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Interfaces with other disciplines

Solidarity to achieve stability☆

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ABSTRACT

Agents may form coalitions. Each coalition shares its endowment among its agents by applying a sharing rule. The sharing rule induces a coalition formation problem by assuming that agents rank coalitions according to the allocation they obtain in the corresponding sharing problem. We characterize the sharing rules that induce a class of stable coalition formation problems as those that satisfy a natural axiom that formalizes the principle of solidarity. Thus, solidarity becomes a sufficient condition to achieve stability.

1. Introduction

According to the Merriam-Webster Dictionary, solidarity is defined as *unity (as of a group or class) that produces or is based on community of interests, objectives, and standards*. It is a fundamental ethical principle that can be traced back to ancient philosophers such as Socrates and Aristotle, and it is also connected to the slogan of the French revolution. It is also one of the six titles of the Charter of Fundamental Rights of the European Union.

The principle of solidarity has been often invoked in the axiomatic approach to economic design. It can be formally stated as follows: "if the environment (e.g., resources, technology, population, or preferences) in which a group of people find themselves changes, and if no one in this group is responsible for the change, the welfare of all of them should be affected in the same direction: either they all end up at least as well off as they were initially, or they all end up at most as well off" (Thomson, 2023). Axioms formalizing this idea have been crucial to characterize egalitarian allocation rules in diverse settings (Martínez & Moreno-Ternero, 2022; Moreno-Ternero & Roemer, 2006; Moulin, 1987a; Moulin & Roemer, 1989; Roemer, 1986). They have also been instrumental to characterize focal egalitarian rules in the axiomatic theory of bargaining and cooperative game theory (Chun & Thomson, 1988; Kalai, 1977; Kalai & Smorodinsky, 1975; Thomson & Myerson, 1980; Young, 1988). Thus, the egalitarian implications of the principle of solidarity have been well explored. Here, we focus on less explored implications, highlighting its role as a means to guarantee stability in the context of coalition formation.

Coalition formation itself has been the object of a large literature dealing with a plethora of social and economic issues such as cartel formation, lobbies, customs unions, conflict resolution, public goods provision, political party formation, etc. (Grabisch & Funaki, 2012; Ray, 2007; Ray & Vohra, 2015). A central concern in this literature has been stability, that is, immunity of a coalitional arrangement to "blocking" (Perry & Reny, 1994; Pulido & Sánchez-Soriano, 2006; Seidmann & Winter, 1998). To be precise, a partition of agents into coalitions is (core) stable if there is no coalition in which each of its members strictly prefers it to the coalition to which they belong in the partition.[1] Our focus here is stability for environments in which coalition members have an endowment to be shared. In these contexts,

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[1] Other stability concepts have been analyzed in the literature. See, for instance, Karakaya (2011) and Gusev (2021).

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we can define for each agent a preference over the possible coalitions she can be part of, depending on the sharing rule used to distribute the endowment of each coalition: if the payoff an agent receives in one coalition is larger than her payoff in another, then this agent will prefer the former to the latter. As preferences are derived from a sharing rule, we can speak of "the coalition formation problem induced by the sharing rule". And a natural question is whether there exist conditions on the sharing rule that guarantee the existence of stable partitions in the coalition formation problems that induces. We show that *solidarity* is such a condition, formalized by the following axiom: the arrival of new agents in a coalition, whether or not it is accompanied by a change in the endowment available for this coalition, should affect the welfare of all incumbents in the same direction.

Our solidarity axiom is reminiscent of the axiom of *population monotonicity* (Thomson, 1983) but it is actually equivalent to the combination of two axioms that appear frequently in the literature: *endowment monotonicity* and *consistency*.[2] The former says that when a bad or good shock changes the endowment of a group, all its members should share in the calamity or the windfall. Thus, it has obvious solidarity underpinnings and it has long been used in axiomatic work (Bergantiños & Moreno-Ternero, 2022; Moreno-Ternero & Vidal-Puga, 2021; Moulin, 1992; Moulin & Thomson, 1988). The latter says that if a sub-group of agents secedes with the endowment allocated to their members, then the rule allocates the remaining endowment to the remaining agents in the same way. As such, it has been referred as a "robustness", often "coherence", or "operational" principle (Balinski, 2005; Gudmundsson et al., 2023; Moreno-Ternero & Roemer, 2012; Thomson, 2011, 2019), although solidarity underpinnings have also been provided (Thomson, 2012). Alternative forms of consistency have indeed been suggested as axioms of stability in related contexts (Harsanyi, 1959; Lensberg, 1987, 1988).

Our main result actually states that a sharing rule satisfies *solidarity* if and only if it induces *non-circular coalition formation problems*. These problems preclude the existence of *rings* (which arise when there is a cycle of coalitions with at least one agent in the intersection of any two consecutive coalitions who prefers the latter to the former, while the rest of the intersection-mates do not have the opposite preference) and satisfy the *weak pairwise alignment* property (if one agent in the intersection of two coalitions ranks them in one way, no other agent in the intersection ranks them in the opposite way). As a non-circular coalition formation problem admits a stable partition, our result implies that requiring solidarity the sharing rule guarantees *stability* of the induced coalition formation problem in the sense that these problems will have a non-empty core.

The closest research to our work is Gallo and Inarra (2018), which combines coalition formation and rationing problems (O'Neill, 1982). Specifically, they assume that agents in each coalition have "claims" over an endowment associated with the coalition. The endowment is not sufficient for claims to be fully honored and, thus, has to be rationed. The rule that is used for that purpose induces agents' preferences over coalitions. Within the domain of continuous rules, they show that the properties of endowment monotonicity and consistency guarantee the existence of stable partitions in the induced coalition formation problems.[3] We generalize that result to any resource allocation situation (i.e., not necessarily rationing) and without imposing continuity.

Another closely related paper is Pycia (2012). We shall be more precise about the connection once we formally introduce our result later in the text. But we mention at least now that Pycia (2012) analyzes a general model of coalition formation (including cases in which not all coalitions are feasible) and shows that, under two regularity conditions, there is a stable partition if agents' preferences are generated by a sharing rule imposing Nash bargaining (Nash, 1950) over coalitional outputs.[4] By contrast, our result characterizes rules inducing stability without resorting to the regularity conditions, but only referring to the case in which all coalitions are feasible.

Our result can be applied to other settings than those mentioned above. We first analyze surplus sharing problems, closely related to rationing problems, and then we develop an application based on the problem of prize allocation in competitions where agents are ranked (Dietzenbacher & Kondratev, 2023). Finally, we show that our result also gives insights into resource allocation situations in which some coalitions are not feasible.

The rest of the paper is organized as follows. In Section 2, we introduce the preliminaries of the model (sharing problems and coalition formation problems). In Section 3, we present our benchmark analysis and result. In Section 4, we present applications of our result to several problems such as bargaining, rationing, surplus sharing, or ranking problems. We conclude in Section 5. The proof of the main result has been relegated to Appendix.

2. Preliminaries

2.1. Sharing problems

Let N be a finite set of agents. Consider a situation where a coalition of agents $C \subseteq N$ has an endowment $E \in \mathbb{R}_+$. A **sharing problem** is a pair (C, E). Let $\mathcal{P}$ denote the class of such problems.

An **allocation** for $(C, E) \in \mathcal{P}$ is a vector $x = (x_i)_{i\in C} \in \mathbb{R}_+^{|C|}$ that satisfies non-negativity, for each $i \in C$, $0 \le x_i$, and efficiency, $\sum_{i\in C} x_i = E$. A **(sharing) rule** is a function F defined on $\mathcal{P}$ that associates with each $(C, E) \in \mathcal{P}$ an allocation $F(C, E)$ for (C, E). The payoff of agent i in problem (C, E) under rule F is denoted by $F_i(C, E)$. We denote by $\mathcal{F}$ the set of all rules.

We now introduce several axioms for rules.

The first axiom states that small changes in the endowment do not lead to large changes in the chosen allocation.

Endowment continuity: For each $(C, E) \in \mathcal{P}$ and each $\{E_j\}_{j=1}^{\infty}$ with $E_j \to E$, $F(C, E_j) \to F(C, E)$.

The second axiom states that if the endowment increases, then each agent receives at least as much as she initially did.

Endowment monotonicity: For each pair $(C, E), (C', E') \in \mathcal{P}$, with $C = C'$ and $E < E'$, and each $i \in C$, $F_i(C, E) \le F_i(C', E')$.

The third axiom states that if some agents leave a coalition with the payoffs assigned to them by the rule, and the situation is reassessed at that point, and the rule reapplies, then each remaining agent receives the same payoff as she initially did.

Consistency: For each $(C, E) \in \mathcal{P}$, each $C' \subset C$, and each $i \in C'$, $F_i(C', \sum_{i\in C'} F_i(C, E)) = F_i(C, E)$.

We finally introduce the axiom of solidarity: the possible arrival of new agents (with or without the endowment changing) does not affect two incumbent agents in opposite directions.

Solidarity: For each pair $(C, E), (C', E') \in \mathcal{P}$, with $C \subseteq C'$, and each pair $i, j \in C$, $F_i(C, E) > F_i(C', E')$ implies $F_j(C, E) \ge F_j(C', E')$.

The next lemma relates the previous axioms.

Lemma 1. *The following statements hold:*

(i) Endowment monotonicity implies endowment continuity.
(ii) Solidarity is equivalent to the conjunction of endowment monotonicity and consistency.

[2] The solidarity axiom we consider was called *population-and-resource monotonicity* in the context of rationing problems (Chun, 1999). See also Moreno-Ternero and Roemer (2006).

[3] Gallo and Inarra (2018) wrongly state that these properties are not only sufficient but also necessary.

[4] From a different perspective, Lensberg (1987) obtains the same functional form.

Proof. (i) Let F be a rule satisfying *endowment monotonicity*. Let $(C, E) \in \mathcal{P}$ and $\{E_j\}_{j=1}^{\infty}$ be a sequence of endowments such that $E_j \to E$. Then, for each j, $|E_j - E| = |\sum_{i\in C} F_i(C, E_j) - \sum_{i\in C} F_i(C, E)|$. Then, by *endowment monotonicity*, for each j,

$$|\sum_{i\in C} F_i(C, E_j) - \sum_{i\in C} F_i(C, E)| = \sum_{i\in C} |F_i(C, E_j) - F_i(C, E)|.$$

Thus, for each $i \in C$, $|F_i(C, E_j) - F_i(C, E)| \leq |E_j - E|$. As $|E_j - E| \to 0$, it follows that for each $i \in C$, $|F_i(C, E_j) - F_i(C, E)| \to 0$. Equivalently, for each $i \in C$, $F_i(C, E_j) \to F_i(C, E)$.

(ii) Let F be a rule satisfyng *solidarity*. Then, it is straightforward to see that it also satisfies *endowment monotonicity*. As for *consistency*, let $(C, E) \in \mathcal{P}$ and $C' \subset C$. Then, by *solidarity*, either for each $i \in C'$, $F_i(C, E) \geq F_i(C', \sum_{i\in C'} F_i(C, E))$, or for each $i \in C'$, $F_i(C, E) \leq F_i(C', \sum_{i\in C'} F_i(C, E))$. Thus, for each $i \in C'$, $F_i(C, E) = F_i(C', \sum_{i\in C'} F_i(C, E))$, as desired.

Conversely, let F be a rule satisfying *endowment monotonicity* and *consistency*. Let $(C, E), (C', E') \in \mathcal{P}$ be such that $C \subseteq C'$. By *endowment monotonicity*, if $C = C'$ and $E \geq E'$, then for each $i \in C$, $F_i(C, E) \geq F_i(C', E')$, whereas if $C = C'$ and $E \leq E'$, then for each $i \in C$, $F_i(C, E) \leq F_i(C', E')$. Thus, either way, *solidarity* holds. Assume next that $C \subset C'$. By *consistency*, for each $i \in C$,

$$F_i(C, \sum_{i\in C} F_i(C', E')) = F_i(C', E').$$

By *endowment monotonicity*, if $E \leq \sum_{i\in C} F_i(C', E')$, then for each $i \in C$, $F_i(C, E) \leq F_i(C, \sum_{i\in C} F_i(C', E')) = F_i(C', E')$, whereas if $E \geq \sum_{i\in C} F_i(C', E')$, then for each $i \in C$, $F_i(C, E) \geq F_i(C, \sum_{i\in C} F_i(C', E')) = F_i(C', E')$. Thus, either way, *solidarity* holds. □

2.2. Coalition formation problems

Consider a situation where each agent ranks the coalitions that she may belong to. Formally, let N be a finite set of agents and $C \subseteq N$ be a coalition. The collection of non-empty coalitions is denoted by 2^N. For each agent $i \in N$, let $\succsim_i$ be a complete and transitive preference relation over coalitions containing i. Given $C, C' \subseteq N$ such that $i \in C \cap C'$, $C \succsim_i C'$ means that agent i finds coalition C at least as desirable as coalition C'. The binary relations $\succ_i$ and $\sim_i$ are defined as usual. A **(hedonic) coalition formation problem** is just a preference profile that consists of a list of preference relations, one for each $i \in N$, $\succsim = (\succsim_i)_{i\in N}$. Let $\mathcal{D}$ denote the class of such problems.

A partition is a set of non-empty coalitions whose union is N and whose pairwise intersections are empty. Formally, a **partition** is a list $\pi = \{C_1, \ldots, C_m\}$ such that *(i)* for each $l = 1, \ldots, m$, $C_l \neq \emptyset$, *(ii)* $\bigcup_{l=1}^{m} C_l = N$, and *(iii)* for each pair $l, l' \in \{1, \ldots, m\}$, with $l \neq l'$, $C_l \cap C_{l'} = \emptyset$. Let Π denote the set of all partitions. For each $\pi \in \Pi$ and each $i \in N$, let $\pi(i)$ denote the coalition in π that contains agent i. A partition $\pi \in \Pi$ is **stable for** $\succsim$ if there is no coalition $T \subseteq N$ such that for each $i \in T$, $T \succ_i \pi(i)$. The set of all stable partitions for $\succsim$ is the **core** of $\succsim$. The literature on coalition formation mostly focuses on identifying properties of the preference profiles that guarantee the existence of stable partitions.

We now introduce several concepts and properties defined for coalition formation problems. First, a ***ring*** is an ordered list of coalitions $(C_0, C_1, \ldots, C_{l-1})$, with $l > 2$, such that for each $k = 0, 1, \ldots, l-1$ (modulo l) and each $j \in C_k \cap C_{k+1}$, $C_{k+1} \succsim_j C_k$, with at least one agent with strict preference in each intersection.[5] That is, in a ring there is at least one agent in the intersection of any two consecutive coalitions who prefers the latter to the former, while those agents who do not prefer the latter to the former are indifferent. It can be easily checked that the lack of rings guarantees the existence of a stable partition.

[5] See, for instance, Inal (2015) and Pycia (2012) for different definitions of rings, under the name of cycles.

The next property, originally introduced by Pycia (2012), requires that all agents in the intersection of two coalitions rank them in the same way.

Pairwise alignment: A coalition formation problem $\succsim \in \mathcal{D}$ is pairwise aligned if for each pair $C, C' \subseteq N$ and each pair $i, j \in C \cap C'$, $C \succsim_i C'$ if and only if $C \succsim_j C'$.

The **common ranking property** (Farrell & Scotchmer, 1988), states that there is a common ranking of all coalitions that agrees with all agents' preferences. Formally, there is an ordering $\succsim$ over 2^N such that for each $i \in N$ and each pair $C, C' \subseteq N$ with $i \in C \cap C'$, $C \succsim_i C' \Leftrightarrow C \succsim C'$. Note that the common ranking property precludes the existence of rings. In addition, when all coalitions are feasible, the common ranking property coincides with the pairwise alignment property (Footnote 6 in Pycia, 2012).

A weakening of the pairwise alignment property, introduced by Gallo and Inarra (2018), requires that if one agent in the intersection of two coalitions ranks them in one way, no other agent in the intersection ranks them in the opposite way.

Weak pairwise alignment: A coalition formation problem $\succsim \in \mathcal{D}$ is weakly pairwise aligned if for each pair $C, C' \subseteq N$ and each pair $i, j \in C \cap C'$, then $C \succ_i C'$ implies $C \succsim_j C'$.

Note that, unlike pairwise alignment, *weak pairwise alignment* allows one agent to have a strict preference over two coalitions while any other agent in the intersection is indifferent between them.

The class of coalition formation problems that satisfy *weak pairwise alignment* and do not have *rings* is dubbed ***non-circular coalition formation problems*** by Gallo and Inarra (2018). This class includes the problems that satisfy the common ranking property (the proof is straightforward). It is also related to the class of problems that satisfy the **top-coalition property** (see Banerjee et al., 2001). Formally, a coalition $C' \subseteq C$ is a **top coalition of** C if for each $i \in C'$ and each $S \subseteq C$ with $i \in S$, we have $C' \succsim_i S$. A coalition formation problem satisfies the top-coalition property if each coalition $C \subseteq N$ has a top coalition. The non-circular coalition formation problems are included in the class of problems satisfying the top-coalition property (see Theorem 1 in Gallo & Inarra, 2018). A weaker version of the top-coalition property guarantees the existence of a stable partition and, in consequence, so does the top-coalition property (Theorem 1 in Banerjee et al., 2001). To complete the relations between the properties, the top-coalition property neither implies *weak pairwise alignment* nor precludes the existence of *rings* (see Examples 1 and 2 in Gallo & Inarra, 2018). Finally, *weak pairwise alignment* does not guarantee the existence of a stable partition (Example 3 in Gallo & Inarra, 2018). All these relations are illustrated in Fig. 1.

3. Benchmark analysis

Given a set of sharing problems, one for each coalition, and a sharing rule, a coalition formation problem is induced as follows: each agent computes her payoff in each problem with the sharing rule, and ranks coalitions accordingly. Formally, given a set of problems $\{(C, E_C)\}_{C\subseteq N, E_C\in\mathbb{R}_+}$, the **coalition formation problem induced by rule** $F \in \mathcal{F}$ is the list of preference relations $\succsim^F = (\succsim_i^F)_{i\in N}$ defined as follows: for each $i \in N$, and each pair $C, C' \subseteq N$ such that $i \in C \cap C'$, $C \succsim_i^F C'$ if and only if $F_i(C, E_C) \geq F_i(C', E_{C'})$.

Our main result characterizes all rules that induce non-circular coalition formation problems. They happen to be those that satisfy the *solidarity* axiom. The proof can be found in Appendix.

Theorem 1. *A sharing rule F satisfies solidarity if and only if for any set of sharing problems $\{(C, E_C)\}_{C\subseteq N, E_C\in\mathbb{R}_+}$, the coalition formation problem induced by F, $\succsim^F$, is non-circular.*

The next result follows from Theorem 1 and the relations among properties presented above (see Fig. 1).

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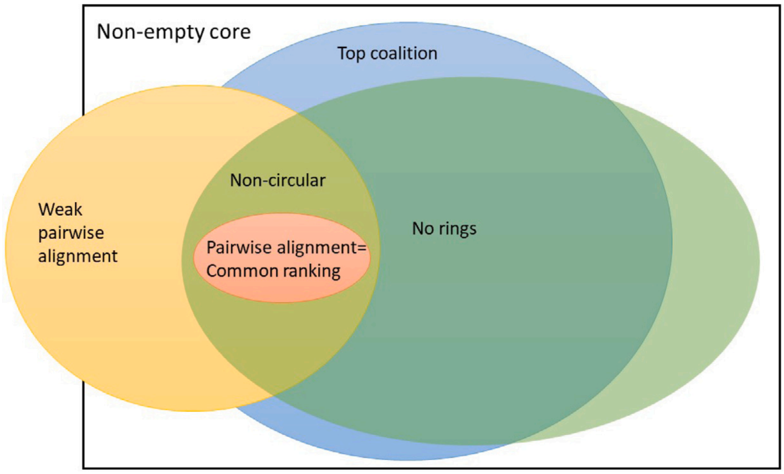

Fig. 1. Relations among properties.

Corollary 1. *If a sharing rule F satisfies solidarity, then for any set of sharing problems $\{(C, E_C)\}_{C \subseteq N, E_C \in \mathbb{R}_+}$, the core of the coalition formation problem induced by F, $\succsim^F$, is non-empty.*

Corollary 1 implies that, when the sharing rule satisfies *solidarity*, then stability is guaranteed. These results are illustrated in Fig. 2.

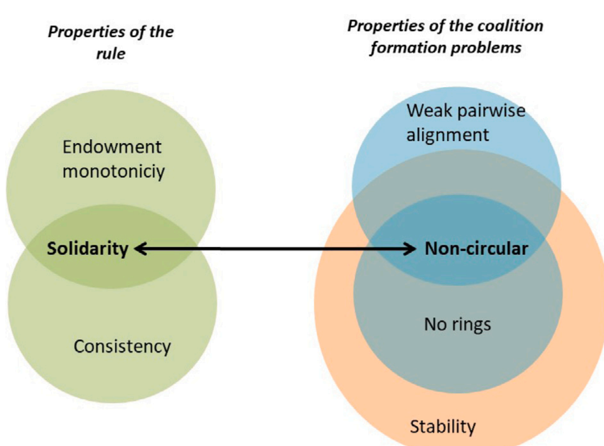

Fig. 2. Sharing rules and their induced coalition formation problems.

Theorem 1 and Corollary 1 relate to Pycia (2012) as follows. On the one hand, note that our results apply to resource allocation situations generating hedonic coalition formation problems (i.e., all elements of 2^N are feasible coalitions) while his results also apply to situations where not all coalitions are feasible, such as matching problems. Nevertheless, we show in Section 4 that our results can also give some insights for any resource allocation situation where only some coalitions are feasible.

On the other hand, Pycia (2012) shows that pairwise alignment is a necessary and sufficient condition to guarantee the non-emptiness of the core in all the situations he analyzes (his Theorems 1 and 2). By contrast, by weakening pairwise alignment and demanding absence of rings, we define a property of non-circularity that is weaker than pairwise alignment and is sufficient to guarantee the non-emptiness of the core in our setting (where all coalitions are feasible). Consequently, this may lead to differences between the sharing rules that yield stability in the induced coalition formation problems in Pycia (2012) and those that do so in our setting. More precisely, Pycia (2012) characterizes the sharing rules obeying strict endowment monotonicity[6] and non-satiability[7] and guarantee that, for any of the resource allocation situations he considers, all induced coalition formation problems have a non-empty core (his Corollary 1).[8] However, we characterize the sharing rules that generate non-circular coalition formation problems by the axiom of *solidarity* (our Theorem 1). Therefore, those sharing rules guarantee stability in the resource allocation situations where all coalitions are feasible (our Corollary 1). The above are not minor technical differences because, as the following example illustrates, some interesting sharing rules inducing stability in the situations where all coalitions are feasible are covered by our result, but not by Pycia's.

Example 1. Consider a sharing rule giving *priority* to some agent over others. Moreover, she receives in each non-singleton coalition at most k units of the endowment, while this bound does not apply for the other agents. Then, F allocates the first k units of the endowment of any coalition to this agent (if she is in the coalition), while the remaining units of the endowment (if any are left) are shared equally among the other agents within the coalition. If the prioritized agent is not in the coalition, the rule simply selects equal sharing of the endowment among all coalition members.

Formally, let $N = \{1, \ldots, n\}$ and consider the following sharing rule:

$$F_i(C, E_C) = \begin{cases} \min\{E_C, k\} & \text{if } i = 1 \in C \text{ and } C \neq \{1\}, \\ \frac{E_C - \min\{E_C, k\}}{|C|-1} & \text{if } 1, i \in C \text{ and } i \neq 1, \\ \frac{E_C}{|C|} & \text{otherwise.} \end{cases}$$

It can be checked that this rule satisfies *solidarity*. Then, by our results, it also guarantees stability. We illustrate this by showing the existence of a stable partition for an example. Let $N = \{1, 2, 3, 4\}$, $k = 10$, and consider the following coalitional endowments:

C	$\{12\}$	$\{13\}$	$\{14\}$	$\{123\}$	$\{124\}$	$\{134\}$	$\{1234\}$	others
E_C	10	12	14	16	20	24	28	0

Then, F yields the following allocations:

[6] Formally, for each pair $(C, E), (C, E') \in \mathcal{P}$, with $E < E'$, and each $i \in C$, $F_i(C, E) < F_i(C, E')$.

[7] Formally, for each $C \subseteq N$ and each $i \in C$, $\lim_{E \to \infty} F_i(C, E) = \infty$.

[8] He also includes the axiom of endowment continuity, but this axiom is implied by strict endowment monotonicity, thanks to our Lemma 1 presented above.

C	{12}	{13}	{14}	{123}
$F(C, E_C)$	(10, 0)	(10, 2)	(10, 4)	(10, 3, 3)

C	{124}	{134}	{1234}	otherwise
$F(C, E_C)$	(10, 5, 5)	(10, 7, 7)	(10, 6, 6, 6)	$(0)_{i\in C}$

As a consequence, the coalition formation problem induced by F, $\succsim^F$, is the following:

$\succsim_1^F$	$\succsim_2^F$	$\succsim_3^F$	$\succsim_4^F$
12 ∼ 13 ∼ 14 ∼	1234	134	134
∼ 123 ∼ 124 ∼	124	1234	1234
∼ 134 ∼ 1234	123	123	124
1	2 ∼ 12 ∼ 23 ∼	13	14
	∼ 24 ∼ 234	3 ∼ 23 ∼	4 ∼ 24 ∼
		∼ 34 ∼ 234	∼ 34 ∼ 234

Note that partitions $\{\{134\},\{2\}\}$ and $\{\{1234\}\}$ are stable. In this example $\succsim^F$ does not satisfy pairwise alignment; for instance, $\{124\} \succ_2^F \{123\}$, whereas $\{124\} \sim_1^F \{123\}$. However, it satisfies *weak pairwise alignment* and it has no *rings*, i.e., it is a non-circular problem. The reason why this rule is not covered by Pycia's results is that it neither satisfies strict endowment monotonicity nor non-satiability.

The next example shows that, although *solidarity* is sufficient to guarantee stability in the induced coalition formation problems, it is not necessary.

Example 2. Consider a variant of Example 1, reflecting a situation in which agent 1 has priority over the rest of the agents, but only when the grand coalition is formed. Moreover, in the grand coalition she receives at most k units of the endowment, as before, while this bound does not apply in other coalitions or for other agents. Then, F allocates the first k units of the endowment of the grand coalition to agent 1, while the remaining units (if any are left) are shared equally among the other agents. If the coalition is not the grand coalition, the endowment is simply shared equally among all coalition members.

Formally, let $N = \{1, \dots, n\}$ and consider the following sharing rule:

$$F_i(C, E_C) = \begin{cases} \min\{E_C, k\} & \text{if } i = 1 \text{ and } C = N, \\ \frac{E_C - \min\{E_C, k\}}{|C|-1} & \text{if } i \neq 1 \text{ and } C = N, \\ \frac{E_C}{|C|} & \text{otherwise.} \end{cases}$$

It can be checked that this rule does not satisfy *solidarity*; in particular (Lemma 1), it is not consistent. However, it never generates rings and, therefore, it induces coalition formation problems with a non-empty core. To see this, note that the rule imposes equal sharing for each coalition $C \neq N$. Thus, all agents agree on the ranking of all subcoalitions of N. Then, the common ranking property restricted to all subcoalitions of N is satisfied. Finally, it is easily checked that, for this rule, bringing coalition N into the picture does not generate any ring.

We illustrate this rule for a particular example. Let $N = \{1, 2, 3\}$, $k = 6$, and consider the following coalitional endowments:

C	{12}	{13}	{23}	{123}	otherwise
E_C	10	8	6	15	0

Then, F yields the following allocations:

C	{12}	{13}	{23}	{123}	otherwise
$F(C, E_C)$	(5, 5)	(4, 4)	(3, 3)	(6, 4.5, 4.5)	$(0)_{i\in C}$

The coalition formation problem induced by F, $\succsim^F$, is the following:

$\succsim_1^F$	$\succsim_2^F$	$\succsim_3^F$
123	12	123
12	123	13
13	23	23
1	2	3

Note that partitions $\{\{12\},\{3\}\}$ and $\{\{123\}\}$ are stable. However, $\succsim^F$ does not satisfy *weak pairwise alignment* as $\{123\} \succ_1^F \{12\}$, whereas $\{12\} \succ_2^F \{123\}$.

4. Applications and related work

In many economic models, agents differ in characteristics (such as utility functions, claims, or ranking positions) that could be taken into account to distribute a given endowment among them. Our results show that, regardless of these characteristics, as long as the sharing rule satisfies *solidarity*, it will induce a coalition formation problem with a non-empty core. In this section, we first relate our results to existing results for bargaining (Pycia, 2012) and rationing (Gallo & Inarra, 2018), and we then develop two novel applications, for surplus sharing and ranking problems. Finally, we illustrate how our results can be applied to resource allocation situations with permissible coalitions.

Bargaining problems

We first consider the problem of coalition formation in bargaining introduced by Pycia (2012). In this model, agents have utility functions. Formally, let $N = \{1, \dots, n\}$ and, for each $i \in N$, let $u_i : \mathbb{R}_+ \to \mathbb{R}_+$ denote agent i's (non-decreasing) utility function. For each $C \subseteq N$, let $u_C = (u_i)_{i\in C}$ be the profile of utility functions of coalition C and $E_C \in \mathbb{R}_+$ the endowment of coalition C. A **bargaining problem** is a triple (C, E_C, u_C). An allocation for (C, E_C, u_C) is a vector $x = (x_i)_{i\in|C|} \in \mathbb{R}_+^{|C|}$ such that $\sum_{i\in C} x_i = E_C$. A **(sharing) rule** is a function F that associates with each (C, E_C, u_C) an allocation. Given a set of bargaining problems $\{(C, E_C, u_C)\}_{C\subseteq N, E_C\in\mathbb{R}_+}$, the **coalition formation problem induced by rule** F, $\succsim^F$, is defined as in Section 3.

The two focal rules in this model are the so-called Nash bargaining solution, N, and the Kalai–Smorodinsky bargaining solution, KS. The first one selects for each problem the solution that maximize the product of agents' utilities (Nash, 1950). The second one equalizes the relative gains – the gain of each player relative to its maximum possible gain – and maximizes this equal value (Kalai & Smorodinsky, 1975). Formally, for each bargaining problem (C, E_C, u_C),

$$N(C, E_C, u_C) = \arg\max \prod_{i\in C} u_i(x_i),$$

and $KS(C, E_C, u_C)$ selects the maximal feasible vector x such that for each pair $i, j \in C$,

$$\frac{u_i(x_i)}{u_i(E_C)} = \frac{u_j(x_j)}{u_j(E_C)}.$$

The Nash bargaining solution guarantees a non-empty core of the induced coalition formation problem (Pycia, 2012). As the Nash bargaining solution satisfies *solidarity*, we know from our Theorem 1 that it induces non-circular coalition formation problems and, thus, stability is guaranteed. The core of the coalition formation problem induced by the Kalai–Smorodinsky bargaining solution can be empty for some coalitional endowments (Pycia, 2012). As this solution violates *solidarity*, we know from our Theorem 1 that it does not always induce non-circular coalition formation problems.

Rationing problems

As we mentioned in the Introduction, our results generalize those obtained by Gallo and Inarra (2018) to any resource allocation situation (beyond rationing problems). We analyze in this subsection how our results apply to the particular case of rationing. In a rationing problem agents have claims over an endowment that is not sufficient to fully honor all claims and sharing rules take those claims into account to generate allocations.[9] Formally, let $N = \{1, \dots, n\}$ and, for each $i \in N$,

[9] This model is renamed as generalized claims problems by Gallo and Klaus (2022).

let $d_i \in \mathbb{R}_+$ denote agent i's claim. For each $C \subseteq N$, let $d_C = (d_i)_{i\in C}$ be the claims vector of coalition C and $E_C \in \mathbb{R}_+$, with $E_C \leq \sum_{i\in C} d_i$, be the endowment of coalition C. A **rationing problem** is a triple (C, E_C, d_C). An allocation for (C, E_C, d_C) is a vector $x = (x_i)_{i\in|C|} \in \mathbb{R}_+^{|C|}$ such that, for each $i \in C$, $x_i \leq d_i$ and $\sum_{i\in C} x_i = E_C$. A **(sharing) rule** F is a function that associates with each (C, E_C, d_C) an allocation. Given a set of claims problems $\{(C, E_C, d_C)\}_{C\subseteq N, 0\leq E_C \leq \sum_{i\in C} d_i}$, the **coalition formation problem induced by rule** F, $\succsim^F$, is defined as in Section 3.

A focal family of sharing rules for rationing problems is the so-called family of *parametric rules* (Young, 1987). The payoff of each agent given by a parametric rule is obtained by a function that only depends on her individual claim and a common parameter. Formally, a rule S is *parametric* if there exists a function $f : [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$, where $[a,b] \subset \mathbb{R} \cup \{\pm\infty\}$, that is continuous and weakly monotonic in its first argument and such that:

(i) $f(a,x) = 0$ and $f(b,x) = x$ for all $x \in \mathbb{R}_+$, and
(ii) for each (C, E_C, d_C), there is $\lambda \in [a,b]$, such that $S_i(C, E_C, d_C) = f(\lambda, d_i)$.

The family of parametric rules includes some well-known rules such as the proportional rule, P, the constrained equal-awards rule, CEA, and the constrained equal-losses rule, CEL (see Thomson, 2019).[10] Proposition 1 in Gallo and Inarra (2018) states that parametric rules induce coalition formation problems with a non-empty core. As these rules satisfy *solidarity* (see Thomson, 2019), our Theorem 1 guarantees that they induce coalition formation problems that are non-circular and, therefore, by our Corollary 1, the induced cores are non-empty.[11]

A focal non-parametric rule is the so-called *random arrival rule*.[12] For this rule, one considers all possible arrival orderings of agents. For each order, agents are fully reimbursed until the endowment runs out. Then, the rule takes the average over all orders (see O'Neill, 1982; Thomson, 2019). Formally,

$$RA_i(C, E_C, d_C) = \frac{1}{|C|!} \sum_{\succ \in \mathcal{O}^C} \min\{d_i, \max\{E - \sum_{j\in C, j\succ i} d_j, 0\}\},$$

where $\mathcal{O}^C$ denotes the set of strict orders in C. Gallo and Inarra (2018) show that the random arrival rule can generate a coalition formation problem with an empty core. As this rule does not satisfy *solidarity*, we know from Theorem 1 that it does not always induce non-circular coalition formation problems.

Surplus sharing problems

We consider here the related case of coalition formation in surplus sharing problems. Surplus sharing problems (Moulin, 2002) are complementary to rationing problems in the sense that endowments exceed the sum of claims. Formally, let $N = \{1,\dots,n\}$ and, for each $i \in N$, let $d_i \in \mathbb{R}_+$ denote agent i's claim. For each $C \subseteq N$, let $d_C = (d_i)_{i\in C}$ be the claims vector of coalition C and $E_C \in \mathbb{R}_+$, with $E_C \geq \sum_{i\in C} d_i$, be the endowment of coalition C. A **surplus sharing problem** is a triple (C, E_C, d_C). An allocation for (C, E_C, d_C) is a vector $x = (x_i)_{i\in C} \in \mathbb{R}_+^{|C|}$ such that, for each $i \in C$, $d_i \leq x_i$ and $\sum_{i\in C} x_i = E_C$. A **(sharing) rule** F is a function that associates with each (C, E_C, d_C) an allocation. Given a set of surplus sharing problems $\{(C, E_C, d_C)\}_{C\subseteq N, E_C \geq \sum_{i\in C} d_i}$, the **coalition formation problem induced by rule** F, $\succsim^F$, is defined as in Section 3.

We define some focal surplus sharing rules. We start with the counterpart of the family of parametric rules (Moulin, 1987b). Formally, a (surplus sharing) rule S is *parametric* if there exists a function $f : [0,+\infty] \times \mathbb{R}_+ \to \mathbb{R}_+$, that is continuous and weakly monotonic in both arguments, such that:

(i) $f(0,x) = 0$ and $f(+\infty,x) = +\infty$ for all $x \in \mathbb{R}_+$, and
(ii) for each (C, E_C, d_C), there is $\lambda \in [0,+\infty]$, such that $S_i(C, E_C, d_C) = f(\lambda, d_i)$.

The family of parametric rules includes some well-known rules such as the proportional rule, P, the uniform gains rule, UG, and the equal surplus rule, ES (see Thomson, 2019).[13] All parametric rules for surplus sharing problems satisfy *solidarity*. Thus, our Theorem 1 guarantees that they induce coalition formation problems that are non-circular and, as our Corollary 1 states, that the core is not empty.[14]

However, there are also (non-parametric) rules for surplus sharing problems that do not satisfy *solidarity*. For instance, let us consider the following extension of the random arrival rule from rationing problems to surplus sharing problems. First, award all agents their claims as many times as the endowment allows. Then, assign the residual endowment (if it exists) sequentially according to an ordering of the agents. The *extended random arrival for surplus sharing problems (ERA)* gives each agent the average of the awards so calculated over all possible orderings. Formally,

$$ERA_i(C, E_C, d_C) = kd_i + \frac{1}{|C|!} \sum_{\succ \in \mathcal{O}^C} \max\{E - k\sum_{j\in C} d_j - \sum_{j\succ i} d_j, 0\},$$

where $k = \left\lfloor \frac{E_C}{\sum_{j\in C} d_j} \right\rfloor$, and $\mathcal{O}^C$ denotes the set of strict orders in C.[15]

The following example shows that this rule does not guarantee stability and also illustrates that parametric rules do.

Example 3. Let $N = \{1,2,3,4\}$ and $d = (1,3,3,10)$. Consider the following coalitional endowments:

C	$\{13\}$	$\{23\}$	$\{124\}$	otherwise
E_C	5	8	17	$\sum_{i\in C} d_i$

Then, the ERA rule yields the following allocations:

C	$\{13\}$	$\{23\}$	$\{124\}$	otherwise
$ERA(C, E_C, d_C)$	$(1.5, 3.5)$	$(4, 4)$	$(1.33, 4.33, 11.33)$	$(d_i)_{i\in C}$

The coalition formation problem induced by ERA, $\succsim^{ERA}$, is the following:

$\succsim_1^{ERA}$	$\succsim_2^{ERA}$	$\succsim_3^{ERA}$	$\succsim_4^{ERA}$
13	124	23	124
124	23	13	$4 \sim 14 \sim 24 \sim$
$1 \sim 12 \sim 14 \sim$	$2 \sim 12 \sim 24 \sim$	$3 \sim 34 \sim$	$\sim 34 \sim 134 \sim$
$\sim 123 \sim 134 \sim$	$\sim 123 \sim 134 \sim$	$\sim 123 \sim 134 \sim$	$\sim 234 \sim 1234$
~ 1234	~ 1234	$\sim 234 \sim 1234$	

[10] Formally,

$f^P(\lambda, d_i) = \lambda d_i$, for all $\lambda \in [0,1]$, $d_i \in \mathbb{R}_+$,

$f^{CEA}(\lambda, d_i) = \min\left\{\lambda, d_i\right\}$, for all λ, $d_i \in \mathbb{R}_+$,

$f^{CEL}(\lambda, d_i) = \max\left\{0, d_i - \frac{1}{\lambda}\right\}$, for all $\lambda \in [0,+\infty]$, $d_i \in \mathbb{R}_+$.

[11] In our benchmark analysis endowments are not constrained, whereas in this application they cannot be above the coalition's aggregate claim. Nevertheless, the proofs of Theorem 1 and Corollary 1 are also valid (with minor modifications) under that premise.

[12] Gallo and Inarra (2018) refer to this rule as the Shapley value.

[13] Formally,

$f^P(\lambda, d_i) = \lambda d_i$, for all $\lambda \in [0,+\infty]$, $d_i \in \mathbb{R}_+$,

$f^{UG}(\lambda, d_i) = \max\{\lambda, d_i\}$, for all $\lambda \in [0,+\infty]$, $d_i \in \mathbb{R}_+$,

$f^{ES}(\lambda, d_i) = d_i + \lambda$, for all $\lambda \in [0,+\infty]$, $d_i \in \mathbb{R}_+$.

[14] A similar caveat to the one made at Footnote 11 applies here.

[15] Note that $ERA_i(C, E_C, d_C) = kd_i + RA_i(C, E_C - k\sum_{j\in C} d_j, d_C)$.

Observe that this problem is not a non-circular coalition formation problem. Although *weak pairwise alignment* is satisfied, $(\{124\},\{13\},\{23\})$ is a ring and it is easily checked that the core is empty.

Consider now the *uniform gains* rule (a parametric rule). This rule yields the following allocations for this problem:

C	$\{13\}$	$\{23\}$	$\{124\}$	otherwise
$UG(C, E_C, d_C)$	$(2,3)$	$(4,4)$	$(3.5, 3.5, 10)$	$(d_i)_{i\in C}$

The coalition formation problem induced by UG, $\succsim^{UG}$, is the following:

$\succsim_1^{UG}$	$\succsim_2^{UG}$	$\succsim_3^{UG}$	$\succsim_4^{UG}$
124	23	23	$4 \sim 14 \sim 24 \sim$
13	124	$3 \sim 13 \sim 34 \sim$	$\sim 34 \sim 124 \sim$
$1 \sim 12 \sim 14 \sim$	$2 \sim 12 \sim 24 \sim$	$\sim 123 \sim 134 \sim$	$\sim 134 \sim 234 \sim$
$\sim 123 \sim 134 \sim$	$\sim 123 \sim 234 \sim$	$\sim 234 \sim 1234$	~ 1234
~ 1234	~ 1234		

Observe that this problem satisfies *weak pairwise alignment* and has no *rings*. Then, it is a non-circular coalition formation problem. In particular, partition $\{\{23\},\{1\},\{4\}\}$ is stable.

Ranking problems

Dietzenbacher and Kondratev (2023) introduce the problem of prize allocation in competitions, in which agents are ranked and a prize endowment has to be shared among the participants of a competition according to their ranking. With this idea in mind, we propose a model where agents are ranked and all coalitions can be formed. Then, the rule may take the ranking of the agents into account to derive the individual payoffs. This model can be applied to any setting where agents can be ordered according to some characteristic (such as their expertise or past performance).

Formally, let $N = \{1,\dots,n\}$ be the set of agents. A **ranking** $\mathcal{R}$ is a bijection $\mathcal{R} : N \longrightarrow N$ that assigns to each agent a position, i.e., $\mathcal{R}(i)$ is the position of agent i in the ranking. We say that agent $i \in N$ has a higher position in the ranking than agent $j \in N$ if $\mathcal{R}(i) < \mathcal{R}(j)$. For each $C \subseteq N$, let E_C denote the endowment of coalition C and $\mathcal{R}_C$ the projection of ranking $\mathcal{R}$ to C. That is, the position of agent i in coalition C is the number of agents, including herself, that have a higher position in that coalition. Formally, for each $i \in C$, $\mathcal{R}_C(i) = |\{j \in C : \mathcal{R}(i) \le \mathcal{R}(j)\}|$. For each coalition $C \subseteq N$, denote by $(C, E_C, \mathcal{R}_C)$ the **ranking problem** faced by coalition C. An allocation for $(C, E_C, \mathcal{R}_C)$ is a vector $x = (x_i)_{i\in C} \in \mathbb{R}_+^{|C|}$ such that $\sum_{i\in C} x_i = E_C$. A **(sharing) rule** F is a function that associates with each $(C, E_C, \mathcal{R}_C)$ an allocation. Given a set of ranking problems $\{(C, E_C, \mathcal{R}_C)\}_{C\subseteq N, E_C\in\mathbb{R}_+}$, the **coalition formation problem induced by rule** F, $\succsim^F$, is defined as in Section 3.[16]

We reformulate the family of *interval rules* considered in Dietzenbacher and Kondratev (2023) for each ranking problem $(C, E_C, \mathcal{R}_C)$. Informally, given a ranking problem $(C, E_C, \mathcal{R}_C)$, each interval rule works as follows: first, a set of disjoint intervals is defined. Then, if the average endowment $\frac{E_C}{|C|}$ does not belong to any of the intervals, the endowment E_C is equally split among the agents. Otherwise, the agents get the lower bound of the interval to which $\frac{E_C}{|C|}$ belongs. If there is endowment left, each agent is allocated up to the upper bound of that interval following the ranking $\mathcal{R}_C$. The formal definition is as follows.

Interval rule for ranking problems: Let $A = \{(a_1, b_1), (a_2, b_2), \dots\}$ with $a_1, a_2, \dots \in \mathbb{R}_+$ and $b_1, b_2, \dots \in \mathbb{R}_+ \cup \{+\infty\}$ be a family of disjoint intervals. The interval rule associated with A, I^A, is such that for each $(C, E_C, \mathcal{R}_C)$ and each $i \in C$,

$$I_i^A(C, E_C, \mathcal{R}_C) = \begin{cases} a_k & \text{if} \quad |C|a_k \le E_C \le (|C| - \beta)a_k + \beta b_k; \\ x & \text{if} \quad (|C| - \beta)a_k + \beta b_k \le E_C \le (|C| - \mathcal{R}_C(i))a_k + \mathcal{R}_C(i) b_k; \\ b_k & \text{if} \quad (|C| - \mathcal{R}_C(i))a_k + \mathcal{R}_C(i) b_k \le E_C \le |C| b_k; \\ \frac{E_C}{|C|} & \text{otherwise,} \end{cases}$$

where $\beta = \mathcal{R}_C(i) - 1$ and $x = E - (|C| - \mathcal{R}_C(i))a_k - \beta b_k$.

As Dietzenbacher and Kondratev (2023) mention, the interval rule with $a_k = b_k = 0$ for each k coincides with Equal Division while the interval rule with $a_1 = 0$ and $b_1 = +\infty$ coincides with Winner Takes All, both well-known rules.

Theorem 1 in Dietzenbacher and Kondratev (2023) states that these are the only order-preserving[17] rules that satisfy *solidarity*.[18] Consequently, our Theorem 1 yields the following.

Corollary 2. *The interval rules are the only order-preserving rules for ranking problems that induce non-circular coalition formation problems.*

Corollary 2 also implies that the interval rules guarantee stability in the induced coalition formation problems. Other interesting rules proposed by Dietzenbacher and Kondratev (2023) do not yield stability. The following example illustrates this fact.

Example 4. Let $N = \{1,2,3\}$ and $\mathcal{R}(i) = i$ for each $i \in N$. Consider the following coalitional endowments:

C	$\{12\}$	$\{13\}$	$\{23\}$	$\{123\}$	otherwise
E_C	20	15	14	21	0

Consider first the interval rule I^A with $A = \{(a_1,b_1),(a_2,b_2),(a_3,b_3)\} = \{(2,9),(9,10.5),(10.5,+\infty)\}$. This rule yields the following allocations:

C	$\{12\}$	$\{13\}$	$\{23\}$	$\{123\}$	otherwise
$I^A(C, E_C, \mathcal{R}_C)$	$(10.5, 9.5)$	$(9,6)$	$(9,5)$	$(9,9,3)$	$(0)_{i\in C}$

The coalition formation problem induced by I^A, $\succsim^{I^A}$, is the following:

$\succsim_1^{I^A}$	$\succsim_2^{I^A}$	$\succsim_3^{I^A}$
12	12	13
$13 \sim 123$	$23 \sim 123$	23
1	2	123
		3

Observe that this coalition formation problem satisfies *weak pairwise alignment* and has no *rings*. Then, it is a non-circular coalition formation problem. In particular, partition $\{\{12\},\{3\}\}$ is stable.

We finally consider a class of rules for ranking problems based on the family of proportional rules defined in Dietzenbacher and Kondratev (2023).

Proportional rule for ranking problems: Let $\Lambda = \{\lambda^1, \lambda^2, \dots \lambda^{|N|}\}$ with $\lambda^1 > 0$ and $\lambda^k \ge \lambda^{k+1}$ for each $k \in \{1,\dots,|N|-1\}$. The proportional rule associated with Λ, P^Λ, is such that for each $(C, E_C, \mathcal{R}_C)$ and each $i \in C$,

$$P_i^\Lambda(C, E_C, \mathcal{R}_C) = \frac{\lambda^{\mathcal{R}_C(i)}}{\sum_{j\in C} \lambda^{\mathcal{R}_C(j)}} \cdot E_C.$$

A proportional rule assigns an allocation to the agent in position j in the ranking proportionally to λ^j. There exist rules within this family that do not satisfy *solidarity* and do not guarantee stability. An instance is the rule obtained when $|N| = 3$, $\lambda^1 = 3$, and $\lambda^2 = \lambda^3 = 1$. It yields the following allocations for this problem:

[16] Lucchetti et al. (2022) consider a different approach to induce coalition formation problems using a ranking of agents (which is not exogenous, but rather deduced from a ranking over the different coalitions). They also focus on the study of core stable partitions. However, neither coalitional endowments nor sharing rules do appear in their model and, thus, our results cannot be applied therein.

[17] If $\mathcal{R}(i) < \mathcal{R}(j)$, then $F_i(C, E_C, \mathcal{R}_C) \ge F_j(C, E_C, \mathcal{R}_C)$.

[18] Dietzenbacher and Kondratev (2023) do not use the axiom of *solidarity*, but the separate axioms of *endowment monotonicity* and *consistency*.

C	$\{12\}$	$\{13\}$	$\{23\}$	$\{123\}$	otherwise
$P^A(C, E_C, \mathcal{R}_C)$	$(15,5)$	$(11.25, 3.75)$	$(10.5, 3.5)$	$(12.6, 4.2, 4.2)$	$(0)_{i\in C}$

The coalition formation problem induced by P, $\succsim^P$, is the following:

$\succsim_1^{P^A}$	$\succsim_2^{P^A}$	$\succsim_3^{P^A}$
12	23	123
123	12	13
13	123	23
1	2	3

Note that this problem is not a non-circular coalition formation problem. In particular, $(\{12\}, \{23\}, \{13\})$ is a *ring* and, as $23 \succ_2 123$, while $123 \succ_3 23$, *weak pairwise alignment* is also violated. Moreover, it can easily be shown that this problem has an empty core.

Sharing problems with permissible coalitions

In some resource allocation situations not all coalitions are feasible and, consequently, our results cannot be directly applied. Instances are matching problems (e.g., Demange & Gale, 1985; Roth & Sotomayor, 1990), network games (e.g., Jackson, 2005; Jackson & Wolinsky, 2003), river sharing problems (e.g., Alcalde-Unzu et al., 2015; Ambec & Sprumont, 2002) or legislative bargaining problems (e.g., Le Breton et al., 2008; Puy, 2013). However, we propose a way to partially circumvent this issue. Suppose that the resource allocation situation involves just a collection of permissible coalitions $\mathcal{K} \subset 2^N$ and, therefore, sharing problems are only defined for these coalitions.[19] We propose to define, for each non-permissible coalition, an auxiliary sharing problem with a zero endowment. We then define an enlarged model that encompasses the sharing problems for permissible coalitions as well as the auxiliary sharing problems for non-permissible coalitions. Then, Theorem 1 and Corollary 1 can be applied to this enlarged model. Thus, the induced coalition formation problem has a non-empty core if the sharing rule satisfies *solidarity*. If so, note that there is a stable partition formed by coalitions of $\mathcal{K}$, given that only those may have a positive endowment. Clearly, such a partition is also stable in the original model. To summarize, given a resource allocation situation where not all coalitions are permissible, we can always obtain a stable partition by constructing such an enlarged model, and then applying a sharing rule that satisfies *solidarity* therein. Observe that *solidarity* refers to the sharing rule in the enlarged model, as our results require that all coalitions are feasible. However, the application of a sharing rule satisfying *solidarity* in the original model does not guarantee stability. That is illustrated with the following example.

Example 5. Let $N = \{1, 2, 3\}$ and $\mathcal{K} = \{\{12\}, \{13\}, \{23\}, \{1\}, \{2\}, \{3\}\}$ be the set of permissible coalitions. Consider the following coalitional endowments:

C	$\{12\}$	$\{13\}$	$\{23\}$	$\{i\}$
E_C	9	9	9	0

Now, consider the following sharing rule F: for each $(C, E_C) \in \mathcal{P}$, with $C \in \mathcal{K}$,

$$F(C, E_C) = \begin{cases} \left(\frac{2}{3}E_C, \frac{1}{3}E_C\right) & \text{if } C = \{12\}, \\ \left(\frac{2}{3}E_C, \frac{1}{3}E_C\right) & \text{if } C = \{23\}, \\ \left(\frac{1}{3}E_C, \frac{2}{3}E_C\right) & \text{if } C = \{13\}, \\ E_C & \text{otherwise.} \end{cases}$$

Note that this rule gives priority to one agent within each pair, but whoever has priority depends on the coalition. It can be checked that this rule satisfies *solidarity*. And it yields the following allocations:

C	$\{12\}$	$\{13\}$	$\{23\}$	$\{i\}$
$F(C, E_C)$	$(6,3)$	$(3,6)$	$(6,3)$	(0)

The coalition formation problem induced by F, $\succsim^F$, is the following:

$\succsim_1^F$	$\succsim_2^F$	$\succsim_3^F$
12	23	13
13	12	23
1	2	3

Observe that $(\{12\}, \{23\}, \{13\})$ is a *ring* and that this coalition formation problem has an empty core.

Consider now the enlarged model in which the grand coalition gets a zero endowment:

C	$\{12\}$	$\{13\}$	$\{23\}$	$\{i\}$	$\{123\}$
E_C	9	9	9	0	0

We know, by Corollary 1, that the application to this enlarged model of any sharing rule that satisfies *solidarity* guarantees the non-emptiness of the core of the induced coalition formation problem. Specifically, we can guarantee that there is a stable partition in which the grand coalition is not included. Consider, for instance, the *uniform sharing* rule, US, the rule that divides the endowment of each coalition equally among its members (and, thus, it obviously satisfies *solidarity*).[20] Then, US yields the following allocations:

C	$\{12\}$	$\{13\}$	$\{23\}$	$\{i\}$	$\{123\}$
$US(C, E_C)$	$(4.5, 4.5)$	$(4.5, 4.5)$	$(4.5, 4.5)$	(0)	$(0, 0, 0)$

The coalition formation problem induced by US, $\succsim^{US}$, is the following:

$\succsim_1^{US}$	$\succsim_2^{US}$	$\succsim_3^{US}$
$12 \sim 13$	$12 \sim 23$	$13 \sim 23$
$1 \sim 123$	$2 \sim 123$	$3 \sim 123$

Observe that this problem has three stable partitions: $\{\{12\}, \{3\}\}$, $\{\{13\}, \{2\}\}$ and $\{\{23\}, \{1\}\}$, all of them formed by permissible coalitions.

5. Concluding remarks

We have studied coalition formation problems in a context in which coalitions have to share collective resources. We have characterized the sharing rules that induce non-circular coalition formation problems as those satisfying a natural axiom formalizing the principle of solidarity. This implies that such a solidarity axiom guarantees the existence of (core) stable partitions in the induced coalition formation problems. Our result can be applied to canonical problems of resource allocation long studied such as bargaining, rationing, or surplus sharing problems as well as to other problems recently considered, such as ranking problems.

Although our benchmark model requires that all coalitions are feasible, we have also seen that we can partially apply the results to resource allocation situations where not all coalitions are feasible. By contrast, a similar argument cannot be applied to situations where agents are equipped with individual endowments such as revenue sharing in hierarchies (e.g., Harless, 2020; Hougaard et al., 2017). Exploring whether the connection between solidarity (in the resource allocation problem) and stability (in the corresponding coalition formation problem) extends to these cases is left for further research.

The class of non-circular coalition formation problems is not the only class in which stability is guaranteed. As mentioned already, various other properties have been introduced to guarantee the existence of stable partitions in hedonic games. Most of them, like the so-called *top-coalition property* and *ordinal balance* (Bogomolnaia &

[19] To ensure the existence of partitions, we assume that singletons are permissible (i.e., for each $i \in N$, $\{i\} \in \mathcal{K}$).

[20] Formally, for each $(C, E_C) \in \mathcal{P}$ and each $i \in C$, $US_i(C, E_C) = \frac{E_C}{|C|}$.

Jackson, 2002) are sufficient conditions for the non-emptiness of the core. The somewhat related notion of *pivotal balance* (Iehlé, 2007), is both necessary and sufficient for the existence of a core stable partition. Nevertheless, all these properties guarantee the non-emptiness of the core even though they allow for the presence of rings.[21] A natural research question would be to identify properties of sharing rules that induce coalition formation problems satisfying, for instance, ordinal balance or pivotal balance. However, this is a challenging question as these properties allow for the existence of rings in preferences. As it is known, the non-emptiness of the core when rings exist is contingent on various factors, including the number of coalitions in the ring and their positions within the preferences of the agents involved (see Bonifacio et al., 2022). This last factor is relevant because the higher these positions are, *ceteris paribus*, the more difficult it is to have a non-empty core.[22] In our model, these positions depend significantly on the coalitional endowments. Note also that we give full flexibility to the values of these coalitional endowments. Therefore, if a particular sharing rule can generate a ring for some coalitional endowments, changing the endowments of all the coalitions outside the ring to 0 would suffice to generate a coalition formation problem (induced by that rule) with all the coalitions of the ring occupying the first positions in the agents' preferences. Therefore, it seems very difficult to formulate general properties of sharing rules that only generate rings compatible with non-empty core coalition formation problems. A hypothetical characterization of sharing rules that allow for rings but guarantee a non-empty core would be based on restricting the values of coalitional endowments. To summarize, the search for a characterization of the sharing rules that induce coalition formation problems satisfying any of the above conditions (that allow for rings but guarantee a non-empty core) is a daunting task that is beyond the scope of this paper. We have focused instead on the non-circular property, which conveys the appealing feature of excluding rings altogether (an aspect that can be naturally linked to sharing rules).

Finally, another interesting question is the study of the computational complexity of coalition formation problems. Ballester (2004) studies the complexity of coalition formation games and shows that the computation of stable partitions is NP-complete. The computational complexity of stable partitions in additive coalition formation problems has been studied by Sung and Dimitrov (2010). More recently, Gairing and Savani (2019) focus on symmetric additively separable coalition formation problems. The study of the computational complexity of non-circular coalition formation problems is left for further research.

Appendix. Proof of Theorem 1

Lemma 2. *If F satisfies solidarity, then for each $\{E_C\}_{C\subseteq N}$, $\succsim^F$ satisfies weak pairwise alignment.*

Proof. Let F be a sharing rule that satisfies *solidarity*. Then, by Lemma 1, F satisfies *endowment monotonicity* and *consistency*. Let $C, C' \subseteq N$, $E_C, E_{C'} \in \mathbb{R}_+$, and $i, j \in C \cap C'$.

If $C \subset C'$ or $C' \subset C$, then by *solidarity*, either [for each $k \in \{i,j\}$, $F_k(C, E_C) \le F_k(C', E_{C'})$] or [for each $k \in \{i,j\}$, $F_k(C, E_C) \ge F_k(C', E_{C'})$]. Therefore, agents i and j do not rank C and C' in opposite ways.

Otherwise, $C \not\subset C'$ and $C' \not\subset C$. Then, for each $k \in C$, let $x_k = F_k(C, E_C)$, and for each $k' \in C'$, let $x'_{k'} = F_{k'}(C', E_{C'})$. Consider the sharing problems $(\{i,j\}, x_i + x_j)$ and $(\{i,j\}, x'_i + x'_j)$. By *consistency*,

$$(x_i, x_j) = F(\{i,j\}, x_i + x_j) \text{ and } (x'_i, x'_j) = F(\{i,j\}, x'_i + x'_j).$$

Assume, without loss of generality, that $x_i + x_j \ge x'_i + x'_j$. Then, by *endowment monotonicity*, for each $k \in \{i,j\}$, $x_k = F_k(\{i,j\}, x_i + x_j) \ge F_k(\{i,j\}, x'_i + x'_j) = x'_k$. Therefore, for each $k \in \{i,j\}$, $F_k(C, E_C) \ge F_k(C', E_{C'})$. Consequently, agents i and j do not rank C and C' in opposite ways.

Hence, $\succsim^F$ satisfies *weak pairwise alignment*, as desired. □

We now show that if F satisfies *solidarity*, then $\succsim^F$ has no *rings*. We need the following auxiliary lemma.

Lemma 3. *Let F be a sharing rule that satisfies solidarity. Let $C \subseteq N$ and $\{C_1, \ldots, C_m\}$ be a set of coalitions such that $\bigcup_{k=1}^m C_k = C$. Then, there is $C_l \in \{C_1, \ldots, C_m\}$ such that for each $E_{C_l} \in \mathbb{R}_+$, there exists $E_C \in \mathbb{R}_+$ for which $F_i(C_l, E_{C_l}) = F_i(C, E_C)$ for each $i \in C_l$.*

Proof. Let F be a sharing rule that satisfies *solidarity*. Then, by Lemma 1, F satisfies *endowment continuity* and *consistency*. Let $C \subseteq N$ and $\{C_1, \ldots, C_m\}$ be such that $\bigcup_{k=1}^m C_k = C$.

We first prove that there exists $j \in C$ such that $\lim_{E_C \to \infty} F_j(C, E_C) = \infty$. Suppose otherwise. Then, for each $j \in C$, there exists $M_j \in \mathbb{R}_+$ such that for each $E \in \mathbb{R}_+$ arbitrarily large, $F_j(C, E) < M_j$. Let $M \equiv \max_{j\in C} M_j$. Then, for each $j \in C$, $F_j(C, |C|M) < M$ and, therefore, $\sum_{j\in C} F_j(C, |C|M) < |C|M$, which contradicts the definition of an allocation.

Now, as $\bigcup_{k=1}^m C_k = C$, there is $C_l \in \{C_1, \ldots, C_m\}$ such that $j \in C_l$. We then construct $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each $\xi \in \mathbb{R}_+$, $\alpha(\xi) = \sum_{i\in C_l} F_i(C, \xi)$. Then, $\alpha(0) = \sum_{i\in C_l} F_i(C, 0) = 0$ and $\lim_{\xi\to\infty} \alpha(\xi) = \lim_{\xi\to\infty} \sum_{i\in C_l} F_i(C, \xi) = \infty$. As F satisfies *endowment continuity*, α is continuous. Then, for each $E_{C_l} \in \mathbb{R}_+$, there exists $E_C \in \mathbb{R}_+$ for which $\alpha(E_C) = E_{C_l}$. By *consistency*, for each $i \in C_l$, $F_i(C_l, E_{C_l}) = F_i(C, E_C)$, as desired. □

Lemma 4. *If F satisfies solidarity, then for each $\{E_C\}_{C\subseteq N}$, $\succsim^F$ has no rings.*

Proof. Let F be a sharing rule that satisfies *solidarity*. Suppose by contradiction that there exists $\{E_C\}_{C\subseteq N}$ such that $\succsim^F$ has a ring $(C_0, C_1 \ldots, C_{l-1})$. Then, for each $k = 0, \ldots, l-1$ (modulo l), there is at least one agent, say agent $j_{k+1} \in C_{k+1} \cap C_k$, such that $C_{k+1} \succ^F_{j_{k+1}} C_k$.

Let $\hat{C} \equiv \bigcup_{k=0}^{l-1} C_k$. By Lemma 3, there is $C_k \in \{C_0, C_1, \ldots, C_{l-1}\}$ and $\hat{E}_{\hat{C}} \in \mathbb{R}_+$ such that for each $i \in C_k$, $F_i(C_k, E_{C_k}) = F_i(\hat{C}, \hat{E}_{\hat{C}})$. Assume without loss of generality that $C_k = C_0$.

Consider now $\{E'_C\}_{C\subseteq N}$ such that $E'_C = E_C$ for each $C \in 2^N \setminus \{\hat{C}\}$, and $E'_{\hat{C}} = \hat{E}_{\hat{C}}$. We denote by $\succsim^{F'}$ the coalition formation problem when F is applied and the endowments are $\{E'_C\}_{C\subseteq N}$.

By construction, for each $i \in C_0$, $F_i(\hat{C}, E'_{\hat{C}}) = F_i(C_0, E'_{C_0})$ and, therefore, $C_0 \sim^{F'}_i \hat{C}$. In particular, $C_0 \sim^{F'}_{j_0} \hat{C}$ and $C_0 \sim^{F'}_{j_1} \hat{C}$ (possibly $j_0 = j_1$). Similarly, for each $i' \in C_k \cap C_{k+1}$ with $k = 0, 1, \ldots, l-1$ (modulo l), $F_{i'}(C_k, E'_{C_k}) = F_{i'}(C_k, E_{C_k})$ and $F_{i'}(C_{k+1}, E'_{C_{k+1}}) = F_{i'}(C_{k+1}, E_{C_{k+1}})$, which implies $C_{k+1} \succ^{F'}_{i'} C_k \Leftrightarrow C_{k+1} \succ^F_{i'} C_k$ and $C_{k+1} \sim^{F'}_{i'} C_k \Leftrightarrow C_{k+1} \sim^F_{i'} C_k$. In particular, for each $k = 0, 1, \ldots, l-1$ (modulo l), $C_{k+1} \succ^{F'}_{j_{k+1}} C_k$. Then, by transitivity, $\hat{C} \succ^{F'}_{j_0} C_{l-1}$ and $C_1 \succ^{F'}_{j_1} \hat{C}$. As F satisfies *solidarity*, it follows by Lemma 2 that $\succsim^{F'}$ satisfies *weak pairwise alignment* and, therefore, $C_1 \succsim^{F'}_{j_2} \hat{C}$. As $C_2 \succ^{F'}_{j_2} C_1$, it follows by transitivity that $C_2 \succ^{F'}_{j_2} \hat{C}$. Similarly, we have that for each $k \in \{3, \ldots, l-1\}$, $C_k \succ^{F'}_{j_k} \hat{C}$. Then, we have deduced that $\hat{C} \succ^{F'}_{j_0} C_{l-1}$ and $C_{l-1} \succ^{F'}_{j_{l-1}} \hat{C}$. If $j_{l-1} = j_0$, this contradicts transitivity. Otherwise, this implies that $\succsim^{F'}$ does not satisfy *weak pairwise alignment*, which contradicts Lemma 2. □

Lemmas 2 and 4 prove one implication of Theorem 1, while the other is proven by the following lemma.

[21] See the example at page 213 in Bogomolnaia and Jackson (2002) for a coalition formation problem satisfying *ordinal balance* and having a ring but also a non-empty core. The same example is valid for *pivotal balance*.

[22] To see the importance of these positions in the non-emptiness of the core when rings exist, see, for instance, Examples 2 and 3 in Gallo and Inarra (2018).

Lemma 5. *If F does not satisfy solidarity, then there is $\{E_C\}_{C\subseteq N}$ such that $\succsim^F$ does not satisfy weak pairwise alignment.*

Proof. Let F be a sharing rule that does not satisfy *solidarity*. Then, there exist $C, C' \subseteq N$, with $C' \subset C$, $i, j \in C'$ and $E_C, E_{C'} \in \mathbb{R}_+$ such that $F_i(C, E_C) > F_i(C', E_{C'})$ and $F_j(C, E_C) < F_j(C', E_{C'})$. Then, for $\{E_C\}_{C\subseteq N}$, we have that $C \succ_i^F C'$ and $C' \succ_j^F C$. Hence, $\succsim^F$ does not satisfy*weak pairwise alignment*. □

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