GENERALIZED PICKANDS CONSTANTS AND STATIONARY MAX-STABLE PROCESSES

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Abstract. Pickands constants play a crucial role in the asymptotic theory of Gaussian processes. They are commonly defined as the limits of a sequence of expectations involving fractional Brownian motions and, as such, their exact value is often unknown. Recently, [1] derived a novel representation of Pickands constant as a simple expected value that does not involve a limit operation. In this paper we show that the notion of Pickands constants and their corresponding Dieker–Yakir representations can be extended to a large class of stochastic processes, including general Gaussian and Lévy processes. We furthermore develop a link to extreme value theory and show that Pickands-type constants coincide with certain constants arising in the study of max-stable processes with mixed moving maxima representations. Brown–Resnick process and fractional Brownian motion and Gaussian process and generalized Pickands constant and Lévy process and max-stable process and mixed moving maxima representation; Dieker-Mikosch M3 representation.

1. Introduction

The seminal contribution [2] establishes the tail asymptotics of a centered stationary Gaussian process X with continuous sample paths and unit variance under the restriction that its correlation function r satisfies for some $\alpha \in (0,2]$

$$1-r(t) \sim |t|^{\alpha}, \quad t \to 0, \quad r(t)<1, \forall t>0.$$

Specifically, for any $T>0, \delta \ge 0$ we have (set $\delta\mathbb{Z}=\mathbb{R}$ if $\delta=0$)

$$\mathbb{P}\left\{ \sup_{t \in u^{-2/\alpha}\delta\mathbb{Z}\cap[0,T]} X(t) > u \right\} \sim \mathcal{H}_W^{\delta} T u^{2/\alpha} \mathbb{P}\left\{X(0)>u\right\}, \quad u \to \infty, \tag{1}$$

where the *Pickands constant* $\mathcal{H}_W^{\delta}$ introduced in [3] is given by the following limit

$$\mathcal{H}_W^{\delta} = \lim_{T\to\infty} T^{-1} \mathbb{E}\left\{ \sup_{t\in\delta\mathbb{Z}\cap[0,T]} e^{W(t)} \right\} \in (0,\infty), \quad W(t)=\sqrt{2}B_\alpha(t) - |t|^{\alpha}, \tag{2}$$

with $\{B_\alpha(t), t\ge 0\}$ a centered fractional Brownian motion with Hurst index $\alpha/2 \in (0,1]$, that is, a mean zero Gaussian process with continuous sample paths and covariance function

$$\mathrm{Cov}\{B_\alpha(s), B_\alpha(t)\} = \frac{1}{2}\Big(|t|^{\alpha}+|s|^{\alpha}-|t-s|^{\alpha}\Big), \quad s,t \ge 0.$$

The only known values of $\mathcal{H}_W^{\delta}$ are for $\delta=0$ if $\alpha=1,2$. Numerous papers have considered the calculation of Pickands constants, with particular focus on the case $\delta=0$; see for instance [4–13].
Recently, the seminal contribution [1] derived an alternative representation for $\mathcal{H}_W^{\delta}$

$$\mathcal{H}_W^{\delta} = \mathbb{E}\left\{ \frac{M^{\delta}}{S^{\eta}} \right\}, \quad \forall \delta=\eta>0, \text{ or } \delta=0, \eta\ge 0, \tag{3}$$

where

$$M^{\delta} = \sup_{t\in\delta\mathbb{Z}} e^{W(t)}, \quad S^{\eta} = \eta \sum_{t\in\eta\mathbb{Z}} e^{W(t)}, \quad S^0 = \int_{\mathbb{R}} e^{W(t)}\, dt. \tag{4}$$

The principal advantage of *Dieker–Yakir representation* (3) is that it is given as an expectation rather than as a limit, which is particularly useful for Monte Carlo simulations of $\mathcal{H}_W^{\delta}$.
Pickands constants traditionally appear in Gumbel limit theorems, see e.g., [14–16]. Such limit theorems are recently formulated for max-stable processes and provide a first link of classical Gaussian tail asymptotics to extreme value theory. Specifically, [17] showed that [see also 18, 19]

$$\lim_{T\to\infty} \mathbb{P}\left\{ \sup_{t\in\delta\mathbb{Z}\cap[0,T]} \xi_W(t) \le x + \ln T \right\} = \exp\Big(-\mathcal{H}_W^{\delta}\exp(-x)\Big), \quad x\in\mathbb{R}, \tag{5}$$

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where the so-called Brown–Resnick process ξ_W is defined as

(6) $$\xi_W(t) = \max_{i\geq 1}(P_i + W_i(t)), \quad t \in \mathbb{R}.$$

Here $\Pi = \sum_{i=1}^{\infty} \varepsilon_{P_i}$ is a Poisson point process on $\mathbb{R}$ with intensity $e^{-x}dx$, and $W_i, i \geq 1$, are independent copies of W, also independent of Π. We denote by ε_x the unit Dirac measure at $x \in \mathbb{R}$. The Brown–Resnick process ξ_W is both max-stable and stationary [20–24]. The stationarity means that the processes $\{\xi_W(t), t \in \mathbb{R}\}$ and $\{\xi_W(t+h), t \in \mathbb{R}\}$ have the same distribution for any $h \in \mathbb{R}$. Moreover, the process ξ_W arises naturally as the limit of suitably normalized pointwise maxima of independent copies of stationary Gaussian processes [20, Theorem 17]. This makes this class of processes a widely-used model in the risk assessment of spatial extreme events. The result in (5) states that $\mathcal{H}_W^\delta$ coincides with the so-called *extremal index* of the stationary, max-stable process ξ_W, a quantity that summarizes the temporal extremal dependence [c.f., 25].
Under certain conditions the process ξ_W in (6) admits a *mixed moving maxima representation* (for short M3)

(7) $$\xi_W(t) \stackrel{d}{=} \max_{i\geq 1}(P_i + F_i(t - T_i)), \quad t \in \mathbb{R},$$

with $\sum_{i=1}^{\infty} \varepsilon_{(P_i,T_i)}$ a Poisson point process in $\mathbb{R}^2$ with intensity $C_F e^{-p}dpdt$, where $C_F \in (0,\infty)$ is a suitable normalizing constant and the shape functions F_i are independent copies of a measurable càdlàg process $\{F(t), t \in \mathbb{R}\}$. Restricting the class of possible shape distributions F makes the M3 representation unique and the constant C_F is well-defined. Efficient simulation of ξ_W relies on equality (6), see [26], and the unknown constant C_F has to be estimated numerically for this purpose. Surprisingly, it turns out that C_F is in fact equal to the Dieker–Yakir constant (3). Since the latter equals the Pickands constant $\mathcal{H}_W$ under conditions derived in this paper, this also holds for C_F. Similar results are also shown for the discretized versions $\mathcal{H}_W^\delta$ and C_F^δ. This underlines the connection between max-stable processes and classical asymptotic theory of Gaussian processes. As a side product of our results, we derive a new M3 representation based on the recent work by [17] that might be of independent interest.
The objective of this paper is twofold. On the one hand, we consider generalized Pickands constants $\mathcal{H}_W^\delta$ in (2), where W is replaced by more general stochastic processes than fractional Brownian motions, which are not necessarily Gaussian. We are then interested in finding conditions for the existence and positiveness of the limit in (2), and in deriving equivalent representations of these constants. More precisely, we show that for W chosen such that ξ_W is max-stable and stationary, generalized Pickands constants can be defined in $(0,\infty)$, and, most notably, that they admit a Dieker–Yakir type representation (3) under certain conditions. On the other hand, we explore the connection between mixed moving maxima processes and generalized Pickands constants. Our findings are beneficial for both the theory of extremes of max-stable stationary processes, and the asymptotic theory of random processes.
The paper is organized as follows. In Section 2 we introduce generalized Pickands constants $\mathcal{H}_W^\delta$ and give conditions under which they admit a Dieker–Yakir type representation. Examples for the process W will be general Gaussian processes with stationary increments and Lévy processes. In Section 3 we show that the Dieker–Yakir constant (3) conincides with the constant C_F in the intensity of the classical M3 representation (7). Combining the results of Sections 2 and 3, a new representation of Pickands constants $\mathcal{H}_W^\delta$ is provided in Section 3. This link gives a simple proof of the positivity of generalized Pickands constants. All proofs are given in Section 5. The Appendix comprises some facts on discrete mixed moving maxima representations which are needed in Section 3.

2. Generalized Pickands constants

Let $\{B(t), t \in \mathbb{R}\}$ be a stochastic process with sample paths in the space D of càdlàg functions $f : \mathbb{R} \to \mathbb{R}$ with $B(0) = 0$ and finite $\mathbb{E}\left\{e^{B(t)}\right\} < \infty$, for all $t \in \mathbb{R}$. We introduce the drifted process

(8) $$W(t) = B(t) - \ln \mathbb{E}\left\{e^{B(t)}\right\}, \quad t \in \mathbb{R}$$

and note that it satisfies $\mathbb{E}\left\{e^{W(t)}\right\} = 1$. We can therefore define the corresponding max-stable process ξ_W by the construction (6) which has standard Gumbel margins. Hereafter we shall assume that W is chosen such that the process ξ_W is stationary and has càdlàg sample paths; see Proposition 6 in [20] for a general stationarity criterion.
Next, for the process W on the grid $\delta\mathbb{Z}$ for $\delta \geq 0$ we introduce the *generalized Pickands constant* as

(9) $$\mathcal{H}_W^\delta = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left\{\sup_{t\in\delta\mathbb{Z}\cap[0,T]} e^{W(t)}\right\}.$$

The existence of the expected value in (9) when $\delta = 0$ follows from the assumption that ξ_W has càdlàg sample paths, since then for some large $z > 0$

$$0 < \mathbb{P}\left(\sup_{t\in[0,T]} \xi_W(t) \le z\right) = \exp\left(-e^{-z}\mathbb{E}\left\{\sup_{t\in[0,T]} e^{W(t)}\right\}\right).$$

However, the existence and finiteness of the limit (9) is not obvious. In the sequel of the paper, we investigate:

a) the existence of the constant $\mathcal{H}_W^\delta$,
b) its finiteness and positivity,
c) equivalent representations that can for instance be used for efficient approximations.

In Section 2.1 we discuss question a) in a general setting. For question b) and c) we will concentrate on two important examples for W such that the above assumptions are satisfied. In Section 2.2 we consider the general Gaussian case, where

- ◇ B is a sample continuous centered Gaussian process with stationary increments and variance function $\sigma^2(t), t \in \mathbb{R}$. With
$$W(t) = B(t) - \sigma^2(t)/2, \quad t \in \mathbb{R}$$
the process ξ_W is max-stable and stationary. Its law depends only on the variogram $\gamma(t) = \mathrm{Var}(B(t) - B(0))$ and we can therefore assume without loss of generality that $W(0) = 0$; see [20, 22] for details.

The generalized Pickands constant can also be defined for non-Gaussian processes. In Section 2.3 we investigate the case where

- ◇ $\{B(t), t \ge 0\}$ is a Lévy process such that $\Phi(\theta) = \ln \mathbb{E}\left\{e^{\theta B(1)}\right\}$ is finite for $\theta = 1$ and set
$$W(t) = B(t) - \Phi(1)t, \quad t \ge 0.$$
If $\{W(t), t \le 0\}$ is defined as an exponentially transformed version of the corresponding $\{W(t), t \ge 0\}$, then ξ_W can be shown to be stationary and max-stable; see [27, 28] for details.

Clearly, these are not the only examples. For instance, in the Gaussian case, a slight generalization is to introduce an independent mixing random variable $S > 0$ and taking $W(t) = SB(t) - S^2\sigma^2(t)/2$ in (8). We retrieve the variance-mixed Brown–Resnick process ξ_W, which is both max-stable and stationary [29, 30].

2.1. **Existence and positivity of $\mathcal{H}_W^\delta$.** In order to prove the existence of the generalized Pickands constant $\mathcal{H}_W^\delta$ we do not need any further assumptions on the process W. In fact, the stationarity of the process ξ_W and the existing theory of max-stable processes are sufficient to give an immediate answer to a) and partially to b) above. Indeed, for any compact $E \subset \mathbb{R}$ we define $H_W(E) = \mathbb{E}\left\{\sup_{t\in E} e^{W(t)}\right\}$ and observe that

$$-\ln \mathbb{P}\left\{\sup_{t\in E} \xi_W(t) \le x\right\} = H_W(E)e^{-x}, \quad x \in \mathbb{R}. \tag{10}$$

Consequently, by stationarity of ξ_W for any $a \in \mathbb{R}$, we have $H_W(a+E) = H_W(E)$, where $a+E := \{a+x : x \in E\}$. Since for any disjoint, non-empty compact sets $E_1, E_2 \subset \mathbb{R}$

$$H_W(E_1 \cup E_2) = \mathbb{E}\left\{\sup_{t\in E_1\cup E_2} e^{W(t)}\right\} \le \mathbb{E}\left\{\sup_{t\in E_1} e^{W(t)}\right\} + \mathbb{E}\left\{\sup_{t\in E_2} e^{W(t)}\right\} = H_W(E_1) + H_W(E_2),$$

the set-function $H_W(\cdot)$, restricted on the sets $\delta\mathbb{Z} \cap [0, T], T > 0$, is subadditive and by Fekete's Lemma

$$\mathcal{H}_W^\delta = \lim_{T\to\infty} \frac{H_W(\delta\mathbb{Z} \cap [0, T])}{T} = \inf_{T>0} \frac{H_W(\delta\mathbb{Z} \cap [0, T])}{T} \in [0, \infty). \tag{11}$$

Therefore, the limit in (9) as $T \to \infty$ exists and is finite. Furthermore, in the case that $\delta > 0$, (11) immediately implies $\mathcal{H}_W^\delta \le 1/\delta$.

The following lemma is crucial for investigating the structure of $\mathcal{H}_W^\delta$ and establishing Dieker–Yakir type representations. It extends Lemma 5.2 in [17], where it was considered for the case that $W(t) = B(t) - \sigma^2(t)/2$ with B a centered Gaussian process with stationary increments and variance function σ^2.

Lemma 2.1. *Suppose that W is such that the process ξ_W in (6) is max-stable and stationary, and $W(t_0) = 0$ for some $t_0 \in \mathbb{R}$. If Γ is a Borel measurable, positive functional on D that is invariant under addition of any constant function, then, given that the expectations below exist,*

$$\mathbb{E}\left\{e^{W(t_0+t)}\Gamma(W)\right\} = \mathbb{E}\left\{\Gamma(\theta_t W)\right\}, \quad t \in \mathbb{R}, \tag{12}$$

where θ_t is the shift operator, that is, $\theta_t W(s) = W(s - t)$.

An application of equation (12) yields a way of rewriting the expectation in (9); see Corollary 2 in [1].

Lemma 2.2. *If μ is the Lebesgue measure on $\mathbb{R}$ or the counting measure on $(k\delta)\mathbb{Z}\cap[0,T]$ with $k\in\mathbb{N},\delta>0$, then*

$$\frac{1}{T}\mathbb{E}\left\{\sup_{t\in\delta\mathbb{Z}\cap[0,T]} e^{W(t)}\right\} = \int_0^1 \mathbb{E}\left\{\frac{\sup_{s\in\delta\mathbb{Z}\cap[-uT,(1-u)T]} e^{W(s)}}{\int_{-uT}^{(1-u)T} e^{W(s)}\mu(ds)}\right\}\mu^T(du), \tag{13}$$

with $\mu^T(du)=\mu(Tdu)/T, T>0$.

Using the result of Lemma 2.2, we establish a Dieker–Yakir representation of $\mathcal{H}_W^\delta$ for $\delta>0$ and then show that $\mathcal{H}_W^\delta$ is strictly positive for $\delta\ge 0$.

Theorem 2.1. *Let W be such that the corresponding max-stable Brown–Resnick process ξ_W is stationary and has càdlàg paths. If for a given $\delta>0$ we have that $\mathbb{P}\left\{S^\delta<\infty\right\}=1$, then with M^δ, S^δ defined in (4), we have*

$$\mathcal{H}_W^\delta=\mathbb{E}\left\{\frac{M^\delta}{S^\delta}\right\}>0. \tag{14}$$

Further, if $\delta\ge 0$ and $\eta=k\delta$ for some $k\in\mathbb{N}$, then

$$\mathcal{H}_W^\delta\ge\mathbb{E}\left\{\frac{M^\delta}{S^\eta}\right\}>0. \tag{15}$$

The restriction $\delta>0$ in (14) is somehow unsatisfactory. Moreover, it is of interest to have equality in (15) for large classes of processes W. In the sequel we therefore consider two important special cases where we can strengthen the above results to

$$\mathcal{H}_W^\delta=\mathbb{E}\left\{\frac{M^\delta}{S^\eta}\right\}\in(0,\infty),\quad \delta=0,\eta\ge 0 \text{ or } \delta>0,\eta=k\delta,k\in\mathbb{N}, \tag{16}$$

which is motivated by the findings of [1] for $W(t)=\sqrt{2}B_\alpha(t)-|t|^\alpha$. Therein (16) is shown if W is a fractional Brownian motion and $\delta=0,\eta>0$ or $\delta=\eta>0$.

2.2. **Gaussian case.** First, we consider the case where $W(t)=B(t)-\sigma^2(t)/2$, with B a centered, sample continuous Gaussian process that has stationary increments, variance function σ^2 and $W(0)=0$ almost surely. In view of [20], the corresponding ξ_W is max-stable and stationary. In order to apply Theorem 2.1 we have to ensure that $S^\delta<\infty$ almost surely. To this end, we shall impose the assumption that

$$\liminf_{|t|\to\infty}\frac{\sigma^2(t)}{\ln t}>8, \tag{17}$$

which by Corollary 2.4 in [31] implies

$$\lim_{|t|\to\infty}W(t)=-\infty. \tag{18}$$

Theorem 6.1 in [32] then yields that $S^\delta<\infty$ almost surely. Consequently, under (17) and by Theorem 2.1 we obtain both the positivity and Dieker–Yakir representation of $\mathcal{H}_W^\delta,\delta>0$.
In order to deal with the case $\delta=0$, we need slightly stronger conditions on σ^2, namely we shall assume that there is an ultimately monotone, non-decreasing function $\ell:[0,\infty)\to[0,\infty)$ and a constant $c\in(0,1]$ such that for all t large

$$c\ell(t)\le\sigma^2(t)\le\ell(t),\qquad \text{where } \lim_{t\to\infty}\frac{\ell(t)}{\ell(t+k)}=1,\quad \forall k\in\mathbb{N}, \tag{19}$$

holds. Clearly, (19) is satisfied for σ^2 being a regularly varying function with index $\alpha>0$. Note in passing that the stationarity of increments implies that $\alpha\le 2$, see also Lemma 2.1 in [31] for the existence of such Gaussian processes.

Theorem 2.2. *Let W be a Gaussian process as above whose variance function σ^2 satisfies condition (19) with $c\in(0,1]$ such that $c^2+8c-8>0$. If further*

$$\liminf_{t\to\infty}\frac{\ell(t)}{\ln t}>\frac{8}{c^2+8c-8}, \tag{20}$$

then the generalized Pickands constant $\mathcal{H}_W^\delta$ possesses a Dieker–Yakir representation

$$\mathcal{H}_W^\delta=\mathbb{E}\left\{\frac{M^\delta}{S^\eta}\right\}\in(0,\infty), \tag{21}$$

which is valid for $\delta=0$ and $\eta\ge 0$, or $\delta>0$ and $\eta=k\delta,k\in\mathbb{N}$.

Remark 2.3. a) *Conditions (19) and (20) are much weaker than the assumption that σ^2 is regularly varying at infinity. In [33] the positivity and finiteness of $\mathcal{H}_W^0$ is shown under the two conditions C1 and C2 therein, which imply that σ^2 is a smooth, regularly varying function at both infinity and zero.*
b) *Note that if $c = 1$, then (20) agrees with (17).*

2.3. **Lévy case.** In [28], the so-called Lévy–Brown–Resnick processes were introduced as ξ_W, where W is constructed from two independent Lévy processes. More precisely, suppose that $\{B^+(t), t \ge 0\}$ is a Lévy process such that its Laplace exponent $\Phi(\theta) = \ln \mathbb{E}\left\{\exp\big(\theta B^+(1)\big)\right\}$ is finite for $\theta = 1$. Define $-W^-$ to be the exponentially tilted version (with tilting parameter $\theta = 1$) of

$$W^+(t) = B^+(t) - \Phi(1)t, \quad t \ge 0,$$

that is, the Laplace exponent of W^- is

$$\ln \mathbb{E}\left\{e^{\theta W^-(1)}\right\} = \Phi(1-\theta) - (1-\theta)\Phi(1).$$

For two independent processes W^+ and W^- we define $W(t) = W^+(t), t \ge 0$, and $W(t) = W^-(-t)$ if $t < 0$. With this definition the corresponding process ξ_W is indeed max-stable and stationary; for details see [28] and [34]. We remark that in this construction there are no further requirements on the mean of the Lévy process and it might for instance be $-\infty$.
In the case where B^+ is a spectrally negative Lévy process, [28] computed the extremal index of the corresponding max-stable process ξ_W explicitly. In view of (5), this index coincides with Pickands constant of the process B^+, and it is therefore given as

$$\mathcal{H}_W^0 = \Phi'(1).$$

For more general examples than spectrally negative Lévy processes, we show below that the Pickands constant $\mathcal{H}_W$ in the Lévy case possesses a Dieker–Yakir type representation. In fact, by [28] it follows that the conditions of Theorem 2.1 are satisfied and thus $\mathcal{H}_W^\delta$ exists and is strictly positive. In what follows we suppose that B^+ is not a compound Poisson process that has $\delta\mathbb{Z}$ as the support of the jump distribution.

Theorem 2.3. *Let $B^+(t), t \in [0,\infty)$ and $W(t), t \in \mathbb{R}$ be as above.*
(1) If $\mathbb{E}\left\{e^{(2+\varepsilon)|W(1)|}\right\} < \infty$ and $\mathbb{E}\left\{e^{(2+\varepsilon)|W(-1)|}\right\} < \infty$ for some $\varepsilon > 0$, then

$$\mathcal{H}_W^0 = \mathbb{E}\left\{\frac{M^0}{S^0}\right\} \in (0,\infty). \tag{22}$$

(2) If $\mathbb{E}\left\{e^{(1+\varepsilon)|W(1)|}\right\} < \infty$ and $\mathbb{E}\left\{e^{(1+\varepsilon)|W(-1)|}\right\} < \infty$ for some $\varepsilon > 0$, then

$$\mathcal{H}_W^\delta = \mathbb{E}\left\{\frac{M^\delta}{S^\eta}\right\} \in (0,\infty), \quad \delta = 0, \eta > 0 \text{ or } \delta > 0, \eta = k\delta, k \in \mathbb{N}. \tag{23}$$

Remark 2.4. a) *Theorem 2.3 holds if both the left and the right tail probability of $W(1)$ is sufficiently light; for example if $\Phi(\theta) < \infty$ for $\theta \in (-2-\varepsilon, 3+\varepsilon)$ for scenario (1) and $\theta \in (-1-\varepsilon, 2+\varepsilon)$ for scenario (2). We conjecture that the claim of Theorem 2.3 is true under weaker assumptions on W.*
b) *Discrete Pickands-type constants appeared already in [35], see for more details Lemma 5.6 therein.*
c) *In [36] an alternative representation of the classical Pickands constants is derived heuristically.*
d) *Pickands constants related to semi-min-stable processes are recently calculated in [37].*

3. Mixed moving maxima processes and Dieker–Yakir constants

In this section we introduce a new mixed moving maxima (M3) representation for a general stationary max-stable process ξ_W, inspired by a construction in [17]. A normalizing constant appearing in this construction turns out to be equal to the Dieker–Yakir constant in (3), and we show that it coincides with the constant arising in the classical M3 representation.
As in the previous section, let W with $W(0) = 0$ a.s. be a càdlàg process such that the corresponding ξ_W is max-stable and stationary. We recall from the introduction that the process ξ_W is said to admit an M3 representation if

$$\xi_W(t) \stackrel{d}{=} \max_{i\ge1}(P_i + F_i(t - T_i)), \quad t \in \mathbb{R}, \tag{24}$$

with $\sum_{i=1}^\infty \varepsilon_{(P_i,T_i)}$ a Poisson point process in $\mathbb{R}^2$ with intensity $C_F e^{-p} dp dt$, where $C_F \in (0,\infty)$ is a suitable normalizing constant and the shape functions F_i's are independent copies of a measurable càdlàg process $\{F(t), t \in \mathbb{R}\}$. The representation (24) is not unique since the distributional equality can hold for different processes F and constants C_F. The standard Gumbel margins of ξ_W directly imply that

$$C_F = \left(\mathbb{E}\left\{\int_{\mathbb{R}} \exp(F(t))\, dt\right\}\right)^{-1} \in (0,\infty). \tag{25}$$

Throughout this section we assume that ξ_W possesses an M3 representation which amounts to assuming one of the equivalent conditions below; for details see [32] and Theorem 2 in [38].

Condition 1. *We assume that one of the following equivalent conditions holds:*

(1) *The max-stable process* $\{\xi_W, t \in \mathbb{R}\}$ *possesses an M3 representation.*
(2) *The process* $\{W(t), t \in \mathbb{R}\}$ *satisfies*

$$\lim_{|t|\to\infty} W(t) = -\infty, \quad a.s.$$

(3) *The process* $\{W(t), t \in \mathbb{R}\}$ *fulfills*

$$\int_{\mathbb{R}} e^{W(t)} dt < \infty, \quad a.s.$$

In applications, it is typically required that almost surely

$$\sup_{t\in\mathbb{R}} e^{F(t)} = 1, \quad \arg\max_{t\in\mathbb{R}} F(t) = 0 \tag{26}$$

since then exact simulation of ξ_W is possible [39, Theorem 4]. Moreover, for any ξ_W satisfying Condition 1, there is a unique M3 representation with shape distribution, say F, that satisfies assumption (26). The corresponding normalizing constant C_F plays a crucial role in the simulation of the process ξ_W but its numerical evaluation is time intensive and the exact value is, apart from special cases, unknown [26].

This classical M3 representations with shape functions safisfying conditions (26) have the drawback that the distribution of the corresponding shape function F is often complicated. Indeed, if W is a Brownian motion or a Lévy process (cf., Sections 2.2 and 2.3), then the shape functions F correspond to the respective process conditioned to stay negative [34, 40]. For non-Markov processes such as fractional Brownian motions there are no known results.

We introduce a new, much simpler class of M3 representations, which might be of independent interest. It is based on the result by [17] that if ξ_W is a Gaussian Brown–Resnick process, then for any probability measure μ on $\mathbb{R}$, it can be written in distribution as

$$\xi_W(t) = \max_{i\geq 1} \left(P_i + W_i(t - T_i) - \ln \int_{\mathbb{R}} e^{W_i(s-T_i)} \mu(ds) \right), \quad t \in \mathbb{R}, \tag{27}$$

where $\sum_{i=1}^{\infty} \varepsilon_{(P_i,T_i)}$ is a Poisson point process in $\mathbb{R}^2$ with intensity $e^{-p} dp\mu(dt)$, and W_i are independent copies of W. As noted by a referee, Lemma 2.1 can be directly used to extend representation (27) to general stationary max-stable processes ξ_W satisfying Condition 1 and, most notably, to arbitrary σ-finite measures μ on $\mathbb{R}$.

Theorem 3.1. *Suppose that* ξ_W *is a stationary max-stable process satisfying Condition 1. For any* σ*-finite measure* μ*,* ξ_W *admits a representation of the form* (27)*. In particular, choosing* $\mu(dt) = C_{DM} dt$*, for any constant* $C_{DM} > 0$*, yields a new M3 representation with*

$$F_{DM}(t) = W(t) - \ln \left(C_{DM} \int_{\mathbb{R}} e^{W(s)} ds \right) \tag{28}$$

and normalizing constant $C_{F_{DM}} = C_{DM}$*.*

We omit the proof of this theorem since in view of Lemma 2.1 it goes along the lines of the proof of Theorem 2.1 in [17]. Note that for the validity of (28) the assumption

$$\int_{\mathbb{R}} e^{W(s)} ds < \infty \ \text{ a.s.},$$

is in view of Condition 1 equivalent to the existence of the M3 representation. The choice of C_{DM} is arbitrary and in order to define a canonical Dieker–Mikosch M3 representation, we require $\mathbb{E}\left\{\sup_{t\in\mathbb{R}} e^{F_{DM}(t)}\right\} = 1$, or equivalently

$$C_{DM} = \mathbb{E}\left\{ \frac{\sup_{t\in\mathbb{R}} e^{W(t)}}{\int_{\mathbb{R}} e^{W(t)}\, dt} \right\}. \tag{29}$$

The right-hand side is nothing else than the Dieker–Yakir constant appearing in (3). The following auxiliary lemma provides a relation between the normalizing constants (25) of different M3 representations.

Lemma 3.1. *Suppose that the process* ξ_W *has two different M3 representations* (24) *with shape functions* F *and* G*, and corresponding constants* C_F *and* C_G *in the intensities. Then* $C_F \mathbb{E}\left\{\sup_{t\in\mathbb{R}} e^{F(t)}\right\} = C_G \mathbb{E}\left\{\sup_{t\in\mathbb{R}} e^{G(t)}\right\}$*, that is,*

$$\frac{\mathbb{E}\left\{\sup_{t\in\mathbb{R}} e^{F(t)}\right\}}{\mathbb{E}\left\{\int_{\mathbb{R}} e^{F(t)}\, dt\right\}} = \frac{\mathbb{E}\left\{\sup_{t\in\mathbb{R}} e^{G(t)}\right\}}{\mathbb{E}\left\{\int_{\mathbb{R}} e^{G(t)}\, dt\right\}}. \tag{30}$$

In particular, $\mathbb{E}\left\{\sup_{t\in\mathbb{R}} e^{F(t)}\right\} = \mathbb{E}\left\{\sup_{t\in\mathbb{R}} e^{G(t)}\right\}$ *implies* $C_F = C_G \in (0, \infty)$*.*

Since the shape functions of the classical and the new canonical Dieker–Mikosch M3 representation, F and F_{DM}, satisfy

$$\mathbb{E}\left\{\sup_{t\in\mathbb{R}} e^{F(t)}\right\} = \mathbb{E}\left\{\sup_{t\in\mathbb{R}} e^{F_{DM}(t)}\right\} = 1$$

the above Lemma readily yields the main results of this section.

Theorem 3.2. *The constant C_{DM}, and therefore the Dieker–Yakir constant, conincides with normalizing constant in the classical M3 representation with shape functions satisfying conditions (26), that is,*

$$C_F = C_{DM} = \mathbb{E}\left\{\frac{\sup_{t\in\mathbb{R}} e^{W(t)}}{\int_{\mathbb{R}} e^{W(t)}\,dt}\right\}. \tag{31}$$

Let us remark that even though the normalizing constants in the classical and the new Dieker–Mikosch M3 representations are the same, the distribution of the respective shape functions F and F_{DM} are in general quite different. For instance, it seems impossible to apply a simple transformation on F_{DM} in order to safisfy conditions (26) and thereby to recover F. Indeed, taking the example of W being a Brownian motion with drift, then F_{DM} is a randomly shifted version of this process, but, on the other hand, F is a three-dimensional Bessel process of drifted Brownian motion [40].

4. An alternative representation of Pickands constants

In the previous section we proposed a new M3 representation with normalizing constant equal to the Dieker–Yakir constant, which turn out to be identical with the normalizing constant C_F in the classical M3 representation (cf., Theorem 3.2). In this section we restate this results for any discretization level $\delta \ge 0$; an proof is provided in the Appendix. Most importantly, together with the results established in Section 2, in many cases we can identify C_F with the Pickands constant $\mathcal{H}_W^0$.
Since we are also interested in the case $\delta > 0$, we show in the Appendix how to derive an M3 representation for the discretized process $\xi_W^\delta = \{\xi_W(t), t \in \delta\mathbb{Z}\}$, with shape functions F^δ and constant

$$C_F^\delta = \left(\mathbb{E}\left\{\int_{\delta\mathbb{Z}} e^{F^\delta(t)}\nu_\delta(dt)\right\}\right)^{-1} \in (0,\infty), \quad \delta > 0.$$

Here, ν_δ/δ for $\delta > 0$ is the counting measure on $\delta\mathbb{Z}$, and $\nu = \nu_0$ is the Lebesgue measure. In the sequel the superscript is omitted if it is 0, for instance we write C_F and $\mathcal{H}_W$ instead of C_F^0 and $\mathcal{H}_W^0$, respectively.

Proposition 4.1. *If ξ_W^δ possesses an M3 representation, then for any $\delta \ge 0$*

$$0 < C_F^\delta = \mathbb{E}\left\{\frac{M^\delta}{S^\delta}\right\} \le \mathcal{H}_W^\delta. \tag{32}$$

In particular, in this case the Pickands constant H_W^δ is strictily positive. Moreover, if the conditions in Theorem 2.1 ,2.2 or 2.3 on W and δ are satisfied, then $C_F^\delta = \mathcal{H}_W^\delta$.

Remark 4.2. *If ξ_W possesses an M3 representation, then (14) and (32) imply for any $\delta > 0$*

$$\mathcal{H}_W = \mathcal{H}_W^0 \ge \mathcal{H}_W^\delta = C_F^\delta > 0.$$

Except for few special cases, the exact value of Pickands constant is unknown. There are several attempts to assess its value by Monte Carlo simulation, most notably via the recent Dieker–Yakir representation in [1]. The above Proposition states that the simulation problem of Pickands constant $\mathcal{H}_W^\delta$ is equivalent to the problem of evaluating constants C_F^δ, which are needed for simulation of max-stable processes, provided that ξ_W admits an M3 representation and the Dieker–Yakir representation for $\mathcal{H}_W^\delta$ holds. This is a fruitful observation since there is active research on the simulation of max-stable processes [17, 41] and even of the constant C_F^δ [26]. We conclude this section with three examples.

Example 4.3. *If $W(t) = \sqrt{2}Zt - t^2$, $t \in \mathbb{R}$, where Z is an $N(0,1)$ random variable it is known [32] that ξ_W has an M3 representation with deterministic shape functions*

$$F(t) = -t^2, t \in \mathbb{R}.$$

Thus

$$C_F = \left(\int_{\mathbb{R}} e^{-t^2}\,dt\right)^{-1} = \frac{1}{\sqrt{\pi}},$$

and consequently, by Theorem 2.2 and Remark 4.2 we recover the well-known fact $\mathcal{H}_W = 1/\sqrt{\pi}$.
If $W(t) = \sqrt{2}B(t) - |t|, t \in \mathbb{R}$, where B is a standard Brownian motion, then it follows by [40] that ξ_W has an M3 representation whose shape functions F are given by a three-dimensional Bessel process and that $C_F = 1$. Thus, again by Theorem 2.2 and Remark 4.2 we recover $\mathcal{H}_W = C_F = 1$, see e.g., [42].

Example 4.4. *Suppose that W is a sample continuous Gaussian process with stationary increments that fulfills the assumptions of Theorem 2.2. Since in this case (17) holds and thus Condition 1 is satisfied, ξ_W admits an M3 representation. Finding an explicit form for the corresponding shape distribution F is an open problem in the general case, with the exception of the two special cases in the previous example. Nevertheless, Theorem 2.2 and Proposition 4.1 imply that $\mathcal{H}_W^\delta$ is positive for any $\delta > 0$ and*

$$C_F^\delta = \mathcal{H}_W^\delta.$$

Furthermore, we have

$$\lim_{\delta \downarrow 0} C_F^\delta = C_F = \mathcal{H}_W = \mathbb{E}\left\{\frac{M^0}{S^0}\right\}.$$

Example 4.5. *If W is as in Section 2.3, [34] shows that the Lévy-Brown–Resnick process ξ_W admits an M3 representation where the constant C_F is explicitly given by*

$$C_F = \frac{\underline{k}(0,1)}{\underline{k}'(0,0)} > 0,$$

where $\underline{k}$ is the bivariate Laplace exponent of the descending ladder process corresponding to W. In particular, this implies that for $\delta = 0$ by Proposition 4.1 $\mathcal{H}_W \geq C_F$ and thus

$$\mathcal{H}_W \geq \frac{\underline{k}(0,1)}{\underline{k}'(0,0)} > 0. \tag{33}$$

In order to have equality in the equation above, it is sufficient that the process W satisfies the conditions of Theorem 2.3, since then

$$\mathcal{H}_W = \mathbb{E}\left\{\frac{M^0}{S^0}\right\} = C_F.$$

5. Proofs

Proof of Lemma 2.1: It is well-known that the stationarity of ξ_W is equivalent to the fact that for arbitrary $h \in \mathbb{R}$ the two Poisson point processes $\{U_i + W_i : i \in \mathbb{N}\}$ and $\{U_i + \theta_h W_i : i \in \mathbb{N}\}$ on D have the same intensity; see [20]. The latter holds if and only if for any Borel subset A

$$\int_{\mathbb{R}} e^{-u} \mathbb{P}\{u + W \in A\}\, du = \int_{\mathbb{R}} e^{-u} \mathbb{P}\{u + \theta_h W \in A\}\, du.$$

Let $B \subset D$ be a shift-invariant Borel set in the sense that $B + x = B$ for any $x \in \mathbb{R}$, and recall that $W(t_0) = 0$ almost surely. Consequently, for any $h \in \mathbb{R}$ we have

$$\begin{aligned}
\mathbb{E}\left\{e^{W(t_0+h)}\mathbf{1}\{W \in B\}\right\} &= \mathbb{E}\left\{\int_{\mathbb{R}} e^{-u}\mathbf{1}\{u + W(t_0+h) > 0\}\mathbf{1}\{W \in B\}du\right\} \\
&= \int_{\mathbb{R}} e^{-u}\mathbb{P}\{u + W(t_0+h) > 0, u + W \in B\}\, du \\
&= \int_{\mathbb{R}} e^{-u}\mathbb{P}\{u + W(t_0) > 0, u + \theta_h W \in B\}\, du \\
&= \int_{\mathbb{R}} e^{-u}\mathbf{1}\{u > 0\}\mathbb{P}\{u + \theta_h W \in B\}\, du \\
&= \mathbb{P}\{\theta_h W \in B\}.
\end{aligned}$$

Furthermore, the above readily extends to Borel measurable, positive functionals Γ on D that are invariant under addition of a constant function and, thus, the assertion follows. □

Proof of Lemma 2.2: Define the translation invariant functional

$$\Gamma(f) = \frac{\sup_{s \in \delta\mathbb{Z} \cap [0,T]} e^{f(s)}}{\int_0^T e^{f(s)}\mu(ds)}, \quad T > 0.$$

Clearly, we have that for any $t \in (k\delta)\mathbb{Z}$

$$\Gamma(\theta_t f) = \frac{\sup_{s \in \delta\mathbb{Z} \cap [0,T]} e^{f(s-t)}}{\int_0^T e^{f(s-t)}\mu(ds)} = \frac{\sup_{s \in \delta\mathbb{Z} \cap [-t,T-t]} e^{f(s)}}{\int_0^T e^{f(s-t)}\mu(ds)} = \frac{\sup_{s \in \delta\mathbb{Z} \cap [-t,T-t]} e^{f(s)}}{\int_{-t}^{T-t} e^{f(s)}\mu(ds)},$$

where the last equality follows by the translation invariance of μ. Hence, as in the proof of Corollary 2 in [1] a direct application of Lemma 2.1 yields

$$\begin{aligned}\frac{1}{T}\mathbb{E}\left\{\sup_{t\in\delta\mathbb{Z}\cap[0,T]} e^{W(t)}\right\} &= \frac{1}{T}\int_0^T \mathbb{E}\left\{e^{W(t)}\frac{\sup_{s\in\delta\mathbb{Z}\cap[0,T]} e^{W(s)}}{\int_0^T e^{W(s)}\mu(ds)}\right\}\mu(dt)\\ &= \frac{1}{T}\int_0^T \mathbb{E}\left\{e^{W(t)}\Gamma(W)\right\}\mu(dt)\\ &= \frac{1}{T}\int_0^T \mathbb{E}\left\{\Gamma(\theta_t W)\right\}\mu(dt)\\ &= \frac{1}{T}\int_0^T \mathbb{E}\left\{\frac{\sup_{s\in\delta\mathbb{Z}\cap[-t,T-t]} e^{W(s)}}{\int_{-t}^{T-t} e^{W(s)}\mu(ds)}\right\}\mu(dt).\end{aligned}$$

Consequently, (13) follows by changing the variable $t = uT$. □

Proof of Theorem 2.1: Let first $\eta = \delta > 0$, then if λ_δ denotes the counting measure on $\delta\mathbb{Z}$, then applying (13) with $\mu = \lambda_\delta$ and $T > 0$ we obtain

$$\frac{1}{T}\mathbb{E}\left\{\sup_{t\in\delta\mathbb{Z}\cap[0,T]} e^{W(t)}\right\} = \int_0^1 \mathbb{E}\left\{\frac{\sup_{s\in\delta\mathbb{Z}\cap[-uT,(1-u)T]} e^{W(s)}}{\delta\int_{-uT}^{(1-u)T} e^{W(s)}\mu(ds)}\right\}\delta\mu^T(du).$$

By the assumption that $S^\delta = \delta\int_{\mathbb{R}} e^{W(s)}\mu(ds) < \infty$ it follows that $\sup_{s\in\delta\mathbb{Z}} e^{W(s)} < \infty$ and $\lim_{|n|\to\infty, n\in\mathbb{Z}} W(n\delta) = -\infty$ almost surely. Hence the almost sure convergence

$$g_{T,\delta}(u) = \frac{\sup_{s\in\delta\mathbb{Z}\cap[-uT,(1-u)T]} e^{W(s)}}{\delta\int_{-uT}^{(1-u)T} e^{W(s)}\mu(ds)} \to \frac{\sup_{s\in\delta\mathbb{Z}} e^{W(s)}}{\delta\int_{\mathbb{R}} e^{W(s)}\mu(ds)} = \mathcal{Q}_\delta \le \frac{1}{\delta}, \quad T\to\infty$$

holds for any $u \in (0,1)$. Clearly, the above convergence remains true if we replace u by a sequence $u_T, T > 0$ such that $\lim_{T\to\infty} u_T = u \in (0,1)$. Since for any $u \in (0,1), T > 0$ we have $g_{T,\delta}(u) \le 1/\delta$ we obtain for any $u \in (0,1)$ by dominated convergence

$$\lim_{T\to\infty}\mathbb{E}\left\{g_{T,\delta}(u_T)\right\} = \mathbb{E}\left\{\mathcal{Q}_\delta\right\}.$$

Since $\delta\mu^T$ converges weakly to the Lebesgue measure as $T\to\infty$, Theorem 5.5 in [43] implies that

$$\mathcal{H}_W^\delta = \lim_{T\to\infty}\int_0^1 \mathbb{E}\left\{g_{T,\delta}(u)\right\}\delta\mu^T(du) = \int_0^1 \mathbb{E}\left\{\mathcal{Q}_\delta\right\}du = \mathbb{E}\left\{\mathcal{Q}_\delta\right\}$$

establishing the first claim in (14).

Next, if $\mu = \lambda_\eta$ with $\eta = k\delta, k = 0,1,\ldots,$ or $\eta > 0, \delta = 0$, by (13) and Theorem 1.1 in [44] for any $u \in (0,1), T > 0$

$$\begin{aligned}\mathcal{H}_W^\delta &= \lim_{T\to\infty}\int_0^1 \mathbb{E}\left\{g_{T,\eta}(u)\right\}\nu_\eta^T(du)\\ &\ge \int_0^1 \liminf_{T\to\infty, v\to u}\mathbb{E}\left\{g_{T,\eta}(v)\right\}du\\ &\ge \int_0^1 \mathbb{E}\left\{\liminf_{T\to\infty, v\to u} g_{T,\eta}(v)\right\}du\\ &= \mathbb{E}\left\{\mathcal{Q}_\eta\right\} > 0,\end{aligned}$$

hence (15) follows. □

Proof of Theorem 2.2: Our assumptions on σ^2 imply that (17) holds, and thus $\delta\sum_{t\in\delta\mathbb{Z}} e^{W(t)} < \infty$ almost surely for any $\delta \ge 0$. Recall that we interpret $\delta\sum_{t\in\delta Z} e^{W(t)}$ as $\int_{\mathbb{R}} e^{W(t)}\,dt$ when $\delta = 0$. Consequently, for any $\delta, \eta \ge 0$, we have the almost sure convergence

$$R_{u,T}^{\delta,\eta} = \frac{M^\delta[-uT,(1-u)T]}{S^\eta[-uT,(1-u)T]} := \frac{\sup_{s\in\delta\mathbb{Z}\cap[-uT,(1-u)T]} e^{W(s)}}{\eta\sum_{s\in\eta\mathbb{Z}\cap[-uT,(1-u)T]} e^{W(s)}} \to \frac{\sup_{s\in\delta\mathbb{Z}} e^{W(s)}}{\eta\sum_{s\in\eta\mathbb{Z}} e^{W(s)}} \in (0,\infty) \tag{34}$$

for all $u \in (0,1), T\to\infty$. Together with (13), the claim of the theorem therefore follows if we can show the uniform integrability

$$\lim_{A\to\infty}\sup_{T>0}\sup_{u\in(0,1)}\mathbb{E}\left\{R_{u,T}^{\delta,\eta}; R_{u,T}^{\delta,\eta} > A\right\} = 0.$$

In order to give a self-contained proof (which follows along the same ideas as in [1]) we introduce the same notation as therein. Namely, we let $a_j = j$ and we define

$$J_j = [a_j, a_{j+1}), \quad J_{-j} = (-a_{j+1}, -a_j], \quad S_j^\eta = \eta \sum_{k:\eta k \in J_j} e^{W(\eta k)}, \quad M_j^\eta = \sup_{k:\eta k \in J_j} e^{W(\eta k)},$$

with S^η, M^η as in (4). Note that in the aforementioned paper our W corresponds to Z.
Fix some $\lambda > 0$ and define $W^\lambda(t) = W(\lambda\lfloor t/\lambda \rfloor), t > 0$, and $W^\lambda(t) = W(\lambda\lceil t/\lambda\rceil)$ otherwise. We have

$$\frac{M_j^\lambda}{S_j^\lambda} \le \frac{1}{\lambda}, \quad \frac{M_j^\delta}{M_j^\lambda} = e^{\sup_{s\in J_j} W^\delta(s) - \sup_{s \in J_j} W^\lambda(s)} \le e^{\sup_{s\in J_j}\left|W^\delta(s) - W^\lambda(s)\right|}$$

and

$$\frac{S_j^\lambda}{S_j^\eta} = \frac{\int_{a_j}^{a_{j+1}} e^{W^\lambda(t) - W^\eta(t)} e^{W^\eta(t)}\, dt}{\int_{a_j}^{a_{j+1}} e^{W^\eta(t)}\, dt} \le e^{\sup_{s\in J_j}\left|W^\eta(s) - W^\lambda(s)\right|}.$$

On the event $\{M^\delta = M_j^\delta\}$ for some $j \in \mathbb{Z}$ we have (assume that $uT, (1-u)T \in \mathbb{Z}$)

$$\begin{aligned} R_{u,T}^{\delta,\eta} &\le \frac{M_j^\delta}{S_j^\eta} = \frac{M_j^\lambda}{S_j^\lambda}\frac{M_j^\delta}{M_j^\lambda}\frac{S_j^\lambda}{S_j^\eta} \\ &\le \frac{1}{\lambda} e^{\sup_{s\in J_j}\left|B^\delta(s) - B^\lambda(s)\right| + \sup_{s\in J_j}\left|B^\eta(s)-B^\lambda(s)\right| + \kappa_\lambda(j)} =: R_{u,T}^{\delta,\eta}(j), \end{aligned} \tag{35}$$

where $B(t) = W(t) + \sigma^2(t)/2$ is a centered Gaussian process and

$$\kappa_\lambda(j) := \sup_{s\in J_j}\left|Var(W(s)) - Var(W^\lambda(s))\right|.$$

Since $M^\delta[-uT, (1-u)T] \ge 1$, we have

$$\begin{aligned} &\mathbb{E}\left\{R_{u,T}^{\delta,\eta}; R_{u,T}^{\delta,\eta} > A\right\} \\ &= \sum_{j\in\mathbb{Z}} \mathbb{E}\left\{R_{u,T}^{\delta,\eta}; R_{u,T}^{\delta,\eta} > A, M_j^\delta = M^\delta\right\} \\ &\le \mathbb{E}\left\{R_{u,T}^{\delta,\eta}; R_{u,T}^{\delta,\eta} > A, M_0^\delta = M^\delta\right\} + 2\sum_{j\ge1}\mathbb{E}\left\{R_{u,T}^{\delta,\eta}(j); R_{u,T}^{\delta,\eta}(j) > A, M_j^\delta \ge 1\right\} \\ &\le \mathbb{E}\left\{M_0^\delta/S_0^\eta; M_0^\delta/S_0^\eta > A\right\} + 2\sum_{j\ge1}\mathbb{E}\left\{R_{u,T}^{\delta,\eta}(j); R_{u,T}^{\delta,\eta}(j) > A, \sup_{s\in J_j} B^\delta(s) \ge \inf_{s\in J_j}\sigma^2(\delta\lfloor s/\delta\rfloor)/2\right\} \\ &=: \mathbb{E}\left\{M_0^\delta/S_0^\eta; M_0^\delta/S_0^\eta > A\right\} + 2\sum_{j\ge1}\pi_j(A). \end{aligned}$$

In the following $C > 0$ may change from line to line. Since $\mathbb{E}\left\{M_0^\delta/S_0^\eta\right\} < \infty$, then

$$\lim_{A\to\infty}\mathbb{E}\left\{M_0^\delta/S_0^\eta; M_0^\delta/S_0^\eta > A\right\} = 0.$$

For all $t, s \in J_j$ and by (19) for all a_j large enough, by the monotonicity of ℓ

$$\inf_{s\in J_j}\sigma^2(\delta\lfloor s/\delta\rfloor)/2 \ge c\ell(\delta\lfloor a_j/\delta\rfloor), \quad \sup_{s\in J_j} Var(B^\delta(s)) \le \ell(\delta\lfloor(a_j+1)/\delta\rfloor).$$

Since for all $j \in \mathbb{Z}$

$$\begin{aligned} \mathbb{E}\left\{\sup_{s\in J_j} B^\delta(s)\right\} &= \mathbb{E}\left\{\sup_{s\in J_j}[B^\delta(s) - B^\delta(j) + B^\delta(j)]\right\} \\ &= \mathbb{E}\left\{\sup_{s\in[0,1]}(B^\delta(s+j) - B^\delta(j))\right\} \\ &\le C, \end{aligned}$$

where the last inequality is consequence of

$$\sup_{s\in[0,1]} Var(B^\delta(s+j) - B^\delta(j)) = \sup_{s\in[0,1]}\sigma^2(\delta[\lfloor(s+j)/\delta\rfloor - \lfloor j/\delta\rfloor]) < C,$$

then by Borell-TIS inequality, see e.g., [45]

$$\begin{aligned}\mathbb{P}\left\{\sup_{s\in J_j} B^\delta(s)\geq \inf_{s\in J_j}\sigma^2(\delta\lfloor s/\delta\rfloor)/2\right\} &\leq \mathbb{P}\left\{\sup_{s\in J_j} B^\delta(s)\geq c\ell(\delta\lfloor a_j/\delta\rfloor)/2\right\}\\ &\leq C\exp\left((1-\varepsilon_1)\frac{c^2\ell^2(\delta\lfloor a_j/\delta\rfloor)}{8\ell(\delta\lfloor(a_j+1)/\delta\rfloor)}\right)\\ &\leq C\exp\left(-(1-\varepsilon_2)\frac{c^2\ell(a_j+1)}{8}\right)\end{aligned}$$

for some $\varepsilon_1,\varepsilon_2$ positive arbitrary small and all $j\geq 1$. Further, the fact that

$$\sup_{s\in J_j} Var(B^\delta(s)-B^\lambda(s))=\sup_{s\in J_j}\sigma^2(\delta\lfloor s/\delta\rfloor-\lambda\lfloor s/\lambda\rfloor)<C$$

for all j, that is, the variance is bounded implies (using again Borell-TIS inequality)

$$\mathbb{E}\left\{e^{p\sup_{s\in J_j}\left|B^\delta(s)-B^\lambda(s)\right|}\right\}\leq C$$

for any $p>1$ and all j. Consequently, by the Hölder inequality for $q=1+1/(p-1)$ and $\varepsilon>0$ sufficiently small

$$\begin{aligned}\pi_j(A) &\leq \frac{1}{\lambda}e^{\kappa_\lambda(j)}\left(\mathbb{E}\left\{e^{p\sup_{s\in J_j}\left|B^\delta(s)-B^\lambda(s)\right|+p\sup_{s\in J_j}\left|B^\eta(s)-B^\lambda(s)\right|}\right\}\right)^{1/p}\\ &\quad\times\left(\mathbb{P}\left\{R^{\delta,\eta}_{u,T}(j)>A\right\}\right)^{1/(pq)}\left(\mathbb{P}\left\{\sup_{s\in J_j}B^\delta(s)\geq\inf_{s\in J_j}\sigma^2(\delta\lfloor s/\delta\rfloor)/2\right\}\right)^{1/q^2}\\ &\leq C\left(\mathbb{P}\left\{R^{\delta,\eta}_{u,T}(j)>A\right\}\right)^{1/(pq)}\exp\left(\kappa_\lambda(j)-(1-\varepsilon_2)\frac{c^2\ell(a_j+1)}{8q^2}\right).\end{aligned}$$

Further, by our assumptions on ℓ and c, for all j large and $\varepsilon_3>0$ sufficiently small

$$\begin{aligned}\kappa_\lambda(j) &= \sup_{s\in J_j}\left|\sigma^2(s)-\sigma^2(\lambda\lfloor s/\lambda\rfloor)\right|\\ &\leq \max\Big(\ell(a_j+1)-c\ell(\lambda\lfloor a_j/\lambda\rfloor),\ell(\lambda\lfloor(a_j+1)/\lambda\rfloor)-c\ell(a_j)\Big)\\ &\leq (1-c+\varepsilon_3)\ell(a_j+1).\end{aligned}$$

Choose $q>1$ sufficiently close to 1. Then, by the assumption $c^2+8c-8>0$ and in view of (20), we can find a constant $B>1$ and take $\varepsilon_i>0, i\leq 4$, sufficiently small such that

$$\begin{aligned}\sum_{j\geq 1}\exp\left(\kappa_\lambda(j)-(1-\varepsilon_2)\frac{c^2}{8q^2}\ell(a_j+1)\right) &\leq \sum_{j\geq1}\exp\left(-\Big((1-\varepsilon_2)\frac{c^2}{8q^2}-(1-c+\varepsilon_3)\Big)\ell(a_j+1)\right)\\ &\leq \sum_{j\geq 1}\exp\left(-(c^2+8c-8-\varepsilon_4)\ell(a_j)\right)\\ &\leq \sum_{j\geq 1}e^{-B\ln a_j}=\sum_{j\geq 1}\frac{1}{a_j^B}<\infty.\end{aligned}$$

Consequently, as $A\to\infty$

$$\sum_{j\geq 1}\pi_j(A)\leq\sum_{j\geq 1}C\left(\mathbb{P}\left\{R^{\delta,\eta}_{u,T}(j)>A\right\}\right)^{1/(pq)}\frac{1}{a_j^B}\to 0$$

establishing the proof. □

Proof of Theorem 2.3: The idea of the proof is similar to the proof of Theorem 2.2, with slight modifications which we analyze below. We use the same notation as in the proof of the aforementioned theorem and focus on the case that $uT,(1-u)T\in\mathbb{Z}$.

Case $\eta=0$. Since $\delta=0$ in this case, we set $\lambda=1$ and observe that, on the event $\{M^\delta=M^\delta_j\}$,

$$R^{0,0}_{u,T}\leq e^{2\sup_{s\in J_j}\left|W(s)-W^1(s)\right|}=:\widehat{R}^{0,0}_{u,T}(j)$$

and

$$\begin{aligned}\mathbb{E}\left\{R_{u,T}^{0,0}; R_{u,T}^{0,0} > A\right\} &\le \mathbb{E}\left\{R_{u,T}^{0,0}; R_{u,T}^{0,0} > A, M_0^0 = M^0\right\} + \sum_{j\in\mathbb{Z}\setminus\{0\}} \mathbb{E}\left\{\widehat{R}_{u,T}^{0,0}(j); \widehat{R}_{u,T}^{0,0}(j) > A, M_j^0 \ge 1\right\} \\ &=: \mathbb{E}\left\{R_{u,T}^{0,0}; R_{u,T}^{0,0} > A, M_0^0 = M^0\right\} + \sum_{j\in\mathbb{Z}\setminus\{0\}} \widehat{\pi}_j(A).\end{aligned}$$

As in the proof of Theorem 2.2 since $\mathbb{E}\left\{M_0^0/S_0^0\right\} < \infty$ we have

$$\lim_{A\to\infty} \mathbb{E}\left\{M_0^0/S_0^0; M_0^0/S_0^0 > A\right\} = 0.$$

Thus we focus on an upper bound for $\widehat{\pi}_j(A)$. By the same argument as given in the proof of Theorem 2.2, for any $p > 1$ and $q = 1 + 1/(p-1)$

$$\widehat{\pi}_j(A) \le \left(\mathbb{E}\left\{e^{2p\sup_{s\in J_j}\left|W(s)-W^1(s)\right|}\right\}\right)^{1/p}\left(\mathbb{P}\left\{\widehat{R}_{u,T}^{0,0}(j) > A\right\}\right)^{1/(pq)}\left(\mathbb{P}\left\{\sup_{s\in J_j} W(s) \ge 1\right\}\right)^{1/q^2}.$$

Next, suppose that $j \ge 1$. By (2.1) in [46] (see also Lemma 9.1 in [47]), for each $u > u_0 > 0$

$$\mathbb{P}\left\{\sup_{s\in[0,1)} W(s) > u\right\} \le \frac{\mathbb{P}\left\{W(1) > u - u_0\right\}}{\mathbb{P}\left\{\inf_{s\in[0,1)} W(s) > -u_0\right\}}$$

and

$$\mathbb{P}\left\{\inf_{s\in[0,1)} W(s) < -u\right\} \le \frac{\mathbb{P}\left\{-W(1) > u - u_0\right\}}{\mathbb{P}\left\{\inf_{s\in[0,1)} -W(s) > -u_0\right\}},$$

which implies that

$$\mathbb{E}\left\{e^{2p\sup_{s\in J_j}\left|W(s)-W^1(s)\right|}\right\} = \mathbb{E}\left\{e^{2p\sup_{s\in[0,1)}|W(s)|}\right\} \le C_1\mathbb{E}\left\{e^{2p|W(1)|}\right\} < \infty \tag{36}$$

for sufficiently small $p > 1$ and some $C > 0$.
In order to derive a tight upper bound for $\mathbb{P}\left\{\sup_{t\in J_j} W(t) > 1\right\}$, as $j\to\infty$, let us recall that $W(t) = B^+(t) - \Phi(1)t$, for $t \ge 0$, and observe that $\mathbb{E}\{W(1)\} = \mathbb{E}\{B^+(1) - \Phi(1)\} < 0$.
Let $\varepsilon = \frac{1}{2}(\Phi(1) - \mathbb{E}\{B^+(1)\}) > 0$ and introduce the following Lévy process $L(t) := W(t) + \varepsilon t$. It is straightforward to check that $\mathbb{E}\{L(1)\} < 0$ and for $\Phi_L(\theta) := \ln\mathbb{E}\left\{e^{\theta L(1)}\right\}$ we have

$$\Phi_L(0) = 0, \quad \Phi_L'(0) = \mathbb{E}\{L(1)\} < 0$$

and $\Phi_L(1) = \varepsilon > 0$. Hence, there exists $1 > \gamma > 0$ such that $\Phi_L(\gamma) = 0$ and $\Phi_L'(\gamma) < \infty$. Now, following, e.g., Theorem 2.6 from [48]

$$\mathbb{P}\left\{\sup_{t\in J_j} W(t) > 1\right\} \le \mathbb{P}\left\{\sup_{t\in[0,\infty)} (B^+(t) - t(\Phi(1)-\varepsilon)) > \varepsilon j\right\} \le \mathrm{C}e^{-\gamma\varepsilon j} \tag{37}$$

for some $\mathrm{C} \in (0,\infty)$ and all $j \ge 1$. Therefore, combining (36) with (37), we get

$$\lim_{A\to\infty}\sum_{j\ge 1}\widehat{\pi}_j(A) = 0.$$

The proof that $\lim_{A\to\infty}\sum_{j\le -1}\widehat{\pi}_j(A) = 0$ follows by the same argument utisilzing further the fact that $W(t) = W^-(-t)$ if $t < 0$, with

$$\ln\mathbb{E}\left\{e^{\theta W^-(1)}\right\} = \Phi(1-\theta) - (1-\theta)\Phi(1).$$

<u>Case $\eta > 0$</u>. We set $a_j := \eta j$ and observe that, on the event $\{M^\delta = M_j^\delta\}$ (assume that $uT, (1-u)T \in \mathbb{Z}$)

$$R_{u,T}^{\delta,\eta} \le \frac{M_j^\delta}{S_j^\eta} = \frac{M_j^\delta}{M_j^\eta}\frac{M_j^\eta}{S_j^\eta} \le \frac{1}{\eta}e^{\sup_{s\in J_j}\left|W^\delta(s)-W^\eta(s)\right|} \le \frac{1}{\eta}e^{\sup_{s\in J_j}|W(s)-W^\eta(s)|}.$$

The rest of the proof goes line by line the same as the proof of case $\eta = 0$, with the use of the fact that if $\eta > 1$, then

$$\mathbb{E}\left\{e^{p|W(\eta)|}\right\} \le \left(\mathbb{E}\left\{e^{p|W(1)|}\right\}\right)^{\lceil\eta\rceil}.$$

This completes the proof. □

Proof of Proposition 4.1: For an M3 process as above, the finite dimensional distributions of ξ_W^δ for $t_i \in \delta\mathbb{Z}, x_i \in \mathbb{R}, 1 \le i \le n, n \in \mathbb{N}$ can be written as

$$-\ln\mathbb{P}\left\{\xi_W^\delta(t_1) \le x_1, \ldots, \xi_W^\delta(t_n) \le x_n\right\} = C_F^\delta\mathbb{E}\left\{\int_{r\in\mathbb{R}}\max_{j=1,\ldots,n}\exp\left(F^\delta(t_j - r) - x_j\right)\nu_\delta(dr)\right\}.$$

Since ξ_W has càdlàg paths by assumption, we have for any compact set $E \subset \mathbb{R}$

$$-\ln \mathbb{P}\left\{\sup_{t\in\delta\mathbb{Z}\cap E} \xi_W^\delta(t) \le 0\right\} = C_F^\delta \mathbb{E}\left\{\int_{\delta\mathbb{Z}} \sup_{t\in\delta\mathbb{Z}\cap E} \exp\left(F^\delta(t-r)\right) \nu_\delta(dr)\right\},$$

which, in view of equation (5), implies

$$\mathcal{H}_W^\delta = \lim_{T\to\infty} \frac{C_F^\delta}{T} \mathbb{E}\left\{\int_{\delta\mathbb{Z}} \sup_{t\in\delta\mathbb{Z}\cap[0,T]} \exp\left(F^\delta(t-r)\right) \nu_\delta(dr)\right\}. \tag{38}$$

Set $T_\delta = T$ if $\delta = 0$ and $T_\delta = \delta\lfloor T/\delta\rfloor$ otherwise. For any fixed $T > 0$

$$\mathbb{E}\left\{\int_{\in\mathbb{R}} \sup_{t\in\delta\mathbb{Z}\cap[0,T]} \exp\left(F^\delta(t-r)\right) \nu_\delta(dr)\right\} \ge \mathbb{E}\left\{\int_{-T_\delta}^0 \sup_{t\in\delta\mathbb{Z}\cap[0,T]} \exp\left(F^\delta(t-r)\right) \nu_\delta(dr)\right\} dr = T_\delta,$$

since by the assumption $\sup_{t\in\delta\mathbb{Z}} F^\delta(t) = F^\delta(0) = 0$ almost surely, for any $r \in [-T_\delta, 0]$ we have

$$\sup_{t\in\delta\mathbb{Z}\cap[0,T]} \exp\left(F^\delta(t-r)\right) = \exp\left(F^\delta(0)\right) = 1.$$

Consequently,

$$\mathcal{H}_W^\delta \ge C_F^\delta \lim_{T\to\infty} \frac{T_\delta}{T} = C_F^\delta.$$

We show next (32) for $\delta = 0$. Recall that we write C_F instead of C_F^0. Theorem 4.1 in [49] implies that the process W with $W(0) = 0$ almost surely, can be obtained by the M3 representation in terms of the shape function as

$$\mathbb{P}\{W \in L\} = C_F \int_D \int_{\mathbb{R}} \mathbf{1}\{f(\cdot+s) - f(s) \in L\} e^{f(s)} ds \mathbb{P}_F(df),$$

where L is an arbitrary Borel subset of D. Consequently, for any $\mathbb{P}_W$-integrable functional $\Gamma : D \to \mathbb{R}$ we have

$$\mathbb{E}\{\Gamma(W)\} = C_F \int_D \int_{\mathbb{R}} \Gamma(f(\cdot+s) - f(s)) e^{f(s)} ds \mathbb{P}_F(df).$$

Let now Γ be given by the mapping (on a suitable subspace of D with full $\mathbb{P}_W$ measure)

$$f \mapsto \frac{\sup_{t\in\mathbb{R}} e^{f(t)}}{\int_{\mathbb{R}} e^{f(t)} dt},$$

and observe

$$\begin{aligned}
\mathbb{E}\left\{\frac{\sup_{t\in\mathbb{R}} e^{W(t)}}{\int_{\mathbb{R}} e^{W(t)} dt}\right\} &= C_F \int_D \int_{\mathbb{R}} \frac{\sup_{t\in\mathbb{R}} e^{f(t+s)-f(s)}}{\int_{\mathbb{R}} e^{f(t+s)-f(s)} dt} e^{f(s)} ds \mathbb{P}_F(df) \\
&= C_F \int_D \int_{\mathbb{R}} \frac{e^{f(s)}}{\int_{\mathbb{R}} e^{f(t+s)} dt} ds \mathbb{P}_F(df) \\
&= C_F \in (0, \infty),
\end{aligned}$$

where the second last equality follows from the assumption that $\sup_{t\in\mathbb{R}} F(t) = 0$ a.s. In the case $\delta > 0$ we can use the same arguments together with Theorem 6.1. ☐

Proof of Lemma 3.1: Let $\sum_{i=1}^\infty \varepsilon_{(P_i,T_i)}$ and $\sum_{i=1}^\infty \varepsilon_{(\tilde{P}_i,\tilde{T}_i)}$ be Poisson point processes on $\mathbb{R}^2$ with intensity $C_F e^{-p} dpdt$ and $C_G e^{-p} dpdt$, respectively. Since, by assumption, the processes $\{\max_{i\ge1}(P_i + F_i(t-T_i)), t \in \mathbb{R}\}$ and $\{\max_{i\ge1}(\tilde{P}_i + G_i(t-\tilde{T}_i)), t \in \mathbb{R}\}$ are equal in distribution, it follows that also the Poisson point processes $\Pi_F = \{P_i + F_i(t-T_i), i \in \mathbb{N}\}$ and $\Pi_G = \{P_i + G_i(t-T_i), i \in \mathbb{N}\}$ with values in the space D of càdlàg functions have the same distribution. Applying the measurable mapping

$$f \mapsto (\sup f, \arg\sup f, f(\cdot - \arg\sup f) - \sup f)$$

to Π_F and Π_G yields two equally distributed Poisson point processes on $\mathbb{R}\times\mathbb{R}\times D$ with intensity measures Λ_F and Λ_G. We compute for $z \in \mathbb{R}$ and $t_1 < t_2$

$$\begin{aligned}
\Lambda_F([z,\infty)\times[t_1,t_2]\times D) &= C_F \int_{\mathbb{R}} e^{-p} \int_{\mathbb{R}} \mathbb{P}(p + \sup F \ge z, \arg\sup F(\cdot - t) \in [t_1,t_2]) dpdt \\
&= C_F \mathbb{E}\left\{\int_{\mathbb{R}} e^{-(z-\sup F)} \mathbf{1}\{\arg\sup F(\cdot - t) \in [t_1,t_2]\}\} dt\right\} \\
&= C_F e^{-z}(t_2 - t_1) \mathbb{E}\{e^{\sup F}\}.
\end{aligned}$$

By the same calculation for Λ_G and the fact that the intensities are equal, the claim follows. ☐

6. Appendix

The notion of a mixed moving maxima process on $\mathbb{R}$ defined in (24) can be extended to the lattice $\delta\mathbb{Z}$; see for instance Remark 7 in [26]. Suppose that $\{\xi^\delta_W(t), t \in \delta\mathbb{Z}\}$ is a stationary max-stable process (with standard Gumbel margins) given by the construction (6) with a process W, restricted to $\delta\mathbb{Z}$. Further suppose that $W(0) = 0$ almost surely and let F^δ_i be independent copies of a process F^δ on $\delta\mathbb{Z}$ with

$$\sup_{t\in\delta\mathbb{Z}} F^\delta(t) = F^\delta(0) = 0$$

almost surely, and

$$C^\delta_F = \left(\mathbb{E}\left\{\sum_{t\in\delta\mathbb{Z}} \exp(F^\delta(t))\right\}\right)^{-1} \in (0,\infty). \tag{39}$$

We say that ξ^δ_W admits an M3 representation, if

$$\xi^\delta_W(t) = \max_{i\geq 0}(F^\delta_i(t - T^\delta_i) + P^\delta_i), \quad t \in \delta\mathbb{Z},$$

where $\sum_{i=1}^\infty \epsilon_{(P^\delta_i, T^\delta_i)}$ is a Poisson point process with intensity $C^\delta_F e^{-p} dp \nu_\delta(dt)$. Here ν_δ/δ is the counting measure on $\delta\mathbb{Z}$. Below we present the counterpart of Theorem 4.1 in [49] for M3 processes on lattices. We omit its proof since it follows with the same arguments as the aforementioned one.

Theorem 6.1. *Suppose that the max-stable and stationary process ξ_W has càdlàg sample paths. The process $W^\delta, \delta > 0$, the restriction of W to $\delta\mathbb{Z}$, can be expressed in terms of the spectral function F^δ as*

$$\mathbb{P}(W^\delta \in L) = C^\delta_F \mathbb{E}\left\{\sum_{t\in\delta\mathbb{Z}} \mathbf{1}\left\{F^\delta(\cdot + t) - F^\delta(t) \in L\right\} \exp(F^\delta(t))\right\},$$

which is well-defined probability measure by (39).

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