On maxima of chi-processes over threshold dependent grids *

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Abstract: In this paper, with motivation from [30] and the considerable interest in stationary chi-processes, we derive asymptotic joint distributions of maxima of stationary strongly dependent chi-processes on a continuous time and an uniform grid on the real axis. Our findings extend those for Gaussian cases and give three involved dependence structures via the strongly dependence condition and the sparse, Pickands and dense grids.

Key Words: stationary chi-processes; normal comparison lemma; discrete time process; Piterbarg max-discretization theorem; Pickands constant.

AMS Classification: Primary 60F05; secondary 60G15

1 Introduction

Consider a stationary chi-process $\{\chi_m(t), t \ge 0\}$ with $m, m \in \mathbb{N}$ degrees of freedom as follows

$$\chi_m(t) = \left(X_1^2(t) + \cdots + X_m^2(t)\right)^{1/2} = \|\boldsymbol{X}(t)\|, \quad t \ge 0,$$

where $\boldsymbol{X}(t) = (X_1(t), \ldots, X_m(t))$ is a vector Gaussian process which components are independent copies of a standard (zero-mean and unit-variance) stationary Gaussian process $\{X(t), t \ge 0\}$ with almost surely (a.s.) continuous sample paths and correlation function $r(t) = \mathbb{E}\{X(0)X(t)\}, t \ge 0$.

In this paper, we are concerned with the dependence of extremes of the continuous time and discrete time of chi-processes. Specifically, assuming that the process $\{\chi_m(t), t \in [0,T]\}$ is observed at time $t \in \mathfrak{R}(\delta) = \{k\delta, k \in \mathbb{N}\}$ with frequency $\delta = \delta_T > 0$, of interest is the asymptotic joint distributions of $(M_m(T), M_m(\delta, T))$ as $T \to \infty$ (after normalization) with

$$M_m(T) := \sup_{t \in [0,T]} \chi_m(t), \quad M_m(\delta, T) := \sup_{t \in \mathfrak{R}(\delta) \cap [0,T]} \chi_m(t). \tag{1}$$

The impetus for this investigation comes from numerical simulations of high extremes of continuous time random processes, see e.g., [15, 30, 37] for Gaussian processes, [16] for the storage process with fractional Brownian motion, [13, 38, 39] for stationary vector Gaussian processes and standardized stationary Gaussian processes, and [41] for stationary processes. It is shown in the aforementioned contributions that the dependence between continuous time extremes and discrete time extremes is determined strongly by the sampling frequency δ and the normalization constants, see also for related discussions [5, 20, 31, 32, 41] in the financial and time series literature. Another motivation is that since the chi-processes appear naturally as limiting processes which have

*Research supported by the Swiss National Science Foundation grant 200021-140633/1, RARE -318984 (an FP7 Marie Curie IRSES Fellowship), Zhongquan Tan also acknowledges National Science Foundation of China (No. 11326175) and Natural Science Foundation of Zhejiang Province of China (No. LQ14A010012).

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attracted considerable interest from both theoretical and practical fields, see e.g., [12, 25, 11, 10] for deeply theoretical discussions involved in the continuous time extremes of various χ-processes, and [1, 4, 17, **?**] for statistics test applications concerning the maxima over the chosen time points set and the continuous time intervals. Therefore, of crucial importance is to understand the underlying asymptotic behavior of the extremes for different grids.

The principle challenge for χ-processes increases significantly due to no counterpart of *Berman's Normal Comparison Lemma* for chi-distributions. However, with the technical methodology from [12, 22, 23, 24, 29, 35, 27, 26], and assuming certain locally and long range dependence on the common Gaussian process $X(\cdot)$, namely (see for its extensional utilizations [26, 19, 36])

$$r(t) = 1 - |t|^{\alpha} + o(|t|^{\alpha}), \quad t \to 0 \ \text{ for some } \alpha \in (0, 2] \tag{2}$$

and

$$\lim_{T\to\infty} r(T)\ln T = r \in [0, \infty), \tag{3}$$

we establish our findings in Theorem 2.1 extending those for weakly dependent stationary Gaussian processes in [30], corresponding to $m = 1$ and $r = 0$ in (3) in our setting, represent asymptotically completely dependence, max-stable dependence and conditional independence according to the three different types of grids in the terminology of [30], namely the dense grid $\mathfrak{R}(\delta)$ with $\delta(T) = o((2\ln T)^{-1/\alpha}), T \to \infty$, the Pickands grid $\mathfrak{R}(\delta)$ with $\delta(T) = D(2\ln T)^{-1/\alpha}$ for some $D \in (0, \infty)$, and the sparse grid $\mathfrak{R}(\delta)$ with $\lim_{T\to\infty} \delta(T)(2\ln T)^{1/\alpha} = \infty$ and $\delta(T) \le \delta_0$ for some $\delta_0 > 0$.

We note in passing that our methodology is different from that in [41] which is strongly based on the Albin's methodology wherein the verification of technical Albin's conditions requires in general a lot of efforts. Moreover, our theoretical results, which do not seem possible to be guessed, are of interest for simulation studies, and give to some extent certain recommendations how tight a simulation grid should be when high extremes are important in simulations of the chi-processes under consideration.

The rest of the paper is organized as follows. Our main results are presented in the next section. All the proofs are relegated to Section 3 which is followed by an Appendix including some technical auxiliary results.

2 Main results

This section is devoted to the asymptotic properties of $(M_m(T), M_m(\delta, T))$ given in (1) for the three different types of grids $\delta = \delta(T)$ in the terminology of [30]. Before giving our main result (see Theorem 2.1 below), we shall first recall some asymptotic results of the considered chi-processes and introduce some notation concerning the Pickands type constants.

As we know from [28] or Corollary 7.3 in [29] that, if the correlation function $r(t)$ satisfies (2) and in addition $r(t) < 1$ for all $t \neq 0$, then for any fixed $T > 0$

$$\mathbb{P}\left\{M_m(T) > u\right\} = T\frac{2^{1-m/2}\mathcal{H}_{\alpha}}{\Gamma(m/2)}u^{2/\alpha+m-2}\exp\left(-\frac{u^2}{2}\right)(1+o(1)), \quad u \to \infty, \tag{4}$$

where $\Gamma(\cdot)$ is the Euler Gamma function and $\mathcal{H}_{\alpha} \in (0, \infty)$ denotes the Pickands constant, see [14, 9, 29, 19, 33, 8] for details and various discussions. The asymptotic properties of $M_m(T)$ have been extensively studied in the literature; see [2, 3, 7, 24, 28, 34, 36, 40] for various results. Moreover, if additionally condition (3) holds for some $r \in [0, \infty)$, then the mixed Gumbel limit theorem holds as follows (see e.g., Theorem 3.1 in [35])

$$\mathbb{P}\left\{a_T(M_m(T) - b_T) \le x\right\} \to \mathbb{E}\left\{\exp\left(-e^{-x-r+\sqrt{2r}\chi_m}\right)\right\}, \quad T \to \infty, \tag{5}$$

with χ_m positive such that χ_m^2 a chi-square random variable with m degrees of freedom, and a_T, b_T given by

$$a_T = \sqrt{2\ln T}, \quad b_T = a_T + \frac{\ln\left(2^{1-m/2}(\Gamma(m/2))^{-1}\mathcal{H}_\alpha a_T^{2/\alpha+m-2}\right)}{a_T}. \tag{6}$$

Next, we shall state our main result which is a type of Piterbarg's max-discretisation theorems for chi-processes in terms of [38]. To this end, two Picaknds type constants (see (7) below) are needed.
Let $B_{\alpha/2}^*(t) := \sqrt{2}B_{\alpha/2}(t) - t^\alpha$ with $B_H(\cdot)$ a fractional Brownian motion (fBm) with Hurst index $H \in (0,1]$, and thus define a m-parameter fBm $B_{\boldsymbol{\alpha}/2}^*(\mathbf{t}) = \sum_{i=1}^m B_{\alpha_i/2}^*(t_i), \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (0,2]^m, \mathbf{t} = (t_1, \ldots, t_m) \in [0,\infty)^m$ with mutually independent fBms $B_{\alpha_i/2}^*(\cdot), i \le m$, see e.g., [21, 29] for related discussions on the m-parameter fBm $B_{\boldsymbol{\alpha}/2}^*(\cdot)$. We define thus, for any $D > 0$ and $\boldsymbol{\alpha}_0 = (\alpha, 2, \ldots, 2) \in (0,2]^m$

$$\mathcal{H}_{D,\alpha} = \lim_{\lambda\to\infty} \frac{\mathbb{E}\exp\left(\max_{kD\in[0,\lambda],k\in\mathbb{N}} B_{\alpha/2}^*(kD)\right)}{\lambda}, \quad \mathcal{H}_{D,\boldsymbol{\alpha}_0}^{x,y} := \lim_{\lambda\to\infty} \frac{\mathcal{H}_{D,\boldsymbol{\alpha}_0}^{x,y}(\lambda)}{\lambda^m}, \ x, y \in \mathbb{R}, \tag{7}$$

which are finite and positive by Theorem 2 in [30] and Lemma 3.3, respectively, here

$$\mathcal{H}_{D,\boldsymbol{\alpha}}^{x,y}(\lambda) = \int_{-\infty}^{+\infty} e^s \mathbb{P}\left\{\max_{\mathbf{t}\in[0,\lambda]^m} B_{\boldsymbol{\alpha}/2}^*(\mathbf{t}) > s + x, \max_{\mathbf{t}\in[0,\lambda]^m\cap(\{kD,k\in\mathbb{N}\}\times\mathbb{R}^{m-1})} B_{\boldsymbol{\alpha}/2}^*(\mathbf{t}) > s + y\right\} ds,$$

and further

$$b_{\delta,T} = \begin{cases} a_T + \dfrac{\ln\left(2^{1-m/2}(\Gamma(m/2))^{-1}\mathcal{H}_{D,\alpha} a_T^{2/\alpha+m-2}\right)}{a_T}, & \mathfrak{R}(\delta) \text{ a Pickands grid;} \\ a_T + \dfrac{\ln\left(2^{1-m/2}(\Gamma(m/2))^{-1}\delta^{-1} a_T^{m-2}\right)}{a_T}, & \mathfrak{R}(\delta) \text{ a sparse grid.} \end{cases} \tag{8}$$

Theorem 2.1. *Let $(M_m(T), M_m(\delta,T))$ be given as in (1). Suppose that the correlation $r(\cdot)$ satisfies condition (2) and (3), we have, with involved quantities given by (5)–(8), as $T \to \infty$ and $x, y \in \mathbb{R}$*
(a) For the sparse grid $\mathfrak{R}(\delta)$

$$\mathbb{P}\left\{a_T\big(M_m(T) - b_T\big) \le x, a_T\big(M_m(\delta,T) - b_{\delta,T}\big) \le y\right\} \to \mathbb{E}\exp\left(-\big(e^{-x} + e^{-y}\big)e^{-r+\sqrt{2r}\chi_m}\right). \tag{9}$$

(b) For the Pickands grid $\mathfrak{R}(\delta) = \mathfrak{R}(D(2\ln T)^{-1/\alpha})$ with $D > 0$

$$\begin{aligned} &\mathbb{P}\left\{a_T\big(M_m(T) - b_T\big) \le x, a_T\big(M_m(\delta,T) - b_{\delta,T}\big) \le y\right\} \\ &\quad \to \mathbb{E}\exp\left(-\big(e^{-x} + e^{-y} - \pi^{(m-1)/2}\mathcal{H}_{D,\boldsymbol{\alpha}_0}^{\ln\mathcal{H}_\alpha+x,\ln\mathcal{H}_{D,\alpha}+y}\big)e^{-r+\sqrt{2r}\chi_m}\right). \end{aligned} \tag{10}$$

(c) For any dense grid $\mathfrak{R}(\delta)$

$$\mathbb{P}\left\{a_T\big(M_m(T) - b_T\big) \le x, a_T\big(M_m(\delta,T) - b_T\big) \le y\right\} \to \mathbb{E}\exp\left(-e^{-\min(x,y)-r+\sqrt{2r}\chi_m}\right). \tag{11}$$

Remark 2.2. *(a) A straightforward application of Theorem 2.1 (a) with $\delta(T) \equiv 1$ yields that*

$$\mathbb{P}\left\{a_T\big(M_m(1,T) - b_{1,T}\big) \le x\right\} \to \mathbb{E}\exp\left(-e^{-x-r+\sqrt{2r}\chi_m}\right), \quad x \in \mathbb{R},$$

which may have independent interest in viewpoint of statistics applications, see e.g., [6] for utilizations of the above limit with $m = 1$ concerning test for additive outliers.
(b) From our results we see that the joint convergence is determined by the choice of the grids and the normalization constants a_T, b_T and $b_{\delta,T}$, which is helpful in simulation studies and statistical applications, see related discussions for vector Gaussian processes in [13].
(c) Clearly, the marginal distributions are the same, i.e., the mixed Gumbel distributions, and our results extend those for the Gaussian processes, see [13, 30]. Moreover, the joint limit distribution for the Pickands grid is more involved due to the complication of the Pickands type constant $\mathcal{H}_{D,\boldsymbol{\alpha}_0}^{\ln\mathcal{H}_\alpha+x,\ln\mathcal{H}_{D,\alpha}+y}$, which calculation and simulation are open problems.

(d) It might be possible to allow $X_i's$ to be dependent with condition (2) *stated in a slightly general form such as $r_i(t) = 1 - C_i |t|^{\alpha_i} (1 + o(1)),\ t \to 0$ as well. Results for extremes of chi-type processes for such generalizations can be found in [1, 4, 24, 25].*
(e) It might be interesting to investigate the limit theorems for different grids as in [13]. Another possibility is to relax $r \in [0, \infty]$ in (3)*; see e.g., [26, 13] for similar discussions.*
(f) Following our arguments, it might be possible to consider the same problem for locally stationary chi-processes and cyclo-stationary chi-processes which are considered in [15, 10] and [18, 36], respectively.

3 Further results and proofs

We present first four lemmas followed then by the proofs of Theorem 2.1 for $m \geq 2$ since the claim for $m = 1$, the stationary Gaussian processes follows immediately from [13]. In what follows, we shall keep the notation as in Section 1, and denote further by $\overline{\Phi}$ and φ the survival distribution function and probability density function of a standard normal variable, respectively. We write C for a positive constant whose values may change from line to line. All the limits are taken as T and u tend to infinity in this coordinated way (unless otherwise stated)

$$u^2 = 2 \ln T + (2/\alpha + m - 2) \ln \ln T + O(1).$$

Note that, in view of [29], for any closed non-empty set $E \subset [0, T]$ and $\mathcal{S}_{m-1}$ the unit sphere in $\mathbb{R}^m$ (with respect to L_2-norm)

$$\sup_{t \in E} \chi_m(t) = \sup_{(t,\mathbf{v}) \in E \times S_{m-1}} Y(t, \mathbf{v}),$$

where the Gaussian field $\{Y(t, \mathbf{v}), (t, \mathbf{v}) \in [0, T] \times \mathcal{S}_{m-1}\}$ is given by

$$Y(t, \mathbf{v}) = X_1(t)v_1 + X_2(t)v_2 + \cdots + X_m(t)v_m, \quad (t, \mathbf{v}) \in [0, T] \times \mathcal{S}_{m-1}.$$

Note in passing that the covariance function of $Y(t, \mathbf{v})$, denoted by $r(t, \mathbf{v}, s, \mathbf{w})$, is as follows

$$r(t, \mathbf{v}, s, \mathbf{w}) = r(t - s)A(\mathbf{v}, \mathbf{w}), \quad A(\mathbf{v}, \mathbf{w}) = 1 - \frac{\|\mathbf{v} - \mathbf{w}\|^2}{2}, \quad \mathbf{v}, \mathbf{w} \in \mathcal{S}_{m-1}. \tag{12}$$

Therefore, crucial in the following is to construct as in [27] the grids $\mathfrak{R}_b^\alpha, b > 0$ over the cylinder $[0, T] \times \mathcal{S}_{m-1}$ (see (18) for details) and to deal with the random field $Y(t, \mathbf{v})$ in terms of $\xi_T(t, \mathbf{v})$ defined below in (16).
Let $\vartheta(x) = \sup_{x \leq |t| \leq T} r(t)$ for any $x > 0$. In view of (2), we choose some small $\varepsilon \in (0, 2^{-1/\alpha})$ such that for all $|t| \leq \varepsilon < 2^{-1/\alpha}$

$$\frac{1}{2}|t|^\alpha \leq 1 - r(t) \leq 2|t|^\alpha. \tag{13}$$

It follows further from (3) that $\vartheta(\varepsilon) < 1$ holds for all sufficiently large T (see p. 86 in [19]). Therefore, we choose some constants c and a such that

$$0 < c < a < \frac{1 - \vartheta(\varepsilon)}{1 + \vartheta(\varepsilon)} < 1. \tag{14}$$

Next, we introduce a Gaussian field $\xi_T(t, \mathbf{v}), (t, \mathbf{v}) \in [0, T] \times \mathcal{S}_{m-1}$ via $Y(t, \mathbf{v})$ and condition (3), which is crucial in our proof, see the technical Lemma 3.2. Following [30], divide $[0, T]$ into intervals with length T^a alternating with shorter intervals with length T^c and write

$$I_i := [(i - 1)(T^a + T^c), (i - 1)(T^a + T^c) + T^a], \quad E_i := [(i - 1)(T^a + T^c), i(T^a + T^c)), \tag{15}$$

for $1 \leq i \leq n, n = \lfloor T/(T^a + T^c) \rfloor$. Here $\lfloor x \rfloor$ stands for the integer part of x. We will see from Lemma 3.4 below that, the asymptotic joint distribution of $(M_m(T), M_m(\delta, T)$ is determined totally by that of the maxima over the closed set $\mathcal{I} = \cup_{i=1}^n I_i$.

Further, let $Y_i(t,\mathbf{v}),(t,\mathbf{v})\in[0,T]\times\mathcal{S}_{m-1}, i\le n$ be independent copies of $\{Y(t,\mathbf{v}),(t,\mathbf{v})\in[0,T]\times\mathcal{S}_{m-1}\}$ and $Z_i, 1\le i\le m$ be standard Gaussian random variables so that the components of the $(n+m)$-dimension random vector

$$(Y_1(t,\mathbf{v}),\ldots,Y_n(t,\mathbf{v}),Z_1,\ldots,Z_m)$$

are mutually independent. We define, with $\rho(T)=r/\ln T$ and Gaussian random field $Z(\mathbf{v})=Z_1v_1+Z_2v_2+\cdots+Z_mv_m,\ \mathbf{v}\in\mathcal{S}_{m-1}$,

$$\xi_T(t,\mathbf{v})=\sqrt{1-\rho(T)}Y_i(t,\mathbf{v})+\sqrt{\rho(T)}Z(\mathbf{v}),\quad (t,\mathbf{v})\in E_i\times\mathcal{S}_{m-1},\ 1\le i\le n, \tag{16}$$

which covariance function $\varrho(t,\mathbf{v},s,\mathbf{w})$ is given by

$$\varrho(t,\mathbf{v},s,\mathbf{w})=r^*(t,s)A(\mathbf{v},\mathbf{w}),$$

where

$$r^*(t,s)=\begin{cases} r(t-s)+(1-r(t-s))\rho(T), & (t,s)\in E_i\times E_i;\\ \rho(T), & (t,s)\in E_i\times E_j, i\neq j.\end{cases} \tag{17}$$

Lemma 3.1. *For the grid $\mathfrak{R}(\delta)$ is a sparse grid or a Pickands grid, there exists a grid $\mathfrak{R}_b^\alpha=\widetilde{\mathfrak{R}}_b\times\mathfrak{R}_b$ on the cylinder $[0,T]\times\mathcal{S}_{m-1}$ such that for any $B>0$, we have for all $x,y\in[-B,B]$*

$$\Bigg|\mathbb{P}\left\{a_T\big(\max_{t\in\mathcal{I}}\chi_m(t)-b_T\big)\le x, a_T\big(\max_{t\in\mathfrak{R}(\delta)\cap\mathcal{I}}\chi_m(t)-b_{\delta,T}\big)\le y\right\}$$
$$-\mathbb{P}\left\{a_T\big(\max_{(t,\mathbf{v})\in\mathfrak{R}_b^\alpha\cap(\mathcal{I}\times\mathcal{S}_{m-1})}Y(t,\mathbf{v})-b_T\big)\le x, a_T\big(\max_{(t,\mathbf{v})\in(\mathfrak{R}(\delta)\times\mathfrak{R}_b)\cap(\mathcal{I}\times\mathcal{S}_{m-1})}Y(t,\mathbf{v})-b_{\delta,T}\big)\le y\right\}\Bigg|\to 0$$

as $T\to\infty$ and $b\downarrow 0$, subsequently.

For the proof of Lemma 3.1, one can follow similar arguments as for Lemma 3 in [27] and thus we omit here. Since the grid $\mathfrak{R}_b^\alpha$ is crucial for our proofs, we provide the details on its construction.

For any given $\varepsilon>0$ we partition the sphere $\mathcal{S}_{m-1}$ onto $N(\varepsilon)$ parts $A_1,\ldots,A_{N(\varepsilon)}$ in the following way. With a polar-coordinate transformation, any point $\mathbf{x}$ on the sphere $\mathcal{S}_{m-1}$ is given in terms of angle $\boldsymbol{\varphi}=(\varphi_1,\ldots,\varphi_{m-1})\in[0,\pi)^{m-2}\times[0,2\pi)$ and divide all the intervals $[0,\pi]$ into intervals of length ε (or less for the last interval), do the same for the interval $[0,2\pi]$. This partition of the parallelepiped $[0,\pi]^{m-2}\times[0,2\pi]$ generates the partition $A_j, 1\le j\le N(\varepsilon)$ of the sphere. For a fixed u, choose in every A_j an inner point B_j and consider the tangent plane to the cylinder $[0,T]\times\mathcal{S}_{m-1}$ at the chosen point. Introduce in the tangent plane rectangular coordinates, with origin at the tangent point; the first coordinate is assigned to the direction t. In the so-constructed space $\mathbb{R}^m$, consider the grid of points

$$\mathfrak{R}_{b,u,\varepsilon}^{j,\alpha,P}:=\left(bl_1u^{-\frac{2}{\alpha}},bl_2u^{-1},\ldots,bl_mu^{-1}\right),\quad j=1,2,\ldots,N(\varepsilon)$$

and

$$\mathfrak{R}_{b,u,\varepsilon}^{j,P}:=\left(bl_2u^{-1},\ldots,bl_mu^{-1}\right),\ \ \widetilde{\mathfrak{R}}_{b,u,\varepsilon}^{j,P}:=\left(bl_1u^{-\frac{2}{\alpha}}\right),\ \ j=1,2,\ldots,N(\varepsilon),$$

where $(l_1,l_2,\ldots,l_m)\in\mathbb{Z}^m$. Suppose that ε is so small that the orthogonal projections of all $[0,T]\times A_j$ onto the corresponding tangent plane are one-to-one. Hence the distance between any two points in $[0,T]\times\mathcal{S}_{m-1}$ has the same order with that of their orthogonal projections on the tangent planes. Denote by A_j^P the projection of A_j at the tangent plane, and by $\mathfrak{R}_{b,u,\varepsilon}^{j,\alpha}$, $\mathfrak{R}_{b,u,\varepsilon}^{j}$ and $\widetilde{\mathfrak{R}}_{b,u,\varepsilon}^{j}$, the prototype of $\mathfrak{R}_{b,u,\varepsilon}^{j,\alpha,P}$, $\mathfrak{R}_{b,u,\varepsilon}^{j,P}$ and $\widetilde{\mathfrak{R}}_{b,u,\varepsilon}^{j,P}$, respectively, under this projection. The grids

$$\mathfrak{R}_b^\alpha=\mathfrak{R}_{b,u,\varepsilon}^\alpha=\bigcup_{j=1}^{N(\varepsilon)}\mathfrak{R}_{b,u,\varepsilon}^{j,\alpha},\quad \mathfrak{R}_b=\mathfrak{R}_{b,u,\varepsilon}=\bigcup_{j=1}^{N(\varepsilon)}\mathfrak{R}_{b,u,\varepsilon}^{j},\quad \widetilde{\mathfrak{R}}_b=\widetilde{\mathfrak{R}}_{b,u,\varepsilon}=\bigcup_{j=1}^{N(\varepsilon)}\widetilde{\mathfrak{R}}_{b,u,\varepsilon}^{j} \tag{18}$$

with an appropriate choice of their parameters, satisfy the assertion of Lemma 3.1.

Next, we will introduce three technical lemmas which proofs will be relegated in the Appendix. We will see that Lemmas 3.2 and 3.3 are crucial for the proof of Theorem 2.1.

Lemma 3.2. *Let the grid $\mathfrak{R}(\delta)$ be a sparse grid or Pickands grid, and $\mathfrak{R}_b^\alpha$ as in Lemma 3.1. For any $B>0$ we have for all $x,y\in[-B,B]$,*

$$\Delta_{T,b} := \Bigg|\mathbb{P}\left\{a_T\big(\max_{(t,\mathbf{v})\in\mathfrak{R}_b^\alpha\cap(\mathcal{I}\times\mathcal{S}_{m-1})} Y(t,\mathbf{v})-b_T\big)\le x, a_T\big(\max_{(t,\mathbf{v})\in(\mathfrak{R}(\delta)\times\mathfrak{R}_b)\cap(\mathcal{I}\times\mathcal{S}_{m-1})} Y(t,\mathbf{v})-b_{\delta,T}\big)\le y\right\}$$
$$-\mathbb{P}\left\{a_T\big(\max_{(t,\mathbf{v})\in\mathfrak{R}_b^\alpha\cap(\mathcal{I}\times\mathcal{S}_{m-1})} \xi_T(t,\mathbf{v})-b_T\big)\le x, a_T\big(\max_{(t,\mathbf{v})\in(\mathfrak{R}(\delta)\times\mathfrak{R}_b)\cap(\mathcal{I}\times\mathcal{S}_{m-1})} \xi_T(t,\mathbf{v})-b_{\delta,T}\big)\le y\right\}\Bigg|\to 0$$

uniformly for $b>0$, as $T\to\infty$.

In the following, we denote (recall $b_{\delta,T}$ in (8))

$$\begin{aligned} v_T &:= \frac{b_T+x/a_T-\sqrt{\rho(T)}\,\|\mathbf{z}\|}{(1-\rho(T))^{1/2}} = b_T+\frac{x+r-\sqrt{2r}\,\|\mathbf{z}\|}{a_T}+o(a_T^{-1})\\ v_T^* &:= \frac{b_{\delta,T}+y/a_T-\sqrt{\rho(T)}\,\|\mathbf{z}\|}{(1-\rho(T))^{1/2}} = b_{\delta,T}+\frac{y+r-\sqrt{2r}\,\|\mathbf{z}\|}{a_T}+o(a_T^{-1}). \end{aligned} \tag{19}$$

Lemma 3.3. *Under the conditions of Theorem 2.1, we have, with $\mathcal{H}_{D,\alpha},\mathcal{H}_{D,\boldsymbol{\alpha}_0}^{x,y},\mathfrak{R}_b$ and v_T,v_T^* given by (7), (18) and (19), respectively,*

$$\begin{aligned} &\mathbb{P}\left\{\max_{(t,\mathbf{v})\in\mathfrak{R}_b^\alpha\cap([0,T^a]\times\mathcal{S}_{m-1})} Y(t,\mathbf{v})>v_T\right\} = T^{a-1}e^{-x-r+\sqrt{2r}\|\mathbf{z}\|}(1+o(1))\\ &\mathbb{P}\left\{\max_{(t,\mathbf{v})\in(\mathfrak{R}(\delta)\times\mathfrak{R}_b)\cap([0,T^a]\times\mathcal{S}_{m-1})} Y(t,\mathbf{v})>v_T^*\right\} = T^{a-1}e^{-y-r+\sqrt{2r}\|\mathbf{z}\|}(1+o(1)) \end{aligned} \tag{20}$$

hold for sufficiently large T and sufficiently small $b>0$. And

$$\begin{aligned} &\mathbb{P}\left\{\max_{(t,\mathbf{v})\in\mathfrak{R}_b^\alpha\cap([0,T^a]\times\mathcal{S}_{m-1})} Y(t,\mathbf{v})>v_T, \max_{(t,\mathbf{v})\in(\mathfrak{R}(\delta)\times\mathfrak{R}_b)\cap([0,T^a]\times\mathcal{S}_{m-1})} Y(t,\mathbf{v})>v_T^*\right\}\\ &=\begin{cases} o(T^{a-1}), & \mathfrak{R}(\delta)\text{ is a sparse grid;}\\ T^{a-1}\pi^{\frac{m-1}{2}}\mathcal{H}_{D,\boldsymbol{\alpha}_0}^{\ln\mathcal{H}_\alpha+x,\ln\mathcal{H}_{D,\alpha}+y}e^{-r+\sqrt{2r}\|\mathbf{z}\|}(1+o(1)), & \mathfrak{R}(\delta)\text{ is a Pickands grid}\end{cases} \end{aligned} \tag{21}$$

hold for sufficiently large T and sufficiently small $b>0$.

Lemma 3.4. *Suppose that the grid $\mathfrak{R}(\delta)$ is a sparse grid or a Pickands grid. For any $B>0$, we have for all $x,y\in[-B,B]$, as $T\to\infty$*

$$\Bigg|\mathbb{P}\left\{a_T\big(M_m(T)-b_T\big)\le x, a_T\big(M_m(\delta,T)-b_{\delta,T}\big)\le y\right\}$$
$$-\mathbb{P}\left\{a_T\big(\max_{t\in\mathcal{I}}\chi_m(t)-b_T\big)\le x, a_T\big(\max_{t\in\mathfrak{R}(\delta)\cap\mathcal{I}}\chi_m(t)-b_{\delta,T}\big)\le y\right\}\Bigg|\to 0. \tag{22}$$

Proof of Lemma 3.4: The proof is similar to that of Lemma 6 in [30]. Clearly, the left-hand side of (22) is bounded from above by

$$\mathbb{P}\left\{\max_{t\in[0,T]\setminus\mathcal{I}}\chi_m(t)>b_T+x/a_T\right\}+\mathbb{P}\left\{\max_{t\in\mathfrak{R}(\delta)\cap[0,T]\setminus\mathcal{I}}\chi_m(t)>b_{\delta,T}+y/a_T\right\} =: J_{T,1}+J_{T,2}. \tag{23}$$

It follows from (4) and (6) and the construction of $\mathcal{I}$ that (recall that (4) holds also for $T=T(u)\to\infty$ with suitable speed, see Theorem 7.2 of [29]), with $mes(\cdot)$ the Lebesgue measure

$$J_{T,1}\le C mes([0,T]\setminus\mathcal{I})(b_T+x/a_T)^{2/\alpha+m-1}\overline{\Phi}(b_T+x/a_T)\le C\frac{mes([0,T]\setminus\mathcal{I})}{T}\le C\frac{nT^c}{T}\to 0$$

as $T\to\infty$. Similarly, using (20) in Lemma 3.3 with v_T^* and $\exp\left(-y-r+\sqrt{2r}\,\|\mathbf{z}\|\right)$ replaced by $u_T^*=b_{\delta,T}+y/a_T$ and e^{-y}, respectively, we have $\lim_{T\to\infty}J_{T,2}=0$, hence the proof is complete. □

Proof of Theorem 2.1. First, by (16) we have

$$\mathbb{P}\left\{a_T\big(\max_{(t,\mathbf{v})\in\mathfrak{R}_b^\alpha\cap(\mathcal{I}\times\mathcal{S}_{m-1})}\xi_T(t,\mathbf{v})-b_T\big)\le x, a_T\big(\max_{(t,\mathbf{v})\in(\mathfrak{R}(\delta)\times\mathfrak{R}_b)\cap(\mathcal{I}\times\mathcal{S}_{m-1})}\xi_T(t,\mathbf{v})-b_{\delta,T}\big)\le y\right\}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-\frac{1}{2}\|\mathbf{z}\|^2} \mathbb{P}\Big\{ a_T\big(\max_{(t,\mathbf{v})\in\mathfrak{R}_b^\alpha\cap(\mathcal{I}\times\mathcal{S}_{m-1})} \xi_T(t,\mathbf{v}) - b_T\big) \le x, \\
&\quad a_T\big(\max_{(t,\mathbf{v})\in(\mathfrak{R}(\delta)\times\mathfrak{R}_b)\cap(\mathcal{I}\times\mathcal{S}_{m-1})} \xi_T(t,\mathbf{v}) - b_{\delta,T}\big) \le y | Z_1 = z_1, \ldots, Z_m = z_m \Big\} dz_1 \cdots dz_m \\
&= \int_{\|\mathbf{z}\|\ge 0} \Bigg(\mathbb{P}\Bigg\{ \max_{(t,\mathbf{v})\in\mathfrak{R}_b^\alpha\cap([0,T^a]\times\mathcal{S}_{m-1})} Y(t,\mathbf{v}) \le \frac{b_T + x/a_T - \sqrt{\rho(T)}\,\|\mathbf{z}\|}{(1-\rho(T))^{1/2}}, \\
&\quad \max_{(t,\mathbf{v})\in(\mathfrak{R}(\delta)\times\mathfrak{R}_b)\cap([0,T^a]\times\mathcal{S}_{m-1})} Y(t,\mathbf{v}) \le \frac{b_{\delta,T} + y/a_T - \sqrt{\rho(T)}\,\|\mathbf{z}\|}{(1-\rho(T))^{1/2}} \Bigg\}\Bigg)^n d\mathbb{P}\{\chi_m \le \|\mathbf{z}\|\}. \qquad (24)
\end{aligned}
$$

Denote

$$
P_{n,b}(x,y) := \mathbb{P}\left\{ \max_{(t,\mathbf{v})\in\mathfrak{R}_b^\alpha\cap([0,T^a]\times\mathcal{S}_{m-1})} Y(t,\mathbf{v}) \le v_T, \max_{(t,\mathbf{v})\in(\mathfrak{R}(\delta)\times\mathfrak{R}_b)\cap([0,T^a]\times\mathcal{S}_{m-1})} Y(t,\mathbf{v}) \le v_T^* \right\}. \qquad (25)
$$

Next, we deal with the grid $\mathfrak{R}(\delta)$ being a sparse, Pickands and dense grid in turn.
(a) For the sparse grid $\mathfrak{R}(\delta)$. Using Lemmas 3.1–3.4 and (24), the first claim in Theorem 2.1 will follow if we show that

$$
\left| (P_{n,b}(x,y))^n - \exp\left(-\left(e^{-x-r+\sqrt{2r}\|\mathbf{z}\|} + e^{-y-r+\sqrt{2r}\|\mathbf{z}\|} \right) \right) \right| \to 0.
$$

Since $\lim_{T\to\infty} P_{n,b}(x,y) = 1$ uniformly for all $x, y \in \mathbb{R}$ and thus

$$
(P_{n,b}(x,y))^n = \exp\left(n \ln P_{n,b}(x,y)\right) = \exp\left(-n(1-P_{n,b}(x,y))(1+o(1))\right).
$$

Finally, using Lemma 3.3 for sparse grids, we get that

$$
n\left(1 - P_{n,b}(x,y)\right) = nT^{a-1}\left(e^{-x-r+\sqrt{2r}\|\mathbf{z}\|} + e^{-y-r+\sqrt{2r}\|\mathbf{z}\|} \right)(1+o(1)),
$$

which together with the fact that $n = T/(T^a + T^c), 0 < c < a < 1$ and the dominated convergence theorem completes the proof for sparse grid.
(b) For the Pickands grid $\mathfrak{R}(\delta)$ with $\delta(T) = D(2\ln T)^{-1/\alpha}$. Similarly as for the sparse grid, it suffices to show that

$$
n\left(1 - P_{n,b}(x,y)\right) = \left(e^{-x} + e^{-y} - \pi^{(m-1)/2} \mathcal{H}_{a,\boldsymbol{\alpha}_0}^{\ln \mathcal{H}_{a,\alpha}+x, \ln \mathcal{H}_\alpha + y} \right) e^{-r+\sqrt{2r}\|\mathbf{z}\|}(1+o(1))
$$

with $P_{n,b}(x,y)$ defined in (25). This is verified by Lemma 3.3 for Pickands grids.
(c) For the dense grid $\mathfrak{R}(\delta)$. In view of Lemma 3 of [27], we have

$$
\begin{aligned}
&\Bigg| \mathbb{P}\left\{ a_T\big(\max_{(t,\mathbf{v})\in[0,T]\times\mathcal{S}_{m-1}} Y(t,\mathbf{v}) - b_T\big) \le x, a_T\big(\max_{(t,\mathbf{v})\in\mathfrak{R}(\delta)\cap[0,T]\times\mathcal{S}_{m-1}} Y(t,\mathbf{v}) - b_T\big) \le y \right\} \\
&\quad - \mathbb{P}\left\{ a_T\big(\max_{(t,\mathbf{v})\in[0,T]\times\mathcal{S}_{m-1}} Y(t,\mathbf{v}) - b_T\big) \le x, a_T\big(\max_{(t,\mathbf{v})\in[0,T]\times\mathcal{S}_{m-1}} Y(t,\mathbf{v}) - b_T\big) \le y \right\} \Bigg| \\
&\le \left| \mathbb{P}\left\{ a_T\big(\max_{(t,\mathbf{v})\in\mathfrak{R}(\delta)\cap[0,T]\times\mathcal{S}_{m-1}} Y(t,\mathbf{v}) - b_T\big) \le y \right\} - \mathbb{P}\left\{ a_T\big(\max_{(t,\mathbf{v})\in[0,T]\times\mathcal{S}_{m-1}} Y(t,\mathbf{v}) - b_T\big) \le y \right\} \right| \to 0.
\end{aligned}
$$

Further, by (5)

$$
\begin{aligned}
&\mathbb{P}\left\{ a_T\big(\max_{(t,\mathbf{v})\in[0,T]\times\mathcal{S}_{m-1}} Y(t,\mathbf{v}) - b_T\big) \le x, a_T\big(\max_{(t,\mathbf{v})\in[0,T]\times\mathcal{S}_{m-1}} Y(t,\mathbf{v}) - b_T\big) \le y \right\} \\
&= \mathbb{P}\left\{ a_T\big(\max_{(t,\mathbf{v})\in[0,T]\times\mathcal{S}_{m-1}} Y(t,\mathbf{v}) - b_T\big) \le \min(x,y) \right\} \\
&\to \mathbb{E}\exp\left(-e^{-\min(x,y)-r+\sqrt{2r}\chi_m} \right),
\end{aligned}
$$

the required claim (c) follows. Consequently, Theorem 2.1 is proved. □

4 Appendix

In this section, we give the proofs of Lemmas 3.2 and 3.3, respectively. Before we proceed the proof, let us recall some basic quantities which will be repeatedly used below. For simplicity of notation, we write $u_T = b_T + x/a_T, u_T^* = b_{\delta,T} + y/a_T$ with $a_T, b_T, b_{\delta,T}$ given by (6) and (8). Thus

$$u_T^2 = 2\ln T + (2/\alpha + m - 2)\ln\ln T + O(1), \tag{26}$$

which implies that

$$T^{-1} = Cu^{2/\alpha+m-2}\exp\left(-\frac{u^2}{2}\right)(1+o(1)). \tag{27}$$

Further, denote by $\varpi(t,s) = \max\{|r(t-s)|, |r^*(t,s)|\}$ with r^* given in (17), and define

$$\theta(t_0) = \sup_{0\le t,s\le T, |t-s|>t_0} \varpi(t,s), \quad t_0 > 0.$$

Since $\theta(t_0) \ge \vartheta(t_0) := \sup_{t_0\le|t|\le T} r(t)$, the constants c and a given in (14) hold also for $\theta(\cdot)$, i.e.,

$$0 < c < a < \frac{1-\theta(\varepsilon)}{1+\theta(\varepsilon)} < 1.$$

Note that, from the construction of $\mathfrak{R}_b$ given by (18), the number of points in $\mathfrak{R}_b \cap \mathcal{S}_{m-1}$ does not exceed $Cb^{-(m-1)}u^{m-1}(1+o(1)) = Cb^{-(m-1)}(2\ln T)^{(m-1)/2}(1+o(1))$.

Proof of Lemma 3.2: It follows by Berman's inequality (see e.g., [29]) that, with $\mathcal{D}_i = \mathfrak{R}_b^\alpha \cap (I_i \times \mathcal{S}_{m-1}), \widetilde{\mathcal{D}}_i = (\mathfrak{R}(\delta)\times\mathfrak{R}_b)\cap(I_i\times\mathcal{S}_{m-1}), i\le n$

$$\begin{aligned}
\Delta_{T,b} \quad &\le \sum_{\substack{(t,\mathbf{v})\in\mathfrak{D}_i,(s,\mathbf{w})\in\mathfrak{D}_j,\\(t,\mathbf{v})\neq(s,\mathbf{w}),1\le i,j\le n}} |\Upsilon_{r,\varrho}| \int_0^1 \frac{1}{\sqrt{1-r^{(h)}(t,\mathbf{v},s,\mathbf{w})}}\exp\left(-\frac{u_T^2}{1+r^{(h)}(t,\mathbf{v},s,\mathbf{w})}\right)dh\\
&\quad + \sum_{\substack{(t,\mathbf{v})\in\widetilde{\mathfrak{D}}_i,(s,\mathbf{w})\in\widetilde{\mathfrak{D}}_j,\\(t,\mathbf{v})\neq(s,\mathbf{w}),1\le i,j\le n}} |\Upsilon_{r,\varrho}| \int_0^1 \frac{1}{\sqrt{1-r^{(h)}(t,\mathbf{v},s,\mathbf{w})}}\exp\left(-\frac{(u_T^*)^2}{1+r^{(h)}(t,\mathbf{v},s,\mathbf{w})}\right)dh\\
&\quad + \sum_{\substack{(t,\mathbf{v})\in\mathfrak{D}_i,(s,\mathbf{w})\in\widetilde{\mathfrak{D}}_j,\\(t,\mathbf{v})\neq(s,\mathbf{w}),1\le i,j\le n}} |\Upsilon_{r,\varrho}| \int_0^1 \frac{1}{\sqrt{1-r^{(h)}(t,\mathbf{v},s,\mathbf{w})}}\exp\left(-\frac{u_T^2+(u_T^*)^2}{2(1+r^{(h)}(t,\mathbf{v},s,\mathbf{w}))}\right)dh\\
&=: \quad \Delta_{T,b}^{(1)} + \Delta_{T,b}^{(2)} + \Delta_{T,b}^{(3)},
\end{aligned} \tag{28}$$

where $\Upsilon_{r,\varrho} = r(t,\mathbf{v},s,\mathbf{w}) - \varrho(t,\mathbf{v},s,\mathbf{w})$ and $r^{(h)}(t,\mathbf{v},s,\mathbf{w}) = hr(t,\mathbf{v},s,\mathbf{w}) + (1-h)\varrho(t,\mathbf{v},s,\mathbf{w})$. Next, we shall show that $\Delta_{T,b}^{(i)} = o(1), i=1,2,3$ for sufficiently large T and small $b>0$, respectively.

We shall present first the proof for $\Delta_{T,b}^{(1)} = o(1)$. To this end, we consider below the sum with $(t,\mathbf{v}),(s,\mathbf{w})$ in the same $\mathfrak{D}_i, 1\le i\le n$, denoted by $\Delta_{T,b}^{(1,1)}$, and split further the sum into two parts as follows

$$\Delta_{T,b}^{(1,1)} = \sum_{\substack{(t,\mathbf{v}),(s,\mathbf{w})\in\mathcal{D}_i,\\1\le i\le n,|t-s|\le\varepsilon}} + \sum_{\substack{(t,\mathbf{v}),(s,\mathbf{w})\in\mathcal{D}_i,\\1\le i\le n,|t-s|>\varepsilon}} =: J_{T,1} + J_{T,2} \tag{29}$$

for some small $\varepsilon>0$ such that (13) and (14) hold. For $J_{T,1}$, note that in this case, it follows from (17) that $|\Upsilon_{r,\varrho}| = \rho(T)(1-r(t-s))A(\mathbf{v},\mathbf{w})$, and by (2) that, we can choose small $\varepsilon>0$ such that

$$r^{(h)}(t,\mathbf{v},s,\mathbf{w}) = (r(t-s) + (1-h)(1-r(t-s))\rho(T))A(\mathbf{v},\mathbf{w}) = r(t-s)A(\mathbf{v},\mathbf{w})(1+o(1))$$

holds for sufficiently large T and $|t-s|\le\varepsilon$. Consequently, we have (recall that $A(\mathbf{v},\mathbf{w})\le 1$)

$$J_{T,1} \quad \le \quad C\sum_{\substack{(t,\mathbf{v}),(s,\mathbf{w})\in\mathcal{D}_i,\\1\le i\le n,|t-s|\le\varepsilon}} \rho(T)\sqrt{1-r(t-s)}\exp\left(-\frac{u_T^2}{1+r(t-s)|A(\mathbf{v},\mathbf{w})|}\right)$$

$$
\begin{aligned}
&\le CTb^{-1}u_T^{2/\alpha}\rho(T)\exp\left(-\frac{u_T^2}{2}\right)\sum_{\substack{\mathbf{v},\mathbf{w}\in\mathfrak{R}_b\cap\mathcal{S}_{m-1},\\ t\in\widetilde{\mathcal{R}}_b\cap[0,T],|t|\le\varepsilon}}\sqrt{1-r(t)}\exp\left(-\frac{(1-r(t)|A(\mathbf{v},\mathbf{w})|)u_T^2}{2(1+r(t)|A(\mathbf{v},\mathbf{w})|)}\right)\\
&= CTb^{-1}u_T^{2/\alpha}\rho(T)\exp\left(-\frac{u_T^2}{2}\right)\sum_{\substack{\mathbf{v},\mathbf{w}\in\mathfrak{R}_b\cap\mathcal{S}_{m-1},\\ t\in\widetilde{\mathcal{R}}_b\cap[0,T],|t|\le\varepsilon}}\sqrt{1-r(t)}\exp\left(-\frac{(1-r(t))u_T^2}{2(1+r(t))}\right)\\
&\quad\times\exp\left(-\frac{r(t)(1-|A(\mathbf{v},\mathbf{w})|)u_T^2}{2(1+r(t))(1+r(t)|A(\mathbf{v},\mathbf{w})|)}\right)\\
&\le CTb^{-m}u_T^{2/\alpha}u_T^{m-1}\rho(T)\exp\left(-\frac{u_T^2}{2}\right)\sum_{\substack{\mathbf{v}\in\mathfrak{R}_b\cap\mathcal{S}_{m-1},\\ t\in\widetilde{\mathcal{R}}_b\cap[0,T],|t|\le\varepsilon}}\sqrt{1-r(t)}\exp\left(-\frac{(1-r(t))u_T^2}{2(1+r(t))}\right)\\
&\quad\times\exp\left(-\frac{r(t)(1-|A(\mathbf{v},\mathbf{w}_0)|)u_T^2}{2(1+r(t))(1+r(t)|A(\mathbf{v},\mathbf{w}_0)|)}\right),
\end{aligned}
$$

where $\mathbf{w}_0$ is any fixed point on $\mathfrak{R}_b\cap\mathcal{S}_{m-1}$. Since

$$
\begin{aligned}
&\sum_{\substack{\mathbf{v}\in\mathfrak{R}_b\cap\mathcal{S}_{m-1},\\ t\in\widetilde{\mathcal{R}}_b\cap[0,T],|t|\le\varepsilon}}\exp\left(-\frac{r(t)(1-|A(\mathbf{v},\mathbf{w}_0)|)u_T^2}{2(1+r(t))(1+r(t)|A(\mathbf{v},\mathbf{w}_0)|)}\right)\\
&\le\sum_{\mathbf{v}\in\mathfrak{R}_b\cap\mathcal{S}_{m-1}}\exp\left(-Cu_T^2\left\|\mathbf{v}-\mathbf{w}_0\right\|^2\right)\le C,
\end{aligned}
$$

it follows further by (13), (27) and $\rho(T)=r/\ln T=O(u_T^{-2})$ that

$$
\begin{aligned}
J_{T,1} &\le CTb^{-m}u_T^{2/\alpha}u_T^{m-1}\rho(T)\exp\left(-\frac{u_T^2}{2}\right)\sum_{t\in\widetilde{\mathcal{R}}_b\cap[0,T],|t|\le\varepsilon}\sqrt{1-r(t)}\exp\left(-\frac{(1-r(t))u_T^2}{2(1+r(t))}\right)\\
&\le Cb^{-m}u_T^{-1}\sum_{t\in\widetilde{\mathcal{R}}_b\cap[0,T],|t|\le\varepsilon}\sqrt{2}|t|^{\alpha/2}\exp\left(-\frac{|t|^\alpha u_T^2}{8}\right)\\
&\le Cb^{-m}u_T^{-1}\sum_{k=1}^{\infty}\exp\left(-\frac{1}{4}(kb)^\alpha\right)\\
&\le Cb^{-m}u_T^{-1},
\end{aligned}
$$

which implies that $J_{T,1}=o(1)$ uniformly for $b>0$ as $T\to\infty$.
Using the fact that $u_T=a_T(1+o(1))$, we obtain

$$
\begin{aligned}
J_{T,2} &\le C\sum_{\substack{(t,\mathbf{v}),(s,\mathbf{w})\in\mathcal{D}_i,\\ 1\le i\le n,|t-s|>\varepsilon}}\exp\left(-\frac{u_T^2}{1+|r(t-s)|}\right)\\
&\le CT^{1+a}b^{-2m}u_T^{4/\alpha}u_T^{2m-2}\exp\left(-\frac{u_T^2}{1+\theta(\varepsilon)}\right)\\
&\le CT^{1+a}b^{-2m}u_T^{4/\alpha}u_T^{2m-2}T^{-\frac{2}{1+\theta(\varepsilon)}}\\
&\le CT^{a-\frac{1-\theta(\varepsilon)}{1+\theta(\varepsilon)}}b^{-2m}(\ln T)^{2/\alpha+m-1}.
\end{aligned}\tag{30}
$$

Thus, $J_{T,2}=o(1)$ uniformly for $b>0$ as $T\to\infty$ since $a<(1-\theta(\varepsilon))/(1+\theta(\varepsilon))$.
Next, we consider the sum $\Delta_{T,b}^{(1)}$ with $(t,\mathbf{v}),(s,\mathbf{w})$ in $\mathfrak{D}_i,\mathfrak{D}_j$ with $1\le i\neq j\le n$, denoted by $\Delta_{T,b}^{(1,0)}$. Note that in this case, $|t-s|\ge T^c$ and $\varrho(s,\mathbf{v},t,\mathbf{w})=\rho(T)A(\mathbf{v},\mathbf{w})$. Choose β such that $0<c<a<\beta<(1-\theta(\varepsilon))/(1+\theta(\varepsilon))$ and split the sum$\Delta_{T,b}^{(1,0)}$ into two parts as follows

$$
\Delta_{T,b}^{(1,0)}=\sum_{\substack{(t,\mathbf{v})\in\mathcal{D}_i,(s,\mathbf{w})\in\mathcal{D}_j,\\ 1\le i\neq j\le n,|t-s|\le T^\beta}}+\sum_{\substack{(t,\mathbf{v})\in\mathcal{D}_i,(s,\mathbf{w})\in\mathcal{D}_j,\\ 1\le i\neq j\le n,|t-s|>T^\beta}}=:S_{T,1}+S_{T,2}.\tag{31}
$$

For $S_{T,1}$, with the similar derivation as for (30), we have

$$
S_{T,1}\le C\sum_{\substack{(t,\mathbf{v})\in\mathcal{D}_i,(s,\mathbf{w})\in\mathcal{D}_j,\\ 1\le i\neq j\le n,|t-s|\le T^\beta}}\exp\left(-\frac{u_T^2}{1+r(t-s)}\right)
$$

$$
\begin{aligned}
&\leq CT^{1+\beta}b^{-2m}u_T^{4/\alpha}u_T^{2m-2}\exp\left(-\frac{u_T^2}{1+\theta(\varepsilon)}\right)\\
&\leq CT^{1+\beta}b^{-2m}u_T^{4/\alpha}u_T^{2m-2}T^{-\frac{2}{1+\theta(\varepsilon)}}\\
&\leq CT^{\beta-\frac{1-\theta(\varepsilon)}{1+\theta(\varepsilon)}}b^{-2m}(\ln T)^{2/\alpha+m-1} \qquad (32)
\end{aligned}
$$

implying that $S_{T,1} = o(1)$ uniformly for $b > 0$, since $\beta < (1-\theta(\varepsilon))/(1+\theta(\varepsilon))$.
For $S_{T,2}$, we need some more precise estimation. By condition (3), there exists some constant $K > 0$ such that

$$
\theta(t)\ln t \leqslant K
$$

for t, T sufficiently large. Thus $\theta(t) \leq K/\ln T^\beta,\ t \geq T^\beta$ holds for T large enough. Now using (26), we obtain

$$
\begin{aligned}
T^2 u_T^{4/\alpha+2m-2}(\ln T)^{-1}\exp\left(-\frac{u_T^2}{1+\theta(T^\beta)}\right) &\leq T^2 u_T^{4/\alpha+2m-2}(\ln T)^{-1}\exp\left(-\frac{u_T^2}{1+K/\ln T^\beta}\right)\\
&\leq C\left(T^2(\ln T)^{2/\alpha+m-2}\right)^{1-\frac{1}{1+K/\ln T^\beta}} \leq C. \qquad (33)
\end{aligned}
$$

Therefore, by similar arguments as for Lemma 6.4.1 of [19] we have

$$
\begin{aligned}
S_{T,2} &\leq C \sum_{\substack{(t,\mathbf{v})\in\mathcal{D}_i,(s,\mathbf{w})\in\mathcal{D}_j,\\ 1\leq i\neq j\leq n,|t-s|>T^\beta}} |r(t-s)-\rho(T)|\exp\left(-\frac{u_T^2}{1+\theta(T^\beta)}\right)\\
&\leq CTb^{-(2m-2)}u_T^{2/\alpha}u_T^{2m-2}\exp\left(-\frac{u_T^2}{1+\theta(T^\beta)}\right)\sum_{t\in\widetilde{\mathfrak{R}}_b\cap[0,T],t>T^\beta}|r(t)-\rho(T)|\\
&= CT^2(\ln T)^{-1}u_T^{4/\alpha}u_T^{2m-2}\exp\left(-\frac{u_T^2}{1+\theta(T^\beta)}\right)\cdot b^{-(2m-2)}\frac{\ln T}{Tu_T^{2/\alpha}}\sum_{t\in\widetilde{\mathfrak{R}}_b\cap[0,T],t>T^\beta}|r(t)-\rho(T)|\\
&\leq Cb^{-(2m-2)}\frac{\ln T}{Tu_T^{2/\alpha}}\sum_{t\in\widetilde{\mathfrak{R}}_b\cap[0,T],t>T^\beta}|r(t)-\rho(T)|\\
&\leq Cb^{-(2m-2)}\frac{1}{\beta Tu_T^{2/\alpha}}\sum_{t\in\widetilde{\mathfrak{R}}_b\cap[0,T],t>T^\beta}|r(t)\ln t - r| + Cb^{-(2m-2)}\frac{r}{Tu_T^{2/\alpha}}\sum_{t\in\widetilde{\mathfrak{R}}_b\cap[0,T],t>T^\beta}\left|1-\frac{\ln T}{\ln t}\right|,
\end{aligned}
$$

where, by (3), the first term is $o(1)$ uniformly for $b > 0$, and the second term is also $o(1)$ uniformly for $b > 0$ following an integral estimate below (see also the proof of Lemma 6.4.1 in [19])

$$
\begin{aligned}
Cb^{-(2m-1)}\frac{r}{Tu_T^{2/\alpha}}\sum_{t\in\widetilde{\mathfrak{R}}_b\cap[0,T],t>T^\beta}\left|1-\frac{\ln T}{\ln t}\right| &\leq Cb^{-(2m-1)}\frac{r}{Tu_T^{2/\alpha}}\frac{1}{\ln T^\beta}\sum_{t\in\widetilde{\mathfrak{R}}_b\cap[0,T],t>T^\beta}|\ln t-\ln T|\\
&= Cb^{-(2m-1)}\frac{r}{\ln T^\beta}\int_0^1|\ln x|dx.
\end{aligned}
$$

Consequently, combining the assertions for $J_{T,i}, S_{T,i}, i = 1,2$ in (29), (31), we have $\Delta_{T,b}^{(1)} = o(1)$.
The proof of $\Delta_{T,b}^{(2)} = o(1)$ is similar as that for $\Delta_{T,b}^{(1)} = o(1)$ with minor modifications by replacing $u_T, \mathfrak{D}_i, i \leq n$ by $u_T^*, \widetilde{\mathfrak{D}}_i, i \leq n$, we omit thus the details.
It remains to prove $\Delta_{T,b}^{(3)} = o(1)$. Recall that $\mathfrak{R}(\delta)$ can be a sparse grid or a Pickands grid. We only show below the proof for $\mathfrak{R}(\delta)$ a sparse grid by following the main arguments as for $\Delta_{T,b}^{(1)}$. The Pickands grid case can be shown similarly for the sparse grid and thus we omit it here.
Consider first the sum $\Delta_{T,b}^{(3)}$ with t, s in the same $I_i, i \leq n$, which is further split into two parts as

$$
\Delta_{T,b}^{(3,1)} := \sum_{\substack{(t,\mathbf{v})\in\mathcal{D}_i,(s,\mathbf{w})\in\widetilde{\mathcal{D}}_i,\\ 1\leq i\leq n,|t-s|\leq\varepsilon}} + \sum_{\substack{(t,\mathbf{v})\in\mathcal{D}_i,(s,\mathbf{w})\in\widetilde{\mathcal{D}}_i,\\ 1\leq i\leq n,|t-s|>\varepsilon}} =: \widetilde{J}_{T,1} + \widetilde{J}_{T,2}. \qquad (34)
$$

Note that

$$
\widetilde{u}_T^2 := \frac{1}{2}(u_T^2 + (u_T^*)^2) = 2\ln T + \ln a_T^{2/\alpha+m-2} + \ln(\delta^{-1}a_T^{m-2}) + O(1) \qquad (35)
$$

and the grid $\mathfrak{R}(\delta)$ is a sparse grid, i.e., $\lim_{T\to\infty}\delta(2\ln T)^{1/2}=\infty$, the remaining proof of $\widetilde{J}_{T,1}+\widetilde{J}_{T,2}=o(1)$ is similar to that for $J_{T,1}$ and thus we omit it here.

Next, for the remaining sum $\Delta_{T,b}^{(3)}-\Delta_{T,b}^{(3,1)}$, i.e., the summand with t,s in the different intervals $I_i,I_j, 1\le i\neq j\le n$, one can show that (recall (35))

$$T^2u_T^{2/\alpha+2m-2}\delta^{-1}(\ln T)^{-1}\exp\left(-\frac{\widetilde{u}_T^2}{1+\theta(T^\beta)}\right)=O(1).$$

The rest proof of $\Delta_{T,b}^{(3)}-\Delta_{T,b}^{(3,1)}=o(1)$ is the same as that for (31). Consequently, we have $\Delta_{T,b}^{(3)}=o(1)$, which together with (28) and the proved $\Delta_{T,b}^{(i)}=o(1), i=1,2$, completes the proof of Lemma 3.2. □

Proof of Lemma 3.3: First, noting that

$$\mathbb{P}\left\{\max_{(t,\mathbf{v})\in\mathfrak{R}_b^\alpha\cap([0,T^a]\times\mathcal{S}_{m-1})}Y(t,\mathbf{v})>v_T\right\}\le\mathbb{P}\left\{\max_{(t,\mathbf{v})\in[0,T^a]\times\mathcal{S}_{m-1}}Y(t,\mathbf{v})>v_T\right\}=\mathbb{P}\left\{M_m(T^a)>v_T\right\},$$

the first assertion follows thus by (4) with elementary calculations, since (4) holds also for $T=T(u)\to\infty$ with suitable speed, see Theorem 7.2 of [29].

Next, we will show the proofs of (20) and (21) with $\mathfrak{R}(\delta)$ a Pickands and sparse grid, respectively.

Proof of (20) with $\mathfrak{R}(\delta)$ a Pickands or sparse grid. The proof for the Pickands grid is similar as that for Corollary 7.3 in [29] with minor modification (replacing $\mathcal{H}_\alpha$ by $\mathcal{H}_{D,\alpha}$), and thus we omit the details here.

Now, we consider $\mathfrak{R}(\delta)$ a sparse grid. For simplicity, we denote in the following $\widetilde{\mathcal{D}}=(\mathfrak{R}(\delta)\times\mathfrak{R}_b)\cap([0,T^a]\times\mathcal{S}_{m-1})$ and

$$P_{T,b}^{(1)}:=\sum_{t\in\mathfrak{R}(\delta)\cap[0,T^a]}\mathbb{P}\left\{\max_{\mathbf{v}\in\mathfrak{R}_b\cap\mathcal{S}_{m-1}}Y(t,\mathbf{v})>v_T^*\right\},\quad P_{T,b}^{(2)}:=\sum_{(t,\mathbf{v})\neq(s,\mathbf{w})\in\widetilde{\mathfrak{D}}}\mathbb{P}\left\{Y(t,\mathbf{v})>v_T^*,Y(s,\mathbf{w})>v_T^*\right\}.$$

By Bonferroni's inequality, we have

$$P_{T,b}^{(1)}-P_{T,b}^{(2)}\le\mathbb{P}\left\{\max_{(t,\mathbf{v})\in\widetilde{\mathfrak{D}}}Y(t,\mathbf{v})>v_T^*\right\}\le P_{T,b}^{(1)},$$

therefore, it suffices to show that

$$P_{T,b}^{(1)}=T^{a-1}e^{-y-r+\sqrt{2r}\|\mathbf{z}\|}(1+o(1)),\quad P_{T,b}^{(2)}=o(P_{T,b}^{(1)})$$

hold for sufficiently large T and small $b>0$. Clearly,

$$\begin{aligned}P_{T,b}^{(1)} &= (1+o(1))\sum_{t\in\mathfrak{R}(\delta)\cap[0,T^a]}\mathbb{P}\left\{\max_{\mathbf{v}\in\mathcal{S}_{m-1}}Y(t,\mathbf{v})>v_T^*\right\}\\ &= (1+o(1))T^a\delta^{-1}\mathbb{P}\left\{\chi_m(t)>v_T^*\right\}\\ &= (1+o(1))T^{a-1}e^{-y-r+\sqrt{2r}\|\mathbf{z}\|}\end{aligned}$$

following by elementary calculations. It remains to deal with $P_{T,b}^{(2)}$. Split the term $P_{T,b}^{(2)}$ into two parts as

$$P_{T,b}^{(2)}=\sum_{(t,\mathbf{v})\neq(s,\mathbf{w})\in\widetilde{\mathfrak{D}},|t-s|<\epsilon}+\sum_{(t,\mathbf{v})\neq(s,\mathbf{w})\in\widetilde{\mathfrak{D}},|t-s|\ge\epsilon}=:P_{T,21}+P_{T,22}.\tag{36}$$

Using the well-known results for bivariate Gaussian tail probability (see e.g., p. 225 in [19]), we have

$$P_{T,21}\le\sum_{(t,\mathbf{v})\neq(s,\mathbf{w})\in\widetilde{\mathfrak{D}},|t-s|<\epsilon}\left[\overline{\Phi}(v_T^*)\overline{\Phi}\left(v_T^*\frac{\sqrt{1-r(t,\mathbf{v},s,\mathbf{w})}}{\sqrt{1+r(t,\mathbf{v},s,\mathbf{w})}}\right)\right].$$

By (12) and (13), we can choose $\epsilon>0$ small enough such that

$$\frac{1-r(t,\mathbf{v},s,\mathbf{w})}{1+r(t,\mathbf{v},s,\mathbf{w})}\ge\frac{1}{4}|t-s|^\alpha+\frac{1}{8}\|\mathbf{v}-\mathbf{w}\|^2$$

and we thus have

$$
\begin{aligned}
P_{T,21} &\leq C \sum_{(t,\mathbf{v})\neq(s,\mathbf{w})\in\widetilde{\mathfrak{D}},|t-s|<\epsilon} \left[\overline{\Phi}(v_T^*)\overline{\Phi}\left(v_T^*\sqrt{\frac{|t-s|^\alpha}{4}+\frac{\|\mathbf{v}-\mathbf{w}\|^2}{8}}\right)\right] \\
&\leq C\overline{\Phi}(v_T^*) \sum_{(t,\mathbf{v})\neq(s,\mathbf{w})\in\widetilde{\mathfrak{D}},|t-s|<\epsilon} \frac{1}{|t-s|^{\alpha/2}v_T^*}\exp\left(-\frac{1}{8}|t-s|^\alpha(v_T^*)^2\right)\exp\left(-\frac{1}{16}\|\mathbf{v}-\mathbf{w}\|^2(v_T^*)^2\right) \\
&= C\overline{\Phi}(v_T^*) \sum_{t,s\in\mathfrak{R}(\delta)\cap[0,T^a],0<|t-s|<\epsilon} \frac{1}{|t-s|^{\alpha/2}v_T^*}\exp\left(-\frac{1}{8}|t-s|^\alpha(v_T^*)^2\right) \\
&\quad\times \sum_{\mathbf{v}\neq\mathbf{w}\in\mathfrak{R}_b\cap\mathcal{S}_{m-1}} \exp\left(-\frac{1}{16}\|\mathbf{v}-\mathbf{w}\|^2(v_T^*)^2\right) \\
&\leq C\overline{\Phi}(v_T^*) \sum_{t,s\in\mathfrak{R}(\delta)\cap[0,T^a],0<|t-s|<\epsilon} \frac{1}{|t-s|^{\alpha/2}v_T^*}\exp\left(-\frac{1}{8}|t-s|^\alpha(v_T^*)^2\right) \\
&\quad\times b^{-(m-1)}u_T^{m-1} \sum_{\mathbf{v}\in\mathfrak{R}_b\cap\mathcal{S}_{m-1}} \exp\left(-\frac{1}{16}\|\mathbf{v}-\mathbf{w}_0\|^2(v_T^*)^2\right).
\end{aligned}
$$

where $\mathbf{w}_0$ is any fixed point on $\mathfrak{R}_b\cap\mathcal{S}_{m-1}$. Since

$$
\sum_{\mathbf{v}\in\mathfrak{R}_b\cap\mathcal{S}_{m-1}} \exp\left(-\frac{1}{16}\|\mathbf{v}-\mathbf{w}_0\|^2(v_T^*)^2\right)\leq C,
$$

for sufficiently large T. Using further the definition of v_T^* we obtain

$$
\begin{aligned}
P_{T,21} &\leq CT^a\delta^{-1}b^{-(m-1)}u_T^{m-1}\overline{\Phi}(v_T^*)\sum_{0<k\delta\leq\epsilon}\frac{1}{(k\delta)^{\alpha/2}v_T^*}\exp\left(-\frac{1}{8}(k\delta)^\alpha(v_T^*)^2\right) \\
&= CT^{a-1}b^{-(m-1)}\sum_{0<k\delta\leq\epsilon}\frac{1}{[k\delta(\ln T)^{1/\alpha}]^{\alpha/2}}\exp\left(-\frac{1}{4}[k\delta(\ln T)^{1/\alpha}]^\alpha\right)(1+o(1)) \\
&\leq CT^{a-1}b^{-(m-1)}\frac{1}{[(\ln T)^{1/\alpha}\delta]^{\alpha/2}}\sum_{0<k\leq\lfloor\epsilon/\delta\rfloor+1}\exp\left(-\frac{1}{4}[k\delta(\ln T)^{1/\alpha}]^\alpha\right)(1+o(1)) \\
&\leq CT^{a-1}b^{-(m-1)}\frac{1}{[(\ln T)^{1/\alpha}\delta]^{\alpha/2}}(1+o(1)) \\
&= T^{a-1}b^{-(m-1)}o(1),
\end{aligned}
$$

where we used additionally the fact that $\lim_{T\to\infty}(\ln T)^{1/\alpha}\delta=\infty$, since $\mathfrak{R}(\delta)$ is a sparse grid. Thus, we have $P_{T,21}=o(T^{a-1})$ uniformly for $b>0$ as $T\to\infty$.

For the second term $P_{T,22}$ in (36), by Normal Comparison Lemma, we have

$$
\begin{aligned}
P_{T,22} &\leq \sum_{(t,\mathbf{v})\neq(s,\mathbf{w})\in\widetilde{\mathfrak{D}},|t-s|\geq\epsilon}\left[\overline{\Phi}^2(v_T^*)+C\exp\left(-\frac{(v_T^*)^2}{1+|r(t,\mathbf{v},s,\mathbf{w})|}\right)\right] \\
&\leq CT^a\delta^{-1}b^{-2(m-1)}u_T^{2(m-1)}\sum_{\epsilon\leq k\delta\leq T^a}\left[\overline{\Phi}^2(v_T^*)+C\exp\left(-\frac{(v_T^*)^2}{1+|r(k\delta)|}\right)\right] \\
&\leq CT^{2a}\delta^{-2}b^{-2(m-1)}u_T^{2(m-1)}\left[\overline{\Phi}^2(v_T^*)+C\exp\left(-\frac{(v_T^*)^2}{1+\vartheta(\epsilon)}\right)\right] \\
&=: P_{T,221}+P_{T,222}.
\end{aligned}
$$

By (19), we have $v_T^*=u_T(1+o(1))$. Therefore,

$$
\begin{aligned}
P_{T,221} &\leq CT^{2a}\delta^{-2}b^{-2(m-1)}u_T^{2(m-1)}\frac{\varphi^2(v_T^*)}{(v_T^*)^2} \\
&\leq CT^{2a}\delta^{-2}b^{-2(m-1)}u_T^{2(m-2)}\exp\left(-(v_T^*)^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\le CT^{2a}\delta^{-2}b^{-2(m-1)}u_T^{2(m-2)}[T^{-1}\delta u_T^{-(m-2)}]^2\\
&= o(T^{a-1})
\end{aligned}
$$

uniformly for $b>0$ as $T\to\infty$. Since $u_T=v_T^*(1+o(1))=(2\ln T)^{1/2}(1+o(1))$

$$
\begin{aligned}
P_{T,222} &\le CT^{2a}\delta^{-2}b^{-2(m-1)}u_T^{2(m-1)}\exp\left(-\frac{(v_T^*)^2}{1+\vartheta(\epsilon)}\right)\\
&\le CT^{2a}\delta^{-2}b^{-2(m-1)}u_T^{2(m-1)}T^{-\frac{2}{1+\vartheta(\epsilon)}}\\
&\le CT^{a-1}T^{a-\frac{1-\vartheta(\epsilon)}{1+\vartheta(\epsilon)}}\delta^{-2}b^{-2(m-1)}(\ln T)^{m-1}.
\end{aligned}
$$

Both (14) and $(\ln T)^{1/\alpha}\delta=\infty$ imply $S_{T,22}=o(T^{a-1})$ uniformly for $b>0$ as $T\to\infty$. This completes the proof of the second assertion.

Proof of (21) with $\mathfrak{R}(\delta)$ a sparse grid For simplicity, we denote below $\mathfrak{D}:=\mathfrak{R}_b^\alpha\cap([0,T^a]\times\mathcal{S}_{m-1})$. Obviously, we have

$$
\begin{aligned}
&\mathbb{P}\left\{\max_{(t,\mathbf{v})\in\mathfrak{D}}Y(t,\mathbf{v})>v_T,\ \max_{(t,\mathbf{v})\in\widetilde{\mathfrak{D}}}Y(t,\mathbf{v})>v_T^*\right\}\\
&=\sum_{(t,\mathbf{v})\in\mathfrak{D},(s,\mathbf{w})\in\widetilde{\mathfrak{D}},|t-s|<\epsilon}+\sum_{(t,\mathbf{v})\in\mathfrak{D},(s,\mathbf{w})\in\widetilde{\mathfrak{D}},|t-s|\ge\epsilon}=:Q_{T,21}+Q_{T,22}.
\end{aligned}
$$

By the same argument as for the term $P_{T,21}$, we have for $\mathbf{w}_0$ fixed on $\mathfrak{R}_b\cap\mathcal{S}_{m-1}$

$$
\begin{aligned}
Q_{T,21} &\le C\sum_{(t,\mathbf{v})\in\mathcal{D},(s,\mathbf{w})\in\widetilde{\mathfrak{D}},|t-s|<\epsilon}\left[\overline{\Phi}(v_T)\overline{\Phi}\left(v_T^*(\frac{1}{4}|t-s|^\alpha+\frac{1}{8}\|\mathbf{v}-\mathbf{w}\|^2)^{1/2}\right)\right]\\
&\le C\overline{\Phi}(v_T)\sum_{(t,\mathbf{v})\in\mathcal{D},(s,\mathbf{w})\in\widetilde{\mathfrak{D}},|t-s|<\epsilon}\frac{1}{|t-s|^{\alpha/2}v_T^*}\exp\left(-\frac{1}{8}|t-s|^\alpha(v_T^*)^2\right)\exp\left(-\frac{1}{16}\|\mathbf{v}-\mathbf{w}\|^2(v_T^*)^2\right)\\
&\le C\overline{\Phi}(v_T)\sum_{\substack{t\in\widetilde{\mathfrak{R}}_b\cap[0,T^a]\\ s\in\mathfrak{R}(\delta)\cap[0,T^a],|t-s|<\epsilon}}\frac{1}{|t-s|^{\alpha/2}v_T^*}\exp\left(-\frac{1}{8}|t-s|^\alpha(v_T^*)^2\right)\\
&\quad\times b^{-(m-1)}u_T^{(m-1)}\sum_{\mathbf{v}\in\mathfrak{R}_b\cap\mathcal{S}_{m-1}}\exp\left(-\frac{1}{16}\|\mathbf{v}-\mathbf{w}_0\|^2(v_T^*)^2\right)\\
&\le CT^ab^{1-m}u_T^{2/\alpha}u_T^{(m-1)}\overline{\Phi}(v_T)\sum_{0<k\delta\le\epsilon}\frac{1}{(k\delta)^{\alpha/2}v_T^*}\exp\left(-\frac{1}{8}(k\delta)^\alpha(v_T^*)^2\right)\\
&\le CT^{a-1}b^{1-m}\sum_{0<k\delta\le\epsilon}\frac{1}{(k\delta)^{\alpha/2}(\ln T)^{1/2}}\exp\left(-\frac{1}{4}(k\delta)^\alpha\ln T\right)\\
&\le CT^{a-1}b^{1-m}\frac{1}{(\ln T)^{1/2}\delta^{\alpha/2}}\sum_{0<k\le[\epsilon/\delta]+1}\exp\left(-\frac{1}{4}(k\delta)^\alpha\ln T\right)\\
&\le CT^{a-1}b^{1-m}\frac{1}{[(\ln T)^{1/\alpha}\delta]^{\alpha/2}}\\
&= T^{a-1}o(1),
\end{aligned}
$$

uniformly for $b>0$, where we used additionally the fact that $\lim_{T\to\infty}(\ln T)^{1/\alpha}\delta=\infty$, since $\mathfrak{R}(\delta)$ is a sparse grid.

To bound the term $Q_{T,22}$, using again Normal Comparison Lemma, with the same arguments as for the term $P_{T,22}$, we have

$$
\begin{aligned}
Q_{T,22} &\le \sum_{(t,\mathbf{v}),(s,\mathbf{w})\in\widetilde{\mathfrak{D}},|t-s|\ge\epsilon}\left[\overline{\Phi}(v_T)\overline{\Phi}(v_T^*)+C\exp\left(-\frac{v_T^2+(v_T^*)^2}{2(1+|r(t,\mathbf{v},s,\mathbf{w})|)}\right)\right]\\
&\le CT^a\delta^{-1}b^{-2(m-1)}u_T^{2(m-1)}\sum_{\epsilon\le k\delta\le T^a}\left[\overline{\Phi}(v_T)\overline{\Phi}(v_T^*)+C\exp\left(-\frac{v_T^2+(v_T^*)^2}{2(1+|r(k\delta)|)}\right)\right]
\end{aligned}
$$

$$\begin{aligned} &\leq \quad CT^{2a}\delta^{-2}b^{-2(m-1)}u_T^{2(m-1)}\left[\overline{\Phi}(v_T)\overline{\Phi}(v_T^*)+C\exp\left(-\frac{v_T^2+(v_T^*)^2}{2(1+\vartheta(\epsilon))}\right)\right] \\ &=: \quad Q_{T,221}+Q_{T,222}. \end{aligned}$$

By the same arguments as for $P_{T,221}$ and $P_{T,222}$, we can show that $Q_{T,221}=o(T^{a-1})$ and $Q_{T,222}=o(T^{a-1})$ uniformly for $b>0$ as $T\to\infty$, respectively. Consequently, (21) holds for $\mathfrak{R}(\delta)$ a sparse grid.

Proof of (21) with $\mathfrak{R}(\delta)$ a Pickands grid. We shall use below some notation and results from [29]. Let $\mathcal{A}$ be a set in $\mathbb{R}^m$ and $\mathbf{d}=(d_1,\ldots,d_m)$ with all $d_i>0, i\leq m$; denote

$$\mathbf{d}\mathcal{A}=\left(\mathbf{x}=(x_1,x_2,\ldots,x_m):\left(\frac{x_1}{d_1},\frac{x_2}{d_2},\ldots,\frac{x_m}{d_m}\right)\in\mathcal{A}\right)$$

and with $\lambda>0$ a constant

$$\mathbf{g}_u=(u^{-2/\alpha_1},u^{-2/\alpha_2},\ldots,u^{-2/\alpha_m}),\quad \mathcal{K}=[0,\lambda]^m.$$

Let $Z(\mathbf{t}),\mathbf{t}\in\mathbb{R}^m$ be a homogeneous Gaussian random field with correlation function $r_Z(\mathbf{t})$ such that, for some $\alpha_i\in(0,2], i\leq m$

$$r_Z(\mathbf{t})=1-\sum_{i=1}^m|t_i|^{\alpha_i}(1+o(1)),\quad \|\mathbf{t}\|\to 0\ \text{ and }\ r_Z(\mathbf{t})<1,\quad \forall \mathbf{t}\neq\mathbf{0}.$$

Then it follows by similar arguments as for Lemma 6.1 in [29] that

$$P\left(\max_{\mathbf{t}\in\mathbf{g}_u\mathcal{K}}Z(\mathbf{t})>u+\frac{x}{u},\max_{\mathbf{t}\in\mathbf{g}_u(\widehat{\mathfrak{R}}_D\times[0,\lambda]^{m-1})}Z(\mathbf{t})>u\right)=\mathcal{H}^{x,0}_{D,\boldsymbol{\alpha}}(\lambda)\Psi(u)(1+o(1))$$

as $u\to\infty$, where $\widehat{\mathfrak{R}}_D=\{kDu^{-2/\alpha}:kDu^{-2/\alpha}\leq\lambda,k\in\mathbb{N}\}$ with $D>0$ is a Pickands grid in $\mathbb{R}$ and $\mathcal{H}^{x,0}_{D,\boldsymbol{\alpha}}(\lambda)$ is defined by (7). It also can be proved in a similar way as for Lemma 7.1 of [29] that

$$\mathcal{H}^{x,0}_{D,\boldsymbol{\alpha}}:=\lim_{S\to\infty}\frac{\mathcal{H}^{x,0}_{D,\boldsymbol{\alpha}}(\lambda)}{\lambda^m}\in(0,\infty).$$

It is easy to check that

$$1-r(t,\mathbf{v},s,\mathbf{w})=(1+o(1))\left(|t-s|^\alpha+\sum_{i=1}^{m-1}\left(\frac{1}{\sqrt{2}}(w_i-v_i)\right)^2\right)$$

as $|t-s|\to 0$ and $\|\mathbf{v}-\mathbf{w}\|\to 0$. Now, by similar arguments as for Theorem 7.1 and Corollary 7.3 of [29], we have for sufficiently large T and small $b>0$

$$\begin{aligned} &\mathbb{P}\left\{\max_{(t,\mathbf{v})\in\mathcal{D}}Y(t,\mathbf{v})>v_T+\frac{x}{v_T},\max_{(t,\mathbf{v})\in\widetilde{\mathfrak{D}}}Y(t,\mathbf{v})>v_T\right\} \\ &=(1+o(1))\mathbb{P}\left\{\max_{(t,\mathbf{v})\in[0,T^a]\times\mathcal{S}_{m-1}}Y(t,\mathbf{v})>v_T+\frac{x}{v_T},\max_{(t,\mathbf{v})\in\mathfrak{R}(\delta)\cap([0,T^a]\times\mathcal{S}_{m-1})}Y(t,\mathbf{v})>v_T\right\} \\ &=(1+o(1))\cdot 2^{(3-m)/2}\pi^{m/2}(\Gamma(m/2))^{-1}T^a\mathcal{H}^{x,0}_{D,\boldsymbol{\alpha}_0}v_T^{2/\alpha+m-1}\Psi(v_T). \end{aligned}$$

Using further (19), we get

$$\begin{aligned} v_T &= \frac{x+r-\sqrt{2r}\,\|\mathbf{z}\|}{a_T}+b_T+o(a_T^{-1}) \\ &= v_T^*+b_T-b_{\delta,T}+(x-y)/a_T+o(a_T^{-1}) \\ &= v_T^*+\frac{\ln\mathcal{H}_\alpha-\ln\mathcal{H}_{D,\alpha}+x-y}{v_T^*}+O\left((\ln\ln(T))^2(\ln T)^{-3/2}\right). \end{aligned}$$

Observing that $v_T^*=(2\ln T)^{1/2}(1+o(1))$, we see that the reminder $O(\cdot)$ plays a negligible role. Therefore, using again (19), we have

$$\mathbb{P}\left\{\max_{(t,\mathbf{v})\in[0,T^a]\times\mathcal{S}_{m-1}}Y(t,\mathbf{v})>v_T,\max_{(t,\mathbf{v})\in\widetilde{\mathfrak{D}}}Y(t,\mathbf{v})>v_T^*\right\}$$

$$= 2^{(3-m)/2}\pi^{m/2}\Gamma^{-1}(m/2)T^a\mathcal{H}_{D,\boldsymbol{\alpha}_0}^{Z_{x,y},0}(v_T^*)^{2/\alpha+m-1}\Psi(v_T^*)(1+o(1))$$
$$= T^{a-1}\pi^{(m-1)/2}\mathcal{H}_{D,\boldsymbol{\alpha}_0}^{Z_{x,y},0}\mathcal{H}_{D,\alpha}^{-1}e^{-y-r+\sqrt{2r}\|\mathbf{z}\|}(1+o(1)),$$

where $Z_{x,y} = \ln\mathcal{H}_\alpha - \ln\mathcal{H}_{D,\alpha} + x - y$. Next, changing the variables in the definition of $\mathcal{H}_{D,\boldsymbol{\alpha}_0}^{x,y}$ we get that $\mathcal{H}_{D,\boldsymbol{\alpha}_0}^{Z_{x,y},0}\mathcal{H}_{D,\alpha}^{-1}e^{-y} = \mathcal{H}_{D,\boldsymbol{\alpha}_0}^{\ln\mathcal{H}_\alpha+x,\ln\mathcal{H}_{D,\alpha}+y}$, which completes the proof of the lemma. □

Acknowledgement: We would like to thank Enkelejd Hashorva for several valuable suggestions and discussions.

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