ECOMOR and LCR Reinsurance with Gamma-like Claims

Enkelejd Hashorva* and Jinzhu Li†

May 15, 2013

Abstract

Assuming that the claim sizes of an insurance company have a common distribution with gamma-like tail, we study the asymptotic tail behaviour of the reinsured amounts under ECOMOR and LCR reinsurance treaties, respectively. Our novel results include a precise asymptotic expansion for the tail probability of the reinsured amounts under the ECOMOR treaty, and tight asymptotic bounds for the LCR case. As a by-product we derive a precise asymptotic expansion for the tail of the product of regularly varying random variables.

Keywords: Asymptotics; Gamma-tail distributions; Reinsurance; LCR and ECOMOR treaties; Tail probabilities

Mathematics Subject Classification: Primary 62P05; Secondary 62E10, 91B30

1 Introduction

In our framework the claim sizes of an insurance portfolio, $X_k,\ k = 1, 2, \ldots$, form a sequence of positive independent and identically distributed (iid) random variables, and arrive according to a counting process $\{N(t); t \geq 0\}$. We assume hereafter that $\{X_k; k = 1, 2, \ldots\}$ and $\{N(t); t \geq 0\}$ are mutually independent. Denote by $X_{1,N(t)} \geq \cdots \geq X_{N(t),N(t)}$ the order statistics of the claims occurring up to time t. Then the total loss amounts covered by LCR (largest claims reinsurance) and ECOMOR (excédent du coût moyen relatif) reinsurance treaties up to time $t > 0$ are, respectively,

$$L_l(t) = \sum_{i=1}^{l} X_{i,N(t)} \mathbf{1}_{\{N(t) \geq l\}}, \quad l \geq 1, \tag{1.1}$$

and

$$E_l(t) = \sum_{i=1}^{l-1} \left(X_{i,N(t)} - X_{l,N(t)}\right) \mathbf{1}_{\{N(t) \geq l\}}, \quad l \geq 2, \tag{1.2}$$

where $\mathbf{1}_{\{.\}}$ is the indicator function. In both treaties the first l largest claims are important; for LCR the reinsurer pays the sum of the first l largest claims reported until time t, and for ECOMOR the reinsurer covers the first $l-1$ largest claims in excess of the lth largest one.

The ECOMOR treaty was considered in theoretical actuarial models first by Thépaut (1950), whereas Ammeter (1964) pioneered discussions on LCR. In reinsurance practice, both treaties have a limited use and popularity, however for actuarial theory and applied probability, these two models are interesting and challenging. Asymptotic properties of these treaties have been studied by several authors, see e.g., Ladoucette and Teugels (2006) and Hashorva (2007) where the case that $N(t) \to \infty$ as $t \to \infty$ was analysed using extreme value theory. Asimit

*Department of Actuarial Science, Faculty of Business and Economics, University of Lausanne, UNIL-Dorigny 1015 Lausanne, Switzerland

†School of Mathematical Science and LPMC, Nankai University, Tianjin 300071, P.R. China. E-mail: lijinzhu@nankai.edu.cn

and Jones (2008) proposed to fix time t and investigate the tail asymptotics of these treaties. Precise results in this direction are derived by Jiang and Tang (2008) for exponential claims and claims with a convolution-equivalent tail, i.e., claims in the class $\mathcal{S}(\gamma)$. The latter paper motivates this contribution, which is concerned with the asymptotic analysis of both treaties with gamma-like claims (see the definition below).

A nice property of iid gamma distributed random variables is that their sum is again gamma distributed. Lemma 7.1 in Rootzén (1986) (see also Rootzén (1987)) showed that if instead independent random variables have distribution functions tail-equivalent to a gamma distribution, then the distribution of their sum is also tail-equivalent to a gamma distribution. A crucial novel result of this contribution (see Lemma 2.1 below) is the extension of Rootzén's result to the larger class of random variables with gamma-like tails. Utilizing the derived closure property under addition for gamma-like random variables, we obtain the precise asymptotic behaviour of the ECOMOR treaty with gamma-like claims. For the LCR treaty which is more complicated to deal with, a precise asymptotic formula is derived when $l = 2$ and some tight asymptotic bounds are given when $l \geq 3$. Moreover, we obtain for the LCR treaty a precise asymptotic formula with closed-form coefficients when the claims follow the gamma distribution with shape parameter greater than 1.

The rest of this paper is organized as follows. After presenting some preliminary results in Section 2, we display our principal findings in Section 3. Section 4 discusses our results and provides some extensions for the weighted ECOMOR treaty. Two auxiliary lemmas and all the proofs are displayed in Section 5.

2 Preliminaries

In this paper, all limit relations hold as x tends to ∞ unless otherwise specified. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim b(x)$ or $b(x) \gtrsim a(x)$ if $\limsup_{x\to\infty} a(x)/b(x) \leq 1$, and write $a(x) \sim b(x)$ if $a(x) \lesssim b(x)$ and $a(x) \gtrsim b(x)$ hold simultaneously.

A distribution F on $(0,\infty)$ is said to have a gamma-like tail with shape parameter $\alpha > 0$ and rate parameter $\gamma > 0$ if there exists a slowly varying function $\ell(\cdot) : (0,\infty) \mapsto (0,\infty)$ such that

$$\overline{F}(x) = 1 - F(x) \sim \ell(x) x^{\alpha-1} \mathrm{e}^{-\gamma x}. \tag{2.1}$$

By the definition, $\ell(\lambda x) \sim \ell(x)$ for all $\lambda > 0$; see Bingham et al. (1987) or Embrechts et al. (1997) for the main properties of slowly varying functions.

Remark 2.1. *Clearly,* (2.1) *can be rewritten in a compact way as* $\overline{F}(x) \sim h(x)\mathrm{e}^{-\gamma x}$*, where* $h(x) = \ell(x)x^{\alpha-1}$ *is a regularly varying function with index* $\alpha - 1$*, i.e.,* $h \in \mathcal{R}_{\alpha-1}$*. This alternative expression of* $\overline{F}$ *will be used in some proofs below to simplify the presentation. However, we do not plug it into our results because it will conceal the gamma-like properties of* $\overline{F}$ *with respect to (w.r.t.) shape parameter* α*.*

A canonical example of the gamma-like distribution with parameters $\alpha, \gamma > 0$ is the gamma distribution with the corresponding parameters, i.e.,

$$\overline{F}(x) = \frac{\gamma^\alpha}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-1} \mathrm{e}^{-\gamma y} \mathrm{d}y, \quad x > 0, \tag{2.2}$$

where $\Gamma(\cdot)$ is the Euler Gamma function. In this case we have

$$\overline{F}(x) \sim \frac{\gamma^{\alpha-1}}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-\gamma x}. \tag{2.3}$$

Note in passing that if $\alpha = 1$, then the gamma distribution coincides with the exponential distribution, for which relation (2.3) holds exactly, i.e.,

$$\overline{F}(x) = \mathrm{e}^{-\gamma x}, \quad \forall x > 0. \tag{2.4}$$

A key interesting finding of this paper is the following result, which will be used in the proofs of several main theorems below. Its proof is postponed to Section 5.

Lemma 2.1. *Let* $X_1,\dots,X_l$ $(l \geq 2)$ *be* l *independent positive random variables with gamma-like tails* $\overline{F_i}(x) \sim \ell_i(x) x^{\alpha_i - 1} \mathrm{e}^{-\gamma x}$ *for some positive constants* γ, α_i *and some slowly varying functions* $\ell_i(x)$, $i = 1,\dots,l$. *Then we have*

$$\mathbb{P}\left(\sum_{i=1}^{l} X_i > x\right) \sim \frac{\gamma^{l-1} \prod_{i=1}^{l} \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{l} \alpha_i\right)} \left(\prod_{i=1}^{l} \ell_i(x)\right) x^{\sum_{i=1}^{l} \alpha_i - 1} \mathrm{e}^{-\gamma x}. \tag{2.5}$$

Lemma 2.1 establishes for gamma-like random variables not only the closure property under addition, but also a gamma-like property w.r.t. shape parameters α_i's. A special case of gamma-like distributions are those satisfying (2.1) with some constant $\ell(x) \equiv c > 0$. For such distributions, the claim of Lemma 2.1 reduces to the crucial result of Lemma 7.1 (iii) in Rootzén (1986).
It turns out that Lemma 2.1 is of particular interest also for the tail asymptotics of the product of independent regularly varying random variables, which is recently studied in Hashorva et al. (2010) and Hashorva (2012). Indeed, as a by-product of the aforementioned lemma we immediately obtain the following result:

Corollary 2.1. *Let* $Y_1,\dots,Y_l$ *be positive independent random variables. If* $\mathbb{P}(Y_i > x) \sim \ell_i(\ln x)(\ln x)^{\alpha_i - 1} x^{-\gamma} \in \mathcal{R}_{-\gamma}$ *with* $\alpha_i, \ell_i(\cdot), i \leq l$, *and* γ *as in Lemma 2.1, then we have*

$$\mathbb{P}\left(\prod_{i=1}^{l} Y_i > x\right) \sim \frac{\gamma^{l-1} \prod_{i=1}^{l} \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{l} \alpha_i\right)} \left(\prod_{i=1}^{l} \ell_i(\ln x)\right) (\ln x)^{\sum_{i=1}^{l} \alpha_i - 1} x^{-\gamma}. \tag{2.6}$$

We remark that if $\ell_i(x) \equiv c^{\gamma}$ and $\alpha_i = 1$ for all $i \leq l$, then (2.6) reduces to

$$\mathbb{P}\left(\prod_{i=1}^{l} Y_i > x\right) \sim \frac{\gamma^{l-1} c^{l\gamma}}{(l-1)!} (\ln x)^{l-1} x^{-\gamma},$$

which is the result of Lemma 4.1(4) in Jessen and Mikosch (2006).

3 Main Results

Denote by $Q_t(z) = \mathbb{E}\left\{z^{N(t)}\right\}$ the probability generating function of $N(t)$. Hereafter t will be fixed and has no particular role. If Q_t is analytic at $z > 0$, then

$$Q_t^{(r)}(z) = \sum_{n=r}^{\infty} \mathbb{P}(N(t) = n) \frac{n!}{(n-r)!} z^{n-r}, \quad r = 1, 2, \dots.$$

As in Jiang and Tang (2008) we shall use the notation $Q_t^{(r)}(1)$ to denote the series at $z = 1$, and do not require that $Q_t^{(r)}(z)$ is analytic at $z = 1$. We assume in the following that $l \geq 2$ since in view of Lemma 1 of Ladoucette and Teugels (2006) the tail behaviour of $L_1(t)$ has been studied for all claim distributions with infinite right tails.

Theorem 3.1. *Consider the ECOMOR treaty defined in (1.2) with* $l \geq 2$ *and* $t > 0$ *fixed. Assume that the claims* $X_1, X_2, \dots$ *are iid with the common continuous distribution* F *on* $(0, \infty)$ *such that* $\overline{F}(x) \sim \ell(x) x^{\alpha - 1} \mathrm{e}^{-\gamma x}$ *for some* $\gamma, \alpha > 0$ *and some slowly varying function* $\ell(\cdot)$. *If* $\mathbb{E}\left\{(N(t))^{l-1}\right\} < \infty$, *then*

$$\mathbb{P}\left(E_l(t) > x\right) \sim \frac{\gamma^{l-2} \left(\Gamma(\alpha)\right)^{l-1}}{(l-1)! \Gamma((l-1)\alpha)} \int_0^{\infty} e^{-(l-1)\gamma y} Q_t^{(l-1)}\left(F(\mathrm{d}y)\right) \cdot \left(\ell(x)\right)^{l-1} x^{(l-1)\alpha - 1} \mathrm{e}^{-\gamma x}. \tag{3.1}$$

Clearly, if the claim sizes have the exponential distribution given as (2.4), then (3.1) can be written as

$$\mathbb{P}\left(E_l(t)>x\right)\sim\frac{\gamma^{l-2}}{(l-2)!}\mathbb{P}\left(N(t)\geq l\right)\cdot x^{l-2}\mathrm{e}^{-\gamma x}, \tag{3.2}$$

which coincides with Theorem 2.2(i) of Jiang and Tang (2008). As we mention in Section 4, the asymptotic relation (3.2) can also be obtained by a well-known property of order statistics; see (4.3) below.

Theorem 3.2. *Consider the LCR treaty defined in (1.1). In addition to the other conditions of Theorem 3.1, if* $\mathbb{E}\left\{(N(t))^l\right\}$*, then*

$$k(l,\gamma,\alpha)Q_t^{(l)}(1)\cdot(\ell(x))^l\,x^{l\alpha-1}\mathrm{e}^{-\gamma x}\lesssim\mathbb{P}\left(L_l(t)>x\right)\lesssim K(l,\gamma,\alpha)Q_t^{(l)}(1)\cdot(\ell(x))^l\,x^{l\alpha-1}\mathrm{e}^{-\gamma x},$$

where

$$k(l,\gamma,\alpha)=\gamma^{l-1}\left(\prod_{i=2}^{l}\int_0^{1/i}(1-z)^{\alpha-1}z^{(i-1)\alpha-1}\mathrm{d}z\right),\quad K(l,\gamma,\alpha)=\frac{\gamma^{l-1}\left(\Gamma(\alpha)\right)^l}{(l-1)!\Gamma(l\alpha)}.$$

If $l=2$*, then*

$$\mathbb{P}\left(L_2(t)>x\right)\sim\frac{\gamma\left(\Gamma(\alpha)\right)^2}{2\Gamma(2\alpha)}Q_t^{(2)}(1)\cdot(\ell(x))^2\,x^{2\alpha-1}\mathrm{e}^{-\gamma x}. \tag{3.3}$$

Intuitively, $L_l(t)$ should have also a gamma-like tail when $l\geq3$, i.e., $\mathbb{P}\left(L_l(t)>x\right)\sim C\left(\ell(x)\right)^l x^{l\alpha-1}\mathrm{e}^{-\gamma x}$, however the constant C seems to be technically too involved, and hence its determination is out of the scope of this contribution. Indeed, as shown below for a gamma claims with shape parameter greater than 1 the constant C is even in that special case very involved.

Theorem 3.3. *Consider the LCR treaty defined in (1.1) with* $l\geq2$ *and* $t\geq0$ *fixed. Assume that the claims* X_1*,* X_2*,... are iid with the common gamma distribution specified in (2.2) for* $\gamma>0$ *and* $\alpha>1$*. If* $\mathbb{E}\left\{(N(t))^l\right\}<\infty$*, then*

$$\mathbb{P}\left(L_l(t)>x\right)\sim\frac{Q_t^{(l)}(1)\gamma^{l\alpha-1}}{(l-1)!}\sum_{k=0}^{l-1}\frac{\binom{l-1}{k}J_{l,\alpha,k}}{\left(\Gamma(\alpha)\right)^{k+1}\left(\Gamma\left(\alpha-1\right)\right)^{l-k-1}}\cdot x^{l\alpha-1}e^{-\gamma x}, \tag{3.4}$$

where

$$J_{l,\alpha,k}=\begin{cases}\displaystyle\int_0^{1/(l-k-1)}\frac{z^{(\alpha-1)(k+1)}}{\left(1+(k+1)z\right)^{l\alpha}}\underset{\substack{s_1+\cdots+s_{l-k-1}\leq1\\ s_1,\ldots,s_{l-k-1}>z}}{\int\cdots\int}\left(1-\sum_{i=1}^{l-k-1}s_i\right)^{l-2}\prod_{i=1}^{l-k-1}s_i^{\alpha-2}\prod_{i=1}^{l-k-1}\mathrm{d}s_i\mathrm{d}z, & k=0,\ldots,l-2;\\[2ex] \displaystyle\frac{\Gamma(l-1)\Gamma(l(\alpha-1)+1)}{\Gamma(l\alpha)l^{l(\alpha-1)+1}}, & k=l-1.\end{cases}$$

When $l=2$, it follows that (3.4) is a special form of (3.3) with $\ell(x)\equiv\gamma^{\alpha-1}/\Gamma(\alpha)$. Theorems 3.1–3.3 together with Lemma 2.1 reveal that, for the gamma-like claims and any fixed l and t, the tails of $E_l(t)$ and $L_l(t)$ (the sums concerning order statistics) have the same decay rate as (are proportionally or weakly equivalent to) the tails of the sums of corresponding iid claims.

4 Discussions and Extensions

A general class of claim size distributions can be defined asymptotically by

$$\overline{F}(x)\sim\ell(x)x^{\alpha-1}\mathrm{e}^{-\gamma x^p} \tag{4.1}$$

for some slowly varying function $\ell(\cdot)$ and constants $\alpha \in \mathbb{R}, \gamma \geq 0, p > 0$. Thus, the case of gamma-like claims dealt with in this paper corresponds to the special choice of parameters $\alpha, \gamma \in (0, \infty)$ and $p = 1$.

In view of Lemma 2.3 of Pakes (2004), if $p = 1$, $\ell(\cdot)$ is a normalized slowly varying function; see formula (1.3.4) of Bingham et al. (1987), and either $\alpha < 0$ or $\alpha = 0$ with $\int_1^\infty \ell(x)/x\,dx < \infty$, then $\overline{F} \in \mathcal{S}(\gamma)$. Clearly, the "normalization" assumption is redundant, since every slowly varying function has a normalized version tail-equivalent to it and the closure property of the class $\mathcal{S}(\gamma)$ under tail equivalence. As mentioned before, for the claims belonging to $\mathcal{S}(\gamma)$, the results for LCR and ECOMOR treaties have been established by Jiang and Tang (2008).

It is well-known that (see e.g., Foss et al. (2011)) if (4.1) holds for $p \in (0, 1)$, then $\overline{F} \in \mathcal{S}(0)$ with $\mathcal{S}(0)$ the class of subexponential distributions. The aggregation of subexponential risks is a well-studied topic; see the aforementioned reference. Recently, some remarkable second order asymptotic expansions for the tail of the sum of subexponential risks are derived in Kortschak (2011).

When (4.1) holds with $p > 1$, then F has super-exponential tails; see Lemma 7.1 in Rootzén (1986) for results covering this general case with F being absolutely continuous. Particularly, if $p = 2$ then F is referred to as a Gaussian-like distribution. A similar result to Lemma 2.1 for Gaussian-like risks is shown in Farkas and Hashorva (2012), see also Lemma 8.6 in Piterbarg (1996).

Under the case $p = 1$ currently considered, $\overline{F}$ has an exponential behaviour, and in particular $\overline{F}$ belongs to the class $\mathcal{L}(\gamma)$, which means simply that F is in the Gumbel max-domain of attraction with a constant scaling function equal to γ. By definition (see e.g., Resnick (1987)) F is in the Gumbel max-domain of attraction with positive scaling function $\sigma(\cdot)$ if, for every $s \geq 0$,

$$\overline{F}(x + s/\sigma(x)) \sim \overline{F}(x)\mathrm{e}^{-s}. \tag{4.2}$$

Both our results and those of Jiang and Tang (2008) for $\overline{F} \in \mathcal{S}(\gamma)$ show that both $L_l(t)$ and $E_l(t)$ with fixed l and t have distribution functions in the Gumbel max-domain of attraction with constant scaling function $\sigma(\cdot) \equiv \gamma$. The recent contribution Asimit et al. (2012) derived the tail asymptotics of weighted sum of order statistics of dependent risks, where in particular the risks have distribution functions in the Gumbel max-domain of attractions. The results of the aforementioned paper are however not applicable in our setup since the scaling function $\sigma(\cdot)$ therein tends to infinity as $x \to \infty$, while in the case of gamma-like distributions the scaling function is constant.

The exponential distribution is of particular interest since relation (4.2) holds exactly for any positive x and s with $\sigma(\cdot) \equiv \gamma$. Therefore, in the rest of this section we analyse in details the exponential claims. Suppose that the claims X_k, $k = 1, 2, \ldots$, are iid with the common exponential distribution given by (2.4) for some $\gamma > 0$. For every $n \geq l$, the first l largest order statistics $X_{1,n} \geq \cdots \geq X_{l,n}$ selected from n iid exponential random variables have the following useful probabilistic representation (see Example 4.1.10 of Embrechts et al. (1997))

$$(X_{i,n})_{i=1,\ldots,l} \stackrel{d}{=} \left(\sum_{j=i}^{n} \frac{1}{j} X_j \right)_{i=1,\ldots,l}, \tag{4.3}$$

where $\stackrel{d}{=}$ means equality of the distribution functions.

The merit of relation (4.3) lies in that it transforms the order statistics of an exponential sample to linear combinations of iid exponential copies and hence makes the problems under consideration much more tractable. It is clear that using (4.3) the kernels of the LCR and ECOMOR treaties, $\sum_{i=1}^{l} X_{i,n}$ and $\sum_{i=1}^{l-1} (X_{i,n} - X_{l,n})$, can be written as the weighted sum of iid exponential random variables with certain non-negative weights $u_1, \ldots, u_n$:

$$S_n = \sum_{i=1}^{n} u_i X_i. \tag{4.4}$$

Relation (4.4) can also be regarded as the sum of independent exponential random variables with different parameters γ/u_i, $i = 1, \ldots, n$.

Assuming that there are m ($1 \le m \le n$) positive and distinct u_i's denoted by $v_1 > \cdots > v_m > 0$, and each v_k, $k = 1, \ldots, m$, corresponds to n_k u_i's with $n_1 + \cdots + n_m = n$, Amari and Misra (1997) gave the following closed-form expression

$$\mathbb{P}(S_n > x) = \left(\prod_{i=1}^{m} \left(\frac{\gamma}{v_i} \right)^{n_i} \right) \sum_{k=1}^{m} \sum_{j=1}^{n_k} \frac{\Psi_{kj}\left(-\gamma/v_k\right)}{(n_k - j)!\,(j-1)!} x^{n_k - j} \mathrm{e}^{-\gamma x / v_k}, \quad \forall x > 0, \tag{4.5}$$

where

$$\Psi_{kj}\left(z\right) = -\frac{\partial^{j-1}}{\partial y^{j-1}} \left(\prod_{j=0, j \neq k}^{m} \left(\frac{\gamma}{v_j} + y \right)^{-n_j} \right) \Bigg|_{y=z} \quad \text{with } n_0 = 1, \ \frac{\gamma}{v_0} \triangleq 0.$$

It is clear that from the asymptotic point of view we only need to take care of the terms in (4.5) with $x^{n_1 - 1} \mathrm{e}^{-\gamma x / v_1}$. Hence, after rearrangement formula (4.5) implies

$$\mathbb{P}(S_n > x) \sim \frac{(\gamma/v_1)^{n_1 - 1}}{(n_1 - 1)!} \prod_{j=2}^{m} \left(\frac{v_1}{v_1 - v_j} \right)^{n_j} \cdot x^{n_1 - 1} \mathrm{e}^{-\gamma x / v_1}. \tag{4.6}$$

Next, we re-consider the LCR treaty in terms of the above analysis. By representation (4.3), it holds for every $n \ge l$ that

$$\sum_{i=1}^{l} X_{i,n} \stackrel{d}{=} \sum_{i=1}^{l} \sum_{j=i}^{n} \frac{1}{j} X_j = \sum_{i=1}^{l} X_i + \sum_{i=l+1}^{n} \frac{l}{i} X_i.$$

Corresponding to the previous notation, we have in this case $m = n - l + 1$, $v_1 = 1$, $v_j = l/(l + j - 1)$ for $j = 2, \ldots, m$, $n_1 = l$, and $n_2 = \cdots = n_m = 1$. Plugging all of these into (4.6) leads to

$$\mathbb{P}\left(\sum_{i=1}^{l} X_{i,n} > x \right) \sim \frac{n!}{(n-l)!} \frac{\gamma^{l-1}}{l!(l-1)!} \cdot x^{l-1} \mathrm{e}^{-\gamma x}. \tag{4.7}$$

For every $n \ge l$, it is well-known that the l largest order statistics selected from n iid random variables with common continuous distribution function F have the following joint probability density function (pdf):

$$\mathbb{P}\left(\bigcap_{i=1}^{l} \left(X_{i,n} \in \mathrm{d}x_i \right) \right) = \frac{n!}{(n-l)!} F^{n-l}(x_l) \prod_{i=1}^{l} F(\mathrm{d}x_i), \quad x_1 > \cdots > x_l. \tag{4.8}$$

Consequently, for any $x > 0$,

$$\mathbb{P}\left(L_l(t) > x \right) = \sum_{n=l}^{\infty} \mathbb{P}(N(t) = n) \frac{n!}{(n-l)!} \int \cdots \int_{\substack{x_1 + \cdots + x_l > x \\ x_1 > \cdots > x_l}} F^{n-l}(x_l) \prod_{i=1}^{l} F(\mathrm{d}x_i). \tag{4.9}$$

It follows from (4.9) that there exists some constant C_1 such that, for all large x,

$$\frac{\mathbb{P}\left(L_l(t) > x \right)}{x^{l-1} \mathrm{e}^{-\gamma x}} \le \sum_{n=l}^{\infty} \mathbb{P}(N(t) = n) \frac{n!}{(n-l)!} \frac{\mathbb{P}\left(\sum_{i=1}^{l} X_i > x \right)}{x^{l-1} e^{-\gamma x}} \le C_1 \sum_{n=l}^{\infty} \mathbb{P}(N(t) = n) n^l.$$

Hence, when $\mathbb{E}\left\{ (N(t))^l \right\} < \infty$, the dominated convergence theorem and (4.7) imply

$$\mathbb{P}\left(L_l(t) > x \right) = \sum_{n=l}^{\infty} \mathbb{P}(N(t) = n) \mathbb{P}\left(\sum_{i=1}^{l} X_{i,n} > x \right) \sim \frac{\gamma^{l-1} Q_t^{(l)}(1)}{l!(l-1)!} \cdot x^{l-1} \mathrm{e}^{-\gamma x},$$

which coincides with Theorem 2.1(i) of Jiang and Tang (2008).

Finally, we discuss briefly the weighted ECOMOR treaty with positive constant weights $w_1, \ldots, w_{l-1}$ defined as

$$E_l^w(t) = \sum_{i=1}^{l-1} w_i \left(X_{i,N(t)} - X_{l,N(t)} \right) \mathbf{1}_{\{N(t)\geq l\}}, \quad l \geq 2. \tag{4.10}$$

By (4.3), it holds for every $n \geq l$ that

$$\sum_{i=1}^{l-1} w_i \left(X_{i,n} - X_{l,n} \right) \stackrel{d}{=} \sum_{i=1}^{l-1} w_i \sum_{j=i}^{l-1} \frac{1}{j} X_j = \sum_{i=1}^{l-1} \frac{\sum_{j=1}^{i} w_j}{i} X_i. \tag{4.11}$$

Recalling (4.4), the right-hand side of the above relation is S_{l-1} with $u_i = \sum_{j=1}^{i} w_j / i$, $i = 1, \ldots, l-1$. Then it follows from (4.5) that, for every $n \geq l$

$$\mathbb{P}\left(\sum_{i=1}^{l-1} w_i \left(X_{i,n} - X_{l,n} \right) > x \right) = \left(\prod_{i=1}^{m} \left(\frac{\gamma}{v_i} \right)^{n_i} \right) \sum_{k=1}^{m} \sum_{j=1}^{n_k} \frac{\Psi_{kj}\left(-\gamma/v_k\right)}{(n_k - j)!\,(j-1)!} x^{n_k - j} \mathrm{e}^{-\gamma x / v_k},$$

where $m, n_1, \ldots, n_m \leq l-1$. Hence, there exists some constant C_2 not related to n such that

$$\mathbb{P}\left(\sum_{i=1}^{l-1} w_i \left(X_{i,n} - X_{l,n} \right) > x \right) \leq C_2 x^{n_1 - 1} \mathrm{e}^{-\gamma x / v_1}$$

is valid for all large x. Consequently, applying the dominated convergence theorem and (4.6) to

$$\mathbb{P}\left(E_l^w(t) > x \right) = \sum_{n=l}^{\infty} \mathbb{P}(N(t) = n) \mathbb{P}\left(\sum_{i=1}^{l-1} w_i \left(X_{i,n} - X_{l,n} \right) > x \right)$$

we arrive at the following theorem:

Theorem 4.1. *Consider the weighted ECOMOR treaty defined in (4.10) with $w_1, \ldots, w_{l-1} > 0$, $l \geq 2$, and $t \geq 0$ fixed. Assume that the claims $X_1, X_2, \ldots$ are iid with the common exponential distribution given by (2.4) for some $\gamma > 0$. Denote by $u_i = \sum_{j=1}^{i} w_j / i$ for $i = 1, \ldots, l-1$. If there are m distinct u_i's denoted by $v_1 > \cdots > v_m$, and each v_k, $k = 1, \ldots, m$, corresponds to n_k u_i's with $n_1 + \cdots + n_m = l-1$, then*

$$\mathbb{P}\left(E_l^w(t) > x \right) \sim \frac{(\gamma/v_1)^{n_1 - 1}}{(n_1 - 1)!} \prod_{j=2}^{m} \left(\frac{v_1}{v_1 - v_j} \right)^{n_j} \mathbb{P}\left(N(t) \geq l \right) \cdot x^{n_1 - 1} \mathrm{e}^{-\gamma x / v_1}. \tag{4.12}$$

Remark 4.1. *In the non-weighted case i.e., $w_1 = \cdots = w_{l-1} = 1$, one can obtain by (4.11) that, for every $n \geq l$,*

$$\sum_{i=1}^{l-1} \left(X_{i,n} - X_{l,n} \right) \stackrel{d}{=} \sum_{i=1}^{l-1} X_i.$$

Then relation (3.2) holds since $\sum_{i=1}^{l-1} X_i$ follows the gamma (Erlang) distribution with shape parameter $l-1$. Ignoring the probabilistic insight, one can also directly obtain (3.2) by (4.12) with $m = v_1 = 1$ and $n_1 = l-1$. We note that (4.12) was derived without imposing any moment conditions on the counting process $\{N(t); t \geq 0\}$, while in Jiang and Tang (2008) $\mathbb{E}\left\{ (N(t))^{l-1} \right\} < \infty$ is required.

5 Further Results and Proofs

We present next Lemma 5.1 which is crucial to prove both Lemma 2.1 stated in Section 2 and Lemma 5.2 below. We proceed then with the proofs of our theorems stated in Section 3.

We mention first the following crucial result in the theory of regularly varying function, namely if $h \in \mathcal{R}_\rho$ for some real ρ, then for every $A > 1$ and $\delta > 0$ there exists some x_0 such that, whenever $x, y \geq x_0$,

$$\frac{1}{A} \min\left(\left(\frac{y}{x} \right)^{\rho + \delta}, \left(\frac{y}{x} \right)^{\rho - \delta} \right) \leq \frac{h(y)}{h(x)} \leq A \max\left(\left(\frac{y}{x} \right)^{\rho + \delta}, \left(\frac{y}{x} \right)^{\rho - \delta} \right), \tag{5.1}$$

which are Potter's bounds for $h(\cdot)$; see Theorem 1.5.6(iii) of Bingham et al. (1987).

Lemma 5.1. *Let $\overline{F}_1$ and $\overline{F}_2$ be two distribution functions with gamma-like tails $\overline{F_i}(x) \sim \ell_i(x)x^{\alpha_i-1}\mathrm{e}^{-\gamma x}$ for some positive constants γ, α_i and some slowly varying functions $\ell_i(\cdot)$, $i = 1, 2$. For any two non-negative functions $a(\cdot)$ and $b(\cdot)$ such that $\lim_{x\to\infty} a(x)/x = a$ and $\lim_{x\to\infty} b(x)/x = b$ for some $0 \le a \le b < 1$ we have*

$$\lim_{x\to\infty} \frac{\int_{a(x)}^{b(x)} \overline{F}_1(x-y)\mathrm{d}F_2(y)}{\ell_1(x)\ell_2(x)x^{\alpha_1+\alpha_2-1}\mathrm{e}^{-\gamma x}} = \gamma \int_a^b (1-z)^{\alpha_1-1} z^{\alpha_2-1}\mathrm{d}z. \tag{5.2}$$

Proof. We only need to prove the assertion for $a = 0$. Then the case of $a > 0$ follows from decomposing the integral on the left-hand side of (5.2) as $\int_{a(x)}^{b(x)} = \int_0^{b(x)} - \int_0^{a(x)}$. Noting that $0 \le b < 1$, we choose ε small enough such that $0 < \varepsilon < \min\left\{b\mathbf{1}_{\{b>0\}}/2 + \mathbf{1}_{\{b=0\}}, 1-b\right\}$. By the assumptions on both $a(\cdot)$ and $b(\cdot)$, for this choice of ε and large x, it holds that

$$\mathbf{1}_{\{b>0\}} \cdot \int_{\varepsilon x}^{(b-\varepsilon)x} \overline{F}_1(x-y)\mathrm{d}F_2(y) \le \int_{a(x)}^{b(x)} \overline{F}_1(x-y)\mathrm{d}F_2(y) \le \int_0^{(b+\varepsilon)x} \overline{F}_1(x-y)\mathrm{d}F_2(y) \triangleq V(x). \tag{5.3}$$

As mentioned in Remark 2.1, we write $h_i(x) = \ell_i(x)x^{\alpha_i-1} \in \mathcal{R}_{\alpha_i-1}$ for $i = 1, 2$. Using $\overline{F_1}(x) \sim h_1(x)\mathrm{e}^{-\gamma x}$ leads to

$$V(x) \sim \mathrm{e}^{-\gamma x} \int_0^{(b+\varepsilon)x} h_1(x-y)\mathrm{e}^{\gamma y}\mathrm{d}F_2(y) = h_1(x)\mathrm{e}^{-\gamma x} \int_0^{(b+\varepsilon)x} \frac{h_1(x-y)}{h_1(x)}\mathrm{e}^{\gamma y}\mathrm{d}F_2(y).$$

By (5.1), it holds for large x that

$$V(x) \lesssim (1+\varepsilon)h_1(x)\mathrm{e}^{-\gamma x} \int_0^{(b+\varepsilon)x} \left(1-\frac{y}{x}\right)^{\alpha_1-1-\varepsilon} \mathrm{e}^{\gamma y}\mathrm{d}F_2(y).$$

Next, partial integration implies

$$\begin{aligned} V(x) \ &\lesssim\ (1+\varepsilon)h_1(x)\mathrm{e}^{-\gamma x}\left(-\left(1-\frac{y}{x}\right)^{\alpha_1-1-\varepsilon} \mathrm{e}^{\gamma y}\overline{F_2}(y)\Big|_0^{(b+\varepsilon)x} + \gamma\int_0^{(b+\varepsilon)x}\left(1-\frac{y}{x}\right)^{\alpha_1-1-\varepsilon}\mathrm{e}^{\gamma y}\overline{F_2}(y)\mathrm{d}y - \right. \\ &\qquad \left. (\alpha_1-1-\varepsilon)\int_0^{(b+\varepsilon)x}\frac{1}{x-y}\left(1-\frac{y}{x}\right)^{\alpha_1-1-\varepsilon}\mathrm{e}^{\gamma y}\overline{F_2}(y)\mathrm{d}y\right) \\ &\triangleq\ (1+\varepsilon)h_1(x)\mathrm{e}^{-\gamma x}\Big(-V_1(x) + \gamma V_2(x) - (\alpha_1-1-\varepsilon)\, V_3(x)\Big). \end{aligned} \tag{5.4}$$

Since $\overline{F_2}(x) \sim h_2(x)\mathrm{e}^{-\gamma x}$, we have

$$V_1(x) = o(1)h_2(x)x. \tag{5.5}$$

Using $\overline{F_2}(x) \sim h_2(x)\mathrm{e}^{-\gamma x}$ and (5.1) again, for every $\delta > 0$ there exists some $y_0 > 0$ such that, for $y \ge y_0$,

$$(1-\delta)h_2(y)\mathrm{e}^{-\gamma y} \le \overline{F_2}(y) \le (1+\delta)h_2(y)\mathrm{e}^{-\gamma y}, \tag{5.6}$$

and, for $y' \ge y'' \ge y_0$

$$\frac{h_2(y'')}{h_2(y')} \le 2\left(\frac{y''}{y'}\right)^{\alpha_2/2-1}. \tag{5.7}$$

By such y_0, we further split $V_2(x)$ into two parts as

$$V_2(x) = \left(\int_0^{y_0} + \int_{y_0}^{(b+\varepsilon)x}\right)\left(1-\frac{y}{x}\right)^{\alpha_1-1-\varepsilon} \mathrm{e}^{\gamma y}\overline{F_2}(y)\mathrm{d}y \triangleq V_{21}(x) + V_{22}(x). \tag{5.8}$$

It is obvious that

$$V_{21}(x) = o(1)h_2(x)x. \tag{5.9}$$

Applying (5.6) to $V_{22}(x)$ and then using the variable substitution $z = y/x$ leads to the following inequalities

$$(1-\delta)h_2(x)xJ(x) \le V_{22}(x) \le (1+\delta)h_2(x)xJ(x), \tag{5.10}$$

where

$$J(x) = \int_0^{b+\varepsilon} (1-z)^{\alpha_1-1-\varepsilon} \frac{h_2(xz)}{h_2(x)} \mathbf{1}_{\{z>y_0/x\}} \mathrm{d}z.$$

By (5.7), for large x, the integrand of $J(x)$ is bounded by $2\,(1-z)^{\alpha_1-1-\varepsilon}\, z^{\alpha_2/2-1}$, which is integrable. Hence, by the dominated convergence theorem, we have

$$\lim_{x\to\infty} J(x) = \int_0^{b+\varepsilon} (1-z)^{\alpha_1-1-\varepsilon}\, z^{\alpha_2-1} \mathrm{d}z.$$

Combining this result with (5.10) and noting the arbitrariness of δ, we obtain

$$V_{22}(x) \sim h_2(x)x \int_0^{b+\varepsilon} (1-z)^{\alpha_1-1-\varepsilon}\, z^{\alpha_2-1} \mathrm{d}z. \tag{5.11}$$

Consequently,

$$V_2(x) \sim h_2(x)x \int_0^{b+\varepsilon} (1-z)^{\alpha_1-1-\varepsilon}\, z^{\alpha_2-1} \mathrm{d}z. \tag{5.12}$$

In addition, it is clear that

$$V_3(x) \leq \frac{V_2(x)}{(1-b-\varepsilon)x} = o(1)h_2(x)x. \tag{5.13}$$

Plugging (5.5), (5.12), and (5.13) into (5.4) and then combining the obtained result with (5.3), we have

$$\limsup_{x\to\infty} \frac{\int_{a(x)}^{b(x)} \overline{F}_1(x-y)\mathrm{d}F_2(y)}{h_1(x)h_2(x)x\mathrm{e}^{-\gamma x}} \leq (1+\varepsilon)\gamma \int_0^{b+\varepsilon} (1-z)^{\alpha_1-1-\varepsilon}\, z^{\alpha_2-1} \mathrm{d}z.$$

Similarly, we can derive that

$$\liminf_{x\to\infty} \frac{\int_{a(x)}^{b(x)} \overline{F}_1(x-y)\mathrm{d}F_2(y)}{h_1(x)h_2(x)x\mathrm{e}^{-\gamma x}} \geq \mathbf{1}_{\{b>0\}} \cdot (1-\varepsilon)\gamma \int_\varepsilon^{b-\varepsilon} (1-z)^{\alpha_1-1+\varepsilon}\, z^{\alpha_2-1} \mathrm{d}z.$$

Noting the arbitrariness of ε, we obtain relation (5.2) for $a=0$ and hence complete the proof. □

Proof of Lemma 2.1. We only need to prove relation (2.5) for $l = 2$, and then the assertion holds by mathematical induction. For $l=2$ and $x>0$ we have

$$\begin{aligned} \mathbb{P}(X_1+X_2>x) &= \mathbb{P}\left(X_1>\frac{x}{2}, X_2>\frac{x}{2}\right) + \mathbb{P}\left(X_1+X_2>x, X_2\leq\frac{x}{2}\right) + \mathbb{P}\left(X_1+X_2>x, X_1\leq\frac{x}{2}\right) \\ &\triangleq I_1(x)+I_2(x)+I_3(x). \end{aligned}$$

It is clear that

$$I_1(x) = \overline{F_1}\left(\frac{x}{2}\right)\overline{F_2}\left(\frac{x}{2}\right) = o(1)\ell_1(x)\ell_2(x)x^{\alpha_1+\alpha_2-1}\mathrm{e}^{-\gamma x}.$$

Applying Lemma 5.1 to $I_2(x)$ and $I_3(x)$ leads to

$$I_2(x) = \int_0^{x/2} \overline{F_1}\,(x-y)\,\mathrm{d}F_2(y) \sim \gamma \int_0^{1/2} (1-z)^{\alpha_1-1}\, z^{\alpha_2-1}\mathrm{d}z \cdot \ell_1(x)\ell_2(x)x^{\alpha_1+\alpha_2-1}\mathrm{e}^{-\gamma x},$$

and

$$I_3(x) = \int_0^{x/2} \overline{F_2}\,(x-y)\,\mathrm{d}F_1(y) \sim \gamma \int_0^{1/2} z^{\alpha_1-1}\,(1-z)^{\alpha_2-1}\,\mathrm{d}z \cdot \ell_1(x)\ell_2(x)x^{\alpha_1+\alpha_2-1}\mathrm{e}^{-\gamma x}.$$

Consequently,

$$\mathbb{P}(X_1+X_2>x) = I_1(x)+I_2(x)+I_3(x) \sim \frac{\gamma\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\ell_1(x)\ell_2(x)x^{\alpha_1+\alpha_2-1}\mathrm{e}^{-\gamma x},$$

which proves relation (2.5) for $l=2$. □

The next lemma is crucial for the proof of Theorem 3.2.

Lemma 5.2. *Under the assumptions of Lemma 2.1, for every non-negative function $b(\cdot)$ satisfying $\lim_{x\to\infty} b(x) = b$, we write*

$$q_l(x, b(x)) = \mathbb{P}\left(\sum_{i=1}^{l} X_i > x + lb(x),\ \ X_l > \cdots > X_1 > b(x)\right).$$

Then it holds that

$$q_l(x, b(x)) \gtrsim \gamma^{l-1}\mathrm{e}^{-l\gamma b}\left(\prod_{i=2}^{l}\int_0^{1/i}(1-z)^{\alpha_i-1}z^{\sum_{j=1}^{i-1}\alpha_j-1}\mathrm{d}z\right)\left(\prod_{i=1}^{l}\ell_i(x)\right)x^{\sum_{i=1}^{l}\alpha_i-1}\mathrm{e}^{-\gamma x}. \tag{5.14}$$

Particularly, when $l = 2$ the two sides of relation (5.14) are asymptotically equivalent, i.e.,

$$q_2(x, b(x)) \sim \gamma\mathrm{e}^{-2\gamma b}\int_0^{1/2}(1-z)^{\alpha_2-1}z^{\alpha_1-1}\mathrm{d}z\cdot\ell_1(x)\ell_2(x)x^{\alpha_1+\alpha_2-1}\mathrm{e}^{-\gamma x}. \tag{5.15}$$

Proof. As above we define $h_i(x) = \ell_i(x)x^{\alpha_i-1}$ for $i = 1, \ldots, l$. By the uniform convergence theorem for regularly varying functions (see Theorem 1.2.1 of Bingham et al. (1987)), for every $i \le l$, $h_i(x+y) \sim h_i(x)$ holds uniformly on each compact y-set in $\mathbb{R}$. Consequently, for every real λ and $i \le l$

$$\overline{F_i}(x + \lambda b(x)) \sim h_i(x + \lambda b(x))\mathrm{e}^{-\gamma(x+\lambda b(x))} \sim \mathrm{e}^{-\gamma\lambda b}h_i(x)\mathrm{e}^{-\gamma x} \sim \mathrm{e}^{-\gamma\lambda b}\overline{F_i}(x). \tag{5.16}$$

We first prove relation (5.15), i.e., the case $l = 2$, and then proceed by mathematical induction. According to whether $X_1 > x/2 + b(x)$ or not, we split $q_2(x)$ into two parts as $q_2(x) = J_1(x) + J_2(x)$. It follows from (5.16) that

$$J_1(x) \le \mathbb{P}\left(X_1 > \frac{x}{2} + b(x)\right)\mathbb{P}\left(X_2 > \frac{x}{2} + b(x)\right) = o(1)h_1(x)h_2(x)x\mathrm{e}^{-\gamma x}.$$

Applying (5.16) and Lemma 5.1, in turn, to $J_2(x)$ gives

$$\begin{aligned} J_2(x) &= \int_{b(x)}^{x/2+b(x)}\overline{F_2}(x + 2b(x) - y)\mathrm{d}F_1(y) \\ &\sim \mathrm{e}^{-2\gamma b}\int_{b(x)}^{x/2+b(x)}\overline{F_2}(x - y)\mathrm{d}F_1(y) \\ &\sim \gamma\mathrm{e}^{-2\gamma b}\int_0^{1/2}(1-z)^{\alpha_2-1}z^{\alpha_1-1}\mathrm{d}z\cdot h_1(x)h_2(x)x\mathrm{e}^{-\gamma x}. \end{aligned}$$

Hence the claim in (5.15) follows.

Next we assume that relation (5.14) holds for $l - 1$. For every $\varepsilon > 0$, since $\lim_{x\to\infty} b(x) = b$, it holds for large x that

$$b(x) \le b + \varepsilon \triangleq b_\varepsilon.$$

Hence, we have for large x that

$$\begin{aligned} q_l(x, b(x)) &\ge q_l(x, b_\varepsilon) \\ &\ge \mathbb{P}\left(\sum_{i=1}^{l} X_i > x + lb_\varepsilon, X_l > \cdots > X_1 > b_\varepsilon, X_l > \frac{(l-1)x}{l} + (l-1)b_\varepsilon, \sum_{i=1}^{l-1} X_i \le \frac{x}{l} + b_\varepsilon\right) \\ &= \mathbb{P}\left(\sum_{i=1}^{l} X_i > x + lb_\varepsilon, X_{l-1} > \cdots > X_1 > b_\varepsilon, \sum_{i=1}^{l-1} X_i \le \frac{x}{l} + b_\varepsilon\right). \end{aligned} \tag{5.17}$$

Define a random variable Z as

$$Z = \left(\sum_{i=1}^{l-1}(X_i - b_\varepsilon)\right)\mathbf{1}_{\{X_{l-1} > \cdots > X_1 > b_\varepsilon\}}.$$

Then (5.17) can be further written as

$$\begin{aligned} q_l(x, b(x)) &\geq \mathbb{P}\left(X_l + Z > x + b_\varepsilon, 0 < Z \leq \frac{x}{l} - (l-2)b_\varepsilon\right) \\ &= \int_0^{x/l-(l-2)b_\varepsilon} \overline{F_l}(x + b_\varepsilon - y)\mathbb{P}(Z \in \mathrm{d}y) \\ &\sim \mathrm{e}^{-\gamma b_\varepsilon}\mathrm{e}^{-\gamma x}\int_0^{x/l-(l-2)b_\varepsilon} h_l(x-y)\mathrm{e}^{\gamma y}\mathbb{P}(Z \in \mathrm{d}y), \end{aligned}$$

where in the last step we used $\overline{F_l}(x) \sim h_l(x)\mathrm{e}^{-\gamma x}$ and the regular variation of $h_l(x)$. By (5.1), it holds for large x that

$$q_l(x, b(x)) \gtrsim (1-\varepsilon)\mathrm{e}^{-\gamma b_\varepsilon}h_l(x)\mathrm{e}^{-\gamma x}\int_0^{x/l-(l-2)b_\varepsilon}\left(1 - \frac{y}{x}\right)^{\alpha_l - 1 + \varepsilon}\mathrm{e}^{\gamma y}\mathbb{P}(Z \in \mathrm{d}y).$$

Using partial integration and noting that $\mathbb{P}(Z > y) = q_{l-1}(y, b_\varepsilon)$ gives

$$\begin{aligned} & q_l(x, b(x)) \\ \gtrsim\ & (1-\varepsilon)\mathrm{e}^{-\gamma b_\varepsilon}h_l(x)\mathrm{e}^{-\gamma x}\left(-\left(1-\frac{y}{x}\right)^{\alpha_l-1+\varepsilon}\mathrm{e}^{\gamma y}q_{l-1}(y, b_\varepsilon)\Big|_0^{x/l-(l-2)b_\varepsilon} + \right. \\ & \left.\gamma\int_0^{x/l-(l-2)b_\varepsilon}\left(1-\frac{y}{x}\right)^{\alpha_l-1+\varepsilon}\mathrm{e}^{\gamma y}q_{l-1}(y,b_\varepsilon)\mathrm{d}y - (\alpha_l - 1 + \varepsilon)\int_0^{x/l-(l-2)b_\varepsilon}\frac{1}{x-y}\left(1-\frac{y}{x}\right)^{\alpha_l-1+\varepsilon}\mathrm{e}^{\gamma y}q_{l-1}(y,b_\varepsilon)\mathrm{d}y\right) \\ \triangleq\ & (1-\varepsilon)\mathrm{e}^{-\gamma b_\varepsilon}h_l(x)\mathrm{e}^{-\gamma x}\Big(-K_1(x) + \gamma K_2(x) - (\alpha_l - 1 + \varepsilon)K_3(x)\Big). \end{aligned} \tag{5.18}$$

It follows from Lemma 2.1 that

$$K_1(x) \leq 1 + \left(\frac{l-1}{l} + \frac{(l-2)b_\varepsilon}{x}\right)^{\alpha_l-1+\varepsilon}\mathrm{e}^{\gamma x/l}\mathbb{P}\left(\sum_{i=1}^{l-1}X_i > \frac{x}{l} - (l-2)b_\varepsilon\right) = o(1)\left(\prod_{i=1}^{l-1}h_i(x)\right)x^{l-1}.$$

By the induction assumption on $q_{l-1}(x, b_\varepsilon)$ and the similar procedures as in dealing with $V_2(x)$ in the proof of Lemma 5.1 before (see (5.8)–(5.12)), we obtain

$$K_2(x) \gtrsim \gamma^{l-2}\mathrm{e}^{-(l-1)\gamma b_\varepsilon}\left(\prod_{i=2}^{l-1}\int_0^{1/i}(1-z)^{\alpha_i-1}z^{\sum_{j=1}^{i-1}\alpha_j - 1}\mathrm{d}z\right)\int_0^{1/l}(1-z)^{\alpha_l-1+\varepsilon}z^{\sum_{j=1}^{l-1}\alpha_j-1}\mathrm{d}z\cdot\left(\prod_{i=1}^{l-1}h_i(x)\right)x^{l-1}. \tag{5.19}$$

Finally, it holds for large x that

$$K_3(x) \leq \frac{|\alpha_l - 1| + 1}{x/l}\int_0^{x/l-(l-2)b_\varepsilon}\left(1-\frac{y}{x}\right)^{\alpha_l-1+\varepsilon}\mathrm{e}^{\gamma y}\mathbb{P}\left(\sum_{i=1}^{l-1}X_i > y\right)\mathrm{d}y,$$

which can be proved to be $o(1)\left(\prod_{i=1}^{l-1}h_i(x)\right)x^{l-1}$ by Lemma 2.1 and the similar procedures as in dealing with $V_2(x)$ in the proof of Lemma 5.1. Plugging (5.19) and $K_1(x) + K_3(x) = o(1)\left(\prod_{i=1}^{l-1}h_i(x)\right)x^{l-1}$ into (5.18) and noting the arbitrariness of ε, we complete the proof of Lemma 5.2. □

Remark 5.1. *Before proceeding with the proofs of our theorems, we mention that in the above lemmas we do not require that F_i's are continuous distribution functions, which is imposed in the statements of our theorems.*

Proof of Theorem 3.1. In view of (4.8), we can derive that

$$\mathbb{P}(E_l(t) > x) = \sum_{n=l}^{\infty}\mathbb{P}(N(t) = n)\frac{n!}{(n-l+1)!}P_E(x, n) \tag{5.20}$$

with

$$P_E(x,n) = \mathbb{P}\left(\sum_{i=1}^{l-1}(X_i - Y) > x, X_1 > \cdots > X_{l-1} > Y\right), \tag{5.21}$$

where Y is a positive random variable independent of $\{X_k; k=1,2,\ldots\}$ and $\{N(t); t\geq 0\}$ with distribution function F^{n-l+1}. For any x, y positive we have

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=1}^{l-1} X_i > x+(l-1)y, \bigcap_{i=1}^{l-1}(X_i>y)\right) \\
= \ &\mathbb{P}\left(\sum_{i=1}^{l-1} X_i > x+(l-1)y\right) \\
&-\sum_{k=1}^{l-2}\binom{l-1}{k}\mathbb{P}\left(\sum_{i=1}^{l-1} X_i > x+(l-1)y, \bigcap_{i=1}^{k}(X_i\leq y), \bigcap_{j=k+1}^{l-1}(X_i>y)\right) \\
\triangleq \ & I(x,y)-\sum_{k=1}^{l-2}\binom{l-1}{k}J_k(x,y).
\end{aligned}$$

Consequently, by conditioning on Y, we obtain

$$\begin{aligned}
P_E(x,n) &= \int_0^\infty \mathbb{P}\left(\sum_{i=1}^{l-1} X_i > x+(l-1)y, X_1>\cdots>X_{l-1}>y\right)\mathrm{d}F^{n-l+1}(y) \\
&= \frac{1}{(l-1)!}\int_0^\infty \mathbb{P}\left(\sum_{i=1}^{l-1} X_i > x+(l-1)y, \bigcap_{i=1}^{l-1}(X_i>y)\right)\mathrm{d}F^{n-l+1}(y) \\
&= \frac{1}{(l-1)!}\int_0^\infty \left(I(x,y)-\sum_{k=1}^{l-2}\binom{l-1}{k}J_k(x,y)\right)\mathrm{d}F^{n-l+1}(y). \qquad (5.22)
\end{aligned}$$

Then Lemma 2.1 implies

$$\int_0^\infty I(x,y)\mathrm{d}F^{n-l+1}(y) \sim \frac{\gamma^{l-2}\left(\Gamma(\alpha)\right)^{l-1}}{\Gamma((l-1)\alpha)}\left(\ell(x)\right)^{l-1}x^{(l-1)\alpha-1}\mathrm{e}^{-\gamma x}\widetilde{I}(x),$$

where

$$\widetilde{I}(x)=\int_0^\infty\left(\frac{\ell\left(x+(l-1)y\right)}{\ell(x)}\right)^{l-1}\left(1+\frac{(l-1)y}{x}\right)^{(l-1)\alpha-1}\mathrm{e}^{-(l-1)\gamma y}\mathrm{d}F^{n-l+1}(y).$$

By (5.1), for large x, the integrand of $\widetilde{I}(x)$ is bounded by $2\left(1+(l-1)y\right)^{(l-1)(\alpha+1)-1}\mathrm{e}^{-(l-1)\gamma y}$, which is integrable. Hence, using the dominated convergence theorem gives

$$\lim_{x\to\infty}\widetilde{I}(x)=\int_0^\infty \mathrm{e}^{-(l-1)\gamma y}\mathrm{d}F^{n-l+1}(y).$$

Consequently,

$$\int_0^\infty I(x,y)\mathrm{d}F^{n-l+1}(y)\sim\frac{\gamma^{l-2}\left(\Gamma(\alpha)\right)^{l-1}}{\Gamma((l-1)\alpha)}\int_0^\infty \mathrm{e}^{-(l-1)\gamma y}\mathrm{d}F^{n-l+1}(y)\cdot\left(\ell(x)\right)^{l-1}x^{(l-1)\alpha-1}\mathrm{e}^{-\gamma x}. \qquad (5.23)$$

With similar arguments as above, for all $k=1,\ldots,l-2$, we obtain

$$\int_0^\infty J_k(x,y)\mathrm{d}F^{n-l+1}(y)\leq\int_0^\infty\mathbb{P}\left(\sum_{i=k+1}^{l-1}X_i>x+(l-k-1)y\right)\mathrm{d}F^{n-l+1}(y)=o(1)\left(\ell(x)\right)^{l-1}x^{(l-1)\alpha-1}\mathrm{e}^{-\gamma x}. \qquad (5.24)$$

Substituting (5.23) and (5.24) into (5.22) yields

$$P_E(x,n)\sim\frac{\gamma^{l-2}\left(\Gamma(\alpha)\right)^{l-1}}{(l-1)!\Gamma((l-1)\alpha)}\int_0^\infty\mathrm{e}^{-(l-1)\gamma y}\mathrm{d}F^{n-l+1}(y)\cdot\left(\ell(x)\right)^{l-1}x^{(l-1)\alpha-1}\mathrm{e}^{-\gamma x}. \qquad (5.25)$$

It follows from (5.21) and Lemma 2.1 that, for large x, there exists some constant C_3 not related to n such that

$$\frac{n!}{(n-l+1)!}\frac{P_E(x,n)}{\left(\ell(x)\right)^{l-1}x^{(l-1)\alpha-1}\mathrm{e}^{-\gamma x}}\leq\frac{n!}{(n-l+1)!}\frac{\mathbb{P}\left(\sum_{i=1}^{l-1}X_i>x\right)}{\left(\ell(x)\right)^{l-1}x^{(l-1)\alpha-1}\mathrm{e}^{-\gamma x}}\leq C_3 n^{l-1}.$$

Recalling $\mathbb{E}\left\{(N(t))^{l-1}\right\} < \infty$, the proof is completed by applying the dominated convergence theorem and (5.25) to (5.20). □

Proof of Theorem 3.2. Recalling (4.9),

$$\mathbb{P}\left(L_l(t) > x\right) = \sum_{n=l}^{\infty} \mathbb{P}(N(t) = n) \frac{n!}{(n-l+1)!} P_L(x,n) \tag{5.26}$$

holds with

$$P_L(x,n) = \mathbb{P}\left(\sum_{i=1}^{l-1} X_i + Y > x, X_1 > \cdots > X_{l-1} > Y\right), \tag{5.27}$$

where Y is the random variable specified in the proof of Theorem 3.1. Next, by the definition of the random variable Y, we have

$$\mathbb{P}\left(Y > x\right) = 1 - F^{n-l+1}(x) \sim (n-l+1)\overline{F}(x).$$

Consequently, (5.27) and Lemma 2.1 imply

$$\begin{aligned} P_L(x,n) &= \frac{1}{(l-1)!}\mathbb{P}\left(\sum_{i=1}^{l-1} X_i + Y > x, \bigcap_{i=1}^{l} (X_i > Y)\right) \\ &\leq \frac{1}{(l-1)!}\mathbb{P}\left(\sum_{i=1}^{l-1} X_i + Y > x\right) \\ &\sim \frac{(n-l+1)\,\gamma^{l-1}\,(\Gamma(\alpha))^l}{(l-1)!\Gamma(l\alpha)} \cdot (\ell(x))^l\, x^{l\alpha-1}\mathrm{e}^{-\gamma x}. \end{aligned} \tag{5.28}$$

We can also obtain by (5.27) and Lemma 2.1 that, for large x, there exists some constant C_4 not related to n such that

$$\frac{n!}{(n-l+1)!}\frac{P_L(x,n)}{(\ell(x))^l\, x^{l\alpha-1}\mathrm{e}^{-\gamma x}} \leq \frac{n!}{(n-l+1)!}\frac{\mathbb{P}\left(\sum_{i=1}^{l-1} X_i + Y > x\right)}{(\ell(x))^l\, x^{l\alpha-1}\mathrm{e}^{-\gamma x}} \leq C_4 n^l. \tag{5.29}$$

Recalling $\mathbb{E}\left\{(N(t))^l\right\} < \infty$, the asymptotic upper bound for $\mathbb{P}\left(L_l(t) > x\right)$ can be obtained by applying Fatou's lemma and (5.28) to (5.26).

On the other hand, applying Lemma 5.2 with $b(x) = 0$ to relation (5.27) leads to

$$P_L(x,n) \gtrsim (n-l+1)\, k(l,\gamma,\alpha) \cdot (\ell(x))^l\, x^{l\alpha-1} e^{-\gamma x},$$

where asymptotic equivalence holds when $l = 2$. Then the asymptotic lower bound for $\mathbb{P}\left(L_l(t) > x\right)$ and relation (3.3) are consequences of Fatou's lemma and the dominated convergence theorem applied to (5.26), respectively. This completes the proof of Theorem 3.2. □

In what follows, given the rate parameter $\gamma > 0$ fixed, we denote by G_α the gamma distribution function with the shape parameter $\alpha > 0$ and rate parameter γ.

Proof of Theorem 3.3. Recalling (5.27) with Y distributed by G_α^{n-l+1}, we split it into two parts according to whether or not $Y > x/l$ and rewrite it as $P_L(x,n) = I_1(x,n) + I_2(x,n)$. It follows from (2.3) that

$$I_1(x,n) \leq \mathbb{P}\left(X_1, \ldots, X_{l-1}, Y > \frac{x}{l}\right) = o(1)x^{l\alpha-1}e^{-\gamma x}. \tag{5.30}$$

For every $i = 1, \ldots, l-1$, we have $X_i \overset{d}{=} E_i + Y_i$, where E_i and Y_i are independent and respectively follow G_1 and $G_{\alpha-1}$. Since we can take the random pairs (E_i, Y_i), $i = 1, \ldots, l-1$ to be mutually independent, $I_2(x,n)$ can be rewritten as

$$\begin{aligned} I_2(x,n) &= \frac{1}{(l-1)!}\int_0^{x/l} \mathbb{P}\left(\sum_{i=1}^{l-1}(E_i + Y_i) > x - y, \bigcap_{i=1}^{l-1}(E_i + Y_i > y)\right) \mathrm{d}G_\alpha^{n-l+1}(y) \\ &\triangleq \frac{1}{(l-1)!}\int_0^{x/l} p(x,y)\mathrm{d}G_\alpha^{n-l+1}(y). \end{aligned} \tag{5.31}$$

For the conciseness on writing in the subsequence, we set some notational conventions as follows:

$$\left|{}_i^j y\right| \triangleq \sum_{k=i}^{j} y_k, \text{ for any reals } y_i, \ldots, y_j,$$

$$\mathrm{d}f({}_i^j y) \triangleq \prod_{k=i}^{j} \mathrm{d}f(y_k), \text{ for any differentiable } f,$$

$$\int_{{}_i^j y \in A} \triangleq \int \cdots \int_{y_i,\ldots,y_j \in A}, \text{ if } i \le j; \qquad \int_{{}_i^j y \in A} \triangleq \int_{\varnothing}, \text{ if } i > j, \quad \text{for any Borel set } A,$$

and we agree to the following convention that whenever $\int_{\varnothing}$ occurs it is regarded as 1, i.e., the notation $\int_{\varnothing}$ will automatically disappear from formulas. In addition, we admit the traditional conventions that $\sum_{i\in\varnothing} y_i = 0$, $\prod_{i\in\varnothing} y_i = 1$, and $\bigcap_{i\in\varnothing} A_i = \Omega$. Hence, combining with the previous conventions, it holds in our setup that if $i > j$ then $\left|{}_i^j y\right| = 0$ and $\mathrm{d}F({}_i^j y) = 1$.

According to the exact number of Y_i's which belongs to $(0, y]$, we further write $p(x, y)$ in (5.31) as

$$p(x,y) = \sum_{k=0}^{l-1}\binom{l-1}{k}\int_{{}_1^k y \le y}\int_{{}_{k+1}^{l-1} y > y} \mathbb{P}\left(\sum_{i=1}^{k}(E_i - (y-y_i)) + \sum_{j=k+1}^{l-1} E_j > x-(k+1)y-\left|{}_{k+1}^{l-1} y\right|, \bigcap_{i=1}^{k}(E_i > y-y_i)\right)\mathrm{d}G_{\alpha-1}({}^{l-1}_{1} y).$$

By the memoryless property of $E_1, \ldots, E_k$, it holds that

$$p(x,y) = \sum_{k=0}^{l-1}\binom{l-1}{k}\int_{{}_1^k y \le y}\int_{{}_{k+1}^{l-1} y > y} \mathbb{P}\left(\sum_{i=1}^{l-1} E_i > x-(k+1)y-\left|{}_{k+1}^{l-1} y\right|\right)\prod_{i=1}^{k}\overline{G_1}\,(y-y_i)\,\mathrm{d}G_{\alpha-1}({}^{l-1}_{1} y). \tag{5.32}$$

Observing that the inner integral w.r.t. $y_{k+1}, \ldots, y_{l-1}$ and the outer one w.r.t. $y_1, \ldots, y_k$ are separable now, we have

$$p(x,y) = \sum_{k=0}^{l-1}\binom{l-1}{k}\left(\overline{G_\alpha}(y) - \overline{G_{\alpha-1}}(y)\right)^k \left\{ \int_{\substack{\left|{}_{k+1}^{l-1} y\right| > x-(k+1)y \\ {}_{k+1}^{l-1} y > y}} \mathbf{1}_{\{k \ne l-1\}} + \int_{\substack{\left|{}_{k+1}^{l-1} y\right| \le x-(k+1)y \\ {}_{k+1}^{l-1} y > y}} \overline{G_{l-1}}(x-(k+1)y-\left|{}_{k+1}^{l-1} y\right|) \right\} \mathrm{d}G_{\alpha-1}({}_{k+1}^{l-1} y). \tag{5.33}$$

The last summand of $k = l-1$ in (5.32) is

$$\left(\overline{G_\alpha}(y) - \overline{G_{\alpha-1}}(y)\right)^{l-1}\overline{G_{l-1}}(x-ly).$$

Hence we equipped the indicator $\mathbf{1}_{\{k\ne l-1\}}$ in (5.33) to make it accurate under our conventions.

Unfolding the brace of (5.33) and plugging the two items obtained into (5.31), and then replacing $\mathrm{d}G_\alpha^{n-l+1}(y)$ by $(n-l+1)G_\alpha^{n-l}(y)g_\alpha(y)\mathrm{d}y$, where g_α is the pdf of G_α, we can rewrite $I_2(x, n)$ as the sum of the corresponding items:

$$I_2(x,n) \triangleq \frac{(n-l+1)}{(l-1)!}\sum_{k=0}^{l-1}\binom{l-1}{k}\left(I_{21}(x,n,k) + I_{22}(x,n,k)\right). \tag{5.34}$$

It is clear that $I_{21}(x,n,l-1)=0$, and for $k=0,\dots,l-2$,

$$\begin{aligned} I_{21}(x,n,k) &\le \int_0^{x/l} \left(\overline{G_\alpha}(y)\right)^k \overline{G_{(l-k-1)(\alpha-1)}}\Big(x-(k+1)y\Big) g_\alpha(y)\mathrm{d}y \\ &\sim \text{const}\cdot \mathrm{e}^{-\gamma x}\int_0^{x/l}\left(\int_0^\infty (u+y)^{\alpha-1}\mathrm{e}^{-\gamma u}\mathrm{d}u\right)^k \Big(x-(k+1)y\Big)^{(l-k-1)(\alpha-1)-1} y^{\alpha-1}\mathrm{d}y. \end{aligned}$$

Using the variable substitution $z=y/x$ and the dominated convergence theorem, we obtain

$$\begin{aligned} I_{21}(x,n,k) &\lesssim \text{const}\cdot x^{l\alpha-l}\mathrm{e}^{-\gamma x}\int_0^{1/l}\left(\int_0^\infty \left(\frac{u}{x}+z\right)^{\alpha-1}\mathrm{e}^{-\gamma u}\mathrm{d}u\right)^k \\ &\quad \times (1-(k+1)z)^{(l-k-1)(\alpha-1)-1} z^{\alpha-1}\mathrm{d}z \\ &= o(1)x^{l\alpha-1}\mathrm{e}^{-\gamma x}. \end{aligned} \tag{5.35}$$

For $I_{22}(x,n,k)$, we have

$$\begin{aligned} I_{22}(x,n,k) &= \int_0^{x/l}\left(\overline{G_\alpha}(y)\right)^k\left(1-\frac{\overline{G_{\alpha-1}}(y)}{\overline{G_\alpha}(y)}\right)^k \int_{\substack{\left|{}_{k+1}^{l-1}y\right|\le x-(k+1)y \\ {}_{k+1}^{l-1}y>y}} \overline{G_{l-1}}(x-(k+1)y-\left|{}_{k+1}^{l-1}y\right|)\mathrm{d}G_{\alpha-1}({}_{k+1}^{l-1}y) \\ &\quad \times G_\alpha^{n-1}(y)g_\alpha(y)\mathrm{d}y \\ &\triangleq \frac{\gamma^{l\alpha+k}}{(\Gamma(\alpha))^{k+1}\left(\Gamma(\alpha-1)\right)^{l-k-1}}\mathrm{e}^{-\gamma x}\widetilde{I_{22}}(x,n,k), \end{aligned} \tag{5.36}$$

where

$$\begin{aligned} \widetilde{I_{22}}(x,n,k) &= \int_0^{x/l}\left(\int_0^\infty (u+y)^{\alpha-1}\mathrm{e}^{-\gamma u}\mathrm{d}u\right)^k\left(1-\frac{\overline{G_{\alpha-1}}(y)}{\overline{G_\alpha}(y)}\right)^k \\ &\quad \times \int_{\substack{\left|{}_{k+1}^{l-1}y\right|\le x-(k+1)y \\ {}_{k+1}^{l-1}y>y}} \left(\int_0^\infty \left(v+x-(k+1)y-\left|{}_{k+1}^{l-1}y\right|\right)^{l-2}\mathrm{e}^{-\gamma v}\mathrm{d}v\right)\prod_{i=k+1}^{l-1} y_i^{\alpha-2}\mathrm{d}({}_{k+1}^{l-1}y)G_\alpha^{n-1}(y)y^{\alpha-1}\mathrm{d}y. \end{aligned} \tag{5.37}$$

For $k=0,\dots,l-2$, we apply the variable substitutions $w=y/x$, $s_i=y_i/(x-(k+1)y)$, $i=k+1,\dots,l-1$, with the Jacobian determinant $x^{l-k}(1-(k+1)w)^{l-k-1}$. Plugging the substitutions into (5.37), we obtain after rearrangement that, for $k=0,\dots,l-2$

$$\begin{aligned} &\widetilde{I_{22}}(x,n,k) \\ &= x^{l\alpha-1}\int_0^{1/l}\left(\int_0^\infty\left(\frac{u}{x}+w\right)^{\alpha-1}\mathrm{e}^{-\gamma u}\mathrm{d}u\right)^k\left(1-\frac{\overline{G_{\alpha-1}}(xw)}{\overline{G_\alpha}(xw)}\right)^k(1-(k+1)w)^{(l-k-1)\alpha+k-1} \\ &\quad\times \int_{\substack{\left|{}_{k+1}^{l-1}s\right|\le 1 \\ {}_{k+1}^{l-1}s>w/(1-(k+1)w)}}\left(\int_0^\infty\left(\frac{v}{x(1-(k+1)w)}+1-\left|{}_{k+1}^{l-1}s\right|\right)^{l-2}\mathrm{e}^{-\gamma v}\mathrm{d}v\right)\prod_{i=k+1}^{l-1}s_i^{\alpha-2}\mathrm{d}({}_{k+1}^{l-1}s)G_\alpha^{n-1}(xw)w^{\alpha-1}\mathrm{d}w. \end{aligned}$$

Since $\overline{G_{\alpha-1}}(x)=o(1)\overline{G_\alpha}(x)$ it follows from the dominated convergence theorem that

$$\begin{aligned} \widetilde{I_{22}}(x,n,k) &\sim \gamma^{-k-1}x^{l\alpha-1}\int_0^{1/l}(1-(k+1)w)^{(l-k-1)\alpha+k-1}w^{(\alpha-1)(k+1)} \\ &\quad\times\int_{\substack{\left|{}_{1}^{l-k-1}s\right|\le 1 \\ {}_{1}^{l-k-1}s>w/(1-(k+1)w)}}\left(1-\left|{}_{1}^{l-k-1}s\right|\right)^{l-2}\prod_{i=1}^{l-k-1}s_i^{\alpha-2}\mathrm{d}({}_{1}^{l-k-1}s)\mathrm{d}w \\ &= \gamma^{-k-1}J_{l,\alpha,k}\cdot x^{l\alpha-1}, \end{aligned} \tag{5.38}$$

where in the last step we used the variable substitution $z = w/(1-(k+1)w)$ to the integral. When $k = l-1$, $\widetilde{I_{22}}(x, n, l-1)$ has the following simple expression

$$\begin{aligned}\widetilde{I_{22}}(x,n,l-1) &= \int_0^{x/l}\left(\int_0^{\infty}(u+y)^{\alpha-1}\mathrm{e}^{-\gamma u}\mathrm{d}u\right)^{l-1}\left(1-\frac{\overline{G_{\alpha-1}}(y)}{\overline{G_{\alpha}}(y)}\right)^{l-1}\\ &\quad\times\left(\int_0^{\infty}(v+x-ly)^{l-2}\mathrm{e}^{-\gamma v}\mathrm{d}v\right)G_{\alpha}^{n-1}(y)y^{\alpha-1}\mathrm{d}y.\end{aligned}$$

Similarly, using the variable substitution $z = y/x$ and applying the dominated convergence theorem, we obtain

$$\widetilde{I_{22}}(x,n,l-1)\sim\gamma^{-l}x^{l\alpha-1}\int_0^{1/l}(1-lz)^{l-2}z^{l(\alpha-1)}\mathrm{d}z=\gamma^{-l}J_{l,\alpha,l-1}\cdot x^{l\alpha-1}. \tag{5.39}$$

Plugging (5.38) and (5.39) into (5.36), we have

$$I_{22}(x,n,k)\sim\frac{\gamma^{l\alpha-1}J_{l,\alpha,k}}{(\Gamma(\alpha))^{k+1}\left(\Gamma\left(\alpha-1\right)\right)^{l-k-1}}x^{l\alpha-1}\mathrm{e}^{-\gamma x}. \tag{5.40}$$

Substituting (5.40) and (5.35) into (5.34) and then combining the obtained relation with (5.30) leads to

$$P_L(x,n)\sim\frac{(n-l+1)\gamma^{l\alpha-1}}{(l-1)!}\sum_{k=0}^{l-1}\frac{\binom{l-1}{k}J_{l,\alpha,k}}{(\Gamma(\alpha))^{k+1}\left(\Gamma\left(\alpha-1\right)\right)^{l-k-1}}\cdot x^{l\alpha-1}e^{-\gamma x}. \tag{5.41}$$

Recalling relation (5.29) and applying the dominated convergence theorem to (5.26) with relation (5.41), we obtain relation (3.4). This completes the proof of Theorem 3.3. □

Acknowledgments. E. Hashorva was partially supported by the Swiss National Science Foundation Grant 200021-1401633/1.

References

[1] Amari, S. V.; Misra, R. B. Closed-form expressions for distribution of sum of exponential random variables. IEEE Trans. Reliability 46, (1997), 519–522.

[2] Ammeter, H. The rating of largest claim reinsurance covers. Quart. Algem. Reinsur. Comp. Jubilee. (1964), 5–17.

[3] Asimit, A. V.; Hashorva, E.; Kortschak, D. Exact tail asymptotics of the total loss of largest claim reinsurance treaties. Preprint, (2012).

[4] Asimit, A. V.; Jones, B. L. Asymptotic tail probabilities for large claims reinsurance of a portfolio of dependent risks. Astin Bull. 38, (2008), 147–159.

[5] Bingham, N.; Goldie, C.M.; Teugels, J.L. Regular Variations. Cambridge, Cambridge University Press, 1987.

[6] Embrechts, P.; Klüppelberg, C.; Mikosch, T. Modelling Extremal Events for Insurance and Finance. Springer-Verlag, Berlin, 1997.

[7] Farkas, J.; Hashorva, E. Extremal behaviour of Gaussian-like risks. Preprint, (2012).

[8] Foss, S.; Korshunov, D.; Zachary, S. An Introduction to Heavy-tailed and Subexponential Distributions. Springer-Verlag, New York, 2011.

[9] Hashorva, E. On the asymptotic distribution of certain bivariate reinsurance treaties. Insurance: Mathematics Math. Econom. 40, (2007), 200–208.

[10] Hashorva, E. Exact tail asymptotics in bivariate scale mixture models. Extremes. 15, (2012), 109–128.

[11] Hashorva, E.; Pakes, A.G.; Tang, Q.; Asymptotics of random contractions, Insurance: Math. Econom. 47, (2010), 405–414.

[12] Jessen, A.H.; Mikosch, T. Regularly varying functions. Publications de l'Institut Mathématique, Nouvelle Série, 80, (2006), 171–192.

[13] Jiang, J.; Tang, Q. Reinsurance under the LCR and ECOMOR treaties with emphasis on light-tailed claims. Insurance Math. Econom. 43, (2008), 431–436.

[14] Kortschak, D. Second order tail asymptotics for the sum of dependent, tail-independent regularly varying risks. Extremes, 15, (2011), 353–388.

[15] Ladoucette, S.A.; Teugels, J. L. Reinsurance of large claims. J. Comput. Appl. Math. 186, (2006), 163–190.

[16] Pakes, G. Convolution equivalence and infinite divisibility, J. Appl. Probab. 41, (2004), 407–424.

[17] Piterbarg, V.I. Asymptotic Methods in the Theory of Gaussian Processes and Fields. AMS, Providence, 1996.

[18] Resnick, S.I. Extreme Values, Regular Variation and Point Processes. Springer, New York, 1987.

[19] Rootzén, H. Extreme value theory for moving average processes. Ann. Probab. 14, (1986), 612–652.

[20] Rootzén, H. A ratio limit theorem for the tails of weighted sums. Ann. Probab. 15, (1987), 728–747.

[21] Thépaut, A. Une nouvelle forme de réassurance: le traité d'excédent du coût moyen relatif (ECOMOR). Bull. Trimestriel Inst. Actuair Français, 49, (1950), 273–343.