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Local versus nonlocal barycentric interactions in 1D agent dynamics

Max-Olivier Hongler

Ecole Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland

Roger Filliger

Bern University of Applied Sciences, CH-2501 Biel, Switzerland

Olivier Gallay

IBM Zurich Research Laboratory, CH-8803 Rueschlikon, Switzerland

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Abstract

The mean-field dynamics of a collection of stochastic agents evolving under local and nonlocal interactions in one dimension is studied via analytically soluble models. The nonlocal interactions result from a barycentric modulation of the observation range of the agents. Our modeling framework is based on a discrete two-velocity Boltzmann dynamics which can be analytically discussed. Depending on the span and the modulation of the interaction range, we analytically observe a transition from a purely diffusive regime without definite pattern to a flocking evolution represented by a solitary wave traveling with constant velocity.

Keywords: Self-organized systems, Transport processes, Kinetic theory.

1 Introduction

Since several decades, the fascination for flocking dynamics and specifically, the detailed mechanisms which induce a collection of interacting stochastic agents to exhibit an emergent collective behavior, stimulates a fruitful activity on both experimental [5, 34] and modeling sides [40, 43]. The recent pioneering and truly engrossing contributions from T. Vicsek et al. [43] and F. Cucker and S. Smale [17] unveil, from a simple and synthetic modeling point of view, some of the basic features underlying the formation of robust collective motions. Recently, a wealth of numerical and analytical explorations based on the Vicsek's and related models, describing self-propelled particles with collective motions, arose in the literature. In particular, in the framework of agent-based modeling, cohesive motions have been studied via a statistical mechanics point of view in [7, 8, 13, 24, 25, 33]. As intuitively expected, a central role is played by the interactions connecting the agents. Both the interaction range and the interaction strength are control parameters which, for critical values, may trigger a flocking (phase) transition. Heuristically, with strong enough interactions, the noise-induced detuning tendency, can be overcome and ultimately produce an emergent synchronized evolution. In the models introduced by Vicsek et al. and by Cucker and Smale, it is indeed the interaction range that directly tunes the capability of agents to self-organize. Alternatively, recent observations for birds demonstrate that a fixed critical number of neighbors rather than a pure metric spatial range is used by birds in order to flock together to platoons [9, 12]. This shows that the interaction range itself may depend on the instantaneous distribution of agents. Such non-metric interactions have been recently taken explicitly into account in [15, 39]. Here, we contribute to this widely open topic by proposing a very simple class of models, where the interaction range depends in an effective way on the agent distribution.

Societies of agents can be composed either of dynamically homogeneous or heterogeneous individuals and each case requires a drastically different modeling approach. Recent analytical results have been derived for heterogeneous agents like ranked-based interacting Brownian motions [3, 14, 38] or bucket brigades dynamics [1, 4]. We here focus on homogeneous populations of agents which offer a wide potential for *analytical* approaches. In an homogeneous population, any randomly selected individual is likely to be a dynamical representative of any other fellow of the society. This very basic feature together with the fact that for large populations, the relative importance of fluctuations diminish, enable, in the thermodynamic limit $N \to \infty$, to adopt a purely macroscopic description. This assumes that the normalized distribution of agents can be "hydrodynamically" described by a probability density which solves a deterministic, yet nonlinear, evolution equation. The resulting equation is usually called the *mean-*

field (MF) *equation*.

The class of MF models introduced in this contribution shows flocking transitions for large enough interaction ranges and belongs to the rare instances of exactly solvable interacting agent dynamics. By a flocking pattern, we understand the self-organized capability of agents to create, via their mutual interactions, a persistent probability density taking the form of a traveling wave. The price to be paid for such an analytical objective is to conveniently reduce the individual agents' *state space* and *decision space*. As a minimal model, we shall consider *random two velocity models* involving agents traveling on the real line with two discrete velocities, say $\{v_\pm\}$ with $v_- < v_+$. The usefulness of one dimensional models has been proven in recent experimental and theoretical contributions, including [11, 18]. Here, the effective agent autonomous decisions will consist in selecting, at non-homogeneous Poisson random times, one out of the two possible velocities. The velocity updating process will, via density-dependent parameters of the underlying Poisson switching times, depend on the observation range. This naturally leads to nonlinear Boltzmann-like dynamics, which is commonly adopted for traffic flows modeling [6, 28, 30], and which recently also appeared in models more closely related to our present contribution, [7, 8, 13, 33]. The observation range, which will be the key control parameter for our discussion, may depend on the distribution of the whole agent society via a *barycentric* modulation. Such a barycentric modulation offers, in an effective manner, the possibility to model agent dynamics with configuration-dependent interaction ranges. This approach enables us to construct several solvable nonlinear models for which critical interaction modulations – required for flocking – can be obtained explicitly.

Although different from our present contribution, related MF models exist in the literature. Exactly solvable MF models have been proposed in different contexts including portfolio theory [21], coupled phase oscillators [10], traffic dynamics [6, 28], self-propelled organisms [7, 8, 24, 42] and [19], this last contribution being particularly relevant for our present approach. Related MF models with barycentric self-interactions are studied [2] and [16]. Our present study provides an explicit and exactly solvable class of models which is closely related to the class of dynamics discussed in [20, 35] (*n.b.* see particularly eqs.(2, 19) in [35] and eq.(1) in [20]). In the present case, the solvability arises since our models are amenable to a discrete velocity Boltzmann equation of the Ruijgrok-Wu type, [41]. Citing the recent detailed review by R. Eftimie [19],

"Because of their complexity, these nonlocal hyperbolic models have not been the subject of very thorough analytical investigations. For this reason it is quite difficult to identify the exact mathematical mechanisms that can explain the formation of these patterns in nonlocal models [...]",

one realizes that exact solutions are presently particularly welcome. As originally discussed in [26, 36], for large populations of agents, our discrete velocities Boltzmann dynamics can be made to converge towards the generalized Burgers equation for which exact results *and* stability issues for traveling waves can be discussed explicitly, [37].

2 Two velocity agent model

Consider a collection $\mathcal{A}$ composed of N autonomous agents which are in a migration process on the real line $\mathbb{R}$. At any time $t \in \mathbb{R}^+$, we assume that the complete population is composed of two types of agents $\mathcal{A}_+$ and $\mathcal{A}_-$, ($\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_-$), characterized by two associated migration velocities v_+ and v_- on $\mathbb{R}$. Agents traveling with velocity v_+ (resp. v_-) belong to $\mathcal{A}_+$ (resp. $\mathcal{A}_-$). The time-dependent positions of the agents $X_k(t)$, $k = 1, 2, ..., N$, can be written as a set of coupled stochastic differential equations (SDEs):

$$\dot{X}_k(t) = I_k(\mathbf{X}(t)), \qquad k = 1, 2, ..., N, \tag{1}$$

with $\mathbf{X}(t) = (X_1(t), ..., X_N(t))$ and where $I_k(\mathbf{X}(t))$ stands for a two-states Markov process with state space $\Omega := \{v_+, v_-\}$. The associated transition rates of agent k, $\alpha = \alpha\big(X_k(t), \mathbf{X}(t)\big) > 0$ from v_+ to v_- and $\beta = \beta\big(X_k(t), \mathbf{X}(t)\big) > 0$ from v_- to v_+ are here state-dependent. For a given configuration $\mathbf{X}(t)$ at time t, they represent the inverse of the average sojourn times of agent k in the velocity states. This dependency of the switching rates on the agent population effectively allows agent k to observe his/her environment and to react accordingly. In the sequel, we consider large populations of homogeneous agents, *i.e.* $N \to \infty$, so that a MF description of the dynamics holds. Using a phenomenological MF representation, we shall write $P_+(x, t \mid x_0)$ (resp. $P_-(x, t \mid x_0)$) for the conditional probability density to find agents at position x at time t with velocity v_+ (resp. v_-), knowing that at time $t = 0$ the density was given by the initial distribution $p_+(x)$ (resp. $p_-(x)$). Due to the assumed Markov character of the transitions, the nonlinear evolution for $P_+(x, t \mid x_0)$ and $P_-(x, t \mid x_0)$ can be written as a discrete Bolzmann-type equation (*n.b.* we drop the x_0-dependence for typographic ease):

$$\dot{P}_\pm(x,t) + v_\pm \partial_x P_\pm(x,t) = \mp\alpha(x,t)P_+(x,t) \pm \beta(x,t)P_-(x,t), \tag{2}$$

with interaction kernels:

$$\alpha(x,t) = \alpha - \int_{x-U}^{x+V} g\left[z - \langle X(t)\rangle\right] P_-(z,t)dz, \tag{3}$$

$$\beta(x,t) = \beta + \int_{x-U}^{x+V} g\left[z - \langle X(t)\rangle\right] P_+(z,t)dz, \tag{4}$$

and where now α and β are two positive constants, $g(x) \geq 0$ is a smooth function and integrable with respect to the P_+ and P_- probability densities and where

$$\langle X(t)\rangle = \int_{\mathbb{R}} x\left[P_+(x,t) + P_-(x,t)\right]dx \tag{5}$$

is the *barycenter*. According to eqs. (2-5), agents modify their velocity by the following dynamic rules:

(i) Agents with velocity v_+. An agent at position x and with velocity v_+ changes spontaneously to v_- with constant rate α. This agent is allowed to observe the environment in the observation interval $\mathcal{O} = [x-U, x+V]$. The α-rate is reduced in a weighted proportion to the number of v_- agents present in $\mathcal{O}$. When $g \neq$ constant, the weight is modulated by the barycentric position $\langle X(t)\rangle$ of the population via the g-dependence.

(ii) Agents with velocity v_-. An agent at position x and with velocity v_- changes spontaneously to v_+ with constant rate β. This agent is allowed to observe the environment in the observation interval $\mathcal{O} = [x-U, x+V]$. The β-rate is enlarged in a weighted proportion to the number of v_+ agents present in $\mathcal{O}$. When $g \neq$ constant, the weight is modulated by the barycentric position $\langle X(t)\rangle$ of the population via the g-dependence.

The following remarks can be drawn:

(1) Agents interact non-locally with their neighbors in two ways: *(a)* via the finite extension of the $\mathcal{O}$ interval and *(b)* via the g-modulation which depends on the barycenter $\langle X(t)\rangle$. Hence, for $\mathcal{O} \neq \mathbb{R}$, one could find an inconstancy between the two mechanisms. This however is not so as the agents may simultaneously base their decisions on real-time information delivered by physically different sensors like *vision, sound, olfaction*, etc.

(2) Observe that the sum of the right-hand-sides of the two equations given in eqs. (2) is equal to zero. This is an expression of the continuity equation which guarantees the conservation of the number of agents.

(3) Using the rescaling $v_\pm \mapsto v_\pm/\epsilon$, $\alpha \mapsto \alpha/\epsilon^2$ and $\beta \mapsto \beta/\epsilon^2$ in eqs. (2), our two velocity model converges for $\epsilon \to 0$ to a Burgers' type dynamics of the form:

$$\partial_t \Psi(x,t) = -\partial_x \left\{ \left(f(x,t) + \gamma \left[\int_{x-V}^{x+U} \Psi(z,t)\,dz \right] \right) \Psi(x,t) \right\} + \frac{D}{2}\partial^2_{xx}\Psi(x,t), \tag{6}$$

which is a MF nonlinear Fokker-Planck equation for interacting diffusion processes driven by White Gaussian Noise, [23, 31]. We exemplify in appendix A how to transfer our findings valid for the Boltzmann-type dynamics to this Burgers' type limiting model.

We now explicitly give the solutions to the set of two nonlinearly coupled equations given in eqs. (2) for the following regimes:

(A) Follow the immediate leaders : $U = 0$, $V = \varepsilon$, with ε infinitesimal small, and $g \equiv 1$,

(B) Follow all the leaders: $U = 0$, $V = \infty$ and $g \equiv 1$,

(C) Follow the leaders with barycentric modulation: $U = 0$, $V = \infty$ and $g = g(x - \langle X(t)\rangle)$.

Studying the regimes (A) and (B), we quantify the importance played by the size of the observation interval $\mathcal{O}$. Comparing the regimes (B) and (C) we may appreciate how the influence of the g-barycentric modulation will give rise to flocking phase transitions.

3 Discrete velocity dynamics

Using a simple rescaling of the state and time variables (*e.g.* see [30] for details), we can without loss of generality set $v_+ = -v_- = 1$ (and similarly, $\gamma = 1$ in eq. (6)) and rewrite eqs. (2) in canonical form:

$$\dot{P}_\pm(x,t) \pm \partial_x P_\pm(x,t) = \mp\alpha(x,t)P_+(x,t) \pm \beta(x,t)P_-(x,t). \tag{7}$$

Based on eq. (7), we now explore the above mentioned regimes.

(A) Follow the immediate leaders

In the myopic case where $V = \varepsilon$ is very small and $U = 0$, we may Taylor expand up to first order the quadrature in eqs. (3) and (4). The resulting dynamics from eqs. (7) reads as:

$$\dot{P}_\pm(x,t) \pm \partial_x P_\pm(x,t) = \pm 2\varepsilon P_+(x,t)P_-(x,t) \mp \alpha P_+(x,t) \pm \beta P_-(x,t). \tag{8}$$

This is an exactly solvable, discrete Boltzmann-type equation discovered by Th. Ruijgrok and T. T. Wu, [41]. Potential applications for this type of dynamics have been recently considered in [27, 32]. Using the boundary conditions $\lim_{|x|\to\infty} P_\pm(x,t) = 0$, the solution reads as:

$$P_+(x,t) = -\beta + \partial_t \log H(x,t) - \partial_x \log H((x,t), \tag{9}$$

$$P_-(x,t) = \alpha - \partial_t \log H(x,t) - \partial_x \log H(x,t), \tag{10}$$

where $H(x,t)$ solves the linear *Telegraphist equation*:

$$\partial_{tt}H(x,t) - \partial_{xx}H(x,t) - \alpha\beta H(x,t) = 0 \tag{11}$$

and whose explicit solution is recalled in the appendix B. The diffusive solution does not converge towards a finite stationary density, implying that, for any initial condition, the agents will ultimately be spread over the whole line $\mathbb{R}$. This explicitly indicates that local interactions are *not* strong enough to create a cooperative motion (*i.e.* no flock is formed).

(B) Follow all the leaders

Instead of the myopic regime (A) given in eqs. (8), let us now consider case (B), namely:

$$\partial_t P_{\pm}(x,t) \pm \partial_x P_{\pm}(x,t) = \mp \left[\alpha - \int_x^{\infty} P_- dz\right] P_+(x,t) \pm \left[\beta + \int_x^{\infty} P_+ dz\right] P_-(x,t). \tag{12}$$

Introducing the notations

$$F_{\pm}(x,t) = \int_x^{\infty} P_{\pm}(z,t)\, dz, \tag{13}$$

we can rewrite eqs. (12) as:

$$\partial_{xt} F_{\pm}(x,t) \pm \partial_{xx} F_{\pm}(x,t) = \pm \partial_x \left[F_-(x,t) F_+(x,t)\right] \mp \alpha \partial_x F_+(x,t) \pm \beta \partial_x F_-(x,t). \tag{14}$$

After integration with respect to x, eqs. (14) exhibit the same structure than eqs. (8) with respect to the fields $F_+(x,t)$ and $F_-(x,t)$. Here however, the boundary conditions are:

$$\lim_{x\to\infty} F_{\pm}(x,t) = 0, \quad \text{and} \quad \lim_{x\to-\infty} F_{\pm}(x,t) = \rho_{\pm} \tag{15}$$

and the normalization imposes that $(\rho_+ + \rho_-) = 1$. With these boundary conditions, the solution reads as, [41]:

$$F_{\pm}(x,t) = \frac{\rho_{\pm}}{2}\left\{1 - \tanh\left[\frac{x - wt}{4}\right]\right\}, \tag{16}$$

with

$$2\rho_{\pm} = 1 \pm w > 0, \tag{17}$$

and where w is the velocity defined by :

$$w = \sqrt{[\alpha + \beta - 1]^2 + 4\beta} - (\alpha + \beta). \tag{18}$$

From eq. (16), the probability densities in eqs. (12) exhibit the form of solitary waves:

$$P_{\pm}(x,t) = -\partial_x F_{\pm}(x,t) = \frac{\rho_{\pm}}{8\cosh^2\left(\frac{x-wt}{4}\right)}. \tag{19}$$

The solutions in eqs. (19) explicitly show that in case (B), the long-range imitation process generates stationary, finite probability densities. Hence contrary to what happens in the myopic regime (A), here *flocking* results from the long-range of the agents' mutual interactions. An illustration of the emerging stationary flocking behavior is given in Fig. 1.

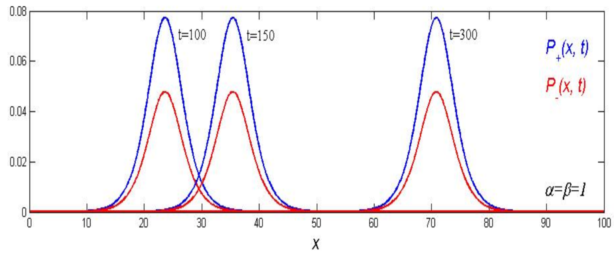

Figure 1: Illustration of the stationary probability densities $P_-(x,t)$ and $P_+(x,t)$ of the agent populations $\mathcal{A}_-$ and $\mathcal{A}_+$ respectively, when $\alpha = \beta = 1$ (*i.e.* the constant transition rates are equal).

(C) Follow the leaders with barycentric modulation

The above analysis of the regimes (A) and (B) suggest that for a certain critical interaction range, one should be able to observe a phase transition from a purely diffusive regime with no stationary patterns to a stationary dispersive flocking regime. A very simple, yet exact solution to this problem is shown here, where we now introduce an explicit barycentric modulation function. Recent contributions using barycentric interactions are discussed in [2, 16, 22, 38]. The class of models we consider here are of the following form:

$$\dot{P}_{\pm}(x,t) \pm \partial_x P_{\pm}(x,t) = \mp P_+(x,t)\left[\alpha - \mathbb{F}_-(x,t)\right] \pm P_-(x,t)\left[\beta + \mathbb{F}_+(x,t)\right], \tag{20}$$

with

$$\mathbb{F}_{\pm}(x,t) = \int_x^{\infty} g\left[z - \langle X(t)\rangle\right] P_{\pm}(z,t) dz, \tag{21}$$

and where $\langle X(t)\rangle$, defined in eq. (5), is the barycenter position at time t.

Let us from now on consider traveling wave solutions to eqs. (20) of the form $f(x - wt)$ for some velocity w. In such a traveling wave regime and for large t, we have $\langle X(t)\rangle = w\,t$. Introducing the variable $\xi = (x - w\,t)$ we may, by a slight abuse of notations, rewrite $P_{\pm}$ in this regime as $P_{\pm}(x,t) = P_{\pm}(\xi)$ and eq. (5) now reads as:

$$0 = \int_{\mathbb{R}} \xi \left[P_+(\xi) + P_-(\xi)\right] d\xi, \tag{22}$$

meaning that in the traveling wave case, the stationary distribution, as seen from the center of mass, has zero mean. As the right-hand-sides in eqs. (20) sum to zero, we immediately find, by simple integration with respect to ξ, the following relation:

$$(1 - w) P_+(\xi) = (1 + w) P_-(\xi) + \kappa, \tag{23}$$

where the velocity of the center of mass w will be chosen below, so as to be consistent with the agent distribution. The integration constant κ will be set to zero in order to match natural (*i.e.* vanishing) boundary conditions at infinity. Therefore, we shall have:

$$P_-(\xi) = \frac{1 - w}{1 + w} P_+(\xi). \tag{24}$$

Using eq.(24) and writing eqs. (20) with the variable ξ, we obtain the MF evolution equation:

$$\partial_\xi P_+(\xi) =$$

$$P_+(\xi)\Big(\frac{2}{1+w}\int_\xi^\infty g(\zeta)P_+(\zeta)d\zeta + \frac{\beta}{1+w} - \frac{\alpha}{1-w}\Big). \quad (25)$$

In view of our analytical objective, we now introduce the class of symmetric interaction modulations (see Fig. 2):

$$g(x) = \Delta \cosh^\eta(x), \quad (26)$$

with $\Delta > 0$ and $\eta \in \mathbb{R}$, together with the *Ansatz*:

$$P_+(\xi) = \mathcal{N}(m)\cosh^m(\xi), \quad (27)$$

where $m < 0$ is some negative constant and where $\mathcal{N}(m)$ stands for the probability normalization, explicitly :

$$\mathcal{N}(m) = \frac{\sqrt{\pi}\,\Gamma(|m|/2)}{2^{|m|}\Gamma\left[(|m|+1)/2\right]}. \quad (28)$$

Introducing eqs. (26) and (27) into the integral equation (25), an elementary calculation shows that eq. (27) is actually a solution provided we impose:

$$m + \eta = -2, \quad (29)$$

in which case we end up with the relation:

$$\left[m + \frac{2\Delta\mathcal{N}(m)}{1+w}\right]\tanh(\xi) = \frac{2\Delta\mathcal{N}(m)+\beta}{1+w} - \frac{\alpha}{1-w}. \quad (30)$$

Clearly, eq. (30) is realized only if the following equalities hold

$$\frac{m}{\mathcal{N}(m)} = -\frac{2\Delta}{1+w} \quad \text{and} \quad \frac{1-w}{1+w} = \frac{\alpha}{\beta + 2\Delta\mathcal{N}(m)}, \quad (31)$$

implying the self-consistency equation for the traveling speed:

$$w = 1 + \frac{2\alpha\Delta\mathcal{N}(m)}{m\left[\beta + 2\Delta\mathcal{N}(m)\right]}. \quad (32)$$

Hence, given the positive input modeling data α, β and Δ, together with the modulation η of the barycentric interaction g, eq. (29) determines the probability density decay m and eq. (32) determines the resulting solitary wave velocity w. Note that the normalization factor in eqs. (27, 28) exists only for $m < 0$, or equivalently for $\eta > -2$, which yields the critical power η_c for the range of the modulation decay:

$$\eta > \eta_c = -2. \quad (33)$$

The condition on the modulation range power η in eq. (33) leads to conclude that for strongly localized interactions, $\eta < \eta_c$ (see Fig. 2 for an illustration), no stable finite stationary probability density (*i.e.* agents spread to ultimately be dispersed over the entire space). In this regime, no stable flocking emerges from the agent interactions. Conversely, when $\eta > \eta_c$, a finite stable stationary probability density exists, thus showing that comparatively long-range interactions ultimately drive the population to a persistent flocking behavior characterized here by a traveling wave.

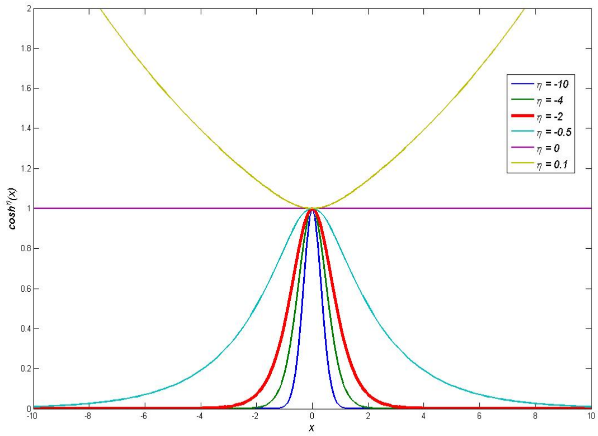

Figure 2: Considered class of barycentric modulations $\cosh^\eta(x)$ (here $\Delta = 1$). The flocking transition emerges at $\eta_c = -2$.

4 Conclusion and perspectives

The models we have exposed in this contribution simultaneously involve two types of nonlinear sources: first an imitation interaction mechanism of the quadratic type which reflects the observation of the state of the agents by their fellows and secondly a barycentric modulation for the strength of the interactions. The possibility to derive an exactly solvable model for the flocking transition dynamics relies partially in the fact that one nonlinearity can be removed. Indeed, the quadratic nonlinearity can be "removed" by using a logarithmic transformation, but the second type of nonlinearity remains as it is truly intrinsic. In actual agent models like birds or fishes, interactions are characterized by both of these nonlinearities: by imitation (stylized here by the quadratic nonlinearity) and by tuning the observation range according to the agents' local density (effectively stylized here by a barycentric modulation). Another important simplification allowing the derivation of exactly solvable models is the mean-field limit adopted here. In this limit, the law of large numbers reduces the influence of the fluctuations to insignificancy and this leads to deterministic evolution equations for the agent distribution. For many applications in perspective, finite-size population effects will manifest via the presence of fluctuations (*i.e.* a *mesoscopic description*). Noise will definitely affect the dynamics and could potentially destroy the flocking capability of the agents. We hope that the basic models discussed in this paper could provide some analytical clues for this truly challenging issue.

Appendix A

The complete program performed in this paper can be repeated for the Burgers' dynamics given in eq.(6). Let us only focus here on the interactions given in case (C) for which the dynamics reads as:

$$\partial_t \Psi(x,t) = -\partial_x \left[-\Psi(x,t) \int_x^\infty g\left(z - \langle X(t)\rangle\right) \Psi(z,t) dz \right.$$
$$\left. - \frac{D}{2} \partial_x \Psi(x,t) \right], \tag{34}$$

where:

$$\langle X(t) \rangle = \int_{\mathbb{R}} x \Psi(x,t) dx. \tag{35}$$

Concerned with traveling wave solutions with constant velocity w, we introduce the notation $\xi = (x - wt)$. Accordingly, eq. (34) takes the form:

$$0 = \partial_\xi \left[\Psi(\xi) \left\{ w - \int_\xi^\infty g\left(z\right) \Psi(z) dz \right\} - \frac{D}{2} \partial_\xi \Psi(\xi) \right] \tag{36}$$

and w is implicitly determined by the equation:

$$\int_{\mathbb{R}} \xi \Psi(\xi) d\xi = 0. \tag{37}$$

Eq.(36) can be reduced to the form:

$$\frac{D}{2} \partial_\xi \log\left[\Psi(\xi)\right] = \left\{ w - \int_\xi^\infty g\left(z\right) \Psi(z) dz \right\} + \kappa, \tag{38}$$

where κ is a constant which will be taken to be zero as no steady probability flow exists in the stationary regime. Assume now a symmetric barycentric modulation of the form:

$$g(x) = \Delta \cosh^\eta(x), \tag{39}$$

with $\Delta > 0$ a constant and $\eta \in \mathbb{R}$. In view of eq.(39), we introduce the *Ansatz*:

$$\Psi(\xi) = \mathcal{N}(m) \cosh^m(\xi), \tag{40}$$

where again, for $m < 0$, the normalization factor $\mathcal{N}(m)$ is given by eq. (28). Note from eq. (40) that $\Psi(\xi) = \Psi(-\xi)$ and therefore eq. (37) is automatically satisfied. Using eqs. (40) and (28) into eq. (38), we end up with:

$$\frac{D}{2}\, m \tanh(\xi) = \left[w - \Delta \mathcal{N}(m) \int_\xi^\infty \cosh^{\eta+m}(\xi) d\xi \right]. \tag{41}$$

Using the identity $\int \cosh(x)^{-2} dx = 1 - \tanh(x)$, we see that eq.(41) is solved provided we have:

$$\frac{m}{\mathcal{N}(m)} = \frac{2\Delta}{D} \quad \text{and} \quad w = \Delta \mathcal{N}(m). \tag{42}$$

From eqs.(41) and (42) , one concludes that for $\eta > -2$, a traveling solitary wave with velocity w is created via the agents' interactions. Hence for $\eta > -2$ the flocking mechanism is triggered. Conversely, for $\eta < -2$, normalization cannot be achieved and this shows that too weak interactions at long-range preclude the formation of a flock.

Appendix B

The solution $H(x,t)$ to the linear *Telegraphist equation*,

$$\partial_{tt} H(x,t) - \partial_{xx} H(x,t) - \alpha\beta H(x,t) = 0 \tag{43}$$

is of the form, [29, 41]:

$$H(x,t) = \frac{1}{2}\left[A(x+t) + A(x-t)\right]$$
$$+ \frac{1}{2}\mathcal{B}_1(x,t) + \frac{\nu t}{2}\mathcal{B}_2(x,t), \tag{44}$$

where $\nu = \frac{\sqrt{\alpha\beta}}{2}$ and where we have the following definitions:

$$\mathcal{B}_1(x,t) = \int_{x-t}^{x+t} \mathbb{I}_0\left(\nu\sqrt{t^2 - (x-z)^2}\right) B(z) dz, \tag{45}$$

and

$$\mathcal{B}_2(x,t) = \int_{x-t}^{x+t} \left(\frac{1}{\sqrt{t^2 - (x-z)^2}} \right)$$
$$\times\, \mathbb{I}_1\left(\nu\sqrt{t^2 - (x-z)^2}\right) A(z) dz, \tag{46}$$

with $\mathbb{I}_n(\cdot)$ being integer-order modified Bessel functions of the first kind and where $B(\cdot)$ and $A(\cdot)$ are short for:

$$B(x) = \frac{1}{2}\left[P_0(x) - Q_0(x) + \alpha + \beta\right] A(x), \tag{47}$$

and

$$A(x) = \exp\left\{ -\frac{1}{2} \int_0^x \left[P_0(z) + Q_0(z) - \alpha + \beta\right] dz \right\}. \tag{48}$$

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References

[1] D. Armbruster, D. Gel and J. Murakami. Eur. J. of Oper. Res. **147**:123-159, 2009.

[2] M. Balázs, M. Rácz and B. Tóth. arXiv.org math arXiv:1107.3289v1, 2011.

[3] A. D. Banner, R. Fernholz and I. Karatzas. Ann. of App. Prob. **15**:2296-2330, 2005.

[4] J. J. Bartholdi III, D. D. Eisenstein and R. D. Foley. Oper. Res. **49**:710-719, 2001.

[5] S. Bazazi et al. Proc. of the Roy. Soc. of Lond. Ser. B. **278**:356-363, 2011.

[6] N. Bellomo and C. Dogbe. SIAM Rev. **53**:409-463, 2011.

[7] E. Bertin, M. Droz and G. Grégoire. Phys. Rev. E **74**, 022101, 2006.

[8] E. Bertin, M. Droz nad G. Grégoire. J. Phys. A: Math. Theor. **42**, 445001, 2009.

[9] W. Bialek, A. Cavagna, I. Giardina, T. Mora, E. Silvestri, M. Viale and A. M. Walczak. Proc. Nat. Amer. Soc. **March**:1118633109, 2012.

[10] L. L. Bonilla, C. J. Perez-Vicente, F. Ritort and J. Soler. Math. Mod. and Meth. in App. Sci. **16**:1919-1959, 2006.

[11] J. Buhl et al. Science. **312**:1402-6, 2006.

[12] A. Cavagna, A. Cimarelli, I. Giardina, G. Parisi, R. Santagati and F. Stefanini. Proc. Nat. Amer. Soc. **June**: 1005766107, 2010.

[13] H. Chaté, F. Ginelli, G. Grégoire, F. Raynault. Phys. Rev. E **77**, 046113, 2008.

[14] S. Chatterjee and S. Pal. Prob. Th. and Rel. Fiel. **147**:123-159, 2009.

[15] Y.-L. Chou, R. Wolfe and T. Ihle. arXiv.org math arXiv: 1205.o830v3, 2012.

[16] F. Comets, M. V. Menshikov, S. Volkov and R. Wade. J. Stat Phys. **143**:855-888, 2011.

[17] F. Cucker and S. Smale. IEEE Trans. on Autom. Control **52**:852-862, 2007.

[18] V. Dosetti. J. Phys. A: Math. Theor. **45**:035003, 2012.

[19] R. Eftimie. J. Math. Biol. **65**:35-75, 2012.

[20] R. Eftimie, G. de Vries, M. A. Lewis and F. Lutscher. Bull. Math. Biol. **69**:1537-1565, 2007.

[21] E. R. Fernholz and I. Karatzas. *Handbook of Numerical Analysis. Mathematical Modeling and Numerical Methods in Finance (ed. A. Bensoussan)*:89-168, Elsevier, Amsterdam, 2009.

[22] T. D. Frank. *Nonlinear Fokker Planck equations*, Springer, 2005.

[23] E. Gambetta and B. Perthame. Math. Mod. and Meth. in App. Sci. **24**:949-967, 2001.

[24] G. Grégoire, H. Chaté, Y. Tu. Physica D **181**:157-170, 2003.

[25] G. Grégoire and H. Chaté. Phys. Rev. Lett. **92**, 025702, 2004.

[26] E. Gutkin and M. Kac. SIAM J. Appl. Math. **43**:971-980, 1983.

[27] F. Hashemi, M.-O. Hongler and O. Gallay. Theor. Econ. Lett. **2**:1-9, 2010.

[28] D. Helbing. *Traffic flow: Encyclopedia of Nonlinear Science (ed. Alwyn Scott)*, Routledge, New York, 2005.

[29] P. C. Hemmer. Physica A **27**:79-82, 1961.

[30] M.-O. Hongler and R. Filliger. Phys. Lett. A **301**:408-412, 2002.

[31] M.-O. Hongler and L. Streit. Europhys. Lett. **12**193-198, 1990.

[32] M.-O. Hongler, O. Gallay, M. Hülsmann, P. Cordes and R. Colmorn. Phys. A **389**:4162-4171, 2010.

[33] T. Ihle. Phys. Rev E **83**, 030901, 2011.

[34] P. D. Lorch, G. A. Sword, D. T. Gwynne and G. A. Anderson. Ecol. Entom. **30**:548-555, 2005.

[35] F. Lutscher and A. Stevens. J. Nonlin. Sci. **12**:619-640, 2002.

[36] H. P. McKean. *Stochastic Differential Equations: Lecture Series in Differential Equations, Session 7, Catholic Univ* **MR 38**:41-57, Air Force Office Sci. Res., Arlington, 1967.

[37] K. Nishihara. Japan J. Appl. Math. **2**:27-35, 1985.

[38] S. Pal and J. Pitman. Ann. of App. Prob. **18**2179-2207, 2008.

[39] A. Peshkov, S. Ngo, E. Bertin, H. Chaé and F. Ginelli. arXiv.org cond-mat.soft arXiv:1203.6853v1, 2012.

[40] S. Ramaswamy. Ann. Rev. of Cond. Matt. Phys. **1**:323-345, 2010.

[41] T. W. Riukgrok and T. T. Wu. Phys. A **113**:401-416, 1982.

[42] J. Toner and Y. Tu. Phys. Rev. E **58**:4828-4858, 1998.

[43] T. Vicsek, A. Czirók, E. Ben-Jacob and O. Schochet. Phys. Rev. Lett. **75**:1226-1229, 1995.