and the stocks. For the government bonds and the real estate investments the logistic distribution is suited best. For the hedge funds, the normal distribution achieves the best fit of the data. For the Cauchy distribution, the results are mixed. While it is better than the logistic and the NIG distribution for the hedge funds, it is not better than the normal distribution for any of the asset classes.

The simulation of the assets over one period is done using multivariate random variables with a copula. A copula C allows us to simulate the dependence structure of the returns (see, e.g., Korn et al., 2010). It is defined as a distribution function on $[0,1]^n$ for $n \in \mathbb{N}$ (in our case $n = 5$) with uniform distributed marginals, i.e.

$$C\left(1, \ldots, 1, x_i, 1, \ldots, 1\right) = x_i, \quad \forall i \in \{1, \ldots, n\}. \tag{4.29}$$

With Sklar's theorem (see, e.g., Sklar, 1959), it is possible to separately calculate the dependence structure and the marginal distributions of the different random variables. This way, we are able to first simulate the dependence structure with a copula using univariate distribution functions and then calculate the marginal distributions. In our work, we model the dependencies between the different asset classes with the help of a Gaussian copula (see, e.g., McNeil et al., 2015). For random variables $X_1, \ldots, X_n$, with Φ_Σ^n denoting the joint n-dimensional normal distribution function and Φ being the standard normal marginal distribution, the Gaussian copula C_{Gauss} with correlation matrix Σ is defined as $C_{\text{Gauss}}\left(x_1, \ldots, x_n\right) = \Phi_\Sigma^n\left[\Phi^{-1}\left(x_1\right), \ldots, \Phi^{-1}\left(x_n\right)\right]$. Using the Gaussian copula C_{Gauss}, we generate multivariate uniform random variables with Σ-dependence structure. The best-fit marginals (cf. Table 4.3) allow to derive adequate random sets of returns. Together with the distribution parameters from fitting the data, we are thus able to simulate the multivariate returns of our different asset classes. Going forward, we simulate the portfolio return with the help of both, the normal distribution as well as the best fit ones, and compare the outcomes to each other.

4.4 Numerical Application

In the following, we present the results of our simulations for a fund operating in Switzerland. Therein, we initially look at the reference case of multivariate normally distributed assets in a minimum variance portfolio under the utility function $V_1(\boldsymbol{\alpha}, \boldsymbol{r})$. We present the efficient frontier for our data as well as the corresponding intersections for target funding ratios and underfunding probabilities. Following that, we look at the case of the portfolio weights being chosen in line with the extended utility function $V_2(\boldsymbol{\alpha}, \boldsymbol{r})$ and the returns following the fitted distributions. We examine how the asset allocation changes and analyze the outcome of the simulations. The risk factors λ_1, λ_2 and λ_3 are set to one in these cases. Going forward, we perform a sensitivity analysis on the values of the different factors. As risk preference is usually treated as a one-dimensional variable, it is of interest to study how the extended utility function reacts to changes in each of the three factors. The numerical application is performed by simulating the efficient portfolios for target values between $\min(\mu_i) = 1.14\%$ and the maximum attainable return of 6.73% using a step size of 0.1% and $N = 10^7$ realizations for every simulation.

Parameter	Variable	Value
Investment limits		
Maximum share MM	$\alpha_1^{\max}$	100%
Maximum share GB	$\alpha_2^{\max}$	100%
Maximum share RE	$\alpha_3^{\max}$	30%
Maximum share ST	$\alpha_4^{\max}$	50%
Maximum share HF	$\alpha_5^{\max}$	15%
Pension fund governance		
Asset value at time $t = 0$	A_0	110
Liability value at time $t = 0$	L_0	100
Funding ratio at time $t = 0$	F_0	110%
Minimum interest rate	r_{L}	1.25%
Risk preference with respect to		
Volatility	λ_1	1
Skewness	λ_2	1
Kurtosis	λ_3	1

Table 4.4: Input parameters for the reference case.

With respect to the investment limits, $\alpha_i^{\max}$, there are no specific limitations for the shares invested in the money market and the government bonds. For the real estate, the maximum share, that can be invested, amounts to 30%. Conversely, the proportion of the assets, that is made up by the stocks, can be up to 50%. The strictest limit is imposed on the hedge funds. For them, the maximum share amounts to 15%. For our simulations, we therefore use the vector $\boldsymbol{\alpha}^{\max} = (100\%, 100\%, 30\%, 50\%, 15\%)'$ for the investment limits (see BVV2, 2017, Art. 55). Furthermore, we assume that the fund starts in a "healthy" situation with assets of $A_0 = 110$ and liabilities of $L_0 = 100$ at time $t = 0$. The funding ratio $F_0 = A_0/L_0$ consequently equals 110% (see Table 4.4 for the parameters), which corresponds to the average value for private pension funds in Switzerland over the past years (see Swisscanto, 2016). While the return on the assets r_{A} is random following the distributions laid out in Section 4.3, the interest rate r_{L} for the liabilities is set to the legal minimum. For 2016, it equals 1.25% (see BVV2, 2017, Art. 12).

4.4.1 Classic Markowitz Optimization with Normally Distributed Returns

Serving as a reference case, we first simulate the pension fund using the asset allocation derived from minimum variance theory. Therein, the asset classes are multivariate normally distributed and the risk aversion coefficient $\lambda_1 = 1$. Figure 4.2 depicts the efficient frontier for the asset portfolios in the cases with and without investment restrictions.[5] Namely, the solid line represents the restricted case while the dashed one shows the unrestricted setting. Additionally, dashed isolines for selected funding ratios at time $t = 1$ put the (μ, σ) asset portfolio characteristics and the resulting funding ratio $\mathbb{E}[F_1]$ into relation.

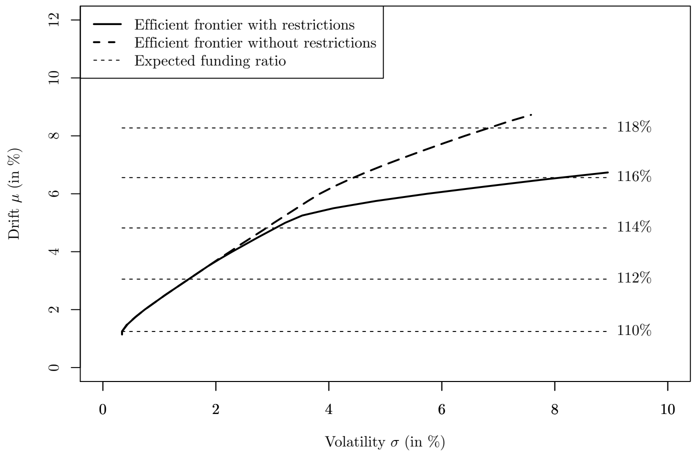


Figure 4.2: Plot of the efficient frontier for the cases with and without investment restrictions as well as isolines for the expected funding ratio $\mathbb{E}[F_1]$ at time $t = 1$.

The case with investment restrictions is derived from the unrestricted one by setting the maximum asset shares to $\boldsymbol{\alpha}^{\max}$. Consequently, the achievable returns are lower (see the solid line). While it can only reach a maximum return of 6.73%, the dashed line of unrestricted investments continues further and allows for returns of up to 8.72% by investing only in hedge funds. Conversely, the maximum volatility is higher for the restricted case, as it has a 15% investment limit for the hedge funds. A higher share for the stocks is therefore required in order to reach high returns. As a results of that, there is a strong increase in the portfolio volatility and the efficient frontier runs flatter. The solid line thus reaches further to the right, while staying lower than the dashed one. Focusing on the restricted case, we observe that the highest funding ratio that can be reached with an efficient portfolio after one year is about 116%. On the other end, the cases with the lowest returns (1.14%) yield a funding ratio slightly below 110% after one period. We can therefore deduce that there is an overall improvement in the funding ratio with the assets chosen in our case. On average, a return of around 1.25% on the assets must be met in order to keep the funding ratio constant. Examining the shape of the two frontiers, we can observe that they are identical up to a volatility of about 2% and start to differ from thereon. The reason for this is that the investment in hedge funds reaches its upper limit $\alpha_5^{\max}$ (cf. Figure 4.4). Consequently, portfolios with higher returns can still be achieved up to a certain point, but the optimal portfolio has a higher volatility than in the unrestricted case. Additionally, a kink in the frontier is observed for the restricted case for a volatility of about 3.5%. The reason for this is that the share of government bonds experiences a steeper decrease from this point onwards.

Staying with the same graphs of the efficient frontiers with and without investment restrictions, Figure 4.3 shows selected isolines for the probability of underfunding. As the pension fund starts with a funding ratio of 110% at time zero, the probability $\mathbb{P}[F_1 < 100\%]$ at time $t = 1$ is low. In the case without investment restrictions, it exceeds 1% only in the most extreme cases. As can be seen in the plot, these cases are the ones with the highest drift and volatility: Increased returns come at the price of a higher volatility, which causes the underfunding probability to increase as well. The case with investment

[5] When using resampling methods for obtaining the efficient frontier, the confidence regions of the resampled efficient frontiers need to be considered (see, e.g., Michaud, 1998).

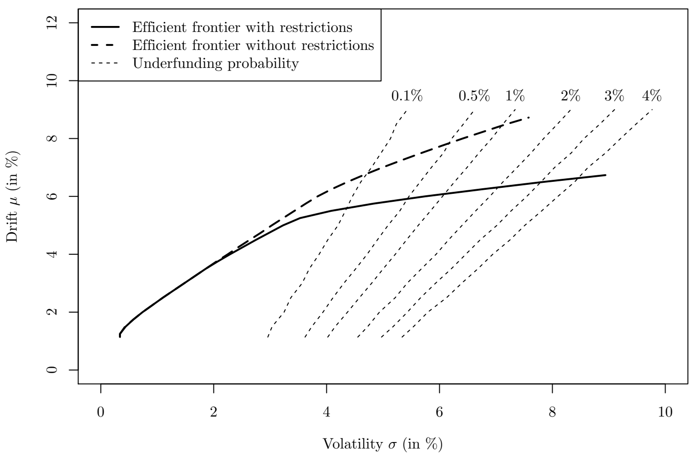


Figure 4.3: Plot of the efficient frontier for the cases with and without investment restrictions as well as underfunding probability isolines.

restrictions reaches a value of about 4% for $\mathbb{P}[F_1 < 100\%]$ in its most extreme case. The depicted isolines for underfunding probabilities of 0.1%, 0.5% and 1% show that for low returns μ the probability of becoming underfunded is even smaller. We conclude that the risk of an average pension fund (starting from $F_0 = 110\%$) becoming underfunded is very low. As the returns rise and consequently the efficient frontier flattens, the volatility σ and the underfunding probability start to grow strongly. Going more into detail, Figure 4.4 displays the asset allocations in the efficient portfolios for the restricted case.

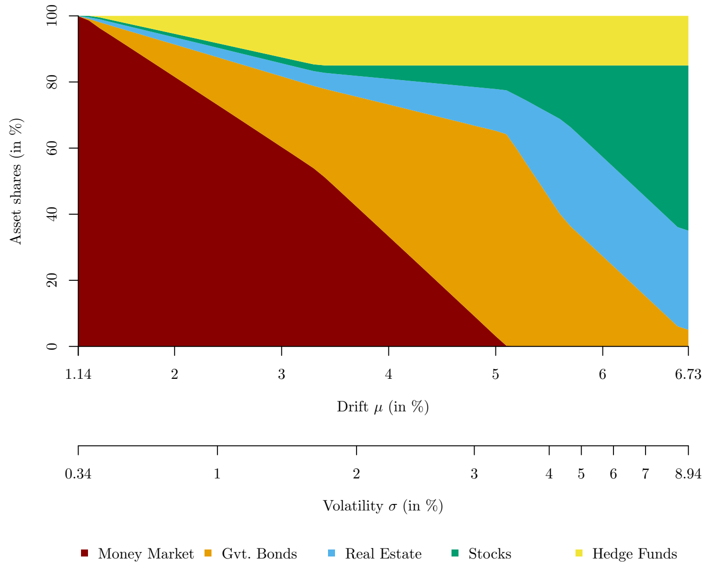
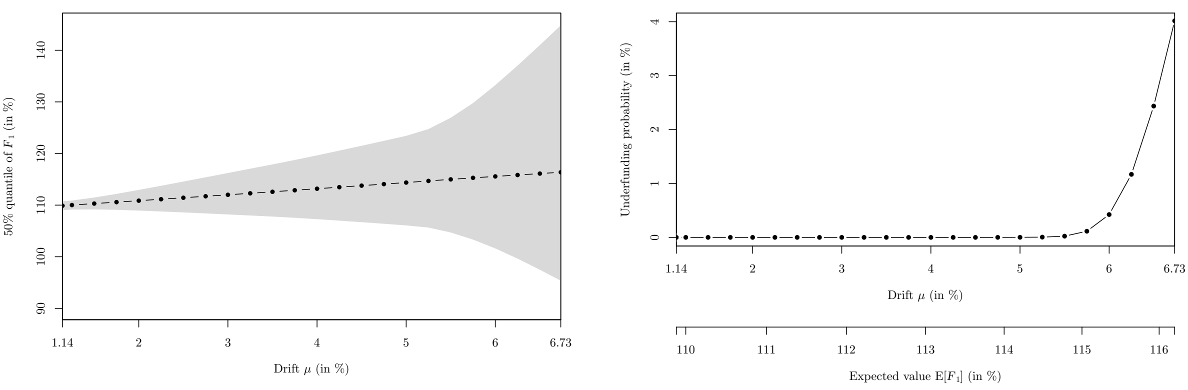


Figure 4.4: Optimal asset allocations for target return μ for $V_1(\boldsymbol{\alpha}, \boldsymbol{r})$ and normally distributed returns.

For every combination of target returns and corresponding minimum variances, it depicts the optimal shares $\boldsymbol{\alpha}^*$ of the five asset classes. It can be seen that for low target returns the efficient portfolio mainly consists of money market investments (the portfolio with the lowest variance yields a drift of 1.14%, corresponding to 100% money market). As the drift μ increases, this share decreases while the shares of the remaining assets increase, with the government bonds having the second-highest percentage. As μ grows further, the share of the hedge funds and the government bonds evolve similarly at first.

The stocks are hardly represented in the portfolio. As they have a relatively high volatility of 15.57% together with a mean return of 7.27% (cf. Table 4.1), it is more advantageous to invest in the hedge funds instead. The fact that the share of the stocks is not equal to zero can be explained by the correlation of 0.304 with the hedge funds (cf. Table 4.2). The changes in the asset shares are linear up to a drift of almost 3.5%, where the hedge funds reach their maximum of $\alpha_5^{\max} = 15\%$. Subsequently, they remain at this value. Comparing with Figures 4.2 and 4.3, this point corresponds to the curves for the restricted and unrestricted case starting to differ from each other (where the first asset class reaches the restriction ceiling). Subsequently, the share of the money market begins to decrease steeply and the portfolio weights shift more towards the government bonds, the real estate and the stocks. For the stocks, it can be seen that an increase of its share is taking place as a result of the hedge funds hitting their investment limit. This effect enhances further when the share of the money market drops to zero. The same holds true for the real estate. In Figure 4.4, these changes appear as kinks in the course of the asset shares. For the government bonds a decrease takes place for high returns. For the portfolio with the highest return, which is located on the right end of the graph, the hedge funds, the stocks and the real estate achieve their investment limits of 15%, 50% and 30%. The remaining proportion of 5% is attributed to the government bonds. The portfolio weights thus are $\boldsymbol{\alpha} = (0\%, 5\%, 30\%, 50\%, 15\%)'$, which means that the highest achievable portfolio return is 6.73% along with a volatility of 8.94%. Overall, changes of the shares take place with kinks in the graphs when assets hit their investment limits or disappear from the portfolio.

Figure 4.5(a) displays the 50% quantile $q_{50\%}(F_1)$ of F_1, which is simulated from the multivariate normal distribution of the asset classes at time $t = 1$ as a function of the target return μ^*. In addition to this, the borders of the gray areas mark the 5% and 95% quantiles of the funding ratio, $q_{5\%}(F_1)$ and $q_{95\%}(F_1)$. We observe that for low returns the 50% quantile stays close to 110% (cf. Figure 4.2), which is the same as at time zero. With increasing drift, $q_{50\%}(F_1)$ also experiences an increase. Its magnitude corresponds to the extent to which the market returns exceed the return r_{L} credited to the liabilities. The plot shows that F_1 grows linearly in μ. As a higher drift also leads to a growth in volatility, the shaded area between the 5% and 95% quantiles gets larger and grows uniformly up to a drift of about 5%. At this point, the outer quantiles also exhibit kinks in their curves. Consequently, they increase, respectively decrease, much more and the gray area grows a lot quicker. This can be connected to the observations made in Figures 4.2 and 4.4. As the share invested in government bonds starts to decrease, a higher return needs to be compensated by investing in more volatile asset classes. This causes the efficient frontier to change its slope and consequently be flatter. As a result, the volatility grows considerably more for the same rise in expected return.



(a) $q_{50\%}(F_1)$, gray area: $q_{5\%}(F_1) - q_{95\%}(F_1)$ (b) Underfunding probability $\mathbb{P}[F_1 < 100\%]$.

Figure 4.5: Plots of the 50% quantile of the funding ratio $q_{50\%}(F_1)$ and the underfunding probability $\mathbb{P}[F_1 < 100\%]$ at time one.

Along with the strong increase of the volatility comes a higher probability $\mathbb{P}[F_1 < 100\%]$ for underfunding. Figure 4.5(b) shows the probability for underfunding for different values for the drift μ and the corresponding 50% quantile $q_{50\%}(F_1)$ of F_1. It can be seen that up to a value of about 5.75% for the drift, the value remains close to zero. Together with the higher share of risky assets that is needed for higher returns, the underfunding probability increases as well. From 5.75% up to the maximum achievable return, the graph experiences a sharp increase that ends at approximately 4%. This corresponds to the results seen in Figure 4.3.
The numerical results for selected target values are given in Table 4.5. In the three parts of the table, we fix target values for the mean return (part I), the expected funding ratio $\mathbb{E}[F_1]$ at time one (part II) and the one-year underfunding probability $\mathbb{P}[F_1 < 100\%]$ (part III). These objectives correspond to the conditions described in Equations (4.22), (4.25) and (4.26).

Condition on	I: Mean return			II: Expected funding ratio			III: Underfunding probability		
μ	**2.00**	**3.00**	**4.00**	1.25	3.05	4.82	5.73	6.04	6.21
σ	0.74	1.48	2.29	0.34	1.52	3.06	4.75	5.89	6.58
$\bar{\gamma}$ [$\times 10^{-2}$]	0.11	0.13	0.12	0.01	0.13	0.11	0.25	0.32	0.35
$\bar{\kappa}$ [$\times 10^{-2}$]	0.03	0.03	0.03	0.00	0.04	0.02	0.13	0.21	0.24
α_1	81.56	60.18	33.28	97.21	59.17	8.48	0.00	0.00	0.00
α_2	9.78	21.57	39.90	1.42	22.02	58.18	35.48	26.18	21.06
α_3	2.16	3.94	7.75	0.76	4.04	11.75	30.00	30.00	30.00
α_4	1.07	1.85	4.07	0.56	1.85	6.59	19.52	28.82	33.94
α_5	5.44	12.46	15.00	0.06	12.92	15.00	15.00	15.00	15.00
$\mathbb{E}[F_1]$	110.83	111.94	113.07	**110.00**	**112.00**	**114.00**	115.04	115.39	115.59
$q_{1\%}(F_1)$	108.94	108.19	107.28	109.14	108.15	106.30	103.46	101.34	100.00
$q_{50\%}(F_1)$	110.86	112.00	113.18	110.01	112.06	114.16	115.26	115.60	115.78
$q_{99\%}(F_1)$	112.95	116.22	119.64	110.89	116.39	122.73	129.48	133.78	136.35
$\mathbb{P}[F_1 < 100\%]$	0.00	0.00	0.00	0.00	0.00	0.00	**0.10**	**0.50**	**1.00**

Table 4.5: Simulation results using minimum variance and normal returns. Values in bold face correspond to the target values. All reported values are given in %.

In part I, we set values for the target return μ^* of 2%, 3% and 4%. The corresponding portfolio volatility σ increases with the return, growing from 0.74% for $\mu^* = 2\%$ up to 2.29% for a target return of $\mu^* = 4\%$. Meanwhile, the skewness and kurtosis remain almost unchanged, taking values of about $\bar{\gamma} = 0.12$ for the skewness and $\bar{\kappa} = 0.03$ for the kurtosis. The asset shares are dominated by the money market with about 81.56% for a return of 2%. For a return of 3%, it can be seen that the weights shift as the money market drops to around 60% and the government bonds increase to more than 20%. For the remaining assets, we observe that the hedge funds obtain a share of about 12%, whereas the real estate and stocks reach less than 6% when taken together. This trend continues when we look at the values for a target return of 4%. There, the money market only makes up a share of about 33%. The government bonds accumulate about 40% of the overall shares. The hedge funds have reached their maximum share of $\alpha_5^{\max} = 15\%$ and cannot grow any further. Therefore, increased returns are obtained by investing more in the riskier asset classes. The expected value and the 50% quantile of the funding ratio, $\mathbb{E}[F_1]$ and $q_{50\%}(F_1)$ grow by about 1% for every increase of the target return μ^* by 1%. While from $\mu^* = 2\%$ to $\mu^* = 4\%$, the 1% quantile $q_{1\%}(F_1)$ decreases from about 109% to 107.28%, the 99% quantile $q_{99\%}(F_1)$ grows considerably stronger by about 7%. Since the 50% quantile grows by about 2% at the same time, this results in the 1% and 99% being approximately symmetrical around the 50% quantile. We thus observe that there is an increase in the general funding level despite the increased volatility. As the volatility stays comparatively low for all return values, the resulting underfunding probability remains zero (cf. Figure 4.5).

In part II, we set target values for the expected funding ratio $\mathbb{E}[F_1]$ at time $t = 1$ to $\bar{F} = 110\%$, 112% and 114%. Following this, the drift μ grows from 1.25% to 4.82%. Similarly, the volatility rises from less than 0.5% to more than 3% while the skewness and kurtosis remain roughly unchanged. While for a target value of $\mathbb{E}[F_1] = 110\%$ the portfolio contains almost only the money market, higher values lead to a shift that results in the government bonds making up more than half of the portfolio and the hedge funds reaching their maximum share of 15%. For the outer quantiles $q_{1\%}(F_1)$ and $q_{99\%}(F_1)$, we observe that they stay fairly symmetrical around the 50% quantile, being separated by almost 1% for $\mathbb{E}[F_1] = 110\%$. This gap then increases to about 4% for $\mathbb{E}[F_1] = 112\%$ and 8% for $\mathbb{E}[F_1] = 114\%$. The median $q_{50\%}(F_1)$ stays very close to the expected value. With the volatility reaching 3.06% at most, the dispersion still remains at a level that causes the underfunding probability to amount to zero.
Turning to part III, we set target values ϵ for the underfunding probability $\mathbb{P}[F_1 < 100\%]$. This way, we focus on the asset allocations that yield probabilities of financial distress that are equal to $\epsilon = 0.1\%$, 0.5% and 1%. As can be observed in Figure 4.5(b), these values correspond to portfolios with much higher values for the drift μ and the volatility σ. In fact, since $F_0 = 110\%$, there is room for risky investment. From the simulation results, it can be noted that a probability of financial distress of 0.1% causes the mean return to rise up to 5.73% and the volatility to 4.75%. For all of the three cases, the weight of the money market in the asset portfolios is zero and the hedge funds and the real estate are at their respective maximum shares of 15% and 30%. The proportion invested in the stocks is comparatively high, ranging approximately between 20% and 30%. The share of the government bonds decreases from about 35% for $\mathbb{P}[F_1 < 100\%] = 0.1\%$ down to almost 20% for a target value of 1%. From the high volatility, it follows that the outer quantiles of the funding ratio F_1 spread out very far. While the 1% quantile decreases to 100%, the 99% quantile reaches more than 136%. At the same time, the expected funding ratio $\mathbb{E}[F_1]$ and the 50% quantile $q_{50\%}(F_1)$ vary by less than 0.5% and remain mostly around 115%.

4.4.2 Asset Allocation with Best-Fit Distributed Returns

In the previous section, we have analyzed the resulting investment portfolios when using the utility function $V_1(\boldsymbol{\alpha}, \boldsymbol{r})$ and normally distributed returns. In the following, we depart this approach and look at the results when using the utility function $V_2(\boldsymbol{\alpha}, \boldsymbol{r})$ together with the return distributions that fit the historic returns best (cf. Section 4.3.2). This way, we want to find out how using moments of higher order in the utility function and more suitable return distributions alter the previously obtained results in Section 4.4.1. In order to ensure comparability, the parameter values at time zero and the risk factors remain unchanged (cf. Table 4.4). Figure 4.6 depicts the asset shares in the optimal portfolios for $V_2(\boldsymbol{\alpha}, \boldsymbol{r})$ with returns that are in agreement with the best-fit distributions. It can directly be compared with Figure 4.4.
As before, the portfolio with the smallest drift consists only of the money market. It thus accomplishes a mean return of 1.14%, which is equal to the mean return μ_1 of the money market (cf. Table 4.1). When increasing the portfolio return from this point on, we can see that at first only the government bonds are added to the portfolio, quickly obtaining a share of about 8%. After this, an investment in stocks takes place. Up to a drift of 4%, their share grows up to about 25%. Meanwhile, the real estate and the hedge funds remain at a single-digit value. From $\mu = 3.5\%$, the share of the hedge funds begins to increase quickly. Together with the real estate growing as well, this leads to the share of the stocks remaining constant. Once the hedge funds reach their maximum share at a portfolio return of about 5%, the stocks increase their share again. This continues until they reach their maximum share of $\alpha_4^{\max} = 50\%$. Subsequently, the real estate replaces the government bonds up to their respective investment limit of $\alpha_3^{\max} = 30\%$. The portfolio with the highest drift and return consequently is the same as in the minimum variance case. The course of the portfolio volatility σ is given on the second horizontal axis

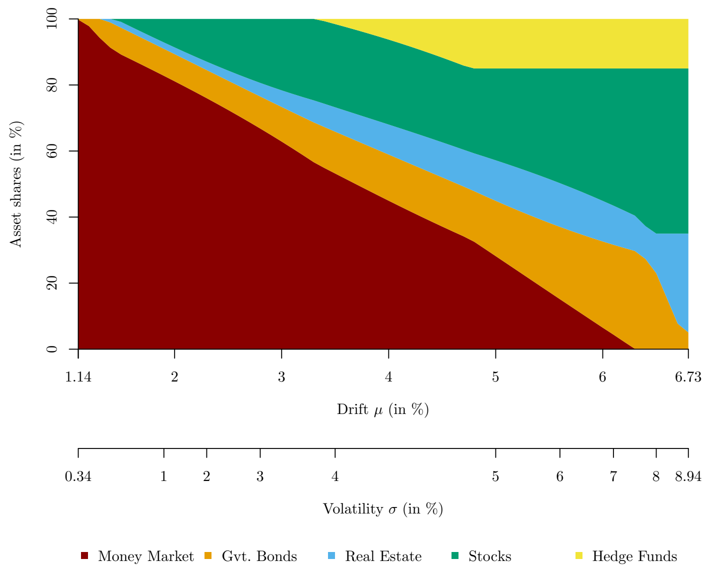
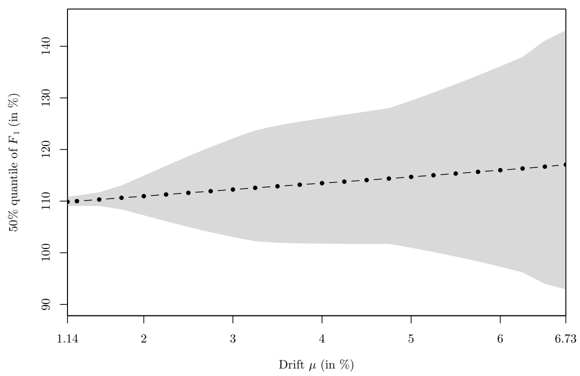
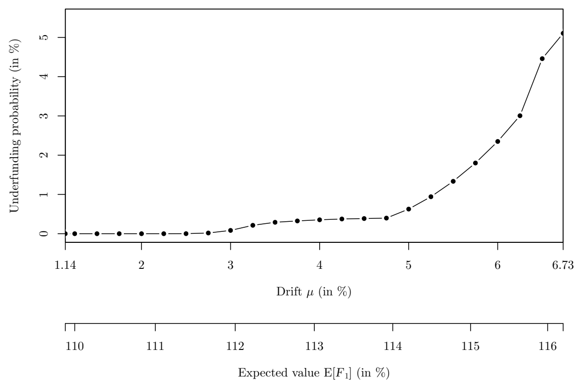


Figure 4.6: Optimal asset allocations for target return μ for $V_2(\boldsymbol{\alpha}, \boldsymbol{r})$ and best-fit distributed returns.

in Figure 4.6. There, it can be seen that the increase of σ is not linear. Up to a portfolio return of about 3.5%, when mainly the stocks are added to the portfolio, the volatility increases uniformly. Past that point, more weight is put on the hedge funds which have a lower dispersion. The growth of the portfolio volatility consequently slows down. Once the hedge funds reach the maximum share $\alpha_5^{\max}$, the percentages of the remaining assets in the optimal portfolio increase again. As a consequence, the portfolio volatility returns to increasing more strongly. We can also see this when looking at the 50% quantile of the funding ratio $q_{50\%}(F_1)$ and the underfunding probability $\mathbb{P}[F_1 < 100\%]$ which are given in Figure 4.7(a) and Figure 4.7(b), respectively.



(a) $q_{50\%}(F_1)$, gray area: $q_{5\%}(F_1) - q_{95\%}(F_1)$



(b) Underfunding probability $\mathbb{P}[F_1 < 100\%]$.

Figure 4.7: Plots of the 50% quantile of the funding ratio $q_{50\%}(F_1)$ and the underfunding probability $\mathbb{P}[F_1 < 100\%]$ at time one.

When looking at the 50% quantile of the funding ratio F_1 at time $t = 1$, we can see that it grows linearly with the portfolio drift μ. As for the minimum variance case, $q_{50\%}(F_1)$ grows from about 110% for a return of 1.14% up to about 116% for the highest value of the drift μ. Looking at the 5% and 95% quantiles $q_{5\%}(F_1)$ and $q_{95\%}(F_1)$ in Figure 4.7(a), we can see a course that corresponds to the one of the volatility. The shaded area that they mark, grows hardly at the beginning when only the government bonds are added to the portfolio. Subsequently, the rise of the stock share leads to a stronger expansion of that area, which is only interrupted by the hedge fund investment between the portfolio returns of $\mu = 3.5\%$ and 5%. A significant change also takes place in the probability for underfunding in Figure 4.7(b). In contrast to the minimum variance case depicted in Figure 4.5(b), it already takes non-zero values for a portfolio drift of about 3% and does not grow as steadily. We can see that the curve instead rises between values of $\mu = 3\%$ and 3.5% because of the rise of stock shares. After that it almost remains constant at a value of $\mathbb{P}[F_1 < 100\%] = 0.5\%$ due to the stronger investment in hedge funds. The rise that takes place for portfolio drifts higher than 5% follows from the higher investment in stocks. For a drift of more than 6%, the probability to become underfunded experiences a steep increase. This is due to the strong increase of the real estate share in the portfolio that we saw when looking at Figure 4.6. Overall, we observe that the underfunding probability is higher when using utility function $V_2(\boldsymbol{\alpha}, \boldsymbol{r})$ and best-fit distributed returns compared to $V_1(\boldsymbol{\alpha}, \boldsymbol{r})$ and normally distributed returns. Not only does the curve attain non-zero values for lower values of μ, but it also achieves a higher maximum value of about 5%.

Condition on	I: Mean return			II: Expected funding ratio			III: Underfunding probability		
μ	**2.00**	**3.00**	**4.00**	1.24	3.03	4.76	3.04	4.88	5.29
σ	1.36	3.40	4.32	0.37	3.41	4.62	3.47	4.85	5.55
$\bar{\gamma}$ $[\times 10^{-2}]$	0.09	0.10	0.10	0.24	0.10	0.10	0.10	0.10	0.10
$\bar{\kappa}$ $[\times 10^{-2}]$	0.09	0.10	0.09	0.15	0.10	0.09	0.10	0.09	0.09
α_1	81.12	62.86	44.96	96.32	62.12	32.99	62.06	30.78	21.87
α_2	8.22	10.48	13.97	3.68	10.62	15.03	10.66	15.86	19.06
α_3	2.04	5.03	9.10	0.00	5.68	12.25	5.23	11.81	13.06
α_4	8.62	21.63	25.72	0.00	21.58	24.92	22.05	26.54	31.01
α_5	0.00	0.00	6.26	0.00	0.00	14.81	0.00	15.00	15.00
$\mathbb{E}[F_1]$	110.83	111.94	113.07	**110.00**	**112.00**	**114.00**	111.99	114.07	114.54
$q_{1\%}(F_1)$	107.26	103.05	101.76	109.09	103.06	101.83	102.91	101.36	100.00
$q_{50\%}(F_1)$	110.96	112.25	113.48	109.99	112.29	114.39	112.30	114.54	115.07
$q_{99\%}(F_1)$	114.93	122.15	126.08	111.06	122.21	127.91	122.41	128.73	131.30
$\mathbb{P}[F_1 < 100\%]$	0.00	0.08	0.35	0.00	0.08	0.37	**0.10**	**0.50**	**1.00**

Table 4.6: Simulation results using extended utility and alternative returns. All values are given in %.

This also becomes clear when we look at the simulation results in Table 4.6. As before, we have three parts where we set target values for the mean return, the expected funding ratio and the underfunding probability. In order to ensure the comparability of the results, we utilize the same target values as in Table 4.5. Looking at the results for a target return of $\mu = 2\%$, we are able to see that the volatility amounts to 1.36%. When we increase the target return, it leads to a growth of the volatility by about 2%. Comparing the simulation results with the ones in Table 4.5, we can see a general increase in the volatility. This is caused by departing from the minimum variance setting. At the same time, the skewness remains unchanged at a value of $\bar{\gamma} = 0.1$, which is the same as before. For the kurtosis $\bar{\kappa}$, a change of the values has taken place. Instead of being close to 0.03, it now equals approximately 0.10 for all the simulations. As a consequence to this increase, the portfolio return can be said to have become more peaked. For the asset weights, the values mirror the observations made in Figure 4.6. While the share of the money market decreases from 81.12% down to 44.96%, the stocks experience the strongest growth, rising from 8.62% to 25.72%. At the same time, the money market only grows by 5.75%, while the real estate stays below 10% and the hedge funds achieve at most 6%. Due to the increase in the drift, the expected funding ratio $\mathbb{E}[F_1]$ grows from 110.83% for $\mu = 2\%$ up to 113.07% for $\mu = 4\%$, with the 50% quantile $q_{50\%}(F_1)$ being close to those values. Due to the increased volatility of the returns, the outer quantiles shift considerably. The 1% quantile $q_{1\%}(F_1)$ decreases from a value of 107.26% down to 101.76%, and thus almost becomes underfunded. The 99% quantile $q_{99\%}(F_1)$ meanwhile grows by 11% and exceeds 125% for a target portfolio drift of $\mu = 4\%$. The strong decrease of the 1% quantile of F_1

also impacts the underfunding probability. Despite being zero for a target return of 2%, $\mathbb{P}[F_1 < 100\%]$ equals 0.08% for $\mu = 3\%$ and even exceeds 0.3% for another increase of the return by 1%.
When setting target values for the expected funding ratio $\mathbb{E}[F_1]$ of 110%, 112% and 114%, the drift μ increases from 1.24% to almost 5%. Similarly, the volatility reaches 4.62%. While being equal to 0.1% for the remaining values, the skewness $\bar{\gamma}$ increases to 0.24% for $\mathbb{E}[F_1] = 110\%$. In a similar fashion, the kurtosis $\bar{\kappa}$ amounts to 0.15%. Thus, the return distribution is more right skewed and peaked for an expected funding ratio of 110%. Looking at the asset shares, we can see that the percentage of the money market drops from nearly 96% down to 33%. In the meantime, the government bonds and the real estate both grow, but stay at about 15%. The strongest growth is taking place for the stocks, which increase from zero up to more than 25%. For $\mathbb{E}[F_1] = 114\%$, the hedge funds reach almost 15%, while being zero for lower values. For the quantiles of the funding ratio F_1, we can make observations similar to the ones when setting target returns. With the quantiles $q_{1\%}(F_1)$ and $q_{99\%}(F_1)$ reaching 101.83% and 127.91% for a value of $\mathbb{E}[F_1] = 114\%$, the gap between the most extreme outcomes widens. Consequently, the resulting probability of financial distress reaches nearly 0.4%.
The simulation results for an underfunding probability of $\mathbb{P}[F_1 < 100\%] = 0.5\%$ are almost identical to the ones for an expected funding ratio of 114%. Increasing the probability of underfunding to 1% increases the drift μ to about 5.30%, but the volatility σ to 5.55% as well. As a consequence, the shares of the asset classes are distributed more evenly, with every value being approximately between 15% and 30%. Due to the underfunding probability being 1%, $q_{1\%}(F_1)$ equals exactly 100%. At the same time, the expected funding ratio exceeds 114.5% and the 99% quantile reaches about 131%. For an underfunding probability of 0.1%, the results for the return distribution turn out less extreme. We can see a clear shift in the asset weights since the biggest part of the portfolio is made up by the money market and the stocks. We are therefore able to see that an increased underfunding probability leads to an increased dispersion and drift, corresponding to an investment in assets with both higher returns and volatility.
When comparing the optimal asset allocations to the ones obtained by using minimum variance theory (cf. Table 4.5), we can see a clear shift in the percentages. In particular, the shares allocated to hedge funds decrease. While they previously reached their maximum value of $\alpha_5 = 15\%$ for a target return of $\mu^* = 4\%$, they now achieve about half of that. Their replacement by stocks is linked to the moments of the return distributions. The higher historical variance of the stock return compared to the hedge funds is detrimental in the mean-variance case. In a similar fashion, a decrease in the percentages invested in government bonds takes place. Depending on the target condition, this can even exceed 40%. While the real estate shares change only little for target values for the mean return μ^* and the expected funding ratio $\mathbb{E}[F_1]$, its shares more than half for a target underfunding ratio. The money market experiences gains of up to 25% for target returns and funding ratios and are even larger ones for target underfunding probabilities. There, the shares change from a value of zero to more than 60% for $\mathbb{P}[F_1 < 100\%] = 0.1\%$. The other asset class that experiences an increase are the stocks. In the minimum variance portfolios, their portion always remains fairly small as they are the asset class that has by far the highest volatility. For the extended utility function, the stocks often reach between 20% and 30%. This way, they cause the strong increase in the portfolio volatility σ that we observed. Overall, we conclude that there is a shift towards the money market and the stocks when using the extended utility function and the alternative returns. A possible explanation for this can be seen in Table 4.1 where the first four moments of the asset classes are given. We observe that for the skewness $\bar{\gamma}_i$ and the kurtosis $\bar{\kappa}_i$, the money market and the stocks take the most extreme values. While for the skewness, the money market takes the highest value and the stocks the lowest one, this relation is reversed for the kurtosis. The stock returns achieve the highest percentage while the money market has the lowest one. It could therefore be inferred that a shift to assets with more extreme skewness and kurtosis takes place when looking at the extended utility

function.

4.4.3 Sensitivity Analysis on Risk Factors

In the previous section, we have examined how the optimization and simulation results change when using the utility function $V_2(\boldsymbol{\alpha}, \boldsymbol{r})$ and assets distributed according to the best fitting distributions. In this part, we want to go further into detail and analyze the effects of risk preferences. The utility function $V_2(\boldsymbol{\alpha}, \boldsymbol{r})$ incorporates the risk coefficients λ_1, λ_2 and λ_3, regarding the volatility, the skewness and the kurtosis. We aim to study their influence on the outcome of the utility optimization. To this end, we vary the parameters one after the other, taking values of $\lambda_i = 0$, 2 and 5, while fixing a target portfolio drift of $\mu = 3\%$.[6] The results from this are shown in Table 4.7.

	I	II			III			IV		
λ_1	1	**0**	**2**	**5**	1	1	1	1	1	1
λ_2	1	1	1	1	**0**	**2**	**5**	1	1	1
λ_3	1	1	1	1	1	1	1	**0**	**2**	**5**
μ	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00
σ	3.40	3.49	3.31	2.75	2.37	2.15	2.33	2.33	2.87	2.56
$\bar{\gamma}$ $[\times 10^{-2}]$	0.10	0.10	0.10	0.10	0.12	0.12	0.12	0.12	0.10	0.11
$\bar{\kappa}$ $[\times 10^{-2}]$	0.10	0.10	0.10	0.10	0.23	0.06	0.07	0.07	0.11	0.15
α_1	62.86	63.41	62.33	62.39	55.90	72.43	73.98	73.98	59.73	57.60
α_2	10.48	9.89	11.06	11.83	16.51	2.82	0.01	0.00	13.39	15.16
α_3	5.03	4.34	5.67	6.35	16.04	0.00	0.00	0.00	9.66	13.13
α_4	21.63	22.35	20.94	16.55	11.55	9.75	11.02	11.02	17.23	14.11
α_5	0.00	0.00	0.00	2.87	0.00	15.00	15.00	15.00	0.00	0.00
$q_{1\%}(F_1)$	103.05	102.79	103.29	104.75	105.85	106.26	105.79	105.80	104.47	105.31
$q_{50\%}(F_1)$	112.25	112.25	112.25	112.22	112.23	112.09	112.10	112.10	112.25	112.25
$q_{99\%}(F_1)$	122.15	122.43	121.89	120.24	119.20	118.46	118.97	118.97	120.62	119.73
$\mathbb{P}[F_1 < 100\%]$	0.08	0.11	0.06	0.00	0.00	0.00	0.00	0.00	0.01	0.00

Table 4.7: Results using extended utility and best-fit returns in %. Testing different risk preferences for $\mu = 3\%$ and $\mathbb{E}[F_1] = 111.94\%$.

For guidance, the reference case of $\lambda_1 = \lambda_2 = \lambda_3 = 1$ is given in part I. Following that, we vary one coefficient at a time while keeping the other ones unchanged. This way, only the impact of the respective coefficient λ_i is analyzed. Looking at part II, where we vary λ_1, it can be seen that there is only little change in the simulation results. From $\lambda_1 = 0$ to $\lambda_1 = 5$, the volatility decreases by less than 1%. The skewness and kurtosis of the returns even remain constant at 0.1. For the asset weights, an increase of λ_1 puts more weight on the portfolio variance. Consequently, the resulting asset weights start to converge towards the outcome of the minimum variance case (cf. Table 4.5). For the outer quantiles $q_{1\%}(F_1)$ and $q_{99\%}(F_1)$, the slight decrease of the volatility results in a change of less than 2%. Consequently, the underfunding probability reduces from 0.11% to 0%. We can see noticeably bigger differences when varying λ_2. Setting it to zero causes the volatility to decrease to 2.37%. Furthermore, there is an increase in both, the skewness and kurtosis. For the skewness $\bar{\gamma}$, we observe a value of 0.12, while the kurtosis $\bar{\kappa}$ reaches 0.23. Thus the returns are more right skewed and peaked. For the asset weights, the government bonds and the real estate have shares of about 16%, while the stocks remain at a lower value and the hedge funds at zero. The remainder is accounted for by the money market. The shares change much

[6] While the sensitivity analysis on the risk preferences in the utility function is performed by using arbitrary numbers, it has been shown in the literature that there are interrelations between the preferences for the different return moments. As this interdependence needs to taken into account, it is difficult to set values for the different factors. For further information, see, e.g., Kraus and Litzenberger (1976) and Harvey et al. (2010).

more when altering λ_2 to a value of two. The investment in the government bonds and real estate almost go down to zero, while the stocks decrease to 9.75%. The money market makes up more than 70% of the portfolio while the portion of the hedge funds is 15%. As a consequence of this, the volatility decreases slightly. The kurtosis returns to a value of 0.06, while the skewness remains at 0.12. The 99% quantile $q_{99\%}(F_1)$ reaches 118.46%, while the 1% quantile $q_{1\%}(F_1)$ increases to 106.26%. As a result, the underfunding probability $\mathbb{P}[F_1 < 100\%]$ remains at zero. Another change of the outcome can be seen when setting $\lambda_2 = 5$. For this value, the resulting portfolio consists mostly of the money market and the hedge funds. The only other asset to be invested in, is the stocks. Due to this, the volatility increases back to 2.33%, which causes an increase of the confidence interval. We can also see variations in the simulation results when varying λ_3 in part III. When we alternate the factor, the volatility changes by less than 0.5%. Alterations in the quantiles of the funding ratio and the underfunding probability thus remain small as well. For the kurtosis and the asset weights, there are more pronounced differences. The kurtosis increases from 0.07 to 0.11 and 0.15. At the same time, the portfolio becomes more diversified. While mostly consisting of the money market and hedge funds for $\lambda_3 = 0$, the shares of the remaining three asset classes already make up more than 40% for $\lambda_3 = 2$. At the same time, the share of the hedge funds drops to zero and the proportion invested in the money market decreases by about 14%. For $\lambda_3 = 5$, this trend continues, albeit in alleviated terms.
Overall, we are able to see that changes in the risk factors lead to considerable differences in the outcomes. This does not involve a uniform trend. For changes with respect to the portfolio volatility, we observe only little variation in the optimal portfolio for small values. For higher values of λ_1, it can be seen that the asset weights start to converge towards the outcome of the minimum variance case. The results are relatively robust with respect to risk aversion linked to the second moment of the portfolio distribution. The differences are more pronounced when we vary the second and third coefficient λ_2 and λ_3. As a result of this, there are strong shifts in both, the portfolio weights and the return characteristics. We conclude that preference with respect to the different moments plays an important role in simulations and needs to considered with care.

4.5 Conclusion

In this research, we look at the impact that higher moments have on the optimal asset allocation of a pension fund. The research question was: Do higher moments lead to a change in the optimal portfolio weights and are different random distributions able to provide an improved fit of the return data. To this end, we compare how the Markowitz efficient portfolio asset weights change when using a more complex utility function that also involves the third and fourth moment of the historic asset returns. Using the so obtained portfolio weights, we simulate the assets of the fund in a one-period model. Therein, normally distributed returns are compared to alternative ones that fit the data in a better way. Furthermore, it is examined how a more complex risk preference with respect to several return moments impacts the optimization results. In order to perform this, we carry out a sensitivity analysis in which we vary the different factors and analyze the changes in the outcome. In our work we consider three different key indicators that are relevant for pension fund management in practice: the target return, the expected one-period funding ratio and the one-period underfunding probability.

Expectedly, our results indicate that the use of an extended utility function does indeed lead to a shift in the optimal portfolio weights. Consequently, main portfolio characteristics change. Among others, a strong increase of the volatility takes place. Following this, the outer quantiles of the funding ratio spread further and the underfunding probability increases. We are therefore able to say that using the minimum variance portfolio can cause misleading security. Working with an extended utility function that departs from the minimum variance framework and incorporates higher moments of returns consequently allows companies to assess their risk taking more adequately.

Along with the utility function, pension funds should also consider looking at the adequate distributions for their asset classes. As previous research shows, the normal distribution is not able to fully reflect the characteristics of asset returns. Comparing it to three other distributions, we are able to find a distribution that fits the historic return data better for almost every type of assets. Using these, we are able to simulate returns that are not symmetric and put more weight on the tails, properties that are characteristic for capital market returns. Consequently, the characteristics of the optimal portfolios such as the skewness and kurtosis change in our simulations. In particular, the return distribution alters to having a longer right tail and being less peaked. We are therefore of the opinion that pension funds, and financial institutions in general, need to consider using more suitable distributions for their assets.

Having analyzed the effects of an extended utility function and different asset distributions, we examine the impact of risk preferences. To this end, we look at a vector of risk factors with respect to the second, third and fourth moment of the portfolio return. The results show that the outcome changes only little when varying the parameter related to the volatility. In contrast to this, we see strong variations when changing the coefficients of the skewness and kurtosis. As a consequence of this, the optimal portfolio weights shift together with its characteristics. It is thus necessary to use a more complex risk preference when optimizing utility functions.

This work analyzes the impact of an extended utility function together with different asset distributions and a more complex risk preference. We observe that there are strong changes in the resulting optimal portfolios and their characteristics. It can therefore be concluded that pension funds would profit considerably from utilizing the aforementioned changes. While the "best" asset allocation is still to be interpreted in the light of the used hypotheses and objectives, the investment shares – differing little or a lot – from the different methods give information about the stability and robustness of the calculated allocations. While being focused on the pension fund system, we believe that our results hold true for financial institutions in general. Possible extensions of our work would be to study a higher number of asset classes relevant for various institutions. With respect to this, more detailed work on preferences with respect to different moments would also be needed. Limitations of our work lie within the complexity of solving the optimization problem for a higher number of assets. Practical implementation is often impeded by the high number of parameters that must be estimated and set as input.

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Chapter 5

Optimal Calibration of Annuities in Swiss Pension Funds under Consideration of Financial and Biometric Risks

The financing of the retirement phase plays a particularly important role for the insureds of a pension fund. Since pension payments often represent the only source of income after retirement, securing the payment of annuities for the remainder of the lifetime is crucial. In recent years, pension funds have been under pressure from the longevity of individuals and lower and more volatile interest rates. Furthermore, parameters like the conversion rate used to calculate the annuity from the available capital at retirement need revision and are to some extend fixed by the regulator. In our research, we study the retirement phase of a defined contribution (DC) Swiss pension scheme (second pillar), where annuities are calculated from the available individual stock of capital using a conversion rate. Our study involves examining the impact of external changes, such as longevity, financial market performance and the technical interest rate. We analyze the impact of variations in the different risk factors using analytical expressions as far as possible. Looking at the sensitivity of the results, we quantify and compare the impact of the biometric and financial uncertainties and evaluate the parameters required to keep a long-term balance. We find that both, the capital market returns and the lifetime, considerably impact the conversion rate. Comparing the two factors, we see that the influence of the investment returns is significantly greater than the one of longevity. This holds particularly when looking at scenarios and random distributions for the return. Pension funds as well as regulators can profit from incorporating the dependency of the conversion rate on financial and biometric factors into their procedures.

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5.1 Introduction

The increasing lifetime of individuals has a strong impact on financial organizations. As their members live longer, insurance companies and pension funds are facing challenges with respect to the duration of their products. As they usually involve specific payments and guarantees, contracts are very sensitive to changes in the expected lifetime of the members. For pension funds, this is especially true for the retirement phase. As insureds enter the retirement phase, they are entitled to regular guaranteed annuity payments until they die. Consequently, it is crucial for the fund that the accumulated savings suffice in order to cover the future pension payments. With respect to this, investing the earnings on the capital market plays an important role. As the pension fund is able to invest the savings and consequently generate interest returns, it is able to make a significant contribution to the available funds. Regarding the calibration of the pension fund system, it is therefore of importance to choose the parametrization in a way, such that sufficient reserves are available in order to meet future obligations.
In this work, we study the retirement phase of a defined contribution Swiss pension scheme. To this end, we study the adequate choice of the conversion rate with respect to the curtate expected lifetime of the member and the expected capital market return. Using historic data, we quantify the impact of both factors and compare their effects. For the life expectancy of the client, this involves using mortality data in order to model the future development of the survival probability. For the investment return, we utilize the past returns of a Swiss pension fund index. Furthermore, we make use of predefined capital market scenarios and a simulation of random returns. This way, it is analyzed to what extend the pension fund system might need to respond to certain financial scenarios and distributions. The research question is: How do the life expectancy of the members, the return from capital market investments and the technical interest rate influence the choice of the conversion rate?

The model that has been developed by Lee and Carter (1992) has quickly become a popular tool for modeling the mortality of individuals and longevity in particular. As it takes the historic development of the death rates into account, it provides a method to forecast the future survival probabilities (Lee and Miller, 2001). Over the years, the original model has been applied in various cases and has been extended for different purposes (Lee, 1998, 2000; Renshaw and Haberman, 2006). For the population in Switzerland, research has been conducted looking at the main causes of death as well as comparing mortality forecasts from academia to the ones used by practitioners (Arnold-Gaille and Sherris, 2013).
The ever-increasing lifetime of individuals is posing challenges for insurance institutions in general and pension funds in particular (Albrecher et al., 2016; Macdonald et al., 1998). As the members are living longer and longer, they receive more annuity payments over their lifetime. This consequently impacts the transfers between active and retired fund members (Eling, 2013; Avanzi and Purcal, 2014). Capital market returns have further been decreasing considerably over the past years. Along with this, an increased volatility poses difficulties for pension funds to achieve stable return streams (Devolder, 2011; Bikker et al., 2012; Gerber and Weber, 2007; Hainaut and Devolder, 2007). It has been studied in the literature how financial institutions can respond to these new circumstances (Berdin and Gründl, 2015; Yao et al., 2014). For pensions funds, there has additionally been extensive research on how pension systems could be reformed on a general scale (Blake, 2000; Poterba et al., 2007; Bodie et al., 1988) as well as for specific countries (Börsch-Supan and Wilke, 2004; Chang, 1999; Mao et al., 2008).
For the second pillar pension fund system in Switzerland, there have been efforts in recent years to reform the system, albeit with differing degrees of success (Swisscanto, 2018; Bütler, 2009). As Swiss retirees seek to increase their savings and pension payments, there has been a trend to go from early retirement to working in old age (Dorn and Sousa-Poza, 2003, 2005; Hanel and Riphahn, 2012). Additionally, individuals who enter into retirement mostly favor annuities over lump-sum payments (Bütler

and Teppa, 2007; Avanzi, 2010). Considering the financial state of the funds, it has further been analyzed how to measure and assess the solvency situation of a fund and how additional contributions as well as surplus distributions affect its stability (Braun et al., 2011; Müller and Wagner, 2017).

In our work, we focus on the retirement phase of a pension fund in Switzerland. To this end, we take particular interest in the pension amount that is regularly paid out to a cohort of members. In the Swiss system, this amount is calculated as a percentage of the savings at retirement, called the conversion rate. The right choice of this parameter is the main focus of our work. We study how the conversion rate depends on the lifetime of the insured, the investment returns and the technical interest rate. Using closed-form expressions, we are able to show that the choice is independent of the volume of the savings at retirement. With respect to the mortality of the clients, we study the historical development of the curtate expected lifetime at retirement. Using a Lee-Carter model, we forecast mortality, survival probabilities and the life expectancy for future decades in order to quantify how the conversion rate would need to be adjusted due to longevity. Looking at the historic investment returns of pension funds, we analyze the historic annual returns of a common Swiss pension fund index. The values indicate that there is a strong dispersion in the capital market returns achieved in the past years. The technical interest rate represents the discounting factor that is used to calculate the technical reserves of a pension fund. It is therefore important to set the technical interest in a reasonable and appropriate way, as changes can lead to strong fluctuations in the required reserves. We consequently compute the conversion rate for a wide range of investment returns. In addition, we consider predefined return scenarios and a random return that follows a specific distribution. This way, we aim to assess how the conversion rate needs to be adjusted in response to return developments.

The remainder of the paper is structured as follows. Section two presents the model framework and the derivation of the analytic expression for the conversion rate. The third Section involves the introduction of the Lee-Carter model. Based on the survival probabilities, we develop a mortality multiplication model in order to compute target values for the curtate expected lifetime. In addition, we present the historic performance of capital investments using the example of a Swiss pension fund index. In Section four, we present the results of our computations. Conducting a sensitivity analysis, we quantify and compare the impact of the investment returns and the lifetime of the members on the conversion rate. In addition, random returns as well as return scenarios are considered. The final Section concludes.

5.2 Model Framework

Our modeling is interested in putting the various parameters used in defined contribution (DC) pension funds for calculating the pensions of a cohort of insureds into relation using a sustainable method. In a typical DC pension fund the actives have collected a savings amount valued A_0 at the moment of retirement. This value lays the basis for the calculation of the yearly pension benefit. Over the retirement phase, while a pension payment is deducted periodically, the assets remaining in the fund earn investment returns. The pension is calculated as a share of the initially available capital amount, taking into account future returns, mortality assumptions and a discounting factor for the liabilities. First, considering the available pension assets at retirement, we link the pension payments to the death probabilities, the investment returns and the technical interest rate. Thereby, we derive a closed-form solution for the conversion rate.

5.2.1 Available Pension Assets during Retirement

We consider the retirement phase of a DC pension fund where the retirement pension is calculated from the value of the savings at retirement. Therein, we assume a cohort of homogeneous members entering into retirement at age x (typically 65 years for men and 64 years for women in Switzerland) with a cumulated savings account balance of A_0 (at time $t = 0$ all members bring in the same amount). The annual pension that the individual receives, is calculated by multiplying the savings A_0 with the so-called conversion rate cr (see BVG, Art. 14). In Switzerland, this rate is regulated for the mandatory part of second pillar savings and amounts to $cr = 6.8\%$ in 2018. The sum of the annuities paid out to the pool of members, consequently equals $A_0 \cdot cr$, i.e. it is constant and calculated as a percentage of the account value at retirement. In our model, we assume that the pension is paid out at the beginning of every period, to the share of members that is still alive. For the beginning of the first year of retirement, it thus holds that

$$A_{0^+} = A_0 - A_0 \cdot cr. \tag{5.1}$$

The probability that an individual of age x is still alive after t periods is denoted by ${}_tp_x$. We have ${}_0p_x = 1$ and ${}_tp_x < 1$ for $t > 0$. The amount that is available for investment at the beginning of each year becomes

$$A_{(t-1)^+} = A_{t-1} - A_0 \cdot cr \cdot {}_{t-1}p_x, \quad t = 1, 2, \ldots, \tag{5.2}$$

where A_t, $t = 0, 1, 2, \ldots$ is the asset value at the end of year t.

During the year, the savings are invested on the capital market by the fund. The investment return in period t is denoted by r_t. With $R_t = 1 + r_t$, it consequently follows for the savings account at the end of the year that

$$A_t = A_{(t-1)^+} \cdot (1 + r_t) = (A_{t-1} - A_0 \cdot cr \cdot {}_{t-1}p_x) \cdot R_t, \quad t = 1, 2, \ldots \tag{5.3}$$

Consequently, the remaining value A_t of the savings account can be obtained from the previous one, A_{t-1} by subtracting the annuity $cr \cdot A_0$ and crediting the investment return r_t. The annuity payment is thereby weighted with the survival probability ${}_{t-1}p_x$ since it is only paid out to the surviving individuals. From the above formulas, the available assets at times $t = 1, 2, \ldots$ can be explicitly written out as follows

$$\begin{aligned}
A_1 &= (A_0 - A_0 \cdot cr) \cdot R_1, \\
A_2 &= (A_1 - A_0 \cdot cr \cdot {}_1p_x) \cdot R_2, \\
A_3 &= (A_2 - A_0 \cdot cr \cdot {}_2p_x) \cdot R_3, \\
&\vdots \\
A_t &= (A_{t-1} - A_0 \cdot cr \cdot {}_{t-1}p_x) \cdot R_t. \qquad (5.4)\\
&\vdots
\end{aligned}$$

The formula for A_t in Equation (5.4) can be rewritten in a non-recursive way. By plugging A_{t-1} into A_t, A_{t-2} into A_{t-1}, and so on, we find

$$A_t = A_0 \cdot R_t \cdot \left(\prod_{k=1}^{t-1} R_k - cr \cdot \sum_{k=0}^{t-1} {}_{t-k-1}p_x \prod_{j=1}^{k} R_{t-j} \right), \quad t = 1, 2, \ldots. \tag{5.5}$$

5.2.2 Derivation of the Conversion Rate

Naturally, an appropriate choice of the conversion rate cr is crucial to the functioning of the fund. If it is set too high, the savings of the pensioners might not suffice on average in order to make the annuity payments over the entire lifetime. A way to balance this would be to invest in assets with higher returns. This, in turn, would be connected to taking more risks and come with a typically higher volatility in the outcome. Choosing the conversion rate too low is not favorable to the members either. If there are savings left at the end of the lifetime, it implies that higher payments and consequently a higher living standard would have been possible. Our goal is therefore to discuss the conversion rate cr depending on three main "ingredients":

1. the life expectancy through mortality tables,
2. the capital market returns,
3. and the technical interest rate for discounting the liabilities.

For the adequate value of cr, the sum of the discounted contributions to the asset account should equal the sum of the discounted payments to the pensioners. The sum of the pension payments discounted to time zero amounts to

$$\sum_{t\geq 0} A_0 \cdot cr \cdot {}_tp_x \cdot \nu^t = A_0 \cdot cr \cdot \sum_{t\geq 0} {}_tp_x \cdot \nu^t, \tag{5.6}$$

with ν denoting the discounting factor which is defined as $\nu = \frac{1}{1+z}$. In this, z denotes the technical interest rate (SKPE, 2015).

The contributions to the account consist of the savings A_0 at the time of retirement and the investment earnings during the following years. For the present value PV of the contributions C_t to the account at time t during the retirement phase, it holds that

$$\begin{aligned}
PV(C_0) &= A_0, \\
PV(C_1) &= (A_0 - A_0 \cdot cr \cdot {}_0p_x) \cdot r_1 \cdot \nu^1 = A_0 \cdot r_1 \cdot \nu \cdot (1 - cr \cdot {}_0p_x), \\
PV(C_2) &= (A_1 - A_0 \cdot cr \cdot {}_1p_x) \cdot r_2 \cdot \nu^2 = A_0 \cdot r_2 \cdot \nu^2 \cdot (R_1 - cr \cdot {}_0p_x \cdot R_1 - cr \cdot {}_1p_x), \\
&\vdots \\
PV(C_t) &= A_0 \cdot r_t \cdot \nu^t \cdot \left(\prod_{k=1}^{t-1} R_k - cr \cdot \sum_{k=0}^{t-1} {}_{t-k-1}p_x \prod_{j=1}^{k} R_{t-j} \right). \qquad (5.7) \\
&\vdots
\end{aligned}$$

It follows that the sum of the contributions discounted to time zero amounts to

$$\sum_{t\geq 0} PV(C_t) = A_0 \cdot \left(1 + \sum_{t\geq 1} r_t \cdot \nu^t \cdot \left(\prod_{k=1}^{t-1} R_k - cr \cdot \sum_{k=0}^{t-1} {}_{t-k-1}p_x \prod_{j=1}^{k} R_{t-j} \right) \right). \tag{5.8}$$

Setting the sum of the discounted contributions from Equation (5.8) equal to the sum of the discounted pension payments expressed in Equation (5.6) and solving for cr, we get

$$cr = \frac{1 + \sum_{t \geq 1} r_t \cdot \nu^t \cdot \prod_{k=1}^{t-1} R_k}{\sum_{t \geq 0} {}_t p_x \cdot \nu^t + \sum_{t \geq 1} r_t \cdot \nu^t \cdot \sum_{k=0}^{t-1} {}_{t-k-1} p_x \cdot \prod_{j=1}^{k} R_{t-j}}. \tag{5.9}$$

It is thus possible to derive a closed-form expression for the contribution rate cr in our framework, provided that the survival probabilities and the returns are known and the technical interest rate is given. Additionally, we observe that the value of cr is independent of the savings at retirement A_0.
In actuarial practice, it is assumed that individuals can only reach a certain maximum age (very high, representing the limit of available statistics), denoted ω. Consequently, one cannot survive past that age. Taking this into account, Equation 5.9 simplifies to

$$cr = \frac{1 + \sum_{t=1}^{\omega-x} r_t \cdot \nu^t \cdot \prod_{k=1}^{t-1} R_k}{\sum_{t=0}^{\omega-x} {}_t p_x \cdot \nu^t + \sum_{t=1}^{\omega-x} r_t \cdot \nu^t \cdot \sum_{k=0}^{t-1} {}_{t-k-1} p_x \cdot \prod_{j=1}^{k} R_{t-j}}. \tag{5.10}$$

5.3 Model Calibration

Having introduced the research setup, we provide the model calibration for the various parameters in this Section. We numerically study the problem on the basis of the population of Switzerland and by using historic investment returns of an index followed by many Swiss pension funds. This encompasses modeling the mortality of the cohort of the insured with the help of the Lee-Carter model. We present and discuss the historic data that we use for modeling and forecasting. In addition to this, we introduce a mortality multiplication model, with the help of which we vary the mortality in order to calibrate target values for the life expectancy. This value and its variation are often in the center of the political discussion. Aside from the mortality of the members, our model depends on the returns from investing the capital savings on the markets. We present historic return data in the Swiss pension sector and discuss the implications that variations have in our calculations.

5.3.1 Force of Mortality and Survival Probability

In our model, we assume that individuals are exposed to mortality. In the following, we introduce some notations following Bowers et al. (1989). If we denote the lifetime of an individual by X, a continuous random variable, then the probability that a newborn child is still alive at age x is given by

$$S_0(x) = \mathbb{P}[X > x]. \tag{5.11}$$

The function $S_0(x)$ is called the survival function. It follows that the cumulative distribution function of X is defined as $F_0(x) = \mathbb{P}[X \leq x]$. For an individual that is already x years old, the probability of still being alive at age $x + t$ is defined as

$${}_t p_x \equiv S_x(t) = \mathbb{P}[X > x + t | X > x] = \frac{S_0(x+t)}{S_0(x)}. \tag{5.12}$$

Conversely, the probability for an individual of age x to die within t years is

$$ {}_tq_x \equiv 1 - {}_tp_x = \mathbb{P}[X \leq x + t | X > x]. \tag{5.13} $$

In order to measure the probability of death in the near future, the force of mortality is used. For an individual at age x, it is defined as

$$ \mu(x) = \lim_{\Delta \to 0} \frac{\mathbb{P}[x < X \leq x + \Delta | X > x]}{\Delta} \tag{5.14} $$

For the time period Δ converging to zero, $\mu(x)$ describes the probability of instant death, i.e. the force of mortality which can be written with the help of the survival function as

$$ \mu(x) = -\frac{1}{S_0(x)} \cdot \lim_{\Delta \to 0} \frac{\mathbb{P}[X \geq x + \Delta] - \mathbb{P}[X > x]}{\Delta} = -\frac{1}{S_0(x)} \cdot \frac{d}{dx} S_0(x) = -\frac{d}{dx} \log\left(S_0(x)\right). \tag{5.15} $$

We are thus able to link the survival probability ${}_tp_x$ introduced in Equation 5.12 to the force of mortality $\mu(x)$. Using Equations (5.12) and (5.15) and solving for ${}_tp_x$, we get

$$ \begin{aligned} & -\mu(y)dy = d\log\left(S_0(y)\right) \\ \Leftrightarrow \quad & -\int_x^{x+t} \mu(y)dy = \log\left(\frac{S_0(x+t)}{S_0(x)}\right) \\ \Leftrightarrow \quad & -\int_x^{x+t} \mu(y)dy = \log\left({}_tp_x\right) \\ \Leftrightarrow \quad & {}_tp_x = \exp\left[-\int_x^{x+t} \mu(y)dy\right]. \end{aligned} \tag{5.16} $$

Thus, if the force of mortality $\mu(x)$ is known, we are able to simulate the survival probabilities ${}_tp_x$.

5.3.2 Lee-Carter Mortality Modeling

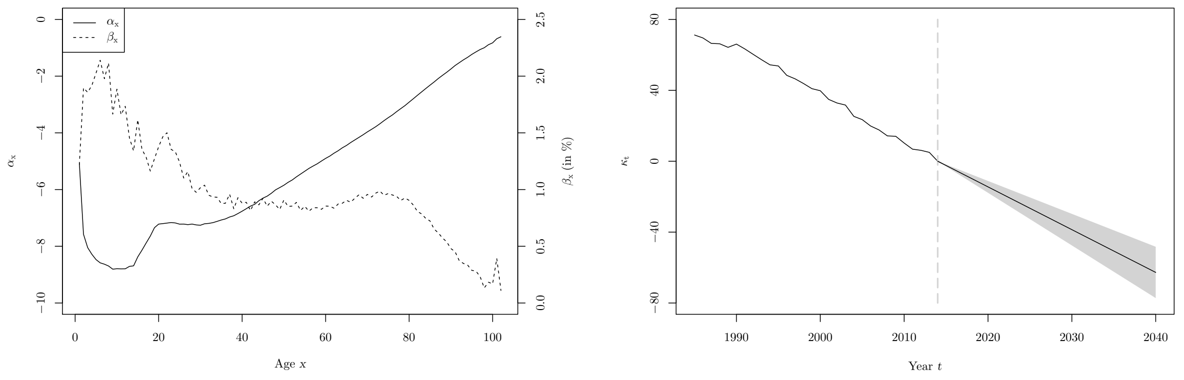
For simulating the mortality rates, we use the model of Lee and Carter (see Lee and Carter, 1992). In the Lee-Carter-model, historic death rates are used to calibrate the model and to estimate the future mortality. Utilizing a discrete time series model, we assume that in the future the mortality will change in the same way that it did in the past. Past longevity developments imply that the life expectancy increases over time. Thus, the force of mortality is not only estimated in dependence of the age x, but also as a function of the time t. The fitted force of mortality at time t for individuals of age x is consequently denoted by $\mu(x,t)$ and defined as

$$ \ln \mu(x,t) = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t}. \tag{5.17} $$

In the model, the constants α_x and β_x represent age-specific constants. The parameter α_x represents the average shape of the age profile. The constant β_x is an age-specific factor which indicates how the death rates deviate from the age profile in response to changes of the parameter κ_t. The time-dependent factor κ_t in turn, represents an index of the level of mortality. Finally, $\epsilon_{x,t}$ stands for the error term containing age-specific effects that cannot be modeled by the other factors. When using the Lee-Carter model, we conduct two steps. In the first one, we fit the factors from Equation (5.17) on the historic data. In the second step, we use the fitted model to forecast future mortality rates. This way, we obtain future death rates for every age group.

We use historic death rates from the Swiss population where data for the years 1970 to 2014 is available from the Human Mortality Database (see `http://www.mortality.org/`). On this basis, we fit our model and compute the mortality rates for times after 1985, the year in which the current pension fund system was introduced. For years between 1985 and 2014, we utilize the entire historical information that is available up to the respective year to fit the model parameters. Once exceeding this point, we are limited to using the entire dataset for predicting the future mortality.

Figure 5.1 presents the fitted values for the model parameters α_x, β_x and κ_t. More specifically, Fig-

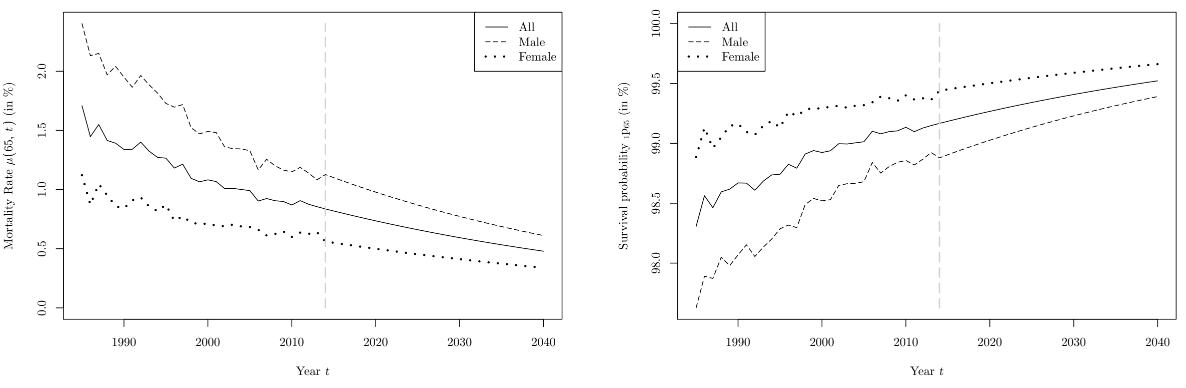


(a) Values of α_x and β_x for the Swiss population. (b) Values of κ_t from 1985 to 2040 for Switzerland.

Figure 5.1: Plots of the coefficients α_x, β_x and κ_t of the Lee-Carter model.

ure 5.1(a) displays the age-specific constants α_x and β_x for ages from $x = 0$ up to $\omega = 101$ years. The time-dependent parameter κ_t is displayed in Figure 5.1(b). For the years from 1985 up to 2014, the fitted values are shown, whereas from 2015 to 2040 the simulated results are given (the dashed line informs about this frontier). Together with the forecasted values, the 60% confidence interval is depicted in gray in order to visualize the volatility of the projections.

The plots of the force of mortality $\mu(65, t)$ and the survival probabilities ${}_1p_{65}$ at retirement for the years $t = 1985$ to 2040 are given in the graphs of Figure 5.2. In them, the historical values are used



(a) Mortality rates $\mu(65, t)$ from 1985 to 2040. (b) Survival probabilities ${}_1p_{65}$ from 1985 to 2040.

Figure 5.2: Plots of the mortality rate $\mu(65, t)$ and the survival probability ${}_1p_{65}$.

again up to the year 2014 and the forecasts from the Lee-Carter model for the years thereafter. As a consequence, the course of the graphs is more unsteady before 2014 and smoother for the simulated values. We separately plot the values for men, women and the overall population. Expectedly, the mortality rate for men at the age of 65 years is the highest, amounting to almost 2.5% in 1985. For women, the values are less than half of that, amounting to about 1% in 1985. The values of the overall population is around 1.5%. Unsurprisingly, we observe a strong decrease in the mortality rates

over the years. The three graphs have a downward slope and reach values between 0.5% and 1% for the year 2040. Additionally, the distances between the curves narrow. Our results indicate that the differences in mortality between men and women decrease over time.
We observe a similar pattern when looking at the survival probabilities in Figure 5.2(b). For ${}_1p_{65}$, the male population has the lowest values, starting with a value of 97.63% for the year 1985. For the same year, women have a survival probability of 98.89%, while for the whole population, the value amounts to 98.31%. Subsequently, the values improve and the curves increase. As before, the historic values show some fluctuation, which causes some dispersion in the plots up to the year 2014. For that specific year, the three curves are closer to each other, with the male population reaching a value of ${}_1p_{65} = 98.88\%$, which is only 0.56% lower than the one for women. This trend continues further with a monotonous increase up to the year 2040. For this last year of the simulation, the values are the closest, differing from each other by less than 0.3%. The aggregated survival probabilities ${}_tp_{65}$ for years $t = 0, \ldots, 37$ are presented in Figure 5.3, where we plot the values for the years 1985, 2014 and 2040. Comparing the

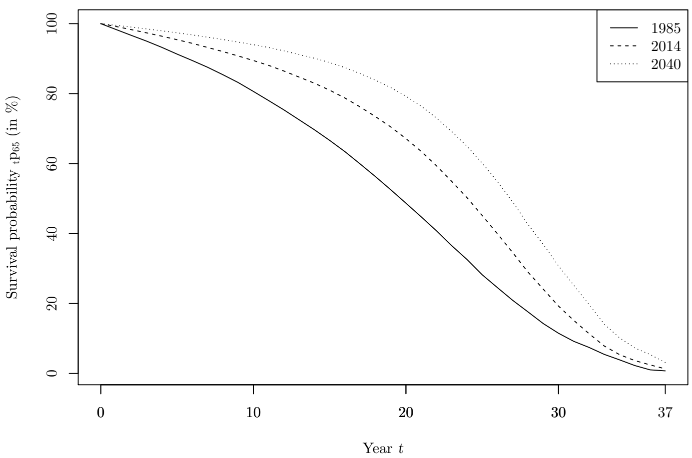
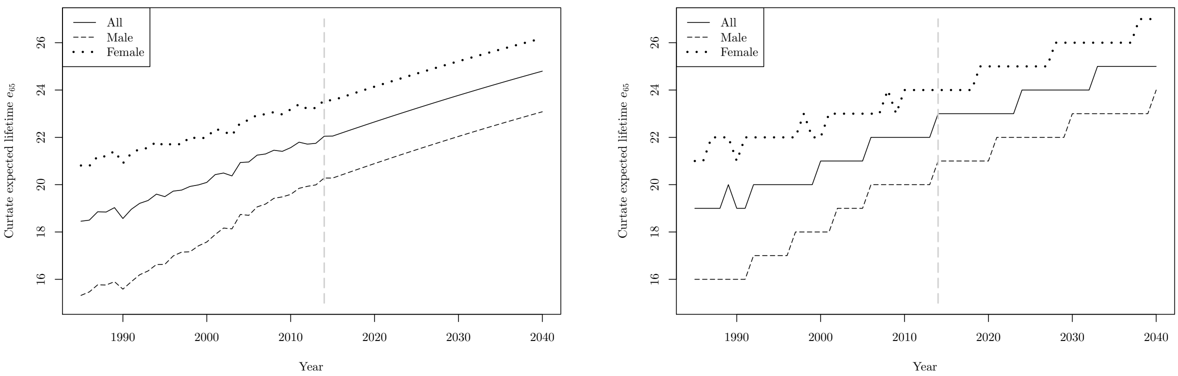


Figure 5.3: Survival probability ${}_tp_{65}$ from age 65 up to the maximum age of $\omega = 101$ years for the years 1985, 2014 and 2040.

different curves, we see the effect that longevity trends have on the survival probability. The probability to survive ten years after retirement, ${}_{10}p_{65}$, equals about 83% in 1985. For 2014 and 2040, this value is 7.5% and 11.5% higher, respectively. If we look at the probability to still be alive after 20 years, the survival probability equals about 70% for 2014. This is almost 18% higher than it is for an individual in 1984 and 11.2% lower than for someone in 2040. This can also be seen in the plot, where the distance between the three curves widens. Once individuals approach the maximum age, the distance between the three curves decreases again. For ${}_{30}p_{65}$, i.e. the probability to still be alive 30 years after age 65, the value for the year 2014 amounts to 24.1%. This is only 9.8% higher than the value for the year 1985. For the year 2040, the distance has slightly increased. For individuals that are 65 in that year, the probability is 12.8% higher than the one in 2014. Going towards the maximum age, all the graphs converge to zero. Overall, we can see that longevity leads to increased survival probabilities for all ages, predicting individuals to live longer.
Figure 5.4 shows the curtate expected lifetime e_{65} at retirement as well as the rounded values of e_{65} to integers. The curtate expected lifetime at age x, e_x, is defined here as

$$e_x = \sum_{t=1}^{\omega - x} {}_tp_x, \tag{5.18}$$

For the years up to 2040, the values for the male, the female and the overall population are given. We use



(a) Values of e_{65}.

(b) Rounded values of e_{65}.

Figure 5.4: Curtate expected lifetime e_{65} at the age of 65 years.

the historic values up to the year 2014 and apply the forecast from the Lee-Carter model starting 2015. The life expectancy is the highest for women and the lowest for men. The value for the overall population lies between the two. While at the age of 65 years, the male population has an curtate expected lifetime of only 15.3 years in 1985, female individuals are expected to live about 5.5 years longer at that point. In the following years, this gap becomes smaller. In 2005, the female population has a value of 22.7 for e_{65}, whereas the male reaches 18.7. The increase for the entire population is on a similar scale, achieving a value of 21 for that year. Past that point, we observe that the three graphs grow almost

Period	$\lfloor e_{65} \rfloor$
1985 – 1991	19
1992 – 1999	20
2000 – 2005	21
2006 – 2013	22
2014 – 2023	23
2024 – 2032	24
2033 – 2040	25

Table 5.1: Development of $\lfloor e_{65} \rfloor$ for the entire population over time.

linearly and with similar slopes, thus keeping the same distance between each other. This is especially true for the values after 2014 that have been estimated with the Lee-Cater model. Consequently, the curtate expected lifetime at age 65 for the entire population in 2040 reaches almost 25 years, which corresponds to individuals turning 90 years old on average, an increase of two years over 2014 and an increase of six years over 1985, the inception year of the second pillar system. The values for the male and female population vary about 1.5 years from this, amounting to 23.1 for men and 26.2 for women. Additionally, we display the rounded values of e_{65} in Figure 5.4(b) and report the development of $\lfloor e_{65} \rfloor$ for the entire population in Table 5.1, since they are of importance for our simulation. Thereby, the notation $\lfloor e_x \rfloor = \max\{k \in \mathbb{Z} \,|\, k \leq e_x\}$ denotes the largest integer smaller or equal to e_x. Naturally, the main findings are as for the absolute values. Due to the rounding, the graphs exhibits jumps between which the curve remains constant, leading to a stairstep course. As the expected lifetime of men grows stronger over time, upward steps occur more often, every 6–7 years on average. On the contrary, the life expectancy of women is high and increases at a slower rate. For the considered period, we observe an additional year of lifetime about every 10 years. As before, the overall population is located between these extremes, experiencing a gain of one year of lifetime every 7–8 years. Looking at the rounded values of the curtate expected lifetime in Table 5.1, we see that the growth slows down over time. In fact, while $\lfloor e_{65} \rfloor$ equals 21 over the course of 6 years, it remains at 22 for 8 years and at 23 for 10 years.

5.3.3 Modeling Variations in Longevity

In a second step, we depart the historical mortality data and focus on the curtate expected lifetime e_x. To this end, we take the entire historic mortality data that is available to us, i.e. from the year 1970 until 2014. Based on this, we compute the mortalities ${}_tq_x$, the survival probabilities ${}_tp_x$ and the curtate expected lifetime $e_x = \sum_{t=1}^{\omega-x} {}_tp_x = \sum_{t=1}^{\omega-x} (1 - {}_tq_x)$ for the year 2015 with the help of the Lee-Carter model. As it is our goal to analyze the effects of longevity on the technical parameters of a pension scheme, we look at e_x as a parameter independently from its development over time. For various values of e_x, we aim to be able to make calculations using the underlying mortality structure (age/time-wise). By amending the mortality rates ${}_tq_x$, different values for e_x can be obtained. Consequently, we choose to use a multiplication model for the mortality (see, e.g., Alonso-García and Sherris, 2018). In this, we take the simulated mortality rates for the year 2015 as a starting point. In order to achieve certain target values for the lifetime, we introduce a multiplication factor c (mortality loading). By multiplying the mortalities with this factor, we vary the lifetime and reach desired target values. For achieving this, we numerically solve the equation

$$e_x \stackrel{!}{=} \sum_{t=1}^{\omega-x} (1 - c \cdot {}_tq_x) \,. \tag{5.19}$$

Consequently, given a target value of e_x, we obtain a value for c. In order to reach higher values for the lifetime, c needs to be smaller than one. Conversely, shortening the life expectancy corresponds to $c > 1$. The resulting values for the multiplication factor are displayed in Figure 5.5 and Table 5.2. In them,

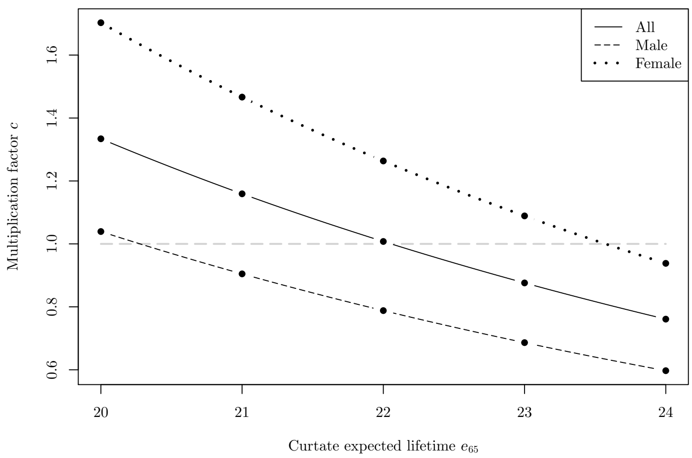


Figure 5.5: Mortality multiplication factors c for the entire population.

the values of c are given for a curtate expected lifetime between $e_{65} = 20$ and 24. We perform this on the entire population level as well as for male and female individuals. Serving as a reference for the initial values of e_{65}, a horizontal dashed line is given at a level of $c = 1$ in the graph. The curve for the female population is located on top of the others, while the one for the male population is at the bottom. This follows from women having a higher expected lifetime than men (cf. Figure 5.4, higher loading c needed for given target). We observe that the graphs do not follow a linear course and have a parabolic shape instead. Additionally, the distance between the graphs is bigger for smaller values of e_{65} and shrinks when looking at larger target values. The expected lifetime of women amounts to 23.6 for a value of $c = 1$ (cf. Figure 5.4(a) and Section 5.3.2) and conversely reaches $e_{65} = 22$ for a multiplication factor that equals 1.26 (cf. Table 5.2). Consequently, c needs to be larger than one for shorter life expectancies, going up to a value of 1.70 for $e_{65} = 20$, meaning that the average mortality for women needs to be almost doubled. For men, this trend is reversed. As for $c = 1$, the lifetime only amounts to 20.3 and 22

e_{65}	All	Male	Female
20	1.33	1.04	1.70
21	1.16	0.91	1.47
22	1.01	0.79	1.26
23	0.88	0.69	1.09
24	0.76	0.60	0.94

Table 5.2: Mortality multiplication factors c for selected target lifetimes e_{65}.

for a value of $c = 0.79$, the multiplication factor needs to be below one for most values of e_{65} that are illustrated in the graph. This subsequently leads to a multiplication factor of 0.60 for a curtate expected lifetime of 24 years, meaning that the mortalities need to almost be halved in order to achieve such a long lifetime for men. The overall population is located in between these two extremes. For $c = 1$, the curtate lifetime equals 22.05. Lowering the target value for e_{65} to 20 results in a multiplication factor of 1.33. At the other end of the scale, we reach 24 years when multiplying the mortalities by 0.76. This uneven change in the multiplication factor corresponds to the shape of the curve being non-linear.

5.3.4 Asset Allocation and Financial Returns

One of the central tasks of pension funds lies in investing the assets of their members on the capital markets (see BVV2, Art. 11). This way, returns on the savings are earned, adding to the available capital. For the pension system in Switzerland, there is no explicit minimum interest rate during the retirement phase. However, information on the expected returns and the technical interest rate are important for the life-long guaranteed pension payments.

In our work, we consider a commonly used pension fund index in the Swiss market. This way, we assess the returns that have historically been achieved by pension funds. We consider the annual returns of the Pictet BVG 40 index. It is composed of a mixture of 50% bonds, 40% stocks, 5% real estate and 5% hedge funds. For all asset classes, a mixture of investments from Switzerland as well as other countries is used.[1] For the historic values, Figure 5.6 depicts the annual returns from 1994 to 2014, i.e. over 21 years, while Table 5.3 presents the corresponding numerical values. We observe a high volatility in the returns. While a maximum return of 16.77% is achieved in 2009, the largest loss took place the year before, amounting to –18.58%. The investment returns of pension funds are closely connected to the developments in the financial markets. Going more into detail, it can be seen that for the majority of the years, the returns have been positive.

The average annual return amounts to 5.75% and the standard deviation to 9.4%. For the five-year moving average a similar pattern can be observed (cf. Figure 5.6). While none of the returns are negative, there is still a strong fluctuation with the maximum and minimum values amounting to 13.42% and 0.47%, respectively. We conclude that pension funds face a strong volatility challenge when investing the savings of their members on the financial market. As a result, the contribution of investment earnings to the savings of the insured is subject to strong uncertainty. Although the long investment horizon

[1] See `https://www.am.pictet/en/switzerland/articles/lpp-indices` for further information.

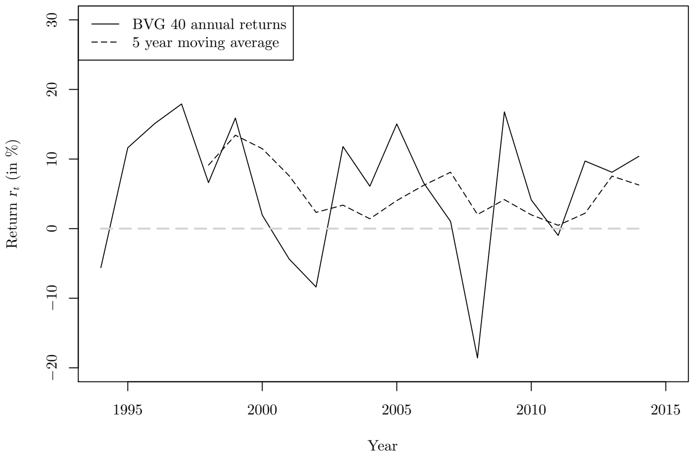


Figure 5.6: Historic annual returns r_t of the Pictet BVG 40 pension fund index together with the five-year moving average.

Year	Return r_t	Year	Return r_t	Year	Return r_t
1994	–5.61	2001	–4.37	2008	–18.58
1995	11.62	2002	–8.39	2009	16.77
1996	15.09	2003	11.79	2010	4.13
1997	17.91	2004	6.09	2011	–0.98
1998	6.61	2005	15.04	2012	9.71
1999	15.88	2006	6.57	2013	8.10
2000	1.94	2007	1.03	2014	10.39

Table 5.3: Annual Pictet BVG 40 returns r_t (in %).

smoothens the overall return, single years may affect the reporting, the funding ratio and the decision taking significantly.

5.4 Numerical Results

In this Section, we turn our attention to the results obtained from our pension fund model. First, we present the numerical implementation. Having presented our model, we then discuss the results of our simulation. In this, we analyze and quantify to what extent the investment return, the lifetime and the technical interest rate influence the conversion rate. We extend our framework by looking at capital market scenarios and random investment returns. In doing so, we assess how variations in the returns influence the savings account value and thus the conversion rate that pension funds may propose.

5.4.1 Model Implementation

An outline of our model is shown in Figure 5.7. For given values of the survival probabilities, investment return and discounting factor, we calculate the adequate conversion rate with the help of Equation (5.10). In order to compute the conversion rate cr corresponding to the mortality, return and discounting assumptions, we require information about the survival probabilities ${}_tp_x$ and the investment returns r_t. Thereby, the survival probabilities stem from the Lee-Carter model as laid out in Section 5.3.2, while the investment returns are takent from a range of values (cf. historical returns in Figure 5.6 and Table 5.3).

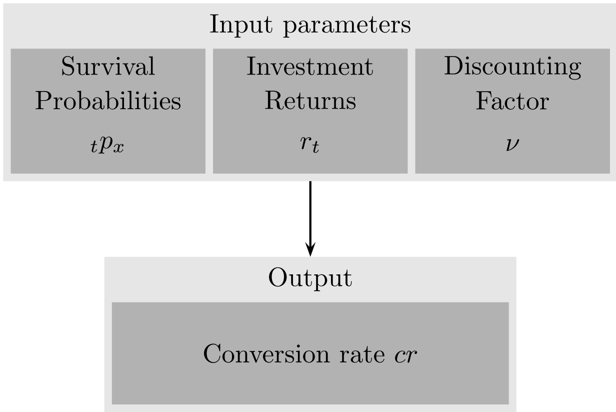


Figure 5.7: Synopsis of the input and output of the model.

As shown in Section 5.2, the value for cr is independent of the savings at retirement A_0. We therefore normalize its value to $A_0 = 1$ for our simulations.

5.4.2 Results with Constant Return

Having discussed our model and the optimization process that it involves, we now turn to our results. A contour plot of the computed value of the conversion rate cr is presented in Figure 5.8. It displays cr in

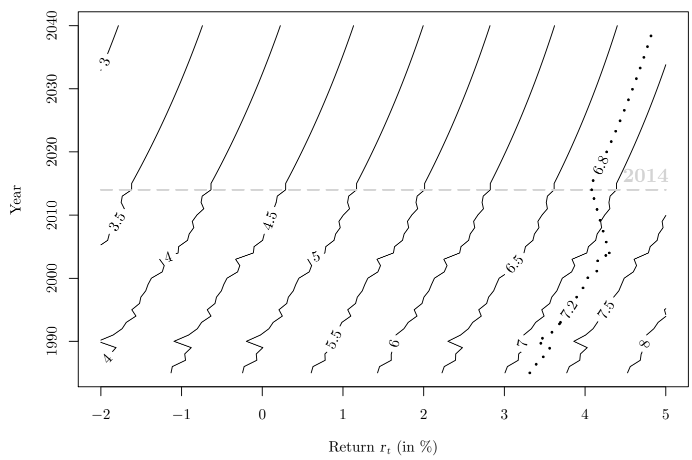


Figure 5.8: Conversion rate cr (in %) for the entire population for the years 1985 to 2040.

dependence of the capital market return r_t and the curtate expected lifetime e_{65} per calendar year. While we vary the returns between –2% and 5% and assume them to be constant over the entire retirement period, the years range from 1985 up to 2040. As the interest rate of ten-year Swiss government bonds is close to zero for 2014, we choose a technical interest rate of $z = 0$ and consequently the discounting factor is $\nu = 1$. Additionally, the conversion rates that are required by the legislator are shown as dotted lines, i.e. 7.2% up to 2004 and 6.8% from 2014, with a transition in the years between. We also display a horizontal dashed gray line for the year 2014, in order to emphasize the change between historical data and simulated values for the mortality rates. Selected results are provided in Table 5.4, where the values for the year 2014 are highlighted. For the conversion rate, we observe that the resulting value ranges between 8.31% and 2.90%. The change in the results takes place in a relatively linear way, and the isolines for the conversion rates have equal distances between each other. Overall, we can see that the impact

	Return $r_t \equiv r$							
Year	–2%	–1%	0%	1%	2%	3%	4%	5%
1990	4.03	4.55	5.11	5.70	6.31	6.94	7.59	8.25
2000	3.68	4.20	4.74	5.32	5.92	6.54	7.18	7.84
2010	3.40	3.90	4.43	5.00	5.59	6.21	6.84	7.50
2014	3.32	3.81	4.34	4.90	5.49	6.11	6.75	7.40
2020	3.22	3.71	4.23	4.79	5.38	5.99	6.63	7.28
2030	3.04	3.52	4.04	4.59	5.17	5.78	6.42	7.07
2040	2.90	3.37	3.88	4.43	5.00	5.61	6.24	6.89

Table 5.4: Conversion rates cr (in %) depending on the return and the year.

of the return is bigger than the one of the expected lifetime. For an increase of 1% of the conversion rate, the capital market return needs to improve by less than 2% (cf. Table 5.4, Figure 5.10(a)). In comparison to that, the same change in cr corresponds to the mortality improvements from almost 30 years. The improvement in the mortality of members still has a strong impact, however. We can see this when looking at the dotted line for the values that have been in force over the years. While in 1990, a return of about 3% sufficed in order to achieve a conversion rate that exceeded the value proposed by the legislator today, this has changed. With the growth in expected lifetime, the same return would only suffice to provide a conversion rate of about 6.11% in 2014. This trend continues further downwards and reaches a value for cr that would be at 5.61% in 2040. In order to meet the conversion rate of 6.8% that is required by the legislator today, an asset return of more than 4% would be required. In the currently prevailing low-interest rate environment, this is becoming more and more challenging for pension funds. Going forward to the simulation of future years, an average annual return of about 5% would be necessary in 2040 for achieving the conversion rate proposed by the legislator today. Conversely, if we look at a capital market return r_t of zero, the conversion rate in 2018 would be about 4.27%. For future years, the results indicate a decrease of the value connected to $r_t = 0$ down to 3.88%.

Departing from the previous setting, we now want to look at the curtate expected lifetime e_{65} in direct connection to the conversion rate cr, i.e. without considering the development of e_{65} over time. To this end, Figure 5.9 and Table 5.5 show the conversion rate in dependence of the capital market return r_t and the curtate expected lifetime at retirement e_{65}. In this, the values of e_{65} are derived using the mortality multiplication factor c that we introduced in Section 5.3.1. Based on the mortality rates for the year 2014, we derive the multiplication factors that are needed in order to achieve selected target values for e_{65} (cf. Figure 5.5). Consequently, we are able to calculate the conversion rate in dependence of the expected lifetime for a discounting factor of $\nu = 1$. As in Figure 5.8, we plot a dotted line for a conversion rate of 6.8%, and a dashed gray line at $e_{65} = 22.05$, the curtate lifetime in 2014. In addition to the results for $z = 0\%$, Table 5.6 presents the optimal conversion rates for a risk-free interest rate of $z = 2\%$.

We see that the impact of changes in the capital market return remains the same as in Figure 5.8: In order to achieve an increase of cr by 1%, the pension fund needs to increase the return r_t by less than 2%, e.g. cr goes from 4.35% to 5.50% as r_t grows from 0% to 2% for $e_{65} = 22$ (cf. Table 5.5). Such variations in the return occur regularly on short-term but do also appear on average returns when comparing returns over five to ten years. For the lifetime, we are now able to directly connect changes in cr to ones in e_{65}. According to the graph, an increase of the lifetime by more than four years causes

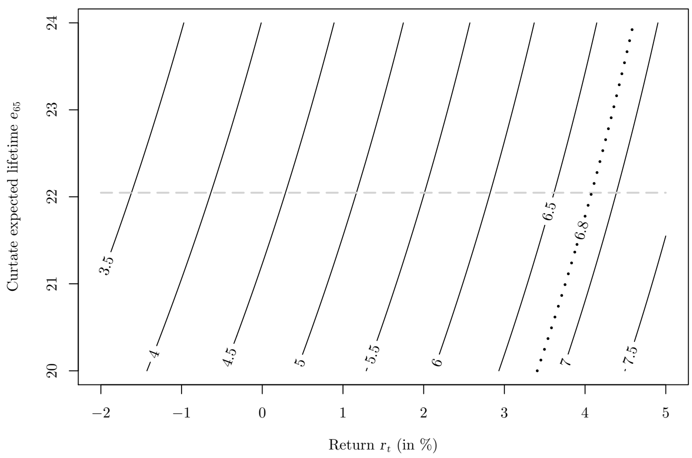


Figure 5.9: Conversion rate *cr* (in %) for the entire population for the curtate expected lifetime e_{65} ranging from 20 to 24 years.

	Capital market return $r_t \equiv r$							
e_{65}	–2%	-1%	0%	1%	2%	3%	4%	5%
20	3.71	4.22	4.76	5.33	5.93	6.54	7.18	7.83
21	3.51	4.01	4.55	5.11	5.70	6.32	6.96	7.61
22	3.32	3.82	4.35	4.91	5.50	6.12	6.76	7.41
23	3.16	3.65	4.17	4.73	5.32	5.93	6.57	7.23
24	3.00	3.49	4.01	4.56	5.15	5.77	6.41	7.06

Table 5.5: Conversion rates *cr* (in %) depending on the return and the curtate expected lifetime for a risk-free interest rate of $z = 0\%$.

a decrease of *cr* by 1%. As we were able to see that on average, e_{65} increases by one unit every 7–8 years (cf. Section 5.3.1), it takes more than 30 years in order to achieve such an increase in lifetime. Therefore, we can say that while longevity does have a considerable impact on the conversion rate, it is a lot smaller than the one that originates from the capital market returns.

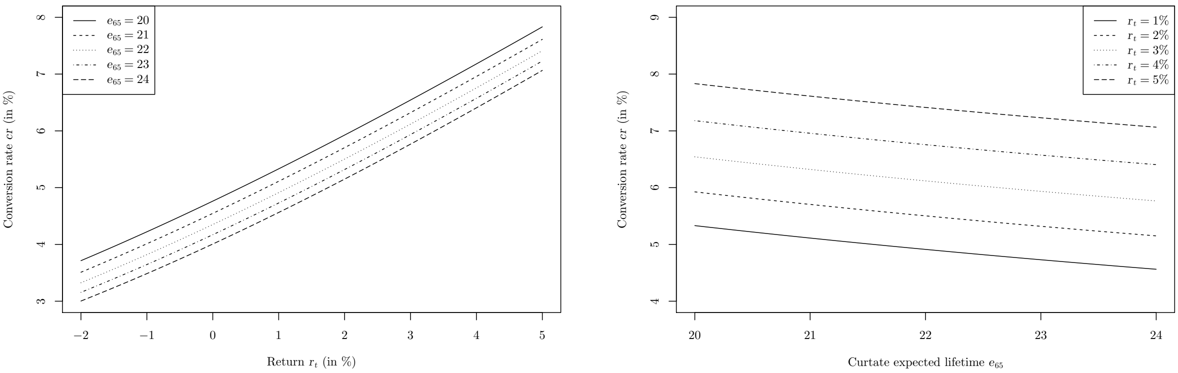
Looking at the outcome when using a risk-free interest rate of $z = 2\%$, the results for the conversion rate are higher. As all future payments are discounted with a higher interest rate, the corresponding conversion rate can be increased. On average, this change is about 1% for a change in z of 2%. It depends of course on the other factors (return and life expectancy) as well.

Taking a more detailed look at the results, Figure 5.10 displays the optimal conversion rate in dependence, both, of the return and the curtate expected lifetime. Figure 5.10(a) presents *cr* depending on the return r_t. We plot the lines for the curtate expected lifetime e_{65} ranging from 20 to 24 years for a range of r_t from –2% up to 5% and $\nu = 1$ as before. The graph can be understood as a plot along the horizontal in Figure 5.9 at the chosen values of e_{65}. Looking at the curvature of the graphs, we can see that the conversion rate grows almost linearly with the return. The growth goes from a value of $cr = 3.32\%$ for $r_t = –2\%$ and a lifetime of 22 years up to almost 7.5% for a return of 5% and the same life expectancy. Looking at the lines for the different values of e_{65}, it can be observed that they are parallel and have only small differences between each other. Examining the results more closely, we can see that the growth of e_{65} from 20 to 21 years leads to a decrease of the conversion rate of

	Capital market return $r_t \equiv r$							
e_{65}	–2%	–1%	0%	1%	2%	3%	4%	5%
20	4.92	5.41	5.93	6.46	7.02	7.59	8.18	8.78
21	4.71	5.19	5.70	6.24	6.79	7.36	7.95	8.55
22	4.51	5.00	5.50	6.03	6.58	7.16	7.74	8.35
23	4.34	4.82	5.32	5.85	6.40	6.97	7.55	8.16
24	4.18	4.65	5.15	5.68	6.22	6.79	7.38	7.98

Table 5.6: Conversion rates cr (in %) depending on the return and the curtate expected lifetime for a risk-free interest rate of $z = 2\%$.

about 0.2%. This difference subsequently decreases for larger values of the curtate expected lifetime. For the graphs of $e_{65} = 23$ and 24, it only averages 0.17%. Summing up the distances, there is a difference of about 0.76% between the lines for $e_{65} = 20$ and $e_{65} = 24$. Consequently, four years of growth in the lifetime come along with an increase of 1% for the conversion rate, as we saw earlier in Figure 5.9.



(a) Conversion rate cr in dependence of the return r_t for selected curtate expected lifetimes e_{65}.

(b) Conversion rate cr in dependence of the curtate expected lifetime e_{65} for selected returns r_t.

Figure 5.10: Conversion rate cr (in %) in dependence of the return r_t and the curtate expected lifetime e_{65}, respectively.

The impact of the curtate expected lifetime e_{65} on the conversion rate is studied more closely in Figure 5.10(b), where we plot cr in dependence of e_{65}. The results are shown for capital market returns of $r_t = 1\%$ up to 5% and $\nu = 1$. In a similar fashion to the previous plot, the graph displays the values of cr along the vertical from Figure 5.9 for the chosen returns. We observe an almost linear decrease of the conversion rate. The reason for this is the improvement of the mortality that takes place. It causes a trend of individuals living longer due to an increased survival probability ${}_tp_{65}$. Consequently, the mean volume of pension payments increases at higher ages. Due to this, the conversion rate needs to be decreased. For an expected lifetime of 20 years, the conversion rate amounts to $cr = 6.54\%$ for a return of 3%. Staying with the same return r_t, the value of cr decreases to 5.76% when the insured is assumed to live another 24 years after being retired. As the graphs for the different returns are parallel to each other, this trend holds true independently of the return r_t. Additionally, we can see that the distance between the graphs is almost constant, meaning that the conversion rate changes nearly linearly with the return. While for an increase of r_t from 1% to 2% the conversion rate improves by 0.59% on average, growing from 4% to 5% leads to an average increase of 0.66%. Overall, we can again see that the impact of the capital market return on the conversion rate is stronger than the one of the expected

lifetime. For an expected lifetime of 22 years, for example, a decrease of r_t from 5% to 1% causes cr to decrease by about 2.5%. This is in line with Figure 5.9, where we mentioned that an increase of the return by less than two percent corresponds to a one percent improvement of the conversion rate.
Up to now, we assumed the technical interest rate z to be equal to zero. In the following, we relax this assumption in order to study the impact that the technical interest rate has on the conversion rate. In Figure 5.11 we display the conversion rate for the technical interest rate z ranging from 0% up to 2.5%. This performance is carried out for selected capital market returns and a curtate expected

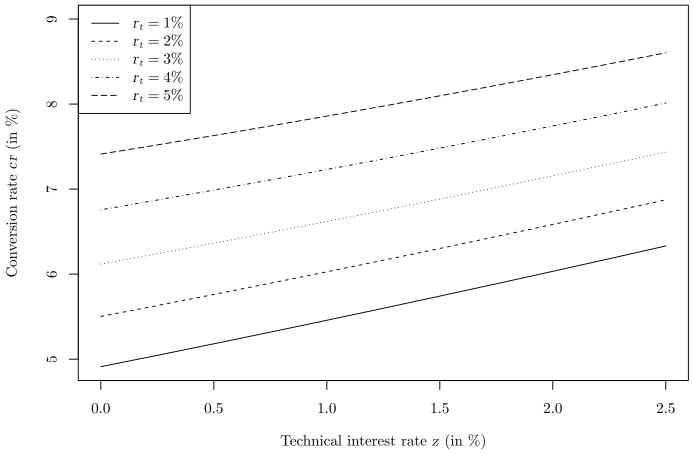
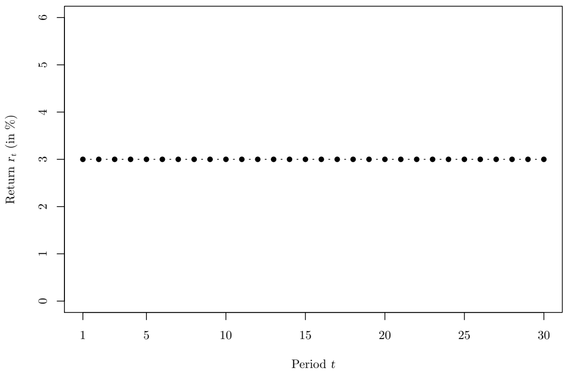


Figure 5.11: Conversion rate cr (in %) in dependence of the technical interest rate z for selected returns r_t and a curtate expected lifetime of $e_{65} = 22$ years.

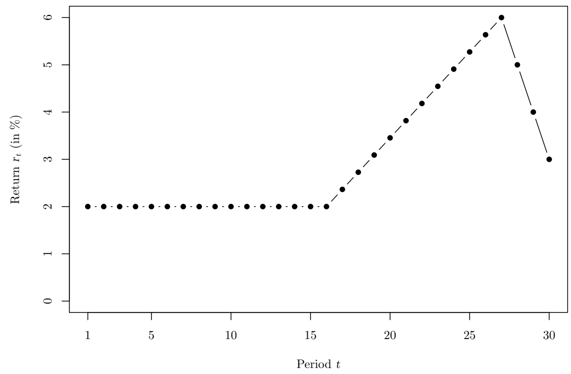
lifetime of $e_{65} = 22$. The conversion rate grows linearly for an increase of the risk-free interest rate. The reason for this is that for an increase of z, the value of the pension payments at time zero decreases. Consequently, a higher conversion rate can be chosen. The graphs for different returns r_t are equidistant and almost parallel to each other. For higher values of z, the distances between the graphs decrease. Comparing with the previous Figures, we can see that the conversion rates for $z = 0\%$ are equal to the ones in Figure 5.10(b) for $e_{65} = 22$. For a return of $r_t = 1\%$, the conversion rate equals about 4.91% for a technical interest rate of zero. As z grows, the conversion rates increases as well, reaching 5.46% for $z = 1\%$ and 6.03% for $z = 2\%$. For the highest depicted value of z of 2.5%, the conversion rate reaches 6.33%. We can therefore conclude that for an increase of z by 1%, the conversion rate increases by about 0.56%. For higher returns, this effect shrinks. For a return of $r_t = 3\%$, the conversion rate equals 6.62% when the risk-free interest rate equals 1% and 7.16% when it is equal to 2%, resulting in a difference of 0.54%. For a capital market return of 5%, this difference decreases further down to 0.49%. We can therefore conclude that a decrease of the technical interest rate leads to a strong decrease of the conversion rate and that the impact of z decreases for higher asset returns. This conclusion is critical for pension funds and the level of annuities in periods where capital market returns are low.

5.4.3 Scenarios for the Asset Returns

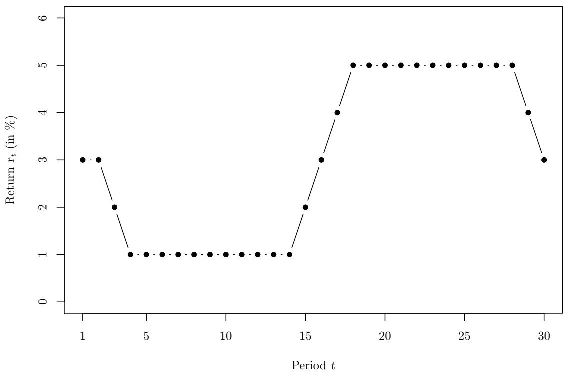
Until now, we have assumed the capital market return r_t to be constant over time. In the following, we relax this assumption and use scenarios for the development of r_t. This way, we try to assess what impact certain developments on the capital market have on the pension fund and the optimality of the conversion rate. Overall, we consider four different market scenarios. For them, it is assumed that the market return r_t follows a specific path over the first 30 periods after retirement. Past that point, we assume the return to be equal to 3% for the remainder of the lifetime of the member. The paths are chosen in a way, such that the return over the whole 30 years equals 3% on average for every scenario. The courses of the scenario returns are given in Figure 5.12. For the first scenario, that is displayed in



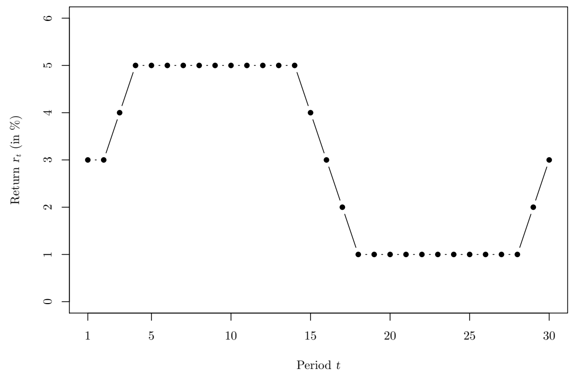
(a) Scenario 1.



(b) Scenario 2.



(c) Scenario 3.



(d) Scenario 4.

Figure 5.12: Plot of the scenario returns r_t.

Figure 5.12(a), we assume a return that is constant and equal to 3% over the whole time. It consequently serves as a reference case for the remaining scenarios. For the second scenario (cf. Figure 5.12(b)), we assume a prolonged time of low market returns. Consequently, r_t remains at 2% for 16 periods and then recovers with a linear growth until it reaches 6% in period 27. From there, it then decreases linearly down to 3% until the last period. The third scenario assumes a phase of low returns at the beginning of the retirement phase that is then followed by an equally long phase of high returns (cf. Figure 5.12(c)). We assume the low returns to be equal to 1% over the course of eleven periods and the high ones to amount to 5% over the same time. This pattern is mirrored in scenario four (cf. Figure 5.12(d)). This way, we try to study what impact an early phase of low returns, that is followed by high ones, has, and vice-versa.

Looking at the results, Figure 5.13 shows the conversion rates for the four scenarios for the curtate expected lifetime e_{65} ranging from 20 to 24 years. We see that the course of the different graphs resembles the one seen in Figure 5.10(b). For an increase in the expected lifetime, the conversion rate decreases almost linearly. As scenario one serves as a reference case, its course is equal to the one in Figure 5.10(b) for $r_t = 3\%$. Comparing the different graphs, we see that the results for the second and third scenario are close to each other, with the second scenario being 0.13% higher on average than the third one. Additionally, the conversion rates for the two scenarios are smaller than the ones of the reference case. Comparing them with the results in Figure 5.10(b), we see that their conversion rates

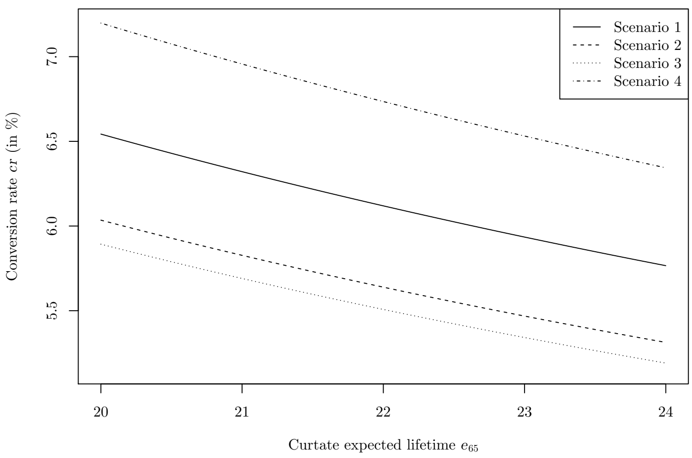
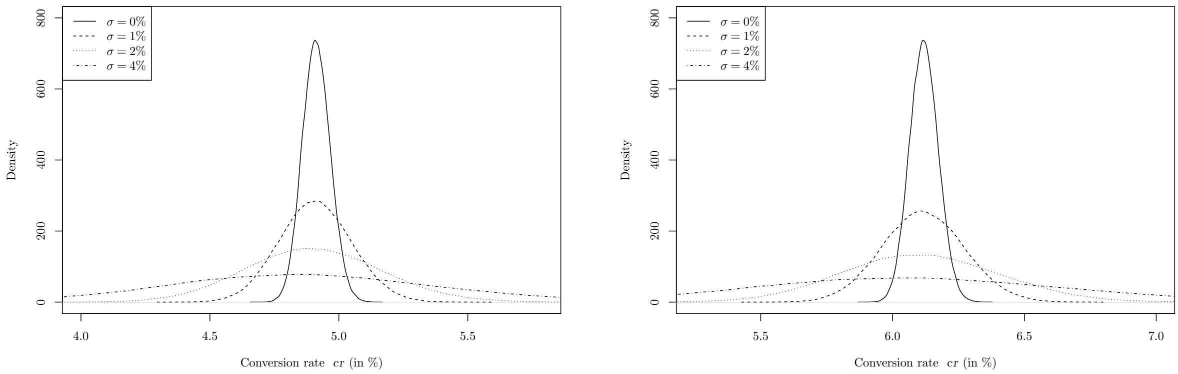


Figure 5.13: Conversion rate *cr* (in %) in dependence of the curtate expected lifetime e_{65} for the scenarios $1-4$.

equal the ones for a constant return of about 2.2% for scenario two and 2% for scenario three. These lowered values for *cr* are caused by the low returns at the beginning of the retirement. Consequently, the earnings that can be accumulated with the help of the capital market returns are smaller. We can therefore conclude that in our model higher returns have a strong impact on the resulting conversion rate at the beginning of the retirement. For scenario four, the periods of high and low returns from the third scenario are switched. Consequently, the investment return r_t equals 5% over eleven periods at the beginning of the retirement (when available assets are still high) and drops down to 1% towards the second half of the time frame. This change results in a conversion rate that is distinctly higher than in the previous scenario. As the savings on the account of the member are a lot higher for earlier years, there are more earnings to be made from higher returns. Compared to that, the impact of lower returns in later periods remains rather small. As a large portion of the savings has already been paid out to the individual, the gains that can be made with the remaining part remain comparatively small. This way, the fourth scenario achieves a conversion rate that is on average about 0.62% higher than that of the reference case. Reaching a value of $cr = 6.74\%$ for $e_{65} = 22$, it achieves a conversion rate that equals the one of a constant return of almost 4%. Overall, the results for the different scenarios are distributed over a wide range, with a difference of up to 1.23% between the scenarios. We conclude that the capital market return has a strong impact on the conversion rate, even more so for varying returns over various scenarios.

5.4.4 Conversion Rate with Survival and Return Distributions

In the previous Sections, we assumed the capital market return to be deterministic. In this Section, we want to analyze the impact of a random return r_t. Going into detail, we aim to examine the effect of the volatility of the returns. In order to do this, we set the capital market return to be normally distributed, i.e. $r_t \sim N(\mu, \sigma)$. For the mean μ, we choose a value of 1% and a value of 3%, allowing us to compare the outcomes with our previous results. For the volatility σ of the return, we select values of 0%, 1%, 2% and 4%. This way, we are able to analyze the impact of volatilities that also occur in practice (cf. Section 5.3.4). For a value of $\sigma = 0\%$ (deterministic case), the only source of randomness consequently lies in the mortality values that are simulated by the Lee-Carter model. We are therefore also able to examine how the mortality model leads to dispersion in the conversion rate. As before, we assume the discounting factor to be equal to one again. The results are shown in Figure 5.14, where 5.14(a) displays the results for a mean return μ of 1% and 5.14(b) the ones for $\mu = 3\%$. The



(a) Plot for $\mu = 1\%$.

(b) Plot for $\mu = 3\%$.

Figure 5.14: Distribution of the conversion rate cr in dependence of the return volatility σ for $r_t \sim N(\mu, \sigma)$ for $\mu = 1\%$ and $\mu = 3\%$.

shapes of the curves are similar for both values of μ. While for a mean return of 1% the conversion rate is equal to 4.91% on average, it amounts to 6.12% on average for $\mu = 3\%$, which is equal to the results for a constant return (cf. Section 5.4.2). For a volatility of zero, the distribution is very peaked and the and the results are close to the mean value. Consequently, the density function has a very high peak at that point, reaching a value of about 750 and the volatility of the conversion rate amounts to 0.05%. The volatility that is introduced by the Lee-Carter model is particularly small. For a return volatility of $\sigma = 1\%$, the peak at the mean is lower. The results are distributed wider around the mean with a volatility of 0.14%. This continues further for volatilities of 2% and 4%. For those values, the density curves flatten further, as the results for the conversion rate spread over an even wider range. Consequently, the cr sample volatilities are equal to 0.27% and 0.52%. We are therefore able to say that the volatility of the capital market return has a considerable influence on the conversion rate. However, while the dispersion in the investment returns affects the conversion rate, it takes place in a weakened way. Still, pension funds need to take the volatility of the investment returns into account with great care.

5.5 Conclusion

In this work, we assess the influence that financial and biometric risks have on the choice of parameters of a pension fund. Our research question was: How do the lifetime of the members, the investment returns and the technical interest rate influence the choice of the conversion rate? To this end, we study the retirement phase of a typical Swiss DC pension fund. Analyzing the changes in the savings of a retired individual, we examine the effects that the capital market return, the survival probability and the technical interest rate have on the conversion rate. The conversion rate is of particular interest in the Swiss pension fund system. As it is used in order to calculate the annuity that is paid out to members every year from retirement to death, its choice has a significant influence on the stability of the entire pension system. As an extension of our research, we study the effect of capital market scenarios and random investment returns. As low interest rates have been prevailing on the financial markets over the last years, it is of strong interest to study how pension funds need to respond to conditions like these. Likewise, we look at the effects of increased returns as well as cases of more fluctuating conditions.

With respect to the survival probability, we look at the development of the historic values for the Swiss population. Using historic data, we fit a Lee-Carter model and forecast the development of the life expectancy. As individuals tend to live longer, the pension funds need to take the increased volume of annuity payments at higher ages into account that are connected to this. In our study, we find a considerable increase in the lifetime of Swiss individuals. While members entering into retirement were expected to live for another 19 years in 1990, this value is expected to increase up to 24 in the following 50 years. For a constant market return, this leads to a drop of the conversion rate. As expected, an increased lifetime makes a decrease of the conversion rate necessary. Our results show that on average a decrease of the conversion rate by one percent is caused by the expected lifetime improving by more than four years. Since the lifetime is supposed to improve by one year about every 7 − 8 years, a decrease of one percent point in the conversion rate equates to the mortality improvement of more than 30 years. This needs to be taken into account, as pensions are a long-term business and require foresighted planning. However, longevity has still a small impact on the conversion rate by comparison to the capital market return.

For the capital market returns, we analyze the returns of a Swiss pension fund index. Looking at the annual data from over 20 years, we are able to observe a strong volatility. From the year with the largest gain to the one with the biggest loss, we are able to see a difference of about 35%. We therefore calculate our model outcomes for a wide range of return values. Looking at the results, we are able to see that an increase of the investment return leads to a higher conversion rate as more savings can be distributed to the members. Conversely, a lower return causes the conversion rate to decrease. A change in the investment return by less than two percent on average for a given life expectancy induces a change by one percent in the conversion rate. The capital market return has a strong impact on the chosen conversion rate, especially with respect to the high dispersion of the historic values.

In light of the high volatility of market returns, we extend our simulations by looking at scenarios for the capital market return. By using predefined courses for the investment return, we analyze how the conversion rate needs to be adjusted in response. The different scenarios involve long periods of low investment returns as well as ones with higher values and a mixture of high and low returns. Summarizing the outcomes, we observe that the beginning of the retirement phase is especially sensitive to different capital market returns. As the savings of the members are still comparatively high, the monetary amounts that can be earned with the help of good investments are rather large. Conversely, low returns lead to small contributions to the savings account which results in a penalty with respect to the conversion rate. For later years, these effects are still present, albeit with a smaller impact. With respect to the volatility of the capital market investment, we analyze the effect of utilizing normally distributed returns. We find that the volatility has a considerable influence on the range of the resulting conversion rate. However, while there is an impact, the dispersion in the conversion rate turns out to be smaller than the one of the return distribution. Overall, we can conclude that the conversion rate is very sensitive to changes to the course of the capital market return during the retirement.

While our model takes a simplified view on the processes within a pension fund, it can give guidance for decisions in practice. In particular, it allows funds to directly measure the effects of changes in the mortality of its members, the investment returns and the technical interest rate. This makes it possible for the pension fund to estimate e.g. the impact of a change of the used mortality table. The effect of a modified investment strategy and a different technical interest rate can also be assessed more easily. As funds differ from each other with respect to the structure of their members and their investment strategies, different choices for their conversion rates would seem natural. A way for the legislator to take this into account could be to set a range for the admissible conversion rate rather than a single value.

Our work is relevant with respect to the pension fund system in Switzerland and other countries with a comparable setting. Our work is limited to the pension fund system of Switzerland and its specific characteristics. By quantifying and comparing the effects of longevity and financial returns on the conversion rate, we contribute to the optimal calibration of annuities in connection to pension funds. The results are consequently relevant for theory and practice alike. Possible extensions include a more extensive model that includes the savings phase as well as a more complex parameter setting also using existing cohorts of retirees.

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