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Article

Random Shifting and Scaling of Insurance Risks

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Abstract: Random shifting typically appears in credibility models whereas random scaling is often encountered in stochastic models for claim sizes reflecting the time-value property of money. In this article we discuss some aspects of random shifting and random scaling of insurance risks focusing in particular on credibility models, dependence structure of claim sizes in collective risk models, and extreme value models for the joint dependence of large losses. We show that specifying certain actuarial models using random shifting or scaling has some advantages for both theoretical treatments and practical applications.

Keywords: Random shifting and scaling; Credibility premium; Elliptically symmetric distribution; L_p Dirichlet distribution; Archimedean copula; Infinite dimensions; Joint tail dependence

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1. Introduction

Random shifting and random scaling in insurance applications are natural phenomena for latent unknown risk factors, time-value of money, or the need of allowing financial risks to be dependent. In this contribution, we are concerned with three principal stochastic models related to credibility theory, ruin theory, and extreme value modeling of large losses.

In credibility theory (e.g., [9]) often stochastic models are defined via a conditional argument. As an illustration, consider the classical Gaussian model assuming that the conditional random variable $X|\Theta = \theta$ has the normal distribution $\mathcal{N}(\theta, \sigma^2)$. If further the random variable Θ has the normal distribution

$\mathcal{N}(\mu, \tau^2)$, we obtain the credibility premium formula for the Bayesian premium (calculated under the L_2 loss function)

$$\boldsymbol{E}\{\Theta|X = x\} \quad = \quad x + \frac{\sigma^2}{\sigma^2 + \tau^2}(\mu - x) \tag{1}$$

for any $x, \mu \in \mathbb{R}$ and σ, τ positive. The relation explained by (1) can be directly derived by introducing a random shift. Indeed, let Y be an independent of Θ random variable with $\mathcal{N}(0, \sigma^2)$ distribution. We have the equality in distribution

$$(X, \Theta) \stackrel{d}{=} (\Theta + Y, \Theta). \tag{2}$$

Consequently, (1) follows immediately by the fact that the conditional random variable $\Theta|(\Theta + Y) = x$ is normally distributed for any $x \in \mathbb{R}$.

The random shifting in this approach is related to Θ which shifts Y. The random shift model given in (2) has natural extensions. For instance, Y can be a d-dimensional normally distributed random vector with Θ being some d-dimensional random vector; a more general case is recently discussed in [15]. Another extension is to consider Y having an elliptical distribution; see Section 4.

In ruin (or risk) theory, realistic stochastic models for claim sizes (or risks) $X_i, i \geq 1$ should allow for dependence among them. Furthermore, dependent claim sizes need to have a tractable and transparent dependence structure. In several contributions (see [9,16] and the references therein) dependent claim sizes (or risks) are introduced by resorting to a dependence structure implied by the Archimedean copula. Recall that an Archimedean copula in d-dimension (denoted by C_ψ) is defined by

$$C_\psi(u_1, \ldots, u_d) = \psi(\sum_{i=1}^{d} \psi^{-1}(u_i)), \quad u_1, \ldots, u_d \in [0, 1], \tag{3}$$

where ψ is called the generator of C_ψ required to be positive, strictly decreasing, and continuous with $\psi(0) = 1$ and $\lim_{s\to\infty} \psi(s) = 0$, and $\psi^{-1}(x) := \inf\{t : \psi(t) \leq x\}$; see e.g., [8] and the references therein.

A similar idea was used in the context of ruin theory in [1] where conditional on the positive random variable Θ

$$\boldsymbol{P}\{X_1 > x_1, \ldots, X_n > x_n|\Theta = \theta\} = \Big(\prod_{i=1}^{n} \exp(-x_i)\Big)^\theta \tag{4}$$

holds for any positive constants $\theta, x_1, \ldots, x_n$. Proposition 1 of the aforementioned paper shows the link of such dependence structure (determined by (4)) with the Archimedean copula. In fact, instead of dealing with the conditional random model defined in (4) we can consider the following equivalent random scale model

$$(X_1, \ldots, X_n) \stackrel{d}{=} (Y_1/\Theta, \ldots, Y_n/\Theta), \tag{5}$$

where $Y_i, i \geq 1$ are independent random variables with unit exponential distribution being further independent of the positive random variable Θ. Clearly, $(X_1, \ldots, X_n)|\Theta = \theta$ has joint survival function given by (4). The random scale model (5) is interesting since it leads to certain simplifications; see [8].

At this point, we emphasise one extreme important issue, which seems to have been very often overlooked in the literature. Claim sizes for an infinite sequence of random variables, therefore, specialising a particular finite dimensional distribution is not enough for completely defining the random sequence. Of course, if the claim sizes are independent, say $Y_i, i \geq 1$ no caution is needed for specialising the dependence for any n. However, if the claims sizes are assumed to be dependent, then particular dependence structures for finite n (like say the one in (5)) lead to randomly scaled independence structures, this is further illustrated below in our Theorem 1.

In view of the above discussions, some possible approaches for modeling dependent claim sizes (or risks) include:

- copula-based models (here one needs to be careful since dependence structures for infinite sequences are needed!);
- conditional dependence models;
- random scale models;
- transformation of simple independence models.

The last point above means that if $Y_i, i \geq 1$ are independent claim sizes, then $X_i = f(Y_{i_1}, \ldots, Y_{i_m})$ with f some given deterministic function and $i_1, \ldots, i_m$ indices form a dependent sequence of claim sizes. One important example in this direction is the multivariate Pareto distribution of the second kind dealt with in [2,3]. Many other dependence models, like m dependence or common shock models can be introduced by this simple transformation of independent risks.

Of course these are only a few possibilities which lead to tractable dependence structures with certain appeal to actuarial applications; see also [4,9,11–14,21,23,24] and the references therein.

Finally, we mention that there are several other aspects of actuarial models where random shifting and scaling are intrinsically present. For instance, in [23] a new interesting copula model was studied, which can be alternatively introduced by a random scale of independent risks; see discussions in Section 4.

The principal goal of this contribution is to discuss various aspects of random shift and random scale paradigms in actuarial models. Our analysis leads to new derivations and insights concerning the calculation of the Bayesian premium. Furthermore, we show that modeling claim sizes by a class of Dirichlet random sequences can be done in the framework of a tractable random scale model. Further, we point out that random scaling approach is of interest for modeling large losses as in the setup of [23]. As a by-product a new class of L_P Dirichlet random vectors is introduced.

Organisation of the paper: In Section 2 we consider the Bayesian premium through certain random shift model. Our main finding is presented in Section 3 which generalizes Theorem 1 in [8]. Section 4 is dedicated to discussions and extensions.

2. Credibility Premium in Random Shift Models

For a given d-dimensional distribution function F we define a shift family of distribution functions $F(\boldsymbol{x}; \boldsymbol{\theta}) = F(\boldsymbol{x} - \boldsymbol{\theta}), \boldsymbol{x}, \boldsymbol{\theta} \in \mathbb{R}^d$. Typically, the assumption on a loss random vector $\boldsymbol{X}$ is that $\boldsymbol{X} | \Theta = \boldsymbol{\theta}$

follows a distribution function parametrised by $\boldsymbol{\theta}$, say it follows $F(\boldsymbol{x};\boldsymbol{\theta})$. A direct way to formulate this model is via the random shift representation

$$(\boldsymbol{X},\boldsymbol{\Theta}) \stackrel{d}{=} (\boldsymbol{Y}+\boldsymbol{\Theta},\boldsymbol{\Theta}), \tag{6}$$

where $\boldsymbol{Y}$ has distribution function F and is independent of $\boldsymbol{\Theta}$. If $\boldsymbol{\Theta}$ possesses a probability density function (pdf) h, then clearly $\boldsymbol{X}$ also possesses a pdf given by $\boldsymbol{E}\{h(\boldsymbol{x}-\boldsymbol{Y})\}$. Consequently, the Bayesian premium (under a L_2 loss function) when it exists, is given by

$$\begin{aligned} \boldsymbol{E}\{\boldsymbol{\Theta}|\boldsymbol{X}=\boldsymbol{x}\} &= \boldsymbol{E}\{\boldsymbol{\Theta}|(\boldsymbol{\Theta}+\boldsymbol{Y})=\boldsymbol{x}\} \\ &= \boldsymbol{x}-\frac{\boldsymbol{E}\{\boldsymbol{Y}h(\boldsymbol{x}-\boldsymbol{Y})\}}{\boldsymbol{E}\{h(\boldsymbol{x}-\boldsymbol{Y})\}}, \end{aligned} \tag{7}$$

where for the derivation of the last equality (7) we assumed additionally that $\boldsymbol{Y}$ also possesses a pdf. Clearly, if $\boldsymbol{Y} \stackrel{d}{=} -\boldsymbol{Y}$ we have further

$$\boldsymbol{E}\{\boldsymbol{\Theta}|\boldsymbol{X}=\boldsymbol{x}\} = \boldsymbol{x}+\frac{\boldsymbol{E}\{\boldsymbol{Y}h(\boldsymbol{x}+\boldsymbol{Y})\}}{\boldsymbol{E}\{h(\boldsymbol{x}+\boldsymbol{Y})\}}. \tag{8}$$

The random shift model (6) is transparent and offers a clear advantage in comparison with the conditional model, if the joint distribution of $(\boldsymbol{\Theta}+\boldsymbol{Y},\boldsymbol{\Theta})$ (or $(\boldsymbol{\Theta}+\boldsymbol{Y},\boldsymbol{Y})$) can be easily found as illustrated below.

Example 1. Suppose that $\boldsymbol{X}|\boldsymbol{\Theta} \sim \mathcal{N}_d(\boldsymbol{\Theta},\Sigma)$ with $\boldsymbol{\Theta} \sim \mathcal{N}_d(\boldsymbol{\mu},\Sigma_0)$ (here $\mathcal{N}_d(\boldsymbol{\nu},A)$ stands for the d-dimensional normal distribution with mean $\boldsymbol{\nu}$ and covariance matrix A). Suppose further that $\Sigma+\Sigma_0$ is positive definite. It follows that $(\boldsymbol{X},\boldsymbol{\Theta}) \stackrel{d}{=} \boldsymbol{Z} = (\boldsymbol{\Theta}+\boldsymbol{Y},\boldsymbol{\Theta})$ with $\boldsymbol{Y} \sim \mathcal{N}_d(\boldsymbol{0},\Sigma)$ independent of $\boldsymbol{\Theta}$. Therefore, in the light of [9] the fact that $\boldsymbol{Z}$ is normally distributed in $\mathbb{R}^{2d}$ implies that $\boldsymbol{Y}|(\boldsymbol{\Theta}+\boldsymbol{Y})=\boldsymbol{x}$ is normally distributed with mean

$$\bar{\boldsymbol{\mu}} = \boldsymbol{E}\{\boldsymbol{Y}|(\boldsymbol{\Theta}+\boldsymbol{Y})=\boldsymbol{x}\} = (\boldsymbol{x}-\boldsymbol{\mu})(\Sigma+\Sigma_0)^{-1}\Sigma.$$

Consequently

$$\boldsymbol{E}\{\boldsymbol{\Theta}|\boldsymbol{X}=\boldsymbol{x}\} = \boldsymbol{x}-\boldsymbol{E}\{\boldsymbol{Y}|(\boldsymbol{\Theta}+\boldsymbol{Y})=\boldsymbol{x}\} = \boldsymbol{x}+(\boldsymbol{\mu}-\boldsymbol{x})(\Sigma+\Sigma_0)^{-1}\Sigma. \tag{9}$$

Particularly, if Σ is positive definite

$$\boldsymbol{E}\{\boldsymbol{\Theta}|\boldsymbol{X}=\boldsymbol{x}\} = \boldsymbol{x}+(\boldsymbol{\mu}-\boldsymbol{x})(\Sigma_0\Sigma^{-1}+I_d)^{-1}, \tag{10}$$

where I_d denotes the $d\times d$ identity matrix.

Clearly, (1) is immediately established by the above for the special case that $d=1$ and $\Sigma=\sigma^2, \Sigma_0 = \tau^2$.

It is worth pointing out that (10) was derived by [15] when Σ_0 is non-singular using an indirect (in that case complicated) approach; whereas Example 1 gives a short direct proof for the formula of the Bayesian premium in the random shift Gaussian model, where we can further allow Σ_0 to be singular.

3. Dirichlet Claim Sizes & Random Scaling

A fundamental question when constructing models for claim sizes $X_i, i \geq 1$ is how to introduce tractable dependence structures. As mentioned in the Introduction, one common approach in the actuarial literature is to assume that the survival copula of $X_i, i \leq n$ is a n-dimensional Archimedean copula; see e.g., [1,22] and the references therein. In view of the link between Archimedean copula and Dirichlet distribution explained in [17], we choose the direct approach for modeling claim sizes by a Dirichlet random sequence as in [8].

With motivation from the definition of L_1 Dirichlet random vectors, we introduce next d-dimensional L_p Dirichlet random vectors. Let $Gamma(a, \lambda)$ denote the Gamma distribution with positive parameters a, λ. It is known that the pdf of it is $\lambda^a x^{a-1} \exp(-\lambda x)/\Gamma(a)$, where $\Gamma(\cdot)$ stands for the Euler Gamma function. Fix some positive constants $\alpha_i, i \geq 1$, and p. In the rest of the paper, without special indication, let $Y_i, i \geq 1$ denote a sequence of positive independent random variables defined on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$ such that, for any $i \geq 1$, Y_i^p has $Gamma(\alpha_i, 1/p)$ distribution with parameters α_i and p. It follows easily that the pdf of Y_i is given by

$$f_i(x) = \frac{p^{1-\alpha_i}}{\Gamma(\alpha_i)} x^{p\alpha_i - 1} \exp\left(-\frac{x^p}{p}\right), \quad x > 0.$$

We say that $(X_1, \ldots, X_d)$ is a d-dimensional L_p Dirichlet random vector, if the stochastic representation

$$(X_1, \ldots, X_d) \stackrel{d}{=} \left(R\frac{Y_1}{(\sum_{1 \leq i \leq d} Y_i^p)^{1/p}}, \ldots, R\frac{Y_d}{(\sum_{1 \leq i \leq d} Y_i^p)^{1/p}} \right) =: R\boldsymbol{O} \tag{11}$$

holds with some positive random variable R defined on $(\Omega, \mathcal{A}, \mathbf{P})$ which is independent of the random vector $\boldsymbol{O}$. The reason for the name of L_p Dirichlet random vector (and distribution) is that the angular component $\boldsymbol{O}$ lives on the unit L_p-sphere of $\mathbb{R}^d$, i.e.,

$$\sum_{i=1}^{d} O_i^p = 1.$$

When $p = 1$, $\boldsymbol{O}$ has the Dirichlet distribution on the unit simplex; see [17].

The main result of this section displayed in the next theorem shows that the model with Dirichlet claim sizes can be explained by a random scale model.

Theorem 1. *Let $X_i, i \geq 1$ be positive random variables. If, for any $d \geq 2$, the random vector $(X_1, \ldots, X_d)$ has a d-dimensional L_p Dirichlet distribution with representation $(X_1, \ldots, X_d) \stackrel{d}{=} R_d \boldsymbol{O}_d$, then*

$$\{X_i, i \geq 1\} \stackrel{d}{=} \{SY_i, i \geq 1\}, \tag{12}$$

with S a non-negative random variable defined on $(\Omega, \mathcal{A}, \mathbf{P})$, independent of $Y_i, i \geq 1$.

Proof: By definition, it is sufficient to show that, for any $d \geq 1$

$$(X_1, \ldots, X_d) \stackrel{d}{=} S(Y_1, \ldots, Y_d) \tag{13}$$

for the non-negative random variable S required. Since for any $n \geq d$ the random vector $(X_1, \ldots, X_n)$ has a L_p Dirichlet distribution, then we have the stochastic representation

$$(X_1, \ldots, X_d) \stackrel{d}{=} \frac{R_n}{a_n} \frac{1}{(\sum_{i=1}^n Y_i^p)^{1/p}/a_n}(Y_1, \ldots, Y_d), \tag{14}$$

with $a_n = (\sum_{i=1}^n \alpha_i)^{1/p}$. Therefore, we have the convergence in distribution (denoted here as $\stackrel{d}{\to}$)

$$\frac{R_n}{a_n} \frac{1}{(\sum_{i=1}^n Y_i^p)^{1/p}/a_n}(Y_1, \ldots, Y_d) \stackrel{d}{\to} (X_1, \ldots, X_d)$$

as $n \to \infty$. Clearly, by the strong law of large numbers, as $n \to \infty$ we have the almost sure convergence $(\sum_{i=1}^n Y_i^p)^{1/p}/a_n \to 1$ which entails

$$\frac{R_n}{a_n}(Y_1, \ldots, Y_d) \stackrel{d}{\to} (X_1, \ldots, X_d)$$

as $n \to \infty$, meaning that

$$\ln\left(\frac{R_n}{a_n}\right) + (\ln(Y_1), \ldots, \ln(Y_d)) \stackrel{d}{\to} (\ln(X_1), \ldots, \ln(X_d)), \quad n \to \infty.$$

In the light of Theorem 3.9.4 in [10], by the independence of R_n and $(Y_1, \ldots, Y_d)$ we conclude that

$$\frac{R_n}{a_n} \stackrel{d}{\to} S, \quad n \to \infty,$$

with S some non-negative random variable defined on $(\Omega, \mathcal{A}, \mathbf{P})$ such that

$$(\ln(Y_1) + \ln(S), \ldots, \ln(Y_d) + \ln(S)) \stackrel{d}{=} (\ln(X_1), \ldots, \ln(X_d))$$

implying (13), and thus the claim follows. □

The following corollary is a generalization of Theorem 1 in [8].

Corollary 2. *If the claim sizes $X_i, i \geq 1$ are identically distributed, then under the assumptions and notation of Theorem 1 (12) holds with $Y_i, i \geq 1$ a sequence of independent random variables with common pdf $f(x) = p^{1-\alpha}/\Gamma(\alpha) x^{p\alpha-1}\exp(-x^p/p), x > 0$, for some $\alpha > 0$.*

In view of the well-known Beta-Gamma algebra (see e.g., [25]) if $\alpha \in (0,1)$, then Y_i in Corollary 2 can be re-written as

$$Y_i \stackrel{d}{=} (T_i E_i)^{1/p}, \quad i \geq 1,$$

with T_i a Beta distribution with parameters $\alpha, 1-\alpha$ and E_i being exponential distributed with mean p. Further, $T_i, E_i, i \geq 1$ are mutually independent. Consequently

$$(X_1, \ldots, X_n) \stackrel{d}{=} (S(T_1 E_1)^{1/p}, \ldots, S(T_n E_n)^{1/p}), \quad n \geq 1.$$

Note that $(SE_1, \ldots, SE_d), d > 1$ is a d-dimensional L_1 Dirichlet random vector.

4. Discussions & Extensions

The conditional credibility model considered in Section 2 is simple since we used a single distribution function F to define a shift family of distributions, i.e., $F(\boldsymbol{x},\boldsymbol{\theta}) = F(\boldsymbol{x}-\boldsymbol{\theta})$. Of course, we can consider a more general case that $F = F_{\boldsymbol{\theta}}$ is a family of d-dimensional distributions and assume that $\boldsymbol{X}|\boldsymbol{\Theta} = \boldsymbol{\theta}$ has distribution function $F_{\boldsymbol{\theta}}(\boldsymbol{x}-\boldsymbol{\theta})$. Hence the random shift model is $(\boldsymbol{X},\boldsymbol{\Theta}) \stackrel{d}{=} (\boldsymbol{\Theta}+\boldsymbol{Y},\boldsymbol{\Theta})$, where $\boldsymbol{Y}|\boldsymbol{\Theta} = \boldsymbol{\theta}$ has distribution function $F_{\boldsymbol{\theta}}$. It is clear that the random shift model is again specified via a conditional distribution, so there is no essential simplification by re-writing the conditional model apart from the case that the joint distribution of $(\boldsymbol{Y},\boldsymbol{\Theta})$ is known.

We consider briefly a tractable instance that $(\boldsymbol{Y},\boldsymbol{\Theta})$ has an elliptical distribution in $\mathbb{R}^{2d}$, i.e.,

$$(\boldsymbol{Y},\boldsymbol{\Theta}) \stackrel{d}{=} \left(R\frac{Z_1}{\sqrt{\sum_{i=1}^{2d} Z_i^2}},\ldots,R\frac{Z_{2d}}{\sqrt{\sum_{i=1}^{2d} Z_i^2}}\right)C+\boldsymbol{\nu} =: R\boldsymbol{U}C+\boldsymbol{\nu}, \quad \boldsymbol{\nu}\in\mathbb{R}^{2d}, \tag{15}$$

with $Z_i, i\le 2d$ independent $\mathcal{N}(0,1)$ distributed random variables being further independent of $R>0$, and C a square matrix in $\mathbb{R}^{2d\times 2d}$. For more details and actuarial applications of elliptically symmetric multivariate distributions see [9].

Let $I=\{1,\ldots,d\}$ and $J=\{d+1,\ldots,2d\}$. For any $2d\times 2d$ matrix A, denote $A_{I,J}$ as the sub-matrix of A obtained by selecting the elements with row indices in I and column indices in J. Similarly, for any row vector $\boldsymbol{\nu}\in\mathbb{R}^{2d}$, define $\boldsymbol{\nu}_I$ and $\boldsymbol{\nu}_J$ to be the sub-vectors of $\boldsymbol{\nu}$. Further, denote by $A^\top$ the transpose of matrix A.

By the stochastic representation (15) we obtain that

$$(\boldsymbol{\Theta}+\boldsymbol{Y},\boldsymbol{\Theta}) \stackrel{d}{=} R\boldsymbol{U}C^*+\boldsymbol{\nu}^*,$$

where

$$C^* = \begin{pmatrix} C_{I,I}+C_{I,J} & C_{I,J} \\ C_{J,I}+C_{J,J} & C_{J,J}\end{pmatrix}, \qquad \boldsymbol{\nu}^* = (\boldsymbol{\nu}_I+\boldsymbol{\nu}_J,\boldsymbol{\nu}_J).$$

Set $B=(C^*)^\top C^*$ and assume that B is non-singular. As in the Gaussian case, for the more general class of elliptically symmetric distributions, the conditional random vector $\boldsymbol{\Theta}|(\boldsymbol{\Theta}+\boldsymbol{Y}) = \boldsymbol{x}$ is again elliptically symmetric with stochastic representation (suppose for simplicity $\boldsymbol{\nu}_I=\boldsymbol{0}, \boldsymbol{\nu}_J =: \boldsymbol{\mu}$)

$$\boldsymbol{\Theta}|(\boldsymbol{\Theta}+\boldsymbol{Y}) = \boldsymbol{x} \stackrel{d}{=} \boldsymbol{\mu}+(\boldsymbol{x}-\boldsymbol{\mu})B_{I,I}^{-1}B_{I,J}+R_{\boldsymbol{x}}\boldsymbol{U}D,$$

where D is a square matrix such that $D^\top D = B_{J,J}-B_{J,I}B_{I,I}^{-1}B_{I,J}$, and the random variable $R_{\boldsymbol{x}}>0$ is independent of $\boldsymbol{U}$; see e.g., [6]. Consequently, since $\boldsymbol{U}$ has components being symmetric about 0, we obtain the Bayesian premium formula

$$\boldsymbol{E}\{\boldsymbol{\Theta}|\boldsymbol{X}=\boldsymbol{x}\} = \boldsymbol{\mu}+(\boldsymbol{x}-\boldsymbol{\mu})B_{I,I}^{-1}B_{I,J}, \tag{16}$$

provided that $\boldsymbol{E}\{R_{\boldsymbol{x}}\}<\infty$. In the special case that $C_{I,J}$ and $C_{J,I}$ have all entries equal to 0, and further

$$C_{I,I}^\top C_{I,I} = \Sigma \quad C_{J,J}^\top C_{J,J} = \Sigma_0,$$

we conclude that (9) holds.

The random vector $\boldsymbol{O}$ defined in (11) has components $O_i, i \le d$ such that O_i^p has beta distribution with parameters $\alpha_i, \sum_{j\le d, j\ne i} \alpha_j$; see e.g., [8]. In the special case that $\alpha_i = 1/p$ for any $i \le d$, properties of $\boldsymbol{O}$ and $\boldsymbol{X} = R\boldsymbol{O}$ with $R > 0$ independent of $\boldsymbol{O}$ are studied in [20]. Our result in Corollary 2 agrees with the finding of Theorem 4.4 in the aforementioned paper. Note that, for the case $p = 2$ the corresponding result of Theorem 1 for spherically symmetric random sequences is well-known, see e.g., [5,19].

Weighted L_p Dirichlet random vectors are naturally introduced by using indicator random variables. Specifically, let $R > 0$ and $\boldsymbol{O}$ be given as in (11). Further, let $I_i, i \le d$ be independent Bernoulli random variables defined on $(\Omega, \mathcal{A}, \mathbf{P})$ with $\boldsymbol{P}\{I_i = 1\} = q_i = 1 - \boldsymbol{P}\{I_i = -1\}, q_i \in (0, 1], i \le d$, which is further independent of the random vector $(R, \boldsymbol{O})$. The random vector $\boldsymbol{X}$ with stochastic representation

$$\boldsymbol{X} \stackrel{d}{=} (RI_1O_1, \dots, RI_dO_d) \tag{17}$$

is referred to as a weighted L_p Dirichlet random vector with indicators $I_i, i \le d$ and parameters $\alpha_1, \dots, \alpha_d$.
If I_i's are iid with $\boldsymbol{E}\{I_1\} = 0$ and $\alpha_1 = \dots = \alpha_d = 2 = p$, then

$$(I_1O_1, \dots, I_dO_d) \stackrel{d}{=} \left(\frac{Z_1}{\sqrt{\sum_{i=1}^d Z_i^2}}, \dots, \frac{Z_d}{\sqrt{\sum_{i=1}^d Z_i^2}} \right).$$

Therefore, if R^2 is chi-square distributed with d degrees of freedom then $\boldsymbol{X}$ has a centered Gaussian distribution with $N(0, 1)$ independent components. The introduction of the weighted Dirichlet random vectors is important since it includes the normal distribution as a special case. In addition, weighted Dirichlet random vectors are suitable for modeling claim sizes in certain ruin models with double-sided jumps; see Example 4 in [8].

As in the case of L_p Dirichlet random sequences, in the weighted case the dependence structure can be given through a random scale model as well. More precisely, if the random sequence $X_i, i \ge 1$, is such that, for any fixed $d \ge 2$, $(X_1, \dots, X_d)$ is a weighted L_p Dirichlet random vector with indicators $I_i, i \le d$ and parameters $\alpha_1, \dots, \alpha_d$, then

$$\{X_i, i \ge 1\} \stackrel{d}{=} \{SI_iY_i, i \ge 1\}, \tag{18}$$

with S some non-negative random variable defined on $(\Omega, \mathcal{A}, \mathbf{P})$ which is independent of $I_i, Y_i, i \ge 1$.

With motivation from credibility theory, we propose to consider a new class of multivariate distributions called L_P Dirichlet distributions, which is naturally introduced by letting the parameter p in our definition above to be random (a common feature of credibility models where parameters are random elements).

Specifically, let P be a positive random variable, and let $\alpha_i, i \ge 1$ be positive constants. Further, let $Y_i, i \ge 1$ be independent random variables, which are further independent of P such that Y_i^P has $Gamma(\alpha_i, 1)$ distribution. We say that $(X_1, \dots, X_d)$ is a d-dimensional L_P Dirichlet random vector if

$$(X_1, \dots, X_d) \stackrel{d}{=} \left(R\frac{Y_1}{(\sum_{1\le i\le d} Y_i^P)^{1/P}}, \dots, R\frac{Y_d}{(\sum_{1\le i\le d} Y_i^P)^{1/P}} \right) =: R\boldsymbol{O}^{(P)} \tag{19}$$

holds for some positive random variable R independent of P and $(Y_1, \dots, Y_d)$. Here, for any $\boldsymbol{O}^{(P)} = (O_1, \dots, O_d)$

$$\sum_{i=1}^{d} O_i^P = 1.$$

This multivariate distribution can be used in the context of credibility models, models for large losses, models for risk aggregation, or models for claim sizes. If we assume that $X_i, i \geq 1$ is a sequence of claim sizes such that, for any $n \geq 2$, $(X_1, \dots, X_d)$ has a d-dimensional L_P Dirichlet distribution, then an extension of Theorem 1 for this case is possible. More precisely choosing now $a_n = (\sum_{i=1}^n \alpha_i)^{1/P}$ we conclude that

$$\{X_i, i \geq 1\} \stackrel{d}{=} \{SY_i, i \geq 1\},$$

with some non-negative random variable S independent of $Y_i, i \geq 1$.

In what follows, we consider a new copula class introduced in [23] which is referred to as MGB2 copula. Let Θ be a positive random variable having an inverse Gamma distribution with shape parameter $q > 0$ and a unit scale, i.e., $1/\Theta$ has $Gamma(q, 1)$ distribution. In view of the aforementioned paper $(X_1, \dots, X_n)$ has a MGB2 distribution (or MGB2 copula) if X_i's are positive random variables and $X_1|\Theta = \theta, \dots, X_n|\Theta = \theta$ are independent with pdf $f_{X_i|\Theta}, i \leq n$ given by

$$f_{X_i|\Theta}(x_i|\theta) = \frac{a_i}{\Gamma(p_i) x_i \theta^{p_i}} \left(\frac{x_i}{b_i}\right)^{a_i p_i} e^{-\theta^{-1}(x_i/b_i)^{a_i}}, \qquad x_i > 0, \theta > 0.$$

Here the parameters $a_i, b_i, p_i, i \leq n$ are all positive constants. Instead of using the conditional argument, we can directly define $(X_1, \dots, X_n)$ via a random scale model as follows

$$(X_1, \dots, X_n) \stackrel{d}{=} (\Theta^{1/a_1} W_1, \dots, \Theta^{1/a_n} W_n), \tag{20}$$

with $W_1, \dots, W_n$ being independent positive random variables such that, for any fixed $i \leq n$, $W_i^{a_i}$ has $Gamma(p_i, b_i^{-a_i})$ distribution. One advantage of the random scale model (20) is that, for modeling purposes, it can be re-written as a random shift model

$$(\ln X_1, \dots, \ln X_n) \stackrel{d}{=} \Big(\frac{1}{a_1} \ln \Theta + \ln W_1, \dots, \frac{1}{a_n} \ln \Theta + \ln W_n\Big). \tag{21}$$

Another advantage of the random scale model (20) becomes clearer if of interest is the joint tail asymptotic behaviour of (X_1, X_2), as discussed in [23]. As illustrated below, the regular variation of survival function of Θ is enough for the joint tail asymptotic behaviour of (X_1, X_2); distributional assumptions on Θ are not really necessary.

Example 2. Let (W_1, W_2) be defined as above with the parameters therein and further assume that $a_1 = a_2 = a > 0$. Define (X_1, X_2) through (20) with Θ an independent of (W_1, W_2) random variable having a regularly varying tail behavior at infinity with index $q > 0$, i.e.,

$$\lim_{x \to \infty} \frac{\boldsymbol{P}\{\Theta > tx\}}{\boldsymbol{P}\{\Theta > x\}} = t^{-q}, \quad \forall t > 0.$$

For modeling joint behaviour of large losses of interest is the calculation of the following limit

$$\lim_{t \to \infty} \frac{\boldsymbol{P}\{X_1 > c_1 t, X_2 > c_2 t\}}{\boldsymbol{P}\{X_1 > t\}}$$

for c_1, c_2 positive constants, see e.g., [7] and [9]. In our case we have

$$\begin{aligned}
\frac{\boldsymbol{P}\{X_1 > c_1 t, X_2 > c_2 t\}}{\boldsymbol{P}\{X_1 > t\}} &= \frac{\boldsymbol{P}\{\Theta^{1/a} W_1/c_1 > t, \Theta^{1/a} W_2/c_2 > t\}}{\boldsymbol{P}\{\Theta^{1/a} W_1 > t\}} \\
&= \frac{\boldsymbol{P}\Big\{\Theta \min\Big((W_1/c_1)^a, (W_2/c_2)^a\Big) > t^a\Big\}}{\boldsymbol{P}\{\Theta W_1^a > t^a\}} \\
&\to \frac{\boldsymbol{E}\Big\{\Big(\min(W_1/c_1, W_2/c_2)\Big)^{aq}\Big\}}{\boldsymbol{E}\{W_1^{aq}\}} = I(c_1, c_2) > 0
\end{aligned}$$

as $t \to \infty$ where in the last step we applied Breiman's lemma; see e.g., [18]. Since the asymptotic dependence function $I(c_1, c_2)$ is positive, an appropriate extreme value model for the joint survival function of X_1 and X_2 is the one that allows for Fréchet marginals and asymptotic dependence.

5. Conclusion

This contribution shows that in various insurance applications besides common conditional stochastic models, equivalent random shift or random scale models can be analysed and explored. As explained in the context of credibility models, simple random shift models lead to direct derivations for the calculation of the Bayesian premium. In particular, Example 1 shows that for Gaussian models, the covariance matrix of the prior distribution can be singular without changing the outcome.

Our main result concerning the random scale property of L_p Dirichlet random sequences is not only of theoretical importance but also of practical importance, since in certain models claim sizes can be reduced to random scale of independent random sequences with known marginal distributions.

Example 2 demonstrates the usefulness of random scale models for analysing joint survival functions for large thresholds. As a by-product in Section 4 we suggest a new dependence structure of interest for dependent risks.

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Author Contributions

Both authors contributed to all aspects of this work.

Conflicts of Interest

The authors declare no conflict of interest.

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