ON THE NON-OPTIMALITY OF HORIZONTAL BARRIER STRATEGIES IN THE SPARRE-ANDERSEN MODEL*

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Abstract. In the collective risk theory literature, horizontal dividend barrier strategies are frequently used to account for a profit participation of shareholders in the insurance business. This is mainly motivated by some optimality results of this strategy available in the classical compound Poisson model and in the diffusion setting. In this paper, we show that the optimality of horizontal barrier strategies does not carry over to Sparre Andersen models in general, by explicitly constructing a counter-example. As a by-product, a heuristic upper bound for the optimal dividend payout in the Erlang(n)-model is derived.

Keywords: collective risk theory; Sparre Andersen model; dividend payments; barrier strategies

1. Introduction

The optimal dividend problem has a long history in risk theory. In its classical formulation it consists of finding the payment strategy that maximizes the expected present value of all dividend payments from the surplus of a non-life insurance portfolio before the event of ruin. De Finetti was the first to investigate such a problem, arguing that expected dividend payments might be a more appropriate way to compare insurance portfolios rather than criteria based on ruin probabilities. In [15], he considered a random walk model with step sizes ± 1 as an approximation for the risk process and identified the optimal dividend payment strategy to be of constant barrier type, i.e. whenever the surplus is above a given barrier, this overshoot is immediately paid out as dividends. Since then, the problem has received a great amount of interest: if the surplus process of the portfolio is approximated by a Brownian motion, the optimal strategy again turns out to be of constant barrier type (see Asmussen & Taksar [5], Jeanblanc-Picqué & Shiryaev [24]), whereas the related problem of maximizing the expected present value of the utility of dividends for this diffusion model might lead to different optimal strategies (see e.g. Hubalek & Schachermayer [20]). For the classical Cramér-Lundberg model the optimal dividend problem was studied by various authors, e.g by Gerber [16], Bühlmann [9], Borch [8] and more recently by Azcue & Muller [7] and Gerber & Shiu [18]. For jump diffusion processes, optimal dividend strategies are discussed by Øksendal & Sulem [25]; recently Avram et al. [6] analyzed the problem for spectrally negative Lévy risk processes.

Let us consider the surplus process of an insurance portfolio at time t

$$R(t) = u + c\,t - \sum_{j=1}^{N(t)} X_j,$$

where the claim sizes X_j are independent and identically distributed positive random variables with distribution function F and mean $\mu < \infty$, $c > 0$ denotes the constant premium density and $u > 0$ is the initial capital. Moreover, the stochastic process $N(t)$ represents the number

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of claims in the portfolio up to time t and W_j $(j = 1, 2, \dots)$ is the sequence of inter-occurrence times (i.e. W_j is the time between the $(j-1)$th and the jth claim in the portfolio). Let T denote the time of ruin, i.e. $T = \inf\{t \geq 0 : R(t) < 0\}$.

A dividend strategy π is *admissible*, $\pi \in \Pi$, if the induced process of accumulated dividends $L_\pi(t)$ up to time t is predictable, nondecreasing, càglàd (left continuous with right limits) and

$$L_\pi(t) \leq u + c\,t - \sum_{j=1}^{N(t-)} X_j \quad \text{for all } t \geq 0.$$

The controlled surplus process is then given by

$$R_\pi(t) = u + c\,t - \sum_{j=1}^{N(t)} X_j - L_\pi(t)$$

and the corresponding ruin time is denoted by T^π. Let $\delta > 0$ denote the discount factor and, for every t, $\tau := t - \sum_{j=1}^{N(t)} W_j$ be the time since the last claim occurrence in the portfolio. The optimal dividend problem at time t is to find the strategy $\pi \in \Pi$ that maximizes the expectation

$$J_{u,\tau}(L_\pi(\cdot)) = \mathbb{E}\left[\int_t^{T^\pi} e^{-\delta(s-t)}\,\mathrm{d}L_\pi(s)\,\middle|\,R(t) = u, t - \sum_{j=1}^{N(t)} W_j = \tau\right],$$

where the integration is understood in the ordinary Lebesgue-Stieltjes sense (note that J eventually does not depend on t).

In the classical Cramér-Lundberg model, $N(t)$ is a homogeneous Poisson process and optimal strategies π^* turn out to have a band form (see [16] or [7] for details) and in particular do not depend on t and τ (which is to some extent due to the lack-of-memory property of the exponentially distributed inter-occurrence times). In the special case of exponentially distributed claims, i.e. $F(x) = 1 - e^{-\beta x}$, $(\beta > 0)$, the optimal strategy is a horizontal barrier. That is, there exists a $b \geq 0$, such that the optimal strategy is of the form

$$\mathrm{d}L_b = \begin{cases} u - b & \text{if } u > b, \\ c\,dt & \text{if } u = b, \\ 0 & \text{if } u < b. \end{cases} \tag{1}$$

This fact motivated a series of papers that study properties of the controlled process, see e.g. Lin et al. [23], Dickson & Waters [14] and Paulsen & Gjessing [26]. Other types of barrier strategies in the classical risk model were for instance investigated in Siegl & Tichy [28] and Albrecher et al. [2, 4].

In this paper, we aim to study dividend strategies in the more general Sparre Andersen model, where the inter-occurrence times W_j, $(j = 1, 2, \dots)$ are independent and identically distributed random variables with distribution function G, i. e. the claim number process N_t is modelled by a general renewal process. The renewal assumption allows for more flexibility than the classical risk process (where N_t constitutes a homogeneous Poisson process) and enables to some extent contagion between claim occurrences. This model was introduced by Sparre Andersen in 1957 [29] and since then has received a lot of attention in risk theory (see Rolski et al. [27] for a survey on the subject). Note that, in contrast to the Cramér-Lundberg model, the resulting surplus process is in general not a Lévy process any more. If G is the generalized Erlang(n)

distribution, then each inter-occurrence time can be represented as a convolution of exponential distributions, which allows to utilize the lack-of-memory property of the latter to retain analytical tractability of the model (concretely, by increasing the dimension of the process, it can be "markovized" again). Various properties of the Erlang(n) model were analyzed recently, see for instance Dickson [11], Dickson & Hipp [12], Cheng & Tang [10], Sun & Tang [30], Tsai & Sun [31], Gerber & Shiu [17] and Li & Garrido [21]. For a horizontal dividend barrier in such a model, Li & Garrido [22] investigate the corresponding discounted penalty function and Albrecher et al. [1] provide formulae for the distribution of the discounted dividends.

Whereas for the classical Cramér-Lundberg model the intensity of a jump within the next infinitesimal time step (i.e. the hazard rate of the inter-occurrence distribution) is constant, in the Erlang(n) model this intensity depends on the time τ since the last claim occurrence, so that one may expect the optimal barrier strategy to depend on τ as well.
In this paper we will show that even for exponential claim sizes, one can identify situations in which a horizontal dividend barrier strategy is indeed not optimal. To that end, in Section 2 we first define "virtual" states of the risk process, assume that at each point in time the current state i is known and subsequently consider a piecewise constant barrier strategy, where the barrier height depends on i. For such a model, integro-differential equations for arbitrary moments of dividend payments are derived and explicitly solved for an Erlang(2)-model with exponential claim amounts. The corresponding optimal dividend payouts outperform the horizontal dividend barrier strategy of the corresponding model. More than that, for reasons specified later, this strategy might provide an upper bound for the optimal dividend payout in the Erlang(n)-model in general. In practice, the current state i of the system will not be observable. However, in Section 3 the approach of Section 2 is used as a heuristic to define a time-dependent barrier strategy that indeed outperforms the (optimized) horizontal barrier strategy in this Sparre Andersen model. This fact is illustrated by several stochastic simulations with incorporated variance reduction techniques of control variate type.

2. A heuristic upper bound for the optimal dividend payout

In the sequel we will use the special structure of the Erlang(n) inter-occurrence times explicitly. Let us decompose each W_j into a sum of independent exponential random variables $Y_{j1}, \dots, Y_{jn}$ with parameters $\lambda_1, \dots, \lambda_n$. For that purpose, let us introduce n states of the risk process: After the occurrence of a claim, the risk process is in state 1 and for each j, the process jumps into state i at the time $\tau = \sum_{k=1}^{i-1} Y_{jk}$ after the previous claim ($i = 2, \dots, n$). Finally, at $\tau = \sum_{k=1}^{n} Y_{jk}$, an actual claim with distribution F occurs and the process jumps into state 1 again (similar constructions were used in Gerber and Shiu [17] and Albrecher et al. [1]). Assume for the rest of this section that at every point in time the actual state i of the risk process is known: Then, the corresponding optimal strategy gives an upper bound for the optimal strategy in the Erlang(n) model, where this information is in general not available.
In view of the fact that, given the current state i is known, the intensity of a jump to the next state $i+1$ is constant, it is reasonable to expect that a strategy with constant barriers b_i for each state i is optimal: For $0 \le b_1 \le b_2 \le \dots \le b_n$, let the dynamics of the dividend stream in state i ($i = 1, \dots, n$) be defined as follows

$$\mathrm{d}L_{\mathbf{b}} = \begin{cases} u - b_i & \text{if } u > b_i, \\ c\,dt & \text{if } u = b_i, \\ 0 & \text{if } u < b_i. \end{cases}$$

Here $\mathbf{b} := (b_1, \dots, b_n)$.

2.1. **The moment-generating function of the discounted dividends.** Let $M^{(i)}(u,y,\mathbf{b}) = E(e^{y\,D}|i,\, R_0 = u)$ denote the moment-generating function of the discounted dividends D according to the above piecewise constant barrier strategy, given that the risk process starts in state i ($i = 1,\dots,n$). We will use the differential approach to derive a system of integro-differential equations for $M^{(i)}(u,y,\mathbf{b})$. For $i = 1,\dots,n-1$ and $0 \le u < b_i$, conditioning on the occurrence of a jump within an infinitesimal time interval gives

$$M^{(i+1)}(u,y,\mathbf{b}) = \left(\frac{\delta y\frac{\partial}{\partial y} - c\frac{\partial}{\partial u} + \lambda_i}{\lambda_i}\right) M^{(i)}(u,y,\mathbf{b}).$$

At $u = b_i$, we have

$$M^{(i+1)}(b_i,y,\mathbf{b}) = \left(\frac{\delta y\frac{\partial}{\partial y} - (c\,y - \lambda_i)}{\lambda_i}\right) M^{(i)}(b_i,y,\mathbf{b}).$$

Thus, by continuity we obtain the conditions

$$yM^{(i)}(b_i,y,\mathbf{b}) = \frac{\partial}{\partial u}M^{(i)}(u,y,\mathbf{b})\bigg|_{u=b_i}.$$

An actual claim can only occur in state n, and for $0 \le u < b_n$ the differential approach yields

$$\begin{aligned} M^{(n)}(u,y,\mathbf{b}) &= (1-\lambda_n\,dt)M^{(n)}(u + c\,dt, ye^{-\delta\,dt},\mathbf{b}) \\ &\quad + \lambda_n\,dt\left(\int_0^{(u-b_1)^+} M^{(1)}(b_1, ye^{-\delta\,dt},\mathbf{b})e^{y(u-v-b_1)}\,\mathrm{d}F(v)\right. \\ &\quad\left. + \int_{(u-b_1)^+}^{u} M^{(1)}(u + c\,dt - v, ye^{-\delta\,dt},\mathbf{b})\,\mathrm{d}F(v) + \int_u^\infty \mathrm{d}F(v)\right) + o(dt). \end{aligned}$$

Taylor expansion and collection of terms of order dt yields for $0 \le u < b_n$:

$$\begin{aligned} &\left(\frac{c\frac{\partial}{\partial u} - \delta y\frac{\partial}{\partial y} - \lambda_n}{\lambda_n}\right) M^{(n)}(u,y,\mathbf{b}) + M^{(1)}(b_1,y,\mathbf{b})e^{y(u-b_1)}\int_0^{(u-b_1)^+} e^{-v\,y}\,\mathrm{d}F(v) \\ &\qquad + \int_{(u-b_1)^+}^{u} M^{(1)}(u-v,y,\mathbf{b})\,\mathrm{d}F(v) + 1 - F(v) = 0 \end{aligned}$$

Considering $u = b_n$ and using continuity again, we get the boundary condition

$$yM^{(n)}(b_n,y,\mathbf{b}) = \frac{\partial}{\partial u}M^{(n)}(u,y,\mathbf{b})\bigg|_{u=b_n}.$$

Let us define

$$M^{(i)}(u,y,\mathbf{b}) = \sum_{j=1}^{i} M^{(i,j)}(u,y,\mathbf{b})I_{(b_{j-1}\le u<b_j)}.$$

The system of integro-differential equations now reads as follows:
For $i = 1,\dots,n$ and $b_{i-1} \le u < b_i$,

$$\begin{aligned} \text{(2)}\quad &\left[\prod_{j=i}^{n}\left(\frac{-c\frac{\partial}{\partial u} + \delta y\frac{\partial}{\partial y} + \lambda_j}{\lambda_j}\right)\right] M^{(i,i)}(u,y,\mathbf{b}) + M^{(1,1)}(b_1,y,\mathbf{b})e^{y(u-b_1)}\int_0^{(u-b_1)^+} e^{-v\,y}\,\mathrm{d}F(v) \\ &\qquad + \int_{(u-b_1)^+}^{u} M^{(1,1)}(u-v,y,\mathbf{b})\,\mathrm{d}F(v) + 1 - F(v) = 0 \end{aligned}$$

(note that the product $y\frac{\partial}{\partial y}$ is not commutative).
For $i = 1, \dots, n-1$, $j = 1, \dots, i$ and $b_{j-1} \le u < b_j$,

$$M^{(i+1,j)}(u, y, \mathbf{b}) = \left(\frac{\delta y \frac{\partial}{\partial y} - c\frac{\partial}{\partial u} + \lambda_i}{\lambda_i} \right) M^{(i,j)}(u, y, \mathbf{b}). \tag{3}$$

The boundary conditions, for $i = 1, \dots, n$, are given by

$$y M^{(i,i)}(b_i, y, \mathbf{b}) = \frac{\partial}{\partial u} M^{(i,i)}(u, y, \mathbf{b}) \bigg|_{u=b_i}, \tag{4}$$

$$\lim_{b_1 \to \infty} M^{(i)}(u, y, \mathbf{b}) = 1, \tag{5}$$

and in addition there are continuity conditions: for $i = 2, \dots, n$ and $j = 2, \dots, i$

$$M^{(i,j)}(b_{j-1}, y, \mathbf{b}) = \lim_{u \to b_{j-1}^-} M^{(i,j-1)}(u, y, \mathbf{b}). \tag{6}$$

Define the mth moment $V_m^{(i)}(u, \mathbf{b})$ of the discounted dividend payments due to the above strategy (given that the process starts in state i) and let again

$$V_m^{(i)}(u, \mathbf{b}) = \sum_{j=1}^{i} V_m^{(i,j)}(u, \mathbf{b}) I_{(b_{j-1} \le u < b_j)}.$$

The system (2)-(6) can be converted into integro-differential equations for the moments of the discounted dividends through the representation

$$M^{(i,j)}(u, y, \mathbf{b}) = 1 + \sum_{m=1}^{\infty} \frac{y^m}{m!} V_m^{(i,j)}(u, \mathbf{b}).$$

By comparison of coefficients at powers of y, we obtain for $i = 1, \dots, n$ and $b_{i-1} \le u < b_i$,

(7)

$$\begin{aligned} 0 = &\left[\prod_{j=i}^{n} \left(\frac{-c\frac{\partial}{\partial u} + \delta\, m + \lambda_j}{\lambda_j} \right) \right] V_m^{(i,i)}(u, \mathbf{b}) \\ &+ I_{(u \ge b_1)} \sum_{l_1+l_2+l_3=m} \frac{m!}{l_1! l_2! l_3!} V_{l_1}^{(1,1)}(b_1, \mathbf{b})(u-b_1)^{l_2} \int_0^{u-b_1} (-v)^{l_3}\, \mathrm{d}F(v) + \int_{(u-b_1)^+}^{u} V_m^{(1,1)}(u-v, \mathbf{b})\, \mathrm{d}F(v). \end{aligned}$$

For $i = 1, \dots, n-1$, $j = 1, \dots, i$ and $b_{j-1} \le u < b_j$, we have

$$V_m^{(i+1,j)}(u, \mathbf{b}) = \left(\frac{-c\frac{\partial}{\partial u} + \lambda_i + \delta m}{\lambda_i} \right) V_m^{(i,j)}(u, \mathbf{b}). \tag{8}$$

The boundary conditions are given by ($i = 1, \dots, n$)

$$V_{m-1}^{(i,i)}(b_i, \mathbf{b}) = \frac{\partial}{\partial u} V_m^{(i,i)}(u, \mathbf{b}) \bigg|_{u=b_i}. \tag{9}$$

Finally, the continuity conditions give for $i = 2, \dots, n$ and $j = 2, \dots, i$:

$$V_m^{(i,j)}(b_{j-1}, \mathbf{b}) = \lim_{u \to b_{j-1}^-} V_m^{(i,j-1)}(u, \mathbf{b}). \tag{10}$$

Remark: Note that, technically, dropping the assumption $b_1 \le b_2 \le \cdots \le b_n$ is easily possible by considering the additional payouts at the times of a state change. However, in view of optimality considerations the assumption is natural since in states with lower index one can “afford” a lower dividend barrier, as the next claim will not arrive before the process has jumped from state n to state 1.

2.2. **Erlang(2) interarrival-times and exponential claims.** In principle, the above equations can be solved explicitly for claim size distributions with rational Laplace-Stieltjes transform through an adaptation of the algorithm developed in Section 4.2 in Albrecher et al. [1]. In the following, we illustrate the approach for an Erlang$(2, \lambda)$-model and exponential claim sizes (with parameter η) and derive an explicit solution for the first moment $V_1(u, \mathbf{b})$.

Let us ignore for a moment that $V_1^{(1,1)}$ is only defined in $[0, b_1)$ and take the Laplace transform of the equation

$$\left(\frac{\lambda + \delta - c\frac{\partial}{\partial u}}{\lambda}\right)^2 V_1^{(1,1)}(u, \mathbf{b}) - \int_0^u V_1^{(1,1)}(u - v, \mathbf{b})\eta e^{-\eta v}\,\mathrm{d}v = 0$$

to obtain the structure of the solution. Define $\tilde{V}_1^{(1,1)}(s, \mathbf{b}) := \int_0^\infty e^{-su} V_1^{(1,1)}(u, \mathbf{b})\,\mathrm{d}u$, then

$$\tilde{V}_1^{(1,1)}(s, \mathbf{b}) = \frac{G_1(s)}{(\delta + \lambda - cs)^2 - \frac{\lambda^2\eta}{(s+\eta)}},$$

where $G_1(s)$ is a linear function in s. The inverse Laplace transform yields

$$V_1^{(1,1)}(u) = \sum_{k=1}^{3} \alpha_k(\mathbf{b}) e^{R_k u}, \tag{11}$$

where $\alpha_k(\mathbf{b})$ are constants and R_k are the three (real) roots of the Lundberg fundamental equation

$$P(R) = (R + \eta)(\lambda + \delta - cR)^2 - \lambda^2\eta \tag{12}$$

(for simplicity these are assumed to be distinct). By substituting (11) in (7) and (9), for $i = 1$, and comparing coefficients, we get the conditions

$$\sum_{k=1}^{3} \frac{\alpha_k(\mathbf{b})}{R_k + \eta} = 0, \qquad \sum_{k=1}^{3} \alpha_k(\mathbf{b}) R_k e^{R_k b_1} = 1. \tag{13}$$

From (8), we have

$$V_1^{(2,1)}(u, \mathbf{b}) = \sum_{k=1}^{3} \alpha_k(\mathbf{b}) \frac{(\lambda + \delta - cR_k)}{\lambda} e^{R_k u}. \tag{14}$$

Equation (7) yields for $i = 2$

$$\left(\frac{\lambda + \delta - c\frac{\partial}{\partial u}}{\lambda}\right) V_1^{(2,2)}(u, \mathbf{b}) = u - b_1 - \frac{1}{\eta} + \sum_{k=1}^{3} \alpha_k(\mathbf{b}) e^{R_k b_1} + e^{-\eta(u - b_1)}\left(\frac{1}{\eta} - \sum_{k=1}^{3} \alpha_k(\mathbf{b}) \frac{R_k e^{R_k b_1}}{R_k + \eta}\right).$$

As the homogenous solution is of the form $e^{\frac{\lambda+\delta}{c}u}$, we try to find a solution of the form

$$V_1^{(2,2)}(u, \mathbf{b}) = K_1(\mathbf{b}) e^{\frac{\lambda+\delta}{c}u} + K_2(\mathbf{b})u + K_3(\mathbf{b}) + K_4(\mathbf{b}) e^{-\eta u}. \tag{15}$$

Substitution in (7) for $i=2$ and comparing coefficients then yields

$$K_2(\mathbf{b}) = \frac{\lambda}{\lambda+\delta}, \tag{16}$$

$$K_3(\mathbf{b}) = \frac{\lambda}{\lambda+\delta}\left(\sum_{k=1}^{3}\alpha_k(\mathbf{b})e^{R_k b_1} - b_1 - \frac{1}{\eta} + \frac{c}{\delta+\lambda}\right), \tag{17}$$

$$K_4(\mathbf{b}) = \frac{\lambda e^{\eta b_1}\left(\frac{1}{\eta} - \sum_{k=1}^{3}\alpha_k(\mathbf{b})\frac{R_k e^{R_k b_1}}{R_k+\eta}\right)}{\lambda+\delta+c\eta}. \tag{18}$$

Next, consider the conditions (9), for $i=1$, and (10), for $i=2$, to get

$$K_1(\mathbf{b})\frac{\lambda+\delta}{c}e^{\frac{\lambda+\delta}{c}b_2} + \frac{\lambda}{\lambda+\delta} - \frac{\lambda\eta e^{\eta(b_1-b_2)}}{\lambda+\delta+c\eta}\left(\frac{1}{\eta} - \sum_{k=1}^{3}\alpha_k(\mathbf{b})\frac{R_k e^{R_k b_1}}{R_k+\eta}\right) = 1, \tag{19}$$

$$\begin{aligned} K_1(\mathbf{b})e^{\frac{\lambda+\delta}{c}b_1} - \sum_{k=1}^{3}\alpha_k(\mathbf{b})e^{R_k b_1}\left(1 + \frac{\delta}{\lambda+\delta} - \frac{R_k\lambda}{(R_k+\eta)(\lambda+\delta+c\eta)}\right) \\ = \frac{\lambda}{\eta}\left(\frac{1}{\lambda+\delta} - \frac{1}{\lambda+\delta+c\eta}\right) - \left(\frac{1}{\lambda} + \frac{\lambda}{(\lambda+\delta)^2}\right) \end{aligned} \tag{20}$$

The expected dividends are therefore given by (11), (14) and (15), where the constants are determined by (13) and (16)-(20).

2.3. **Numerical Example.** For illustration, let us consider the model of Section 2.2 and choose $\lambda = 2$, $\eta = 1$, $\delta = 0.03$ and $c = 1.1 > 1$ (so that the net premium condition is fulfilled). Let us further consider the zero-delayed renewal model (that is $\tau = 0$ and $i = 1$).
The analytical formulas of Section 2.2 allow to quickly determine the expected value of the discounted dividends under the strategy $\pi = \mathbf{b} = (b_1, b_2)$ for given values $b_1 \le b_2$. The optimal pair (b_1^*, b_2^*) can for instance be determined using a bisection method. Table 1 depicts the corresponding numbers for $u = 0$ for a representative set of combinations of barrier values, with a finer grid close to the maximum value of V_1. The optimal barrier choice is in this case $b_1^* = 1.2$ and $b_2^* = 2.3$ (with corresponding expected value $V_1 = 1.13329$). Figure 1 illustrates that the maximum of $V_1(b_1, b_2)$ is not very pronounced, but a zoom into its neighborhood identifies a single peak. Figure 2 shows that for other choices of parameters, the maximum can be even flatter.

$b_2 \setminus b_1$	0	1	2	2.2	2.3	2.4	3
0	1.07574	1.10180	1.10309	1.11308	1.10307	1.10306	1.10301
1		1.11745	1.13234	1.13251	1.13252	1.13251	1.13228
1.1			1.13286	1.13310	1.13309	1.13306	1.13282
1.2			1.13300	1.13327	**1.13329**	1.13328	1.13296
1.3			1.13277	1.13308	1.13311	1.13309	1.13296
2			1.12541	1.12478	1.12431	1.12379	1.12105
3							1.09500

TABLE 1. Expected value of discounted dividend payments $V_1(u, (b_1, b_2))$ for $W_j \sim$Erlang(2,2), $X_j \sim$Exp(1), $\delta = 0.03$, $u = 0$ and $c = 1.1$.

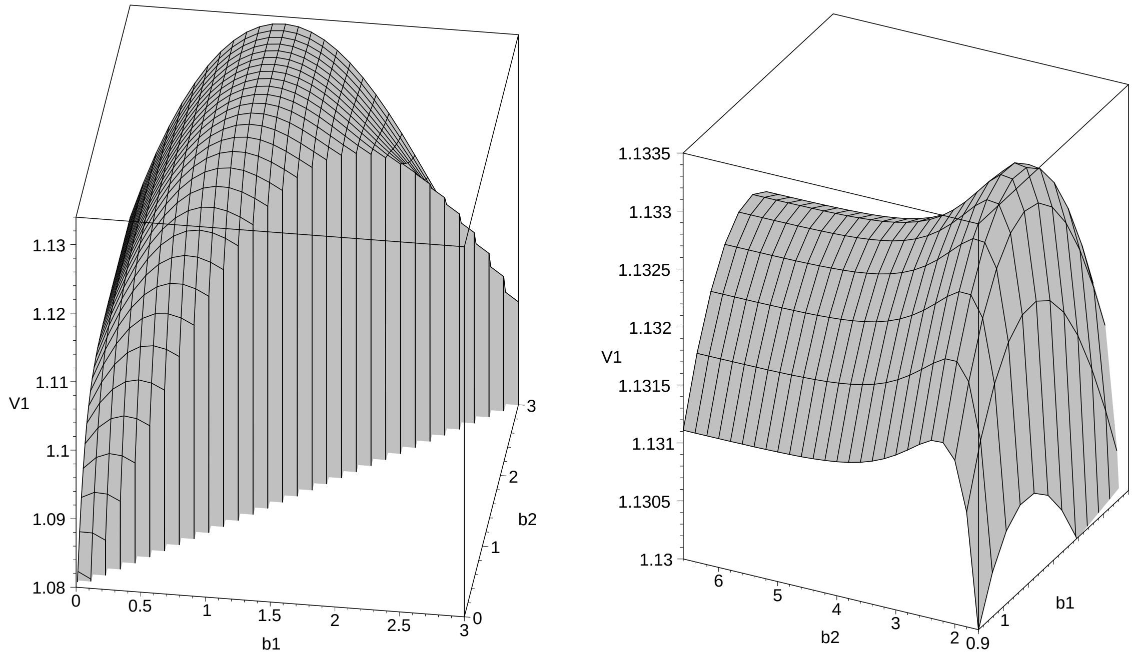

FIGURE 1. Expected present value of dividends in the two barrier strategy: $c = 1.1$, $\delta = 0.03$, $u = 0$. On the right is a zoom into the left plot.

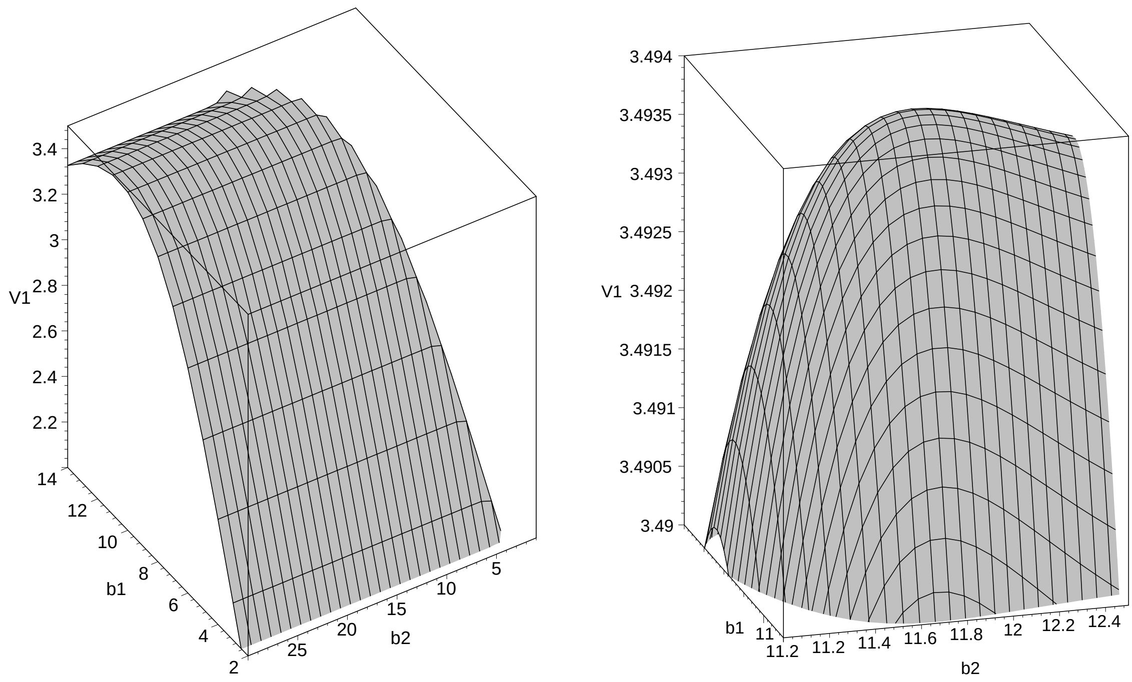

FIGURE 2. Expected present value of dividends in the two barrier strategy: $c = 1.2$, $\delta = 0.01$, $u = 0$. On the right is a zoom into the left plot.

One might expect that, as in the classical model, the optimal barrier levels do not depend on the initial capital u. As an illustration in this direction, Table 2 depicts V_1 for $u = 1$ under otherwise identical parameter values to Table 1, and indeed the maximal value of V_1 is achieved for $b_1^* = 1.2$ and $b_2^* = 2.3$ again. Naturally, (b_1^*, b_2^*) heavily depends on the choice of the discount factor δ and the premium income c (see Table 3).

$b_2 \setminus b_1$	1	2	2.2	2.3	2.4	3
1	2.12045	2.14433	2.14460	2.14463	2.14461	2.14424
1.1		2.14540	2.14577	2.14580	2.14578	2.14533
1.2		2.14568	2.14614	**2.14618**	2.14616	2.14560
1.3		2.14523	2.14578	2.14583	2.14580	2.14511
2		2.13095	2.12977	2.12888	2.12791	2.12276
3						2.07331

TABLE 2. Expected value of discounted dividend payments $V_1(u, (b_1, b_2))$ for $W_j \sim$Erlang(2,2), $X_j \sim$Exp(1), $\delta = 0.03$, $u = 1$ and $c = 1.1$.

$\delta \rightarrow$	0.01		0.02		0.03	
$c \downarrow$	V_1	(b_1^*, b_2^*)	V_1	(b_1^*, b_2^*)	V_1	(b_1^*, b_2^*)
1.025	1.02236	$(0, 1.5)$	1.00982	$(0, 0)$	1.00233	$(0, 0)$
1.05	1.11792	$(2.85, 3.94)$	1.04738	$(0.47, 1.57)$	1.02987	$(0, 0.69)$
1.1	1.55042	$(6.67, 7.78)$	1.22852	$(2.69, 3.80)$	1.13329	$(1.20, 2.30)$
1.2	3.49383	$(10.67, 11.80)$	2.05275	$(5.83, 6.97)$	1.62645	$(3.67, 4.808)$

TABLE 3. Optimal two-barrier strategies and the corresponding optimal expected values of discounted dividend payments V_1 for $W_j \sim$Erlang(2,2), $X_j \sim$Exp(1) and $u = 0$.

2.3.1. *Comparison with horizontal barrier strategy.* The case of a horizontal barrier strategy can be retained from the above formulae by choosing $b = b_1 = b_2$. Alternatively, one can also use the explicit formula for the expected discounted dividend payments $V_1(u)$ derived in Albrecher et al. [1] for this case, which is given by

$$V_1(u) = \sum_{k=1}^{3} \hat{\alpha}_k(b) e^{R_k u}, \quad \text{and} \qquad 0 \le u \le b$$

where R_k $(k = 1, 2, 3)$ are the three roots of the polynomial (12) and the constants are given as solutions of the following system of equations:

$$\sum_{k=1}^{3} \frac{\hat{\alpha}_k(b)}{R_k + \eta} = 0, \qquad \sum_{k=1}^{3} \hat{\alpha}_k(b) R_k e^{R_k b} = 1, \qquad \sum_{k=1}^{3} \hat{\alpha}_k(b) R_k^2 e^{R_k b} = \frac{\delta}{c}.$$

For the parameters chosen in Table 1, the optimal horizontal barrier turns out to be $b^* = 1.7$ and the corresponding expected present value of the dividends is $V_1^*(b^*) = 1.12724$, which has to be compared with $V^*(b_1^*, b_2^*) = 1.13329 > V_1^*(b^*)$. For the parameters of Table 2, the maximal horizontal barrier strategy gives $V_1^*(b^*) = 2.13462$, which is again smaller than $V_1^*(b_1^*, b_2^*) = 2.14618$.

Table 4 depicts the optimal barrier heights b^* and corresponding expected values for the horizontal barrier strategy for the same set of parameters as Table 3 and thus allows to compare the performance of the two strategies.

$\delta \rightarrow$	0.01		0.02		0.03	
$c \downarrow$	$J_{0,0}(L_{b^*})$	b^*	$J_{0,0}(L_{b^*})$	b^*	$J_{0,0}(L_{b^*})$	b^*
1.025	1.02219	1	1.00982	0	1.00233	0
1.05	1.11588	3.3	1.04353	0.9	1.02684	0
1.1	1.54766	7.2	1.224167	3.2	1.12724	1.7
1.2	3.48784	11.2	2.04567	6.4	1.61809	4.2

TABLE 4. Optimal horizontal barrier strategies and the corresponding optimal expected discounted dividend payments for $W_j \sim$Erlang(2,2), $X_j \sim$Exp(1) and $u = 0$.

3. Beating the horizontal barrier strategy

The dividend payment strategy advocated in Section 2 can usually not be implemented in practice, because the knowledge about the state of the risk process is in general not available. However, the results serve as a heuristic to find a natural candidate for a good barrier strategy for the Erlang(n, λ) model. Let $0 \leq b_1^* \leq b_2^* \leq \cdots \leq b_n^*$ denote the barrier levels of the optimal strategy of Section 2. Based on these optimal values, we introduce a strategy that is independent of the knowledge of the state i of the system and yet still turns out to outperform the horizontal barrier strategy. Let τ again denote the time since the last claim occurrence. Then define the time-dependent barrier

$$b(\tau) = \sum_{i=1}^{n} b_i^* \, \mathbb{P}\,(\text{state} = i \,|\, W_1 \geq \tau) = \frac{1}{\sum_{i=0}^{n-1} \frac{(\lambda\tau)^i}{i!}} \sum_{i=1}^{n} b_i^* \frac{(\lambda\tau)^{i-1}}{(i-1)!}. \tag{21}$$

This choice can be interpreted as mimicking the strategy of Section 2 on an "average" basis. The corresponding strategy constitutes a non-linear time-dependent barrier, for which it is usually not possible to obtain analytical formulae for the expected present value of the dividends (see e.g. Albrecher & Kainhofer [3]). Therefore we employ stochastic simulation to assess the performance of this strategy (see Glasserman [19] for a survey on simulation techniques in finance and insurance).

As an example, for $n = 2$ the strategy (21) is given by

$$b(\tau) = \frac{b_1^* + b_2^* \, \lambda \, \tau}{1 + \lambda \, \tau},$$

see Figure 3 for a sample path controlled by this strategy.

3.1. **The simulation algorithm.** The first step is to construct a consistent estimator Z_1 for the expected discounted dividends under the strategy (21). For that purpose, we sample paths of the risk process in the following way: starting with the initial capital $u_0 = u$, we successively generate Erlang$(2, 2)$-distributed random variates w_j to generate the claim arrival epochs, and exponentially distributed random variates x_j for the claim sizes ($j \in \mathbb{N}$). For every $j \in \mathbb{N}_0$, if the surplus u_j at time $\mathcal{T}_j = \sum_{k=1}^{j} w_k$ (after the payment of the jth claim) is non-negative, then

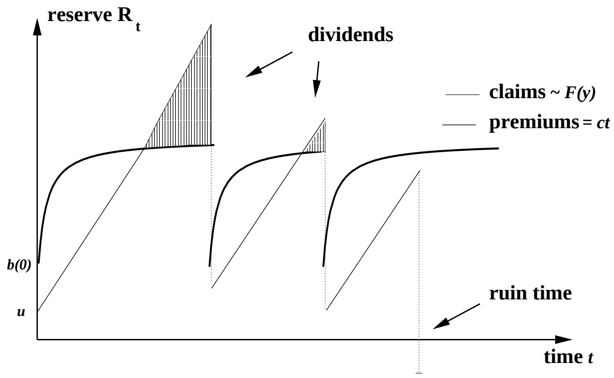

FIGURE 3. Dividend strategy (21) applied to a sample path of R_t with $\lambda = 2$, $\delta = 0.02$, $c = 1.05$.

the suitably discounted dividends d_{j+1} received in the period between $\mathcal{T}_j$ and $\mathcal{T}_{j+1}$ are obtained by

$$d_{j+1} = e^{-\delta \mathcal{T}_j} \left((u_j - b(0))^+ + \left(\int_{t_{j+1}}^{w_{j+1}} e^{-\delta s} \left(c - \frac{\partial}{\partial s} b(s) \right) \mathrm{d}s \right)^+ \right), \tag{22}$$

where $t_{j+1} = \inf\{s \geq 0 | u_j + c\ s \geq b(s)\}$ and $\mathcal{T}_0 = 0$. As soon as $u_K < 0$ for some $K \in \mathbb{N}$, the process is stopped and the present value of the aggregate dividends received for this path is calculated by $z_1 = \sum_{j=1}^K d_j$.
Let $\sigma^2_{Z_1}$ be the variance of the estimator Z_1. We run $N = 10^7$ paths to get samples $z_{1,i}$, $i = 1, \ldots, N$, a point estimate $\hat{J}_{u,0}(L_{b(\tau)}(.)) = \frac{1}{N} \sum_{i=1}^N z_{1,i}$ for $J_{u,0}(L_{b(\tau)}(.))$ under strategy (21), and the sample variance $\hat{\sigma}^2_{Z_1}$. These quantities may be used to establish an asymptotically valid 95%-confidence interval for the expected present value of the dividend stream:

$$\mathbb{P}\left(\hat{J}_{u,0}(L_{b(\tau)}(.)) - 1.96 \frac{\hat{\sigma}_{Z_1}}{N^{1/2}} \leq J_{u,0}(L_{b(\tau)}(.)) \leq \hat{J}_{u,0}(L_{b(\tau)}(.)) + 1.96 \frac{\hat{\sigma}_{Z_1}}{N^{1/2}} \right) = 0.95.$$

3.2. **Variance reduction.** Since the variance $\sigma^2_{Z_1}$ of the estimator Z_1 is very high, this direct approach needs a huge number of sample paths to get sufficiently small confidence intervals. In addition, since the integral in (22) is not available in closed form, the calculation of each path is very time consuming. Therefore, we apply a variance reduction technique based on a control variate (see e.g. [19] for details). Concretely, we use the expected dividends under the horizontal dividend barrier strategy as a control variate, since the latter is available analytically. By simultaneously estimating the expected dividends under the horizontal barrier strategy and strategy (21), one can use the sample error for the horizontal barrier strategy to construct a new estimator with reduced variance, hence significantly reducing the effort to establish reasonable confidence intervals. The efficiency of this variance reduction technique depends on the correlation between the two estimators:
Let H be a consistent estimator of the horizontal barrier strategy with variance σ^2_H and let $\rho(H, Z_1)$ denote the linear correlation coefficient between H and Z_1. Then the new estimator

$$Z_2(\nu) = Z_1 - \nu(H - \mathbb{E}[H]),$$

$\delta \rightarrow$	0.01		0.02		0.03	
$c \downarrow$	direct	control	direct	control	direct	control
1.025	1.02263 $[1.0217, 1.0232]$	1.02268 $[1.0223, 1.02231]$	strategies coincide			
1.05	1.11743 $[1.1158, 1.1191]$	1.11632 $[1.1158, 1.1167]$	1.04373 $[1.0427, 1.0447]$	1.04386 $[1.0434, 1.0443]$	1.2467 $[1.0240, 1.02530]$	1.2476 $[1.0242, 1.0253]$
1.1	1.54870 $[1.5462, 1.513]$	1.54811 $[1.5478, 1.5484]$	1.22410 $[1.2225, 1.22572]$	1.22506 $[1.22469, 1.22546]$	1.12786 $[1.2667, 1.1291]$	1.12840 $[1.1280, 1.1284]$
1.2	3.49070 $[3.4873, 3.49526]$	3.4888 $[3.4844, 3.48916]$	2.04895 $[2.0464, 2.0514]$	2.04721 $[2.0468, 2.0474]$	1.61967 $[1.6180, 1.6201]$	1.61933 $[1.6190, 1.6197]$

TABLE 5. Simulated expected discounted dividend payments (together with 95% confidence intervals) for dividend strategy (21) for $W_j \sim$Erlang(2,2), $X_j \sim$ Exp(1) and $u = 0$.

is again consistent and has variance

$$\sigma^2_{Z_2}(\nu) = \sigma^2_{Z_1} - 2\,\nu\,\sigma_{Z_1}\,\sigma_H\,\rho(H, Z_1) + \nu^2\sigma^2_H.$$

Clearly, the optimal ν is given by

$$\nu^* = \frac{\sigma_{Z_1}}{\sigma_H}\,\rho(H, Z_1),$$

and the corresponding variance improvement is then given by

$$\frac{\sigma^2_{Z_2(\nu^*)}}{\sigma^2_{Z_1}} = 1 - \rho^2(H, Z_1).$$

In our simulations, we estimated the optimal parameter ν^* for H from the same random deviates that were used for the point estimate, which introduces a bias of order $1/N$ (e.g. [19]), but the latter is negligible for our sample size (since ν^* turns out to be close to 1 in all the performed simulations, one can alternatively set $\nu = 1$, in this way getting rid of the bias and the corresponding simulation results turn out to be almost identical). The correlation coefficient $\rho(H, Z_1)$ and hence the variance reduction turns out to be larger for increasing values of horizontal barrier b (e.g. for $c = 1.05$, $\delta = 0.03$ (i.e. $b^* = 0$) $\rho(H, Z_1) \approx 0.65$, whereas for $c = 1.2$, $\delta = 0.01$ (i.e. $b^* = 11.2$) $\rho(H, Z_1) \approx 0.99$). This can intuitively be understood from the fact that the life times of the sample paths of the two strategies are closer for larger values of b.

3.3. **Comparison of dividend payment strategies.** One can now use the discussed algorithm to simulate the performance of the dividend strategy (21). Table 5 shows point estimates together with the corresponding confidence intervals for the direct estimator Z_1 as well as for the improved estimator Z_2 based on the control variate. A comparison of the resulting expected discounted dividend payouts with Table 4 clearly shows that the proposed strategy (21) outperforms the (optimized) horizontal strategy in a large set of scenarios. Note that for the combination $c = 1.025$ and $\delta = 0.02, 0.03$ in the table, we have $b_1^* = b_2^* = b^* = 0$ and hence the two strategies coincide.

We finally would like to remark that the Erlang(2) model was chosen for this comparison because of its analytical tractability. Since the structure of this process is still rather similar to the compound Poisson model (in which case the horizontal barrier is known to be optimal for exponential claim sizes), one can not expect a large improvement of the optimal value. However, this particular example already serves as an illustration of the non-optimality of the horizontal barrier strategy in the Sparre Andersen framework and for other inter-occurrence time distributions one may expect larger differences of the respective optimal values.

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