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ON A GAMMA SERIES EXPANSION FOR THE TIME-DEPENDENT PROBABILITY OF COLLECTIVE RUIN *

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Abstract

In the framework of the extended classical risk model with constant force of real interest i we investigate, when it is suitable to represent the probability of collective survival $U(x,t)$ of an insurance company with initial capital x and time horizon t as a gamma series. Moreover we derive exact analytical solutions for exponentially distributed claim sizes and integer values of λ/i, where λ is the risk parameter. As a by-product we observe that numerical procedures for estimating $U(x,t)$ are very accurate.

Keywords:
Classical risk model, ruin probability, finite time interval, real interest force

1 Introduction

Let $\{N(t) : t \in \mathbb{R}_+\}$ denote the random process that counts the claims of an insurance portfolio of a company and assume that $N(t)$ is a homogeneous Poisson process with intensity λ. Let further $\{X_n : n \in \mathbb{N}\}$ be a sequence of independent identically distributed positive random variables with density function $f(x)$ representing the sizes of the successive claims. If we allow for a constant inflation force δ, then the nth claim is not equal to X_n as in the classical case but rather $e^{\delta T_n} X_n$, where T_n $(n \in \mathbb{N})$ denotes the moment of occurrence of the nth claim. In a time interval $[t, t+dt]$ the company receives the premium $c(t)dt$, where $c(t) = c\,e^{\delta t}$ and $c = c(0) > 0$ is the premium density at $t = 0$. In addition to the premium income, the company also receives interest of its reserves with a constant interest force $\tilde{i}$ (for $\delta = \tilde{i} = 0$ we have the classical risk model). If we introduce the purely discontinuous measure $e^{\delta t} X_{N_t}\,\mathrm{d}N_t$ which puts a weight equal to $e^{\delta T_n} X_{N_t}$ at times T_n $(n \in \mathbb{N})$, then the value of the reserve at time t, denoted by $Z(t)$, satisfies

$$\mathrm{d}Z(t) = c\,e^{\delta t}\,\mathrm{d}t + Z(t)\,\cdot\,\tilde{i}\,\mathrm{d}t - e^{\delta t} X_{N_t}\,\mathrm{d}N_t$$

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(see for example DELBAEN AND HAEZENDONCK [3]). For general background in ruin theory we refer to BÜHLMANN [2] and GERBER [4].
We are now interested in the probability that the company survives through time t

$$U(x,t) = Pr\{Z(s) \geq 0 \ \forall \ 0 \leq s \leq t \,|\, Z(0) = x\},$$

where $x \geq 0$ denotes the initial reserve of the company. Correspondingly the probability of ruin is defined by $\psi(x,t) = 1 - U(x,t)$.
For an infinite time horizon (i.e. $t = \infty$) SUNDT AND TEUGELS [12, 13] have studied equations for $U(x,\infty)$ and derived approximations and bounds using Laplace transformation with respect to x. For arbitrary t the following integro-differential equation for the probability of survival $U(x,t)$ within the finite time interval $[0,t]$ and initial capital x for any claim size density function $f \in C^1$ can be derived:

$$(c+ix)\frac{\partial U}{\partial x} - \frac{\partial U}{\partial t} - \lambda\, U + \lambda \int_0^x U(x-y,t)\, f(y)\, dy \;=\; 0 \tag{1}$$

with initial condition $U(x,0) = 1$, where $i = \tilde{i} - \delta$ denotes the real (constant) interest force and is assumed to be positive. Equation (1) is a generalization of a fundamental result of SEAL [9], see e.g. [5]. In the case of exponential claim size distributions KNESSL AND PETERS [5] studied this equation by Laplace transformation with respect to t and obtained explicit expressions for the Laplace transform of the survival probability. Furthermore the asymptotic behavior of $U(x,t)$ was analysed for $i = 0$ in [5] and for $i > 0$ in [6]. For $i = 0$ and exponentially distributed claims PERVOZVANSKY [8] derived an exact explicit expression for $U(x,t)$ by inverting the Laplace transform using complex analysis techniques.

The idea of representing $U(x,t)$ as a series of incomplete gamma functions goes back to TAYLOR [14], who considered the case of $i = 0$. In this paper we study the possibility of generalizing this approach to arbitrary $i \geq 0$. In Section 2 we derive a recurrence relation for the coefficients of the gamma series for a large class of claim size distributions. This relation is then analysed in Section 3 for the exponential case. For integer values of λ/i we obtain simple exact expressions for $U(x,t)$. Furthermore we show that for arbitrary values of λ/i in the case of an infinite time horizon the gamma series representation leads to a well-known explicit expression for $U(x,\infty)$.

2 A gamma series expansion for $U(x,t)$

We look for a solution of (1) of the form

$$U(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) P(n, \alpha x), \tag{2}$$

where the notation

$$P(n, \alpha x) = \frac{\gamma(n, \alpha x)}{\Gamma(n)} = \frac{1}{\Gamma(n)} \int_0^{\alpha x} y^{n-1} e^{-y} dy, \qquad \alpha > 0,\, n > 0 \tag{3}$$

is chosen according to [1] (the parameter α could of course be omitted w.l.o.g., but will turn out to be useful later on). The existence of such a solution will be discussed later. Suppose that the claim distribution function $F(x)$ can also be expressed as a series of the same incomplete gamma functions

$$F(y) = \sum_{n=1}^{\infty} f_n P(n, \alpha y) \tag{4}$$

(which is e.g. the case if $F(y)$ agrees on $(0, \infty)$ with an entire function $B(\cdot)$ with the property $|B^{(k)}(0)| < K^k$ for some constant $K > 0$ and all $k \geq 1$ (cf. [14])). Now we make use of the fact that for positive and integral n the incomplete gamma function can be written as

$$P(n, \alpha x) = 1 - e^{-\alpha x} \sum_{j=0}^{n-1} \frac{(\alpha x)^j}{j!} \tag{5}$$

and thus it can easily be verified that

$$\frac{\partial P(n, \alpha x)}{\partial x} = \alpha \Big(P(n-1, \alpha x) - P(n, \alpha x) \Big), \quad n = 1, 2, 3, \ldots \tag{6}$$

with the definition $P(0, \alpha x) = 1$. Furthermore

$$x \frac{\partial P(n, \alpha x)}{\partial x} = x \alpha e^{-\alpha x} \frac{(\alpha x)^{n-1}}{(n-1)!} = n e^{-\alpha x} \frac{(\alpha x)^n}{n!} = n \Big(P(n, \alpha x) - P(n+1, \alpha x) \Big). \tag{7}$$

If we now substitute (2),(4), (6) and (7) into (1), we obtain:

$$c \sum_{n=1}^{\infty} a_n(t) \alpha \Big(P(n-1, \alpha x) - P(n, \alpha x) \Big) + i \sum_{n=1}^{\infty} a_n(t) n \Big(P(n, \alpha x) - P(n+1, \alpha x) \Big) =$$

$$= a_0'(t) + \sum_{n=1}^{\infty} a_n'(t) P(n, \alpha x) + \lambda a_0(t) + \lambda \sum_{n=1}^{\infty} a_n(t) P(n, \alpha x)$$

$$- \lambda \int_0^x \Big[a_0(t) + \sum_{n=1}^{\infty} a_n(t) P(n, \alpha(x-y)) \Big] \Big[\sum_{n=1}^{\infty} f_n \, dP(n, \alpha y) \Big]. \tag{8}$$

Equating the coefficients of $P(1, \alpha x)$, $P(2, \alpha x), \ldots$ yields:

$$a_1(t) = \frac{1}{\alpha c} \left(a_0'(t) + \lambda\, a_0(t) \right) \tag{9}$$

and for $n \in \mathbb{N}$

$$a_{n+1}(t) = \frac{1}{\alpha c}\Big((\lambda + \alpha c - i\,n)a_n(t) + a_n'(t) + i\,(n-1)a_{n-1}(t) - \lambda\{\mathbf{a}(t) * \mathbf{f}\}_n\Big), \quad (10)$$

where the sequence $\{\mathbf{a}(t) * \mathbf{f}\}_n = a_0(t)f_n + a_1(t)f_{n-1} + \ldots + a_{n-1}f_1$ $(n \in \mathbb{N})$ is the convolution of the two sequences $\mathbf{a}(t) = a_0(t), a_1(t)$, etc. and $\mathbf{f} = f_1, f_2$, etc.
By setting $x = 0$ in (2) the series can be started with

$$a_0(t) = U(0,t). \quad (11)$$

If the representation of the claim size distribution function $F(y)$ as a series of incomplete gamma functions contains only finitely many terms (that is $f_n = 0$ for $n > N$), then (10) simplifies to the difference equation

$$a_{n+1}(t) = \frac{1}{\alpha c}\Big((\lambda + \alpha c - i\,n)a_n(t) + a_n'(t) + i\,(n-1)a_{n-1}(t) - \\ - \lambda f_1 a_{n-1}(t) - \ldots - \lambda f_N a_{n-N}(t)\Big) \quad (12)$$

In the special case of an exponential claim size distribution $f(z) = \alpha\, e^{-\alpha\, z}$ we have $N = 1, f_1 = 1$ and thus we get the second-order linear recursion formula

$$a_{n+1}(t) = \frac{1}{\alpha c}\Big((\lambda + \alpha c - i\,n)a_n(t) + a_n'(t) + (i\,(n-1) - \lambda)\, a_{n-1}(t)\Big). \quad (13)$$

Remark: If we look at the limit $t \to \infty$, the above equations simplify, as all terms with derivatives w.r.t. t vanish. This leads to the representation

$$U(x,\infty) = a_0(\infty) + \sum_{n=1}^{\infty} a_n(\infty) P(n, \alpha x), \quad (14)$$

where

$$\begin{aligned} a_0(\infty) &= U(0,\infty), \\ a_1(\infty) &= \tfrac{\lambda}{\alpha c} a_0(\infty), \\ a_{n+1}(\infty) &= \tfrac{1}{\alpha c}\Big((\lambda + \alpha c - i\,n)a_n(\infty) + i\,(n-1)\, a_{n-1}(\infty) - \lambda\{\mathbf{a}(\infty) * \mathbf{f}\}_n\Big) \end{aligned} \quad (15)$$

and the infinite-time survival probability $U(0,\infty)$ with initial capital 0 is a function of c, λ, α and i only.
For the exponential claim size distribution recurrence (15) again simplifies to

$$a_{n+1}(\infty) = \frac{1}{\alpha c}\Big((\lambda + \alpha c - i\,n)a_n(\infty) + (i\,n - i - \lambda)\, a_{n-1}(\infty)\Big). \quad (16)$$

3 Analysis of the recurrence relation

Formula (12) enables us to investigate the existence of simple explicit solutions of (1) of certain type by appropriate choice of the involved parameters. Let us concentrate on the case of an exponential claim size distribution, i.e. equation (13), first:

3.1 Some exact solutions for exponentially distributed claims

Theorem 1. *Let the claim size distribution be exponential with parameter α and let the parameters λ, i be chosen such that*

$$\lambda = k\, i \quad (k \in \mathbb{N}). \tag{17}$$

Then the finite-time survival probability is given by

$$U(x,t) = a_0(t) + \sum_{n=1}^{k} a_n(t) P(n, \alpha x), \tag{18}$$

and there is a simple algorithm to calculate $a_n(t)$, $n \geq 0$ in this case.
Moreover, (17) is also a necessary condition for a finite-sum representation of $U(x,t)$ of type (18).

Proof: The only possibility for the coefficients $a_n(t)$ to vanish in (13) on from a certain index n_0, is that $i(n_0 - 1) - \lambda = 0$ holds (considering the fact that $0 < a_0(t) = U(0,t) \leq 1$ and the second order of the recurrence). This simple observation restricts the representation of $U(x,t)$ in the form (2) with finitely many terms to the parameter choice $\lambda = k\, i$ $(k \in \mathbb{N})$.
Furthermore, one can indeed derive an explicit solution for (1) for every $k \in \mathbb{N}:$ For a fixed $k \in \mathbb{N}$ we have to show that $a_{k+1}(t) = 0$ (because then $a_{k+j}(t) = 0 \;\forall\; j \geq 1$). This condition leads to the homogeneous linear differential equation

$$\sum_{j=0}^{k+1} c_j \, a_0^{(j)}(t) = 0 \tag{19}$$

with constant coefficients $c_j \in \mathbb{R}$, where $a_0^{(j)}(t)$ denotes the j-th derivative of $a_0(t)$. One can inductively show that $c_j > 0$ for $j \geq 1$ and $c_0 = 0$. Now let us assume that all the roots of the characteristic equation of (19), which we denote by $-R_j$ $(j = 0 \ldots k+1)$, are pairwise distinct (which is only a technical restriction). Then $a_0(t)$ is of the form

$$a_0(t) = U(0,t) = A_0 + \sum_{j=1}^{k+1} A_j \, e^{-R_j t}. \tag{20}$$

From $c_j > 0$ for $j \geq 1$ it follows that $R_j > 0$, if $R_j \in \mathbb{R}$, as it should be.
In order to obtain the coefficients A_j one now has to calculate $a_j(t)$, $j = 1, \ldots, k$

using (20) and (13). Then by inserting those into (2) we can derive a linear system of equations for A_j by equating the coefficients of x^n in $U(x,0) = 1$. This inhomogeneous system contains exactly $k+1$ equations for $k+1$ variables and thus we can easily calculate the desired coefficients A_j and we finally arrive at (18) as the solution of (1) for exponentially distributed claims and $\lambda = k\,i$ ($k \in \mathbb{N}$), where $a_n(t)$ are determined by (13) and (20). □

This procedure can easily be implemented in a computer program.

Remark: From the recurrence structure of (10) and (12), respectively, it follows that the exponential claim size distribution together with above choice of the corresponding parameters is the only one among all the claim size distributions of type (4) that enables a finite-sum-representation of type (2) for its solution $U(x,t)$.

Examples: (a) In the case $\lambda = i$ (i.e. $k = 1$) KNESSL AND PETERS [5] derived the exact solution

$$U(x,t) = 1 - \frac{i}{i+\alpha c}\, e^{-\alpha x} \left(1 - e^{-(i+\alpha c)t}\right) \tag{21}$$

by means of Laplace transformation w.r.t. t. Using the gamma series approach above we obtain the same result rather easily, with only one iteration step.

(b) As another illustrating example of the above method we give the explicit solution of (1) for $f(x) = \alpha\, e^{-\alpha x}$ and $\lambda = 2\,i$ (see also Fig. 1):

$$U(x,t) = b_0(t) + \Big(1 - e^{-\alpha x}\Big) b_1(t) + \Big(1 - e^{-\alpha x} - \alpha x\, e^{-\alpha x}\Big) b_2(t)$$

with

$$b_0(t) = \frac{\alpha^2 c^2 D + e^{-R_1 t}\Big(i^2 \alpha c + D\,(i\alpha\, c + i^2) - i^3\Big) + e^{-R_2 t}\Big(- i^2 \alpha\, c + i^3 + D\,(i\alpha\, c + i^2)\Big)}{D\,(\alpha^2\, c^2 + 2\, i\alpha\, c + 2\, i^2)},$$

$$b_1(t) = \frac{i\Big(2\, D\, \alpha\, c - e^{-R_1 t}\,\Big(- 4\, i^2 - i\, \alpha\, c + D\, \alpha\, c\Big) - e^{-R_2 t}\,\Big(4\, i^2 + i\, \alpha\, c + D\, \alpha\, c\Big)\Big)}{D\,(\alpha^2\, c^2 + 2\, i\, \alpha\, c + 2\, i^2)},$$

$$b_2(t) = -\frac{i^2\,\Big(e^{-R_1 t}\,\Big(2\, \alpha\, c + 3\, i + D\Big) - 2\, D + e^{-R_2 t}\,\Big(- 2\, \alpha\, c - 3\, i + D\Big)\Big)}{D\,(\alpha^2\, c^2 + 2\, i\, \alpha\, c + 2\, i^2)}$$

where

$$R_1 = \frac{2\, \alpha\, c + 3\, i - D}{2}, \qquad R_2 = \frac{2\, \alpha\, c + 3\, i + D}{2},$$

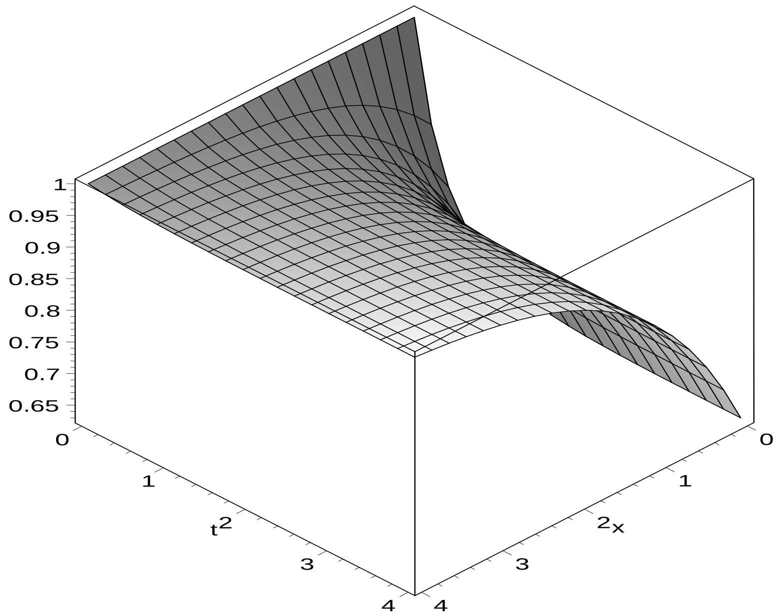

Figure 1: $U(x,t)$ for $c = 2.1$, $\lambda = 1$, $\alpha = 1$, $i = 0.5$

and $D = \sqrt{i\,(4\,\alpha\,c + i)}$.

(c) By fixing λ and taking $i = \frac{\lambda}{k}$, $k \to \infty$, one can study the limit behavior of $U(x,t)$ for $i \to 0$ using analytic solutions. For example, from (18) and (20) it is easily seen that the asymptotic behavior of $U(x,t)$ for large t is determined by the (in absolute value) smallest negative root $-R_{j_0}$ of the characteristic equation of (19), where R_{j_0} decreases with decreasing force of real interest i. This in general means that for smaller interest rates it takes larger values of $t = T$ so that $U(x,t),\ t > T$ is practically identical to $U(x,T)$, in agreement with the results of Knessl and Peters [6]. For given parameters and a specified error bound we can use our exact formula for $U(x,t)$ to explicitly calculate T.

(d) In [6] some estimates for $U(x,t)$ for discrete values of t in the presence of constant real interest force i and exponential claim size distribution were calculated by numerical inversion of the Laplace transform of $U(x,t)$ via truncation of the Bromwich contour integral along the imaginary axis at a given point and then calculating the truncated integral and estimating the thereby produced error. Another application of our exact analytical formulae is the possibility of checking this numerical procedure. Figure 2 shows the exact solution (solid line) and the numerical estimates of [6] (crosses) for the probability of ruin with initial capital $x = 10$ and $c = 2$, $\lambda = \alpha = 1$, $i = 0.1$. Figure 3 depicts the exact function and the discrete numerical estimates for a qualitatively different region of parameter choice, namely $\lambda = 2$, $c = 1, \alpha = 1$, $i = 0.1$ (cf. [6]). Both graphs show that the performance of the numerical procedure described above is very satisfactory.

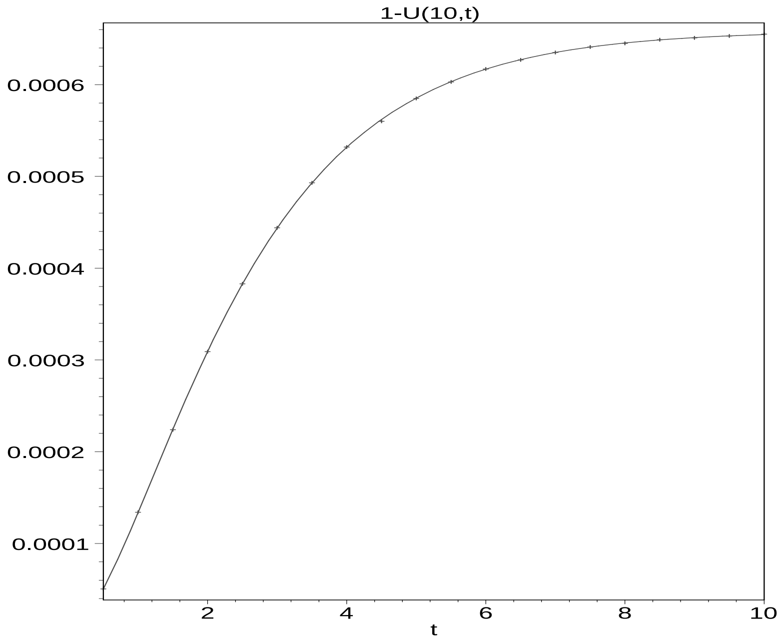

Figure 2: Numerical and analytical results for $\psi(10,t)$ for $c > \lambda/\alpha$

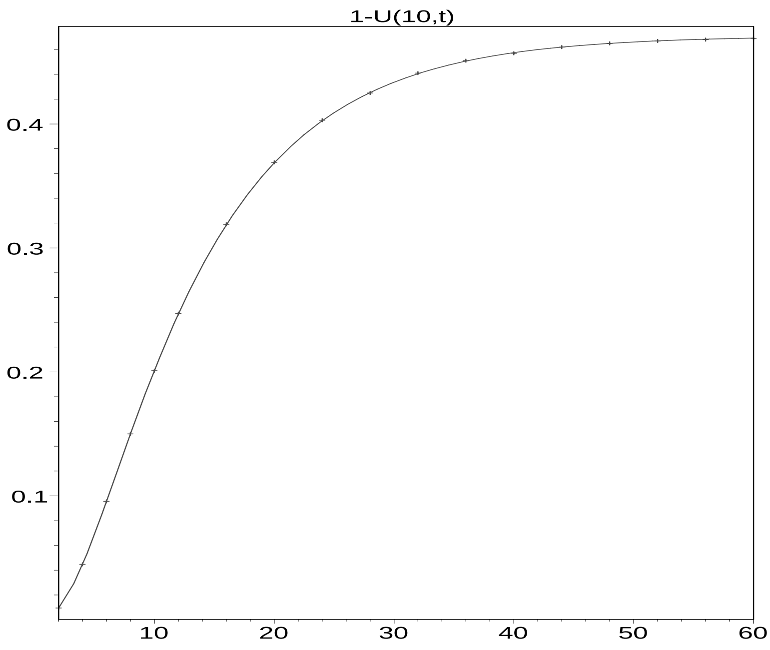

Figure 3: Numerical and analytical results for $\psi(10,t)$ for $c < \lambda/\alpha$

(e) Delbaen and Haezendonck [3] derived upper bounds for $U(x,t)$ (denoted DH-bounds for short) by virtue of martingale methods. In the case of an exponential claim size distribution we can use our exact solutions to test the sharpness of these bounds. A direct qualitative comparison of the exact solution and the DH-bound based on their known analytical expressions is a very time-consuming task due to the complex structure of the exact solution. Thus we confine ourselves to a quantitative comparison here: Figure 4 and 5 illustrate the empirical observation that the DH-bound is about three to ten times the exact value of $\psi(u,t)$ for typical choices of the parameter values (the factor getting smaller as i decreases).

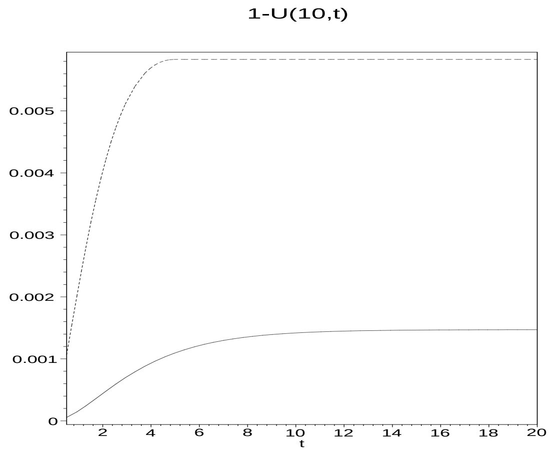

Figure 4: Exact solution (solid line) and DH-bound (dashed line) of $\psi(10,t)$ for $c = 1.6, \alpha = 1, \lambda = 1, i = 0.1$

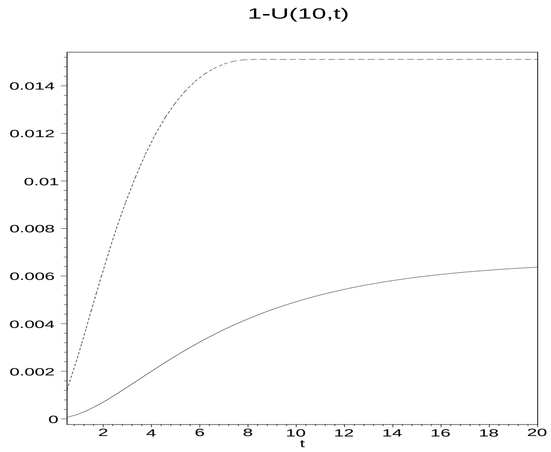

Figure 5: Exact solution (solid line) and DH-bound (dashed line) of $\psi(10,t)$ for $c = 1.6, \alpha = 1, \lambda = 1, i = 0.025$

3.2 Infinite time horizon

The question now arises, whether something can be said about the convergence of (2) in general (i.e. $\lambda \neq k\,i$). If the coefficients $a_n(t)$ are bounded, then the convergence of the series can be shown by elementary techniques. For some general tools for investigating the asymptotic behavior of linear recurrences we refer to Kooman [7]. However, studying the asymptotic behavior of (13) seems very difficult.
If we restrict ourselves to the infinite time horizon case, then the corresponding linear recurrence (16) can be solved:
For that purpose, consider the generating function

$$A(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where we use $a_n := a_n(\infty)$ for short. From (16) one then derives that

$$\alpha c \sum_{n=1}^{\infty} a_{n+1} z^{n-1} = \sum_{n=1}^{\infty} (\lambda + \alpha c - in) a_n z^{n-1} + \sum_{n=1}^{\infty} (i(n-1) - \lambda) a_{n-1} z^{n-1}.$$

With a little bit of calculations we find that

$$i(z-1)A'(z) + A(z)\frac{1}{z^2}\{\lambda + \alpha c - \lambda z - \alpha c\} = a_0 \frac{\lambda + \alpha c}{z} - \frac{a_1 \alpha c}{z} - \frac{a_0 \alpha c}{z^2}.$$

Using the fact that $a_1 = \frac{\lambda}{\alpha c} a_0$, it is straightforward to find that $A(z)$ satisfies the first order non-homogeneous differential equation

$$iz^2 A'(z) + (\alpha c - \lambda z)A(z) - a_0 \alpha c = 0.$$

The homogeneous equation has the solution

$$A_h(z) = z^{\frac{\lambda}{i}} \exp\left(\frac{\alpha c}{iz}\right).$$

Put

$$A(z) = C(z) A_h(z)$$

and determine the function $C(z)$ in order to solve the non-homogeneous equation. Here some care is needed. Clearly for $z \neq 0$ we find that

$$C'(z) = \frac{a_0 \alpha c}{i} z^{-\frac{\lambda}{i} - 2} \exp\left(-\frac{\alpha c}{iz}\right).$$

For some constant K we have that

$$A(z) = \left(K - a_0 \left(\frac{i}{\alpha c}\right)^{\frac{\lambda}{i}} \int_0^{\frac{\alpha c}{iz}} e^{-t} t^{\frac{\lambda}{i}}\, dt\right) z^{\frac{\lambda}{i}} \exp\left(\frac{\alpha c}{iz}\right).$$

Since $A(0) = a_0$, it is necessary that

$$K = a_0 \left(\frac{i}{\alpha c}\right)^{\frac{\lambda}{i}} \Gamma(\frac{\lambda}{i} + 1).$$

This means that

$$A(z) = a_0 \left(\frac{iz}{\alpha c}\right)^{\frac{\lambda}{i}} \exp\left(\frac{\alpha c}{iz}\right) \int_{\frac{\alpha c}{iz}}^{\infty} e^{-t} t^{\frac{\lambda}{i}}\, dt$$

or even better

$$A(z) = a_0 \int_0^{\infty} e^{-w} \left(1 + \frac{izw}{\alpha c}\right)^{\frac{\lambda}{i}} dw.$$

An expansion now gives the explicit expression

$$a_n = a_0\,\lambda(\lambda - i)\dots(\lambda-(n-1)i)(\alpha c)^{-n} = a_0\,n!\binom{\lambda/i}{n}\left(\frac{i}{\alpha c}\right)^n, \quad n \geq 1. \tag{22}$$

It can easily be checked that this expression solves (16). Moreover, the reason why there remains only a finite number of non-zero a_n for integral λ/i becomes clear (cf. (18) for $t=\infty$).
Introducing expression (22) in (14) leads to

$$U(x,\infty) = a_0 + a_0\,\frac{\lambda}{c}\int_0^x e^{-\alpha y}\left(1+\frac{iy}{c}\right)^{\frac{\lambda}{i}-1}\,dy$$

which can be rewritten in terms of incomplete gamma functions in the form

$$U(x,\infty) = a_0 + a_0\frac{\lambda}{i}\left(\frac{i}{\alpha c}\right)^{\frac{\lambda}{i}} e^{\frac{\alpha c}{i}}\left[\Gamma\left(\frac{\lambda}{i},\frac{\alpha c}{i}\right) - \Gamma\left(\frac{\lambda}{i},\frac{\alpha c}{i}+\alpha x\right)\right], \tag{23}$$

where $\Gamma(a,x) = \int_x^\infty e^{-t}t^{a-1}\,dt$.
Now, one can calculate $a_0 = U(0,\infty)$ by taking the limit $x \to \infty$ in (23) (recall that $\lim_{x\to\infty} U(x,\infty) = 1$) and we finally arrive at the pleasant formula

$$U(x,\infty) = 1 - \frac{\Gamma(\frac{\lambda}{i},\frac{\alpha c}{i}+\alpha x)}{\Gamma(\frac{\lambda}{i},\frac{\alpha c}{i}) + \frac{i}{\lambda}\left(\frac{\alpha c}{i}\right)^{\lambda/i} e^{-\alpha c/i}}, \tag{24}$$

which was already given in Segerdahl [11, 10] and proved in various other ways in the literature (cf. [5],[12]).

4 Conclusion

We showed that using a gamma series expansion, simple exact solutions can be obtained for the time-dependent probability of survival $U(x,t)$ for exponentially distributed claim sizes in the presence of a constant interest force and certain model parameter relations. We used our result to test numerical techniques for calculating $U(x,t)$. Moreover we proved that such a gamma series expansion leads to an explicit expression for the infinite time survival probability for arbitrary choices of the parameters.
As this method leads to a recursion based on $U(0,t)$ and this term is sometimes easier to calculate than $U(x,t)$ (especially in connection with Laplace transforms w.r.t. x), this approach might also be useful in other related problems in ruin theory.

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