Optimal Dynamic Management of a Renewable Energy Source under Uncertainty

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Abstract

We consider a risk averse and prudent social planner who has access to different energy sources to produce electricity: hydroelectricity produced with a dam and thermal electricity with unlimited supply at some exogenous cost. The dam is supplied with a random water flow. The presence of constraints on a minimal and on a maximal storage capacity makes electricity consumption smoothing possible only when the quantity of water available to the agent lies in a certain range that we determine. Consumption smoothing is possible even when the dam is almost empty thanks to the alternative costly energy source. Moreover a comparative static analysis reveals that the marginal propensity to produce hydroelectricity is an increasing function of the cost of the second technology. Therefore, the availability at a low cost of the fossil source improves time diversification. Finally, the optimal electric park is composed of a number of dams that is increasing with the cost of the second technology.

Keywords: intertemporal expected utility maximization, hydroelectricity, thermal power, investment under uncertainty, prudence.

JEL Classification: C61, Q25, Q42.

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1 Introduction

The aim of this work is to determine the optimal structure of an electric park that generates power with different energy sources. To preserve the environment and the energy sources that are exhaustible, governments are increasingly concerned with the use of renewable energy sources besides classic thermal power sources. However, renewable energy sources are not easy to use as their availability is not constant over time. Therefore an electric park must be designed by taking into account this random availability and its management has to solve the problem of providing enough electricity even when renewable sources are not available in the short run.

Norway, for instance, is the sixth largest hydropower generator in the world and the biggest in Europe. Hydropower accounts for 99% of the electricity generated and annual production varies to a great extent in line with precipitation levels. Thus, when the country faces dry periods as it was the case in 2002 and 2003, hydropower reservoirs work as buffers between output and consumption. Besides hydropower, electricity is also generated from sources such as natural gas and wind. Indeed, "gas-fired power station" can be started up and closed down at short notice. They are suitable for providing peak-load power but have a relatively high cost. In fact, during dry periods, the loss of hydropower output is offset by increasing thermal power generation[1].

In this work, we mainly focus on two energy sources. The renewable energy source has a random availability whereas the thermal power source is available at an exogenous market price. The possibility to store the renewable energy source in a dam allows to smooth consumption over time. During a dry episode, some of the water stored in the dam is consumed and the water reserve goes down, potentially to the lower limit of the reservoir. In that case, it may be possible that electricity consumption be limited or rationed. On the contrary, when the water inflow is higher, the dam is replenished potentially up to the maximum capacity of the reservoir. Therefore, the capacity of the dam is a key factor of the optimal management policy as the Norwegian example shows us. Besides this renewable energy source, the permanent availability of the thermal power source softens the effect of the uncertainty of the water inflow.

A large body of literature concerning commodity storage presents meaningful results. Williams and Wright [26, 27] developed a model where the supply is stochastic and where production and storage are performed by competitive profit maximizers. They found that "storage is much more effective in eliminating excessive levels of consumption and low prices than in preventing low levels of production and high prices". They explained this result by evoking the non symmetry of storage. Indeed, storage has to be non negative or, said in other words, water cannot be "borrowed" during a drought. Deaton and Laroque [6, 7] also worked on the topic of commodity prices and commodity storage. In the second paper, they tried to explain some stylized facts of commodity price behavior by fitting a competitive storage model directly to the data. They

[1] See Ministry of Petroleum and Energy of Norway [22].

proposed two ways to model productivity shocks: either iid shocks or time dependent (autoregressive) shocks. But finally none of the two models fits the data well. Deaton and Laroque explained this failure as follows: "all the autocorrelation in the data has to be attributed to the underlying processes. Although speculation is capable of increasing the autocorrelation that would otherwise exist in an unmoderated price series, it cannot raise it to the levels that we observe".

Concerning the analysis of the use of energy sources, many directions have been explored. On the one hand, equilibrium models have been developed. So, Garcia et al. [9] analyzed the price formation process and its policy implications in an infinite horizon duopoly model. They focused on two hydroelectricity producers who engage in dynamic Bertrand competition. At each date, water reservoirs are replenished with some strictly positive probability and a price cap affects the opportunity cost of producing electric power. They found that hydroelectricity producers might sell less today to have more capacity tomorrow: they adopt a strategic pricing behavior. They explained that the introduction of a price cap may shift down the entire price distribution. Crampes and Moreaux [4] studied a model where two energy sources are available: hydroelectricity and thermal electricity. They focused on a model of two time periods and did not introduce uncertainty in the hydroelectric technology. They studied the case of a central planner, of a monopoly that is regulated or not and the case of Cournot competition. They concluded that in the presence of hydroelectricity, thermal plants have to be dynamically planned. Moreover the optimized output for the thermal station is determined by the intertemporal specification of costs and utility. On the other hand, Hotelling [17] did not focus on equilibrium situations but rather analyzed the optimal rate of depletion of an exhaustible resource. He found that the rate at which consumption falls over time should equal the ratio of the discount rate and the elasticity of marginal utility of consumption. As Heal [14] noted in his review on the optimal use of exhaustible resource, many extensions to this initial model have been developed. Hoel [15], for instance, introduced uncertainty in a setting with two energy sources: the date when the substitute will become available is known, but its unit cost is uncertain. He found that an increase in uncertainty may increase the consumption depending on the shape of the utility function. Ayong Le Kama [1] studied the problem of the use of one energy source under uncertainty in a finite horizon model. He found that the introduction of two types of constraints, one on the availability of the resource and another on the agent's solvency, modifies the agent's behavior. In order to determine the optimal consumption, the agent takes into account the energy stock but also his anticipations on the realizations of future shocks. In the fifties, different authors addressed the question of minimizing dispatch cost in a hydrothermal system. Thus, Little [21] determined the optimal water use in an uncertain setting close to the one we use, but he did not focus on the way an electric park is valued. Two years later, Koopmans [19] developed a model with two energy sources without uncertainty and aimed at determining the optimal water storage policy that minimizes the operating cost of thermal generation. In a second step, he tried to obtain the value of the power generated and of the water used and/or stored.

A parallel can be drawn between a dam that contains water and the savings of an agent and between the water flow that enters a dam and the random revenue of the agent. In models used to study the consumption/saving behavior of agents, one usually takes into account the fact that the agent is not allowed to borrow at each period of time. Without such liquidity constraints, agents would perfectly smooth their consumption over time. But with liquidity constraints, agents are not able any more to use an anticipated increase in their revenue in the future by increasing today the amount they are allowed to borrow. Therefore, the introduction of such constraints decreases the consumption level even if they are not binding. Agents indeed are afraid of not being able to borrow. Such models have been studied by Deaton [5], Zeldes [29], Carroll [2] and Gollier [12].

In this paper, we aim at determining the optimal electric park that generates power with two energy sources. This model is an extension of the initial Hotelling model in which uncertainty has been added. Indeed, it is assumed here that the level of the stock of one of the energy sources is uncertain and the analysis concerns the optimal management of random stocks. We are in a setting close to the one of Crampes and Moreaux (when they study the case of a central planner) as far as we consider the optimal allocation between two energy sources. However, two main features have been added: not only do we consider an infinite horizon model, but we also introduce uncertainty on the water inflow. A social planner chooses the energy production at each period depending on the state of the system and his expectations on its evolution. He maximizes the discounted sum of the expected utility he gets from the use of different energy sources. In a first step, the optimal production flow when only hydroelectricity is available is analyzed. Hydroelectricity generation comes from water stored in a dam that is supplied with a random inflow. Unlike Ayong Le Kama, we do not consider any solvency constraint since future water inflows are assumed to be always positive. However, we add a second constraint on the availability of the resource since it must be finite. Therefore, a second kind of "liquidity constraint" is introduced: not only is the social planner unable to produce electricity from water not yet fallen in the reservoir, but it is also not possible to store more water in the dam than its capacity. We find that the use of the dam allows electricity smoothing when the quantity z of available water is in a given range $[z^*, z^{**}]$ that we determine. Indeed in this region, the social planner prefers cutting down on total consumption today to let enough water in the dam for the future. But when the quantity of available water is too low (lower than some threshold z^*), it is completely consumed since the social planner expects that future rainfalls will replenish the dam. A second energy source is then added and the optimal combination between the two energy sources allows for a better smoothing of electricity consumption even when the costly energy source is not consumed. Moreover the introduction of the second energy source shifts up water production. Besides the study of the allocation between the two sources, we consider the effect of an increase in uncertainty of the water inflow. For some values of the quantity of available water, more water is consumed under uncertainty than under certainty. Once the optimal allocation has been determined, we consider a long term situation where the characteristics of the electric park have to be determined. We compute the optimal number of dams and find that it is an increasing function of the price of the alternative energy source. Lastly, as an extension, we focus on the efficiency of time diversification when a second random energy source is introduced that is non storable.

In the next section of this paper, the model is presented. A benchmark case is studied in section 3 when there is no uncertainty on the water inflow. Section 4 is devoted to the resolution of the model in a general setting. Section 5 deals with the characteristics of an optimal electric park in the long term. In section 6, we propose as an extension to introduce a third energy source that is uncertain and non-storable. Section 7 concludes.

2 Basic Model

We consider a small economy in which a social planner produces electricity using two different technologies: hydroelectric energy and thermal power. Hydroelectricity is obtained from water extracted from a dam. The dam is supplied with a random water flow $\widetilde{y}_t$ and is characterized by its capacity $\overline{Z}$. Thermal power is available at any time at a constant exogenous market price[2]. In this setting, hydroelectricity is a renewable resource whereas the relative scarcity of the thermal power is expressed in its price: there is no constraint on its availability given its market price. Therefore, thermal power is a backstop technology of hydroelectricity in the sense used by Nordhaus [23] and Heal [14]: "a technology that can provide substitutes for the resource once it is fully depleted, and can provide these substitutes on a very large scale indeed".

We consider the standard setting where a social planner chooses at each period the energy production depending on the state of the system and its expectations on its evolution. He aims at maximizing the expected intertemporal utility he gets from the flow of future production. The use of the dam introduces some constraints. On the one hand, consumption is limited by the quantity of stored water in the dam, but on the other hand, the quantity of water consumed has to be high enough since the amount of water stored is limited by the dam capacity $\overline{Z}$. Concerning thermal power, the only constraint is a non-negativity one. Let us introduce

- w_t the amount of water in the dam at the beginning of period t,
- $\widetilde{y}_t$ the constant flow of water that enters the reservoir,
- z_t the total amount of water that is available to the agent in period t, that is $z_t = w_t + \widetilde{y}_t$. This implies that $z_t \geq \min \widetilde{y} \ \forall t$,
- c_t the amount of water that is extracted from the dam in period t,

[2]In this setting of a competitive market for thermal electricity, the price corresponds to the marginal cost.

- $\overline{Z}$ the dam capacity. w_t, μ, z_t, c_t and $\overline{Z}$ are measured in cubic meters and refer to volumes of water in the reservoir. We assume $\min \widetilde{y} < \overline{Z}$,
- x_t the quantity of electricity produced with thermal power consumed in period t and measured in kWh,
- p the unit price of thermal power,
- R the coefficient that allows to convert a volume of water into a quantity of electricity produced in kWh. It depends on the characteristics of the dam (its height, its flow),
- β the discount factor with $\beta < 1$,
- u the utility function: it is strictly increasing and strictly concave.

The timing is represented on Figure 1.

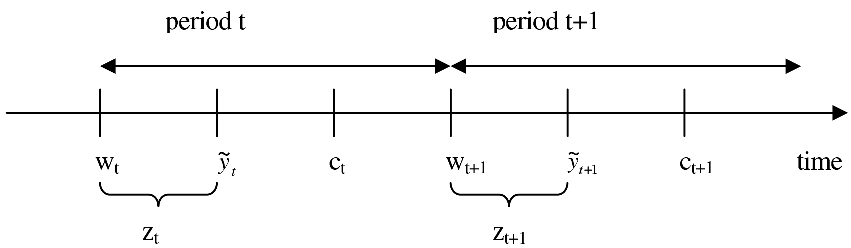

Figure 1: Timing

The dynamics of the water that is available to the social planner is equal to $z_{t+1} = z_t - c_t + \widetilde{y}_{t+1}$. Water consumption is limited by the quantity of available water, therefore the first constraint reduces to $c_t \leq z_t$. Finally, the remaining stock of water has to be lower than the dam capacity: $w_{t+1} \leq \overline{Z}$. But as the dynamics of the quantity of water stored in the dam is $w_{t+1} = w_t + \widetilde{y}_t - c_t = z_t - c_t$, the constraint comes down to $z_t - c_t \leq \overline{Z}$. The social planner's program is

$$\max_{\{c_t, x_t\}_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[u\left(Rc_t + x_t\right) - px_t \right] \tag{1}$$

subject to

$$\begin{aligned} z_{t+1} &= z_t - c_t + \widetilde{y}_{t+1}, & (2)\\ c_t &\leq z_t, & (3)\\ c_t &\geq z_t - \overline{Z}, & (4)\\ c_t &\geq 0, & (5)\\ x_t &\geq 0, & (6) \end{aligned}$$

$$z_0 \text{ given.} \tag{7}$$

We choose a CRRA utility function where the concavity coefficient γ is equal to the inverse of the constant price elasticity of the demand for thermal power $x(p)$ taken in absolute value

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma} + constant.$$

We assume moreover that $\gamma > 1$.

Before solving the model, we state a first result on the shape of both consumption flows.

Proposition 1 *Thermal power is consumed after having consumed all available water in the reservoir:* $x_t > 0 \Rightarrow c_t = z_t$.

This result means that the electricity consumption path can be decomposed into two phases. As in Hotelling, the cheapest energy source is consumed first. Once the water reserve is fully depleted, thermal power is used in combination with the constant water inflow. The driving force for this result is the willingness of the social planner to postpone energy expenditure because $\beta < 1$.

Proof: Suppose the proposition does not hold: $x_t > 0$ and $c_t < z_t$. Let t' be the first time period for which $c_{t'} > 0$ (it exists else constraint 4 would be violated). Consider the following strategy

- $\{\widehat{c}_t, \widehat{x}_t\}$ with $\widehat{c}_t = c_t + \frac{\varepsilon}{R}$ and $\widehat{x}_t = x_t - \varepsilon$,
- $\{\widehat{c}_{t+1}, \widehat{x}_{t+1}\}$ with $\widehat{c}_{t'} = c_{t'} - \frac{\varepsilon}{R}$ and $\widehat{x}_{t'} = x_{t'} + \varepsilon$.

In period t, $\Delta u_t = p$, in period t', $\Delta u_{t'} = -p$. Therefore, the total effect, $\Delta u = \beta^t p - \beta^{t'} p = \beta^t p \left(1 - \beta^{t'-t}\right)$ is strictly positive, and strategy $\{\widehat{c}, \widehat{x}\}$ is strictly preferred to strategy $\{c, x\}$ that is the optimal one. This leads to a contradiction. □

In the light of this result, to solve the initial maximization program, we first determine the optimal consumption of thermal power in the second stage of the process, i.e., when the dam is empty. We then use this information to determine the optimal consumption of hydroelectricity in the first stage. To do so, we introduce function

$$\widehat{u}(Rc; p) = \max_{x \geq 0} u(Rc + x) - px.$$

If ν is the Lagrangian multiplier associated with the constraint $x \geq 0$, the FOC reads $u'(Rc + x) - p + \nu = 0$. With $e^* = u'^{-1}(p)$, two cases occur:

- either $\nu = 0$, what implies $x \geq 0$. It follows that $u'(Rc + x) = p = u'(e^*)$, $x = e^* - Rc \geq 0$, and $Rc \leq e^*$,
- or $\nu > 0$, what implies $x = 0$ and $u'(Rc + x) = p - \nu < u'(e^*)$, $x = 0$, and $Rc > e^*$.

Therefore, function $\widehat{u}$ is equal to

$$\widehat{u}\left(Rc;p\right)=\begin{cases} u\left(e^{*}\left(p\right)\right)-p\left(e^{*}\left(p\right)-Rc\right) & \text{if } Rc\leq e^{*}\left(p\right), \\ u\left(Rc\right) & \text{if } Rc\geq e^{*}\left(p\right). \end{cases}$$

We present the shape of $\widehat{u}$ on Figure 2. When $c\leq e^{*}\left(p\right)/R$, it is a straight line and once this threshold is crossed, it is equal to the utility function u.

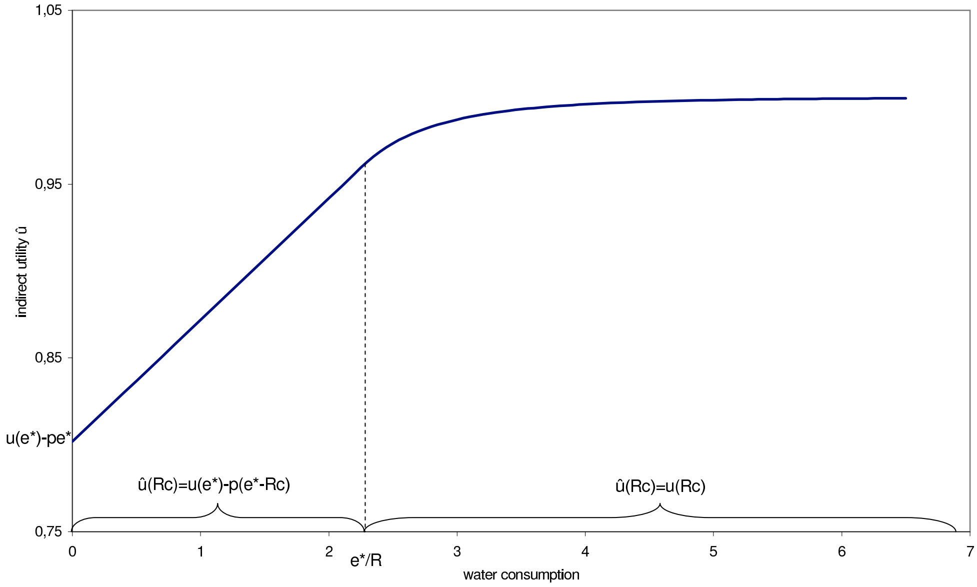

Figure 2: Indirect utility function $\widehat{u}$

The social planner's program reduces to

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \widehat{u}\left(Rc_t\right) \tag{8}$$

subject to

$$z_{t+1} = z_t - c_t + \widetilde{y}_{t+1}, \tag{9}$$

$$c_t \leq z_t, \tag{10}$$

$$c_t \geq z_t - \overline{Z}, \tag{11}$$

$$c_t \geq 0, \tag{12}$$

$$z_0 \text{ given.} \tag{13}$$

Before we characterize the solution of this program, we study, as a benchmark, the case where there is no uncertainty on the water inflow.

3 Benchmark: model under certainty

In this section, the water inflow that fills the dam at each period is assumed to be constant: $\forall t, \widetilde{y}_t = \mu > 0$. Total consumption is at each period greater or equal than μ: $\forall t, c_t \geq \mu$ and the constraint (12) is thus always satisfied.

With this indirect utility function $\hat{u}$, we note that, at each period, electricity production is at least equal to $e^* = u'^{-1}(p)$, since $\forall t, x(t) + Rc(t) \geq e^*$. $x(t) + Rc(t) = e^*$ happens if the constant water inflow is not sufficient (or just sufficient) to satisfy all the demand, i.e. $\mu \leq e^*/R$. Two cases may occur. Either $\mu < z_t \leq e^*/R$: there remains a strictly positive quantity of water in the dam before the precipitation replenishes the reservoir ($w_t > 0$). Or $z_t = \mu \leq e^*/R$: there is no water stock in the dam any more and at each time period, water consumption is equal to the rainfall. $x(t) + Rc(t) > e^*$ implies that no thermal power is consumed ($x(t) = 0$) and electricity is produced using only the hydroelectric technology. Optimal thermal power consumption path is therefore equal to $x(t) = \max(e^*(p) - Rc(t), 0)$.

Recalling that $\forall t, z_t \geq \mu$, two cases may occur:

1. $\frac{e^*}{R} \geq \mu$: either $z_0 \geq \frac{e^*}{R}$ and the water consumption flow is decreasing until it reaches μ. Once it reaches $\frac{e^*}{R}$, thermal power consumption becomes strictly positive and equal to $x(t) = \max(e^*(p) - Rc(t), 0)$. Or $z_0 < \frac{e^*}{R}$ and water consumption is decreasing until it reaches μ. Thermal power consumption is strictly positive from the first period on and is equal to $\frac{e^*}{R} - c_t$,

2. $z_0 \geq \mu > \frac{e^*}{R}$: water consumption flow is decreasing until it reaches μ and it is equal to μ thereafter. No thermal power is consumed.

The resolution of the program is in the Appendix. We present the shape of the optimal water consumption path for the first case on Figure 3. γ is taken to be equal to 5. This is consistent with the estimation of the price elasticity for different European countries found in Söderholm [24] whose mean amounts to -0.2. The constant inflow of water, μ, is equal to 2. Concerning the other parameters, the values chosen are $\beta = 0.95$, $R = 0.7$, $\overline{Z} = 10$, $z_0 = 10$ and $p = 0.05$. $e^*(p)$ is therefore equal to 1.82. These values will be used for all graphical representations if nothing else is mentioned.

Hydroelectricity consumption is decreasing until the dam is empty. Afterwards, it is constant and equal to $R\mu$. Thermal power consumption is equal to $\max(e^*(p) - Rc(t), 0)$. When there is still some water in the dam, total electricity consumption is greater than $e^*(p)$. On the contrary, once thermal power is consumed, the total electricity consumption amounts to $e^*(p)$.

In order to be able to draw comparisons with the case where the inflow of water is uncertain (see following section), we give the shape of the water consumption flow c relative to the quantity of available water z on Figure 4.

There is a kink: indeed, for low levels of stored water, all the available water is consumed. But afterwards, this is a step function because of the definition of the time T from which thermal power is consumed. As we work in discrete time, T has to be integer.

We focus now on the general case.

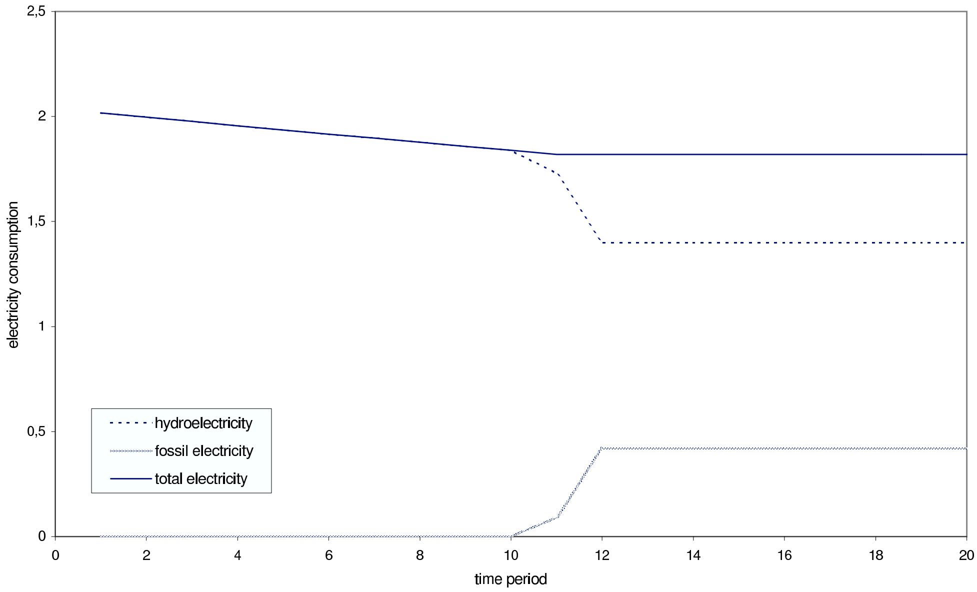

Figure 3: Optimal power consumption path

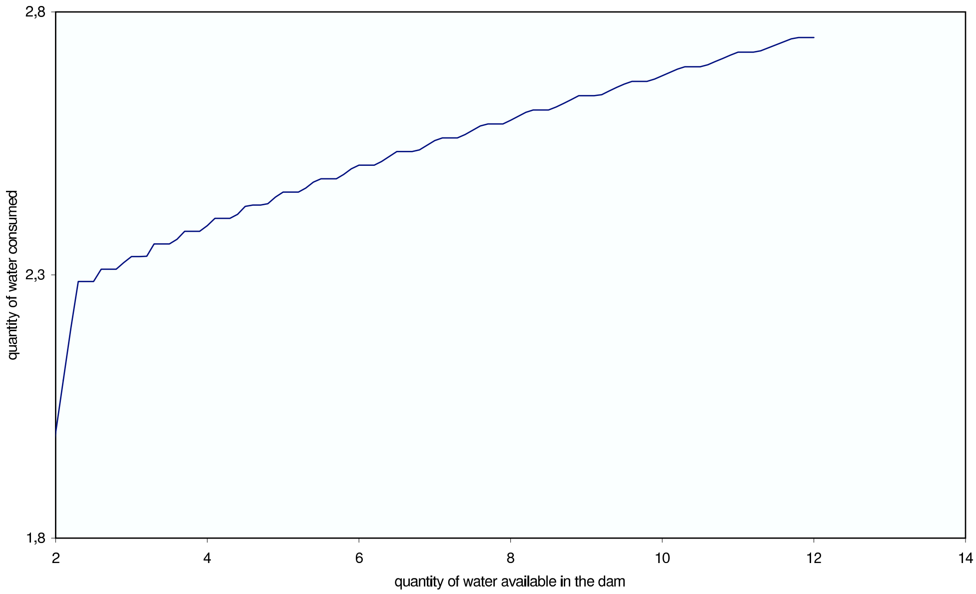

Figure 4: Optimal water consumption path in function of the quantity of water available

4 General case: model with uncertainty

We assume now that each realization of the random variable $\widetilde{y}_t$ is positive. At each time period, we have that $z_t > \min \widetilde{y}$.

The resolution of problem (8) is made more convenient by using the Bellman equation

$$v(z) = \max_c \{\widehat{u}(Rc) + \beta \mathbb{E} v(z - c + \widetilde{y})\} \tag{14}$$

subject to

$$c \leq z, \tag{15}$$
$$c \geq z - \overline{Z}, \tag{16}$$
$$c \geq 0. \tag{17}$$

The following lemma gives a first result on the shape of the value function v.

Lemma 1 *The value function v is strictly concave.*

Proof: See the Appendix. □

This technical result is a first step before obtaining results on the shape of both consumption flows (see the following subsection). If λ, η_1 and η_2 are the Lagrangian multipliers associated with constraints (15), (16) and (17), the FOC reduces the three following cases

$$R\widehat{u}'(Rc) \begin{cases} \geq \beta \mathbb{E} v'(z + \widetilde{y} - c) & \text{if (15) is binding,} \\ \leq \beta \mathbb{E} v'(z + \widetilde{y} - c) & \text{if (16) or (17) is binding,} \\ = \beta \mathbb{E} v'(z + \widetilde{y} - c) & \text{otherwise.} \end{cases}$$

The second order conditions, $\frac{\partial^2 \mathcal{L}}{\partial c^2} = R^2 \widehat{u}''(Rc) + \beta \mathbb{E} v''(z + \widetilde{y} - c) \leq 0$, are satisfied because of the concavity of u and v.

In this case where the water inflow is uncertain, there are two means for the social planner to smooth electricity consumption: consuming thermal power when water extraction is low or storing water in the dam when precipitation is large. Proposition 1 explains how consumption smoothing is possible in this uncertain setting. When water is scarce, the social planner does not use the dam to store water. He prefers consuming all the water available and smoothing electricity consumption with the consumption of thermal power. On the contrary, when there is more water in the dam, the social planner does not use the alternative energy source any more, but the dam to smooth electricity consumption. Analytically, we have:

$-\forall z$ such that $Rz \leq e^*$, $c(z) = z$, and $x(z) = e^*(p) - Rz$

$-\forall z$ such that $Rz \geq e^*$, $x(z) = 0$, and $c(z)$ is the solution of $R\widehat{u}'(Rc) = \beta \mathbb{E} v'(z - c + \widetilde{y}) + \lambda - \eta_1 - \eta_2$,

where λ, η_1 and η_2 are the Lagrange multipliers associated with the constraints (15), (16) and (17).

As in the certain case with two energy sources, a minimum amount of electricity is produced at each period. It is equal to $e^*(p)$: $\forall z, x(z) + Rc(z) \geq e^*$. If $x(z) + Rc(z) = e^*$, electricity is produced through a combination of the thermal power and water. In this case, the role of thermal power is to complete the level of electricity consumed up to e^* because there is not enough water in the dam. When $x(z) + Rc(z) > e^*$, no thermal power is consumed ($x(z) = 0$) and electricity is only produced with hydroelectric technology. It can be the case that all the available water is consumed ($c(z) = z$), but in most cases, water is stored and smoothing is possible (except if z is too high and we are in the case where the constraint on the lower bound is binding).

In the next two subsections, we analyze the consumption flow of each energy source.

4.1 Analysis of the water consumption flow

We begin this section with a result on the shape of function c.

Lemma 2 *Electricity consumption is strictly positive when the quantity of available water is strictly positive:* $\forall z > 0, c(z) > 0$.

Proof: See the Appendix. □

This result tells that constraint $c \geq 0$ in program (14) is never binding when $z > 0$. Therefore, $\eta_2 = 0$ and we replace η_1 with η.

The numerical resolution of the problem is represented on Figure with the usual parameters' values. We consider a random inflow $\widetilde{y}$ that takes the values 0, 3 and 6 with equal probabilities, thus $\forall t, z_t \geq 0$.

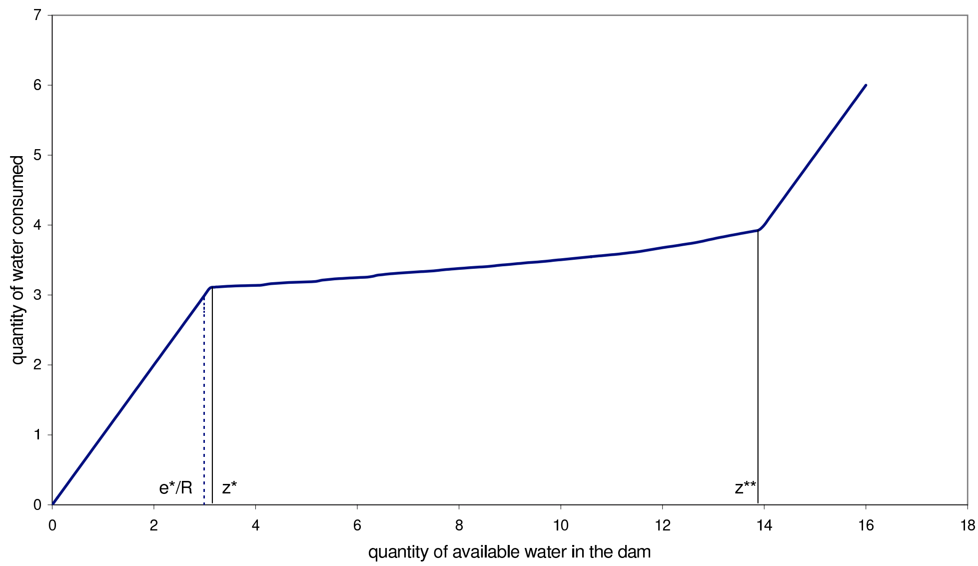

Figure 5: Consumption of water in function of the quantity of available water when $\widetilde{y} \in \{0, 3, 6\}$

According to Figure 5, there exist two thresholds[3] z^* and z^{**} such that

- $\forall z \leq z^*, \quad c(z) = z,$
- $\forall z \geq z^{**}, \quad c(z) = z - \overline{Z}.$

In order to understand how hydroelectricity consumption smoothing is possible for different levels of stored water, we compute the marginal propensity to consume $\partial c/\partial z$ from the first order conditions. It is a measure of the efficiency of intertemporal smoothing: if smoothing were perfect, $c'(z)$ would be equal to zero. On the contrary, when $c'(z) = 1$, there is no smoothing at all since all the water added in the reservoir is immediately consumed.

$$\frac{\partial c}{\partial z} = \begin{cases} 1 & \text{if } z \leq z^*, \\ \frac{\beta \mathbb{E} v''(z+\widetilde{y}-c(z))}{\widehat{u}''(c(z))+\beta \mathbb{E} v''(z+\widetilde{y}-c(z))} & \text{if } z^* < z < z^{**}, \\ 1 & \text{if } z \geq z^{**}. \end{cases}$$

First, $c'(z)$ is positive implying that c is an increasing function. Next, note that when neither (15) nor (16) is binding, the marginal propensity to consume is strictly less than 1 and time diversification is possible. Note that because of Proposition 1, $z^* \geq e^*/R$. Indeed, imagine this were not the case, then the social planner would consume thermal power whereas there is still water left in the reservoir and this would contradict Proposition 1. However, it might be the case that $z^* > e^*/R$ meaning that the social planner empties out the reservoir and consumes more than the minimal amount e^*.[4] He knows indeed that at the next time period there will be a water inflow greater or equal then $\min \widetilde{y}$, there is thus no need to keep water in stock. Once e^*/R (and thus z^*) is crossed, thermal energy is not consumed anymore. On $[z^*, z^{**}]$, as the social planner knows that in the future he could not produce the quantity of water he would like, he prefers cutting down on the production today to let enough water in the dam for the future. When $c(z) \leq z^*$, the social planner modifies the consumption behavior and there is no consumption smoothing. Indeed, the social planner knows that, in the next period, the water flow that will replenish the dam will be greater or equal to $\min \widetilde{y}$. Therefore, he empties the water out of the reservoir.

Note that the marginal propensity to consume is first decreasing and then increasing. At the approach of z^* (when $z > z^*$), the social planner realizes that consumption smoothing is not possible any more. Indeed there is not enough water any more and he is better off emptying out all the stock. Similarly, at the approach of z^{**} (when $z < z^{**}$), the social planner suddenly increases hydroelectricity production because of the risk of a very rainy period for many successive periods. Therefore, function c is successively concave then convex[5].

[3] The existence of z^* has already been proven by Deaton [5] and Deaton and Laroque [6, 7]. In Deaton's model studying the consumption/saving behavior of agents, there is only one constraint on the maximal amount that can be borrowed which corresponds to (15).

[4] In the case where $p \to +\infty$ (the alternative energy source is not available), it can be shown that $z^* \geq \min \widetilde{y}$ (see Appendix C).

[5] The difference with the concavity result of the consumption function proven by Carroll and Kimball [3] comes from the second constraint: consumption has to be high enough in our model.

It is also meaningful to look at the evolution of hydroelectricity production over time. On Figure 6, the consumption flow for 100 time periods is represented together with the water stock. It has been obtained through a simulation of the random variable $\widetilde{y}$ (in this numerical illustration, we consider the extreme case where $p \to +\infty$, meaning that thermal energy is not available).

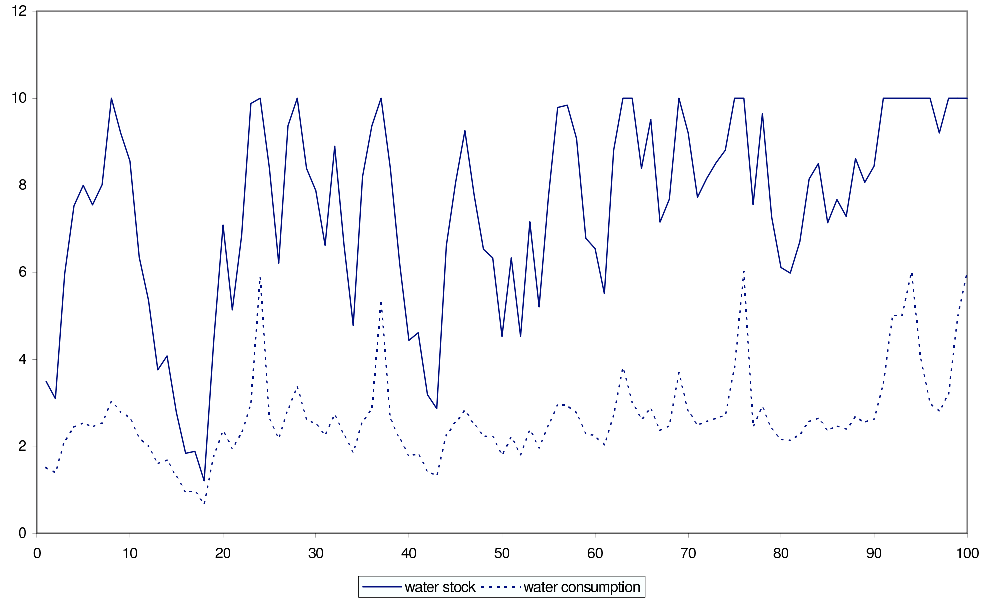

Figure 6: Evolution of the stock and of the consumption with the time

The path of water consumption has a completely different shape than in the certainty case where it was a decreasing function of time. The variations in the consumption flow are indeed smaller than the stock variations. When the water stock approaches the dam capacity (i.e. the constraint on the lower bound is binding), consumption smoothing is less efficient. This is also the case when the dam is almost empty (i.e. the constraint on the upper bound is binding). In this two cases, the dam does not play its smoothing role because of the technical constraints.

4.2 Description of the thermal energy consumption flow

Once the water consumption flow is known, thermal power consumption is straightforward to obtain as we noted at the beginning of the section that $x(z) = \max(e^*(p) - Rz, 0)$. The production of thermal power decreases linearly with z down to 0. The next step is to see how both flows evolve according to p.

4.3 Comparative static relative to price

On Figures 7 and 8, we present the production of thermal power and water in function of the water stock for different values for p (for $p = 0.1$ to $p = 0.01$).

It is straightforward to note that thermal electricity production is decreasing with p.

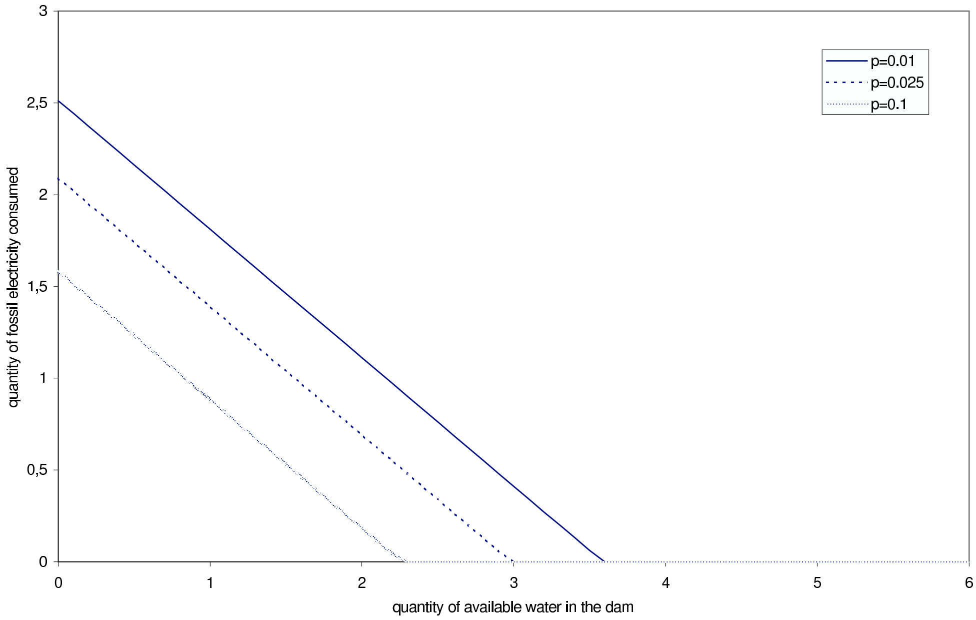

Figure 7: Consumption of thermal power in function of the quantity of available water for different p

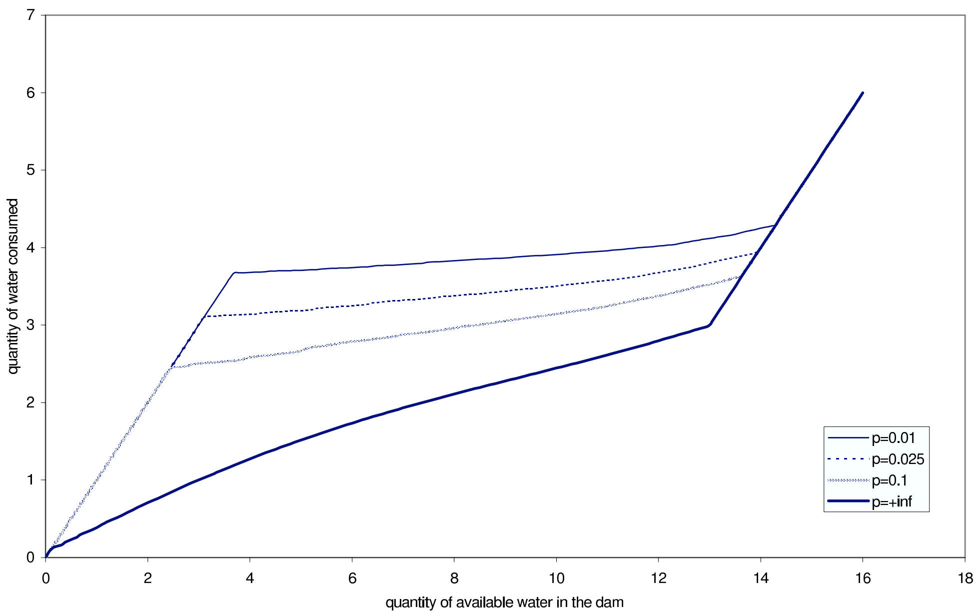

Figure 8: Consumption of water in function of the quantity of available water for different p

The numerical resolution reveals that $c(z;p)$ is decreasing with p. When p increases, the social planner knows that he is going to decrease thermal power production since it is more expensive. That is why, for a given value of z, he prefers decreasing hydroelectricity production too in order to keep water stock for the future. This shift of the consumption flow when the price of the second energy source increases expresses a precautionary behavior of the social planner. Note that as p decreases, $\partial c/\partial z$ decreases: the existence of thermal power at a low price improves intertemporal diversification even when the fossil source is not consumed (for values of z such that $e^*(p) - Rz = 0$).

Table 1 presents the proportion of thermal power and water in the total amount of electricity consumed for different values for p. These values have been obtained by simulating the random variable $\widetilde{y}$ 10000 times: one obtains a path for the water stock for 10000 periods and consequently both consumption flows. Then, we compute the proportion of electricity from water and from thermal power for those 10000 periods.

p	electricity (from water)	electricity (from thermal power)
0.1	98.51%	1.49%
0.075	97.73%	2.27%
0.05	95.89%	4.11%
0.025	92.42%	7.58%
0.01	86.77%	13.23%

Table 1: Proportion of hydroelectricity and thermal power in the total consumption for different values for p

As p decreases, the proportion of thermal power increases and the proportion of hydroelectricity decreases. This happens in an exponential way. This result completes the result on the precautionary behavior of the social planner developed at the beginning of the subsection. When p increases, although, for a given quantity z of available water, the social planner reduces hydroelectricity production to keep water stock for the future, the share of hydroelectricity production relative to thermal power production increases.

4.4 Comparative static relative to uncertainty

We study the impact of an increase in uncertainty of the water inflow on electricity production. We consider three mean preserving spreads of random water inflows: where there is no uncertainty $\left(\widetilde{y} = \left\{2, 2, 2; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}\right)$, when there is a low level of uncertainty $\left(\widetilde{y} = \left\{1, 2, 3; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}\right)$ and when there is a higher level of uncertainty $\left(\widetilde{y} = \left\{0, 2, 4; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}\right)$. In Figure 9, we zoom on the low levels of available water.

According to Leland [20], an agent is prudent (in the sense where he consumes less today and saves more) if and only if the marginal utility of future consumption is convex. With the Bellman formulation, the maximization problem reads $\max_c \widehat{u}(c) + \beta \mathbb{E} v(z - c + \widetilde{y})$. As the maximization

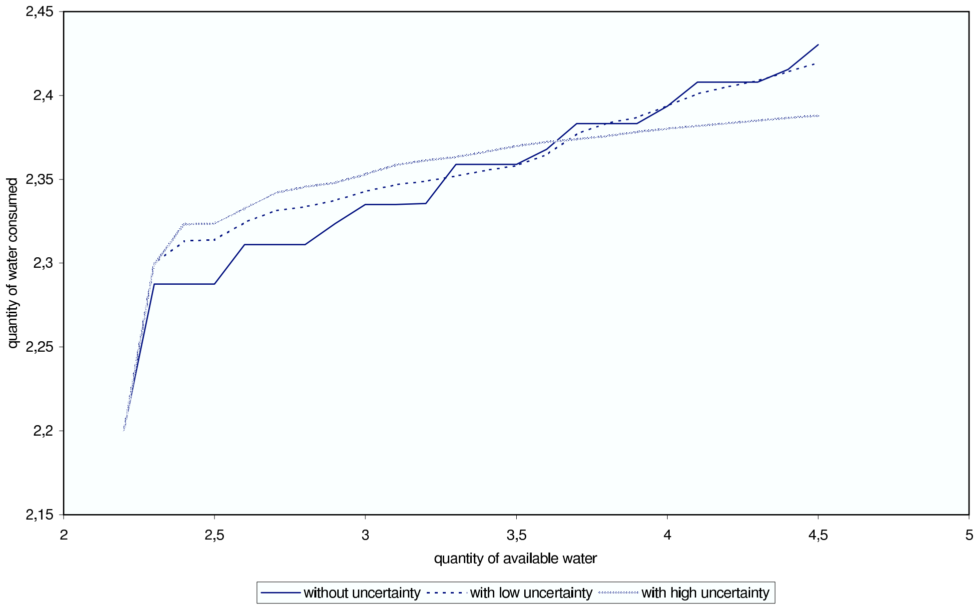

Figure 9: Water consumption for different levels of uncertainty when the dam is almost empty

operator preserves prudence[6], in this case an agent is prudent if and only if $\widehat{u}'$ is convex. Recall that $\widehat{u}'$ that is represented on Figure 10 is equal to

$$\widehat{u}'(Rc;p) = \begin{cases} Rp & \text{if } Rc \leq e^*(p), \\ Ru'(Rc) < Rp & \text{if } Rc \geq e^*(p). \end{cases}$$

Function $\widehat{u}'$ is neither concave nor convex. Therefore, no general conclusion can be drawn about the effect of uncertainty on the optimal extraction. However, it is locally concave for values of c close to $\frac{e^*}{R}$ and is convex for higher values of c.

As we focus on the effect of an increase in uncertainty at date $t+1$ on consumption at date t, we look at values of z satisfying the following equation: $c(z - c(z) + \mu) = \frac{e^*}{R}$. The solution to this equation is $z = 2.55$. For values of z around 2.55, the marginal utility of future consumption is concave, therefore consumption is larger under uncertainty than under certainty. Intuitively, for values of z around the kink of the marginal utility function, if there is an increase in uncertainty, two cases occur:

- either the random flow is very low. In this case, the marginal utility is constant and electricity consumption is equal to the minimum demand $e^*(p)$,
- or the random flow is very high and thus consumption is increased.

The presence of the alternative energy source makes the social planner less prudent for this state of the system. But this increase in consumption does not hold any more for values of z higher than this threshold because, in this case, the marginal utility of the future consumption is convex. Therefore, for a given quantity of available water z, consumption is lower under

[6] See Gollier [12], chapter 14.

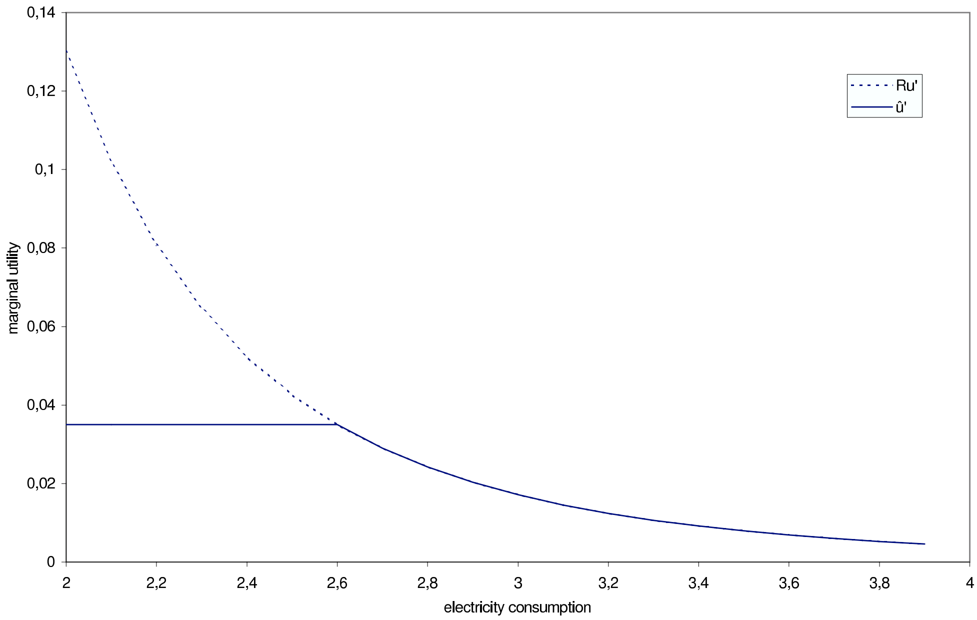

Figure 10: The marginal value of the indirect utility function $\widehat{u}^{\prime}$

uncertainty. And for values of z lower than this threshold, consumption in both cases are equal since the marginal utility of future consumption is linear and all the water is consumed (at least in the certain case). Let us just note that this result on the prudence of the social planer is close to the one obtained by Hoel [15] when he studied the optimal exhaustible resource extraction when the future substitute has an uncertain cost. Indeed, he found that if the marginal utility of future consumption is concave, the resource extraction will be increased by a mean cost preserving increase in uncertainty.

We now turn to the determination of the optimal size of an electric park that generates power with two kinds of power plants: a hydroelectric power plant and a thermal power plant.

5 What is the optimal infrastructure?

With the previous section, we are able to compute the value the social planner gets from the use of the dam and from the production of thermal power. The next step is to compute the optimal number of dams[7]. Let us first introduce the number of dams as a parameter in program (14)

$$v(z;\alpha,p) = \max_{c,x}\left\{u(Rc+x) - px + \beta\mathbb{E}v(z-c+\alpha\widetilde{y};\alpha,p)\right\} \tag{18}$$

subject to

$$c \leq z, \tag{19}$$

$$c \geq z-\alpha\overline{Z}, \tag{20}$$

[7]We choose to focus on the optimal number of dams although other interpretations would have been possible (optimal size of the dams...).

$$x \geq 0. \tag{21}$$

Variable α stands for the number of dams. If α dams are used, the total flow of water that fills then amounts to $\alpha\widetilde{y}$ and the total capacity is equal to $\alpha\overline{Z}$. $\widetilde{y}$ and $\overline{Z}$ are exogenous parameters. We slightly modify the values taken until now: we choose $\widetilde{y} = \{0.25, 0.75, 1.25\}$ and $\overline{Z} = 2.5$. Thus, we recover the former values for $\alpha = 4$. As α increases, two opposite effects appear:

- as α increases, it is as if the size of the dam increased. If there were no increase in the flow of water at the same time, this would tend to decrease the consumption flow. Indeed, with a larger dam, the social planner can smooth consumption in a more efficient way and therefore is tempted to decrease total production today and to store more water for the future.

- as α increases, the quantity of rainfalls that fills the dams increases also. If this increase was taken independently without considering the increases in the size of the dam, it would clearly increase total production. Indeed, the social planner knows that at each period, there will be more water.

The second effect dominates for value of z comprised between 0 and $\alpha\overline{Z}$, therefore $c(z, \alpha)$ increases as α increases $\forall z \in \left[0, \alpha\overline{Z}\right]$. Once $v(z; \alpha, p)$ has been obtained, the social planner is going to focus on the utility extracted from α empty dams $v(0; \alpha, p)$.

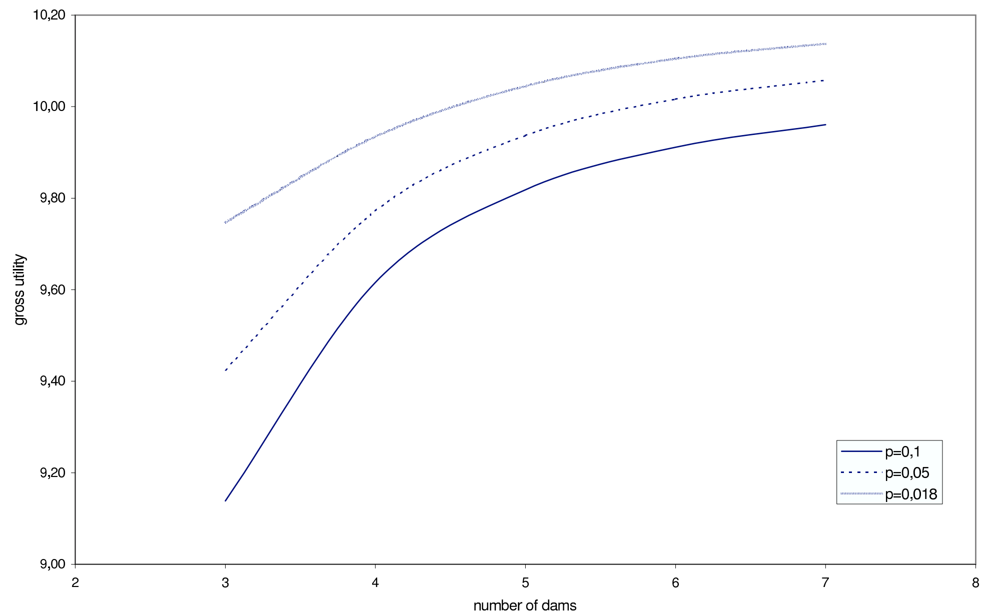

Figure 11: Utility retired from empty dams: $v(0; \alpha, p)$ for different thermal power prices

$v(0; \alpha, p)$ is an increasing function of α and it is straightforward to prove that $v(0; \alpha, p)$ is a decreasing function of p (see Figure 11). In order to obtain the optimal number of dams, the social planner maximizes the utility he gets from α empty dams, from which he subtracts the cost of building α dams. This maximization program reads

$$\max_{\alpha} v(0;\alpha,p) - C(\alpha) \tag{22}$$

We choose a linear cost function: $C(\alpha) = k\alpha + K$ where K is a constant. The numerical resolution whose results are reported on Table 2 reveals that $\alpha^*(p)$ is an increasing function of p.

p	number of dams
0.1	5
0.075	4
0.05	4
0.025	3
0.018	3

Table 2: Optimal number of dams for different values for p

When thermal power price increases, it is optimal to build more dams. Indeed when p increases, the social planner uses less thermal power. In order to smooth electricity consumption, the second way of smoothing (the dam) will be more used.

6 Extensions

In this part, we suppose that a third energy source is available, wind energy for instance. We focus on hydro power and wind power. Indeed, we know from section 2 that introducing thermal power will be equivalent to using utility function $\widehat{u}$ instead of utility function u.

Suppose the social planner can provide hydroelectricity and wind electricity. The dam is supplied with a random inflow. Wind energy is also random. Two cases occur: either there is no wind, in which case no wind electricity can be consumed (which happens with probability q), or there is wind and the quantity of wind electricity amounts to $\overline{W}$ (which happens with probability $(1-q)$). Let c_0 (respectively c_1) be the water consumption flow when wind electricity is available (respectively not available) and $v_0(z)$ (respectively $v_1(z)$) be the value function when wind electricity is available (respectively not available). The social planner's program is then the following

$$\begin{cases} v_0(z) = \max_{c_0} u(Rc_0) + \beta\left[q\mathbb{E}v_0(z - c_0 + \widetilde{y}) + (1-q)\mathbb{E}v_1(z - c_0 + \widetilde{y})\right] \\ \text{subject to } c_0 \le z, c_0 \ge 0, c_0 \ge z - \overline{Z}, \\ v_1(z) = \max_{c_1} u\left(Rc_1 + \overline{W}\right) + \beta\left[q\mathbb{E}v_0(z - c_1 + \widetilde{y}) + (1-q)\mathbb{E}v_1(z - c_1 + \widetilde{y})\right] \\ \text{subject to } c_1 \le z, c_1 \ge 0, c_1 \ge z - \overline{Z}. \end{cases}$$

Let us denote λ_0, μ_0 and ν_0 (resp. λ_1, μ_1 and ν_1) the three Lagrange multipliers associated with the constraints defining c_0 (resp. c_1). We first have some results on the shape of the consumption flows.

Lemma 3 *Consumption flows $c_0(z)$ and $c_1(z)$ have the following properties:*

1. *$\forall z > 0, c_0(z) > 0$,*

2. *if $c_1(z) = 0$, then $Rc_0(z) < \overline{W}$,*

3. *if $c_1(z) = z$, then $c_0(z) = z$. The opposite is not true,*

4. *if $c_0(z) = z - \overline{Z}$, then $c_1(z) = z - \overline{Z}$. Once more, the opposite is not true.*

Proof: See the Appendix. □

As $\forall z > 0, c_0(z) > 0$, the constraint $c_0(z) \geq 0$ can be omitted. When there is no wind, as soon as there is a strictly positive quantity of available water, it is consumed. Unlike the case where wind power was not available, $c_1(z)$ might be equal to zero even when the quantity of available water is strictly positive. Indeed, when there is wind power, the social planner is tempted to keep water in stock for the future in case wind power and/or precipitation are low the following time period. Thus, when z is very low, he prefers not producing hydropower. However this only happens for some quantity of available water z such that if there were no wind, the total quantity of hydroelectricity produced would be less than than the quantity of wind electricity $\overline{W}$. This means that, when the quantity of available water is high enough, water is consumed whatever the quantity of wind power. The last two results represent a first step completed with the following proposition. They allow a preliminary ranking of both consumption flows c_0 and c_1 when one of the constraints is binding.

Proposition 2 *For any level of available water in the dam z:*

1. *hydroelectricity consumption is higher when there is no wind: $c_1(z) \leq c_0(z)$,*

2. *electricity consumption is higher when there is wind: $Rc_0(z) \leq Rc_1(z) + \overline{W}$.*

Proof: See the Appendix. □

When wind power is available, the social planner prefers saving water and taking advantage of wind power. However, when looking at the total electricity consumption, it is higher when there is wind power. Let us now introduce an auxiliary problem: the social planner's problem when a quantity W of wind power is available with certainty at each period.

$$v(z; W) = \max_c \{u(Rc + W) + \beta \mathbb{E} v(z - c + \widetilde{y}; W)\} \tag{23}$$

subject to

$$c \leq z \tag{24}$$

$$c \geq z - \overline{Z} \tag{25}$$

Let us denote $c(z; W)$ the solution to this program. The consumption flow $c(z; W)$ is represented for different values of W on Figure 12.

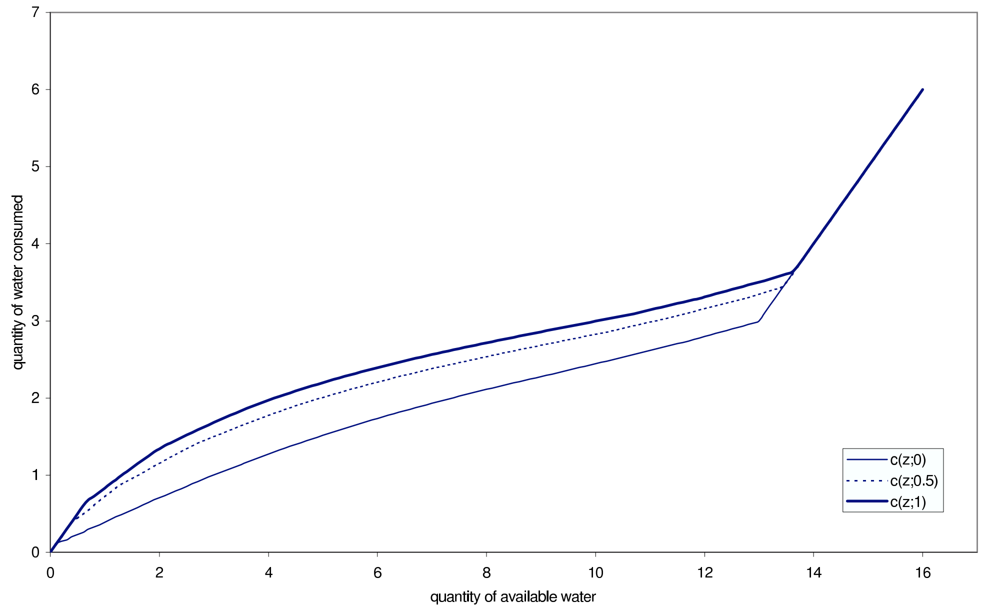

Figure 12: Consumption flows $c(z;W)$ for different values of W

We observe that as the quantity of available wind power increases, the consumption of hydro power also increases. The social planner adopts a precautionary behavior. Knowing that less wind power is available to him, he prefers cutting down today on hydro power production. But contrary to the case where hydroelectricity and thermal power were available in section 4, here the introduction of wind power does not modify a lot the marginal propensity to consume $\partial c/\partial z$: the slope does not significantly decrease when W increases. Note that when $\overline{W} = 0$, we have $c_0(z) = c_1(z) = c(z;0)$. The three consumption flows $c_0(z)$, $c_1(z)$ and $c(z;0)$ are represented on Figure 13.

We observe that $\forall z, c_1(z) \leq c(z;0) \leq c_0(z)$. It is difficult to compare the consumption flows that correspond to the case where there is uncertainty on the quantity of available wind power and the consumption flow when a certain quantity of wind power is available at each period. The introduction of this third energy source that is random but non-storable increases electricity consumption but does not produce a visible effect on time diversification.

7 Concluding remarks

We study the optimal allocation between different energy sources that are uncertain. When only hydroelectricity is available, the central objective of the management of the water resource is to limit the volatility of electricity consumption coming from the uncertain precipitation. Dams should thus be used as a buffer stock. The main aim of the first model was to determine the optimal strategy of water extraction. We showed that the optimal extraction strategy is a function of the quantity z of water that is available to the consumer. It is characterized by two thresholds z^* and z^{**}. When the water stock is smaller than z^*, the social planner is rationed by

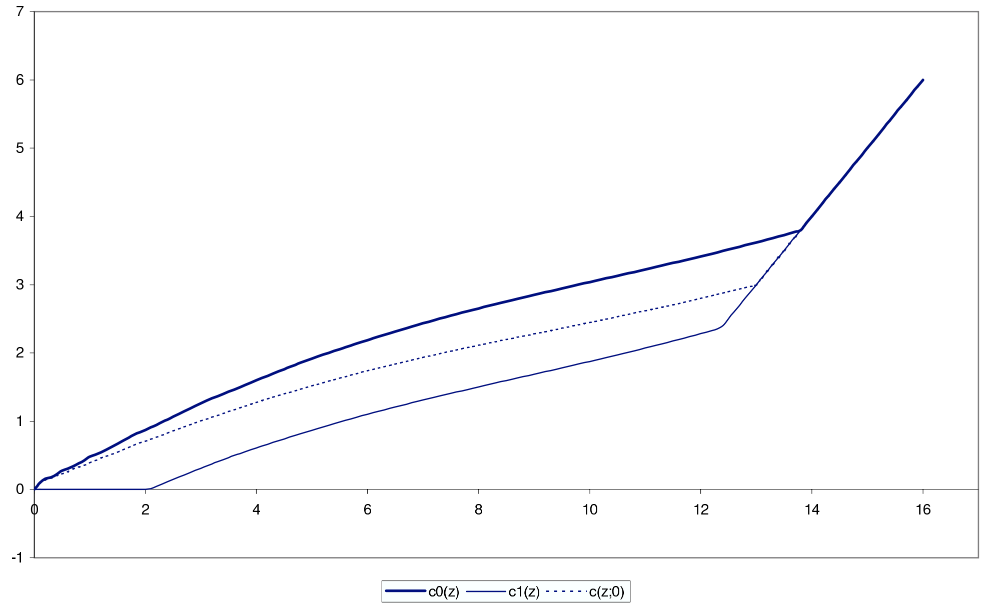

Figure 13: Consumption flows $c_0(z)$, $c_1(z)$ and $c(z;0)$

the limited amount of water in the dam. If the water stock is greater than z^{**}, the social planer immediately consumes the surplus that comes from the random water flow that fills the dam in order not to waste water. If the water stock is in the interval $[z^*, z^{**}]$, the marginal propensity to consume, $\partial c/\partial z$, is positive and strictly less than 1, expressing the intertemporal smoothing of hydroelectricity consumption. The social planner prefers storing water in the dam in order to face a potential unfavorable future in the case of low levels of precipitation.

The introduction of a second energy source improves intertemporal smoothing. When the price of this alternative energy source decreases, the marginal propensity to consume electricity decreases, illustrating an improvement of the time diversification effect. This result even holds on ranges of the water stock for which hydroelectricity is the only energy source that is consumed. Indeed, with two energy sources, there exists a minimum level of electricity that is produced at each period and thermal power is used to reach this level when there is not enough water. But as soon as the quantity of water is sufficient to touch or to exceed this level, the social planner does not use the costly energy source any more and prefers consuming water exclusively. Moreover, the presence of thermal power shifts up the consumption flow. Indeed, when the price of thermal power increases, the social planner adopts a precautionary behavior. He prefers producing less to constitute a greater stock for the future.

Concerning the optimal capacity of the total infrastructure, it is increasing with the thermal power price. The less thermal power is produced (what occurs when p is high), the more dams are needed to smooth consumption.

The introduction of a second uncertain energy source that is not storable allows to increase hydroelectricity consumption for a given quantity of available water. However, water consumption smoothing is not significantly increased since the new energy source is uncertain. It is the certain availability of thermal energy that allows to improve time diversification.

References

[1] Ayong Le Kama, A., (2004), La consommation de ressources environnementales en incertitude, *Recherches Économiques de Louvain*, 70(3), 255-286.

[2] Carroll, C.D., (1997), Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis, *The Quaterly Journal of Economics*, 112, 1-55.

[3] Carroll, D. C., M. S. Kimball, (1996), On the Concavity of the Consumption Function, *Econometrica*, 64 (4), 981-992.

[4] Crampes, C., M. Moreaux, (2001), Water Ressource and Power Generation, *International Journal of Industrial Organization*, 19, 975-997.

[5] Deaton, A., (1991), Saving and Liquidity Constraints, *Econometrica*, 59, 1221-1248.

[6] Deaton, A., G. Laroque, (1992), On the Behaviour of Commodity Prices, *Review of Economic Studies*, 59, 1-23.

[7] Deaton, A., G. Laroque, (1996), Competitive Storage and Commodity Price Dynamics, *Journal of Political Economy*, 104, 896-923.

[8] Edwards, B. K., (2003), *The Economics of Hydroelectric Power*, Edward Elgar.

[9] Garcia, A., J. D. Reitzes, E. Stacchetti, (2001), Strategic Pricing when Electricity is Storable, *Journal of Regulatory Economics*, 20, 223-247.

[10] Garcia, A., E. Campos, J. D. Reitzes, (2002), Dynamic Pricing and Learning in Electricity Markets, Working Paper.

[11] Gaudet, G., P. Howitt, (1989), A Note on Uncertainty and the Hotelling Rule, *Journal of Environmental Economics and Management*, 16, 80-86.

[12] Gollier, C., (2001), *The Economics of Risk and Time*, MIT Press, Cambridge USA.

[13] Gollier, C., (2002), Time Diversification, Liquidity Constraints and Decreasing Aversion to Risk on Wealth, *Journal of Monetary Economics*, 49, 1439-1459.

[14] Heal, G. M., (1993), The Optimal Use of Exhaustible Resources, *Handbook of Natural Resource and Energy Economics*, Volume 3, Edited by A. V. Kneese and J. L. Sweeney.

[15] Hoel, M., (1978), Resource Extraction when a Future Substitute has an Uncertain Cost, *Review of Economic Studies*, 45 (3), 637-644.

[16] Hoel, M., (1978), Resource Extraction, Uncertainty and Learning, *The Bell Journal of Economics*, 9(2), 642- 645.

[17] Hotelling, H., (1931), The Economics of Exhaustible Resources, *The Journal of Political Economy*, 39(2), 137-175.

[18] Kimball, M. S., (1990), Precautionary Saving in the Small and in the Large, *Econometrica*, 58, 53-73.

[19] Koopmans, T. C., (1957), Water storage in a simplified hydroelectric system, *Proceedings of the First International Conference on Operational Research*, Oxford, September, 1-35.

[20] Leland, H. E., (1968), Saving and Uncertainty: The Precautionary Demand for Saving, *Quarterly Journal of Economics*, 82, 465-473.

[21] Little, D. C. E., (1955), The Use of Storage Water in a Hydroelectric System, *Journal of the Operations Research Society of America*, 3, 187-197.

[22] Ministry of Petroleum and Energy, (2004), Facts 2004: The Energy Sector and Water Resources in Norway.

[23] Nordhaus, W., (1979), *The Efficient Use of Energy Sources*, Cowles Foundation Monograph (Yale University Press, New Haven, CT).

[24] Söderholm, P., (2001), Fossil Fuel Flexibility in West European Power Generation and the Impact of System Load Factors, *Energy Economics*, 23, 77-97.

[25] Stockey, N. L., R. E. Lucas, (1984), *Recursive Methods in Economic Dynamics*, Harvard University Press.

[26] Williams, J. C., B. Wright, (1982), The Economic Role of Storage Commodity, *The Economic Journal*, 92, 596-614.

[27] Williams, J. C., B. Wright, (1984), The Welfare Effects of the Introduction of Storage, *The Quarterly Journal of Economics*, 99, 169-192.

[28] Wright, B., (2001), Storage and Price Stabilization, *Handbook of Agricultural Economics*, Volume 1, Edited by B. Gardner and G. Rausser.

[29] Zeldes, S. P., (1989), Consumption and Liquidity Constraints: An Empirical Investigation, *Journal of Political Economy*, 97, 305-346.

A Computation of the optimal consumption flows in the certain case (Section 2)

We present the computations concerning the cases 1 and 2.

Case 1:

Denoting by T the last time period for which no thermal power is consumed, the program of the social planner reads

$$\max_{c(t)} \sum_{t=0}^{T} \beta^t u\left(Rc\left(t\right)\right) + \beta^{T+1}\left[u\left(e^*\right) - p\left(e^* - Rc\left(T+1\right)\right)\right] - \sum_{t=T+2}^{+\infty} \beta^t \left(u\left(e^*\right)\right) - p\left(e^* - R\mu\right))$$

subject to

$$\begin{aligned} z_{t+1} &= z_t - c_t + \mu, \\ c\left(T\right) &> \frac{e^*}{R}, \\ c\left(T+1\right) &= z_0 + \left(T+1\right)\mu - \sum_{t=0}^{t=T} c_t, \\ c\left(T+1\right) &\leq \frac{e^*}{R}, \\ c\left(T+1\right) &\geq \mu, \\ & z_0 \text{ given.} \end{aligned}$$

The first constraint is the usual one on the dynamics of available water, the second inequality means that at time T, no thermal power is consumed yet. The third equation means that at time $T+1$, all the available water increased by the flow of water is consumed. Indeed, if it was not the case, water would remain in the dam in period $T+1$. Therefore no thermal power would be consumed in this time period what would not be consistent with the definition of T. The fourth inequality means that at time $T+1$, thermal power is consumed and the last equality ensures that until T, no more water than the quantity stored in the dam has been consumed.

The resolution of this program leads to the following results:

$$c\left(t\right) = \begin{cases} \beta^{-\frac{T-t}{\gamma}} c\left(T\right) & \text{if } t \leq T \\ z_0 + \left(T+1\right)\mu - c\left(T\right)\beta^{-\frac{T}{\gamma}} \frac{1-\beta^{\frac{T+1}{\gamma}}}{1-\beta^{\frac{1}{\gamma}}} & \text{if } t = T+1 \\ \mu & \text{if } t \geq T+2 \end{cases}$$

where T is the lowest integer t [8] such that:

$$\frac{z_0 + \left(t+1\right)\mu}{1 + \beta^{-\frac{t+1}{\gamma}} \frac{1-\beta^{\frac{t+1}{\gamma}}}{1-\beta^{\frac{1}{\gamma}}}} < \frac{e^*}{R}.$$

[8] Indeed, $t \mapsto f\left(t\right) = \frac{z_0 + \left(t+1\right)\mu}{1 + \beta^{-\frac{t+1}{\gamma}} \frac{1-\beta^{\frac{t+1}{\gamma}}}{1-\beta^{\frac{1}{\gamma}}}}$ is a decreasing function.

and where $c(T)$ is given by the following expression:

$$c(T) = \begin{cases} \frac{e^*}{R}\beta^{-\frac{1}{\gamma}} & \text{if } z_0 + (T+1)\mu - \frac{e^*}{R}\frac{\beta^{-\frac{T+1}{\gamma}}-1}{1-\beta^{\frac{1}{\gamma}}} < \frac{e^*}{R} \\ \frac{(z_0+T\mu)\left(1-\beta^{\frac{1}{\gamma}}\right)}{\beta^{-\frac{T}{\gamma}}-\beta^{\frac{1}{\gamma}}} & \text{else.} \end{cases}$$

Case 2:

We now give the solution of the problem in the third case when $\mu > \frac{e^*}{R}$, that is to say when thermal power is never consumed:

$$c(t) = \begin{cases} \beta^{-\frac{T-t}{\gamma}} c(T) & \text{if } t \leq T \\ \mu & \text{if } t \geq T+1 \end{cases}$$

where T is the highest integer t [9] such that:

$$\frac{(z_0 + t\mu)\left(1-\beta^{\frac{1}{\gamma}}\right)}{\beta^{-\frac{t}{\gamma}} - \beta^{\frac{1}{\gamma}}} > \mu.$$

and where $c(T)$ is given by the following expression:

$$c(T) = \frac{(z_0 + T\mu)\left(1-\beta^{\frac{1}{\gamma}}\right)}{\beta^{-\frac{T}{\gamma}} - \beta^{\frac{1}{\gamma}}}.$$

B Computation of the value function v in the certain case (Section 2)

We begin with the most interesting case when $\frac{e^*}{R} > \mu$.

The water consumption path has the following expression (as we already noted in section 2, this amounts to studying c_0 with respect to z_0:

$$c(z) = \begin{cases} \max\left(\beta^{-\frac{T}{\gamma}} c(T), z - \overline{Z}\right) & \text{if } z > \frac{e^*}{R}\beta^{-\frac{1}{\gamma}} \\ z & \text{else} \end{cases}$$

The shape of the value function depends on the initial quantity of water. Three cases have to be distinguished:

1. $z < \frac{e^*}{R}$

   In this case, in the first period, all the water is consumed and thermal power is consumed

[9] Indeed, $t \mapsto g(t) = \frac{(z_0 + t\mu)\left(1-\beta^{\frac{1}{\gamma}}\right)}{\beta^{-\frac{t}{\gamma}} - \beta^{\frac{1}{\gamma}}}$ is a decreasing function.

in order to reach e^*. In the following periods, water is consumed in quantity μ and thermal power in quantity $e^* - R\mu$. Therefore, the value function takes the following form:

$$\begin{aligned} v(z) &= u(e^*) - p(e^* - Rz) + \sum_{t=1}^{+\infty} \beta^t \left(u(e^*) - p(e^* - R\mu)\right) \\ &= \frac{(e^*)^{1-\gamma}}{1-\gamma} - p(e^* - Rz) + \frac{\beta}{1-\beta}\left(\frac{(e^*)^{1-\gamma}}{1-\gamma} - p(e^* - R\mu)\right) \end{aligned}$$

2. $\frac{e^*}{R} < z < \frac{e^*}{R}\beta^{-\frac{1}{\gamma}}$

   The only difference with the previous case is that no thermal power is consumed in the initial period because there is enough water:

$$\begin{aligned} v(z) &= u(Rz) + \sum_{t=1}^{+\infty} \beta^t \left(u(e^*) - p(e^* - R\mu)\right) \\ &= \frac{(Rz)^{1-\gamma}}{1-\gamma} + \frac{\beta}{1-\beta}\left(\frac{(e^*)^{1-\gamma}}{1-\gamma} - p(e^* - R\mu)\right) \end{aligned}$$

3. $z > \frac{e^*}{R}\beta^{-\frac{1}{\gamma}}$

   In this case there are three periods:

   - until period T, only water is consumed
   - in period $T+1$, all the water that remains in the dam is consumed and thermal power is consumed to reach the level e^* of electricity
   - from period $T+2$ on, there is no water in the dam any more. Only the consumption flow μ is consumed, therefore thermal power is consumed in quantity $e^* - R\mu$.

   Therefore, the value function has the following expression:

$$v(z) = \sum_{t=0}^{T} \left(\beta^t u(Rc_T)\right) + \beta^{T+1}\left(u(e^*) - p(e^* - Rc(T+1))\right) + \sum_{t=T+2}^{+\infty} \left(\beta^t \left(u(e^*) - p(e^* - R\mu)\right)\right)$$

   It follows that:

$$\begin{aligned} v(z) &= \frac{(Rc(T))^{1-\gamma}\beta^{-\frac{1-\gamma}{\gamma}T}}{1-\gamma}\frac{1-\beta^{\frac{T+1}{\gamma}}}{1-\beta^{\frac{1}{\gamma}}} + \beta^{T+1}\left(\frac{(e^*)^{1-\gamma}}{1-\gamma} - p(e^* - Rc(T+1))\right) + \\ &+ \left(\frac{((e^*)^{1-\gamma}}{1-\gamma} - p(e^* - R\mu)\right)\frac{\beta^{T+2}}{1-\beta} \end{aligned}$$

For sake of completeness, we now focus on the case where $\frac{e^*}{R} < \mu$.

First of all, in this case:

$$c(z) = \begin{cases} \max\left(\frac{\left(1-\beta^{\frac{1}{\gamma}}\right)(z+T\mu)}{1-\beta^{\frac{T+1}{\gamma}}}, z - \overline{Z}\right) & \text{if } z > \mu \\ z & \text{else} \end{cases}$$

Following the same steps that above, we have to compute the value function for the different value taken by the initial level of water in the dam.

1. $z < \frac{e^*}{R}$

   In this case, thermal power and water are consumed in the initial period, and from the second period on, water is only consumed in quantity μ.

$$\begin{aligned} v(z) &= u(e^*) - p(e^* - Rz) + \sum_{t=1}^{+\infty} \beta^t (u(R\mu)) \\ &= \frac{(e^*)^{1-\gamma}}{1-\gamma} - p(e^* - Rz) + \frac{\beta}{1-\beta}\frac{(R\mu)^{1-\gamma}}{1-\gamma} \end{aligned}$$

2. $\frac{e^*}{R} < z < \mu$

   Here also, the only difference with the previous case is that in the first period, no thermal power is consumed. But, from the second period on, the consumption path is the same.

$$\begin{aligned} v(z) &= u(Rz) + \sum_{t=1}^{+\infty} \beta^t u(R\mu) \\ &= \frac{(Rz)^{1-\gamma}}{1-\gamma} + \frac{\beta}{1-\beta}\frac{(R\mu)^{1-\gamma}}{1-\gamma} \end{aligned}$$

3. $z > \mu$

   In this case, there exists the threshold T that defines the time from which on thermal power is consumed:

$$v(z) = \sum_{t=0}^{T} \left(\beta^t u(Rc_T)\right) + \sum_{t=T+1}^{+\infty} \beta^t u(R\mu)$$

   It follows that:

$$v(z) = \frac{(Rc(T))^{1-\gamma} \beta^{-\frac{1-\gamma}{\gamma}T}}{1-\gamma} \frac{1-\beta^{\frac{T+1}{\gamma}}}{1-\beta^{\frac{1}{\gamma}}} + \frac{\beta^{T+1}}{1-\beta}\frac{((R\mu)^{1-\gamma}}{1-\gamma}$$

C Lemma

Lemma 4 *When $p \to +\infty$, the threshold z^* is higher than the minimum water inflow* $\min \widetilde{y}$.

Proof: Suppose this result does not hold: $\min \widetilde{y} > z^*$. Denoting $\overline{z} = \min \widetilde{y}$ and $\overline{c} = c(\overline{z})$, we have $\overline{c} < \overline{z}$. The FOC leads to

$$\begin{aligned} Ru'(R\overline{c}) &= \beta \mathbb{E} v'(\overline{z} - \overline{c} + \widetilde{y}), \\ &< \beta v'(2\overline{z} - \overline{c}), \\ &< \beta v'(\overline{z}) \text{ (since } \overline{c} < \overline{z}), \\ &< v'(\overline{z}), \\ &= Ru'(R\overline{c}) \text{ (envelope theorem).} \end{aligned}$$

Therefore, $\min \widetilde{y} \leq z^*$. $\square$

D Proofs

Proof of Lemma 1:

We are going to apply the proof of Theorem 9.8 page 265 in Stokey and Lucas [25]. All the assumptions are satisfied:

1. $X = \left[0, \overline{Z} + \max \widetilde{y}\right]$ is a convex subset of $\mathbb{R}$,
2. $\widetilde{y}$ is a discrete random variable that takes a finite number of values: $\widetilde{y} \in \{y_1, y_2, ..., y_n\}$,
3. the correspondence $\Gamma : X \rightarrow X$ describing the feasibility constraints is non empty, compact valued and continuous (see Figure 14),
4. $\widehat{u}(x)$ is bounded and continuous, and $\beta < 1$,

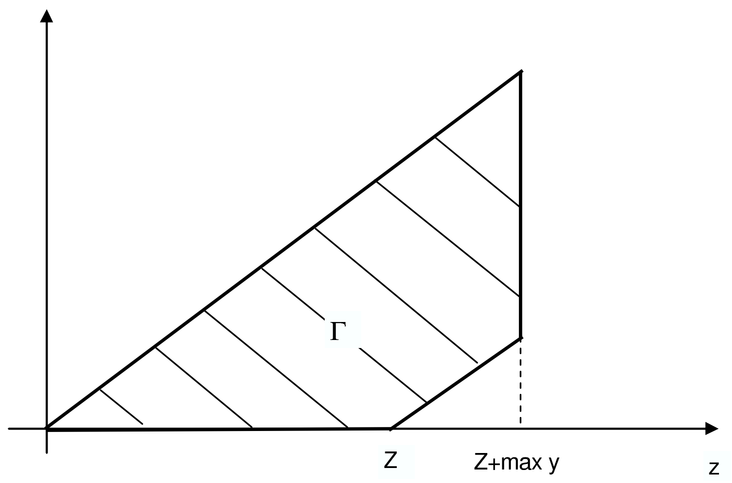

Figure 14: Correspondence Γ

5. u is strictly concave ($\gamma < 1$),
6. Γ is convex (see Figure 14).

Thus, according to Theorem 9.8 p. 265 of Stokey and Lucas [25], function v is strictly concave.

Proof of Lemma 2:

Let us suppose that there exists $z_0 > 0$ such that $c(z_0) = 0$. In this case, constraint (15) does not bind and the FOC reads

$$
\begin{aligned}
R\widehat{u}'(Rc(z_0)) &\leq \beta \mathbb{E} v'(z_0 - c(z_0) + \widetilde{y}), \\
&= \beta \mathbb{E} v'(z_0 + \widetilde{y}), \\
&\leq \beta v'(z_0) \text{ because } v \text{ is concave,}
\end{aligned}
$$

$$
\begin{aligned}
&< v'(z_0) \text{ because } \beta < 1, \\
&= R\widehat{u}'(Rc(z_0)) \text{ because of the envelope theorem.}
\end{aligned}
$$

Therefore there is a contradiction and $c(z_0) > 0$ (and eventually that constraint (15) binds leading to $c(z_0) = z_0 > 0$). □

Proof of Lemma ??:

Consider $p_1 < p_0$ and recall that

$$
v(z_t; p) = \max_{c_t, x_t} \mathbb{E} \sum_{t=0}^{+\infty} \beta^t \left(u(Rc_t + x_t) - px_t \right)
$$

subject to

$$
\begin{aligned}
z_{t+1} &= z_t - c_t + \widetilde{y}_t, \\
c_t &\leq z_t, \\
c_t &\geq z_t - \overline{Z}, \\
x_t &\geq 0.
\end{aligned}
$$

Suppose that we have the optimal policy $c^*(z; p_0)$ and $x^*(z; p_0)$ at price p_0. This allocation is feasible at price p_1. Therefore

$$
\begin{aligned}
v(z, p_0) &\leq \mathbb{E} \sum_{t=0}^{+\infty} \beta^t \left(u(Rc^*(z, p_0) + x^*(z, p_0)) - p_1 x^*(z, p_0) \right), \\
&\leq \mathbb{E} \sum_{t=0}^{+\infty} \beta^t \left(u(Rc^*(z, p_1) + x^*(z, p_1)) - p_1 x^*(z, p_1) \right), \\
&= v(z, p_1). \blacksquare
\end{aligned}
$$

Proof of Lemma 3:

We are going to successively prove the four assertions.

1. Suppose there exists $z_0 > 0$ such that $c_0(z_0) = 0$. This implies that $\mu_0 \geq 0$. The FOC of the maximization program leads to

$$
\begin{aligned}
Ru'(Rc_0) &= \beta \left[q\mathbb{E}v_0'(z_0 - c_0 + \widetilde{y}) + (1-q)\mathbb{E}v_1'(z_0 - c_0 + \widetilde{y}) \right] - \mu_0, \\
&\leq \beta \left[q\mathbb{E}v_0'(z_0 - c_0 + \widetilde{y}) + (1-q)\mathbb{E}v_1'(z_0 - c_0 + \widetilde{y}) \right], \\
&< qv_0'(z_0) + (1-q)v_1'(z_0) \text{ because } v \text{ is concave}, \\
&= qRu'(c_0) + (1-q)Ru'\left(Rc_1 + \overline{W}\right) \text{ because of the envelope theorem.}
\end{aligned}
$$

This leads to $Rc_0 = 0 > Rc_1 + \overline{W}$ what is not possible since $c_1 \geq 0$. Therefore, the constraint $c_0(z) \geq 0$ never binds for strictly positive z and $\mu_0 = 0$.

2. Suppose z_0 is strictly positive and $c_1(z_0)$=0. The FOC of the maximization program gives

$$\begin{aligned} Ru'\left(Rc_1+\overline{W}\right) &< q\mathbb{E}v_0'\left(z_0+\widetilde{y}\right)+\left(1-q\right)\mathbb{E}v_1'\left(z_0+\widetilde{y}\right), \\ &< qRu'\left(Rc_0\right)+\left(1-q\right)Ru'\left(Rc_1+\overline{W}\right). \end{aligned}$$

This leads to $Rc_0(z_0) < \overline{W}$.

3. We suppose $c_1(z) = z$ and $c_0(z) < z$.

$c_1(z) = z$ implies that $Ru'\left(Rz+\overline{W}\right) = \beta\left[q\mathbb{E}v_0'\left(\widetilde{y}\right)+\left(1-q\right)\mathbb{E}v_1'\left(\widetilde{y}\right)\right]+\lambda_1$. $c_0(z) < z$ implies that

$$\begin{aligned} Ru'\left(Rc_0\right) &= \beta\left[q\mathbb{E}v_0'\left(z-c_0+\widetilde{y}\right)+\left(1-q\right)\mathbb{E}v_1'\left(z-c_0+\widetilde{y}\right)\right], \\ &\leq \beta\left[q\mathbb{E}v_0'\left(\widetilde{y}\right)+\left(1-q\right)\mathbb{E}v_1'\left(\widetilde{y}\right)\right], \\ &= Ru'\left(Rz+\overline{W}\right)-\lambda_1, \\ &\leq Ru'\left(Rz+\overline{W}\right). \end{aligned}$$

The concavity of function u implies that $Rc_0 \geq Rz+\overline{W}$, what cannot happen. Therefore there is a contradiction and $c_0(z) = z$.

4. As for the previous result, let us suppose that $c_0(z) = z-\overline{Z}$ and $c_1(z) > z-\overline{Z}$. The equality gives $Ru'\left(Rz-R\overline{Z}\right) = \beta\left[q\mathbb{E}v_0'\left(\overline{Z}+\widetilde{y}\right)+\left(1-q\right)v_1'\left(\overline{Z}+\widetilde{y}\right)\right]-\nu_0$. The inequality implies that

$$\begin{aligned} Ru'\left(Rc_1+\overline{W}\right) &= \beta\left[q\mathbb{E}v_0'\left(z-c_1+\widetilde{y}\right)+\left(1-q\right)v_1'\left(z-c_1+\widetilde{y}\right)\right], \\ &> \beta\left[q\mathbb{E}v_0'\left(\overline{Z}+\widetilde{y}\right)+\left(1-q\right)v_1'\left(\overline{Z}+\widetilde{y}\right)\right], \\ &= Ru'\left(Rz-R\overline{Z}\right)+\nu_0, \\ &> Ru'\left(Rz-R\overline{Z}\right). \end{aligned}$$

This leads to $\overline{W} < 0$, a contradiction. Therefore $c_1(z) = z-\overline{Z}$. □

Proof of Proposition 2:

Concerning the first result, suppose by contradiction there exists z_0 such that $c_0(z_0) < c_1(z_0)$. When neither constraint is binding and by the concavity of u, this implies that: $Ru'\left(Rc_0(z_0)\right) > Ru'\left(Rc_1(z_0)+\overline{W}\right)$. Depending on the first order conditions, we have then:

$$q\left[\mathbb{E}v_0'\left(z_0-c_0(z_0)+\widetilde{y}\right)-\mathbb{E}v_0'\left(z_0-c_1(z_0)+\widetilde{y}\right)\right] > \left(1-q\right)\left[\mathbb{E}v_1'\left(z_0-c_1(z_0)+\widetilde{y}\right)-\mathbb{E}v_1'\left(z_0-c_0(z_0)+\widetilde{y}\right)\right].$$

As $c_0(z_0) < c_1(z_0)$ and as $z \mapsto v'(z;0)$ is a decreasing function, the left hand side is strictly negative. Similarly, the right hand side is strictly positive and we have a contradiction.

When one of the constraint is binding, we know according to the results of Lemma 3 that $c_1(z) \leq c_0(z)$.

Concerning the second point, suppose there exists z_1 such that $Rc_0(z_1) > Rc_1(z_1) + \overline{W}$. When neither constraint is binding, this implies that: $Ru'(Rc_1(z_1) + W) > Ru'(Rc_0(z_1))$, and therefore:
$(1-q)[\mathbb{E}v_0'(z_1 - c_1(z_1) + \widetilde{y}) - \mathbb{E}v_0'(z_1 - c_0(z_1) + \widetilde{y})] > q[\mathbb{E}v_1'(z_1 - c_0(z_1) + \widetilde{y}) - \mathbb{E}v_1'(z_1 - c_1(z_1) + \widetilde{y})]$.
Once more, the assumptions imply that the left hand side is strictly negative and the right hand side strictly positive, we have a contradiction.

We consider rapidly the cases where one of the constraints is binding:

- If $c_1(z) = 0$, then according to the previous lemma, $Rc_0(z) < \overline{W}$.
- If $c_0(z) = z$ and $c_1(z) < z$, then the FOC lead to

$$\begin{aligned} Ru'\left(Rc_1 + \overline{W}\right) &< \beta\left[q\mathbb{E}v_0'(\widetilde{y}) + (1-q)\,\mathbb{E}v_1'(\widetilde{y})\right] \\ &= Ru'(Rc_0) - \lambda_0 \\ &< Ru'(Rc_0) \end{aligned}$$

  Therefore, $Rc_1 + \overline{W} > Rc_0$.
- If $c_1(z) = z - \overline{Z}$ and $c_0(z) > z - \overline{Z}$, a similar reasoning leads to the result. □