DIVIDEND CORRIDORS AND A RUIN CONSTRAINT

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Abstract. We propose a new class of dividend payment strategies for which one can easily control an infinite-time-horizon ruin probability constraint for an insurance company. When the risk process evolves as a spectrally negative Lévy process, we investigate analytical properties of these strategies and propose two numerical methods for finding explicit expressions for the optimal parameters. Numerical experiments show that the performance of these strategies is outstanding and, in some cases, even comparable to the overall-unconstrained optimal dividend strategy to maximize expected aggregate discounted dividend payments, despite the ruin constraint.

1. Introduction

Consider an insurance company whose surplus process evolves according to a spectrally-negative Lévy process. We assume that this process satisfies the safety loading condition, reflecting the idea that, in expectation, the company charges more premiums than the amount of claims to be paid. Under this assumption, however, the process also possesses the unrealistic property that, with probability one, it will diverge to infinity. One way to avoid this issue is to consider dividend payments to shareholders. Since the introduction of this idea in the seminal work of De Finetti (1957), there has been a lot of research activity on establishing optimal strategies for distributing dividends under various objective functions and constraints. For instance, it was established that for the maximization of the expected sum of discounted dividend payments until ruin a *band strategy* is often optimal (see e.g., Gerber (1969), Loeffen (2008), Azcue and Muler (2005) and Avram et al. (2015) as well as Avanzi (2009) and Albrecher and Thonhauser (2009) for surveys). While band strategies maximize expected aggregate dividend payments for a rather general set of assumptions, they also lead to the undesirable property that, with probability one, the surplus process will eventually become negative, i.e., the company will get ruined. Hence, while the introduction of dividend payments makes the model more realistic, the optimal solution is typically unacceptable in practice. In response to this, a growing body of literature examined the trade-off between profitability and safety (avoiding or delaying ruin). Thonhauser and Albrecher (2007) and Loeffen and Renaud (2010) examined the dividend problem with a penalty for early ruin, see also Liang and Young (2012). For a discrete-time model, Hipp (2003) was the first to approach the optimal dividend problem under a ruin constraint, which turns out challenging in view of the resulting *time-inconsistency* of the stochastic control problem. Grandits (2015) studied the problem of optimizing dividend payments in finite time with a constraint on the probability of ruin for the case of the diffusion, providing a solution in terms of a complicated set of differential equations. Similarly, Hernandez et al. (2018) provided a solution to the case where the constraint is a bound on the Laplace transform of the time of ruin, under the assumption that the density of the Lévy measure is completely monotone. Recently, Strini and Thonhauser (2023) used a game-theoretic approach to reformulate the idea of optimality and provided a solution to the case of the diffusion.

Rather than directly addressing the control problem of maximizing expected discounted dividend payments under a ruin constraint whose general solution seems out of reach, Hipp (2018, 2020) started to study particular candidate strategies with an intuitive structure that allow a balancing of profitability and safety with a bottom-up approach, see also Hipp (2019). The contribution of the present paper is a considerable extension and deepening of the latter approach. We define a sort of corridor dividend strategies for which, using scale functions and fluctuation theory, the ruin probability can be easily controlled and at the same time the expected dividend payments can be maximized locally. While we do not prove optimality of such strategies for the general dividend problem under a ruin constraint, the numerical illustrations at the end of the manuscript show that this kind of strategies perform exceptionally well, sometimes even leading to comparable efficiency to the best overall un-constrained (band) strategy, but respecting the pre-given ruin constraint. In order to make the numerical optimization of the involved parameters work, we implement and adapt two numerical schemes to the present problem: a recursive approach inspired by Newtonian optimization techniques and an evolutionary algorithm. The resulting optimized strategies can serve as new benchmarks for both intuition and numerics for the general problem of maximizing dividends under a ruin constraint. For instance, when we apply corridor strategies to the diffusion case and finite time horizon problem studied in Grandits (2015) and pay the remaining surplus as a final dividend lump sum at the end of the time horizon as done in that paper, the optimal corridor strategy in fact outperforms the numerical solution given in Grandits (2015) for the same problem, cf. Section 6. We also show that linear barrier strategies whose slope and intercept is optimized for a given initial capital level perform surprisingly well, and for certain model parameter values outperform the corridor strategies of this paper by a small margin, but for other parameter values are strongly dominated by our corridor strategies. An attractive feature of corridor strategies is indeed that they are broadly applicable across models and represent very competitive performance across the entire parameter range.

The rest of the paper is organized as follows: in Section 2 we establish the basic assumptions for the surplus model and introduce some notation. Section 3 introduces the corridor payment strategies and derives formulas to compute the value associated with them. Section 4 examines analytical properties of the value function associated with the strategies. Since the value function eventually needs to be evaluated and optimal parameters need to be determined, Section 5 introduces several numerical techniques that can be used for this purpose. Section 6 presents the numerical results for a selected number of surplus processes commonly studied in the literature. Finally, Section 7 concludes and provides some directions for future research.

2. The model

Consider a spectrally negative Lévy risk process $(C_t)_{t\geq 0}$ for the surplus process of an insurance portfolio with initial surplus level $C_0 = u$. In the following we will formulate the results first for the general case and then go into more detail for two special cases of interest, namely the case of a diffusion approximation

$$C_t = u + \mu t + \sigma B_t, \quad t \geq 0, \tag{1}$$

where $\mu > 0$ is a constant drift, $\sigma > 0$ and $(B_t)_{t\geq 0}$ denotes a standard Brownian motion, and the Cramér-Lundberg process

$$C_t = u + c\,t - \sum_{i=1}^{N_t} X_i, \quad t \geq 0, \tag{2}$$

where $(N_t)_{t\geq 0}$ is a homogeneous Poisson process with rate $\lambda > 0$, X_i are the individual claim sizes modelled by i.i.d. random variables with cumulative distribution function F_X and finite mean, and $c > \lambda E(X_i)$ is the premium collected per time unit. Denote by $\tau := \inf\{t \geq 0 : C_t < 0\}$ the time of the ruin, by $\psi(u) := \mathbb{P}(\tau < \infty)$ the ruin probability of this risk process and by

$$\phi(u) = 1 - \psi(u)$$

the corresponding survival probability. Let $\kappa(\theta) := \log \mathbb{E}e^{\theta(C_1 - C_0)}$ denote the Laplace exponent of the Lévy process, which has the form

$$\kappa(\theta) = -a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} (e^{\theta x} - 1 - \theta x 1_{\{x>-1\}})\Pi(dx)$$

with Π the Lévy measure and $\Phi_\delta = \psi^{-1}(\delta) > 0$ (see for instance Kyprianou (2014)). The various results obtained throughout the manuscript will be expressed in terms of the scale function of C, which for $x \geq 0$ and any $\delta \geq 0$ is defined as the function $W_\delta(x)$ satisfying the identity

$$\int_0^\infty e^{-\theta x} W_\delta(x)\, dx = \frac{1}{\kappa(\theta) - \delta}, \quad \theta > \Phi_\delta.$$

We also define $W_\delta(x) = 0$ whenever $x < 0$. Assuming that $\kappa'(0)$ is finite, it is then well-known that

$$\phi(u) = \kappa'(0) W_0(u)$$

(see e.g. Kyprianou (2014) or (Asmussen and Albrecher, 2010, Ch.IX)). For the sake of convenience, we will assume that Π accepts a continuous density, so that $\Pi(dx) = f_\Pi(x)dx$ and that this density is sufficiently smooth to ensure that $W_\delta \in C^2(0, \infty)$.

Dividends are now paid out according to a strategy $D = (D_t)_{t\geq 0}$, where D_t represents the aggregate dividends up to time t. The surplus process after dividends is given by

$$C_t^D = C_t - D_t$$

and the expected value of the aggregate discounted dividend payments until ruin are given by

$$V^D(u) = \mathbb{E}\left(\int_0^{\tau^D} e^{-\delta t} dD_t\right)$$

where

$$\tau^D := \inf\{t > 0 : C_t^D < 0\}$$

is the time of the ruin of the resulting surplus process with dividends.

Consider now the following dividend payment strategy: For a fixed $n \in \mathbb{N}$, there is a sequence of surplus levels $a_1, a_2, a_3, \ldots, a_n$. Assume for the moment that $u < a_1$. When the risk process reaches a_i $(i = 1, \ldots, n)$ for the first time (which we denote by τ_i), there is a lump sum dividend payment of $a_i - b_i$ down to a barrier level b_i. Then continuous dividend payments start according to a horizontal barrier strategy with barrier b_i until the surplus process goes below the lower limit $l_i \leq b_i$ for the first time (denoted by $\tau_i^d \geq \tau_i$), at which point the barrier in b_i is dissolved. Dividend payments only continue later in case the surplus process reaches the level $a_{i+1} > l_i$ before ruin, which happens at time τ_{i+1}, given that $\tau_{i+1} < \tau^D$. In that case the next lump sum $a_{i+1} - b_{i+1}$ is paid, followed by dividends according to a horizontal barrier strategy at b_{i+1} until the process goes below l_{i+1} etc. Once the last dividend barrier b_n is dissolved, the surplus process survives according to the classical survival probability (without dividends) with initial surplus level $C^D_{\tau^d_n}$. Note that this formulation of the strategy includes the case of a pure lump sum payment (in case $b_i = l_i$ and infinite variation) as well as the case of a pure 'horizontal dividend corridor' without a lump sum payment at the beginning ($a_i = b_i$), and we will be looking for the optimal values of $0 \le l_i \le b_i \le a_i$, $i = 1, \dots, n$. Figure 1 depicts a sample path of such a strategy for a Cramér-Lundberg process, in which ruin occurs before a_3 is reached.

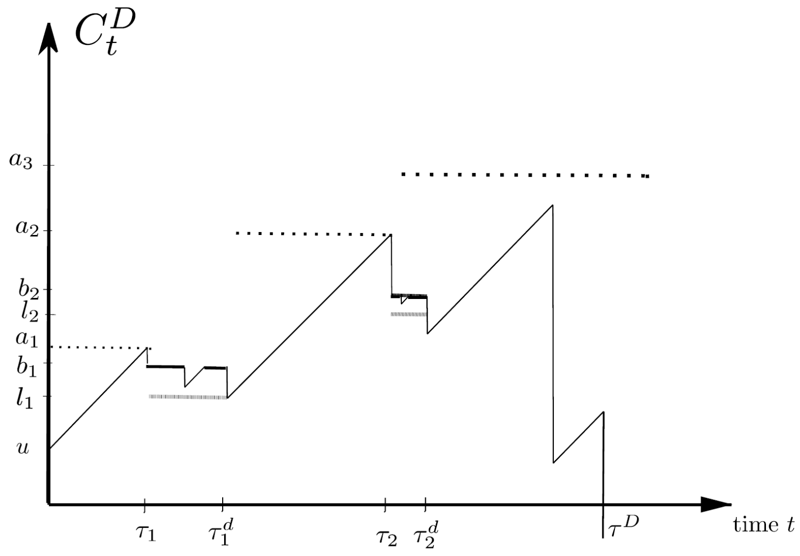


Figure 1. A sample path for a Cramér-Lundberg process with a dividend strategy as described above

For a fixed set of levels, we may allow $u > a_1$ by making a lump-sum payment down to b_1 and proceed as described above.

3. Some results

Denote by $\phi^D(u) := \mathbb{P}(\tau^D = \infty)$ the survival probability of the resulting risk process, and recall that $\phi(u)$ is the classical survival probability of the risk process without any dividend payments.

Theorem 3.1. *For $u \le a_1$, we have*

$$\phi^D(u) = \kappa'(0)\, W_0(u) \prod_{k=1}^{n} A(a_k, b_k, l_k) \tag{3}$$

with

$$A(a_k, b_k, l_k) := \frac{\mathbb{E}(W_0(C^D_{\tau^d_k}))}{W_0(a_k)}.$$

Note that $C^D_{\tau^d_k}$ is the surplus value at the time of the first undershoot of level l_k after paying dividends at barrier b_k, which occurs at the stopping time τ^d_k. We naturally have $\phi(x) = 0$ for $x < 0$, i.e. if the undershoot at the time of dissolving the k-th corridor leads to a negative surplus value, the company is ruined.

Proof. A simple iterative application of the strong Markov property of C gives

$$\phi^D(u) = \frac{\phi(u)}{\phi(a_1)} \left(\prod_{k=1}^{n-1} \mathbb{P}(\tau_{k+1} < \tau^D | \tau_k < \tau^D) \right) \mathbb{E}(\phi(C^D_{\tau^d_n})),$$

which, using the strong Markov property again, can also be expressed as

$$\phi^D(u) = \frac{\phi(u)}{\phi(a_1)} \left(\prod_{k=1}^{n-1} \frac{\mathbb{E}(\phi(C^D_{\tau^d_k}))}{\phi(a_{k+1})} \right) \mathbb{E}(\phi(C^D_{\tau^d_n}))$$
$$= \phi(u) \prod_{k=1}^{n} \frac{\mathbb{E}(\phi(C^D_{\tau^d_k}))}{\phi(a_k)}.$$

Note that $\phi(u)/\phi(a_1) = W_0(u)/W_0(a_1)$ is the probability that the surplus process C_t reaches surplus level a_1 before ruin, when starting at a lower surplus level $u < a_1$. We can hence rewrite the above expression with scale functions as

$$\phi^D(u) = \kappa'(0)\, W_0(u) \prod_{k=1}^{n} \frac{\mathbb{E}(W_0(C^D_{\tau^d_k}))}{W_0(a_k)},$$

establishing the result. □

While simple, Equation (3) expresses the probability of ruin implicitly in terms of expectations. We would like to obtain more formulas expressions for A, for which we make use of the concept of Gerber-Shiu measures (c.f. (Kyprianou, 2014, Ch.X)). Recall the Gerber-Shiu measure of the process, K^δ, which for any $\omega : \mathbb{R}^2 \to [0, \infty)$ such that $\omega(0, \cdot) = 0$, allows us to write

$$\mathbb{E}(e^{-\delta\tau}\omega(-C_\tau, C_{\tau-})) = \int_{(0,\infty)^2} \omega(y, z) K^\delta(dy, dz) \tag{4}$$

with C_τ being the severity of ruin and $C_{\tau-}$ the surplus just before ruin. An explicit expression for K^δ can be given in terms of the Lévy measure and the scale function of the process, i.e.,

$$K^\delta(dy, dz) = (e^{-\Phi_\delta z} W_\delta(u) - W_\delta(u - z)) f_\Pi(-y - z) dy dz. \tag{5}$$

Similarly, the discounted probability of ruin by creeping can be computed through the formula

$$\mathbb{E}(e^{-\delta\tau} 1_{\{C_\tau = C_{\tau-} = 0\}}) = \frac{\sigma^2}{2} \left(W'_\delta(u) - \Phi_\delta W_\delta(u) \right), \tag{6}$$

where the right-hand side is understood as zero whenever $\sigma = 0$. Expectations of discounted penalties of the form $\psi(u) := \mathbb{E}(e^{-\delta\tau} g(C_\tau))$ can therefore be evaluated through equations (5) and (6) for any function g. This is almost what we need, however, to obtain an explicit form for A, we need to compute this expectation assuming dividends have been paid according to a barrier strategy. Denoting by C^b_τ the severity of ruin after dividends have been paid according to a barrier strategy at level b, this means that we require to compute expectations of the form $\psi(u; b) := \mathbb{E}(e^{-\delta\tau} g(C^b_\tau))$. Luckily, this can be easily computed through the dividends-penalty identity (cf. Gerber et al. (2006)),

$$\psi(u; b) = \psi(u) - \frac{W_\delta(u)}{W'_\delta(b)} \psi'(b). \tag{7}$$

Using all these equations, we can reach an explicit expression for the function $A(a, b, l)$.

Proposition 3.2. *The function $A(a, b, l)$ can be written as*

$$A(a, b, l) = \frac{\sigma^2}{2} \frac{W_0(l)}{W_0(a)} \left(W'_0(b - l) - \frac{W_0(b - l) W''_0(b - l)}{W'_0(b - l)} \right)$$
$$+ \int_0^l \int_0^\infty \frac{W_0(l - y)}{W_0(a)} \left(\frac{W_0(b - l) W'_0(b - l - z)}{W'_0(b - l)} - W_0(b - l - z) \right) f_\Pi(-y - z) dz dy.$$

Proof. Observe that $C^D_{\tau_k^d} - l_k$ is equal in distribution to the severity of ruin after dividends have been paid according to a barrier strategy at level $b_k - l_k$ of a process with initial capital $b_k - l_k$, so Equations (5), (6) and (7) fully characterize its distribution. Hence, by setting $g = W_0$ in the definition of ψ and combining (5), (6) and (7) one obtains

$$A(a,b,l) = \frac{\sigma^2}{2}\frac{W_0(l)}{W_0(a)}\left(W_0'(b-l) - \frac{W_0(b-l)W_0''(b-l)}{W_0'(b-l)}\right) + \int_0^l\int_0^\infty \frac{W_0(l-y)}{W_0(a)}\left(\frac{W_0(b-l)W_0'(b-l-z)}{W_0'(b-l)} - W_0(b-l-z)\right) f_\Pi(-y-z)dzdy,$$

where we have omitted the dependence on the index k for simplicity of exposition. □

Let us now turn to the expected value of the sum of the discounted dividend payments. Recall from classical risk theory that the expected discounted dividend payments according to a horizontal dividend barrier strategy at b when starting at an initial surplus level $u < b$ is simply given by $W_\delta(u)/W_\delta'(b)$. This quantity will be a building block of our more complex dividend payment strategy in this paper.

Theorem 3.3. *For $u \le a_1$, the value function $V^D(u)$ of the expected discounted dividend payments can be written as*

$$V^D(u) = W_\delta(u)\sum_{k=1}^n B(a_k, b_k, l_k)\prod_{i=1}^{k-1} G(a_i, b_i, l_i) \tag{8}$$

with

$$B(a,b,l) = \frac{a - b + W_\delta(b-l)/W_\delta'(b-l)}{W_\delta(a)}$$

and

$$G(a,b,l) = \frac{\sigma^2}{2}\frac{W_\delta(l)}{W_\delta(a)}\left(W_\delta'(b-l) - \frac{W_\delta(b-l)W_\delta''(b-l)}{W_\delta'(b-l)}\right) + \int_0^l\int_0^\infty \frac{W_\delta(l-y)}{W_\delta(a)}\left(\frac{W_\delta(b-l)W_\delta'(b-l-z)}{W_\delta'(b-l)} - W_\delta(b-l-z)\right) f_\Pi(-y-z)dzdy.$$

Proof. Consider the scenario in which the $(k-1)$-th corridor has just been dissolved. Once we reach a_k, there will be a lump sum payment $a_k - b_k$ and then dividend payments will start according to a horizontal barrier strategy at barrier b_k until the k-th corridor is dissolved. Denote by D_k the expected present value of all dividend payments at the k-th corridor. By construction of the dividend payment strategy, D_k can also be viewed as the expected discounted dividends according to a barrier strategy collected until ruin in a risk model with initial surplus level $b_k - l_k$ and also barrier level $b_k - l_k$, since the event of ruin in that model will exactly correspond to C^D undershooting l_k for the first time. Correspondingly, $D_k = W_\delta(b_k - l_k)/W_\delta'(b_k - l_k)$. Using the strong Markov property, we can write

$$V^D(u) = \frac{W_\delta(u)}{W_\delta(a_1)}\Big[(a_1 - b_1) + D_1 + \mathbb{E}(e^{-\delta(\tau_2-\tau_1)}1_{\{\tau_2<\tau^D\}})\Big[(a_2 - b_2) + D_2 + \mathbb{E}(e^{-\delta(\tau_3-\tau_2)}1_{\{\tau_3<\tau^D\}})\Big[(a_3 - b_3) + D_3 + \cdots\Big]\Big]\Big], \tag{9}$$

so that we obtain

$$V^D(u) = \frac{W_\delta(u)}{W_\delta(a_1)}\sum_{k=1}^n \left(a_k - b_k + \frac{W_\delta(b_k - l_k)}{W_\delta'(b_k - l_k)}\right)\prod_{i=2}^k \mathbb{E}(e^{-\delta(\tau_i-\tau_{i-1})}1_{\{\tau_i<\tau^D\}}), \tag{10}$$

with the usual convention $\prod_{i=2}^{1} \cdot = 1$. By the strong Markov property, $\tau_{i-1}^d - \tau_{i-1}$ can be seen as the time to ruin of a process with initial surplus $b_i - l_i$ which pays dividends according to a barrier strategy at level $b_i - l_i$. Moreover, on $1_{\{\tau_i < \tau^D\}}$ and given $C^D_{\tau^d_{i-1}}$, $\tau_{i-1}^d - \tau_{i-1}$ is independent of $\tau_i - \tau_{i-1}^d$, and due to

$$\mathbb{E}\left(e^{-\delta(\tau_i - \tau_{i-1}^d)} 1_{\{\tau_i < \tau^D\}} \mid C^D_{\tau^d_{i-1}}\right) = \frac{W_\delta\left(C^D_{\tau^d_{i-1}}\right)}{W_\delta(a_i)},$$

we obtain

$$\mathbb{E}(e^{-\delta(\tau_i - \tau_{i-1})} 1_{\{\tau_i < \tau^D\}}) = \mathbb{E}\left(e^{-\delta(\tau_{i-1}^d - \tau_{i-1})} \frac{W_\delta\left(C^D_{\tau^d_{i-1}}\right)}{W_\delta(a_i)} 1_{\{\tau_i < \tau^D\}}\right).$$

Hence, (10) can be rewritten as

$$V^D(u) = W_\delta(u) \sum_{k=1}^{n} B(a_k, b_k, l_k) \prod_{i=1}^{k-1} G(a_i, b_i, l_i)$$

with

$$B(a, b, l) = \frac{a - b + W_\delta(b-l)/W_\delta'(b-l)}{W_\delta(a)}$$

and

$$G(a_i, b_i, l_i) = \frac{1}{W_\delta(a_i)} \mathbb{E}\left(e^{-\delta(\tau_i^d - \tau_i)} W_\delta\left(C^D_{\tau^d_i}\right) 1_{\{\tau_{i+1} < \tau^D\}}\right).$$

As in the computation of the explicit formula for A, we can use the Gerber-Shiu measure and the dividends-penalty identity to give an explicit form for G, thus obtaining

$$\begin{aligned} G(a, b, l) = & \frac{\sigma^2}{2} \frac{W_\delta(l)}{W_\delta(a)} \left(W_\delta'(b-l) - \frac{W_\delta(b-l) W_\delta''(b-l)}{W_\delta'(b-l)}\right) \\ & + \int_0^l \int_0^\infty \frac{W_\delta(l-y)}{W_\delta(a)} \left(\frac{W_\delta(b-l) W_\delta'(b-l-z)}{W_\delta'(b-l)} - W_\delta(b-l-z)\right) f_\Pi(-y-z) dz dy. \end{aligned}$$

as desired. □

A few comments are in order: Equation (8) shows that we can express V^D in a recursive way as follows: define the sequence $c_1^D, \ldots, c_n^D$ by

$$c_n^D = B(a_n, b_n, l_n)$$

and, for $1 \le j \le n-1$,

$$c_j^D = B(a_j, b_j, l_j) + c_{j+1}^D G(a_j, b_j, l_j).$$

With these definitions, we have, for any $1 \le j \le n-1$,

$$V^D(u) = W_\delta(u) \sum_{k=1}^{j-1} B(a_k, b_k, l_k) \prod_{i=1}^{k-1} G(a_i, b_i, l_i) + W_\delta(u) c_j^D \prod_{i=1}^{j-1} G(a_i, b_i, l_i). \tag{11}$$

In particular, $V^D(u) = W_\delta(u) c_1^D$. The advantage of defining the c_{j+1}^D's in this way is that, once c_{j+1}^D is known, c_j^D depends only on a_j, b_j and l_j, a fact that can be exploited in a constrained optimization setting (cf. Section 6).

Now, while the expressions for A and G are explicit, they are rather complicated. One reason for this is found by examining the proof of the previous theorem. Observe that, while computing G, one encounters the expectation $\mathbb{E}(e^{-\delta(\tau_i-\tau_{i-1})}1_{\{\tau_i<\tau^D\}})$. Rewriting this as

$$\mathbb{E}(e^{-\delta(\tau_i-\tau_{i-1})}1_{\{\tau_i<\tau^D\}}) = \mathbb{E}(e^{-\delta(\tau^d_{i-1}-\tau_{i-1})} \cdot e^{-\delta(\tau_i-\tau^d_{i-1})}1_{\{\tau_i<\tau^D\}}), \tag{12}$$

we see that a reason for the involved expressions is that, in general, the two product terms in (12) are not independent. Exceptions are the diffusion case and the Cramér-Lundberg model with exponential claims, where a decomposition into a product of two expectations is feasible, which significantly simplifies the analysis. To that end, note that for general spectrally negative Lévy processes, a formula for the joint Laplace transform of $\tau_k^d - \tau_k$ and $C^D_{\tau_k^d}$ for any $k = 1, \ldots, n$ is available:

$$\mathbb{E}(e^{-\delta(\tau_k^d-\tau_k)-\theta(l_k-C^D_{\tau_k^d})}) = Z_\delta(b_k-l_k,\theta) + W_\delta(b_k-l_k)\,\frac{W_\delta(b_k-l_k)\,(\kappa(\theta)-\delta) - \theta Z_\delta(b_k-l_k,\theta)}{W'_\delta(b_k-l_k)}, \tag{13}$$

where $Z_\delta(x,\theta)$ denotes the second scale function defined by

$$Z_\delta(x,\theta) = e^{-\theta x}\left(1-(\kappa(\theta)-\delta)\int_0^x e^{-\theta y}W_\delta(y)dy\right), \quad x \geq 0$$

(see Ivanovs and Palmowski (2012)). For $\theta = 0$, one obtains the familiar simpler version

$$Z_\delta(x,0) = 1 + \delta\int_0^x W_\delta(y)dy, \quad x \geq 0,$$

which was for instance used in (Kyprianou, 2014, Ch.8.2). Formula (13) originally goes back to Avram et al. (2004). For the concrete form used here, see (Albrecher et al., 2016, Eq.25). In particular,

$$\begin{aligned} \mathbb{E}(e^{-\delta(\tau_k^d-\tau_k)}) &= Z_\delta(b_k-l_k,0) - \delta\,\frac{(W_\delta(b_k-l_k))^2}{W'_\delta(b_k-l_k)} \\ &= 1 + \delta\int_0^{b_k-l_k} W_\delta(y)dy - \delta\,\frac{(W_\delta(b_k-l_k))^2}{W'_\delta(b_k-l_k)}, \end{aligned} \tag{14}$$

which can now help to simplify the form of G, see below.
The formulas presented so far assume $u \leq a_1$. However, by the description of the strategy given at the end of Section 2, in the case $u > a_1$, one can simply replace a_1 by u in (3) and (8) to obtain the formulas for ϕ^D and V^D.

3.1. **The diffusion case.**

Let us now look at the special case of a diffusion approximation

$$C_t = u + \mu t + \sigma B_t, \quad t \geq 0$$

in more detail, where $\mu > 0$ is a constant drift, $\sigma > 0$ is the volatility and $(B_t)_{t\geq 0}$ denotes a standard Brownian motion. In this case $C^D_{\tau_k^d} = l_k$ (deterministically), so that (3) simplifies to

$$\phi^D(u) = \kappa'(0)\,W_0(u)\prod_{k=1}^n \frac{W_0(l_k)}{W_0(a_k)}. \tag{15}$$

It is well-known that the Laplace exponent for this diffusion case is simply given by

$$\kappa(\theta) = \theta\mu + \frac{1}{2}\theta^2\sigma^2,$$

and correspondingly the (first) scale function is

$$W_\delta(x) = \frac{1}{\sqrt{\mu^2+2\delta\sigma^2}}\,(e^{\theta_1 x} - e^{\theta_2 x}), \quad x \geq 0, \tag{16}$$

where $\theta_1 \geq 0$ and $\theta_2 < 0$ are the two roots of the quadratic equation

$$\frac{1}{2}\sigma^2 z^2 + \mu z - \delta = 0. \tag{17}$$

See e.g. Kyprianou (2014) for details. With the resulting

$$W_0(u) = (1 - e^{-(2\mu/\sigma^2)u})/\mu$$

and $\kappa'(0) = \mu$ we hence obtain the survival probability

$$\phi^D(u) = (1 - e^{-(2\mu/\sigma^2)u}) \prod_{k=1}^{n} \frac{1 - e^{-(2\mu/\sigma^2)l_k}}{1 - e^{-(2\mu/\sigma^2)a_k}}. \tag{18}$$

For numerical purposes later on, we note that in view of (3), in the diffusion case

$$A(a,l) = \frac{1 - e^{-(2\mu/\sigma^2)l}}{1 - e^{-(2\mu/\sigma^2)a}}$$

(note that $A(a,b,l)$ does not depend on b here, so that we suppress it in the notation). For the expected discounted dividends, we note that we are in one of the exceptions where we can factor (13) as

$$\mathbb{E}(e^{-\delta(\tau_i-\tau_{i-1})}1_{\{\tau_i<\tau^D\}}) = \mathbb{E}(e^{-\delta(\tau^d_{i-1}-\tau_{i-1})}) \cdot \mathbb{E}(e^{-\delta(\tau_i-\tau^d_{i-1})}1_{\{\tau_i<\tau^D\}}),$$

which helps seeing that G and B are given by

$$\begin{aligned} G(a,b,l) &= \frac{\sigma^2}{2}\frac{W_\delta(l)}{W_\delta(a)}\left(W'_\delta(b-l) - \frac{W_\delta(b-l)W''_\delta(b-l)}{W'_\delta(b-l)}\right), \\ B(a,b,l) &= \frac{a - b + W_\delta(b-l)/W'_\delta(b-l)}{W_\delta(a)}. \end{aligned}$$

Equivalently, combining (14) and (16) we obtain after algebraic manipulations,

$$G(a,b,l) = \frac{(\theta_1-\theta_2)(e^{\theta_1 l} - e^{\theta_2 l})e^{(\theta_1+\theta_2)b}}{(e^{\theta_1 a} - e^{\theta_2 a})(\theta_1 e^{\theta_1 b+\theta_2 l} - \theta_2 e^{\theta_2 b+\theta_1 l})},$$

and

$$B(a,b,l) = \sqrt{\mu^2 + 2\delta\sigma^2}\,\frac{a - b + (e^{\theta_1(b-l)} - e^{\theta_2(b-l)})/(\theta_1 e^{\theta_1(b-l)} - \theta_2 e^{\theta_2(b-l)})}{e^{\theta_1 a} - e^{\theta_2 a}}.$$

3.2. **The Cramér-Lundberg model with exponential claims.** Consider now the Cramér-Lundberg model (2) with a homogeneous Poisson process of intensity $\lambda > 0$ and exponential claims with parameter $\alpha > 0$. In that case, we have

$$\kappa(\theta) = c\theta - \frac{\lambda\theta}{\theta + \alpha}$$

and (under the positive saftely loading condition $c > \lambda/\alpha$) the scale function is given by

$$W_\delta(x) = \frac{(\alpha + \Phi_\delta)e^{\Phi_\delta x} - (\alpha - R_\delta)e^{-R_\delta x}}{c(\Phi_\delta + R_\delta)}, \quad x \geq 0, \tag{19}$$

where $\Phi_\delta \geq 0$ and $-R_\delta < 0$ are the two roots of the quadratic equation

$$c\rho^2 + (c\alpha - \lambda - \delta)\rho - \alpha\delta = 0, \tag{20}$$

see e.g. Albrecher et al. (2016). As an immediate consequence we have

$$W_0(u) = \frac{\alpha - (\alpha - R_0)e^{-R_0 u}}{c\,R_0} = \frac{\alpha - \frac{\lambda}{c}e^{-(\alpha-\lambda/c)u}}{c\alpha - \lambda}, \quad u \geq 0,$$

and with $\kappa'(0) = c - \lambda/\alpha$ the classical formula

$$\phi(u) = 1 - \frac{\lambda}{\alpha c} e^{-(\alpha-\lambda/c)u}.$$

For our dividend model, observe that

$$\begin{aligned}\mathbb{E}(W_0(C^D_{\tau^d_k})) &= \int_0^{l_k} \left(\frac{\alpha - \frac{\lambda}{c} e^{-(\alpha-\lambda/c)(l_k-y)}}{c\alpha - \lambda} \right) \alpha\, e^{-\alpha y}\, dy \\ &= \frac{\alpha - \alpha e^{(\lambda/c-\alpha)l_k}}{c\alpha - \lambda},\end{aligned} \tag{21}$$

which leads to the survival probability

$$\phi^D(u) = \left(1 - \frac{\lambda}{\alpha c} e^{-(\alpha-\lambda/c)u}\right) \prod_{k=1}^{n} \frac{1 - e^{-(\alpha-\lambda/c)l_k}}{1 - \frac{\lambda}{\alpha c} e^{-(\alpha-\lambda/c)a_k}}. \tag{22}$$

For numerical purposes later on, we note that in view of (3) in this case

$$A(a,l) = \frac{1 - e^{-(\alpha-\lambda/c)l}}{1 - \frac{\lambda}{\alpha c} e^{-(\alpha-\lambda/c)a}}.$$

Due to the lack-of-memory property of the exponential distribution, we can again decompose (13) as a product

$$\mathbb{E}(e^{-\delta(\tau_i-\tau_{i-1})} 1_{\{\tau_i<\tau^D\}}) = \mathbb{E}(e^{-\delta(\tau^d_{i-1}-\tau_{i-1})}) \cdot \mathbb{E}(e^{-\delta(\tau_i-\tau^d_{i-1})} 1_{\{\tau_i<\tau^D\}}),$$

and, after some algebra, obtain

$$G(a,b,l) = \frac{\alpha\lambda(\Phi_\delta - R_\delta)\left(e^{l(\alpha+\Phi_\delta)} - e^{l(\alpha+R_\delta)}\right) e^{b(\Phi_\delta+R_\delta)-l(\alpha+\Phi_\delta+R_\delta)}}{c\left(e^{a\Phi_\delta}(\alpha+\Phi_\delta) - e^{aR_\delta}(\alpha+R_\delta)\right)\left(\Phi_\delta(\alpha+\Phi_\delta)e^{\Phi_\delta(b-l)} - R_\delta(\alpha+R_\delta)e^{R_\delta(b-l)}\right)}.$$

In addition,

$$B(a,b,l) = \frac{c(\Phi_\delta+R_\delta)((\alpha+\Phi_\delta)(1+\Phi_\delta(a-b))e^{\Phi_\delta(b-l)} - (\alpha+R_\delta)(1+R_\delta(a-b))e^{R_\delta(b-l)})}{(\Phi_\delta(\alpha+\Phi_\delta)e^{\Phi_\delta(b-l)} + R_\delta(\alpha-R_\delta)e^{-R_\delta(b-l)})((\alpha+\Phi_\delta)e^{\Phi_\delta a} - (\alpha-R_\delta)e^{-R_\delta a})}.$$

4. Properties of the strategy with optimal parameters

Let $\mathscr{D}_n$ denote the family of dividend strategies with n corridors as defined in Section 2. Observe that $\mathscr{D}_n$ can naturally be identified with the set

$$\{(\bar{a},\bar{b},\bar{l}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \mid l_k \le b_k \le a_k,\ \text{and } l_k < a_{k+1},\ k = 1,\dots,n\},$$

where, for convenience, we set $a_{n+1} = \infty$.
For a given initial surplus $u \ge 0$ and survival constraint $0 \le \varphi < 1$, let

$$V^*_{n,\varphi}(u) = \sup\left\{V^D(u) \mid D \in \mathscr{D}_n,\ \phi^D(u) \ge \varphi\right\}, \tag{23}$$

where we define $V^*_{n,\varphi}(u) = 0$ whenever the set is empty, e.g., whenever $\varphi > \phi(u)$. If clear from the context, we will drop the dependence of $V^*_{n,\varphi}$ on φ and simply write V^*_n in (23). In the following, for a strategy $D \in \mathscr{D}_n$ we will denote by $a^D_i, b^D_i, l^D_i, i = 1,\dots,n$ the levels of D.
We immediately observe that, by continuity of the functions involved, $V^*_{n,\varphi}$ is locally bounded and on the set $\varphi < \phi(u)$, $V^*_{n,\varphi}(u)$ is left-continuous as a function of φ for fixed u. Hence, for each $u > 0$, there exists a strategy $D \in \mathscr{D}_n$ such that $V^*_n(u) = V^D(u)$ and $\phi^D(u) = \varphi$, so the inequality on the right-hand side of (23) could be replaced by an equality. However, in terms of continuity, a stronger result holds.

Proposition 4.1. *The mapping* $(u,\varphi)\mapsto V^*_{n,\varphi}(u)$ *is continuous on the set*

$$S=\{(u,\varphi)\in\mathbb{R}^2 \mid u>0,\ 0\le\varphi<\phi(u)\}.$$

Proof. Define $f:\mathbb{R}_+\times\mathscr{D}_n\to[0,1)$ and $F:\mathbb{R}_+\times\mathscr{D}_n\to\mathbb{R}_+\times\mathbb{R}^{3n}$ by $f(u,\bar{a},\bar{b},\bar{l})=\phi^D(u)$ and

$$F(u,\bar{a},\bar{b},\bar{l})=(u,a_1,\ldots,a_{n-1},\bar{b},\bar{l},f(u,\bar{a},\bar{b},\bar{l})).$$

The D in the definition of f is the strategy associated with $(\bar{a},\bar{b},\bar{l})$. Now, recall that W_0 is a continuous bijection, so by Equation (3), it follows that F is a one-to-one continuous open function. Let $T\subset\mathbb{R}_+\times\mathbb{R}^{3n-1}\times[0,1)$ be the image of F and define $\tilde{V}:T\mapsto\mathbb{R}_+$ as $\tilde{V}(u,z,\varphi)=V^D(u)$, where $(u,D)=F^{-1}(u,z,\varphi)$. From Equation (8), we see that $\tilde{V}$ is continuous and, moreover, for $(u,\varphi)\in S$, we have

$$V^*_{n,\varphi}(u)=\sup\left\{\tilde{V}(u,z,\varphi)\mid(u,z,\varphi)\in T\right\}. \tag{24}$$

The set on the right hand side of (24) is not empty as $\phi^D(u)\to\phi(u)$ whenever the levels of the strategy go to infinity. It is then clear from this last equation and the continuity of $\tilde{V}$, that $V^*_{n,\cdot}$ is jointly continuous in (u,φ). □

Next, we see that in terms of the number of bands, it is better to allow as many bands as possible.

Proposition 4.2. *For every* $n\in\mathbb{N}$ *and* $\phi(u)>\varphi$, *we have* $V^*_n(u)\le V^*_{n+1}(u)$.

Remark 4.3. This proposition is necessary since, in general, $\mathscr{D}_n\not\subset\mathscr{D}_{n+1}$. Indeed, in order to see an n-corridor strategy $D\in\mathscr{D}_n$ as an $(n+1)$-corridor strategy, one is required to introduce a new corridor somewhere (equivalent to including the three remaining parameters to go from $\mathbb{R}^{3n}_+$ to $\mathbb{R}^{3(n+1)}_+$) and by doing this, one could end up changing the probability of ruin. Note, however, that for Lévy processes with unbounded variation, one can still identify $\mathscr{D}_n$ with a subset $\mathscr{D}_{n+1}$ by introducing the "empty corridor" at the end (that is, the corridor for which $a_{n+1}=b_{n+1}=l_{n+1}$), so Proposition 4.2 is immediate in this case. In the case of bounded variation, one cannot just add more corridors without care, as adding a new corridor to a previously defined strategy, whether empty or not, strictly decreases the survival probability. So in order to increase the probability to the minimal level, the previous corridors have to be shrunk or "moved vertically up" – thus potentially decreasing the amount of dividends being paid.

Proof. Given $n\in\mathbb{N}$, let $D\in\mathscr{D}_n$ be such that $V^*_n(u)=V_D(u)$. By continuity of the scale functions and its derivatives, for a given $\varepsilon>0$, there exists an $\epsilon>0$ such that $V^*_n(u)\le V^D(u)+\varepsilon$, where D is strategy with surplus levels $a^D_k=a^{D_n}_k, b^D_k=b^{D_n}_k, l^D_k=l^{D_n}_k$, $k=1,\ldots,n-1$ and remaining surplus levels equal to $a^*_n+\epsilon, b^*_n+\epsilon$ and $l^*_n+\epsilon$. Now, since $\phi^D(u)>\phi^{D_n}(u)\ge\phi_{\min}$ and $A(l+1,l+1,l)\to1$ as $l\to\infty$, we can find $\tilde{l}>a^D_n+\epsilon$ such that $A(\tilde{l}+1,\tilde{l}+1,\tilde{l})>\phi^{D_n}(u)/\phi^D(u)$ and $\tilde{l}>l^D_n$. Letting $D'\in\mathscr{D}_{n+1}$ denote the dividend strategy with the same first n corridors equal to those of D and extra corridor composed by the levels $a_{n+1}=b_{n+1}=\tilde{l}+1$ and $l_{n+1}=\tilde{l}$, we clearly have $V^D(u)\le V^{D'}(u)$. Hence, $V^*_n(u)\le V_{n+1}(u)+\varepsilon\le V^*_{n+1}(u)+\varepsilon$ and, in particular, $V^*_n(u)\le V^*_{n+1}(u)+\varepsilon$. Since the value of $V^*_{n+1}(u)$ is independent of $\varepsilon>0$, we can let $\varepsilon\downarrow0$ in the previous inequality, proving the proposition. □

Remark 4.4. The cases considered in Section 6 seem to indicate that the inequality in Proposition 4.2 is in fact strict, so one is in principle always obliged to add more bands to improve the amount of dividends being paid.

Proposition 4.5. *There exists* $\vartheta^*\ge0$ *and* $D\in\mathscr{D}_n$ *such that* $a^D_1=\vartheta^*$ *and* $V^*_n(u)=V^D(u)$ *for all* $u\ge a^*_n$. *In particular,* $V^{*\prime}_n(u)=1$ *for* $u\ge a^*_n$.

In other terms, the meaning of this proposition is that the strategies maximizing the value function can eventually be taken to be "constant", in the sense that for high values of the initial capital, all but their first levels can be taken to be the same.

Proof. Notice that (see, e.g., Equation (3.15) in Avram et al. (2007))

$$\lim_{x\to\infty} \frac{W_\delta(x)}{W_\delta'(x)} = \frac{1}{\Phi(\delta)},$$

so there exists $x^* > 0$ and $c > 0$ such that $W_\delta(x) \le cW_\delta'(x)$ for all $x \ge x^*$.
Let b_1^*, l_1^* and a_k^*, b_k^*, l_k^*, $k = 2, \dots, n$ be a set of points that maximize the value of the function T given by

$$T(l_1, b_1, l_2, b_2, \dots, a_n) = -b_1 + \frac{W_\delta(b_1 - l_1)}{W_\delta'(b_1 - l_1)} + \sum_{k=2}^{n} B(a_k, b_k, l_k) \prod_{j=1}^{k-1} G(a_j, b_j, l_j)$$

subject to the constraints $l_1 \le b_1$, $l_k \le b_k \le a_k$, $l_k \le a_{k+1}$ and

$$H(b_1, l_1) \prod_{k=2}^{n} A(a_k, b_k, l_k) \ge \varphi,$$

where H is given by $H(b, l) = W_0(1)A(1, b, l)$. Let M be this maximal value. We claim that we can take $\vartheta^* = \max(b_1^*, x^*, c - M)$ and D be given by $a_1^D = \vartheta^*$ and remaining levels given by the a_k^*, b_k^*, l_k^* in the same ordering. Indeed, clearly $D \in \mathscr{D}_n$ and by definition $a_1^D = \vartheta^*$. Now, let $u \ge \vartheta^*$ and $D' \in \mathscr{D}_n$ be such that $\phi^{D'}(u) \ge \varphi$. We need to show that $V^{D'}(u) \le V^D(u)$. We can assume at the outset that $u \le a_1^{D'}$, since otherwise we can replace $a_1^{D'}$ by u, obtaining the same value for the strategy. Now, notice that

$$V^{D'}(u) = \frac{W_\delta(u)}{W_\delta(a_1^{D'})}(a_1^{D'} + C)$$

and

$$V^D(u) = u + M,$$

where $C = T(l_1^{D'}, b_1^{D'}, l_2^{D'}, b_2^{D'}, \dots, a_n^{D'})$. Consider the mapping $x \mapsto (x + M)/W_\delta(x)$. This function has derivative given by

$$\frac{W_\delta(x) - W_\delta'(x)(x + M)}{W_\delta(x)^2}.$$

Hence, for $x \ge \vartheta^*$, we have

$$W_\delta(x) - W_\delta'(x)(x + M) \le W_\delta'(x)(c - M - x) \le 0,$$

implying that the mapping is decreasing on $[\vartheta^*, \infty)$. Since $\vartheta^* \le u \le a_1^{D'}$, we obtain

$$V^{D'}(u) = \frac{W_\delta(u)}{W_\delta(a_1^{D'})}(a_1^{D'} + C) \le \frac{W_\delta(u)}{W_\delta(a_1^{D'})}(a_1^{D'} + M) \le \frac{W_\delta(u)}{W_\delta(u)}(u + M) = V^D(u),$$

finishing the proof. □

We observed before that for each $u > 0$ such that $\phi(u) > \varphi$, there exists a strategy D such that $V_n^*(u) = V^D(u)$. However, as stated in the proof of the previous proposition, if in this case we have $a_1^D \le u$, then any other strategy D' which has $b_1^D \le a_1^{D'} \le u$ and the remaining levels the same as D will also satisfy $V_n^*(u) = V^{D'}(u)$ and so for large values of initial surplus, there will not be a unique strategy. Moreover, this might even be the case for $a_1^D > u$. Therefore, in the following, we will rely on the following assumption.

Assumption. For each $u > 0$, there is a unique strategy $D \in \mathscr{D}_n$ satisfying $a_1^D \geq u$, $\phi^D(u) = \varphi$ and $V_n^*(u) = V^D(u)$. This strategy will be denoted by $D_n^*(u)$ and its levels by $a_{n,k}^*, b_{n,k}^*, l_{n,k}^*$.

With this notation, we have the following:

Proposition 4.6. *Assume $V_n^* < V_{n+1}^*$ for every $n \in \mathbb{N}$. Then, the function D_n^* is continuous on $(\phi^{-1}(\varphi), \infty)$.*

Proof. Given $u > \phi^{-1}(\varphi)$, let $(u_m)_{m\geq 0}$ be a sequence in $(\phi^{-1}(\varphi), \infty)$ converging to u. Without loss of generality, we may assume that there exists an $\alpha > 0$ such that $\phi(u-\alpha) > \varphi$, and the entire sequence is contained in an interval $[u-\alpha, u+\alpha]$. In the notation of the proof of Proposition 4.1, let

$$E = ([u-\alpha, u+\alpha] \times \mathbb{R}^{3n}) \cap F^{-1}(\mathbb{R}_+ \times \mathbb{R}^{3n-1} \times \{\varphi\}).$$

Observe that E is simply the set of pairs (v, D), where $v \in [u-\alpha, u+\alpha]$ and D is a strategy such that $\phi^D(v) = \varphi$. The description given in the previous equation simply shows that E is closed and, by construction, the pairs $(u_m, D_n^*(u_m))$ belong to E. We claim that, further than that, there exists an $M > 0$ such that the ball B_M in $\mathbb{R}^{3n+1}$ of radius M centred in the origin contains the pairs $(u_m, D_n^*(u_m))$, $m \geq 0$, thus showing that these pairs are contained in the compact set $K = E \cap B_M$. Arguing by contradiction, suppose this is not the case. Since the u_m's are clearly bounded, there has to exist at least one coordinate of $D_n^*(u_m)$ that is not bounded. By the description of $\mathscr{D}_n$, it follows that there has to exist at least one k such $(a_{n,k}^*(u_m))_{m\geq 0}$ is unbounded. Let

$$J_U = \{k \in \{1, \dots, n\} \mid (a_{n,k}^*(u_m))_{m\geq 0} \text{ is unbounded}\}$$

be the set of indices producing unbounded sequences of the a's and $J_B = \{1, \dots, n\} \setminus J_U$. Observe that if $k \in J_B$, then also $(b_{n,k}^*(u_m))_{m\geq 0}$ and $(l_{n,k}^*(u_m))_{m\geq 0}$ are bounded. By passing to a subsequence if necessary, we can assume that

- If $k \in J_U$, then $a_{n,k}^*(u_m) \to \infty$ as $m \to \infty$.
- If $k \in J_B$, then there exist a_k, b_k and l_k such that $a_{n,k}^*(u_m) \to a_k$, $b_{n,k}^*(u_m) \to b_k$ and $l_{n,k}^*(u_m) \to l_k$.

Now, since

$$B(a_{n,k}^*(u_m), b_{n,k}^*(u_m), l_{n,k}^*(u_m)) \leq \frac{a_{n,k}^*(u_m)}{W_\delta(a_{n,k}^*(u_m))} + \frac{W_\delta(b_{n,k}^*(u_m) - l_{n,k}^*(u_m))}{W_\delta(a_{n,k}^*(u_m)) W_\delta'(b_{n,k}^*(u_m) - l_{n,k}^*(u_m))},$$

and

$$\lim_{m\to\infty} \frac{a_{n,k}^*(u_m)}{W_\delta(a_{n,k}^*(u_m))} = 0 \quad \text{and} \quad \lim_{m\to\infty} \frac{W_\delta(b_{n,k}^*(u_m) - l_{n,k}^*(u_m))}{W_\delta(a_{n,k}^*(u_m)) W_\delta'(b_{n,k}^*(u_m) - l_{n,k}^*(u_m))} = 0,$$

then $B(a_{n,k}^*(u_m), b_{n,k}^*(u_m), l_{n,k}^*(u_m)) \to 0$ as $m \to \infty$. The first limit is zero because $\lim_{m\to\infty} e^{-\Phi(\delta)x} W_\delta(x) = 1$. For the second we have two cases: the sequence $((b_{n,k}^*(u_m) - l_{n,k}^*(u_m))$ is bounded or unbounded. If it is bounded, the quotient

$$W_\delta(b_{n,k}^*(u_m) - l_{n,k}^*(u_m)) / W_\delta'(b_{n,k}^*(u_m) - l_{n,k}^*(u_m))$$

is bounded by continuity and strict positivity of W_δ', which implies that the limit is zero. If it is unbounded, we simply notice that $W_\delta(a_{n,k}^*(u_m)) / W_\delta(b_{n,k}^*(u_m) - l_{n,k}^*(u_m)) \leq 1$ and since W_δ' diverges to infinity, the limit is again zero. Since G is bounded by 1, we see that, on the one hand

$$\lim_{m\to\infty} V_n^*(u_m) = V_n^*(u),$$

while on the other,

$$\lim_{m\to\infty} V_n^*(u_m) = \lim_{m\to\infty} W_\delta(u) \sum_{k=1}^{n} B(a_{n,k}^*(u_m), b_{n,k}^*(u_m), l_{n,k}^*(u_m)) \prod_{i=1}^{k-1} G(a_{n,i}^*(u_m), b_{n,i}^*(u_m), l_{n,i}^*(u_m))$$
$$\leq W_\delta(u) \sum_{k\in J_B}^{n} B(a_k, b_k, l_k) \prod_{\substack{i=1\\ i\in J_B}}^{k-1} G(a_i, b_i, l_i).$$

It is not hard to see that, if p is the cardinality of J_B, then $p \leq n-1$ and the a_k's, b_k's and l_k's form a strategy $D \in \mathscr{D}_p$ with $\phi^D(u) \geq \varphi$. Therefore, $V_n^*(u) \leq V_p^*(u)$, which is clearly a contradiction to the hypothesis of the proposition. Thus, there exists an $M > 0$ such that $\mid D_n^*(u_m) \mid \leq M$ and $D_n^*(u_m) \in K$ for every $m \geq 0$.
Continuity of D_n^* is readily proven: if the D_n^* were not continuous at u, we would be able to find an $\varepsilon > 0$ and a subsequence, which we can assume to be the original sequence, such that $\mid D_n^*(u_m) - D_n^*(u) \mid > \varepsilon$. By compactness of K, there would exist a subsequence $(u_{m_r})_{r\geq 0}$ such that $((u_{m_r}, D_n^*(u_{m_r})))_{r\geq 0}$ converged to a point in K, say (v, D). Since $u_{m_r} \to u$, we have $v = u$ and, moreover,

$$V_n^*(u) = \lim_{r\to\infty} V_n^*(u_{m_r}) = \lim_{r\to\infty} V^{D_n^*(u_{m_r})}(u_{m_r}) = V^D(u).$$

Since $\phi^D(u) = \varphi$, uniqueness of the strategies would imply that $D = D_n^*(u)$, which would not be possible for large enough r according to the choice of $(u_m)_{m\geq 0}$. Hence, D is continuous. □

The argument to show boundedness of D_n^* in the previous proof is simply a formalization of the intuitive idea that the best strategy cannot have arbitrarily high levels/corridors, as this would imply longer waiting times between dividend payments.

Remark 4.7. In line with the stochastic control literature, we close this section with an (informal) discussion on a potential dynamic-programming formulation of the strategy. Observe first that from the results in Section 6.1, for a finite number of corridors, the multi-corridor strategies are not necessarily optimal, so any optimality result could at most be achieved in the limit. Now, while in general, due to time-inconsistency, a dynamic programming principle cannot be formulated for control problems with ruin probability considerations, in the current setting there is some evidence pointing to a weaker version of this to indeed hold: in the case of the diffusion, from Equations (3) and (8), we obtain

$$a_{n,i+1}^*(u,\varphi) = a_{n-1,i}^*(u_1,\varphi_1), \quad i = 1,\dots,n-1, \tag{25}$$

and similarly for the $b_{n,i+1}^*$'s and $l_{n,i+1}^*$'s, where $u_1 = l_{n,1}^*(u,\varphi)$, $\varphi_1 = \varphi W_0(a_{n,1}^*(u,\varphi))/W_0(u)$ and the second argument now refers to the survival probability under consideration. The meaning of (25) is that, with the knowledge of $a_{n,1}^*(u)$, $b_{n,1}^*(u)$ and $l_{n,1}^*(u)$, one could determine the remaining levels of the best n-corridor strategy with survival probability φ by now considering instead the best $(n-1)$-corridor strategy with initial surplus u_1 and φ_1. Furthermore, iterating (25), we obtain

$$a_{n,i+1}^*(u,\varphi) = a_{n-i,1}^*(u_i,\varphi_i), \quad i = 1,\dots,n-1, \tag{26}$$

where, recursively, we have $u_i = l_{n-i+1,1}^*(u_{i-1})$ and $\varphi_i = \varphi_{i-1} W_0(a_{n-i+1,1}^*(u_{i-1}))/W_0(u_{i-1})$, $i = 2,\dots,n$. The recursive nature of these equations hints towards the existence of a type of dynamic programming principle leading the value of $V_{n,\varphi}^*$. Indeed, observe that

Equation (25) is, in a way, equivalent to the equation

$$V^*_{n,\varphi}(u) = \max_{D\in\mathscr{D}_n} \mathbb{E}_u \left[\int_0^{\tau^D\wedge\tau_1^d} e^{-\delta t}\, dD_t + e^{-\delta(\tau^D\wedge\tau_1^d)} V^*_{n-1}\left(C^D_{\tau^D\wedge\tau_1^d}, \frac{W_0(a_1^D)}{W_0(u)}\varphi \right) \right], \tag{27}$$

and Equation (26) could then be phrased as

$$V^*_{n,\varphi}(u) = \max_{D\in\mathscr{D}_n} \mathbb{E}_u \left[\int_0^{\tau^D\wedge\tau_i^d} e^{-\delta t}\, dD_t + e^{-\delta(\tau^D\wedge\tau_i^d)} V^*_{n-i}\left(C^D_{\tau^D\wedge\tau_i^d}, \frac{W_0(a_1^D)\cdots W_0(a_i^D)}{W_0(u)\cdots W_0(l_{i-1}^D)}\varphi \right) \right]. \tag{28}$$

In both cases one needs to be able to keep track of the survival probability to account for "how much probability has been consumed" and one cannot just plug in arbitrary stopping times, allowing for a classical a dynamic programming principle. For example, we have

$$\begin{aligned} V^*_{n,\varphi}(u) &= \max_{D\in\mathscr{D}_n} \mathbb{E}_u \left[\int_0^{\tau^D\wedge\tau_1} e^{-\delta t}\, dD_t + e^{-\delta(\tau^D\wedge\tau_1)} V^*_{n-1}\left(C^D_{\tau^D\wedge\tau_1-}, \frac{W_0(a_1^D)}{W_0(u)}\varphi \right) \right] \\ &= \max_{D\in\mathscr{D}_n} \mathbb{E}_u \left[e^{-\delta(\tau^D\wedge\tau_1)} V^*_{n-1}\left(C^D_{\tau^D\wedge\tau_1-}, \frac{W_0(a_1^D)}{W_0(u)}\varphi \right) \right], \end{aligned}$$

where the difference is that one now has to account for the value of the process just before the lump-sum payment at a_1, i.e., $C^D_{\tau^D\wedge\tau_1-}$.

These equations, however, suggest the idea of a dynamic programming principle being satisfied *locally* as follows: for any stopping time τ with $\tau_{i-1}^d \le \tau \le \tau_i$ for $i \in \{1,\ldots,n\}$ (and $\tau_0^d = 0$), we have

$$V^*_{n,\varphi}(u) = \max_{D\in\mathscr{D}_n} \mathbb{E}_u \left[\int_0^{\tau^D\wedge\tau} e^{-\delta t}\, dD_t + e^{-\delta(\tau^D\wedge\tau)} V^*_{n-i}\left(C^D_{\tau^D\wedge\tau-}, \frac{W_0(a_1^D)\cdots W_0(C^D_{\tau^D\wedge\tau-})}{W_0(u)\cdots W_0(l_{i-1}^D)}\varphi \right) \right],$$

while for $\tau_i < \tau \le \tau_i^d$,

$$V^*_{n,\varphi}(u) = \max_{D\in\mathscr{D}_n} \mathbb{E}_u \left[\int_0^{\tau^D\wedge\tau} e^{-\delta t}\, dD_t + e^{-\delta(\tau^D\wedge\tau)} V^*_{n-i}\left(C^D_{\tau^D\wedge\tau}, \frac{W_0(a_1^D)\cdots W_0(a_i^D)}{W_0(u)\cdots W_0(l_{i-1}^D)}\varphi \right) \right].$$

Observe, further, that the numerical results from Section 6 suggest that, for n large enough, $a^*_{n,k} = b^*_{n,k}$ for small k, so one does not need to consider the lump sum payment. If one could further justify the exchange of the expectation with the limit, then one would obtain an even stronger formula for $\lim_{n\to\infty} V^*_n$ seen as a function of two parameters.

Although intuitive, we do not attempt to formalize this approach here, as it is beyond the scope of the present paper.

5. Optimization of barrier levels

From the previous considerations, we know that for each $n \in \mathbb{N}$ there exists a strategy $D^* \in \mathscr{D}_n$ such that $V^*_n(u) = V^{D^*}(u)$. We are now interested in identifying this strategy, which is equivalent to finding surplus values a_k, b_k, l_k, $k = 1,\ldots,n$, which maximize $V^D(u)$ subject to the constraint $\phi^D(u) \ge \varphi$. Recall that the objective function and the constraint are of the form

$$V^D(u) = W_\delta(u) \sum_{k=1}^n B(a_k, b_k, l_k) \prod_{i=1}^{k-1} G(a_i, b_i, l_i)$$

and

$$\phi^D(u) = \kappa'(0)\, W_0(u) \prod_{k=1}^{n} A(a_k, b_k, l_k),$$

respectively. We will pursue two different approaches in the sequel.

5.1. **A gradient-inspired method.** Despite the possibly high-dimensional nature of this optimization problem, the particular structure of the above equations makes a classical Lagrange method look feasible. In what follows, we fix $n \in \mathbb{N}$.
It is clear that the constraint is "adversarial" to the objective function, meaning that the surplus values that maximize V will at the same time minimize ϕ^D. Given that the constraint is imposed in terms of an inequality, it follows that the optimal strategy D^* will as well satisfy $\phi^{D^*}(u) = \phi_{\min}$. Using this observation, we consider hence the function

$$L(a_1, \ldots, a_n, b_1, \ldots, b_n, l_1, \ldots, l_n, \Lambda) = V(u) - \Lambda(\phi^D(u) - \phi_{\min}).$$

The normal equations then turn out to be[1]

$$\begin{aligned}
&W_\delta(u) D_i B(a_m, b_m, l_m) \prod_{j=1}^{m-1} G(a_j, b_j, l_j) \\
&+ W_\delta(u) \sum_{k=m+1}^{n} B(a_k, b_k, l_k) D_i G(a_m, b_m, l_m) \prod_{\substack{j=1 \\ j \neq m}}^{k-1} G(a_j, b_j, l_j) \\
&- \Lambda \kappa'(0)\, W_0(u) D_i A(a_m, b_m, l_m) \prod_{\substack{k=1 \\ k \neq m}}^{n} A(a_k, b_k, l_k) = 0, \quad m = 1, \ldots, n,\ i = 1, 2, 3
\end{aligned} \tag{29}$$

$$\kappa'(0)\, W_0(u) \prod_{k=1}^{n} A(a_k, b_k, l_k) - \phi_{\min} = 0.$$

Despite the relatively easy form of these equations, even for the simplest cases no exact solution can be provided in terms of elementary functions. The following example illustrates this point:

Example 5.1. Let us consider $n = 1$, i.e. there is only one level a, at which a lump sum is paid down to a barrier $b \leq a$, and after that dividend payments at this barrier take place until the barrier is dissolved when the surplus value undershoots l. Recalling the formulas for W and A in the diffusion case, the equations

$$W_\delta''(b - l) = 0, \qquad \kappa'(0)\, W_0(u) A(a, l) = \phi_{\min}$$

allow us to write a and b in terms of l, obtaining

$$\begin{aligned}
b &= l - \frac{\log(\theta_1^2) - \log(\theta_2^2)}{\theta_1 - \theta_2}, \\
a &= -\frac{\sigma^2}{2\mu} \left(\log\left(\frac{\phi_{\min}}{\kappa'(0)\, W_0(u)} + e^{-(2\mu/\sigma^2) l} - 1 \right) - \log\left(\frac{\phi_{\min}}{\kappa'(0)\, W_0(u)} \right) \right).
\end{aligned}$$

Equations (29) reduce into

$$\frac{A_a(a, l)}{A_l(a, l)} = \frac{B_a(a, b, l)}{B_l(a, b, l)},$$

[1]Here $D_i A$ designs the partial derivative of A with respect to its i-th argument and similarly $D_i B$ and $D_i G$. We interchangeably use the notation A_a, A_b and A_l to denote the partial derivatives of A.

so, by letting $c = 2\mu/\sigma^2$ and $d = \phi_{\min}\kappa'(0)^{-1} W_0(u)^{-1}$, we obtain

$$(d-1)e^{cl}\left(1-\rho\left(d-1+e^{-cl}\right)^{\alpha}\right) + \left(\gamma - l - \frac{1}{c}\log\left(d-1+e^{-cl}\right)\right)\left(\theta_1 - \theta_2\rho\left(d-1+e^{-cl}\right)^{\alpha}\right) = 0 \tag{30}$$

with

$$\alpha = (\theta_1-\theta_2)/c, \quad \rho = d^{-\alpha}, \quad \xi = \frac{\log(\theta_2^2) - \log(\theta_1^2)}{\theta_1-\theta_2}, \quad \gamma = \frac{1}{c}\log(d) + \frac{W_\delta(\xi)}{W_\delta'(\xi)} + \xi.$$

By making the change of variable $y = e^{-cl}$, after some algebraic manipulations, (30) becomes

$$(d-1)\left(1-\rho\,(d-1+y)^{\alpha}\right) + y\left(\gamma + \frac{1}{c}\log\left(\frac{y}{d-1+y}\right)\right)\left(\theta_1 - \theta_2\rho\,(d-1+y)^{\alpha}\right) = 0.$$

While easily solved by a numerical optimizer, the solution cannot be expressed in terms of elementary functions. □

The last equation in the previous example accepts two possible solutions for y, although only one making $l \le b \le a$. Situations like this arise similarly for a higher number of corridors. While for small n this might be something easy to deal with, for large n this issue might introduce a complexity problem in the numerical solution of the equations resulting from (29). These considerations motivate pursuing different alternatives for obtaining the optimal corridor levels, one of which we explain now.

In what follows we will focus only in the case of the diffusion. Hence, A does not depend on its second argument and we actually have $A(a,l) = W_0(l)/W_0(a)$. Motivated by the constraint $\phi^D(u) = \varphi$, for fixed a_k, we introduce the change of variable $s_k = A(a_k, l_k) = W_0(l_k)/W_0(a_k)$, so that $l_k = W_0^{-1}(s_k W_0(a_k))$, $k = 1, \dots, n$. The constraint can now be phrased in terms of the s_k's, where we have $\kappa'(0)W_0(u)\prod_k s_k = \varphi$. Assume for the moment that the optimal s_k's are known, which we denote by $s^*_{n,1}, \dots, s^*_{n,n}$ in accordance with the notation of Section 4. Since a_n, b_n and l_n appear only as arguments of B in the the last term of the sum in (8), the optimal levels of the last corridor, $a^*_{n,n}, b^*_{n,n}$ and $l^*_{n,n}$, should also maximize the mapping

$$(a,b,l) \mapsto B(a,b,l)$$

subject to the constraint $s^*_{n,n} = W_0(l)/W_0(a)$. Since the inverse of W_0 can be explicitly computed from the formula given in Section 3.1, this is a two-dimensional optimization problem, which can be easily solved by standard optimization techniques. Assume we have found the optimal levels for the last corridor and, motivated by (11), let $c_n = B(a^*_{n,n}, b^*_{n,n}, l^*_{n,n})$. We can move backwards one step and repeat a similar procedure by observing that a_{n-1}, b_{n-1} and l_{n-1} appear only in the last two terms of the sum in (8) so, after dividing by a common factor, we notice that $a^*_{n,n-1}, b^*_{n,n-1}$ and $l^*_{n,n-1}$ should maximize the mapping

$$(a,b,l) \mapsto B(a,b,l) + c_n G(a,b,l)$$

subject to the constraint $s^*_{n,n-1} = W_0(l)/W_0(a)$. We can then set

$$c_{n-1} = B(a^*_{n,n-1}, b^*_{n,n-1}, l^*_{n,n-1}) + c_n G(a^*_{n,n-1}, b^*_{n,n-1}, l^*_{n,n-1})$$

and repeat. Since $V^*_n(u) = W_\delta(u)c_1$, we observe that by proceeding in this fashion, the optimal strategy can be obtained as a result of n consecutive 2-dimensional problems.

Having described an approach to finding the $a^*_{n,k}$'s, $b^*_{n,k}$'s and $l^*_{n,k}$'s given the $s^*_{n,k}$'s, we are only left with the question of finding the appropriate $s^*_{n,k}$'s. Since the sequential optimization part is relatively fast for a given set of the s_k's, optimal or not, we can use a greedy method to solve this last issue. The overall procedure is summarized in Algorithm 1.

Input : Loops L, increment r, optimization function P.
Output: Approximation to optimal levels $a^*_{n,k}, b^*_{n,k}$ and $l^*_{n,k}$, $k = 1, \dots, n$.

```
1  begin
2      initialize({(s_k) | k = 1,...,n});
3      (c_1,...,c_n) := computeC(s_1,...,s_n);
4      l := 1;
5      while l < L do
6          for (m_1, m_2) in {1,...,n}^2 do
7              m := min(m_1, m_2);
8              s̃_k := s_k,  k ≠ m_1, m_2;
9              s̃_{m_1} := r s_{m_1};
10             s̃_{m_2} := s_{m_2} / r;
11             ã_n, b̃_n := P(B, s̃_n);
12             l̃_n := W_0^{-1}(s̃_n W_0(ã_n));
13             c̃_n := B(ã_n, b̃_n, l̃_n);
14             for j := n-1 to m do
15                 ã_j, b̃_j := P(B + c̃_{j+1} G, s̃_j);
16                 l̃_j := W_0^{-1}(s̃_j W_0(ã_j));
17                 c̃_j := B(ã_j, b̃_j, l̃_j) + c̃_{j+1} G(ã_j, b̃_j, l̃_j);
18             end
19             if c̃_m > c_m then
20                 s_k = s̃_k,  k = 1,...,n;
21                 (c_1,...,c_n) := computeC(s_1,...,s_n);
22             end
23         end
24         l = l + 1;
25     end
26     a*_{n,n}, b*_{n,n} := P(B, s_n);
27     l*_{n,n} := W_0^{-1}(s_n W_0(a*_{n,n}));
28     for j := n-1 to 1 do
29         a*_{n,j}, b*_{n,j} := P(B + c_{j+1} G, s_j);
30         l*_{n,j} := W_0^{-1}(s_j W_0(a*_{n,j}));
31     end
32     V = c_1;
33 end
```

ALGORITHM 1: Corridor level optimization algorithm

The idea of the algorithm is to start with an arbitrary set of levels $s_1, \dots, s_n$ satisfying the constraint $\kappa'(0)W_\delta(u)\prod_k s_k = \varphi$ and the value of the c_k's associated with them (Lines 2 and 3). Once these initial values are set, the algorithm iterates through all couples (m_1, m_2) of indices m_1 and m_2 in the set $\{1, \dots, n\}$. The idea is to alter the values of s_{m_1} and s_{m_2} in a multiplicative way so that the constraint is still satisfied, obtaining a new set of s-levels, which we denote by $\tilde{s}_1, \dots, \tilde{s}_n$ (Lines 8 to 10). If after computing the optimal corridor levels associated with the $\tilde{s}_k$'s (Lines 11 to 18) we observe that the alteration

leads to an improvement in $V^D(u)$, we discard the current s-levels and replace them by their tilded versions, moving to the next iteration. Observe that for $m = \min(m_1, m_2)$ one can already notice an improvement if $\tilde{c}_m > c_m$, so it is not necessary to check every value. Once the overall procedure has been repeated L times, we compute the corridor levels associated with the latest version of the s_k's, which are an approximation to the optimal levels $a^*_{n,k}, b^*_{n,k}$ and $l^*_{n,k}$, $k = 1, \dots, n$.
The optimization function P appearing in Lines 11, 15, 26 and 29 is to be understood as any procedure that maximizes the function given in the first argument subject to A being equal to the second argument of P. As explicit formulas are available for the inverse of W_0, we can replace the third arguments of B and G, and let P be any *unconstrained* maximization algorithm.
A few comments are in order: while the algorithm can be used for any value of n, the complexity of its main iterating procedure scales quadratically with the number of corridors. Combined with the optimization procedure, this will typically lead to an excessive computation time. To address this issue, one can restrict the set of couples (m_1, m_2) that are considered for improvement. This, however, requires a good set of initial values for the s_k's. A rule of thumb used during the implementation was to compute the optimal $s^*_{n,k}$'s for a small n and for computing the optimal levels for, say, $N > n$ corridors, initialize the s_k's as $s_k = s^*_{n,k}$, $k = 1, \dots, n$ and $s_k = 1$, $k = n+1, \dots, N$. For these initial values, one would restrict m_1 to the set $\{1, \dots, n\}$ and m_2 to the set $\{n+1, \dots, N\}$. Conversely, one could initialize the s_k's as $s_{N-n+k} = s^*_{n,k}$, $k = 1, \dots, n$ and $s_k = 1$, $k = 1, \dots, N-n$, and restrict m_1 to the set $\{1, \dots, N-n\}$ and m_2 to $\{N-n+1, \dots, N\}$. While both approaches yielded similar results, the latter performed slightly better, with exceptionally good results obtained by taking $N = n+1$ and repeating the procedure several times (see also Section 6.2 for further insights into the initialization procedure).

Remark 5.2. Recall from (25) and (26) that

$$a^*_{n,i+1}(u, \varphi) = a^*_{n-i,1}(u_i, \varphi_i), \quad i = 1, \dots, n-1$$

where $u_i = l^*_{n-i+1,1}(u_{i-1})$ and $\varphi_i = \varphi_{i-1} W_0(a^*_{n-i+1,1}(u_{i-1}))/W_0(u_{i-1})$, $i = 2, \dots, n$. The meaning of these equations is that, to solve the overall optimization problem, we "only" need to learn to determine the optimal first corridor of any strategy. While it might seem that this leads to a more efficient optimization algorithm, the use of (25) and (26) makes implicit use of full knowledge of the functions $a^*_{i,1}$, $b^*_{i,1}$ and $l^*_{i,1}$ for each $i = 1, \dots, n-1$ at all values u and φ, which seems infeasible to numerically achieve for a large amount of corridors[2]. One could still use this idea by optimizing over the values of $\varphi_1, \dots, \varphi_{n-1}, u_{n-1}$, and determining the remaining quantities by using the optimality properties of $a^*_{n,i}, b^*_{n,i}$ and $l^*_{n,i}$. However, this leads precisely to the implementation described in Algorithm 1.

The considerations described so far work equally well for the case of the Cramér-Lundberg model with exponential claims, as evidenced by the results in Section 3.2. For more general processes, we lose the ability of expressing l purely in terms of a, as A usually depends on its second argument. If we want to keep the advantage of transforming the constrained optimization problem into a sequence of n unconstrained ones, we can shift the focus by observing that A can be written in the form $A(a, b, l) = I(b, l)/W_0(a)$ for some function I. Hence, for $0 < s < 1$ we can write $a = W_0^{-1}(I(b, l)/s)$ and then replace the first argument in B and G by this function. The limitation here is that the inverse of W_0 is in general not readily available even if explicit formulas for W_0 exist, so that further methods need to be used in this case. We explain another one in the next section.

[2]Recall that there are no explicit solutions for these functions, so this needs to be done numerically.

5.2. **An evolutionary strategy.** Evolutionary strategies (ES) have been applied with some success in reinsurance problems where the evaluation of the function to optimize is only possible through numerical procedures due to the non-existence of explicit algebraic expressions, see e.g. Salcedo-Sanz et al. (2014) and Román et al. (2018). Within the context of optimization of dividend strategies, ES were recently systematically used in Albrecher and Garcia Flores (2023) for tackling the classical (un-constrained) version of the current problem, and we refer to there for a broader discussion of the corresponding algorithms and background. In this work, we explore a suitably adapted strategy for our purposes as well as a penalized version of the algorithm.

As outlined in Albrecher and Garcia Flores (2023), basic ES's are designed for unconstrained search spaces, so in order to enforce the survival probability condition, some adaptations are needed to the way $V^D(u)$ and $\phi^D(u)$ are evaluated. We begin by identifying the current set of strategies of the form (8) as a subset of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ in the natural way. A point $(a, b, l) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ satisfies the constraints on our dividend problem if and only if:

(i) For every $k = 1, \ldots, n$, $0 \leq l_k \leq b_k \leq a_k$,
(ii) For every $k = 1, \ldots, n-1$, $a_k \leq a_{k+1}$ and
(iii) $\phi^D(u) = \kappa'(0)\, W_0(u) \prod_{k=1}^n A(a_k, b_k, l_k) = \varphi$.

Here $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$ and $l = (l_1, \ldots, l_n)$. Now, mutations occur coordinate by coordinate by adding a normally distributed error to recombinations of the parental population. If we want to maintain this procedure, then, whenever $a_k - l_k$ is small, constraint (i) will be violated with high probability after this addition, which restricts the way in which mutations can be carried out. As for problems with high values for n and φ, $a_k - l_k$ will be small, and additive mutations of this way are infeasible.

We begin by assuming that A does not depend on b, which as explained in the previous section, happens in the case of the diffusion and the Cramér-Lundberg model with exponential jumps. Thus, by strict monotonicity of W_0, for every $a > 0$, the mapping $l \mapsto A(a, l)$ is strictly increasing. With this in mind, we can use constraint (iii) to come around the limitation from the previous paragraph. Indeed, by defining the changes of variable in the l-space

$$q_1 = A(a_1, l_1)$$

and for $k = 1, \ldots, n-1$

$$q_{k+1} = q_k A(a_{k+1}, l_{k+1}),$$

the previous set of constraints is converted to

(i') For every $k = 1, \ldots, n-1$, $q_{k+1} \leq q_k$ and $q_1 < 1$,
(ii') For every $k = 1, \ldots, n-1$, $a_k \leq a_{k+1}$ and
(iii') $q_n = \kappa'(0)^{-1}\, W_0(u)^{-1} \varphi$.

Denoting the product $\kappa'(0)^{-1}\, W_0(u)^{-1} \varphi$ by d, we see that these changes of variable allow us to convert condition (iii) into a univariate condition, making clear the reduction to a search in a $3n - 1$ dimensional space. Constraint (i') can be summarized by the chain of inequalities $d = q_n < q_{n-1} < \cdots < q_1 < 1$. For high values of φ, the value of d will be close to 1, so, once again, with high probability, any addition of a normally distributed error to the parameters q_k will make them not satisfy constraint (i'). Hence, we apply the final transformation

$$y_k = \Phi^{-1}\left(\frac{q_k - d}{1 - d}\right)$$

with Φ the c.d.f. of the normal distribution. With the latter, we see that the only constraints in y_k and a_k are

(i") For every $k = 1, \dots, n-2$, $y_{k+1} \le y_k$,
(ii") For every $k = 1, \dots, n-1$, $a_k \le a_{k+1}$.

The new set of constraints (i") and (ii") can easily be handled by sorting the values within the vector and is handled by the ES, similarly to Albrecher and Garcia Flores (2023).
One should keep in mind that this approach is only viable when A does not depend on its second argument. While one can still use strict monotonicity of W_0 to theoretically argue that the map $a \mapsto A(a, b, l)$ is invertible for each fixed b and l, and proceed in a similar manner, in this situation one often runs into the problem that the inverse of W_0 cannot be explicitly identified, thus limiting the applicability of the approach. In these cases, we suggest a more straightforward procedure applying a penalty function to the value function. More specifically, an adaptive penalty function is used in the algorithm, as e.g. described in Michalewicz and Schoenauer (1996). Then, (8) is replaced by the function

$$\tilde{V}(u) = V(u) - \xi_t 1_{\{\phi^D(u)<\varphi\}}, \tag{31}$$

where $1_{\{\phi^D(u)<\varphi\}}$ denotes the indicator function of the set $\{\phi^D(u) < \varphi\}$ and ξ_t is a parameter that depends on the generation t. This parameter is initially chosen larger than the overall optimal strategy, so that levels a_k, b_k, l_k for which $\phi^D(u) < \varphi$ produce a negative value. Moreover, it is updated for every generation according to the rule

$$\xi_{t+1} = \begin{cases} c_1\xi_t & \text{if best candidate satisfies } \phi^D(u) > \varphi \text{ for } k \text{ generations,} \\ c_2\xi_t & \text{if best candidate satisfies } \phi^D(u) < \varphi \text{ for } k \text{ generations,} \\ \xi_t & \text{otherwise,} \end{cases}$$

where $c_1 < 1$, $c_2 > 1$ and k are predetermined parameters. We require $c_1c_2 \neq 1$ to avoid circularity.
While the optimal set of levels satisfies $\phi^D(u) = \varphi$, we cannot replace the indicator function in (31) by $1_{\{\phi^D(u)=\varphi\}}$, since, by the nature of the algorithm, with probability zero the new candidates will be in the set $\{\phi^D(u) = \varphi\}$, so that using this indicator function would instead produce the overall best strategy of the form (8) without regard to survival probability.

6. Numerical Results

We examine the performance of the strategy for four Lévy processes: the diffusion, the Cramér-Lundberg model with exponential and Erlang claims, as well as the perturbed Cramér-Lundberg model with exponential claims. More specifically, for the diffusion model

$$C_t = u + \mu t + \sigma B_t, \quad t \ge 0,$$

we consider $\mu = 1$, $\sigma = 1$, $\varphi = 0.95$ and $\delta = 0.03$ as well as $\mu = 0.04$, $\sigma = \sqrt{0.02}$, $\varphi = 0.95$ and $\delta = 0.02$. For the case of the Cramér-Lundberg process

$$C_t = u + c\,t - \sum_{i=1}^{N_t} X_i, \quad t \ge 0,$$

we choose $X_i \sim \text{Exp}(1000)$, a Poisson intensity $\lambda = 5 \times 10^5$, $c = 501$, $\delta = 0.03$ and $\varphi = 0.95$; as well as $X_i \sim \text{Erlang}(2, 1)$, a Poisson intensity $\lambda = 10$, $c = 21.4$, $\delta = 0.03$ and $\varphi = 0.1$. Finally, for the perturbed Cramér-Lundberg process,

$$C_t = u + \mu t + \sigma B_t - \sum_{i=1}^{N_t} X_i,, \quad t \ge 0,$$

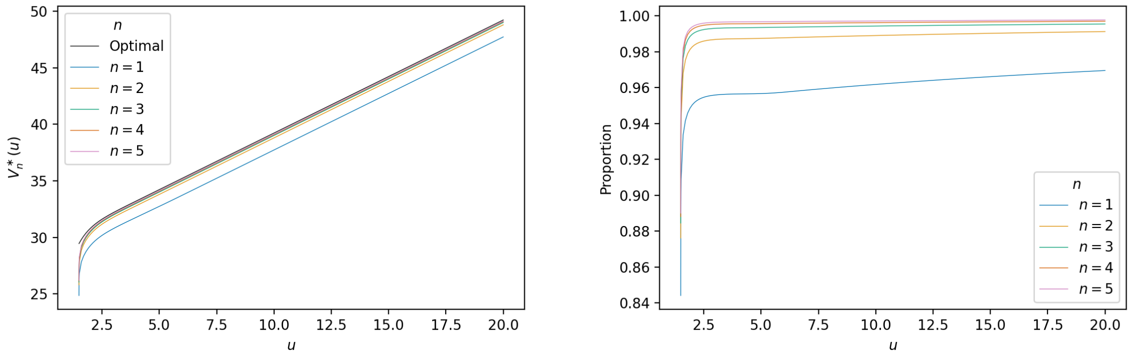
we consider $\mu = 1$, $\sigma = 1$, $\varphi = 0.7$, $X_i \sim \text{Exp}(2)$, a Poisson intensity $\lambda = 1$ and $\delta = 0.03$. While the concrete choice of parameters for the perturbed Cramér-Lundberg process and the first set of parameters for the diffusion is somewhat arbitrary, the second set for the diffusion was chosen in accordance with the parameters used in Grandits (2015). The Cramér-Lundberg process with exponential claims is chosen with a high Poisson intensity λ and a small expectation in such a way that it approximates the diffusion case with $\mu = 1$ and $\sigma = 1$. The parameter choice for the Cramér-Lundberg process with Erlang claims is taken from Azcue and Muler (2005), for which the overall (unconstrained) optimal dividend strategy is known to be a two-band strategy (as opposed to a barrier strategy like in the other three cases). The relevant quantities for the diffusion and Cramér-Lundberg process with exponential claims are evaluated through the formulas obtained in Section 3.1 and 3.2, while the formulas for the Cramér-Lundberg process with Erlang claims and perturbed Cramér-Lundberg process are obtained through the more general formulas from earlier in Section 3. The results are shown in Figures 2–7. In all these cases, and for each relevant u, there was only one optimal strategy D making $V_n^*(u) = V^D(u)$, which allows us to present the information in terms of the functions $a_{n,k}^*, b_{n,k}^*$ and $l_{n,k}^*$ as in Figures 5, 6 and 7. The results in Figure 5 are shown only in the case of the diffusion, since the resulting plots for the other processes are similar and we decided to omit them for the sake of brevity.

We start by observing that the performance of the strategy seems to depend rather strongly on the type of process and the concrete parameters. The values for different n converge rather fast and uniformly to a limiting function $\lim_{n\to\infty} V_n^*$, which is why we only display the results for small n. In Figure 2b, we see that even for values of u close to $\phi^{-1}(0.95) \approx 1.497866$, the process achieves about 90% of the value of the unconstrained optimal strategy (which for the diffusion is a barrier strategy). This is quite remarkable, given that the optimal barrier strategy has a survival probability of zero. Figure 3 shows the corresponding plots for the second set of parameters in the diffusion case. Here the convergence to the solution for large n is slower, and only about 80% of the unconstrained optimal value is achieved, which is nevertheless still noteworthy. Recall that this figure shows the results obtained for the parameters that were also used in the numerical experiments of Grandits (2015) who considered the optimal dividend problem up to a finite time horizon T with a ruin probability constraint and a potential lump sum payout of the remaining surplus at T. Within the current framework one can not directly compare the results from Figure 3 to those obtained in that paper, but a small adaptation adding a lump sum dividend payment at T makes the comparison possible: using D_n^*, we can compute

$$V_n(u,T) = \mathbb{E}\left(\int_0^{\tau\wedge T} e^{-\delta t} dD_t^* + e^{-\delta(\tau\wedge T)} C_{\tau\wedge T}^{D^*}\right) \tag{32}$$

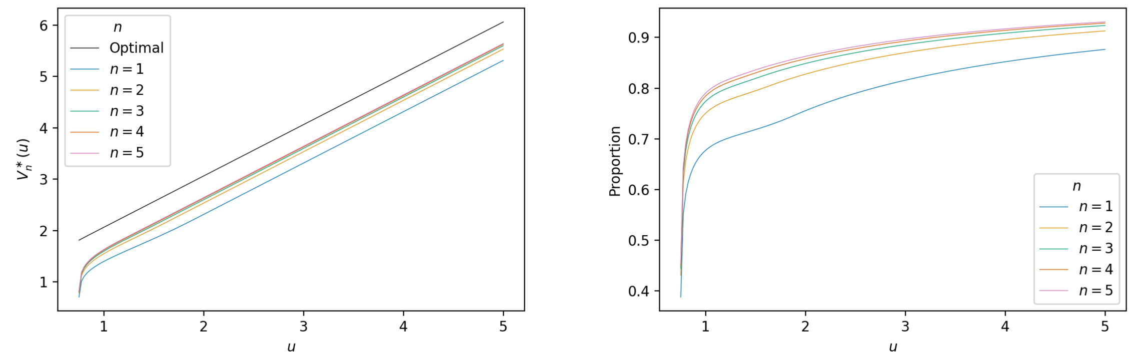
where $\tau = \inf\{t > 0 : C^{D^*} < 0\}$ and D^* is the stochastic process associated with using the strategy represented by $D_n^*(u)$. For $n = 10$, $T = 10$ and $u = 1$, we computed the expectation in (32) through MC simulation using a sample of 100,000 simulations and approximating the diffusion through a random walk with drift and a step size of 10^{-5} time units. These simulations provide an approximating value of 1.158832, which is in fact considerably higher than the value 0.20879 reported in Grandits (2015).

Figure 4 shows that for the Cramér-Lundberg process with Erlang claims, the performance is similar. Recall from Proposition 4.5 that for any $n \geq 1$, $\lim_{u\to\infty} V_n^*(u)/\bar{V}(u) = 1$, where $\bar{V}$ denotes the unconstrained optimal value function. Figures 5, 6 and 7 show the evolution of the optimal strategies D_n^* as a function of u and n for the diffusion and Cramér-Lundberg process with exponential claims, respectively. Due to the imposition



(A) V_n^* as a function of initial surplus. (B) V_n^* relative to the best strategy.

FIGURE 2. $V_n^*(u)$ for $n = 1, 2, 3, 4$ and 5 in absolute terms (left), and relative to the unconstrained optimal dividend strategy (that is also given on the left) for the diffusion with parameters $\mu = 1$, $\sigma = 1$, $\varphi = 0.95$ and $\delta = 0.03$ (right).

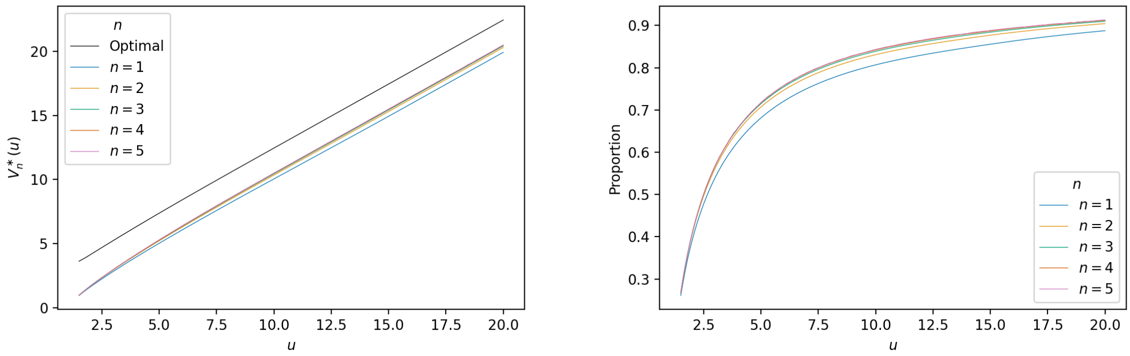


(A) V_n^* as a function of initial surplus. (B) V_n^* relative to the best strategy.

FIGURE 3. $V_n^*(u)$ for $n = 1, 2, 3, 4$ and 5 in absolute terms (left) and relative to the unconstrained optimal dividend strategy (that is also given on the left) for the diffusion with parameters $\mu = 0.04$, $\sigma = \sqrt{0.02}$, $\varphi = 0.95$ and $\delta = 0.02$ (right).

$a_1^D \geq u$ made before Proposition 4.6 to ensure uniqueness of the strategies, for values of u large enough we will have $a_{n,1}^*(u) = u$, which is reflected in all the plots of Figure 5. Moreover, the conclusion of Proposition 4.5 can be traced in Figures 5b and 5c, where, again, for large enough u, all levels but $a_{n,1}^*$ become constant. Finally, it seems that we have the general trends $a_{n+1,k}^* \leq a_{n,k}^*$ and $b_{n+1,k}^* \leq b_{n,k}^*$ for $k \leq n$, as well as $a_{n,k}^* \leq a_{n,k+1}^*$, $b_{n,k}^* \leq b_{n,k+1}^*$ and $l_{n,k}^* \leq l_{n,k+1}^*$, which simply means that $a_{n,k}^*$ and $b_{n,k}^*$ are decreasing in n, but increasing in k, while $l_{n,k}^*$ is only increasing in k as shown in Figures 6 and 7. While this can not be proven explicitly with the current means, it is somewhat intuitive, at least for the last observation: after a corridor $a_{n,k}^*, b_{n,k}^*$ and $l_{n,k}^*$, it seems optimal to wait for a level higher than $a_{n,k}^*$ to start the new corridor $a_{n+1,k}^*$, for if $a_{n+1,k}^* \leq a_{n,k}^*$ we could have exchanged the order of the corridors, which would imply paying dividends earlier and hence on average increasing the amount of dividends paid without changing the overall survival probability.

Finally, we would like to comment two further details about the implementation: first,



(A) V_n^* as a function of initial surplus. (B) V_n^* relative to the best strategy.

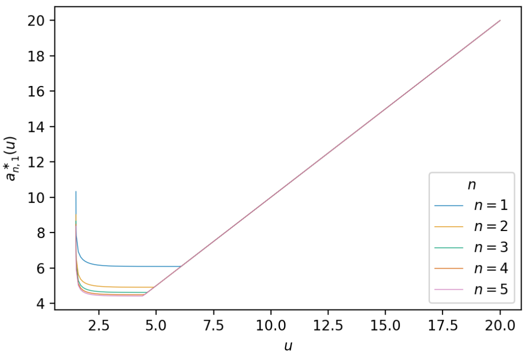
FIGURE 4. $V_n^*(u)$ for $n = 1, 2, 3, 4$ and 5 in absolute terms (left), and relative to the unconstrained optimal dividend strategy (that is also given on the left) for the Cramér-Lundberg process with Erlang claims (right).

all processes except the diffusion exhibited local maxima around the points where $b_k = l_k$ for some index k. Hence, for these particular cases, the algorithms described in Section 5 were not applied exactly as described there but with the extra condition that $l_k < rb_k$ for some $0 < r < 1$, generally $r \approx 0.95$, which seemed to produce more adequate results. Second, while the gradient equations derived at the beginning of Section 5 are hard to deal with, they produce an interesting equation:

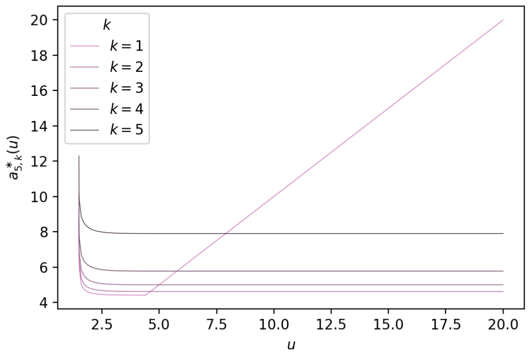
$$W_\delta(b_n - l_n) = 0.$$

Combined with the form of B, one can then deduce that the optimal distance between $b_{n,n}^*$ and $l_{n,n}^*$ equals the barrier level of the optimal barrier strategy with initial capital $a_{n,n}^* - l_{n,n}^*$ (which in most cases equals the overall unconstrained dividend strategy). This observation was used, for example, to solve the first step in Algorithm 1 or as another check for convergence of the ES.

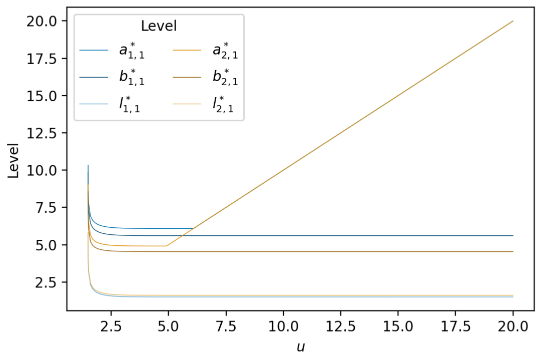
6.1. **Comparison with other strategies.** While the results above show that, for certain sets of parameters, the strategy has a particularly strong performance (relative to the best unconstrained strategy), it is interesting to compare its performance to other dividend strategies studied in the literature. Despite the existing literature addressing the trade-off between dividend optimization and long-term safety of the company, a large part of the optimality research is focused on delaying ruin rather than avoiding it (cf. Thonhauser and Albrecher (2007), Loeffen and Renaud (2010), Hernandez et al. (2018)), as in this case time-consistency still holds. Nevertheless, we now want to compare it with strategies that aim to optimize the value of the strategy at time $t = 0$ subject to a constraint on the probability of ruin at that time, ignoring their non-optimality at a future date (i.e., we will only work with strategies that can be understood in a *precommitment* sense). Concretely, we will compare the multi-corridor strategy against *linear dividend strategies* as well as the refraction strategies developed in Strini and Thonhauser (2023). Recall that a linear dividend strategy is one that pays all incoming surplus as dividends whenever the process is above a pre-determined fixed increasing line $b(t) = b_0(u) + b_1(u)t$ (linear in time t) and does not pay dividends otherwise. The feature that has not been considered in earlier literature on the topic (like Albrecher et al. (2005, 2007)) is that for each initial u the barrier is optimized with respect to the value function and survival constraint (and then kept for the realization of the entire surplus process in time). While more general reflection/refraction strategies have been developed (cf. Albrecher



(A) $a^*_{n,1}$ as function of initial surplus.



(B) $a^*_{5,k}$ as function of initial surplus.



(C) Comparison of levels for $n = 1, 2$.

FIGURE 5. Plot of the change of the optimal levels $a^*_{n,k}$, $b^*_{n,k}$, $l^*_{n,k}$ for different n's and k's as a function of initial surplus for diffusion with parameters $\mu = 1$, $\sigma = 1$, $\varphi = 0.95$ and $\delta = 0.03$.

and Kainhofer (2002); Albrecher and Hartinger (2007); Lin and Sendova (2008)), we restrict the comparison to these two strategies here.
The formulas for the computation of the ruin probability and the expected value of doiscounted dividend strategies under a linear barrier can already be found in Gerber (1981) for the case of a diffusion approximation and the Cramér-Lundberg process with exponential claims. Since in Strini and Thonhauser (2023) only the case of a diffusion approximation is considered, we restrict ourselves to that case here. Figure 8 displays the plots of these strategies together with the best unconstrained strategy for different sets of parameters. The slope and intercept of the linear dividend barrier were chosen in such a way that the survival constraint is met and the value function is maximized (for each initial capital u, the best parameters $b_0(u), b_1(u)$ are determined numerically, so that the plotting of the figure is quite time-intensive). For the strategy developed in Strini and Thonhauser (2023), the parameters where chosen to satisfy the survival constraint. Notice, however, that in their strategy the dividend payment rate needs to be bounded by some level $L_{max} < \mu$, as otherwise the survival constraint cannot be satisfied. In all cases, this bound was chosen to represent 95% of the drift μ. One observes that in all cases the strategy in Strini and Thonhauser (2023) is outperformed by the other two strategies, which is no surprise as that strategy was built with other considerations in mind. The comparison of the best linear strategy and the best multi-corridor strategy is more subtle. For example, for the parameters in Figure 8a, the linear strategy very

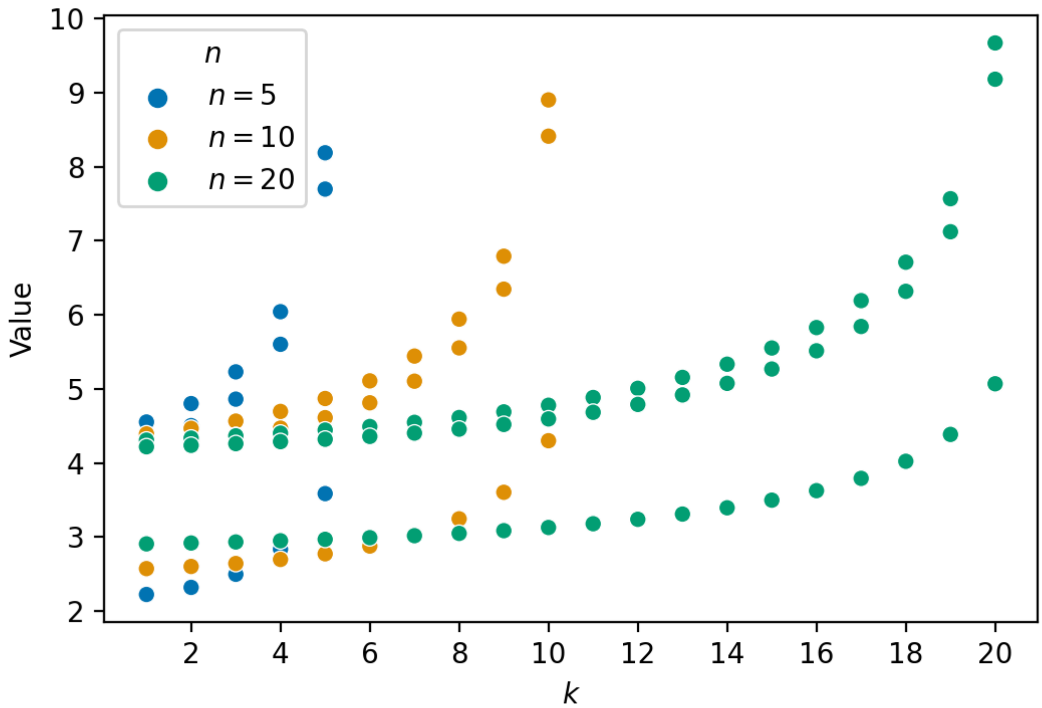


FIGURE 6. Plot of the change of the optimal levels $a^*_{n,k}$, $b^*_{n,k}$, $l^*_{n,k}$ for different n's and k's as a function of k for the Cramér-Lundberg process with exponential claims and initial surplus $u = 2$.

slightly outperforms the multi-corridor strategy for low values of initial capital u and vice versa for large values of u. In any case, both strategies perform surprisingly well, to an extent that one can hardly differentiate the plots from the optimal unconstrained strategy (the largest difference is of the order of 10^{-2}, decreasing with u). For the parameters in Figure 8b, once again the strategies perform very similarly (with a little advantage for the linear barrier strategy of the order 10^{-2}), though slightly worse than the optimal unconstrained strategy. Finally, for the parameters of Figure 8c, it can be seen that the multi-corridor strategy strongly outperforms the best linear strategy.

This pattern is in fact similar for other choices of parameters. In general, the multi-corridor strategy performs considerably better than the linear strategy whenever the drift is small in comparison to the volatility of the process, together with a not-so-restrictive constraint on the survival probability (which often cannot be enforced for small values of initial capital, precisely because μ/σ^2 is small). In the other cases, the linear strategy performs better than the multi-corridor strategy, but that difference is usually very small. Considering that the comparisons where made using only 5 corridors, one can expect that difference to decrease further by letting the number of corridors grow.

6.2. **Asymptotic behaviour of barrier levels.** We will restrict ourselves now to the case of the diffusion and fix the parameters to $\mu = 0.04$, $\sigma = \sqrt{0.02}$, $\varphi = 0.95$, $\delta = 0.02$ and initial surplus $u = 2$. Figure 9 displays the optimal barrier levels for large values of n (80, 100, and 150) computed by means of Algorithm 1. Together with Figures 6 and 7, the results show a sort of common behavior both at the "middle" levels as well as in the last barriers. Figure 10 displays the distances between $a^*_{n,k}$, $b^*_{n,k}$ and $l^*_{n,k}$ for $n = 30$ and $n = 80$ for the last 30 bands, as well as the distances between $a^*_{n,n}$ and $a^*_{n,k}$, with a shift

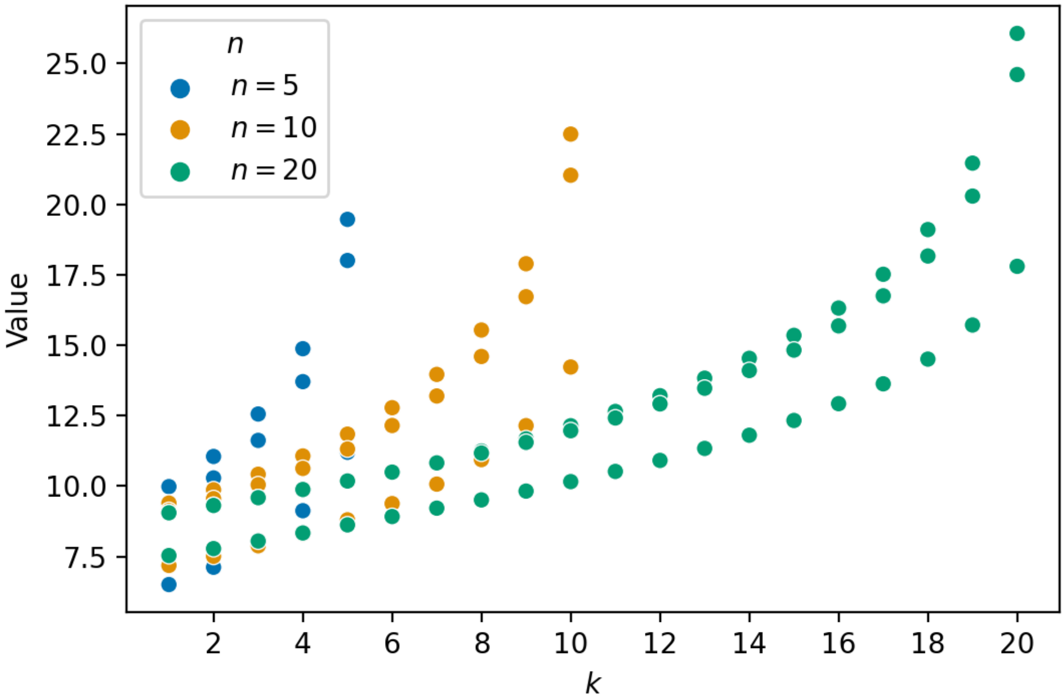


FIGURE 7. Plot of the change of the optimal levels $a^*_{n,k}$, $b^*_{n,k}$, $l^*_{n,k}$ for different n's and k's as a function of k for the perturbed CL process with initial surplus $u = 2$.

in the indices of $n = 80$ to match the last bands. We notice that except for small values of k, the distances are extremely close (one even does not see the difference visually).

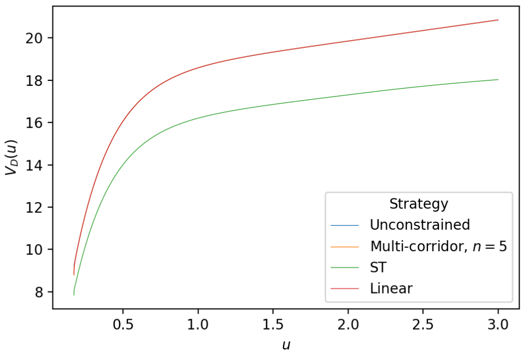
These observations suggest that the distances between the barrier levels converge in the final barriers. Explicitly, the results suggest that the limits

$$\begin{aligned} &\lim_{n\to\infty} a^*_{n,n-M+1} - l^*_{n,n-M+1}, \\ &\lim_{n\to\infty} b^*_{n,n-M+1} - l^*_{n,n-M+1}, \\ &\lim_{n\to\infty} a^*_{n,n} - a^*_{n,n-M+1}, \end{aligned} \tag{33}$$

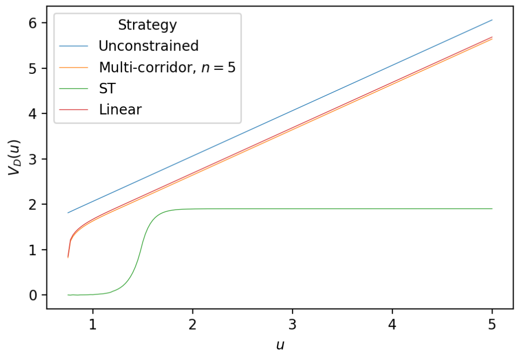
exist for $M \in \mathbb{N}$ (and that in turn several other limits pertaining to the distances among the levels also exist, e.g., the existence of $\lim_{n\to\infty} a^*_{n,n-M+2} - a^*_{n,n-M+1}$). In the following, we assume that the limits in (33) indeed exist and denote them by ζ_M, η_M and ν_M respectively.

In fact, from the Lagrange equations it was observed already earlier that $b^*_{n,n} - l^*_{n,n}$ should always equal the (surplus-independent) optimal barrier level for the diffusion, giving the exact value for η_1. Motivated by this, we use the Lagrange equations to deduce further properties of ζ_M, η_M and ν_M.

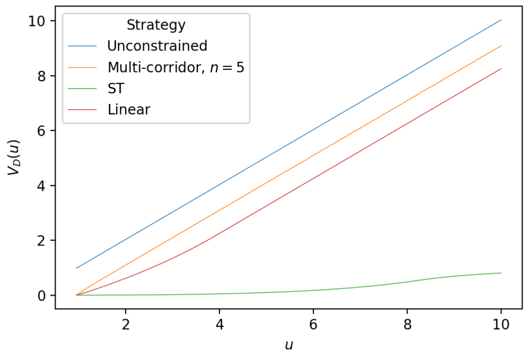
Recall the normal equations (29), as well as the definitions of A, B and G, here specialized



(A) $\mu = 2.0$, $\sigma^2 = 1.0$, $\delta = 0.1$, $\phi(u) = 0.5$ and $L_{max} = 1.9$.



(B) $\mu = 0.04$, $\sigma^2 = 0.02$, $\delta = 0.02$, $\phi(u) = 0.95$ and $L_{max} = 0.038$.



(C) $\mu = 0.1$, $\sigma^2 = 1.0$, $\delta = 0.1$, $\phi(u) = 0.172$ and $L_{max} = 0.095$.

FIGURE 8. Value functions for the best multi-corridor, the best linear and the best strategy in the sense of Strini and Thonhauser (2023) (referred as ST) against the optimal (unconstrained) strategy for various parameter sets.

for the diffusion:

$$\begin{aligned}
A(a, l) &= W_0(l)/W_0(a),\\
B(a, b, l) &= \frac{a - b + W_\delta(b - l)/W_\delta'(b - l)}{W_\delta(a)},\\
G(a, b, l) &= \frac{\sigma^2}{2}\frac{W_\delta(l)}{W_\delta(a)}\left(W_\delta'(b - l) - \frac{W_\delta(b - l)W_\delta''(b - l)}{W_\delta'(b - l)}\right).
\end{aligned}$$

For $m = n$ we obtain, from (29) (with $i = 1$ and $i = 3$),

$$\frac{D_1 B(a^*_{n,n}, b^*_{n,n}, l^*_{n,n})}{D_3 B(a^*_{n,n}, b^*_{n,n}, l^*_{n,n})} - \frac{D_1 A(a^*_{n,n}, , l^*_{n,n})}{D_2 A(a^*_{n,n}, l^*_{n,n})} = 0$$

Using the explicit forms of A and B we obtain

$$\frac{-W_\delta(a^*_{n,n}) + W_\delta'(a^*_{n,n})\left(a^*_{n,n} - b^*_{n,n} + \frac{W_\delta(\eta_1)}{W_{\delta'(\eta_1)}}\right)}{W_\delta(a^*_{n,n})} + \frac{W_0(l^*_{n,n})W_0'(a^*_{n,n})}{W_0'(l^*_{n,n})W_0(a^*_{n,n})} = 0.$$

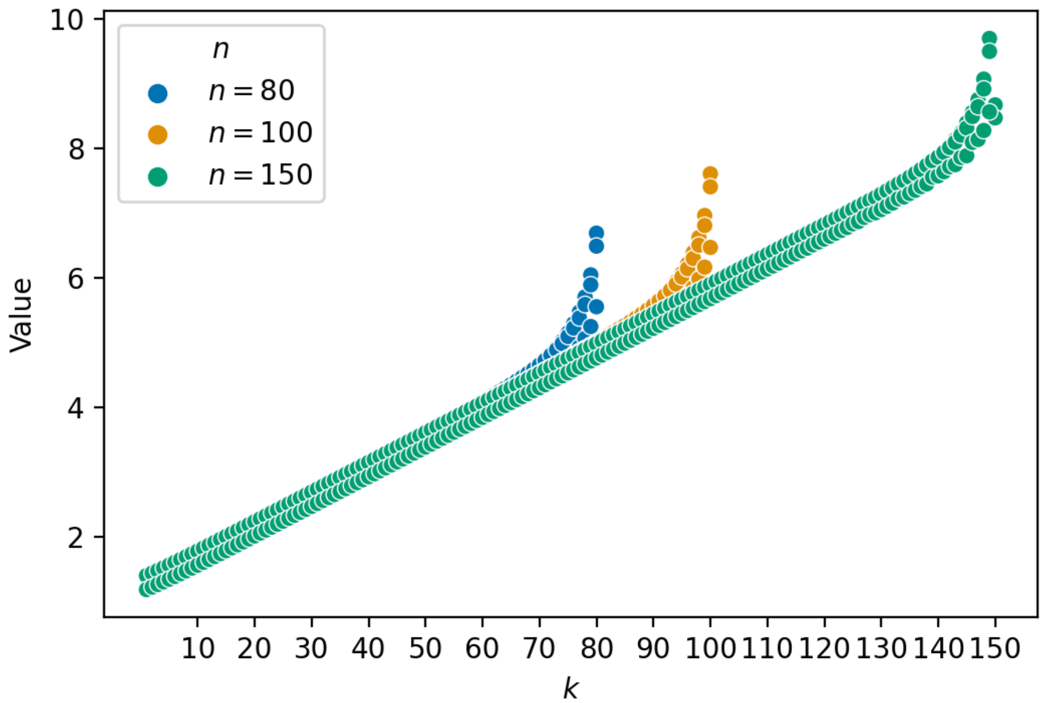


FIGURE 9. Plot of the change of the optimal levels $a^*_{n,k}$, $b^*_{n,k}$, $l^*_{n,k}$ for large n's and k's as a function of k for the diffusion with parameters $\mu = 0.04$, $\sigma = \sqrt{0.02}$, $\varphi = 0.95$, $\delta = 0.02$ and initial surplus $u = 2$.

If we assume that $a^*_{n,n} \to \infty$ as $n \to \infty$, we obtain

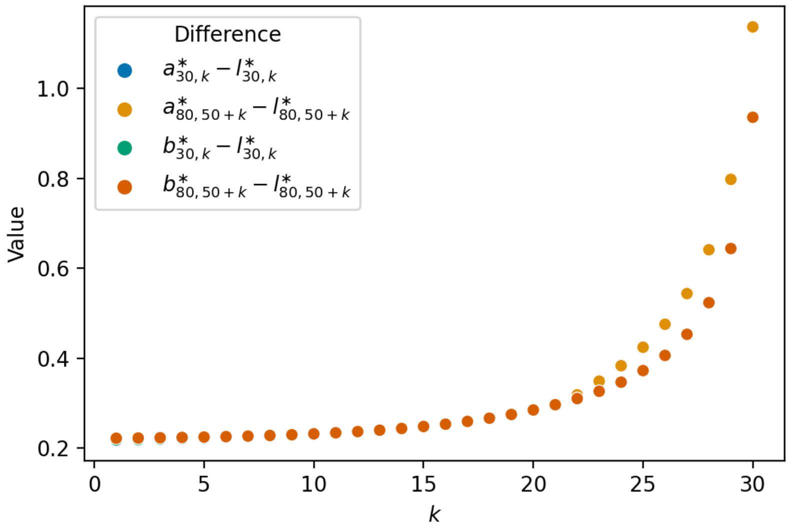
$$-1 + \theta_1 \left(\zeta_1 - \eta_1 + \frac{W_\delta(\eta_1)}{W'_\delta(\eta_1)} \right) + e^{-2\mu\zeta_1/\sigma^2} = 0. \tag{34}$$

This is an implicit equation for ζ_1 which can be easily (numerically) solved given the value of η_1. While the derivation of this equation relied on the (somewhat mild) assumptions made along the way, the value obtained from solving (34) is indeed pleasantly close to the value obtained after computing $a^*_{80,80} - l^*_{80,80}$ with the previous numerical means (1.137077040 against 1.137077050).
Following this line of thought, one could try to obtain simpler equations for ζ_M, η_M and ν_M for $M \geq 1$. Indeed, assuming that (29) is always satisfied, we obtain for $(r,s) \in \{(1,1),(3,2)\}$ and for $m = n - M + 1$,
(35)

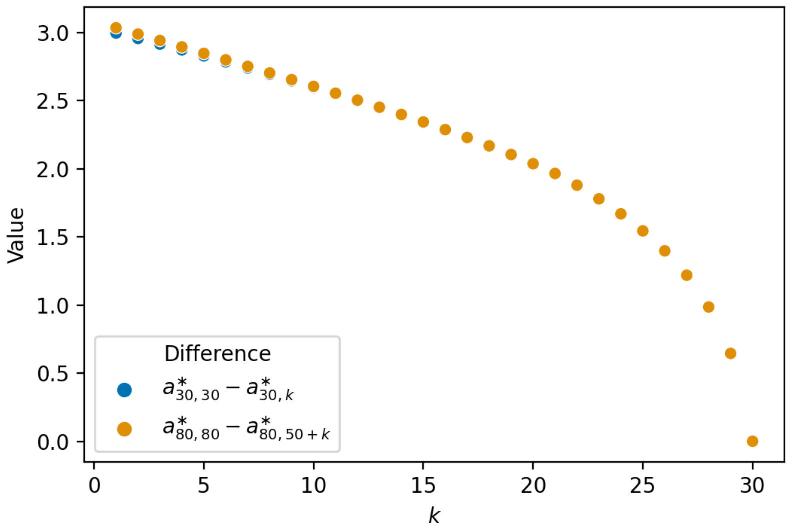
$$\Lambda\kappa'(0)W_0(u)\varphi = \frac{W_\delta(u)D_rB_{n-M+1}\prod_{j=1}^{n-M} G_j + W_\delta(u)D_rG_{n-M+1}\sum_{k=n-M+2}^{n} B_k \prod_{\substack{j=1 \\ j\neq n-M+1}}^{k-1} G_j}{D_sA_{n-M+1}A_{n-M+1}}$$

with

$$B_k = B(a^*_{n,k}, b^*_{n,k}, l^*_{n,k}),$$
$$D_rB_k = D_rB(a^*_{n,k}, b^*_{n,k}, l^*_{n,k})$$



(A) Differences $a^*_{n,k} - l^*_{n,k}$ and $b^*_{n,k} - l^*_{n,k}$ for $n = 30$ and $n = 80$.



(B) Differences $a^*_{n,n} - a^*_{n,k}$ for $n = 30$ and $n = 80$.

FIGURE 10. Plot of the differences between the optimal levels of the last bands. For $n = 80$, there is a shift by of 50 to match the indices with those of $n = 30$.

and similarly for A and G. Plugging (35) in (29) with $m = n - M$, we obtain, after cancellation of common factors,

(36)

$$D_r B_{n-M} + D_r G_{n-M} \sum_{k=n-M+1}^{n} B_k \prod_{j=n-M+1}^{k-1} G_j -$$

$$\frac{D_s A_{n-M} A_{n-M+1}}{D_s A_{n-M+1} A_{n-M}} \left(D_r B_{n-M+1} G_{n-M} + D_r G_{n-M+1} G_{n-M} \sum_{k=n-M+2}^{n} B_k \prod_{j=n-M+2}^{k-1} G_j \right) = 0.$$

Now, observe that the left hand side of (36) converges to zero as $n \to \infty$. However, after multiplication by $W_\delta(a^*_{n,n})$, we can take the limit, obtaining a new set of equations:

(37)
$$\begin{aligned}
&\tilde{B}_1(\zeta_{M+1}, \eta_{M+1}, \nu_{M+1}) + \tilde{G}_1(\zeta_{M+1}, \eta_{M+1}) \sum_{k=1}^{M} \tilde{B}(\zeta_k, \eta_k, \nu_k) \prod_{j=k+1}^{M} \tilde{G}(\zeta_j, \eta_j) \\
&\quad - e^{-2\mu(\nu_{M+1}-\nu_M)/\sigma^2} \tilde{B}_1(\zeta_M, \eta_M, \nu_M) \tilde{G}_1(\zeta_{M+1}, \eta_{M+1}) \\
&\quad - e^{-2\mu(\nu_{M+1}-\nu_M)/\sigma^2} \tilde{G}_1(\zeta_M, \eta_M) \tilde{G}(\zeta_{M+1}, \eta_{M+1}) \sum_{k=1}^{M-1} \tilde{B}(\zeta_k, \eta_k, \nu_k) \prod_{j=k+1}^{M-1} \tilde{G}(\zeta_j, \eta_j) = 0,
\end{aligned}$$

(38)
$$\tilde{B}_2(\eta_{M+1}, \nu_{M+1}) + \tilde{G}_2(\zeta_{M+1}, \eta_{M+1}) \sum_{k=1}^{M} \tilde{B}(\zeta_k, \eta_k, \nu_k) \prod_{j=k+1}^{M} \tilde{G}(\zeta_j, \eta_j) = 0$$

(39)
$$\begin{aligned}
&\tilde{B}_3(\eta_{M+1}, \nu_{M+1}) + \tilde{G}_3(\zeta_{M+1}, \eta_{M+1}) \sum_{k=1}^{M} \tilde{B}(\zeta_k, \eta_k, \nu_k) \prod_{j=k+1}^{M} \tilde{G}(\zeta_j, \eta_j) \\
&\quad - e^{-2\mu(\zeta_{M+1}+\nu_{M+1}-\zeta_M-\nu_M)/\sigma^2} \tilde{B}_3(\eta_M, \nu_M) \tilde{G}_1(\zeta_{M+1}, \eta_{M+1}) \\
&\quad - e^{-2\mu(\zeta_{M+1}+\nu_{M+1}-\zeta_M-\nu_M)/\sigma^2} \tilde{G}_3(\zeta_M, \eta_M) \tilde{G}(\zeta_{M+1}, \eta_{M+1}) \sum_{k=1}^{M-1} \tilde{B}(\zeta_k, \eta_k, \nu_k) \prod_{j=k+1}^{M-1} \tilde{G}(\zeta_j, \eta_j) = 0
\end{aligned}$$

with

$$\begin{aligned}
\tilde{B}(\zeta, \eta, \nu) &= e^{\theta_1 \nu} \left(\zeta - \eta + \frac{W_\delta(\eta)}{W'_\delta(\eta)} \right), \\
\tilde{B}_1(\zeta, \eta, \nu) &= e^{\theta_1 \nu} - \theta_1 \tilde{B}(\zeta, \eta, \nu), \\
\tilde{B}_2(\eta, \nu) &= -e^{\theta_1 \nu} \frac{W_\delta(\eta) W''_\delta(\eta)}{W'_\delta(\eta)^2}, \\
\tilde{B}_3(\eta, \nu) &= -\tilde{B}_2(\eta, \nu) - e^{\theta_1 \nu}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{G}(\zeta, \eta) &= \frac{\sigma^2}{2} e^{-\theta_1 \zeta} \left(W'_\delta(\eta) - \frac{W_\delta(\eta) W''_\delta(\eta)}{W'_\delta(\eta)} \right), \\
\tilde{G}_1(\zeta, \eta,) &= -\theta_1 \tilde{G}(\eta, \nu), \\
\tilde{G}_2(\zeta, \eta) &= \frac{\sigma^2}{2} e^{-\theta_1 \zeta} \left(\frac{W_\delta(\eta) W''_\delta(\eta)^2}{W'_\delta(\eta)^2} - \frac{W_\delta(\eta) W'''_\delta(\eta)}{W'_\delta(\eta)} \right), \\
\tilde{G}_3(\zeta, \eta) &= \theta_1 \tilde{G}(\eta, \nu) - \tilde{G}_2(\eta, \nu).
\end{aligned}$$

These functions can be obtained from W_δ, B, G and their derivatives. Hence, we have

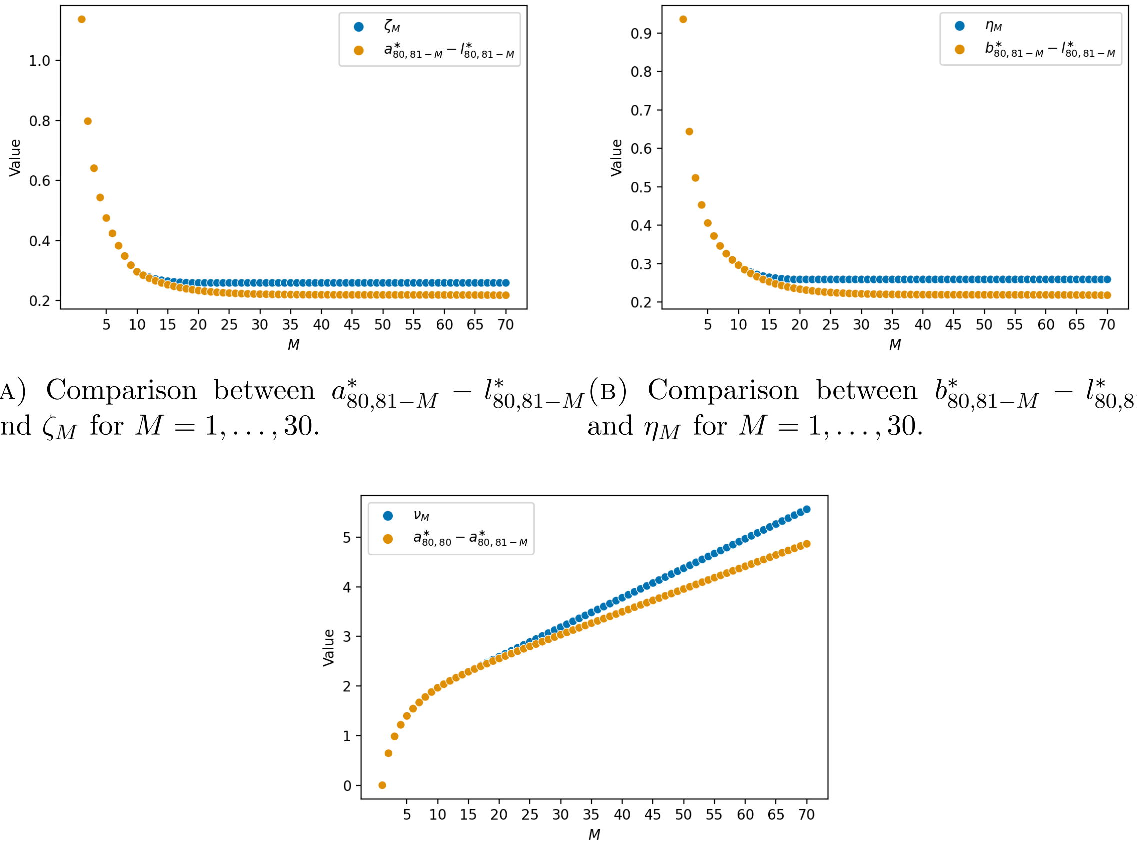
$$\tilde{B}(\zeta, \eta, \nu) = \lim_{x \to \infty} W_\delta(x + \nu) B(x, x + \eta - \zeta, x - \zeta)$$

and similarly for $\tilde{B}_1$, $\tilde{B}_2$ and $\tilde{B}_3$ using $D_1 B$, $D_2 B$ and $D_3 B$ respectively. Likewise, we have

$$\tilde{G}(\zeta, \eta) = \lim_{x \to \infty} G(x + \zeta, x + \eta, x)$$

and similarly for $\tilde{G}_1$, $\tilde{G}_2$ and $\tilde{G}_3$.

Observe that Equations (37) to (39) represent an improvement over the Lagrange equations as by taking the limit we reduce the dimensionality of the problem by eliminating the Lagrange multiplier. Moreover, these equations allow for a truly recursive algorithm since they show that the values of ζ_{M+1}, η_{M+1} and ν_{M+1} depend only on the previous values, and ζ_1 and η_1 can be obtained independently from the equations displayed before. Derivation of (37), (38) and (39) relied on the fact that the gradient equations (29) are always satisfied, which will happen if and only if the optimal strategy is in the interior of $\mathscr{D}_n$. However, as seen by the numerical examples considered before, this is in general not the case as, for example, with the current parameters, one has $a^*_{n,1} = b^*_{n,1}$ for n large enough. This will be reflected in a way that there will exist a minimal M_1 such (37) to (39) will not have a "sensitive" solution (e.g., they will only have a solution with negative values). Numerical experiments for the diffusion show, however, that for $M \geq M_1$ one can simply assume $\eta_M = \zeta_M$ and replace (37) and (38) by the equation obtained after adding their left hand sides (which can be derived from a Lagrange equation after assuming $a^*_{n,n-M+1} = b^*_{n,n-M+1}$), thus obtaining a system of two equations with two unknowns. Curiously enough, the case of the diffusion also shows that there might exist a (minimal) $M_2 \geq M_1$ such that $\eta_M = \zeta_M = \eta_{M_2}$ for all $M \geq M_2$. In this case, one is only left with Equation (39) to obtain the successive values of ν_M, which produces the "linear" behavior observed for the "middle" barriers in Figure 10. Figure 11 shows the results of comparing the results from this procedure with the differences between barrier levels for $n = 80$, where one observes that for these parameters one has $M_1 = 10$ and $M_2 = 19$.



(A) Comparison between $a^*_{80,81-M} - l^*_{80,81-M}$ and ζ_M for $M = 1, \ldots, 30$.

(B) Comparison between $b^*_{80,81-M} - l^*_{80,81-M}$ and η_M for $M = 1, \ldots, 30$.

(C) Comparison between $a^*_{80,80} - a^*_{80,81-M}$ and ν_M for $M = 1, \ldots, 30$.

FIGURE 11. Comparison between the limits in (33) and the distances between barrier levels for $n = 80$.

Now, while the divergence of $a^*_{n,n}$ to infinity implies that a limit strategy does not exist, it might still be useful to compute the values of ζ_k, η_k and ν_k for $k = 1, \ldots, M$ for a large M: as explained before (cf. Section 5), for large n, the algorithms have difficulties finding the optimal levels unless an appropriate set of values is provided for initialization. Since the previous figures indicate that convergence of the limits happens relatively fast, one might want to use the following pseudo-algorithm (Algorithm 2) to approximate the values of $a^*_{n,k}, b^*_{n,k}, l^*_{n,k}$, $k = 1, \ldots, n$ for large n. The idea is to suppose that k is large

Input : Large n and $M < n$ such that the problem with $k := M - n$ barriers can easily be solved (for example, $k = 30$).
Output: Approximation to optimal levels $a^*_{n,k}, b^*_{n,k}$ and $l^*_{n,k}$, $k = 1, \ldots, n$.

1 **begin**
2 $(\zeta_1, \eta_1, \nu_1, \ldots, \zeta_M, \eta_M, \nu_M) :=$ `computeLimits`(M);
3 `initialize`$(\{(a_k, b_k, l_k) \mid k = 1, \ldots, n - M\})$;
4 `initialize`(a_n);
5 **for** $j := n - M + 1$ **to** n **do**
6 $a_j := a_n - \nu_{n-j+1}$;
7 $b_j := a_j - \zeta_{n-j+1} + \eta_{n-j+1}$;
8 $l_j := a_j - \zeta_{n-j+1}$;
9 **end**
10 **while** *not convergence* **do**
11 `improve`$(\{(a_k, b_k, l_k) \mid k = 1, \ldots, n - M\})$;
12 `improve`(a_n);
13 **for** $j := n - M + 1$ **to** n **do**
14 $a_j := a_n - \nu_{n-j+1}$;
15 $b_j := a_j - \zeta_{n-j+1} + \eta_{n-j+1}$;
16 $l_j := a_j - \zeta_{n-j+1}$;
17 **end**
18 **end**
19 **end**

ALGORITHM 2: Corridor level optimization pseudo-algorithm

enough so that convergence of the limits in (33) is already achieved (or close enough to be achieved) and hence one only needs to optimize over $3k + 1$ variables instead of $3n$. Moreover, the constraint on the survival probability might be used to find a suitable value for a_n and hence the `initialize` and `improve` functions on Lines 4 and 12 can be thought as determined by this condition. This pseudo-algorithm is not restricted to work only with the barrier levels but can be adapted to the step-wise survival probabilities instead (the $s^*_{n,k}$'s described in Section 5).

7. Conclusions and further remarks

In this paper we proposed a new kind of dividend strategies which naturally generalize classical barrier strategies, but have the advantage of being adjusted to control for survival probability. While the performance of these strategies turns out to be process- and parameter-dependent, their nature has an easy interpretation and – as observed in the illustrations – typically only a few parameters are needed to reach a remarkable resulting survival probability while not losing much efficiency. It is rather surprising that for a small number of corridors, the performance of these strategies turns out to be typically outstanding, and, as opposed to linear dividend strategies (which sometimes lead to slightly better results), this performance is consistent across the parameter space. Correspondingly, corridor strategies can serve as benchmarks for further studies of the constrained optimization problem.

Much like in the passage from barrier to band strategies, one can think of a further generalization of the strategies proposed here: given sequences of surplus levels $a_1, a_2, a_3, \ldots, a_n$ and $l_1, l_2, l_3, \ldots, l_n$ with $l_n \leq \min(a_n, a_{n+1})$ and stopping times τ_k as defined in Section 2, at time τ_k, one can proceed to pay dividends according to an r-band strategy up until the time when the process controlled in this way reaches the lower limit l_k. It is easy to see that in this case the formula for V is similar to (9), but D_k is replaced by the relevant value. We could go yet one step further and allow the number of bands at each corridor to vary, however, since locally in the time interval $[\tau_k, \tau_k^d]$ the process behaves exactly like a controlled process for a normal band strategy, it seems a priori better to keep the number of bands constant and equal to the number of bands of the band strategy that produces the overall best dividend-payment strategy. Since the dimensionality and complexity of the formulas for this generalization increases greatly with the number of bands considered, we preferred in the present paper to adhere to the simpler case of barrier strategies in each corridor. A further difficulty arises when trying to generalize the results of Section 4, as it is well known that for general band strategies, the value function is not necessarily continuous.

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