An Accuracy Condition for the Finite Element Discretization of Biot's Equations on Triangular Meshes

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ABSTRACT

Finite element solutions of Biot's equations may be characterized by unphysical oscillations for small time-steps or low permeabilities. We analyse these numerical wiggles by comparing the Schur complement of the system with an equivalent reaction-diffusion problem. We show that the non-physical behaviour of the discrete solution is due to the fact that the discrete maximum principle is not satisfied. We provide a sufficient condition for two-dimensional problems of this kind to ensure monotonic solutions on triangular meshes.

INTRODUCTION

In this work, we analyze positivity and monotonicity properties of the finite element (FE) approximation of the solution of Biot's equations (Biot, 1941). It is a time-dependent set of partial differential equations (PDEs) which describes consolidation processes of porous media. Generalizations of this model for large deformations are nowadays also employed in several bio-mechanical applications (Favino et al., 2016). We consider the following linear operator

$$\left|\begin{array}{cc} \mathscr{E} & \mathscr{B}^T \\ \mathscr{B} & -\lambda\mathscr{L} - \mathscr{I} \end{array}\right| \left|\begin{array}{c} \mathbf{u} \\ p \end{array}\right|, \tag{1}$$

which is obtained after time discretization using the implicit Euler method of Biot's equations. Here, $\mathbf{u}$ is the displacement vector, p is the pore pressure, $\mathscr{E}$ is a linear elasticity operator, $\mathscr{B}$ and $\mathscr{B}^T$ are the divergence and gradient operators, respectively, $\mathscr{L}$ and $\mathscr{I}$ are Laplacian-like and identity-like operators, respectively. The parameter λ is the product the hydraulic conductivity κ and the time-step τ. Stokes' problem and incompressible linear elasticity systems can be regarded as particular cases of (1), in which the bottom-right block is null.

It is well known that for small values of λ the FE solution of (1) may exhibit unphysical oscillations with regard to both the pressure and the components of the displacement (Preisig and Prévost, 2010; Favino et al., 2013; Rodrigo et al., 2016). These numerical wiggles, which are usually localized close to the boundaries of the domain and in regions where jumps in the material properties are present, have been explained as being due to violations (i) of the Ladyženskaja-Babuška-Brezzi (LBB) condition (Brezzi and Fortin, 2012) or (ii) of a minimum time-step condition (Vermeer and Verruijt, 1981).

The first explanation suffers from some theoretical flaws, as oscillations are present also when LBB-stable FE spaces are used. Moreover, the Laplacian in the bottom-right entry of (1) plays the same role as the Brezzi-Piktäranta stabilization for Stokes' problem, for which both Taylor-Hood elements (Brezzi and Fortin, 2012) and FE spaces of the same order have been shown to be stable (Brezzi and Pitkäranta, 1984).

The violation of the minimum time-step condition is actually related to the discrete maximum principle (DMP) for the pressure Schur complement of the Biot system, which can be obtained by static condensation of $\mathbf{u}$ in (1)

$$\lambda \mathscr{L} + \mathscr{I} + \mathscr{B}\mathscr{E}^{-1}\mathscr{B}^T. \tag{2}$$

The third term in the Schur complement has been shown to be (i) proportional to the identity operator in the one-dimensional case (Vermeer and Verruijt, 1981; Gaspar et al., 2003) and (ii) spectrally equivalent to an identity operator in the two- and three-dimensional cases (Verfurth, 1984). These properties have been widely exploited for the preconditioning of the discrete poroelasticity equations (Favino et al., 2012) as well as of the Stokes system (Verfurth, 1984).

To study the properties of the pressure Schur complement, we therefore proceed in analogy to a singularly perturbed diffusion-reaction problem where λ plays the role of the diffusion coefficient. The FE discretization of diffusion-reaction problems does not preserve the monotonicity properties of the continuous problem. Several conditions on the maximum mesh size have been derived in order to satisfy the DMP. In particular, on triangular meshes the satisfaction of the DMP introduces bounds on the angles and the areas of the elements in relation with the diffusion and reaction coefficients (Ciarlet and Raviart, 1973; Brandts et al., 2008).

Owing the fact that the third term in (2) is proportional to the identity operator, accuracy conditions derived for diffusion-reaction problems can be directly translated to Biot's equations in the one-dimensional case. For multi-dimensional scenarios, no analytical proof of this equivalence exists, albeit Favino et al. (2013) provided corresponding numerical evidence. Following this strategy, we will provide a sufficient condition to ensure that the corresponding diffusion-reaction problem does not exhibit unphysical oscillations. Moreover, we will show that, as for diffusion-reaction problems, triangular elements whose angles are not strictly acute may provide non-monotonic pressure distributions in a confined compression case.

1 MATHEMATICAL MODEL AND DISCRETIZATION

Let $\Omega \subset \mathbb{E}^d$ be a bounded domain in the d-dimensional Euclidean space and $I = (0, T]$ be a time interval of length $T > 0$. Consider the Biot problem of finding the solid displacement $\mathbf{u}$ and the pore pressure p

$$\begin{array}{rcll} -\nabla \cdot (\boldsymbol{\sigma}_E(\mathbf{u}) + \alpha p \mathbf{I}) & = & \mathbf{0} & \text{in } \Omega \times I, \\ -\frac{1}{M}\dot{p} + \alpha \nabla \dot{\mathbf{u}} + \nabla \cdot (\kappa p) & = & 0 & \text{in } \Omega \times I, \end{array} \tag{3}$$

where $\alpha \in (0, 1]$ is the Biot coefficient, κ the hydraulic conductivity, and M a positive parameter. Biot's equations describe the balance of linear momentum and mass for a fluid-saturated porous medium. A similar system of equation can be obtained from the theory of porous media (Grillo et al., 2012) under the assumption that the solid and fluid phases are incompressible, which implies setting $\alpha = 1$ and $M \to \infty$ in (3).
The effective stress for the porous medium can be defined by splitting the deviatoric and volumetric components as

$$\boldsymbol{\sigma}_E = 2\mu \text{Dev}(\boldsymbol{\epsilon}) + B\text{tr}(\boldsymbol{\epsilon})$$

with $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\mathbf{u}) = \frac{\nabla \mathbf{u} + \nabla \mathbf{u}^T}{2}$ being the symmetric gradient of the displacement, μ the shear modulus, and B the bulk modulus. In order to discretize (3) in time, we introduce a uniform temporal grid with time-step $\tau = T/N$, where N is a positive integer representing the number of time-steps. The symbol $a^m(\mathbf{x}) \simeq a(x, t^m)$ denotes the approximation of a quantity a at $t^m = m\tau$ for $m = 0, 1, \ldots M$. Applying the implicit Euler method to the Biot system, the semi-discretation in time reads

$$\begin{array}{rcll} -\nabla \cdot (\boldsymbol{\sigma}_E(\mathbf{u}^m) + \alpha p^m \mathbf{I}) & = & \mathbf{0} & \text{in } \Omega, \\ -\frac{1}{M}p^m + \alpha \nabla \mathbf{u}^m + \tau \nabla \cdot (\kappa p) & = & -\frac{1}{M}p^{m-1} + \alpha \nabla \mathbf{u}^{m-1} & \text{in } \Omega. \end{array} \tag{4}$$

The weak form of the above problem can be found in Preisig and Prévost (2010) and its FE approximation leads to the following linear system

$$\begin{vmatrix} \mathbf{E} & \alpha \mathbf{B}^T \\ \alpha \mathbf{B} & -\lambda \mathbf{C} - \frac{1}{M}\mathbf{M} \end{vmatrix} \begin{vmatrix} \underline{u} \\ \underline{p} \end{vmatrix} = \begin{vmatrix} \underline{f} \\ \underline{g} \end{vmatrix}, \tag{5}$$

where $\mathbf{E}$ is the algebraic representation of an elasticity operator. Again, exploiting the splitting between the deviatoric and volumetric parts, we assume that

$$\mathbf{E} = \mu \mathbf{A} + B\mathbf{G}.$$

The parameter λ is, as mentioned earlier, the product of the time-step τ and the hydraulic conductivity κ. In the one-dimensional case, the shear component vanishes and the elasticity operator reduces to $\mathbf{E} = \kappa \mathbf{C}$. In this work, we focus on Taylor-Hood FE spaces. In particular, we employ the $\mathbb{P}_2$-$\mathbb{P}_1$ couple, that is, quadratic elements for the displacement and linear elements for the pressure. A study of the monotonicity properties in the one-dimensional case of $\mathbb{P}_1$-$\mathbb{P}_1$ elements and mini-elements can be found in Rodrigo et al. (2016).

DIFFUSION-REACTION MODEL AND DMP

The matrix in (5) is symmetric and indefinite. Since the sub-matrix $\mathbf{E}$ is symmetric and positive definite, we can compute $\underline{u}$ from the first equation obtaining

$$\underline{u} = \mathbf{E}^{-1}\underline{f} - \alpha\mathbf{E}^{-1}\mathbf{B}^T\underline{p}. \tag{6}$$

Substituting this in the second equation of (5) yields

$$\mathbf{S} = -\underline{g} + \mathbf{B}\mathbf{E}^{-1}\underline{f},$$

where $\mathbf{S}$ is the pressure Schur complement of the system, which reads

$$\mathbf{S} = \lambda\mathbf{C} + \frac{1}{M}\mathbf{M} + \alpha^2\mathbf{B}\mathbf{E}^{-1}\mathbf{B}^T. \tag{7}$$

The matrix $\mathbf{S}$ is symmetric and positive definite and hence it admits at each time-step a unique pressure $\underline{p}$. The corresponding displacement can then be obtained from (6). The first two terms of the Schur complement are a diffusion and a mass matrix and can be regarded as the discrete form of a reaction-diffusion problem

$$-\epsilon\Delta p + \sigma u = RHS, \tag{8}$$

with $\epsilon = \lambda$ and $\sigma = \frac{1}{M}$. Hence, time integration with small time-steps and materials characterized by small permeabilities lead to Schur complements associated with the discretization of singularly perturbed diffusion-reaction problems. When $\frac{\epsilon}{\sigma} \ll 1$, the corresponding FE solutions may contain unphysical wiggles, which are indeed related to a violation of the DMP due to too coarse spatial meshes. In the following, we will discuss the relation between pressure oscillations and the DMP in the one- and two-dimensional cases on triangular grids.

One-dimensional case

If $d = 1$, the third term of (7) reduces to $\frac{1}{B}\mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T$, which has been shown to be equal to the mass matrix $\frac{1}{B}\mathbf{M}$ (Vermeer and Verruijt, 1981; Gaspar et al., 2003). Hence, the Schur complement can be seen as the discretization of (8) with

$$\epsilon = \lambda \quad \text{and} \quad \sigma = \frac{1}{M} + \frac{\alpha^2}{B}.$$

At the algebraic level, accurate discretizations of (7) provide stiffness matrices which are M-matrices(Brandts et al., 2008; Quarteroni et al., 2010). For this analysis, we consider the local stiffness matrices employed to assemble the diffusion-reaction problem

$$\mathbf{S}^i = \frac{\lambda}{h_i}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \left(\frac{1}{M} + \frac{\alpha^2}{B}\right)\frac{h_i}{6}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}. \tag{9}$$

The extra-diagonal terms are the sum of a positive and a negative term. Imposing that this sum is negative, in order to ensure the DMP, we obtain the following condition with regard to the element length h_i

$$\left(\frac{1}{M}+\frac{\alpha^2}{B}\right)\frac{h_i^2}{6\tau\kappa}<1.$$

The same condition has been derived in Preisig and Prévost (2010) by imposing the positivity of the discrete solution.

DMP on triangular meshes

The equivalence between $\mathbf{BE}^{-1}\mathbf{B}$ and $\mathbf{M}$ is not valid in the multi-dimensional case, but $\mathbf{BE}^{-1}\mathbf{B} \to \mathbf{M}$ in uniform norm for $\mu \to 0$ (Favino et al., 2013). Hence, the equivalence between the Schur complement and the discretization of a diffusion-reaction problem holds true for the considered case when $B \gg \mu$.

On the other hand, the matrix $\mathbf{BE}^{-1}\mathbf{B}$ has been shown to be spectrally equivalent to $\frac{1}{\mu+B}\mathbf{M}$ for Taylor-Hood FE spaces. In the following, we assume that $\mathbf{S}$ can be seen as a discretization of (8) with

$$\epsilon=\lambda \quad \text{and} \quad \sigma=\frac{1}{M}+\frac{\alpha^2}{\mu+B}$$

and we will employ the DMP for diffusion-reaction problems to derive an accuracy bound for the Biot equations on triangular meshes.

A sufficient condition with regard to the DMP for discretizations of reaction-diffusion problems on triangular elements relies on the angles of the elements (Ciarlet and Raviart, 1973; Brandts et al., 2008). To derive it, we will consider a triangular element T_i of the computational grid with vertices $|X_1, X_2, X_3|$ and area A. The edges of the element are the vectors $\mathbf{e}_1 = X_3 - X_2$, $\mathbf{e}_2 = X_1 - X_3$, $\mathbf{e}_3 = X_2 - X_1$. Hence, the local diffusion operator assembled employing linear basis functions can, based on geometrical information, be simply written as

$$\mathbf{A}^i=\frac{\lambda}{4A}\begin{vmatrix} \mathbf{e}_1\cdot\mathbf{e}_1 & -\mathbf{e}_1\cdot\mathbf{e}_2 & -\mathbf{e}_1\cdot\mathbf{e}_3 \\ -\mathbf{e}_2\cdot\mathbf{e}_1 & \mathbf{e}_2\cdot\mathbf{e}_2 & -\mathbf{e}_2\cdot\mathbf{e}_3 \\ -\mathbf{e}_3\cdot\mathbf{e}_1 & -\mathbf{e}_3\cdot\mathbf{e}_2 & \mathbf{e}_3\cdot\mathbf{e}_3 \end{vmatrix}.$$

The complication here is that the off-diagonal entries may also be positive for purely diffusive problems if an obtuse angle is present in the triangulation. The element matrix for the equivalent reactive term takes the form

$$\mathbf{M}^i=\left(\frac{1}{M}+\frac{\alpha^2}{\mu+B}\right)\frac{A}{12}\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}.$$

In order to have negative extra-diagonal entries, the angles of each element have to be acute and the following condition has to be fulfilled

$$\left(\frac{1}{M} + \frac{\alpha^2}{\mu + B}\right) \frac{1}{3\kappa\tau} \frac{A^2}{\mathbf{e}_i \cdot \mathbf{e}_j} < 1 \quad \text{with } i, j = 1, 2, 3 \text{ and } i \neq j. \tag{10}$$

As in the one-dimensional case, the time-step τ has to be increased or the spatial grid has to be refined in order to ensure the DMP for the equivalent reaction-diffusion problem. The derived bound is actually only a sufficient condition, since, even if a positive entry appears in the local stiffness matrix of a given element, the negative contributions of adjacent elements may generate a stiffness matrix which globally is an M-matrix. For this reason, straight- and obtuse-angled triangles close to the boundary are most likely responsible for a violation of the DMP. The M-matrix condition is actually a sharp bound for the monotonicity of the solution. Diffusion-reaction problems with some non-negative entries may also provide monotonic solutions.

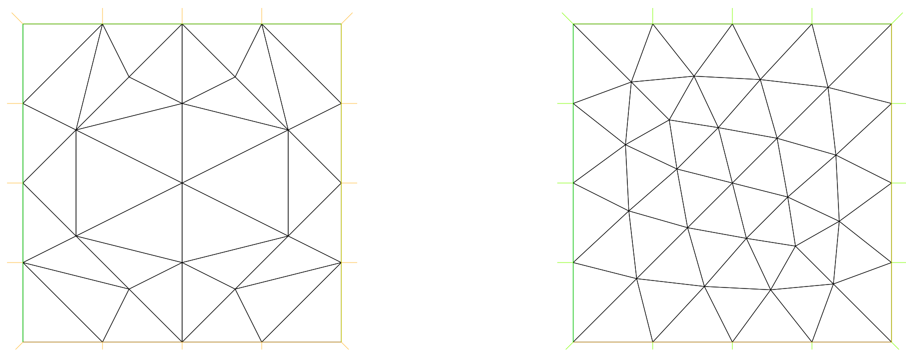
For $d = 3$ on tetrahedral meshes, a relation between element characteristics, time-step and material parameters can also be explicitly derived with a similar approach. However, uniform refinements cannot be applied, since, as opposed to the two-dimensional case, a tetrahedron cannot be divided into regular and similar tetrahedra, hence requiring a re-meshing.

NUMERICAL EXPERIMENTS

We consider two numerical experiments, which show that, for triangulations with right- and obtused-angled triangles, the non-satisfaction of the DMP may lead to oscillatory solutions. In both experiments, the computational domain is $\Omega = (0,1)^2$ and its edges are denoted by Γ_{bottom}, Γ_{top}, Γ_{left}, and Γ_{right}. On Ω, we consider two different families of triangulations (Fig. 1). The first one (family A) is obtained from uniform refinements of the grid illustrated in Fig. 1a. This mesh has been automatically generated with the FE software FreeFem++. It is a Delunay triangulation presenting four right-angled triangles close to the vertices and obtuse triangles in the interior of the domain. The second family (B) has been obtained from uniform refinements and Laplacian smoothing of the mesh shown in Fig. 1b. The corners of Ω are connected to two elements. Each internal vertex is connected to six elements except two vertices which are connected to five elements. We consider a uniform biphasic material composed of incompressible solid and fluid phases, with $\mu = 1$, $B = 2$, $\kappa = 3$, $\alpha = 1$, and $1/M = 0$.

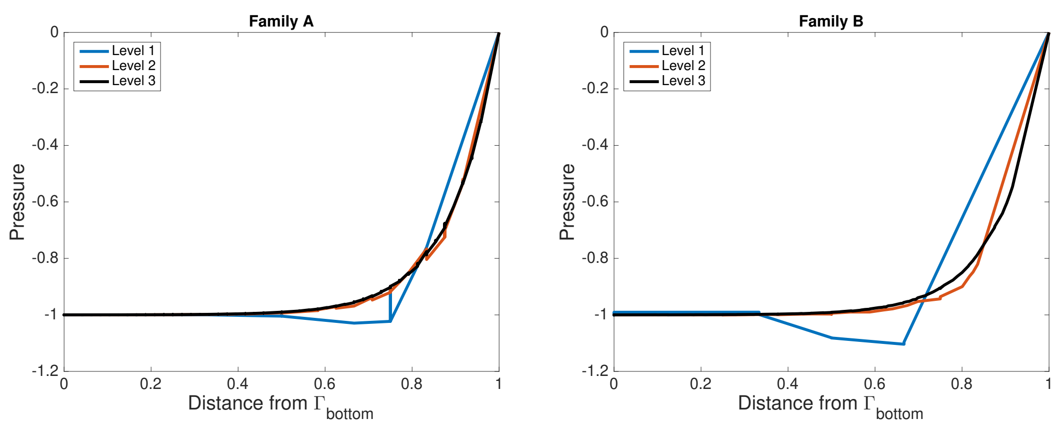
Confined compression

The first numerical experiment is a confined compression test. Displacement is zero on Γ_{bottom}, no horizontal displacement is allowed on Γ_{left} and Γ_{right}, and on the drained part Γ_{top} a uniform force is applied. Given these boundary conditions, this



(a) Family A (b) Family B

Figure 1: Coarsest level of the two families of meshes employed in the numerical experiments. (a) Delaunay mesh with right- and obtuse-angled elements; (b) Delunay mesh with acute-angled elements.



(a) (b)

Figure 2: Vertical pressure distribution after the first time-step in (a) family A and (b) family B for a confined compression test. Levels 1,2, and 3 refer to the degree of mesh refinement.

problem is one-dimensional and the analytical solution is reported in Biot (1941). The vertical pressure distribution is characterized by the presence of a sharp exponential boundary layer close to the drained boundary.

In Fig. 2, we show the pressure after one time-step with $\tau = 0.001$. The results illustrate that, for coarse meshes (level 1), the solutions contain instabilities. After one refinement (level 2), the solution obtained employing a mesh with acute-angled triangles exhibits a monotonic and stable behavior (Fig. 2b), which is not the case for the solution obtained with a mesh with right- and obtuse-angled triangles (Fig. 2a). Applying an additional mesh refinement (level 3) to this latter grid, numerical wiggles in the solution are still present, albeit with a reduced magnitude. This illustrates that the FE

	Family A	Family B
Level 1	-2.89388	-2.63516
Level 2	-8.0846e-05	0
Level 3	0	0

Table 1: Effect of mesh refinement on the minimum of the pressure for the Barry and Mercer test for the two families of triangulations shown in Fig. 1.

discretization is actually convergent.

Barry and Mercer’s source problem

Barry and Mercer’s problem is a two-dimensional problem describing a square porous medium with an oscillating pressure point source (Mercer and Barry, 1999). Tangential displacement is null on all edges of the boundary of $\partial\Omega$ and a draining boundary condition is applied on $\partial\Omega$. In the considered example, the source is a Dirac delta located at $(0.25, 0.25)$. The pressure distribution is monotonic and has the same sign of the source (Rodrigo et al., 2016).

To illustrate the importance of element aspect ratios for poroelastic problems, we compute the minimum of the pressure obtained with the two families of meshes discussed above after the first-time step with $\tau = 0.001$. The obtained values are reported in Table 1. At the coarsest discretizations (level 1), the DMP does not hold for either of the two meshes providing negative values for the minimum of the pressure since condition (10) is violated. For the meshes obtained after the first refinement (level 2), the DMP is respected only for family B. As in the previous case, an additional mesh refinement (level 3) provides solutions with positive pressure distributions for both families showing that the FE solution is convergent and that the condition (10) is actually only sufficient.

CONCLUSIONS

The pressure Schur complement of Biot’s equations can be interpreted as the discretization of a reaction-diffusion problem with a diffusive term proportional to the time-step and the permeability. Numerical pressure and displacement wiggles can be explained as violations of the DMP for this equivalent problem. In order to ensure a positivity condition on the Schur complement, it can be verified that the local stiffness matrix is an M-matrix. In the two-dimensional case, this condition translates into a bound on the length of the time-step and on the element area, provided that the mesh elements are acute triangles. Our numerical results show that this condition is sufficient and that adequate mesh refinements provide oscillation-free solutions.

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