

ON THE DISCOUNTED PENALTY FUNCTION IN A MARKOV-DEPENDENT RISK MODEL*

HANSJÖRG ALBRECHER ^{a, b, †}

ONNO J. BOXMA ^c

^a *Graz University of Technology, Steyrergasse 30, A-8010 Graz, Austria*

^b *University of Aarhus, Ny Munkegade, DK-8000 Aarhus C, Denmark*

^c *Eindhoven University of Technology and EURANDOM, P.O. Box 513, 5600 MB Eindhoven, The Netherlands*

Abstract

We present a unified approach to the analysis of several popular models in collective risk theory. Based on the analysis of the discounted penalty function in a semi-Markovian risk model by means of Laplace-Stieltjes transforms, we rederive and extend some recent results in the field. In particular, the classical compound Poisson model, Sparre Andersen models with phase-type interclaim times and models with causal dependence of a certain Markovian type between claim sizes and interclaim times are contained as special cases.

Keywords: Dependence; classical risk model; Sparre Andersen model; time of ruin; deficit at ruin; surplus before ruin

1 Introduction

Let us consider the following risk model for the surplus process $R(t)$ of an insurance portfolio:

$$R(t) = x + ct - \sum_{j=1}^{N(t)} X_j, \quad (1)$$

where x is the initial capital, c is the premium density which is assumed to be constant, X_j is the size of the j th claim and $N(t)$ is the number of claims up to time t . In classical risk theory, the claims X_j and the claim number process $N(t)$ are assumed to be independent. However, in many applications the independence assumption is too restrictive and recently several authors looked at more general models where this assumption is relaxed in some way (see Asmussen (2000) for a survey on the subject).

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†Corresponding author. Email: albrecher@TU Graz.at; Supported by Fellowship F/04/009 of the K.U. Leuven and the Austrian Science Foundation Project P-18392-N12.

In this paper we will consider a semi-Markovian dependence structure of the following type: Let W_i denote the time between the arrival of the $(i - 1)th$ and the i th claim and $W_0 = X_0 = 0$ a.s. Then

$$\begin{aligned} \mathbb{P}(W_{n+1} \leq x, X_{n+1} \leq y, Z_{n+1} = j | Z_n = i, (W_r, X_r, Z_r), 0 \leq r \leq n) \\ = \mathbb{P}(W_1 \leq x, X_1 \leq y, Z_1 = j | Z_0 = i) = (1 - e^{-\lambda_i x}) p_{ij} B_j(y), \end{aligned} \quad (2)$$

where $\{Z_n, n \geq 0\}$ is an irreducible discrete-time Markov chain with state space $\{1, \dots, M\}$ and transition matrix $P = ((p_{ij}), 1 \leq i, j \leq M)$. Thus at each instant of a claim, the Markov chain jumps to a state j , and the distribution B_j of the claim depends on the new state j . Then the next interarrival time is exponentially distributed with parameter λ_j . Note that given the states Z_{n-1} and Z_n , the quantities W_n and X_n are independent, but there is autocorrelation among consecutive claim sizes and among consecutive interclaim times as well as cross-correlation between W_n and X_n . This semi-Markov process was first considered in Janssen and Reinhard (1985), where a formal solution for the survival probabilities in terms of an infinite series of matrix convolutions was derived.

We considerably generalize the approach in Janssen and Reinhard (1985) and investigate the discounted penalty function in such a risk model by means of Laplace-Stieltjes transforms (LST). This allows us to obtain information on several characteristics of the risk process.

The model considered in this paper is quite general: it contains the compound Poisson model ($M = 1$) and Sparre Andersen models with (generalized) Erlang(n)-interclaim distributions (see e.g. Gerber and Shiu (2005), Li and Garrido (2004)) as well as phase-type interclaim distributions (see Avram and Usabel (2004) and Li and Garrido (2005)) as special cases (just choose appropriate transition probabilities and let B_j be degenerate at 0 for all but one state among $\{1, \dots, M\}$). Moreover, it also covers models with causal dependence structures of the type considered in Albrecher and Boxma (2004), namely that the distribution of the inter-arrival time depends on the size of the previous claim in a specified way. To see this, choose a generic claim size random variable X and for all $i = 1, \dots, M$, $p_{ij} = \mathbb{P}(X \in A_j)$ for some (possibly random) interval $A_j \subset \mathbb{R}$ and $B_j \sim X | X \in A_j$ (cf. Section 6.3). Note that by considering state-dependent transition probabilities $p_{ij} = \mathbb{P}(X \in A_j | \text{current state } i)$, we here arrive at a more general model that also allows the claim size distribution itself to depend on the state of the Markov chain. The purpose of this paper can also be seen as to provide an umbrella to the analysis of all these risk models.

A fluid queue approach for the Laplace transform of the time until ruin (which is a special case of the discounted penalty function) in a related model can be found in Badescu et al. (2005). For an analysis of the time until ruin based on the methodology of piece-wise deterministic Markov processes, see Jacobsen (2003). For a study on the asymptotic behavior of the ruin function in the presence of dependence between interclaim times and claim sizes based on random walk techniques, see Albrecher and Teugels (2004). Adan and Kulkarni (2003) recently considered a

queueing model with dependence structure (2) with λ_i replaced by λ_j . Translated to a risk model setting, the latter means that an interclaim time of state j is always followed by a claim size of state (and thus distribution) j (whereas in model (2) it is the other way round). In principle, a similar analysis can be developed for this model, too. However, in view of applications, (2) seems more appealing.

The paper is organized as follows. In Section 2, an explicit expression for the Laplace transform of the discounted penalty function in model (2) is derived. Section 3 then gives an explicit formula for the discounted penalty function for zero initial capital. In Section 4, the asymptotic behavior of the penalty function is investigated for light-tailed claim sizes. In concrete cases, it is sometimes not possible to explicitly evaluate the occurring expressions. Thus, in Section 5 it is shown how to (at least) obtain arbitrary moments of the time to ruin, surplus before ruin and the deficit at ruin. Finally, in Section 6, we specify examples and use the results of the paper to rederive and extend various formulas from the risk theory literature.

2 An equation for the discounted penalty function

Let $\mu_i^{(j)}$ denote the j th moment of distribution B_i , given it exists and furthermore $\mu_i := \mu_i^{(1)}$. In the sequel we will always assume the net profit condition

$$\sum_{i=1}^M \pi_i \mu_i < c \sum_{i=1}^M \pi_i \lambda_i^{-1}, \quad (3)$$

where $\pi = (\pi_1, \dots, \pi_M)$ is the stationary distribution of $\{Z_n\}$. We are now interested in various characteristics of the risk model (1) together with (2). Gerber and Shiu (1998) introduced the by now classical *discounted penalty function at ruin*

$$m_\delta(x) := \mathbb{E}\left(w(R(T_x^-), |R(T_x)|)e^{-\delta T_x} 1_{\{T_x < \infty\}}\right), \quad (4)$$

where T_x denotes the time of ruin with initial capital x , $R(T_x^-)$ is the surplus immediately before ruin, $|R(T_x)|$ is the deficit at ruin and the penalty $w(x_1, x_2)$ is an arbitrary non-negative function on $[0, \infty) \times [0, \infty)$. $\delta \geq 0$ may be interpreted as a force of interest, but (4) may also be considered in terms of a Laplace transform with δ as its argument. The function $m_\delta(x)$ contains a lot of useful information about the risk process. For example, if $w \equiv 1$, then $m_\delta(x)$ is the LST of the time to ruin given it occurs, and $m_0(x)$ is then simply the ruin probability $\psi(x)$. For $w(x_1, x_2) = 1_{\{x_1 \leq y_1\}} 1_{\{x_2 \leq y_2\}}$, $m_0(x)$ is just the joint distribution of the surplus before ruin and the deficit at ruin.

We will now derive an integro-differential equation for $m_\delta(x)$ for our Markov additive risk process. Let $m_{\delta,i}(x)$ denote the discounted penalty function given that $Z_0 = i$.

Then by conditioning on the time interval $(0, dt)$, we obtain

$$m_{\delta,i}(x) = (1 - \lambda_i dt) e^{-\delta dt} m_{\delta,i}(x + c dt) + \lambda_i dt \sum_{j=1}^M p_{ij} \int_0^{x+c dt} e^{-\delta dt} m_{\delta,j}(x + c dt - y) dB_j(y) \\ + \lambda_i dt \sum_{j=1}^M p_{ij} \int_{x+c dt}^{\infty} e^{-\delta dt} w(x + c dt, y - x - c dt) dB_j(y) + o(dt) \quad (i = 1, \dots, M).$$

Taylor expansion and rearranging yields

$$c \frac{dm_{\delta,i}}{dx}(x) - (\lambda_i + \delta) m_{\delta,i}(x) + \lambda_i \sum_{j=1}^M p_{ij} \int_0^x m_{\delta,j}(x - y) dB_j(y) \\ + \lambda_i \sum_{j=1}^M p_{ij} \int_x^{\infty} w(x, y - x) dB_j(y) = 0 \quad (i = 1, \dots, M). \quad (5)$$

Define, for $\text{Re } s \geq 0$ the Laplace(-Stieltjes) transforms

$$\tilde{m}_{\delta,i}(s) := \int_0^{\infty} e^{-sx} m_{\delta,i}(x) dx, \\ \tilde{b}_i(s) := \int_{x=0}^{\infty} e^{-sx} dB_i(x), \quad (i = 1, \dots, M), \\ \tilde{\omega}_i(s) := \int_{x=0}^{\infty} e^{-sx} \int_x^{\infty} w(x, y - x) dB_i(y) dx, \quad (i = 1, \dots, M).$$

Then we obtain for $i = 1, \dots, M$,

$$cs \tilde{m}_{\delta,i}(s) - cm_{\delta,i}(0) - (\lambda_i + \delta) \tilde{m}_{\delta,i}(s) + \lambda_i \sum_{j=1}^M p_{ij} (\tilde{m}_{\delta,j}(s) \tilde{b}_j(s) + \tilde{\omega}_j(s)) = 0, \quad (6)$$

or in matrix notation,

$$\left((cs - \delta) I - \Lambda + \Lambda P \tilde{B}(s) \right) \vec{\tilde{m}}_{\delta}(s) = c \vec{m}_{\delta}(0) - \Lambda P \vec{\tilde{\omega}}(s), \quad (7)$$

where I is the identity matrix, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_M)$, $\tilde{B}(s) = \text{diag}(\tilde{b}_1(s), \dots, \tilde{b}_M(s))$ and $\vec{\tilde{m}}_{\delta}(s) = (\tilde{m}_{\delta,1}(s), \dots, \tilde{m}_{\delta,M}(s))$.

Thus it remains to solve a system of linear equations. First, the quantities $m_{\delta,i}(0)$ have to be determined. For that purpose, denote

$$A_{\delta}(s) := (cs - \delta) I - \Lambda + \Lambda P \tilde{B}(s).$$

Proposition 2.1. (i) The equation $\det(A_0(s)) = 0$ has one zero $s_1 = 0$ and $M - 1$ zeroes s_2, \dots, s_M with $\text{Re}(s_i) > 0$.

(ii) If $\delta > 0$ then $\det(A_{\delta}(s)) = 0$ has M zeroes s_1, \dots, s_M with $\text{Re}(s_i) > 0$.

Proof: *Case (i):* If $\delta = 0$, then the statement immediately follows from Theorem 3.2 of Adan and Kulkarni (2003) (after transposing their matrix).

Case (ii): We follow an idea in De Smit (1983), see also Adan and Kulkarni (2003). Let C denote a circle with its center at $[\delta + \max_{1 \leq i \leq M} \lambda_i]/c$ and radius $[\delta + \max_{1 \leq i \leq M} \lambda_i]/c$, and let $A_\delta(s, u) := (cs - \delta)I - \Lambda + u\Lambda P\tilde{B}(s)$, $0 \leq u \leq 1$. We first prove, for $0 \leq u \leq 1$, that

$$\det(A_\delta(s, u)) \neq 0 \text{ for } s \in C. \quad (8)$$

The matrix $A_\delta(s, u)$ is diagonally dominant for $0 \leq u \leq 1$, since

$$\begin{aligned} & |cs - \delta - \lambda_i + u\lambda_i p_{i,i} \tilde{b}_i(s)| \geq |\delta + \lambda_i - cs| - |u\lambda_i p_{i,i} \tilde{b}_i(s)| \\ & \geq \delta + \lambda_i - u\lambda_i p_{i,i} \tilde{b}_i(0) > u\lambda_i (1 - p_{i,i} \tilde{b}_i(0)) \\ & = u\lambda_i \sum_{j \neq i} p_{i,j} \tilde{b}_j(0) \geq |u\lambda_i \sum_{j \neq i} p_{i,j} \tilde{b}_j(s)|. \end{aligned} \quad (9)$$

The diagonal dominance implies (cf. Marcus and Minc (1964, pp.146-147)) that $\det A_\delta(s, u) \neq 0$ for $s \in C$.

Now let $f(u)$ denote the number of zeroes of $\det(A_\delta(s, u))$ in C^+ , the interior of C . Then

$$f(u) = \frac{1}{2\pi i} \int_C \frac{\frac{d}{ds} \det(A_\delta(s, u))}{\det(A_\delta(s, u))} ds.$$

Hence $f(u)$ is a continuous function on $[0, 1]$, integer valued, and therefore constant. $f(0) = M$, because $\det(A_0(s)) = \det((cs - \delta)I - \Lambda) = 0$ for $s_i^* := \frac{\delta + \lambda_i}{c}$, $1 \leq i \leq M$. Hence also $f(1) = M$. \square

Remark 2.1. It follows from the above proof for Case (ii), and from Remark 6.2 of Adan and Kulkarni (2003) for Case (i), that the zeroes are located in the interior C^+ ($s_1 = 0$ is located on C in Case (i)).

Remark 2.2. The solutions of

$$\det(A_\delta(s)) = 0 \quad (10)$$

are intimately connected with the behavior of $m_\delta(x)$: zeroes with negative real part determine the asymptotic behavior, whereas zeroes in the right half-plane determine the constants in the exact expressions for $m_\delta(x)$. Since for $M = 1$, this equation is known as the Lundberg fundamental equation (see e.g. Gerber and Shiu (2005)), we call (10) the *generalized Lundberg fundamental equation*.

Now, under the assumption that the functions $m_{\delta,i}(x)$ do not grow super-exponentially fast (which is fulfilled for all penalty functions of practical interest), $\tilde{m}_{\delta,i}(s)$ are analytic functions for $Re(s) \geq 0$, so that for each of the M zeroes s_1, \dots, s_M we can proceed in the following way: Determine a non-trivial solution \vec{k}_i of

$$A_\delta^T(s_i) \vec{k}_i = \vec{0}$$

for each $i = 1, \dots, M$. Since we then have

$$0 = \vec{m}_\delta(s_i)^T A_\delta^T(s_i) \vec{k}_i = (c \vec{m}_\delta(0) - \Lambda P \vec{\omega}(s_i))^T \vec{k}_i, \quad (11)$$

this gives M linear equations for $m_{\delta,1}(0), \dots, m_{\delta,M}(0)$.

Remark 2.3. For $\delta = 0$, the zeroes s_1, \dots, s_M can always be obtained numerically. Moreover, if the involved claim size distributions have a rational Laplace transform, then the discounted penalty function can be obtained explicitly by inversion of the Laplace transform of the solution of (7).

3 Zero initial capital

The following explicit expression for the discounted penalty function with zero initial capital can be obtained:

Proposition 3.1. *Let $K := (\vec{k}_1, \dots, \vec{k}_M)^T$ and let $\det K_{j_2, i}$ denote the minor of K with respect to row j_2 and column i . Then*

$$m_{\delta, i}(0) = \sum_{j_1=1}^M \sum_{j_2=1}^M C_{j_1, j_2}^{(i)}(s_1, \dots, s_M, \delta) \tilde{\omega}_{j_1}(s_{j_2}), \quad (i = 1, \dots, M) \quad (12)$$

where the coefficients $C_{j_1, j_2}^{(i)}(j_1, j_2 = 1, \dots, M)$ are given by

$$C_{j_1, j_2}^{(i)} = \frac{(-1)^{i+j_2} \cdot \det K_{j_2, i} \cdot \sum_{l=1}^M \lambda_l p_{l, j_1} k_{j_2, l}}{c \det K} \quad (13)$$

with $k_{j_2, l}$ denoting the l -th component of vector \vec{k}_{j_2} ($l=1, \dots, M$).

Proof: Equations (11) can be written as

$$c K \vec{m}_\delta(0) = \begin{pmatrix} \sum_{j_1=1}^M d_{j_1}^{(1)} \tilde{\omega}_{j_1}(s_1) \\ \vdots \\ \sum_{j_1=1}^M d_{j_1}^{(M)} \tilde{\omega}_{j_1}(s_M) \end{pmatrix},$$

with $d_{j_1}^{(j_2)} = \sum_{l=1}^M \lambda_l p_{l, j_1} k_{j_2, l}$. An application of Cramér's rule gives

$$m_{\delta, i}(0) = \frac{\begin{vmatrix} k_{1,1} & \cdots & k_{1, i-1} & \sum_{j_1=1}^M d_{j_1}^{(1)} \tilde{\omega}_{j_1}(s_1) & k_{1, i+1} & \cdots & k_{1, M} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ k_{M,1} & \cdots & k_{M, i-1} & \sum_{j_1=1}^M d_{j_1}^{(M)} \tilde{\omega}_{j_1}(s_M) & k_{M, i+1} & \cdots & k_{M, M} \end{vmatrix}}{c \det K}.$$

Expanding the determinant in the numerator along the i -th column then yields the desired result. \square

Let $f_i(y_1, y_2, t|x)$ denote the (defective) joint density function of $R(T_x^-)$, $|R(T_x)|$ and T_x given $Z_0 = i$, i.e.

$$m_{\delta,i}(x) = \int_{y_1=0}^{\infty} \int_{y_2=0}^{\infty} \int_{t=0}^{\infty} w(y_1, y_2) e^{-\delta t} f_i(y_1, y_2, t|x) dt dy_2 dy_1$$

and define the discounted joint density function of surplus prior to and after ruin by $f_i(y_1, y_2|x) = \int_0^{\infty} e^{-\delta t} f_i(y_1, y_2, t|x) dt$. The following result generalizes Formula (8.3) of Gerber and Shiu (2005):

Corollary 3.2. *Assume that the claim size distributions B_i ($i = 1, \dots, M$) are absolutely continuous with density function $b_i(y)$. Then*

$$f_i(y_1, y_2|0) = \sum_{j_1=1}^M \sum_{j_2=1}^M C_{j_1, j_2}^{(i)}(s_1, \dots, s_M, \delta) e^{-s_{j_2} y_1} b_{j_1}(y_1 + y_2), \quad (i = 1, \dots, M) \quad (14)$$

Proof: Choose $w(x_1, x_2)$ to be the Dirac delta function with respect to $x_1 = y_1, x_2 = y_2$ (i.e. $\tilde{\omega}_i(s) = e^{-s y_1} b_i(y_1 + y_2)$). Then the assertion is a direct consequence of Proposition 3.1. \square

Accordingly, we obtain for the discounted marginal density of the surplus prior to ruin

$$f_i(y_1|0) = \int_0^{\infty} f_i(y_1, y_2|0) dy_2 = \sum_{j_1=1}^M \sum_{j_2=1}^M C_{j_1, j_2}^{(i)}(s_1, \dots, s_M, \delta) e^{-s_{j_2} y_1} (1 - B_{j_1}(y_1))$$

and for the discounted marginal density $f_i(y_2|0) = \int_0^{\infty} f_i(y_1, y_2|0) dy_1$ of the deficit at ruin

$$f_i(y_2|0) = \sum_{j_1=1}^M \sum_{j_2=1}^M C_{j_1, j_2}^{(i)}(s_1, \dots, s_M, \delta) e^{s_{j_2} y_2} \left(\tilde{b}_{j_1}(s_{j_2}) - \int_0^{y_2} e^{-s_{j_2} z} b_{j_1}(z) dz \right).$$

4 Asymptotic behavior

From Section 2 it follows that

$$\vec{m}_{\delta}(s) = \frac{A_{\delta, adj}(s) \left(c \vec{m}_{\delta}(0) - \Lambda P \vec{\omega}(s) \right)}{\det A_{\delta}(s)} \quad (15)$$

is a vector of analytic functions for $Re(s) > 0$ (here $A_{\delta, adj}(s)$ denotes the adjunct matrix of $A_{\delta}(s)$).

Let us assume that all the LST $\tilde{b}_i(s)$ of the claim size distributions B_i exist in a neighborhood of the origin. Then, due to the structure of (15), the functions $\vec{m}_{\delta,i}(s)$ are analytic for all s with $Re(s) > -R_{\delta}$, where $-R_{\delta}$ denotes the zero with largest real part in the negative halfplane of $\det A_{\delta}(s)$ (which is the generalized Lundberg

adjustment coefficient). From the damping property of Laplace transforms, we have $\mathcal{L}(e^{R_\delta x} m_{\delta,i}(x)) = \tilde{m}_{\delta,i}(s - R_\delta)$ so that

$$\lim_{x \rightarrow \infty} e^{R_\delta x} \vec{m}_\delta(x) = \lim_{s \rightarrow 0} s \vec{m}_\delta(s - R_\delta) = \vec{C},$$

given that the limit exists (which for instance is guaranteed if $e^{R_\delta x} m_{\delta,i}(x)$ is monotonically increasing in x for each $i = 1, \dots, M$, see e.g. Doetsch (1937)).

For convenience, let $s = -R_\delta$ be a simple pole of $\tilde{m}_{\delta,i}(s)$, then we obtain, using de L'Hospital:

$$\vec{C} = \frac{A_{\delta,adj}(-R_\delta) \left(c \vec{m}_\delta(0) - \Lambda P \vec{\omega}(-R_\delta) \right)}{\left. \frac{\partial}{\partial s} \left(\det A_\delta(s) \right) \right|_{s=-R_\delta}}. \quad (16)$$

Thus the discounted penalty function decays exponentially with initial capital x at rate R_δ and the corresponding constants are given by (16).

5 Moments of three Characteristics of the Ruin Process

5.1 Moments of the Time to Ruin

First take $w(\cdot, \cdot) \equiv 1$ and define $f_{n,i}(s) := \left. \frac{\partial^n \tilde{m}_{\delta,i}(s)}{\partial \delta^n} \right|_{\delta=0}$ which is (up to the sign) the Laplace transform w.r.t. x of the n th moment of the time to ruin. Differentiation of Formula (6) and substitution of $\delta = 0$ gives after some algebraic manipulations:

$$(cs - \lambda_i) f_{n,i}(s) - n f_{n-1,i}(s) + \lambda_i \sum_{j=1}^M p_{ij} \tilde{b}_j(s) f_{n,j}(s) = c \left. \frac{\partial^n m_{\delta,i}(0)}{\partial \delta^n} \right|_{\delta=0},$$

or in matrix form

$$A_0(s) \vec{f}_n(s) = c \left. \frac{\partial^n \vec{m}_\delta(0)}{\partial \delta^n} \right|_{\delta=0} + n \vec{f}_{n-1}(s). \quad (17)$$

Note that $\vec{f}_0(s) = \vec{m}_0(s) = \vec{\psi}(s)$, where $\tilde{\psi}_i(s)$ is the Laplace transform of the ruin probability $\psi_i(x)$ ($i = 1, \dots, M$). The vector $\vec{\psi}(s)$ is available as the solution of (6) for $\delta = 0$. Thus we get a recursion for the n th moment of the time to ruin. In the j th recursive step, we have to determine the M constants $\left. \frac{\partial^j m_{\delta,i}(0)}{\partial \delta^j} \right|_{\delta=0}$ ($i = 1, \dots, M$) in the above way using the M zeroes of $\det A_0(s) = 0$ in the positive halfplane (which can always be obtained numerically).

5.2 Moments of the Surplus prior to Ruin

Next take $w(x, y) \equiv e^{-ax}$ so that $\tilde{\omega}_i(s) = \frac{1 - \tilde{b}_i(s+a)}{s+a}$. Let furthermore $\delta = 0$ and denote $g_{n,i}(s) = \left. \frac{\partial^n \tilde{m}_{0,i}(s)}{\partial a^n} \right|_{a=0}$, which is (up to the sign) the Laplace transform w.r.t.

x of the n th moment of the surplus prior to ruin. Differentiation of (6) w.r.t. a and substitution of $a = 0$ gives

$$A_0(s)\vec{g}_n(s) = c \frac{\partial^n \vec{m}_0(0)}{\partial a^n} \Big|_{a=0} - \Lambda P \text{diag} (\xi_{n,1}(s), \dots, \xi_{n,M}(s)),$$

where

$$\xi_{n,i}(s) := \frac{\partial^n \tilde{\omega}_i(s)}{\partial a^n} \Big|_{a=0} = \frac{(-1)^n n!}{s^{n+1}} \left(1 - \tilde{b}_i(s) - \sum_{j=1}^n \frac{(-s)^j}{j!} \frac{\partial^j \tilde{b}_i(s)}{\partial s^j} \right).$$

Thus the Laplace transform of the n -th moment can be obtained by first determining the M constants $\frac{\partial^n m_{0,i}(0)}{\partial a^n} \Big|_{a=0}$ using the zeroes of the generalized Lundberg's fundamental equation in the positive halfplane in the usual way and then solving the above linear system of equations.

5.3 Moments of the Deficit at Ruin

Finally take $w(x, y) \equiv e^{-ay}$ and thus $\tilde{\omega}_i(s) = \frac{\tilde{b}_i(a) - \tilde{b}_i(s)}{s-a}$. Define $k_{n,i}(s) := \frac{\partial^n \tilde{m}_{0,i}(s)}{\partial a^n} \Big|_{a=0}$, which is (up to the sign) the Laplace transform w.r.t. x of the n th moment of the deficit at ruin. Differentiation of (6) w.r.t. a and substitution of $a = 0$ gives

$$A_0(s)\vec{k}_n(s) = c \frac{\partial^n \vec{m}_0(0)}{\partial a^n} \Big|_{a=0} - \Lambda P \text{diag} (\eta_{n,1}(s), \dots, \eta_{n,M}(s)),$$

where

$$\eta_{n,i}(s) := \frac{\partial^n \tilde{\omega}_i(s)}{\partial a^n} \Big|_{a=0} = -\frac{n!}{s^{n+1}} \left(\tilde{b}_i(s) - \sum_{j=0}^n \frac{(-s)^j}{j!} \mathbb{E}(B_i^j) \right).$$

Thus the Laplace transform of the n -th moment can again be obtained by first determining the M constants $\frac{\partial^n m_{0,i}(0)}{\partial a^n} \Big|_{a=0}$ using the zeroes in the positive halfplane of Lundberg's fundamental equation and subsequently solving the above linear system of equations.

6 Examples

6.1 The classical compound Poisson model

For $M = 1$ we retain the classical compound Poisson risk model, and indeed from (6) it follows that in this case

$$\tilde{m}_\delta(s) = \frac{cm_\delta(0) - \lambda\tilde{\omega}(s)}{cs - \delta - \lambda + \lambda\tilde{b}(s)} = \frac{\lambda(\tilde{\omega}(s_1) - \tilde{\omega}(s))}{cs - \delta - \lambda + \lambda\tilde{b}(s)}, \quad (18)$$

in agreement with Dickson (1998). If the LST of the claim size distribution B exists in a neighborhood of the origin and $\lim_{x \rightarrow \infty} e^{R_\delta x} m_\delta(x)$ exists, then one obtains from (16)

$$\lim_{x \rightarrow \infty} e^{R_\delta x} m_\delta(x) = \frac{\lambda(\tilde{\omega}(s_1) - \tilde{\omega}(-R_\delta))}{c + \lambda\tilde{b}'(-R_\delta)}, \quad (19)$$

where $-R_\delta$ denotes the negative zero of $cs - \delta - \lambda + \lambda\tilde{b}(s) = 0$ (which is unique, cf. Gerber and Shiu (1998)). In the special case $w \equiv 1$ and $\delta = 0$ we have $s_1 = 0$ and $\tilde{\omega}(s) = (1 - \tilde{b}(s))/s$, so that (19) reduces to the Cramér-Lundberg approximation $\lim_{x \rightarrow \infty} e^{R_0 x} \psi(x) = \frac{\lambda\mu - c}{c + \lambda\tilde{b}'(-R_0)}$.

Let us now look at moments of the time to ruin in the classical risk model and assume that $\mu^{(2)} < \infty$. Let, for $n \in \mathbb{N}$, $\psi_n(x) := \mathbb{E}(T_x^n 1_{\{T_x < \infty\}})$ and $\psi_0(x) := \psi(x)$, the ruin probability. Then $\mathbb{E}(T_x^n | T_x < \infty) = \frac{\psi_n(x)}{\psi(x)}$ and the Laplace transform of $\psi_n(x)$ is just $(-1)^n f_n(s)$ defined in Section 5.1. Equation (17) here translates into

$$f_n(s) = \frac{c \left. \frac{\partial^n m_\delta(0)}{\partial \delta^n} \right|_{\delta=0} + n f_{n-1}(s)}{cs - \lambda + \lambda\tilde{b}(s)}. \quad (20)$$

Lemma 6.1. *For the classical compound Poisson model and $w \equiv 1$, the following recursive relation holds for $n \geq 1$ ($n \in \mathbb{N}$):*

$$\left. \frac{\partial^n m_\delta(0)}{\partial \delta^n} \right|_{\delta=0} = \frac{(-1)^n n}{c} \int_0^\infty \psi_{n-1}(u) du.$$

Proof: (20) is an analytic function for $\text{Re}(s) \geq 0$. Since $s = 0$ is the only zero of the denominator in the positive halfplane, it follows that

$$\begin{aligned} \left. \frac{\partial^n m_\delta(0)}{\partial \delta^n} \right|_{\delta=0} &= -\frac{n}{c} \lim_{s \rightarrow 0} \left. \frac{\partial^{n-1} \tilde{m}_\delta(s)}{\partial \delta^{n-1}} \right|_{\delta=0} \\ &= -\frac{n}{c} \int_0^\infty \left. \frac{\partial^{n-1} m_\delta(u)}{\partial \delta^{n-1}} \right|_{\delta=0} du = \frac{(-1)^n n}{c} \int_0^\infty \psi_{n-1}(u) du. \end{aligned}$$

□

Using the Pollaczek-Khintchine formula $\tilde{\psi}(s) = \frac{1}{s} - \frac{c - \lambda\mu}{cs - \lambda + \lambda\tilde{b}(s)}$ and Lemma 6.1, (20) now yields

$$f_n(s) = (-1)^n \tilde{\psi}_n(s) = (-1)^n n \left(\int_0^\infty \psi_{n-1}(u) du + n \tilde{\psi}_{n-1}(s) \right) \frac{\frac{1}{s} - \tilde{\psi}(s)}{c - \lambda\mu}$$

leading to

$$\psi_n(x) = \frac{n}{c - \lambda\mu} \left(\int_0^x \psi(x-u) \psi_{n-1}(u) du + \int_x^\infty \psi_{n-1}(u) du - \psi(x) \int_0^\infty \psi_{n-1}(u) du \right),$$

which is equivalent to Formula (6.29) of Lin and Willmot (2000), where the result was obtained using compound geometric tails. Note that the above derivation is particularly simple.

Using the identity $\int_0^\infty \psi(x) dx = \frac{\lambda\mu^{(2)}}{2(c - \lambda\mu)}$ (which itself is a direct consequence of the Pollaczek-Khintchine formula for $\mu^{(2)} < \infty$), one obtains for the specific case $n = 1$

$$\mathbb{E}(T_x | T_x < \infty) = \frac{\int_0^x \psi(x-u) \psi(u) du + \int_x^\infty \psi(u) du - \frac{\lambda\mu^{(2)}}{2(c - \lambda\mu)} \psi(x)}{(c - \lambda\mu) \psi(x)},$$

which is Formula (6.23) of Lin and Willmot (2000).

Let us now choose $\delta = 0$, then $s_1 = 0$ and it follows from (18) and the Pollaczek-Khintchine formula

$$\tilde{m}_0(s) = \lambda \left(\tilde{\omega}(0) - \tilde{\omega}(s) \right) \frac{\frac{1}{s} - \tilde{\psi}(s)}{c - \lambda \mu}$$

so that

$$m_0(x) = \frac{\lambda}{c - \lambda \mu} \left(\tilde{\omega}(0)(1 - \psi(x)) - \int_0^x (1 - \psi(x-u)) \int_u^\infty w(u, y-u) dB(y) du \right). \quad (21)$$

The latter formula gives rise to a number of nice identities. For instance, the LST of the surplus prior to ruin is obtained for $w \equiv e^{-ax}$, i.e. $\tilde{\omega}(s) = \frac{1 - \tilde{b}(a+s)}{a+s}$, and from (21)

$$\mathbb{E}(e^{-a R_{T_x}^-} 1_{\{T_x < \infty\}}) = \frac{\lambda}{c - \lambda \mu} \left(\frac{1 - \tilde{b}(a)}{a} (1 - \psi(x)) - \int_0^x (1 - \psi(x-u)) e^{-au} \bar{B}(u) du \right). \quad (22)$$

This leads to the defective density of the surplus prior to ruin

$$f(y_1|x) = \frac{\lambda}{c - \lambda \mu} \left(\bar{B}(y_1)(1 - \psi(x)) - 1_{\{y_1 < x\}} \bar{B}(y_1)(1 - \psi(x - y_1)) \right),$$

which already appeared in Dickson (1992). By differentiation of (22) we immediately obtain

$$\mathbb{E}((R_{T_x}^-)^n | T_x < \infty) = \frac{\lambda}{(c - \lambda \mu) \psi(x)} \left(\frac{\mu^{(n+1)}(1 - \psi(x))}{n+1} - \int_0^x u^n (1 - \psi(x-u)) \bar{B}(u) du \right),$$

in agreement with (5.3) of Lin and Willmot (2000) (again, our Laplace transform approach leads to the result in a straight-forward way).

On the other hand, the choice $w \equiv e^{-ay}$ (i.e. $\tilde{\omega}(s) = \frac{\tilde{b}(a) - \tilde{b}(s)}{s-a}$) in (21) leads to the LST of the deficit at ruin

$$\mathbb{E}(e^{-a |R_{T_x}|} 1_{\{T_x < \infty\}}) = \frac{\lambda}{c - \lambda \mu} \left(\frac{1 - \tilde{b}(a)}{a} (1 - \psi(x)) - \int_0^x (1 - \psi(x-u)) \int_u^\infty e^{-a(y-u)} dB(y) du \right) \quad (23)$$

and differentiation gives the moments

$$\mathbb{E}(|R_{T_x}|^n | T_x < \infty) = \frac{\lambda}{(c - \lambda \mu) \psi(x)} \left(\frac{\mu^{(n+1)}(1 - \psi(x))}{n+1} - \int_0^x (1 - \psi(x-u)) \int_u^\infty (y-u)^n dB(y) du \right),$$

which is another way of writing Equation (4.5) in Lin and Willmot (2000).

In particular, it follows from (22) and (23) that for $x = 0$ the distributions of the surplus prior to ruin and of the deficit at ruin coincide (see also Dufresne and Gerber (1988)).

6.2 Renewal models

6.2.1 Generalized Erlang(n)-interclaim times

Let us assume that we start in state 1 and that

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Assume furthermore that $M = n$ and a claim can only occur in state 1 (with claim size distribution $B_1 = B$ and LST \tilde{b}), and the claim size distributions B_2, \dots, B_n of all other states are degenerate at zero, i.e. $\tilde{B}(s) = \text{diag}(\tilde{b}(s), 1, \dots, 1)$ and $\tilde{\omega}(s) = (\tilde{\omega}(s), 0, \dots, 0)^T$.

Then $m_\delta(x) := m_{\delta,1}(x)$ is the discounted penalty function for a renewal model with generalized Erlang(n) interclaim times. Here $A_\delta(s)$ has the simple form

$$A_\delta(s) = \begin{pmatrix} cs - \delta - \lambda_1 & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & cs - \delta - \lambda_2 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & cs - \delta - \lambda_{n-2} & \lambda_{n-2} & 0 \\ 0 & 0 & \cdots & 0 & cs - \delta - \lambda_{n-1} & \lambda_{n-1} \\ \lambda_n \tilde{b}(s) & 0 & \cdots & \cdots & 0 & cs - \delta - \lambda_n \end{pmatrix}$$

so that its determinant is easily calculated yielding

$$\det A_\delta(s) = (-1)^n \left(\prod_{j=1}^n (\lambda_j + \delta - cs) - \tilde{b}(s) \prod_{j=1}^n \lambda_j \right). \quad (24)$$

Due to the simple transition matrix P of this example, it follows from (5) that $m_\delta(x)$ is the solution of the integro-differential equation

$$\prod_{j=1}^n \left(1 + \frac{\delta - cD}{\lambda_j} \right) m_\delta(x) = \int_0^x m_\delta(x-y) dB(y) + \int_x^\infty w(x, y-x) dB(y), \quad (25)$$

where D denotes the differentiation operator w.r.t. x .

This model has recently been studied in detail by Gerber and Shiu (2005) and for $\lambda_1 = \dots = \lambda_n$ by Li and Garrido (2004) (see Dickson and Hipp (2001), Sun and Yang (2004) and Cheng and Tang (2003) for the special case $n = 2$).

Again, our formalism can be used to rederive results for this model in a quite transparent way. For instance, $m_\delta(0)$ follows from Proposition 3.1: Since we start in state $i = 1$ and only in state 1 there is a non-degenerate claim size distribution, (12) simplifies to

$$m_\delta(0) = \sum_{j_2=1}^n C_{1,j_2}^{(1)}(s_1, \dots, s_n, \delta) \tilde{\omega}(s_{j_2}).$$

It remains to determine the constants

$$C_{1,j_2}^{(1)}(s_1, \dots, s_n, \delta) = \frac{(-1)^{1+j_2} \lambda_n k_{j_2,n} \det K_{j_2,1}}{c \det K}.$$

From (11) one obtains

$$K = \begin{pmatrix} \prod_{j=2}^n \frac{\lambda_j + \delta - cs_1}{\lambda_{j-1}} & \prod_{j=3}^n \frac{\lambda_j + \delta - cs_1}{\lambda_{j-1}} & \dots & \frac{\lambda_n + \delta - cs_1}{\lambda_{n-1}} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \prod_{j=2}^n \frac{\lambda_j + \delta - cs_n}{\lambda_{j-1}} & \prod_{j=3}^n \frac{\lambda_j + \delta - cs_n}{\lambda_{j-1}} & \dots & \frac{\lambda_n + \delta - cs_n}{\lambda_{n-1}} & 1 \end{pmatrix}$$

leading to

$$\det K = \frac{c^{\frac{n(n-1)}{2}}}{\lambda_1 \lambda_2^2 \dots \lambda_{n-1}^{n-1}} \begin{vmatrix} 1 & s_1 & \dots & s_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & s_n & \dots & s_n^{n-1} \end{vmatrix} = \frac{c^{\frac{n(n-1)}{2}}}{\lambda_1 \lambda_2^2 \dots \lambda_{n-1}^{n-1}} \prod_{\substack{j,k=1 \\ k>j}}^n (s_k - s_j).$$

Analogously,

$$\det K_{j_2,1} = \frac{c^{\frac{(n-1)(n-2)}{2}}}{\lambda_2 \lambda_3^2 \dots \lambda_{n-1}^{n-2}} \prod_{\substack{j,k=1 \\ k>j, k \neq j_2, j \neq j_2}}^n (s_k - s_j),$$

from which we obtain

$$C_{1,j_2}^{(1)}(s_1, \dots, s_n, \delta) = \frac{\lambda_1 \dots \lambda_n}{c^n} \prod_{\substack{k=1 \\ k \neq j_2}}^n \frac{1}{s_k - s_{j_2}}$$

and finally

$$m_\delta(0) = \sum_{j_2=1}^n \frac{\lambda_1 \dots \lambda_n}{c^n} \prod_{\substack{k=1 \\ k \neq j_2}}^n \frac{1}{s_k - s_{j_2}} \tilde{\omega}(s_{j_2}). \quad (26)$$

As in Corollary 3.2, it for instance follows that the discounted joint defective density function of the surplus prior to ruin and the deficit at ruin with $x = 0$ is given by

$$f(y_1, y_2|0) = \sum_{j_2=1}^n \frac{\lambda_1 \dots \lambda_n}{c^n} \prod_{\substack{k=1 \\ k \neq j_2}}^n \frac{1}{s_k - s_{j_2}} e^{-s_{j_2} y_1} b(y_1 + y_2), \quad (27)$$

which is Formula (8.3) of Gerber and Shiu (2005) (for further details, see Albrecher (2005)). A comparison of (26) with (22) and (23) elucidates that for $n \geq 2$ the presence of strictly positive zeroes s_j distorts the symmetry of the classical model between the distribution of the surplus prior to ruin and of the deficit at ruin for $x = 0$.

A general expression for $\tilde{m}_\delta(s)$ in this model can be obtained by evaluating the first row of the numerator in (15):

Lemma 6.2. *The first row of the vector $A_{\delta,adj}(s)\vec{m}_\delta(0)$ is given by*

$$\left(A_{\delta,adj}(s)\vec{m}_\delta(0)\right)_1 = \frac{(-1)^{n+1}\lambda_1 \cdots \lambda_n}{c} \sum_{j=1}^n \tilde{\omega}(s_j) \prod_{\substack{k=1 \\ k \neq j}}^n \frac{s-s_k}{s_j-s_k}. \quad (28)$$

Proof: The first row of the adjunct matrix $A_{\delta,adj}(s)$ of $A_\delta(s)$ is given by

$$(A_{\delta,adj}(s))_1 = \left(\prod_{i=2}^n (cs - \delta - \lambda_i), -\lambda_1 \prod_{i=3}^n (cs - \delta - \lambda_i), \lambda_1 \lambda_2 \prod_{i=4}^n (cs - \delta - \lambda_i), \dots, (-1)^{n+1} \lambda_1 \cdots \lambda_{n-1} \right).$$

In view of (12), we have

$$\vec{m}_\delta(0) = \frac{\lambda_n}{c \det K} \begin{pmatrix} \det K_{1,1}\tilde{\omega}(s_1) - \det K_{2,1}\tilde{\omega}(s_2) + \dots + (-1)^{n+1} \det K_{n,1}\tilde{\omega}(s_n) \\ -\det K_{1,2}\tilde{\omega}(s_1) + \det K_{2,2}\tilde{\omega}(s_2) + \dots + (-1)^{n+2} \det K_{n,2}\tilde{\omega}(s_n) \\ \vdots \\ (-1)^{n+1} \det K_{1,n}\tilde{\omega}(s_1) + (-1)^{n+2} \det K_{2,n}\tilde{\omega}(s_2) + \dots + \det K_{n,n}\tilde{\omega}(s_n) \end{pmatrix}.$$

Now, collecting all coefficients of $\tilde{\omega}(s_j)$ ($j = 1, \dots, n$) in $\left(A_{\delta,adj}(s)\vec{m}_\delta(0)\right)_1$, we obtain

$$\begin{aligned} & \frac{(-1)^{j+1}\lambda_n}{c \det K} \left(\prod_{i=2}^n (cs - \delta - \lambda_i) \det K_{j,1} + \lambda_1 \prod_{i=3}^n (cs - \delta - \lambda_i) \det K_{j,2} \right. \\ & \quad \left. + \dots + \lambda_1 \cdots \lambda_{n-1} \det K_{j,n} \right) \\ &= \frac{(-1)^{j+1}\lambda_1 \cdots \lambda_n}{c \det K} \left((-1)^{n-1} \prod_{i=2}^n \frac{\lambda_i + \delta - cs}{\lambda_{i-1}} \det K_{j,1} + (-1)^{n-2} \prod_{i=3}^n \frac{\lambda_i + \delta - cs}{\lambda_{i-1}} \det K_{j,2} \right. \\ & \quad \left. + \dots + \det K_{j,n} \right) \\ &= \frac{(-1)^{n-1}\lambda_1 \cdots \lambda_n}{c \det K} \left(-1)^{j+1} \prod_{i=2}^n \frac{\lambda_i + \delta - cs}{\lambda_{i-1}} \det K_{j,1} + (-1)^{j+2} \prod_{i=3}^n \frac{\lambda_i + \delta - cs}{\lambda_{i-1}} \det K_{j,2} \right. \\ & \quad \left. + \dots + (-1)^{j+n} \det K_{j,n} \right). \end{aligned}$$

But the last term in brackets is just the determinant of a matrix K_j^* , which is the matrix K with entries s instead of s_j . Hence (28) follows from $\frac{\det K_j^*}{\det K} = \prod_{\substack{k=1 \\ k \neq j}}^n \frac{s-s_k}{s_j-s_k}$. \square

Since $(A_{\delta,adj}(s)\Lambda P\vec{\omega}(s))_1 = (-1)^{n+1}\lambda_1 \cdots \lambda_n \tilde{\omega}(s)$, (15) now implies that

$$\tilde{m}_\delta(s) = \frac{\tilde{\omega}(s) - \sum_{j_2=1}^n \tilde{\omega}(s_{j_2}) \prod_{\substack{k=1 \\ k \neq j_2}}^n \frac{s-s_k}{s_{j_2}-s_k}}{\prod_{j=1}^n \left(1 + \frac{\delta - cs}{\lambda_j}\right) - \tilde{b}(s)}, \quad (29)$$

which is Equation (7.3) of Gerber and Shiu (2005).

If the LST of the claim size distribution B exists in a neighborhood of the origin and $\lim_{x \rightarrow \infty} e^{R_\delta x} m_\delta(x)$ exists, then (16) yields

$$\lim_{x \rightarrow \infty} e^{R_\delta x} m_\delta(x) = \frac{\sum_{j_2=1}^n \tilde{\omega}(s_{j_2}) \prod_{\substack{k=1 \\ k \neq j_2}}^n \frac{-R_\delta - s_k}{s_{j_2} - s_k} - \tilde{\omega}(-R_\delta)}{\sum_{j=1}^n \frac{c}{\lambda_j} \prod_{\substack{k=1 \\ k \neq j}}^n (1 + \frac{\delta + c R_\delta}{\lambda_k}) + \tilde{b}'(-R_\delta)},$$

where $-R_\delta$ is the largest negative zero of $\det A_\delta = 0$. This formula generalizes Equation (4.10) of Gerber and Shiu (1998).

Finally, in line with the approach leading to (21), one can provide an alternative proof of Identity (3.4) of Dickson and Drekić (2004):

Proposition 6.3. *The joint (defective) density function of the surplus prior to ruin and the deficit at ruin in the Sparre Andersen model with generalized Erlang interclaim times is given by*

$$f(y_1, y_2 | x) = \frac{\lambda_1 \cdots \lambda_n b(y_1 + y_2)}{c^n \phi(0)} \sum_{j_2=1}^n \left(\prod_{\substack{k=1 \\ k \neq j_2}}^n \frac{1}{s_k - s_{j_2}} \right) e^{-s_{j_2}(y_1 - x)} \int_{\max\{0, x - y_1\}}^x e^{-s_{j_2} z} d\phi(z), \quad (30)$$

where $\phi(x) = 1 - \psi(x)$ is the survival probability for initial capital x .

Proof: For $\delta = 0$ and $w \equiv 1$ we have from (29)

$$\tilde{\psi}(s) = \frac{\frac{1 - \tilde{b}(s)}{s} - \sum_{j_2=1}^n \frac{1 - \tilde{b}(s_{j_2})}{s_{j_2}} \prod_{\substack{k=1 \\ k \neq j_2}}^n \frac{s - s_k}{s_{j_2} - s_k}}{\prod_{j=1}^n (1 - \frac{cs}{\lambda_j}) - \tilde{b}(s)}.$$

In terms of the survival probability, this implies

$$\tilde{\phi}(s) = \frac{1}{s} - \tilde{\psi}(s) = \frac{\prod_{j=1}^n (1 - \frac{cs}{\lambda_j}) - 1 + s \sum_{j_2=1}^n \frac{1 - \tilde{b}(s_{j_2})}{s_{j_2}} \prod_{\substack{k=1 \\ k \neq j_2}}^n \frac{s - s_k}{s_{j_2} - s_k}}{s \left(\prod_{j=1}^n (1 - \frac{cs}{\lambda_j}) - \tilde{b}(s) \right)}.$$

Obviously the numerator is a polynomial in s of degree n . But $\tilde{\phi}(s)$ is an analytic function for $Re(s) > 0$ and has a simple pole at $s = 0$, so that the numerator has to have the zeroes s_1, \dots, s_n (since for $\delta = 0$ we have $s_1 = 0$, the latter is a zero of multiplicity 2 in the above denominator). One can deduce that the numerator is of the form $\beta \prod_{j=1}^n (s - s_j)$ for a constant $\beta \in \mathbb{R}$. By taking the limit

$$\phi(0) = \lim_{s \rightarrow \infty} s \tilde{\phi}(s) = \frac{(-1)^n \beta \lambda_1 \cdots \lambda_n}{c^n},$$

we obtain

$$\tilde{\phi}(s) = \frac{(-1)^n c^n \phi(0) \prod_{j=1}^n (s - s_j)}{\lambda_1 \cdots \lambda_n s \left(\prod_{j=1}^n (1 - \frac{cs}{\lambda_j}) - \tilde{b}(s) \right)}. \quad (31)$$

Clearly, $f(y_1, y_2|x)$ is obtained from the discounted penalty function for $\delta = 0$ and w the Dirac delta function at $x_1 = y_1$ and $x_2 = y_2$ (so that $\tilde{\omega}(s) = e^{-s y_1} b(y_1 + y_2)$). Thus the Laplace transform of $f(y_1, y_2|x)$ follows from (29) to be

$$\tilde{f}(y_1, y_2|s) = b(y_1 + y_2) \frac{e^{-s y_1} - \sum_{j_2=1}^n e^{-s_{j_2} y_1} \prod_{\substack{k=1 \\ k \neq j_2}}^n \frac{s - s_k}{s_{j_2} - s_k}}{\prod_{j=1}^n (1 - \frac{cs}{\lambda_j}) - \tilde{b}(s)}.$$

Substituting (31) into the last equation leads to

$$\tilde{f}(y_1, y_2|s) = b(y_1 + y_2) \frac{(-1)^n \lambda_1 \cdots \lambda_n s \tilde{\phi}(s)}{c^n \phi(0)} \left(\frac{e^{-s y_1}}{\prod_{j=1}^n (s - s_j)} - \sum_{j_2=1}^n \frac{e^{-s_{j_2} y_1}}{s - s_{j_2}} \prod_{\substack{k=1 \\ k \neq j_2}}^n \frac{1}{s_{j_2} - s_k} \right).$$

Using partial fractions, the first term in the brackets above can be written as

$$\frac{e^{-s y_1}}{\prod_{j=1}^n (s - s_j)} = \sum_{j_2=1}^n \frac{e^{-s y_1}}{(s - s_{j_2}) \prod_{\substack{k=1 \\ k \neq j_2}}^n (s_{j_2} - s_k)}$$

and hence

$$\tilde{f}(y_1, y_2|s) = b(y_1 + y_2) \frac{\lambda_1 \cdots \lambda_n}{c^n \phi(0)} \left(\sum_{j_2=1}^n s \tilde{\phi}(s) \frac{e^{-s_{j_2} y_1} - e^{-s y_1}}{s - s_{j_2}} \prod_{\substack{k=1 \\ k \neq j_2}}^n \frac{1}{s_k - s_{j_2}} \right), \quad (32)$$

which is just the Laplace transform of (30). \square

6.2.2 Phase-type interclaim times

Our Markov additive process also contains the Sparre Andersen model with a phase-type interclaim time distribution and arbitrary claim size distribution. From the definition, a phase-type distribution is the lifetime of a terminating Markov process $\{J_t\}$ with finite state space E and time homogeneous transition rates (see e.g. Asmussen (2000)). In our setting, we pick out state 1 as the absorbing state of $\{J_t\}$ and let p_{ij} ($i, j = 2, \dots, M$) coincide with the transition probabilities of the embedded Markov chain of J_t . A claim (with distribution B) can then only occur, if our process is in state 1 (i.e. B_i is degenerate at 0 for $i = 2, \dots, M$), so that again $\tilde{B}(s) = \text{diag}(\tilde{b}(s), 1, \dots, 1)$ and $\vec{\omega}(s) = (\tilde{\omega}(s), 0, \dots, 0)$. Moreover, the first row entries of P are given by $p_{11} = 0$ and $p_{1j} = \alpha_j$, where $\vec{\alpha} = (\alpha_2, \dots, \alpha_M)$ is the $(M-1)$ -dimensional vector of initial probabilities of $\{J_t\}$ and p_{21}, \dots, p_{M1} represent the exit probabilities of $\{J_t\}$ into the absorbing state 1. If our Markov additive

process $R(t)$ is in state 1 and a claim has occurred, it immediately jumps to one of the other states according to the vector $\vec{\alpha}$. Thus, $R(t)$ corresponds to a renewal risk model with phase-type interclaim times, if we let $\lambda_1 \rightarrow \infty$. For then, by taking the limit in (6) for $i = 1$, we obtain the discounted penalty function

$$m_\delta(x) := m_{\delta,1}(x) = \sum_{j=2}^M \alpha_j m_{\delta,j}(x). \quad (33)$$

Going to the limit $\lambda_1 \rightarrow \infty$, one can rewrite (7) as

$$A_\delta^{Ph}(s) \vec{m}_\delta(s) = c(0, m_{\delta,2}(0), \dots, m_{\delta,M}(0))^T - \Lambda^{Ph} P \vec{\omega}(s),$$

where $\Lambda^{Ph} = \text{diag}(1, \lambda_2, \dots, \lambda_M)$, $A_\delta^{Ph}(s) := \left((cs - \delta)I - \Lambda^{Ph} + \Lambda^{Ph} P \tilde{B}(s) \right) - (cs - \delta)I \vec{e}_1 \vec{e}_1^T$ (with $\vec{e}_1 = (1, 0, \dots, 0)^T$), i.e.

$$A_\delta^{Ph}(s) = \begin{pmatrix} -1 & \alpha_2 & \alpha_3 & \cdots & \alpha_M \\ \lambda_2 p_{21} \tilde{b}(s) & cs - \delta - \lambda_2(1 - p_{22}) & \lambda_2 p_{23} & \cdots & \lambda_2 p_{2M} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \lambda_M p_{M1} \tilde{b}(s) & \lambda_M p_{M2} & \cdots & \cdots & cs - \delta - \lambda_M(1 - p_{MM}) \end{pmatrix}$$

and

$$\vec{m}_\delta(s) = \frac{A_{\delta,adj}^{Ph}(s) \cdot \begin{pmatrix} 0 \\ m_{\delta,2}(0) - \lambda_2 p_{21} \tilde{\omega}(s) \\ \vdots \\ m_{\delta,M}(0) - \lambda_M p_{M1} \tilde{\omega}(s) \end{pmatrix}}{\det A_\delta^{Ph}(s)}, \quad (34)$$

where $A_{\delta,adj}^{Ph}(s)$ is the adjugate matrix of $A_\delta^{Ph}(s)$.

The equation $\det A_\delta^{Ph}(s) = 0$ has exactly $M - 1$ solutions s_1, \dots, s_{M-1} in the right half-plane (which, for simplicity, we assume to be distinct). Thus, one can determine the unknown quantities $m_{\delta,j}(0)$ ($j = 2, \dots, M$) by (11) (with the obvious adaptation) and (12). Finally, $m_\delta(0)$ then follows from (33).

However, in the concrete situation, $m_\delta(0)$ can also be obtained more directly from (34): A careful analysis of the structure of $A_{\delta,adj}^{Ph}(s)$ reveals that

$$\vec{m}_\delta(s) = \frac{q_\delta(s) - g_\delta(s) \vec{\omega}(s)}{\det A_\delta^{Ph}(s)}, \quad (35)$$

where $q_\delta(s)$ is a polynomial in s of degree $M - 2$, the coefficients of which contain the unknown quantities $m_{\delta,j}(0)$ ($j = 2, \dots, M$). In addition, $g_\delta(s)$ is explicitly given by

$$g_\delta(s) = \det \left((cs - \delta)I - \Lambda^{Ph} + \Lambda^{Ph} P \right) + {}^{(1,1)}A_\delta^{Ph}(s), \quad (36)$$

with ${}^{(1,1)}A_\delta^{Ph}(s)$ denoting the minor of $A_\delta^{Ph}(s)$ w.r.t. row and column 1. In particular, $g_\delta(s)$ is also a polynomial in s of degree $M - 2$. Since $\vec{m}_\delta(s)$ is analytic in the

positive halfplane, the zeroes s_1, \dots, s_{M-1} must also be zeroes of the numerator in (35). By Lagrange interpolation we thus obtain

$$q_\delta(s) = \sum_{j=1}^{M-1} g_\delta(s_j) \tilde{\omega}(s_j) \prod_{k=1, k \neq j}^{M-1} \frac{s - s_k}{s_j - s_k}. \quad (37)$$

Now, $m_\delta(0) = \lim_{s \rightarrow \infty} s \tilde{m}_\delta(s)$ and it just remains to determine the latter limit. Since for $s \rightarrow \infty$, $\tilde{b}(s) \rightarrow 0$ and $\tilde{\omega}(s) = \mathcal{O}(1/s)$ (the latter holds for penalty functions w that do not grow super-exponentially fast), we have to collect the dominating terms in the denominator and numerator of (35), i.e.

$$q_\delta(s) \sim s^{M-2} \sum_{j=1}^{M-1} g_\delta(s_j) \tilde{\omega}(s_j) \prod_{k=1, k \neq j}^{M-1} \frac{1}{s_j - s_k}$$

and

$$\det \left((cs - \delta) I - \Lambda^{Ph} + \Lambda^{Ph} P \right) \sim -c^{M-1} s^{M-1}.$$

This finally leads to

$$m_\delta(0) = \lim_{s \rightarrow \infty} s \tilde{m}_\delta(s) = -\frac{1}{c^{M-1}} \sum_{j=1}^{M-1} g_\delta(s_j) \tilde{\omega}(s_j) \prod_{k=1, k \neq j}^{M-1} \frac{1}{s_j - s_k}, \quad (38)$$

which is Equation (20) of Li and Garrido (2005). For instance, let $w(x_1, x_2)$ be the Dirac delta function with respect to $x_1 = y_1, x_2 = y_2$ (i.e. $\tilde{\omega}(s) = e^{-s y_1} b(y_1 + y_2)$). Then we obtain from (38) the (defective) joint density function of the surplus prior to ruin and the deficit at ruin

$$f(y_1, y_2|0) = (-1)^{M-1} \frac{b(y_1 + y_2)}{c^{M-1}} \sum_{j=1}^{M-1} g_\delta(s_j) e^{-s_j y_1} \prod_{k=1, k \neq j}^{M-1} \frac{1}{s_k - s_j},$$

generalizing Formula (27) and Formula (4.6) of Dickson and Drekcic (2004).

For arbitrary $x > 0$, $f(y_1, y_2|x)$ is, from (35), given as the inverse Laplace transform of

$$\tilde{f}(y_1, y_2|s) = b(y_1 + y_2) \frac{\sum_{j=1}^{M-1} g_\delta(s_j) e^{-s_j y_1} \prod_{k=1, j \neq k}^{M-1} \frac{s - s_k}{s_j - s_k} - g_\delta(s) e^{-s y_1}}{\det A_\delta^{Ph}(s)}, \quad (39)$$

which can be obtained explicitly whenever the claim size distribution B has a rational Laplace transform. On the other hand, for arbitrary B and $\delta = 0$, one can derive the following generalization of (30):

Proposition 6.4. *The joint (defective) density function of the surplus prior to ruin and the deficit at ruin in the Sparre Andersen model with phase-type interclaim times satisfies*

$$f(y_1, y_2|x) = -\frac{b(y_1 + y_2)}{c^{M-1} \phi(0)} \sum_{j=1}^n g_0(s_j) \left(\prod_{\substack{k=1 \\ k \neq j}}^n \frac{1}{s_k - s_j} \right) e^{-s_j(y_1 - x)} \int_{\max\{0, x - y_1\}}^x e^{-s_j z} d\phi(z). \quad (40)$$

Proof: We will proceed in a similar fashion as in Proposition 6.3. First, one observes that

$$\tilde{\phi}(s) = \frac{1}{s} - \tilde{\psi}(s) = \frac{\det A_0^{Ph}(s) - s \sum_{j=1}^{M-1} g_0(s_j) \frac{1-\tilde{b}(s_j)}{s_j} \prod_{k=1, j \neq k}^{M-1} \frac{s-s_k}{s_j-s_k} + g_0(s)(1-\tilde{b}(s))}{s \det A_0^{Ph}(s)}.$$

The polynomial $g_0(s)$ can also be written as

$$g_0(s) = \frac{\det A_0^{Ph}(s) + {}^{(1,1)}A_0^{Ph}(s)}{\tilde{b}(s)},$$

so that we are left with

$$\tilde{\phi}(s) = \frac{- {}^{(1,1)}A_0^{Ph}(s) - s \sum_{j=1}^{M-1} g_0(s_j) \frac{1-\tilde{b}(s_j)}{s_j} \prod_{k=1, j \neq k}^{M-1} \frac{s-s_k}{s_j-s_k} + g_0(s)}{s \det A_0^{Ph}(s)},$$

the numerator of which is again a polynomial in s of degree $M-1$. As in the proof of Proposition 6.3, it follows by analyticity arguments that

$$\tilde{\phi}(s) = \frac{-c^{M-1}\phi(0) \prod_{j=1}^n (s-s_j)}{s \det A_0^{Ph}(s)}.$$

Now, the latter equation can be substituted in (39) which gives

$$\tilde{f}(y_1, y_2|s) = \frac{b(y_1 + y_2) s \tilde{\phi}(s)}{c^{M-1}\phi(0)} \left(\frac{g_0(s) e^{-s y_1}}{\prod_{k=1}^{M-1} (s-s_k)} - \sum_{j=1}^{M-1} \frac{g_0(s_j) e^{-s_j y_1}}{s-s_j} \prod_{k=1, k \neq j}^{M-1} \frac{1}{s_j-s_k} \right).$$

By partial fractions, we have

$$\frac{g_0(s)}{\prod_{k=1}^{M-1} (s-s_k)} = \sum_{j=1}^{M-1} \frac{g_0(s_j)}{\prod_{k=1, k \neq j}^{M-1} (s_j-s_k)} \frac{1}{s-s_j}$$

and hence

$$\tilde{f}(y_1, y_2|s) = \frac{b(y_1 + y_2)}{c^{M-1}\phi(0)} \sum_{j=1}^{M-1} g_0(s_j) s \tilde{\phi}(s) \frac{e^{-s y_1} - e^{-s_j y_1}}{s-s_j} \prod_{k=1, k \neq j}^{M-1} \frac{1}{s_j-s_k}, \quad (41)$$

which is the Laplace transform of (40). \square

Note that for generalized Erlang(n) interclaim times we have $g_0(s) = (-1)^{M+1} \lambda_2 \cdots \lambda_M$ (corresponding to $(-1)^n \lambda_1 \cdots \lambda_n$ in the notation of Subsection 6.2.1), so that indeed in this case (40) reduces to (32).

Finally, our approach leads to an alternative proof of a nice identity that is due to Dickson and Drekić (2004, Equ. 2.4):

Corollary 6.5. For $\delta = 0$ we have

$$f(y_1, y_2|x) = \frac{b(y_1 + y_2)}{\phi(0)} \int_{\max\{0, x-y_1\}}^x \frac{f(y_1 - x + z|0)}{1 - B(y_1 - x + z)} d\phi(z). \quad (42)$$

Proof: The density of the surplus prior to ruin is obtained from $m_\delta(x)$ for $\delta = 0$ and w the Dirac-delta function w.r.t. $x_1 = y_1$ (and thus $\tilde{\omega}(s) = e^{-sy_1}(1 - B(y_1))$). From (38) we then have

$$f(y_1|0) = -\frac{1 - B(y_1)}{c^{M-1}} \sum_{j=1}^{M-1} g_0(s_j) e^{-s_j y_1} \prod_{k=1, k \neq j}^{M-1} \frac{1}{s_j - s_k}.$$

The idea is now to consider the function

$$\frac{f(y_1 - x|0)}{1 - B(y_1 - x)} 1_{\{y_1 \geq x\}},$$

whose Laplace transform w.r.t. x is given by

$$-\frac{1}{c^{M-1}} \sum_{j=1}^{M-1} g_0(s_j) e^{-s_j y_1} \frac{1 - e^{-(s-s_j)y_1}}{s - s_j} \prod_{k=1, k \neq j}^{M-1} \frac{1}{s_j - s_k}.$$

In view of (41), the result easily follows. \square

6.3 A causal dependency model

In Albrecher and Boxma (2004), a causal dependency model was introduced, where the distribution of the interclaim time depends on the actual size of the previous claim in the following way: if the claim exceeds a (possibly random) threshold T , then the next interclaim time is exponentially distributed with rate λ_1 , otherwise the interclaim time is exponentially distributed with rate λ_2 . As mentioned in the introduction, the approach introduced in this paper contains this case for the special choice $M = 2$ and

$$dB_1(y) = \frac{1}{\mathbb{P}(T < B)} T(y) dB(y) \quad \text{and} \quad dB_2(y) = \frac{1}{\mathbb{P}(T > B)} (1 - T(y)) dB(y)$$

for a generic claim size distribution $B(y)$ and a threshold variable T together with the transition probabilities $p_{i1} = \mathbb{P}(B > T)$ and $p_{i2} = \mathbb{P}(T > B)$ for $i = 1, 2$. We can thus extend the analysis of Albrecher and Boxma (2004) considerably, since an investigation of the discounted penalty function is at hand.

For illustration, let us work out explicit expressions for a numerical example. Let $T \sim \text{Exp}(2)$, $B \sim \text{Exp}(1)$, $c = 2$, $\lambda_1 = 3$, $\lambda_2 = 1$ (which is the setup of Example 3 of Albrecher and Boxma (2004)), and thus

$$P = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\tilde{b}_1(s) = \frac{3}{2} \left(\frac{1}{1+s} - \frac{1}{3+s} \right), \quad \tilde{b}_2(s) = \frac{3}{3+s}.$$

Assume in the sequel $\delta = 0$. Then we obtain the determinant

$$\det A_0(s) = 3 - 8s + 4s^2 + \frac{6s-3}{1+s} - \frac{4s}{3+s},$$

which has one zero at 0 and one positive zero ($s_2 = 1.226$, all the other zeroes are negative), as they should be (here and later on, all numbers are rounded to their last digit). Moreover,

$$K = \begin{pmatrix} 2/3 & 1 \\ -0.922 & 0.388 \end{pmatrix}.$$

For an arbitrary penalty function w , one thus obtains from Proposition 3.1

$$\begin{aligned} m_0(0) = & \begin{pmatrix} 0.328 \\ 0.781 \end{pmatrix} \tilde{\omega}_1(0) + \begin{pmatrix} 0.672 \\ -0.448 \end{pmatrix} \tilde{\omega}_1(1.226) \\ & + \begin{pmatrix} 0.164 \\ 0.391 \end{pmatrix} \tilde{\omega}_2(0) + \begin{pmatrix} 0.336 \\ -0.224 \end{pmatrix} \tilde{\omega}_2(1.226). \end{aligned} \quad (43)$$

The largest negative zero of $\det A_0(s) = 0$ is given by $-R_0 = -0.065$. Consequently, under the assumptions of Section 4, it follows from (16) and (43) that

$$\lim_{x \rightarrow \infty} e^{0.065x} \vec{m}_0(x) = \vec{C}$$

with

$$\begin{aligned} \vec{C} = & \begin{pmatrix} 6.014 \\ 5.555 \end{pmatrix} \tilde{\omega}_1(-0.065) - \begin{pmatrix} 5.631 \\ 5.202 \end{pmatrix} \tilde{\omega}_1(0) - \begin{pmatrix} 0.383 \\ 0.354 \end{pmatrix} \tilde{\omega}_1(1.226) \\ & + \begin{pmatrix} 3.007 \\ 2.778 \end{pmatrix} \tilde{\omega}_2(-0.065) - \begin{pmatrix} 2.815 \\ 2.601 \end{pmatrix} \tilde{\omega}_2(0) - \begin{pmatrix} 0.191 \\ 0.177 \end{pmatrix} \tilde{\omega}_2(1.226). \end{aligned} \quad (44)$$

In the special case $w \equiv 1$ (where $m_i(x)$ is the ruin probability $\psi_i(x)$, given $Z_0 = i$), we have

$$\tilde{\omega}_1(s) = \frac{1}{s} \left(1 - \frac{3/2}{1+s} + \frac{3/2}{3+s} \right), \quad \tilde{\omega}_2(s) = \frac{1}{s} \left(1 - \frac{3}{3+s} \right)$$

and from (43) it follows that $\psi_1(0) = 0.945$, $\psi_2(0) = 0.870$. Subsequently, from (7) we then arrive at

$$\psi_1(x) = 0.007 e^{-3.161x} + 0.938 e^{-0.065x}, \quad \psi_2(x) = 0.003 e^{-3.161x} + 0.867 e^{-0.065x},$$

which coincides with (26) in Albrecher and Boxma (2004). Moreover, in this case (44) indeed reduces to $\vec{C} = (0.938, 0.867)^T$.

Let us now choose $w(x_1, x_2)$ as the Dirac delta function with respect to $x_1 = y_1, x_2 = y_2$ (i.e. $\tilde{\omega}_i(s) = e^{-s y_1} b_i(y_1 + y_2)$). Then $m_{0,i}(x)$ is the (defective) joint density function

$f_i(y_1, y_2|x)$ of the surplus prior to ruin and the deficit at ruin. From (43) we obtain $\vec{m}_0(0)$ and then from (7) and inversion of the Laplace transform

$$\vec{f}(y_1, y_2|x) = e^{-y_2} \vec{f}(y_1|x)$$

with

$$\begin{aligned} \vec{f}(y_1|x) = & 1_{\{x \leq y_1\}} \begin{pmatrix} 9 \\ 9 \end{pmatrix} e^{-y_1} + e^{-3.161x} \left(\begin{pmatrix} 0.106 \\ 0.045 \end{pmatrix} e^{-2.226y_1} - \begin{pmatrix} 0.061 \\ 0.026 \end{pmatrix} e^{-y_1} \right) \\ & + e^{-0.065x} \left(\begin{pmatrix} -0.574 \\ -0.531 \end{pmatrix} e^{-2.226y_1} - \begin{pmatrix} 8.446 \\ 7.802 \end{pmatrix} e^{-y_1} \right) + 1_{\{x \leq y_1\}} e^{1.226x - 2.226y_1} \begin{pmatrix} 1.476 \\ -0.186 \end{pmatrix} \\ & + 1_{\{x \geq y_1\}} \left(\begin{pmatrix} 9.020 \\ 8.333 \end{pmatrix} e^{-0.0645x - 0.935y_1} - \begin{pmatrix} 0.045 \\ 0.019 \end{pmatrix} e^{-3.161x + 2.161y_1} \right) \end{aligned}$$

Note that due to the lack-of-memory property of the exponential distribution, the distribution of the deficit at ruin is again exponential and independent of the surplus before ruin. Alternatively, one could have derived the above formula using $w = e^{-ax_1}$ (so that $m(x)$ is the Laplace transform of the surplus before ruin) and then, by virtue of (43) and (7), inverting iteratively the Laplace transform with respect to s and with respect to a . The moments of the surplus before ruin can now be determined either from the density above or by differentiating the above mentioned Laplace transform of the surplus before ruin. For instance,

$$\begin{aligned} \mathbb{E}(R_{T_x}^- 1_{\{T_x < \infty\}}) &= \begin{pmatrix} 1.746 \\ 1.613 \end{pmatrix} e^{-0.065x} - \begin{pmatrix} 0.050 \\ 0.021 \end{pmatrix} e^{-3.161x} - \begin{pmatrix} 1 \\ 0.556 \end{pmatrix} e^{-x} \\ \mathbb{E}((R_{T_x}^-)^2 1_{\{T_x < \infty\}}) &= \begin{pmatrix} 5.041 \\ 4.657 \end{pmatrix} e^{-0.065x} - \begin{pmatrix} 0.095 \\ 0.040 \end{pmatrix} e^{-3.161x} - \begin{pmatrix} 2x + 3.778 \\ 1.111x + 2.395 \end{pmatrix} e^{-x}. \end{aligned}$$

In Figure 1 the density function $f_i(y_1|x)/\psi_i(x)$ of the surplus prior to ruin, given it occurs, is plotted for two specific values of initial capital x . Figure 2 depicts both the expected value

$$\mathbb{E}(R_{T_x}^- | T_x < \infty, Z_0 = i) = \frac{\mathbb{E}(R_{T_x}^- 1_{\{T_x < \infty\}} | Z_0 = i)}{\psi_i(x)}$$

and the standard deviation

$$SD_{R_{T_x}^-} = \sqrt{\mathbb{E}((R_{T_x}^-)^2 | T_x < \infty, Z_0 = i) - \mathbb{E}^2(R_{T_x}^- | T_x < \infty, Z_0 = i)}$$

as a function of initial capital x . Note that from the analytic expressions above, we see that $\lim_{x \rightarrow \infty} \mathbb{E}(R_{T_x}^- | T_x < \infty) = 1.86$ and $\lim_{x \rightarrow \infty} SD_{R_{T_x}^-} = 2.32$ (this holds for both $Z_0 = 1, 2$).

Finally, we briefly illustrate how to use the procedure of Section 5.1 to obtain moments of the time to ruin. From (17), we have for $n = 1$

$$A_0(s) \vec{f}_1(s) = c \frac{\partial \vec{m}_\delta(0)}{\partial \delta} \Big|_{\delta=0} + \vec{\psi}(s). \quad (45)$$

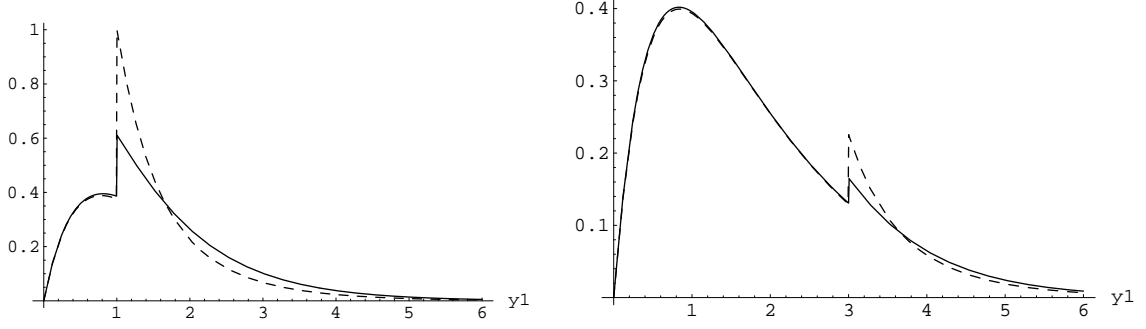


Figure 1: Density function of the surplus prior to ruin, given it occurs, for $x = 1$ (left) and $x = 3$ (right) (initial state $Z_0 = 1$ (dashed line) and $Z_0 = 2$ (solid line)).

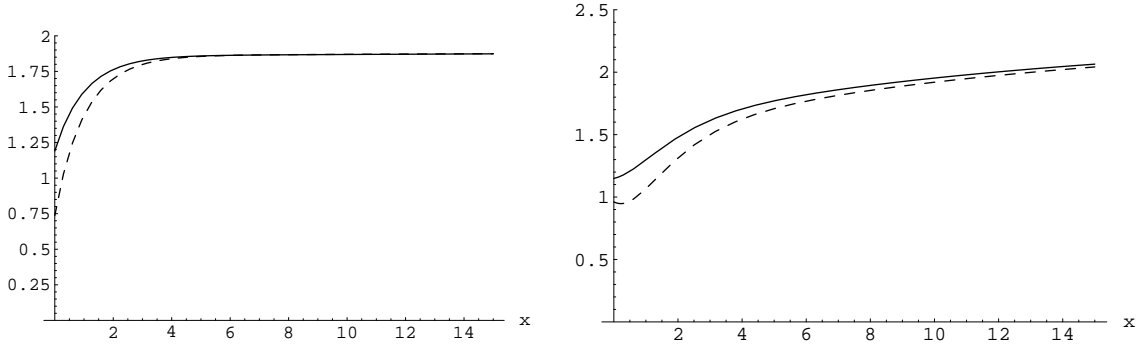


Figure 2: Expected value (left) and standard deviation (right) of the surplus prior to ruin, given it occurs (initial state $Z_0 = 1$ (dashed line) and $Z_0 = 2$ (solid line)).

By analyticity of $\vec{f}_1(s)$ in the right half-plane we thus obtain $c \frac{\partial \vec{m}_\delta(0)}{\partial \delta} \Big|_{\delta=0} = - \begin{pmatrix} 7.949 \\ 17.841 \end{pmatrix}$ and thus, after solving (45) and inverting the Laplace transform, we obtain

$$\mathbb{E}(T_x 1_{\{T_x < \infty\}}) = \begin{pmatrix} 4.330x + 4.431 \\ 4x + 9.114 \end{pmatrix} e^{-0.065x} - \begin{pmatrix} 0.457 \\ 0.193 \end{pmatrix} e^{-3.161x}$$

Analogously, from (17),

$$\mathbb{E}(T_x^2 1_{\{T_x < \infty\}}) = \begin{pmatrix} 19.980x^2 + 711.096x + 681.816 \\ 18.458x^2 + 703.242x + 1469.25 \end{pmatrix} e^{-0.065x} - \begin{pmatrix} 75.458 \\ 32.806 \end{pmatrix} e^{-3.161x}.$$

Figure 3 depicts the expected value $\mathbb{E}(T_x | T_x < \infty, Z_0 = i) = \frac{\mathbb{E}(T_x 1_{\{T_x < \infty\}} | Z_0 = i)}{\psi_i(x)}$ and the standard deviation of the time to ruin, given it occurs, as a function of initial capital x . One observes that the standard deviation of the time to ruin exceeds the expected value, so that in this dependency model it is particularly dangerous to just consider the first moment as an indicator for the riskiness of the portfolio strategy.

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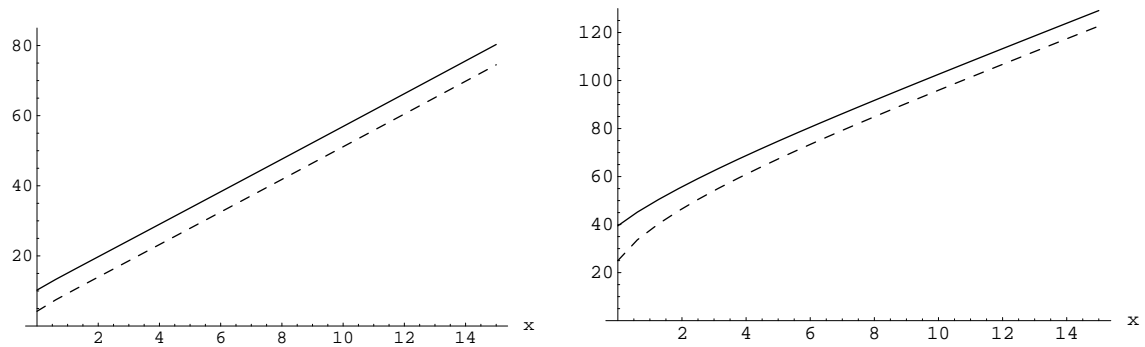


Figure 3: Expected value (left) and standard deviation (right) of the time to ruin, given it occurs (initial state $Z_0 = 1$ (dashed line) and $Z_0 = 2$ (solid line)).

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