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# POVERTY TRAPPING: A RUIN THEORY PERSPECTIVE

FLORES CONTRO José Miguel

UNIL | Université de Lausanne

FACULTÉ DES HAUTES ÉTUDES COMMERCIALES

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THÈSE DE DOCTORAT

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Faculté des Hautes Études Commerciales
de l'Université de Lausanne

pour l'obtention du grade de
Doctorat en sciences actuarielles

par

José Miguel FLORES CONTRO

Directrice de thèse
Prof. Séverine Arnold

Jury

Prof. Boris Nikolov, président
Prof. Hansjoerg Albrecher, expert interne
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José Miguel FLORES CONTRO

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***Poverty Trapping: A Ruin Theory Perspective***

sans se prononcer sur les opinions exprimées dans cette thèse.

Lausanne, le 12.06.2024

Professeure Marianne Schmid Mast, Doyenne

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and have found it to meet the requirements for a doctoral thesis.
All revisions that I or committee members
made during the doctoral colloquium
have been addressed to my entire satisfaction.

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# Summary

This thesis is structured around a key question: To what extent are social protection strategies (and complementary instruments such as insurance) effective in reducing poverty? This work is composed of four articles that attempt to provide an answer to our central question. These articles lie at the intersection of two scientific disciplines: insurance mathematics and development economics.

The first article (Chapter 2) explores the benefits of collaboration between governments and private insurers by examining the effects of insurance (with and without subsidies) on the probability of a household falling into the area of poverty (the trapping probability). Applying concepts from ruin theory, a branch of insurance mathematics, we study a risk process with random-valued losses that models a household's capital over time. For this model, we derive explicit formulas for the trapping probability and the cost of social protection incurred by the government, which is defined in a way in which it incorporates premium subsidies, capital transfers needed to close the poverty gap and an additional fixed cost that ensures, with a certain level of confidence, that households will not return to poverty, should they fall into it. Our analysis reveals the insufficiency of insurance alone (without subsidies) as a means of poverty reduction for the vulnerable non-poor (those with capital levels just above the poverty line), as premium payments can in fact heighten the risk of falling into poverty. It therefore highlights the benefits of subsidising insurance for both households and governments, as they experience a reduction in their trapping probability and the incurred cost of social protection, respectively.

In the second article (Chapter 3), we adapt the risk process to consider random proportional rather than random-valued capital losses. This adaptation changes the approach to the analysis of the risk process, as traditional techniques commonly used in ruin theory are no longer straightforward. Nevertheless, adopting alternative methods, we derive for the first time a closed-form solution for the trapping probability of the risk process under the proportional configuration. In this article, we also assess the impact of insurance (without subsidies) on the trapping probability. Our results suggest that insurance for proportional losses is more affordable than coverage for random-valued losses, which aligns with the idea that premiums are normalised to wealth under the proportional loss structure.

The contributions of the third article (Chapter 4) are mainly concerned with poverty measurement, which is undoubtedly a key point in the monitoring and evaluating phase of a social protection strategy. In this article, we derive an integral equation for the Gerber-Shiu expected discounted penalty function, a mathematical function

widely studied in ruin theory that gives information on three trapping-related quantities: the trapping time (the point in time at which a household falls into the area of poverty), the capital deficit at trapping and the capital surplus prior to trapping. We then note the relationship between the capital deficit and an important group of poverty measures, known as the Foster-Greer-Thorbecke (FGT) index. In addition, for a particular case of the risk process with proportional losses, we derive a microeconomic foundation for the beta of the first kind (B1) as a suitable model to describe the distribution of the capital deficit given that trapping occurs. In particular, this finding allows to interpret the shape parameters of the theoretical distribution, as well as to perform sensitivity analyses of poverty measures to variations in the shape parameters. To conclude this article, we exemplify this notion using data from Burkina Faso's *Enquête Multisectorielle Continue (EMC) 2014*, a survey aimed at generating robust data to monitor the country's sustainable development.

The fourth article (Chapter 5) studies the role of cash transfer programmes in poverty alleviation. In this article, we consider an omega risk process with proportional losses; an extension of the risk process studied in previous articles that now also incorporates direct transfers (capital cash transfers) provided by donors or governments to only those households deemed eligible. This extension allows us to introduce a new event: the event of extreme poverty (the event when a household becomes extremely poor), which only depends on an extreme poverty rate that is a function of the household's current capital. Under this model, we derive closed-form expressions for the trapping probability and then do the same for the probability of extreme poverty (the probability that an event of extreme poverty occurs). Our numerical illustrations expose the ability of cash transfer programmes to keep households out of poverty and extreme poverty. In particular, we outline the role of both the intensity (or frequency) of the transfers and the eligibility threshold in achieving lower probabilities.

# Résumé


Cette thèse s'articule autour d'une question clé : Dans quelle mesure les stratégies de protection sociale (et les instruments complémentaires tels que l'assurance) sont-elles efficaces pour réduire la pauvreté ? Ce travail est composé de quatre articles qui tentent d'apporter une réponse à cette question centrale. Ces articles se situent à l'intersection de deux disciplines scientifiques : les mathématiques de l'assurance et l'économie du développement.

Le premier article (Chapitre 2) explore les avantages de la collaboration entre les gouvernements et les assureurs privés en examinant les effets de l'assurance (avec et sans subventions) sur la probabilité qu'un ménage tombe dans une zone de pauvreté (la probabilité de prise au piège). En appliquant les concepts de la théorie de la ruine, une branche des mathématiques de l'assurance, nous étudions un processus de risque avec des pertes à valeur aléatoire qui modélise le capital d'un ménage au fil du temps. Pour ce modèle, nous dérivons des formules explicites pour la probabilité de prise au piège et le coût de la protection sociale encouru par le gouvernement. Ce coût est défini de manière à incorporer les subventions aux primes accordées aux ménages, et, pour les cas où les ménagent tombent dans la pauvreté, les transferts de capitaux nécessaires pour les aider à en sortir ainsi qu'un coût fixe supplémentaire qui garantit, avec un certain niveau de confiance, que les ménages ne retomberont pas dans la pauvreté. Notre analyse révèle l'insuffisance de l'assurance seule (sans subvention) comme moyen de réduction de la pauvreté pour les non-pauvres vulnérables (ceux dont le niveau de capital se situe juste au-dessus du seuil de pauvreté), car le paiement des primes peut accroître le risque de tomber dans la pauvreté. Nous mettons donc en évidence les avantages de la subvention de l'assurance pour les ménages et les gouvernements, ceux-ci bénéficiant d'une réduction de la probabilité de pauvreté et du coût de la protection sociale, respectivement.

Dans le deuxième article (Chapitre 3), nous adaptons le processus de risque pour considérer des pertes en capital proportionnelles aléatoires plutôt que des pertes en capital à valeur aléatoire. Cette adaptation modifie l'approche de l'analyse du processus de risque, car les techniques traditionnelles couramment utilisées dans la théorie de la ruine ne sont plus directement applicables. Néanmoins, en adoptant des méthodes alternatives, nous dérivons pour la première fois une expression de forme fermée pour la probabilité de prise au piège du processus de risque avec une configuration proportionnelle. Dans cet article, nous évaluons également l'impact de l'assurance (sans subvention) sur la probabilité de prise au piège. Nos résultats suggèrent que l'assurance pour les pertes proportionnelles est plus abordable que la couverture pour les pertes à valeur aléatoire, ce qui correspond à l'idée que les

primes sont normalisées en fonction de la richesse dans le cadre de la structure des pertes proportionnelles.

La contribution du troisième article (Chapitre 4) concerne principalement la mesure de la pauvreté, qui est sans aucun doute un point clé dans la phase de suivi et d'évaluation d'une stratégie de protection sociale. Dans cet article, nous dérivons une équation intégrale pour la fonction de pénalité escomptée de Gerber-Shiu, une fonction mathématique largement étudiée dans la théorie de la ruine qui donne des informations sur trois quantités liées à la prise au piège : le temps de la prise au piège (le moment où un ménage tombe dans la zone de pauvreté), le déficit de capital au moment de la prise au piège et l'excédent de capital avant la prise au piège. Nous notons ensuite la relation entre le déficit de capital et un groupe important de mesures de la pauvreté, connu sous le nom d'indice de Foster-Greer-Thorbecke (FGT). En outre, pour un cas particulier de processus de risque avec des pertes proportionnelles, nous déduisons un fondement microéconomique pour la loi bêta de première espèce (B1) en tant que modèle approprié pour décrire la distribution du déficit de capital étant donné que la prise au piège se produit. En particulier, cette découverte permet d'interpréter les paramètres de forme de la distribution théorique, ainsi que d'effectuer des analyses de sensibilité des mesures de pauvreté aux variations des paramètres de forme. Pour conclure cet article, nous illustrons cette notion en utilisant les données de l'*Enquête Multisectorielle Continue (EMC) 2014* du Burkina Faso, une enquête visant à générer des données robustes pour suivre le développement durable du pays.

Le quatrième article (Chapitre 5) étudie le rôle des programmes de transferts monétaires dans la réduction de la pauvreté. Dans cet article, nous considérons un processus de risque oméga avec des pertes proportionnelles ; une extension du processus de risque étudié dans les articles précédents qui incorpore maintenant des transferts directs (transferts de capitaux en espèces) fournis par des donateurs ou des gouvernements aux seuls ménages jugés éligibles. Cette extension nous permet d'introduire un nouvel événement : l'événement d'extrême pauvreté (l'événement où un ménage devient extrêmement pauvre), qui dépend uniquement d'un taux d'extrême pauvreté qui lui-même dépend du capital actuel du ménage. Sous ce modèle, nous dérivons des expressions de forme fermées pour la probabilité de prise au piège et nous faisons de même pour la probabilité d'extrême pauvreté (la probabilité qu'un événement d'extrême pauvreté se produise). Nos illustrations numériques mettent en évidence la capacité des programmes de transferts monétaires à maintenir les ménages hors de la pauvreté et de l'extrême pauvreté. En particulier, nous soulignons le rôle de l'intensité (ou de la fréquence) des transferts et du seuil d'éligibilité dans l'obtention de probabilités de pauvreté et d'extrême pauvreté plus faibles.

# Contents

# Chapter 1

# Introduction

In 2015, world leaders agreed on seventeen Sustainable Development Goals (SDGs) which engage not only public and private sectors but also society in attaining a better and more sustainable future for all. Among these goals, eradicating extreme poverty by 2030 is at the top of the list of priorities, followed by other targets among which the reduction of at least by half of the proportion of people living in poverty and the implementation of appropriate social protection strategies stand out (SDG 1: End poverty in all its forms everywhere (United Nations, 2015)).

According to the World Bank (2018), the number of people living in extreme poverty declined from 36% in 1990 to 10% of the world's population in 2015. However, this downward trend has been decelerating throughout the years. Indeed, recent research published by the United Nations University World Institute for Development Economics Research (UNU-WIDER) shows that, due to the COVID-19 crisis, global poverty could increase for the first time since 1990 (Sumner et al., 2020), therefore threatening one of the global public's priorities: ending poverty.

Social protection strategies are seen as important mechanisms to contribute to the achievement of SDG 1 by closing the gap of inequalities in income and access to opportunities. The chapters of this thesis are structured around one key question: To what extent are social protection strategies (and complementary instruments such as insurance) effective in reducing poverty?

In addressing this question, Chapters 2 and 3 assess the efficiency of inclusive insurance (microinsurance) in reducing poverty. Specifically, in Chapter 2, the cost-efficiency of government-sponsored inclusive insurance schemes is studied, while a non-subsidised scheme in which the insured pays the full amount of the premium is considered in Chapter 3. Similarly, Chapter 5 presents evidence on the important role of unconditional cash transfer programmes in poverty and extreme poverty alleviation. The contributions in Chapter 4 are mainly concerned with poverty measurement, which is clearly a key point in the monitoring and evaluating phase of a social protection strategy and therefore goes hand in hand with the other chapters and with our aim of finding an answer to our key question.

This chapter provides some background literature and relevant mathematical preliminaries that are used in the following chapters. In this regard, Section 1.1 provides a brief overview of poverty trapping, the main topic of interest in this thesis. Two key instruments to fight against poverty, social protection strategies and inclusive insurance, are discussed in Sections 1.1.1 and 1.1.2, respectively. In particular, Section 1.1.1 introduces unconditional cash transfer programmes more in detail. An overview of poverty measurement and an introduction to the Foster–Greer–Thorbecke (FGT) index, which is a family of poverty measures, is provided in Section 1.1.3. The value of using parametric distributions to model personal income and their usefulness in poverty measurement is discussed in Section 1.1.4. Section 1.2 introduces the mathematical foundations that will be used in subsequent chapters, with special emphasis on ruin theory; the widely-studied risk process, the Cramér-Lundberg model, is discussed in Section 1.2.2. In addition, this section introduces two extensively studied risk measures in ruin theory: the ruin probability and the Gerber-Shiu expected discounted penalty function. Section 1.2.2 also presents classical examples illustrating how these risk measures can be estimated for a particular Cramér-Lundberg model, which will certainly facilitate the understanding of the following chapters. We end our brief introduction to ruin theory by highlighting the fact that there are a large number of variations of the Cramér-Lundberg model that have been studied in the literature. This chapter concludes by providing the main contributions of this thesis in Section 1.3.

## 1.1 Poverty Trapping

There are two main ways of looking at poverty: absolute poverty and relative poverty. We say that an individual lives in absolute poverty when he or she subsists below a minimum of socially accepted living conditions. These conditions may be established with respect to certain nutritional requirements or other essential goods which are considered necessary for a rewarding life. Relative poverty, on the other hand, refers to the situation in which an individual is poor in comparison to other people in the economy. Thus, a person can live in poverty from a relative perspective even though he or she may not be poor in an absolute sense, as he or she might be able to acquire the "basic needs". In this thesis, we will focus on the study of absolute poverty and we will therefore generally omit the term "absolute" when referring to absolute poverty.

Within absolute poverty, there are different levels of poverty, which are usually defined according to the degree of deprivation of an individual. For example, the World Bank uses the International Poverty Line (IPL), set at USD 2.15 per person per day, to measure extreme poverty[1] (Jolliffe et al., 2022). The IPL is also the most relevant poverty line to measure poverty in low-income countries, whereas in other countries, other poverty lines are used to measure poverty. Extreme poverty differs from other levels of poverty in that it has a higher degree of deprivation and a longer duration over time. Moreover, individuals living in extreme poverty are characterised by having greater deficiencies such as higher rates of illiteracy and malnutrition, among others (Emran et al., 2014; Barrett et al., 2019).

[1] Some researchers have also used the term "ultrapoverty" to refer to extreme poverty or an even greater degree of deprivation (see Barrett et al. (2019)).

The depth and persistence of poverty give rise to the so-called *poverty traps*. A poverty trap is a state of poverty from which it is difficult to escape without external help. Individuals living in a poverty trap are characterised by behaviours, stemming from their poverty situation, that perpetuate their low standard of living. Thus, it is essential to differentiate between transitory poverty and persistent poverty in order to identify individuals who have difficulties in escaping poverty and to design appropriate anti-poverty policies that stimulate economic growth among the poor. Poverty traps have been extensively studied in the development economics literature (see, for example, Azariadis and Stachurski (2005), Bowles et al. (2006), Matsuyama (2008), Kraay and McKenzie (2014), Barrett et al. (2016) and references therein). This thesis uses the term "trapping" only to describe the event that an individual (or household) falls into poverty and we will later assign the name *trapping time* to the time at which this event occurs. As a matter of fact, all subsequent chapters are concerned with studying the trapping time, which we will later see is a random variable, since we do not know when, if ever, an individual will fall into poverty.

Poverty affects all of us and is therefore not an individualised condition. Indeed, poverty entails high economic, social and psychological costs for both the poor and the non-poor. Take the example of child poverty: children living in poverty are more likely to commit crime as adults (see, for example, Bjerk (2007)). In turn, higher crime rates mean higher correction costs and a rise in private spending on crime prevention (e.g. in buying alarms and locks). Moreover, growing up in poverty can also have a number of health repercussions in later life (Brooks-Gunn and Duncan, 1997; Case et al., 2002; Ravallion, 2016). This translates into higher spending on the treatment of diseases that could have been avoided (Children's Defense Fund (U.S.), 1994). Indeed, McLaughlin and Rank (2018) have recently estimated that, for every dollar spent on reducing childhood poverty[2], the United States of America would save at least seven dollars with respect the economic costs of poverty. Ending poverty must therefore be a common good, as stipulated in the SDGs, and is therefore the main focus of this thesis.

### 1.1.1 Social Protection Strategies

For the purposes of this thesis and following Slater (2011), we will consider public actions that seek to address risk, vulnerability and poverty as social protection strategies. According to Slater (2011) (see also Harvey (2005), Department for International Development (DFID) (2006) and Farrington and Slater (2006)), social protection strategies can be subdivided into three groups:

(i) Social insurance: pooling of contributions paid by individuals to the state or private organisations so that, if they suffer a loss, they receive financial support (e.g. health and unemployment insurance);

(ii) Social assistance: non-contributory transfers to persons deemed eligible on the basis of their vulnerability (e.g. cash transfer programmes) and;

[2] McLaughlin and Rank (2018) also estimated in 2018 that the aggregate annual cost of child poverty in the United States of America amounts to USD 1.0298 trillion, representing 5.4% of the country's gross domestic product (GDP).

(iii) Standards/regulation: establishment of minimum standards to protect citizens.

Social protection strategies are an important tool for preventing poverty and providing a pathway out of poverty for the poor. We now introduce cash transfer programmes and, in particular, unconditional cash transfer programmes, whose role in the fight against poverty will be analysed in Chapter 5.

### Cash Transfer Programmes

Cash transfer programmes are one of the main social protection strategies to reduce poverty and inequality. In their simplest form, these programmes transfer cash, whether in small, regular amounts, or as lump sums, to people living below the poverty line and are generally funded by governments, international organisations, donors or nongovernmental organisations (NGOs) (Garcia and Moore, 2012). In recent years, cash transfer programmes have reached unprecedented levels of coverage. For example, in 2020, in response to the COVID-19 pandemic, one out of six people in the world received at least one cash transfer payment (Gentilini, 2022).

Entitlements to cash transfers can be unconditional (not requiring beneficiaries to undertake any specific actions nor meet any conditions) or conditional (beneficiaries need to have some specific behavioural conditions in exchange of the cash transfer (Baird et al., 2014), such as enrolling children in school or taking them to regular health check-ups (Handa and Davis, 2006)). Moreover, cash transfers can be universal (all people are entitled to them, although this does not mean that all people will receive them) or targeted according to a specific level of vulnerability or social category (e.g. age or gender).

The role of unconditional cash transfer programmes as a pathway out of extreme poverty for households has been extensively studied under an empirical approach. Handa et al. (2016) find that the Child Grant Programme (CGP) and the Multiple Category Targeted Programme (MCP) in Zambia, in addition to protecting household food security and consumption, have a huge impact on household productive capacity. Similarly, Ambler and De Brauw (2017) evidence how the Benazir Income Support Program (BISP) has increased women empowerment in Pakistan. This is certainly an important finding, as women empowerment has been frequently associated with economic growth (Duflo, 2012), which at the same time has been linked with poverty reduction (Adams, 2003).

Undoubtedly, unconditional cash transfer programmes have recently gained popularity as a cost-effective social protection strategy to attain some public policy objectives, including poverty alleviation (Aker, 2013; Baird et al., 2014; Blattman and Niehaus, 2014; Haushofer and Shapiro, 2016; Jensen et al., 2017; Pega et al., 2022).

As mentioned before, Chapter 5 will analyse the role of unconditional cash transfer programmes in the fight against poverty and will complement the vast empirical literature with a comprehensive analysis based on a rigorous mathematical framework.

### 1.1.2 New Approaches to Social Protection: Inclusive Insurance (Microinsurance)

Churchill (2006) was one of the first to use the term "microinsurance". His definition of microinsurance, found in a compendium on the topic, reads as follows:

*"Microinsurance is the protection of low-income people against specific perils in exchange for regular premium payments proportionate to the likelihood and the cost of the risk involved. This definition is essentially the same as one might use for regular insurance except for the clearly prescribed target market: low-income people... How poor do people have to be for their insurance protection to be considered micro? The answer varies by country, but generally microinsurance is for persons ignored by mainstream commercial and social insurance schemes, persons who have not had access to appropriate products."*

Due to the rapid growth of the microinsurance market over the years and the need for insurance authorities to adapt their regulations to facilitate the expansion of insurance products for the poor, a year later, the International Association of Insurance Supervisors (IAIS) published a paper that sought to identify the issues and challenges in developing and enabling a regulatory framework to promote microinsurance in line with the IAIS Insurance Core Principles (ICPs)[3] (International Association of Insurance Supervisors (IAIS), 2007). In this paper, the definition of microinsurance took on a more normative approach:

*"Microinsurance is insurance that is accessed by low-income population, provided by a variety of different entities, but run in accordance with generally accepted insurance practices (which should include the Insurance Core Principles). Importantly this means that the risk insured under a microinsurance policy is managed based on insurance principles and funded by premiums. The microinsurance activity itself should therefore fall within the purview of the relevant domestic insurance regulator/supervisor or any other competent body under the national laws of any jurisdiction."*

While both of the above are sound definitions and agree, above all, that microinsurance is focused on providing protection to the poor, it is easy to realise that it can be difficult to differentiate microinsurance from traditional insurance. For example, an insurance company may find it difficult to assign tasks to its microinsurance department. How would these tasks differ from those performed by departments focused on traditional insurance products? Based on these two definitions of microinsurance, this is certainly a difficult question to answer. This is one of the main reasons why, years later, in a second volume of the compendium on microinsurance, Churchill and Matul (2012) highlight the vagueness of the definition presented in the first compendium and provide readers with four different ways to define microinsurance, which facilitate its distinction from traditional insurance: (i) microinsurance targets low-income people (target group); (ii) a microinsurance product is characterised by a small sum assured and/or premium (product definition); (iii) apart from formal

[3]The IAIS Insurance Core Principles (ICPs) consist of Principle Statements, Standards and Guidance that represent a globally accepted framework for insurance supervision (International Association of Insurance Supervisors (IAIS), 2019).

insurers, microinsurance is generally provided by burial or friendly societies, mutuals, cooperatives and community-based organisations (provider definition); and (iv) microinsurance products are insurance products that are distributed by microfinance institutions (MFIs), low-cost retailers or other organisations that serve low-income individuals (distribution channel). Churchill and Matul (2012) stress the importance of combining these four definitions and other possible characteristics (e.g. appropriate product design and accessibility) to properly define microinsurance and thus differentiate it from conventional insurance. Nevertheless, they also underline that microinsurance should be defined in a way that responds to the national and corporate objectives of regulators and insurers, respectively, and therefore its definition can vary.

In 2015, the IAIS defined a broader term: "inclusive insurance". *Inclusive insurance* refers to all the insurance products that offer protection to the excluded or underserved market, not just the low-income population (International Association of Insurance Supervisors (IAIS), 2015). Today, the term inclusive insurance is more widely used than microinsurance. In fact, for example, the International Microinsurance Conference, which is the most important event on the subject and has been held since 2005, changed its name to the International Conference on Inclusive Insurance (ICII) in 2015, demonstrating the relevance that this broader term has acquired.

At first glance, one might wonder what the difference between inclusive insurance and social insurance is. In fact, one of the main objectives of inclusive insurance is precisely to offer protection to those who are excluded by formal social protection strategies (e.g. those in the informal economy and the rural workers). Hence, for instance, an inclusive insurance scheme differs from social protection strategies that offer statutory protection to formal workers (see, for instance, Churchill (2006) and Churchill and Matul (2012), which discuss the potential roles of inclusive insurance as a complement to social protection strategies). That is, inclusive insurance can be viewed as a complement to the three groups listed in Section 1.1.1 and thus as a component of social protection strategies.

In recent years, there has been an increase in public-private partnerships (PPPs)[4] and the willingness of governments to subsidise insurance premiums for low-income individuals (Churchill and Matul, 2012). For example, Chinese farmers receive support from central and provincial governments such that they end up paying only about 20% of the premium amount (Wang et al., 2011; Ye et al., 2020). Similarly, Indian farmers enrolled in the Pradhan Mantri Fasal Bima Yojana (PMFBY) crop insurance scheme, pay a maximum premium ranging from 2% to 5% of the sum insured (or the actuarial rate, whichever is lower), with the remaining part of the premium paid on a 50/50 basis by the central and state governments (Kaur et al., 2021). Hill et al. (2014) argue that insurance premium subsidies must be designed with a clearly stated purpose. Moreover, they should target those in need and address market deficiencies or consumer equity concerns. Subsidies that are properly designed have shown to be a powerful and cost-effective tool to achieve public policy objectives such as poverty alleviation. Conversely, poorly designed subsidies can be inefficient and lead to significant economic costs.

[4]In this thesis, public-private partnerships (PPPs) are partnerships involving public- and private-sector actors with complementary resources and functions.

The role of inclusive insurance as an important actor in achieving poverty reduction will be discussed in Chapters 2 and 3. Chapter 2 will examine the important role of PPPs and highlight the value of designing appropriate premium subsidies, while non-subsidised schemes will be explored in Chapter 3. For a collection of previous studies analysing the role of insurance in the fight against poverty, interested readers may wish to consult Dercon (2004).

### 1.1.3 A Class of Poverty Measures

One of the main tools to monitor and evaluate the performance of a social protection strategy are *poverty measures*[5]. According to Haughton and Khandker (2009), there are three main steps that need to be taken in measuring poverty:

(i) Set an indicator of welfare (e.g. income or consumption per capita that is usually obtained from survey data);

(ii) Define a *poverty line*, representing a minimum acceptable level of this welfare indicator to differentiate the poor from the non-poor (in Section 1.1, for instance, we described it as a minimum of socially accepted living conditions) and;

(iii) Generate a summary statistic that describes the distribution of this welfare indicator relative to the poverty line.

In this thesis, we are particularly interested in the third step. Indeed, in Chapter 4 we will see that there is a connection between the Foster–Greer–Thorbecke (FGT) index (Foster et al., 1984), a summary statistic that describes the distribution of income relative to the poverty line, and the Gerber-Shiu expected discounted penalty function (Gerber and Shiu, 1998), a risk measure that has been extensively studied in the insurance mathematics literature (see Section 1.2.2 for more information about this risk measure).

The FGT index has become the standard measure for international poverty assessments and is regularly reported on by individual countries and international organisations such as the World Bank. Economists James Eric Foster, Joel Greer and Erik Thorbecke, then at Cornell University, introduced this type of poverty measure in a 1984 paper. It emerged as an alternative to previously used measures such as the *head-count index*, the *poverty gap index* and the *Sen measure*. The head-count index calculates the proportion of the population living below the poverty line and has been considered as one of the most common indices for measuring poverty since the first studies of poverty were conducted (see, for example, Booth (1889) and Rowntree (1901)). On the other hand, the poverty gap index represents the average income short-fall (the absolute value of the difference between a poor household's income (or consumption) and some poverty line) with respect to the poverty line. Although widely used, the head-count index ignores the depth of poverty, while the

[5]Sometimes also referred to as *poverty metrics*, *poverty indicators* or *poverty indices* in the literature.

poverty gap index ignores the distribution of income among the poor, making them poor indicators of poverty (Sen, 1976). Seeking to overcome these limitations, Sen (1976) introduced a "rank weighting" approach (see Theorem 1 from Sen (1976)) that accounts for the normalised gap and the rank order of a person in the group of the poor. However, as Foster et al. (1984) argue, this Sen measure and its variants (which are all based on a rank-weighted approach) may violate the natural condition where subgroup and total poverty have to move in the same direction (see Footnote 6 from Foster et al. (1984) for a simple example). For all these reasons, Foster et al. (1984) introduced their new class of poverty indicators, which we define now:

**Definition 1.1.1** (Foster–Greer–Thorbecke (FGT) Index)**.** Let $F_X(x)$ be the continuous distribution function of the income variable $X$ from a population and $f_X(x)$ its probability density function (p.d.f.). The FGT class of poverty measures indexed by $\gamma \geq 0$ is defined as

$$FGT_\gamma = \int_0^z \left(\frac{z-x}{z}\right)^\gamma f_X(x)\,dx, \tag{1.1.1}$$

where $z$ is the poverty line.

Particular cases of the FGT class of poverty measures include $FGT_0$, which is simply the head-count index. Other choices are $\gamma = 1$, which gives rise to the poverty gap index[6], and $\gamma = 2$, which is often referred to as the *poverty severity index*. Note that a larger $\gamma$ in (1.1.1) gives greater emphasis to the poorest poor. Hence, this parameter is viewed as a measure of poverty aversion (Foster et al., 1984). For a detailed review of the contributions of the FGT index over the 25 years since its publication, see Foster et al. (2010).

### 1.1.4 Parametric Modelling of Personal Income

Parametric modelling of the empirical distribution of personal income has been one of the main topics of study in economics since the work of the Italian economist Wilfried Fritz Pareto, when he introduced his Pareto law in 1896 (Pareto, 1967). Parametric modelling is a statistical approach that makes it possible to characterise and reconstruct the distribution of personal income from a finite number of parameters and some limited information. Since Pareto's law, several distributions for modelling personal income have been proposed (see Hlasny (2021) for a recent survey). However, as Callealta Barroso et al. (2020) state, when looking for a suitable model of the phenomenon of interest, one should choose the one that best describes the characteristics of the phenomenon itself. Furthermore, Callealta Barroso et al. (2020) also argue that one should base its selection on certain properties such as characteristics of the observed income distribution (e.g. a light or heavy tailed model), desirable mathematical properties (e.g. a model that satisfies some differentiability properties) and economic properties (e.g. a model arising from an economic model itself).

[6] Note that the poverty line $z$ is in the denominator. That is, the poverty gap index is expressed as a percentage of the poverty line.

Poverty measurement has also been one of the main applications of parametric modelling of the empirical distribution of personal income (see, for example, Kleiber and Kotz (2003) and Chotikapanich (2008)). One of the main advantages of parametric estimation of income distributions is that explicit formulas, as functions of the parameters of the theoretical income distribution, are available to measure poverty and inequality (see, for example, Section 1.1.3). This allows, for example, to further interpret the shape parameters of the theoretical income distribution, as well as to carry out sensitivity analyses of poverty measures to variations in the shape parameters (Graf and Nedyalkova, 2014).

In Chapter 4, we will focus on the parametric estimation of income short-fall distributions. Moreover, we also provide a compelling microeconomic foundation for modelling the income short-fall distribution, which is derived from an economic model representing a household's capital over time.

## 1.2 Mathematical Preliminaries

In this section, we outline the main mathematical tools that will be used throughout this thesis. We begin by defining Poisson processes (Section 1.2.1), a topic much studied in probability theory, and then provide a concise introduction to ruin theory (Section 1.2.2), a branch of insurance mathematics that will constitute the main tool to address the key question presented at the beginning of this chapter.

### 1.2.1 The Poisson Process

Since the Poisson process takes its name from the Poisson distribution, we first describe the properties of a Poisson random variable.

**Definition 1.2.1** (Poisson Distribution)**.** The Poisson distribution was introduced by the French mathematician Siméon Denis Poisson in 1838. A random variable $N$ is said to follow a Poisson distribution with parameter $\lambda > 0$ and we write, $N \sim Poisson\,(\lambda)$, if its probability mass function (p.m.f.) is given by:

$$\mathbb{P}\,(N = n) = \frac{\lambda^n}{n!} e^{-\lambda}, \qquad for \qquad n = 0, 1, 2, ...$$

The variance of a Poisson random variable is equal to its mean, that is, $\mathbb{E}\,[N] = \mathrm{Var}\,[N] = \lambda$.

**Definition 1.2.2** (Counting Process)**.** A stochastic process $\{N_t\}_{t \geq 0}$ is called a *counting process* if it represents the number of events that occur in the time interval $[0, t]$. A counting process has the following properties:

(i) $N_t$ is a discrete random variable whose only possible values are the non-negative integers $0, 1, 2, ...$;

(ii) If $s < t$, then $N_s \leq N_t$ and;

(iii) For $s < t$, $N_t - N_s$ is equal to the number of events that have occurred in the time interval $(s, t]$.

**Definition 1.2.3** (Poisson Process)**.** A counting process $\{N_t\}_{t\geq 0}$ is called a *Poisson process* with *rate* (or *intensity*) $\lambda > 0$ if:

(i) $N_0 = 0$;

(ii) The process has independent increments. That is, for any $t_i$, with $i = 0, ..., n$ and $n \geq 1$ such that $0 = t_0 < t_1 < t_2 < ... < t_n$, the increments $N_{t_i} - N_{t_{i-1}}$, $i = 1, ...., n$, are mutually independent and;

(iii) The number of events in any interval of length $t$ is Poisson distributed with mean $\lambda t$. That is, for all $s, t \geq 0$,

$$\mathbb{P}\left(N_{t+s} - N_s = n\right) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad \textit{for} \quad n = 0, 1, 2, ...$$

From Property (iii) above, we have that $\mathbb{E}\left[N_t\right] = \lambda t$, which is the reason why the parameter $\lambda$ is called the rate.

There are other equivalent definitions for the Poisson process (see, for example, Ross (1995)). Nevertheless, for the purposes of this thesis, it is sufficient to stick to the above definition. Moreover, we have so far defined the most popular case of a Poisson process, which, as stated above, is characterised by having a constant rate $\lambda$ and a linear mean ($\mathbb{E}\left[N_t\right] = \lambda t$). In probability theory, this particular case is known as the *homogeneous Poisson process*. It is possible to generalise this assumption to allow the rate to be a function of $t$, in which case it is called *inhomogeneous Poisson process* (see, for example, Ross (1995) and Mikosch (2006)). Throughout this thesis only the homogeneous case is considered, and for the remainder of this thesis we will omit the term "homogeneous" and only write Poisson process to refer to the homogeneous case.

One of the most important features of the Poisson process is that the so-called *inter-arrival times* are exponentially distributed with mean $1/\lambda$. Indeed, consider a Poisson process $N_t$ with rate $\lambda > 0$ and denote the time of the first event with $T_1$. Furthermore, for $i \geq 1$, let $\tilde{T}_i$ denote the time between the $(i-1)$st and the $i$th event. The sequence of random variables $\{\tilde{T}_i\}_{i\in\mathbb{N}}$ is called the sequence of inter-arrival times. One can easily prove that $\tilde{T}_i \sim Exp(\lambda)$ (see, for example, Ross (1995)).

In insurance mathematics, the Poisson process is often used to model claim frequency (the number of claims) in the time interval $[0, t]$, since insurance companies are concerned about the random number of claims that may occur. However, the (random) severity of a claim is also of great interest to insurers. In particular, insurers are interested in estimating the aggregate claims of a portfolio of policies (the total sum of all claims). This is often modelled by what is known as a *compound Poisson process*, which we define below:

**Definition 1.2.4** (Compound Poisson Process)**.** Let $\{N_t\}_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$ and $\{Z_i\}_{i\in\mathbb{N}}$ a sequence of independent and identically distributed (i.i.d.) random variables with common distribution function $G_Z$ that are also independent of $\{N_t\}_{t\geq 0}$. The stochastic process

$$S_t = \sum_{k=1}^{N_t} Z_k,$$

is called a *compound Poisson process.* It is not difficult to show that the mean of $S_t$ is given by $\mathbb{E}[S_t] = \lambda t \mu_z$ while its variance is $\text{Var}[S_t] = \lambda t\,(\sigma_z^2 + \mu_z^2)$, with $\mu_z$ and $\sigma_z$ denoting the mean and variance of the random variable $Z_i$, respectively.

The four definitions reviewed in this section will provide us with the necessary tools to understand the Cramér-Lundberg model, which is the classical risk process in ruin theory and has been widely-studied over the past century. Ruin theory and the Cramér-Lundberg model are discussed more in detail in Section 1.2.2.

### 1.2.2 Ruin Theory

Ruin theory deals with stochastic processes and their fluctuations (Asmussen and Albrecher, 2010). Its main objective is to build mathematical models for the financial reserves of an insurance company, which are characterised by random fluctuations. In insurance mathematics, ruin theory has been an active field of research from Lundberg's time (Lundberg, 1903) to the present day. In general, the stochastic process of interest is the *Cramér-Lundberg model*, originally introduced by the Swedish actuary and mathematician Filip Lundberg in his doctoral dissertation (Lundberg, 1903) and later studied by the Swedish statistician Harald Cramér (Cramér, 1930), from which it derives its name. However, variations of this model have also been widely studied in the actuarial science literature in the last century.

Insurance companies offer protection against certain risks (e.g. theft and loss of property) to policyholders in exchange for premium payments. Typically, policyholders contribute to the insurer's financial reserves by paying premiums in advance at the inception of an insurance policy, while, in return, the insurer provides coverage against losses that the insured may suffer during a defined period of time clearly stated in the policy. Premiums paid by policyholders, generally settled by the insurer, are intended to ensure the viability of the insurance business. In mathematical terms, this means that the expected premium for a given risk must be higher than its expected loss. In other words, there should exist a *positive safety loading* on the premiums, because without a positive safety loading on the premiums, an insurance company would go bankrupt in the long run (Albrecher et al., 2017). The Cramér-Lundberg model, which is a simple stochastic process, attempts to describe the surplus process of an insurance company with these ideas in mind. We now introduce this model, which is also the most intensively studied in ruin theory (Mandjes and Boxma, 2023).

### The Cramér-Lundberg Model

In the classical Cramér-Lundberg model, we consider for an insurance portfolio a non-negative initial capital $U_0 = u \geq 0$ at time $t = 0$. We further assume that the insurer's total premium inflow up to time $t$ is given by an increasing stochastic process, which we denote by $C_t$. In particular, in the Cramér-Lundberg model, $C_t$ is given by the increasing deterministic linear function

$$C_t = ct, \tag{1.2.1}$$

for some constant premium rate $c > 0$.

Let $N_t$ be a Poisson process with rate $\lambda > 0$ which counts the number of claims (or losses) in the time interval $[0, t]$ and $\{T_i\}_{i\in\mathbb{N}_0}$ the set of arrival times of these claims. Hence, we can write $N_t = \max\{i \geq 0 : T_i \leq t\} = \min\{i \geq 0 : T_{i+1} > t\}$, with $T_0 = 0$. Then, the total claim amount up to time $t$ is given by

$$S_t = \sum_{k=1}^{N_t} Z_k, \tag{1.2.2}$$

where $Z_i$, for $i \in \mathbb{N}$, denotes the size of the $i$th claim. The claim sizes $\{Z_i\}_{i\in\mathbb{N}}$ represent a sequence of i.i.d. random variables having common distribution function $G_Z(z) = \mathbb{P}(Z_i \leq z)$, with $G_Z(0) = 0$, mean value $\mathbb{E}[Z_i] = \mu_Z$ and variance $\mathrm{Var}[Z_i] = \sigma_Z$. The Poisson process $\{N_t\}_{t\geq 0}$ and the claim sizes $\{Z_i\}_{i\in\mathbb{N}}$ are assumed to be independent of each other. Therefore, the total claim amount up to time $t$ given by (1.2.2) is a compound Poisson process (see Section 1.2.1).

This gives rise to the classical Cramér-Lundberg model, which, through the previous quantities, describes an insurer's surplus process by

$$U_t = u + C_t - S_t = u + ct - \sum_{k=1}^{N_t} Z_k. \tag{1.2.3}$$

Clearly, due to the randomness of the variables $\{T_i\}_{i\in\mathbb{N}_0}$ and $\{Z_i\}_{i\in\mathbb{N}}$, the surplus (1.2.3) represents a stochastic process.

The "profit" of the insurance business over the time interval $(0, t]$ is given by the difference: $C_t - S_t = ct - \sum_{k=1}^{N_t} Z_k$. This difference contains important information about the credibility of the insurance company. In particular, we are usually interested in studying the so-called *relative safety loading*, which can be interpreted as the expected profit of the insurance company per claim unit. This is given by the following limit:

$$\rho = \lim_{t\to\infty} \frac{\mathbb{E}[C_t - S_t]}{\mathbb{E}[S_t]} = \frac{c - \lambda\mu_Z}{\lambda\mu_Z} = \frac{c}{\lambda\mu_Z} - 1. \tag{1.2.4}$$

The insurer's surplus process (1.2.3) is said to have a *positive relative safety loading* if $\rho > 0$. A positive relative safety loading means that the process has an increasing tendency (or equivalently, that the stochastic process (1.2.3) has a drift to infinity).

One is particularly interested in the behaviour of the surplus process (1.2.3) over time. Specifically, we observe whether this process ever becomes negative (when this happens, we say that ruin has occurred). Thus, we define the *infinite-time ruin probability* as

$$\psi(u) = \mathbb{P}\left(\inf_{t\geq 0} U_t < 0 \mid U_0 = u\right),$$

and the *finite-time ruin probability* up to time $T$ is correspondingly

$$\psi(u,T) = \mathbb{P}\left(\inf_{0\leq t\leq T} U_t < 0 \mid U_0 = u\right).$$

The ruin probability quantifies the likelihood that the insurer will run out of money in the time interval considered. Without a positive relative safety loading (1.2.4), ruin would be certain (see, for example, Rolski et al. (1999)). Therefore, it is natural to assume a positive relative safety loading: $\rho > 0$.

The *ruin time* is denoted by

$$\tau_u = \inf\left\{t \geq 0 : U_t < 0 \mid U_0 = u\right\},$$

hence we can rewrite the infinite-time ruin probability as

$$\psi(u) = \mathbb{P}(\tau_u < \infty).$$

As mentioned previously, under the positive relative safety loading assumption, we expect the surplus process (1.2.3) to go to infinity. Hence, it is possible that the insurer will not experience ruin at all. In other words, the ruin time represents a *defective random variable* and it can take the value infinity.

It is sometimes more convenient to work with the complement function

$$\phi(u) = 1 - \psi(u),$$

which is called the *survival probability* (or non-ruin probability) and has the form

$$\phi(u) = \mathbb{P}\left(\inf_{t\geq 0} U_t \geq 0 \mid U_0 = u\right) = \mathbb{P}(\tau_u = \infty).$$

Throughout the chapters of this thesis, we will be mainly interested in studying three concepts: the ruin probability, the ruin time and the deficit at ruin. Thus, in the next two sections, we provide an introduction on how to estimate the ruin probability in the Cramér-Lundberg model. In addition, we introduce the so-called *Gerber-Shiu expected discounted penalty function*, a function that will allow us to study other quantities of interest such as the ruin time and the deficit at ruin.

Figure 1.1 displays a sample path of the insurer's surplus process (1.2.3) and portrays some of the variables previously introduced.

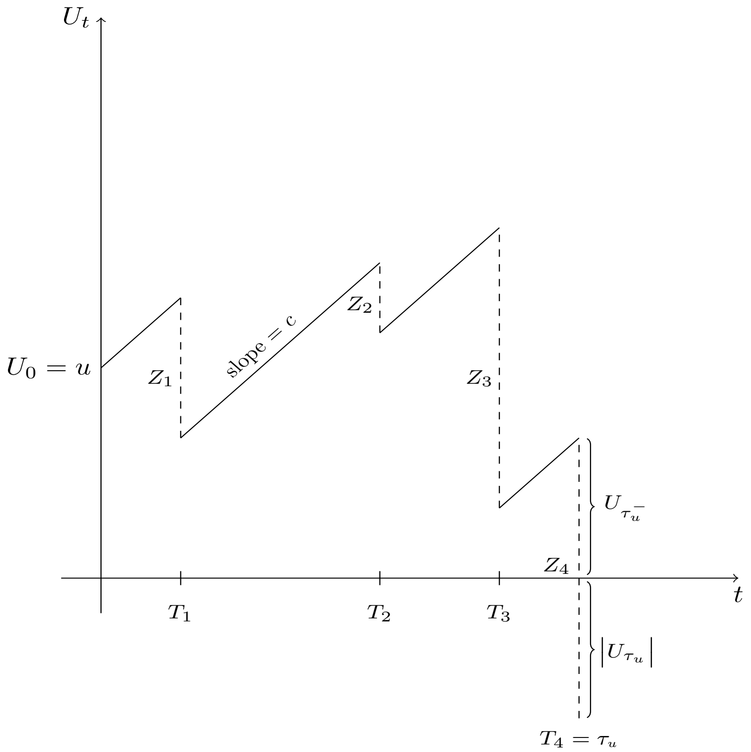



Figure 1.1: Trajectory of an insurer's surplus process $U_t$ with initial capital $U_0 = u$. In this particular example, ruin occurs at $T_4 = \tau_u$.

### The Ruin Probability

We now present two main approaches to estimate an insurer's infinite-time ruin probability when the surplus process is given by the Cramér-Lundberg model (1.2.3). We first derive a functional equation for $\psi(u)$ and we then present a classical example in which the claim sizes follow a particular distribution. The two approaches presented in this section will serve as the fundamental basis of the subsequent chapters, as they will be applied when considering a different model.

The first approach is based on a somehow heuristic "differential" argument[7] (see, for instance, Cramér (1930)). We start by considering a sufficiently small $h > 0$ such that in the time interval $(0, h)$ there are four different possible outcomes:

(i) There is no claim in the time interval $(0, h)$;

(ii) A claim occurs in the time interval $(0, h)$ but its size is not large enough to cause ruin;

(iii) A claim occurs in the time interval $(0, h)$ and its size is large enough such that it causes ruin and;

(iv) More than one claim occurs in the time interval $(0, h)$.

[7]The formal framework of this approach are generators (see, for example, Chapter 2 of Asmussen and Albrecher (2010)).

Since the inter-arrival times of the claims are exponentially distributed, the probability that there is no claim up to time $h$ is $e^{-\lambda h}$, while the probability that the first claim occurs in the infinitesimal interval $(t, t+dt)$ is $e^{-\lambda t}\lambda dt$. Conversely, the probability that there is more than one claim in the time interval $(0, h)$ is negligible. Hence, we obtain

$$\phi(u) = e^{-\lambda h}\phi(u+ch) + \int_0^h \left[\int_0^{u+ct} \phi(u+ct-z)dG_Z(z)\right] e^{-\lambda t}\lambda dt \quad (1.2.5)$$

Differentiating (1.2.5) with respect to $h$ and setting $h = 0$ yields the following integro-differential equation (IDE)

$$\phi'(u) = \frac{\lambda}{c}\left(\phi(u) - \int_0^u \phi(u-z)dG_Z(z)\right). \quad (1.2.6)$$

We will now derive the IDE (1.2.6) using a more rigorous argument. Following Feller (1971), we know that the value of the surplus process at the time of the first claim is given by $U_{T_1} = u + cT_1 - Z_1$. Moreover, observing that ruin cannot occur in the time interval $(0, T_1)$ and using the fact that $N_t$ is a Poisson process we obtain

$$\phi(u) = \mathbb{E}\left[\phi(u+cT_1-Z_1)\right] = \int_0^\infty \left(\int_0^{u+ct} \phi(u+ct-z)dG_Z(z)\right)\lambda e^{-\lambda t}dt.$$

Then, doing the change of variable $v = u + ct$ yields

$$\phi(u) = \frac{\lambda}{c}\int_u^\infty \left(\int_0^v \phi(v-z)dG_Z(z)\right) e^{-\lambda(v-u)/c}dv, \quad (1.2.7)$$

and differentiation of the integral equation (IE) (1.2.7) with respect to $u$ gives again (1.2.6).

Now, our aim is to solve the IDE (1.2.6). Integrating this IDE over $(0, x)$ we obtain

$$\begin{aligned}\phi(x) - \phi(0) &= \frac{\lambda}{c}\left(\int_0^x \phi(v)dv - \int_0^x \left[\int_0^v \phi(v-z)dG_Z(z)\right] dv\right)\\ &= \frac{\lambda}{c}\int_0^x \phi(x-v)\left(1-G_Z(v)\right)dv. \quad (1.2.8)\end{aligned}$$

Then, by monotone convergence, it follows from (1.2.8), as $x \to \infty$, that

$$\phi(\infty) = \phi(0) + \frac{\lambda\mu_z}{c}\phi(\infty). \quad (1.2.9)$$

Under the assumption of a positive relative safety loading, i.e. $\rho = c/(\lambda\mu_z) - 1 > 0$, it follows from the law of large numbers that $\phi(\infty) = 1$ (see, for example Grandell (1991) and Rolski et al. (1999)). Therefore, considering that $\phi(\infty) = 1$ and writing (1.2.9) in the ruin probability notation yields to

$$\psi(0) = \frac{\lambda\mu_z}{c} = \frac{1}{1+\rho}, \quad (1.2.10)$$

which, interestingly, only depends on the relative safety loading $\rho$. That is, the ruin probability of an insurer with zero initial capital depends on the claims only through their mean $\mu_z$.

**Example 1.2.1** (Cramér-Lundberg Model with Exponential Claims)**.** If the claims follow an exponential distribution with parameter $\alpha > 0$, i.e. $Z_i \sim Exp(\alpha)$, (1.2.10) yields $\psi(0) = \lambda/(\alpha c)$. Then, substituting this in (1.2.8) leads to (using the ruin probability notation):

$$\begin{aligned}
\psi(u) &= \frac{\lambda}{\alpha c} - \frac{\lambda}{c}\int_0^u e^{-\alpha v}dv + \frac{\lambda}{c}\int_0^u \psi(v)e^{-\alpha(u-v)}dv \\
&= \frac{\lambda}{c}\int_0^\infty e^{-\alpha v}dv - \frac{\lambda}{c}\int_0^u e^{-\alpha v}dv + \frac{\lambda}{c}\int_0^u \psi(v)e^{-\alpha(u-v)}dv \\
&= \frac{\lambda}{c}\int_u^\infty e^{-\alpha v}dv + \frac{\lambda}{c}e^{-\alpha u}\int_0^u \psi(v)e^{\alpha v}dv \\
&= \frac{\lambda}{\alpha c}e^{-\alpha u} + \frac{\lambda}{c}e^{-\alpha u}\int_0^u \psi(v)e^{\alpha v}dv. \qquad (1.2.11)
\end{aligned}$$

There are several approaches to solve the IE (1.2.11), we present two of them. The first approach consists of differentiating both sides of (1.2.11) with respect to $u$ (see Theorem 5.3.1 of Rolski et al. (1999), where they show that $\phi(u)$ is differentiable everywhere except for the countable set where $G_Z(z)$ is not continuous), which yields

$$\begin{aligned}
\psi'(u) &= -\frac{\alpha\lambda}{c}e^{-\alpha u}\left(\int_0^u \psi(v)e^{\alpha v}dv + \frac{1}{\alpha}\right) + \frac{\lambda}{c}\psi(u) \\
&= \psi(u)\left(\frac{\lambda}{c} - \alpha\right). \qquad (1.2.12)
\end{aligned}$$

Then, integrating both sides of (1.2.12) leads to

$$\psi(u) = \psi(0)e^{-\left(\alpha-\frac{\lambda}{c}\right)u} = \frac{\lambda}{\alpha c}e^{-\left(\alpha-\frac{\lambda}{c}\right)u}. \qquad (1.2.13)$$

The second approach to solve the IE (1.2.11) consists of applying *Laplace transform techniques*. If we denote the Laplace transform of the ruin probability $\psi(x)$ by

$$\mathcal{L}\{\psi(u)\} = F(s) = \int_0^\infty \psi(u)e^{-su}du,$$

and we produce the Laplace transform on both sides of (1.2.11) yields

$$F(s) = \frac{\lambda}{c(\alpha+s)}F(s) + \frac{\lambda}{\alpha c(\alpha+s)},$$

where we used the fact that the Laplace transform of the exponential function is $\mathcal{L}\{e^{au}\} = 1/(s-a)$. Then, solving for $F(s)$ yields

$$F(s) = \frac{\lambda}{\alpha c\left(\alpha - \frac{\lambda}{c} + s\right)}. \qquad (1.2.14)$$

Finally, inverting the Laplace transform (1.2.14) gives us (1.2.13).

In Chapter 3, we will follow in particular the Laplace transform approach introduced earlier in Example 1.2.1 to obtain a solution for the ruin probability.

### The Gerber-Shiu Expected Discounted Penalty Function

We are now not only interested in the case of an insurer's ruin, but also in the conditions under which it occurs. The Gerber-Shiu expected discounted penalty function (Gerber and Shiu, 1998), a concept commonly used in actuarial science, provides us with information on the circumstances under which ruin occurs. Let us assume an insurer's surplus process is given by (1.2.3), such that with a force of interest $\delta \geq 0$, the Gerber-Shiu expected discounted penalty function is defined as

$$m_\delta(u) = \mathbb{E}\left[w(U_{\tau_u^-}, \mid U_{\tau_u} \mid) e^{-\delta \tau_u} \mathbb{1}_{\{\tau_u < \infty\}}\right], \tag{1.2.15}$$

where $\mathbb{1}_{\{A\}}$ is the indicator function of a set $A$ and $w(u_1, u_2)$ for $0 \leq u_1 < \infty$ and $0 \leq u_2 < \infty$, is a non-negative penalty function of $u_1$, the surplus prior to the ruin time, and $u_2$, the surplus deficit at the ruin time. The function (1.2.15) is useful for deriving results in connection with joint and marginal distributions of $\tau_u$, $U_{\tau_u^-}$ and $\left|U_{\tau_u}\right|$. For example, when $\delta$ is serving as the argument, (1.2.15) can be viewed in terms of a Laplace transform. Indeed, setting $w(u_1, u_2) = 1$, (1.2.15) becomes the Laplace transform of the ruin time $\tau_u$[8]. Another choice is $w(u_1, u_2) = e^{-su_1 - zu_2}$ for which (1.2.15) leads to the trivariate Laplace transform of the ruin time, the surplus prior to ruin and the deficit at ruin. Similarly, setting $w(u_1, u_2) = u_1 + u_2$ in (1.2.15) for $\delta = 0$ gives the expected claim size causing ruin. One observes that, by appropriately choosing a penalty function $w(u_1, u_2)$ and a force of interest $\delta$, various risk quantities can be modelled. He et al. (2023) provide a non-exhaustive list of these quantities in their recent survey on the Gerber-Shiu expected discounted penalty function. Interested readers may also consult Kyprianou (2013) for more details on the so-called Gerber-Shiu risk theory. Figure 1.1 shows the three random variables of interest: $\tau_u$, $U_{\tau_u^-}$ and $\left|U_{\tau_u}\right|$.

As was done for the ruin probability and following Gerber and Shiu (1998), our goal is to derive a functional equation for $m_\delta(u)$. We use a "differential" argument in which we consider the time interval $(0, h)$ for a sufficiently small $h > 0$. Hence, by conditioning on the time and the size of the first claim and discounting the expected values to time 0 at the force of interest $\delta$, we obtain

$$m_\delta(u) = e^{-(\delta+\lambda)h} m_\delta(u + ch) + \int_0^h \left[\int_{u+ct}^\infty w\left(u + ct, z - u - ct\right) dG_Z(z)\right] e^{-(\delta+\lambda)t} \lambda dt$$
$$+ \int_0^h \left[\int_0^{u+ct} m_\delta(u + ct - z) dG_Z(z)\right] e^{-(\delta+\lambda)t} \lambda dt. \tag{1.2.16}$$

Differentiating (1.2.16) with respect to $h$ and setting $h = 0$ yields

$$cm'_\delta(u) - (\delta + \lambda) m_\delta(u) + \lambda \int_0^u m_\delta(u - z) dG_Z(z) = -\lambda A(u), \tag{1.2.17}$$

where $A(u) := \int_u^\infty w(u, z - u) dG_Z(z)$.

[8]The Laplace transform of a positive random variable $Y$ with p.d.f. $f_Y$ is given by the expected value $\mathcal{L}\{f_Y\}(s) = \mathbb{E}\left[e^{-sY}\right]$.

**Example 1.2.2** (The Laplace Transform of the Ruin Time in the Cramér-Lundberg Model with Exponential Claims)**.** In this example, we consider the choice $w(u_1, u_2) = 1$ for which (1.2.15) leads to the Laplace transform of the ruin time. Moreover, as in Example 1.2.1, we also assume that $Z_i \sim Exp(\alpha)$. Under these assumptions, Equation (1.2.17) can be rewritten as follows:

$$cm'_\delta(u) - (\delta + \lambda)m_\delta(u) + \alpha\lambda \int_0^u m_\delta(u - z)e^{-\alpha z}dz = -\lambda e^{-\alpha u}. \qquad (1.2.18)$$

Applying the operator $\left(\frac{d}{du} + \alpha\right)$ to both sides of (1.2.18), together with a number of algebraic manipulations, yields the second order homogeneous differential equation

$$cm''_\delta(u) + [\alpha c - (\delta + \lambda)]\, m'_\delta(u) - \alpha\delta m_\delta(u) = 0,$$

which can be rewritten as follows:

$$m''_\delta(u) + am'_\delta(u) + bm_\delta(u) = 0, \qquad (1.2.19)$$

for $a = (\alpha c - (\delta + \lambda))\,/c$ and $b = -(\alpha\delta)\,/c$. In physics, Equation (1.2.19) is called an equation of *damped vibrations* (see, for example, Zaitsev and Polyanin (2003)). A general solution of (1.2.19) is given by

$$m_\delta(u) = e^{-\frac{1}{2}au}\left[A_1 e^{\frac{1}{2}\kappa u} + A_2 e^{-\frac{1}{2}\kappa u}\right], \qquad (1.2.20)$$

for arbitrary constants $A_1, A_2 \in \mathbb{R}$ and $\kappa^2 = a^2 - 4b > 0$. To determine the constants $A_1$ and $A_2$ we consider the boundary conditions for $m_\delta(u)$ at 0 and at infinity. For $u \to \infty$, the first term of (1.2.20) is unbounded, while the second term tends to zero. The boundary condition $\lim_{u\to\infty} m_\delta(u) = 0$, by definition of $m_\delta(u)$ in (1.2.15), thus implies that $A_1 = 0$. Letting $u = 0$ in (1.2.18) yields

$$m'_\delta(0) = \frac{(\delta + \lambda)\, m_\delta(0) - \lambda}{c},$$

which then by (1.2.20) leads to

$$-\frac{(a + \kappa)\, A_2}{2} = \frac{(\delta + \lambda)\, A_2 - \lambda}{c}.$$

Lastly, solving for $A_2$ we obtain

$$A_2 = \frac{2\lambda}{2\,(\delta + \lambda) + (a + \kappa)\, c} = \frac{2\lambda}{\alpha c + \delta + \lambda + \sqrt{(\alpha c - (\delta + \lambda))^2 + 4\alpha\delta c}}.$$

Thus, the Laplace transform of the ruin time in the Cramér-Lundberg model with exponential claims is given by

$$m_\delta(u) = \frac{2\lambda}{\alpha c + \delta + \lambda + \sqrt{(\alpha c - (\delta + \lambda))^2 + 4\alpha\delta c}} e^{-\frac{1}{2}\left(\frac{\alpha c - (\delta+\lambda)}{c} + \frac{1}{c}\sqrt{(\alpha c - (\delta+\lambda))^2 + 4\alpha\delta c}\right)u}. \qquad (1.2.21)$$

In fact, the expression (1.2.21) has been obtained previously using other techniques, such as *integrating factors* (see, for example, Gerber and Shiu (1997b) and Gerber and Shiu (1998)).

In Chapters 2, 4 and 5, we derive an IE for the Gerber-Shiu expected discounted penalty function following the approach presented in Example 1.2.2.

*Remark* 1.2.1. The Laplace transform of the trapping time approaches the trapping probability as $\delta$ tends to zero: $\lim_{\delta\downarrow 0} m_\delta(u) = \mathbb{P}(\tau_u < \infty) \equiv \psi(u)$. For instance, for the Cramér-Lundberg model with exponential claims, we can observe that setting $\delta = 0$ in (1.2.21) yields (1.2.13).

*Remark* 1.2.2. As an application of the Laplace transform of the ruin time, one quantity of interest is the expected ruin time, i.e. the expected time at which the insurer will be ruined. This quantity can be obtained by taking the derivative of $m_\delta(u)$, such that

$$\mathbb{E}\left[\tau_u; \tau_u < \infty\right] = -\left.\frac{d}{d\delta} m_\delta(u)\right|_{\delta=0},$$

where $\mathbb{E}\left[\tau_u; \tau_u < \infty\right]$ is analogous to $\mathbb{E}\left[\tau_u \mathbb{1}_{\{\tau_u<\infty\}}\right]$. Similarly, the expected ruin time given that ruin occurs can be calculated by taking the following ratio: $\mathbb{E}\left[\tau_u \mid \tau_u < \infty\right] = \mathbb{E}\left[\tau_u; \tau_u < \infty\right]/\psi(u)$. In the Cramér-Lundberg model with exponential claims, this is given by (see, for instance, Gerber and Shiu (1998)):

$$\mathbb{E}\left[\tau_u \mid \tau_u < \infty\right] = \frac{\lambda}{c\left(\alpha c - \lambda\right)}\left(\frac{c}{\lambda} + u\right).$$

### Variations of the Cramér-Lundberg Model

It would be a mistake to think that the Cramér-Lundberg model is the only stochastic process studied in ruin theory because, although it is a simple model that manages to capture the main features of the behaviour over time of an insurer's surplus, it is certainly only a stylized description of reality (Mandjes and Boxma, 2023). This is one of the main reasons why multiple variations of the classical risk model have been proposed in the literature. For an overview of the variations of the Cramér-Lundberg model, interested readers may wish to consult Asmussen and Albrecher (2010) and Mandjes and Boxma (2023). In this section, we will briefly discuss some of these.

Representing an insurer's surplus as a *spectrally negative Lévy process* is perhaps one of the best known generalisations. Lévy processes are stochastic processes with stationary independent increments and, in particular, spectrally negative Lévy processes are Lévy processes with negative jumps only (we refer readers interested in the topic of Lévy processes to Bertoin (1996)). Clearly, the Cramér-Lundberg model is a spectrally negative Lévy process. Other examples of Lévy processes include the Brownian motion and the Gamma process. Over the years, Lévy processes have been extensively studied as an alternative to the classical risk model. For instance, Dufresne and Gerber (1991) considered a classical compound Poisson process perturbed by diffusion, which was years later studied by Gerber and Landry (1998). Moreover, the surplus process has also been extended to a Gamma process and other infinitly divisible process models (see, for example, Dufresne et al. (1991), Gerber (1992) and Dufresne and Gerber (1993)). For more details on the ruin probabilities of Lévy processes, see Asmussen and Albrecher (2010).

Perhaps one of the most criticisable points of the Cramér-Lundberg model is that it assumes that claims occur at a constant rate, i.e. it considers a homogeneous Poisson process. This is not consistent with the seasonal behaviour of certain lines of business. For instance, adverse weather conditions (e.g. during winter) and epidemics increase the frequency of claims in motor and life insurance, respectively (Asmussen and Albrecher, 2010). In response to this drawback, several alternatives have been proposed, of which we will mention some of the most important. The most natural step is to replace the homogeneous Poisson process with an inhomogeneous one. In addition, one can consider an inhomogeneous Poisson process with an intensity function coming from some stochastic mechanism, which gives rise to the so-called *Cox process*[9]. Björk and Grandell (1988) model the insurer's surplus under this alternative case and derive some inequalities for ruin probabilities. Years earlier, however, the Poisson process was replaced with a *renewal process*, which allows the inter-arrival times to have any distribution in the positive numbers, i.e. it is not necessary for them to have an exponential distribution as in the classical case. Under the renewal process set up, one derives the so-called *Sparre Andersen process*[10]. Finally, one can also consider a Markov-modulated Poisson process, which allows for "explosive claims arrivals" (Asmussen and Albrecher, 2010).

Another limitation of the classical Cramér-Lundberg model is the independence among claim sizes, claim inter-arrival times and between claim sizes and claim inter-arrival times. Over the years, researchers have introduced variations of the classical model to relax this assumption. For instance, Albrecher and Boxma (2004) derive exact solutions for the survival probability for a generalisation of the classical model in which the distribution of the time between two claim occurrences depends on the previous claim size. Similarly, Albrecher and Teugels (2006) incorporate dependence among the inter-arrival time and its subsequent claim size according to an arbitrary copula structure. A few years later, Albrecher et al. (2011) provided a class of dependence models for which explicit expressions for the ruin probability can be obtained (for more examples of risk processes with a certain degree of dependence, see Chapters 13 and 9 of Asmussen and Albrecher (2010) and Mandjes and Boxma (2023), respectively).

Other extensions of the classical Cramér-Lundberg model include modifying the premium process (1.2.1) to incorporate more realistic scenarios such as stochastic premiums (see, for example, Boikov (2003)) and investments (see, for instance, Segerdahl (1942)).

As we can see, extensions of the classical risk process (and in general, of mathematical models) arise, among other reasons, to try to resemble as closely as possible the surplus of an insurance company (or the reality of the phenomenon of interest). Throughout the chapters of this thesis, and with the aim of obtaining an answer to our key question, we will work with stochastic processes that attempt to portray a household's capital over time. Some of these models can be seen as particular cases of other risk processes previously studied in ruin theory.

[9]This process is named after the British statistician David Roxbee Cox, who was the first to introduce it (Cox, 1955).

[10]The Danish mathematician Erik Sparre Andersen was the first to consider the renewal process in ruin theory in 1957 (Andersen, 1957).

## 1.3 Main Contributions of this Thesis

The main contributions of this thesis lie at the intersection of ruin theory and development economics. The results of Chapters 2 and 5 have already been published in the *North American Actuarial Journal* and the *Scandinavian Actuarial Journal*, respectively. On the other hand, results derived in Chapters 3 and 4 have been submitted for publication and are currently under review.

Motivated by recent concerns among academics and practitioners about the role of (inclusive) insurance in poverty alleviation (see, for example, Kovacevic and Pflug (2011), Carter and Janzen (2018), Liao et al. (2020) and Janzen et al. (2021)), Chapters 2 and 3 aim to attach a rigorous mathematical framework (ruin theory) to previous literature analysing the role of inclusive insurance in poverty reduction. In particular, due to the growing importance of cooperation between governments and insurers (as highlighted in Section 1.1.2), Chapter 2 discusses the role of both government-subsidised and non-subsidised inclusive insurance schemes, while Chapter 3 considers a non-subsidised scheme. Furthermore, understanding the need for "smart" subsidies, which provide maximum social benefits while minimising distortions in the insurance market and the mis-targeting of clients (Hill et al., 2014), Chapter 2 introduces a transparent method to estimate optimal subsidies. From a ruin theory perspective, the main contributions of Chapters 2 and 3 lie in the derivation of closed-form solutions for trapping probabilities[11] of two variations of the classical Cramér-Lundberg model, which describe the dynamics of a household's capital over time. As Asmussen and Albrecher (2010) point out, the ideal in ruin theory is to find analytical expressions for ruin probabilities. In particular, Chapter 3 provides for the first time a closed-form solution for the trapping probability of a household's capital risk process with proportional capital losses[12]. Moreover, it introduces a recursive method to obtain a solution under the assumption that the household purchases insurance protection. In previous works on the risk process considered in Chapter 3, the trapping probability was only estimated numerically, without attempting to find an analytical solution for the probability (see, for instance, Kovacevic and Pflug (2011) and Azaïs and Genadot (2015)). According to Asmussen and Albrecher (2010), numerical approximations are the second best alternative after a closed-form solution.

Using ruin theory, and, in particular, applying the concept of the Gerber-Shiu expected discounted penalty function to the risk process with proportional capital losses also studied in Chapter 3, Chapter 4 derives a functional form to model household income short-fall. This finding is important, as in economics it is well-known that the processes of income generation and distribution must be connected, underpinned by a microeconomic foundation, to the functional form of any model that adequately represents the distribution of personal income (see Section 1.1.4). Interestingly, we find that the obtained model belongs to a family of distributions that has been widely studied in economics for modelling individual income: the Gen-

[11]In following chapters, we will define the concept of "trapping probability", which is analogous to the concept of an insurer's ruin probability (see Section 1.2.2).

[12]In modelling the capital of a household, "capital losses" play a role analogous to that of claims experienced by an insurer (see Section 1.2.2).

eralised Beta (GB) Distribution family, which was introduced by McDonald and Xu (1995). Chapter 4 also provides for the first time an integral equation (IE) for the Gerber-Shiu expected discounted penalty function of the risk process with proportional capital losses, which represents the main contribution of this chapter to the field of ruin theory. As underlined in Section 1.2.2, the Gerber-Shiu expected discounted penalty function is useful for deriving results in connection with joint and marginal distributions of the trapping (ruin) time, the capital surplus prior to the trapping time and the capital deficit at the trapping time. Thus, we further solve the IE for two particular cases from which the Laplace transform of the trapping time and the distribution of the capital deficit at trapping are derived. Chapter 4 concludes with a discussion about the connection between the capital deficit at trapping and the Foster-Greer-Thorbecke (FGT) index, a class of poverty measures that was introduced in Section 1.1.3. As mentioned in Section 1.1.4, one of the main advantages of parametric estimation of income distributions is that explicit formulas, as functions of the parameters of the theoretical income distribution, are available to measure poverty and inequality. Thus, Chapter 4 also validates the adequacy of the GB distribution to model household income short-fall by fitting it to household microdata from Burkina Faso's *Enquête Multisectorielle Continue (EMC) 2014*[13]. It further exemplifies how this allows us to interpret in more detail the shape parameters of the theoretical income short-fall distribution, as well as to conduct sensitivity analyses of poverty measures to variations in the shape parameters.

The last chapter of this thesis focuses on social assistance (see Section 1.1.1) and, in particular, it assesses the effectiveness of unconditional cash transfer programmes in reducing poverty. In doing so, it applies concepts from ruin theory and examines a variation of the capital risk process with proportional capital losses which incorporates capital cash transfers for those households that are deemed eligible. Indeed, in Chapter 5, we consider the same capital risk process also studied in Chapters 3 and 4, but incorporate ideas from the Omega risk process. In classical ruin theory, the Omega risk process distinguishes between ruin (negative surplus) and bankruptcy (going out of business). Thus, it is assumed that, even with negative surplus levels, an insurance company can do business as usual and continue until bankruptcy occurs. The Omega model was first introduced in Albrecher et al. (2011) and has been extensively studied during the last decade in the actuarial science literature, with researchers incorporating the bankruptcy concept into the Cramér-Lundberg model and its variations (see, for example, Gerber et al. (2012), Albrecher and Lautscham (2013) and Wang et al. (2016)). In Chapter 5, we derive explicit formulas for both the trapping probability and the probability of extreme poverty for a particular case of the distribution of the remaining proportion of a household's capital upon experiencing a capital loss. Here, the event of extreme poverty only depends on the current value of a household's capital given by some extreme poverty rate function and is analogous to the bankruptcy event from classical ruin theory. Numerical examples presented in Chapter 5 are in line with previous empirical studies in development economics, which indicate that unconditional cash transfer programmes are an efficient social protection strategy to keep households out of poverty and ex-

[13]The main objective of the Continuous Multisectoral Survey (EMC) was to generate sound data to monitor the country's sustainable development.

treme poverty, as their trapping probability and the probability of extreme poverty, respectively, decrease when they are part of such a strategy.

# Chapter 2

# Subsidising Inclusive Insurance to Reduce Poverty

This chapter is based on the following article:

Flores-Contró, J. M., K. Henshaw, S. H. Loke, S. Arnold, and C. D. Constantinescu (2024). Subsidising Inclusive Insurance to Reduce Poverty. Forthcoming in *North American Actuarial Journal*.

**Abstract.** In this chapter, we assess the benefits of coordination and partnerships between governments and private insurers, and provide further evidence for microinsurance products as powerful and cost-effective tools for achieving poverty reduction. To explore these ideas, we model the capital of a household from a ruin-theoretic perspective to measure the impact of microinsurance on poverty dynamics and the governmental cost of social protection. We analyse the model under four frameworks: uninsured, insured (without subsidies), insured with subsidised constant premiums and insured with subsidised flexible premiums. Although insurance alone (without subsidies) may not be sufficient to reduce the likelihood of falling into the area of poverty for specific groups of households, since premium payments constrain their capital growth, our analysis suggests that subsidised schemes can provide maximum social benefits while reducing governmental costs.

## 2.1 Introduction

In recent years, governments in developing countries have been increasingly involved in the provision of insurance programmes. In countries such as China and India for instance, the agricultural insurance sector has grown significantly thanks to the support (and premium subsidies) provided by central and provincial governments (Kramer et al., 2022). While doubts about the role of insurers in alleviating poverty exist among practitioners, adequate coordination between governments, private insurance companies and other stakeholders (e.g. NGOs, international financial institutions and other donors) has been shown to enhance the development of sustainable, affordable and cost-effective insurance products (Linnerooth-Bayer and Mechler, 2007; Auzzir et al., 2014).

The most common form of government support for insurance are premium subsidies. The central and provincial governments provide Chinese farmers with subsidies exceeding 50% of the premium amount, of which farmers pay only about 20% (Wang et al., 2011; Ye et al., 2020). Similarly, the Pradhan Mantri Fasal Bima Yojana (PMFBY), a government-sponsored multi-stakeholder crop insurance scheme in India, charges farmers a maximum premium ranging from 2% to 5% of the sum insured (or the actuarial rate, whichever is lower), with the remaining part of the premium paid on a 50/50 basis by the central and state governments (Kaur et al., 2021). Insurance premium subsidies must be designed with a clearly stated purpose. They should target those in need and address market deficiencies or consumer equity concerns (see Hill et al. (2014), which is a technical report from the United Nations' International Labour Organization (ILO)). Experience shows that, when designed properly, subsidised insurance schemes represent a powerful and cost-effective way to achieve public policy objectives, while poorly designed insurance premium subsidies can be inefficient and lead to significant economic costs (Hazell and Varangis, 2020).

Adopting the novel ruin-theoretic approach presented by Kovacevic and Pflug (2011), this chapter studies the impact of insurance (both with and without subsidies) on poverty dynamics and the governmental cost of social protection. Through this analysis, we seek to determine the benefits derived from coordination and partnerships between governments and private insurers, and to highlight the cost-effectiveness of government support for insurance. Previous studies have approached the same problem from a dynamic stochastic programming perspective. Ikegami et al. (2018), Carter and Janzen (2018) and Janzen et al. (2021) propose dynamic models of household consumption, investment and risk management, considering a social insurance-type mechanism which first prioritises lending aid to the vulnerable non-poor, contingent on their experience of negative shocks, then to those already below the poverty line. Introduction of an index-based insurance market is found to outperform the asset-based vulnerability-targeted protection in poverty reduction, economic growth and the cost of social protection. Although implementation of a vulnerability-targeted strategy induces a short-term increase in poverty, rates are lower than those associated with both in-kind and cash transfers in the medium- and long-term.

Carter and Janzen (2018) and Janzen et al. (2021) compare the impact of insurance when all costs are paid by the policyholder and when targeted-subsidies are provided to the vulnerable and already poor. In the latter study, those in the neighbourhood of the poverty line do not optimally purchase insurance (without subsidies), instead suppressing their consumption and mitigating the probability of falling into poverty. Observing a greater reduction in poverty in comparison to pure cash transfers, Jensen et al. (2017) provide empirical evidence for the benefits of insurance-based social protection through analysis of safety net and drought-based livestock insurance programmes in northern Kenya. Chantarat et al. (2017) consider the welfare impacts of the same index-based insurance programme, using herd size dynamics to address the vulnerability to poverty associated with covariate livestock mortality such that critical herd size mimics the poverty line. Here, targeted premium subsides are optimised across various herd size groups such that given measures of poverty reduction are maximised. Increases and decreases in household wealth and poverty, respectively, were greater under the optimal strategy than under alternative needs-based subsidisation mechanisms and with no insurance. In the presence of needs-based subsidisation which provides free protection to the most poor, the number of poor continued to increase, thus highlighting the importance of social protection strategies that target those still above but close to the poverty line in addition to the already poor.

The insurance strategies considered in these studies are inclusive insurance mechanisms specifically designed to cater for the most vulnerable. Inclusive insurance, commonly referred to as microinsurance, relates to the provision of insurance services to low-income populations with limited, or no access to mainstream insurance or alternative effective risk management strategies. Targeting low-income individuals living close to or below the poverty line, microinsurance aims to close the protection gap that exists between uninsured and insured losses to life, property and health by providing protection to the poor. However, barriers to microinsurance penetration exist due to constraints on product affordability resulting from fundamental features of the microinsurance environment. These distinct features include the nature of low-income risks, limited consumer financial literacy and experience, product accessibility and data availability. While novel solutions for the supply and distribution of products in this environment exist (e.g. mobile-based business models (Kousky et al., 2021)), it is important to consider the viability of microinsurance uptake for all sectors of the target population, particularly for the most vulnerable.

Premium payments can in fact heighten the risk of falling into poverty for the proportion of the population living just above the poverty line, inducing a balance between protection and loss as a result of insurance coverage which is dependent on the entity's level of capital (see, for example, Kovacevic and Pflug (2011) and Liao et al. (2020), where the latter use a multiple-equilibrium framework to analyse the impact of subsidised and unsubsidised agricultural insurance on poverty rates in rural China). This insufficiency of microinsurance alone as a means for poverty reduction for the most exposed necessitates an alternative solution. For this purpose, as in the aforementioned studies, we consider microinsurance schemes which are supported by social protection strategies, and more specifically, their potential in minimising both the probability of a household falling below the poverty line and the governmental cost of social protection. For thorough discussions of microinsurance, the challenges associated with adapting commercial insurance to serve the poor and the insurability of risks in the market, the interested reader may refer to Dror (2019), Churchill (2007) and Biener and Eling (2012).

Besides reducing the impact on household capital growth, the use of subsidies to lower consumer premium payments has the potential to increase microinsurance uptake, with wealth and product price positively and negatively influencing microinsurance demand, respectively, see Eling et al. (2014) and Platteau et al. (2017). However, this relationship is not transparent (see, for instance, Cole et al. (2013), where the authors find that, in the city of Ahmedabad in India, more than half of households in their sample do not purchase rainfall insurance even when premiums are set significantly below actuarially fair values), with additional factors, including financial education levels, insurer trust and logistical problems in the purchase and renewal of coverage, having significant influence on a household's decision to insure. A comprehensive approach should therefore be adopted by insurance providers such that low-cost subsidisation schemes are complemented by innovative activities improving understanding of and access to insurance products. Focusing specifically on agricultural insurance, Hazell and Varangis (2020) present government and donor incentives for subsidisation. As an example, temporary subsidies can enable low-income farmers to bear the risk of adopting innovative technologies which may bring them out of poverty. However, in addition to improving the economic circumstances of the insured, through the provision of insurance experience this strategy mitigates the uncertainties surrounding insurance common among consumers in the microinsurance environment, while improving the quality of consumer data. The study additionally highlights how subsidisation schemes help to scale up insurance products.

Although important for poverty alleviation, the behaviour of a household below the poverty line is not considered in this study. Households that live or fall below the poverty line are said to be in a poverty trap, where a poverty trap is a state of poverty from which it is difficult to escape without external help. Poverty trapping is a well-studied topic in development economics (the interested reader may refer to Azariadis and Stachurski (2005), Bowles et al. (2006), Kraay and McKenzie (2014), Barrett et al. (2016) and references therein for further discussion; see Matsuyama (2008) for a detailed description of the mechanics of poverty traps), however, for the purpose of this study, we use the term "trapping" only to describe the event that a household falls into poverty, focusing our interest on low-income behaviours above this critical line.

Our study complements the aforementioned studies that analyse the impact of inclusive insurance from both an empirical and dynamic stochastic programming perspective. We introduce a more formal and rigorous mathematical framework that analytically demonstrates the benefits of partnerships between governments and private insurers. For that purpose, we adapt the piecewise-deterministic Markov process proposed by Kovacevic and Pflug (2011) such that households are subject to shocks of random size and we consider a non-discretised capital process. In line with the poverty trap ideology, we assume the area of poverty to be an absorbing state and consider only the state of events above the poverty threshold. Obtaining explicit solutions for the trapping probability and the governments' cost of social protection using classical risk theory techniques (where this is considered the ideal scenario (Asmussen and Albrecher, 2010)), we compare the influence of three structures of microinsurance on these quantities. Specifically, we consider a microinsurance scheme with (i) unsubsidised premiums, (ii) subsidised constant premiums and (iii) subsidised flexible premiums. Unlike previous studies, where the cost of social protection is defined as the present value of government subsidies plus the transfers needed to close the poverty gap for all poor households (see, for example, Ikegami et al. (2018) and Janzen et al. (2021)), the ruin-theoretic perspective adopted in the proposed model allows us to include a supplemental fixed cost that ensures, with a certain level of confidence, that households will not return to poverty, should they fall underneath the threshold. In this way, the likelihood that the government will re-incur these costs for the same household is reduced.

The adopted capital models are special cases of well-studied risk theory models. Therefore, a standard modelling approach with application to poverty trapping is considered. Typically assumed to represent the surplus process of an insurer, our alternative application enforces two key adaptations. First, unlike the barrier at zero considered in the classical setting, where an insurer is deemed to be ruined if their surplus falls below zero, a non-zero critical barrier reflecting the poverty line is assumed. Second, in the classical case, insurers raise capital and rely on access to reinsurance to avoid falling below the critical level. Thus, two mechanisms for escaping ruin exist. On the other hand, the level of capital growth attained by a household is the only mechanism protecting them from ruin in the absence of insurance. In addition, while in the classical setting the initial surplus is typically considered to be large enough to keep the insurer away from ruin, in this study we focus on analysing households with initial capital just above the poverty line.

Under the first premium framework we assume premium payments are made by households. We demonstrate that these payments can constrain households' capital growth and thus increase their trapping probability compared to that of uninsured households, as previous studies have shown. Conversely, under such a scheme, the cost of social protection remains lower than the corresponding uninsured cost. With the need for an alternative solution to address the observed negative impact on poverty dynamics, under the second premium framework we assume governments provide insurance premium subsidies to all households. Reducing premium payments by means of subsidies has a positive impact on household capital growth and their trapping probabilities. Furthermore, the results obtained allow us to estimate optimal subsidies for households with varying degrees of capital such that they preserve a trapping probability equal to that of when uninsured. The proposed subsidy optimisation aligns with the idea of "smart" subsidies, which provide maximum social benefits while minimising distortions in the insurance market and the mistargeting of clients (Hill et al., 2014). Here, the optimal subsidy seeks to reduce the likelihood of a household falling into the area of poverty (clear objective), has a mathematical foundation (transparent), intends to help those in need of assistance (targeted), can be assessed over time (monitoring and evaluation), can be strategically planned (exit strategy/long-term financing) and is capable of being costed (costs contained). Our analysis shows that, under this subsidised microinsurance scheme, while government support is not essential for privileged households, vulnerable households with capital levels close to the poverty line require assistance.

Moreover, the cost of social protection is lower for the most vulnerable than in the corresponding uninsured framework, but is higher for the most privileged.

Mimicking the well-known risk theory dividend barrier strategy, the third framework considers a novel scheme where households pay premiums only when their capital is above some pre-defined capital barrier, with the premium otherwise paid by the government. Granting flexibility on premium payments allows households to attain lower trapping probabilities, since they are assisted by the government when their capital lies close to the poverty trap. Continuing with the idea of "smart" subsidies, we optimise the capital barrier level at which governments should begin providing support. As could be expected, those closest to the poverty line require immediate aid, with optimal barriers lying above their initial capital, whereas those further away from the poverty trap possess the ability to pay premiums themselves once enrolled in the scheme, yielding to optimal barriers lying below their initial capital levels. Under this framework, the cost of social protection remains lower than the corresponding uninsured cost.

Premium subsidies are not phased out over time in the inclusive insurance schemes considered here. Nevertheless, it is often necessary to assess the financial dependence of subsidised inclusive insurance schemes on external support. That is, situations in which governments decide to reduce or end the provision of subsidies, that may lead to the need to raise premiums beyond the reach of their customers, should be taken into account when evaluating the viability of a subsidised scheme, as they expose concerns regarding the scheme's sustainability.

Informal risk-sharing networks are highly prevalent in low-income economies (see, for example, Townsend (1994), Bardhan and Udry (1999), De Weerdt and Dercon (2006)). In particular, community-based networks in Ghana and South Africa gather funds and other contributions to meet funeral expenses (Ramsay and Arcila, 2013). These networks help to mitigate the risk of idiosyncratic losses. Heavily subsidising an insurance product would, however, lessen the need for such networks, as policyholders would be protected from the occurrence of adverse events. Hence, the strength of the networks could suffer as a result, leaving households exposed if the subsidy is eventually removed. Although our definition of "insurance" covers all forms of losses, in reality, an insurance policy typically covers a single peril. It is therefore likely that, even with subsidised insurance, low-income households will still participate in risk-sharing mechanisms to mitigate the losses that are not covered by their insurance policies. Thus, considering the situation in which subsidies cease, the informal networks could be rebuilt on the foundations that remain. Furthermore, if phasing out subsidies, governments could undertake activities to encourage the continuation of informal risk-sharing to prevent the crowding out of informal risk-transfer mechanisms. Evidence does, however, suggest that insurance uptake drops when a subsidy is removed (Platteau et al., 2017), implying that households revert back to informal risk-sharing mechanisms rather than purchasing unsubsidised coverage. Formal insurance and informal risk-sharing networks have, in fact, been found to be complementary (see, for example, Will et al. (2021)).

Government subsidies may induce an increase in moral hazard, since the reduction in premiums diminishes policyholders' sensitivity to the consequences of a loss.

However, as described by Biener and Eling (2012) and Hill et al. (2014) among others, reducing information asymmetries through government investment in data collection can help to alleviate the increased risk by enabling better understanding of a policyholder's true risk exposure. This issue is of particular concern in the health, agricultural and catastrophe insurance markets. Distribution through local enterprises (Dercon et al., 2006), group-based products (Biener and Eling, 2012) and financial education (Biener et al., 2014) have also been found to lessen information asymmetries associated with moral hazard.

The remainder of the chapter is structured as follows. In Section 2.2, we introduce the household capital model and its associated infinitesimal generator. The (trapping) time at which a household falls into the area of poverty is defined in Section 2.3, and subsequently the explicit trapping probability and the expected trapping time are derived for the basic uninsured model. Links between classical ruin models and the trapping model of this chapter are stated in Sections 2.2 and 2.3. Microinsurance is introduced in Section 2.4, where we assume a proportion of household losses are covered by a microinsurance policy. The capital model is redefined and the trapping probability is derived. Sections 2.5 and 2.6 consider the case where households are proportionally insured through a government subsidised microinsurance scheme, with the impact of subsidised flexible premiums discussed in Section 2.6. Optimisation of the subsidy and capital barrier levels are presented in Sections 2.5 and 2.6, alongside the associated governmental cost of social protection. Concluding remarks are provided in Section 2.7.

## 2.2 The Capital Model

The fundamental dynamics of the model follow those of Kovacevic and Pflug (2011), where the growth in accumulated capital $\{X_t\}$ of an individual household is given by

$$\frac{dX_t}{dt} = r \cdot [X_t - x^*]^+ , \tag{2.2.1}$$

where $[x]^+ = \max(x, 0)$. The capital growth rate $r = (1-a) \cdot b \cdot c > 0$ incorporates household rates of consumption $(0 < a < 1)$, income generation $(0 < b)$ and investment or savings $(0 < c < 1)$, while $x^* > 0$ represents the threshold below which a household lives in poverty. Reflecting the ability of a household to produce, accumulated capital $\{X_t\}$ is composed of land, property, physical and human capital, with health as a form of capital in extreme cases where sufficient health services and food accessibility are not guaranteed (Dasgupta, 1997). The notion of a household in this model setting may be extended for consideration of poverty trapping within economic units such as community groups, villages and tribes, in addition to the traditional household structure.

The dynamical process in (2.2.1) is constructed such that consumption is assumed to be an increasing function of wealth (for full details of the model construction see Kovacevic and Pflug (2011)). The poverty threshold $x^*$ represents the amount of capital required to forever attain a critical level of income, below which a household would not be able to sustain their basic needs, facing elementary problems relating to health and food security. Throughout the chapter, we will refer to this threshold as the critical capital or the poverty line. Since (2.2.1) is positive for all levels of capital greater than the critical capital, points less than or equal to $x^*$ are stationary (capital remains constant if the critical level is not met). In this basic model, stationary points below the critical capital are not attractors of the system if the initial capital exceeds $x^*$, in which case the capital process $\{X_t\}$ grows exponentially with rate $r$.

Using capital as an indicator of financial stability over other commonly used measures such as income enables a more effective analysis of a household's wealth and well-being. Households with relatively high income, considerable debt and few assets would be vulnerable if any loss of income was to occur, while low-income households could live comfortably on assets acquired during more prosperous years for a long-period of time (Gartner et al., 2004).

In line with Kovacevic and Pflug (2011), we expand the dynamics of (2.2.1) under the assumption that households are susceptible to the occurrence of capital losses, including severe illness, the death of a household member or breadwinner and catastrophic events such as floods and earthquakes. We assume occurrence of these events follows a Poisson process with intensity $\lambda$, where the capital process follows the dynamics of (2.2.1) in between events. On the occurrence of a loss, the household's capital at the event time reduces by a random amount $Z_i$. The sequence $\{Z_i\}$ is independent of the Poisson process and i.i.d. with common distribution function $G_Z$. In contrast to Kovacevic and Pflug (2011), we assume reduction by a given amount rather than a random proportion of the capital itself. This adaptation enables analysis of a tractable mathematical model that provides, for instance, the possibility of finding an analytical solution for the infinite-time trapping probability (defined in Section 2.3). This differs from previous work in which numerical methods, considered by Asmussen and Albrecher (2010) as the second best alternative to calculating trapping probabilities when closed-form expressions are not available, are employed to estimate such a quantity (see, for example, Kovacevic and Pflug (2011) and Azaïs and Genadot (2015)). The core objective of studying the probability of a household falling into the area of poverty remains.

A household reaches the area of poverty if it suffers a loss large enough that the remaining capital is attracted into the poverty trap. Since a household's capital does not grow beyond the critical capital $x^*$, households that fall into the area of poverty will never escape without external help. Once below the critical capital, households are exposed to the risk of falling deeper into poverty, with the dynamics of the model allowing for the possibility of negative capital. A reduction in a household's capital below zero could represent a scenario where total debt exceeds total assets, resulting in negative capital net worth. The experience of a household below the critical capital is, however, out of the scope of this chapter.

We will now formally define the stochastic capital process, where the process for the inter-event household capital (2.2.2) is derived through the solution of the first order Ordinary Differential Equation (ODE) (2.2.1). This model is an adaptation of the model proposed by Kovacevic and Pflug (2011).

**Definition 2.2.1.** Let $T_i$ be the $i$th event time of a Poisson process $\{N_t\}_{t\geq 0}$ with parameter $\lambda$, where $T_0 = 0$. Let $Z_i \geq 0$ be a sequence of i.i.d. random variables with distribution function $G_Z$, independent of the process $\{N_t\}$. For $T_{i-1} \leq t < T_i$, the stochastic growth process of the accumulated capital $X_t$ is defined as

$$X_t = \begin{cases} \left(X_{T_{i-1}} - x^*\right) e^{r(t-T_{i-1})} + x^* & \textit{if } X_{T_{i-1}} > x^*, \\ X_{T_{i-1}} & \textit{otherwise.} \end{cases} \tag{2.2.2}$$

At the jump times $t = T_i$, the process is given by

$$X_{T_i} = \begin{cases} \left(X_{T_{i-1}} - x^*\right) e^{r(T_i-T_{i-1})} + x^* - Z_i & \textit{if } X_{T_{i-1}} > x^*, \\ X_{T_{i-1}} - Z_i & \textit{otherwise.} \end{cases} \tag{2.2.3}$$

The infinitesimal generator of the stochastic process $\{X_t\}_{t\geq 0}$, which is a piecewise-determinsitic Markov process (Davis, 1984), is given by

$$(\mathcal{A}f)(x) = r(x - x^*)f'(x) + \lambda \int_0^\infty [f(x-z) - f(x)] \, \mathrm{d}G_Z(z), \qquad x \geq x^*.$$

The capital model as defined in (2.2.2) and (2.2.3) is in fact a topic well-studied in ruin theory since the 1940s. As such, well-established techniques can be easily applied to the poverty trapping context of this chapter. In ruin theory, modelling is undertaken from the point of view of an insurance company. Consider the insurer's surplus process $\{U_t\}_{t\geq 0}$ given by

$$U_t = u + pt + \nu \int_0^t U_s \, ds - \sum_{i=1}^{N_t} Z_i, \tag{2.2.4}$$

where $u$ is the insurer's initial capital, $p$ is the constant premium rate, $\nu$ is the risk-free interest rate, $N_t$ is a Poisson process with parameter $\lambda$ which counts the number of claims in the time interval $[0, t]$ and $\{Z_i\}_{i=1}^\infty$ is a sequence of i.i.d. claim sizes with distribution function $G_Z$. This model is also called the insurance risk model with deterministic investment, first proposed by Segerdahl (1942) and subsequently studied by Harrison (1977) and Sundt and Teugels (1995). For a detailed literature review on this model prior to the turn of the century, readers may consult Paulsen (1998).

Observe that when $p = 0$, the insurance model (2.2.4) for positive surplus is equivalent to the capital model (2.2.2) and (2.2.3) above the poverty line $x^* = 0$. Subsequently, the capital growth rate $r$ in our model corresponds to the risk-free investment rate $\nu$ of the insurer's surplus model. More connections between these two models will be made in the next section following introduction of the trapping time.

## 2.3 The Trapping Time

Let

$$\tau_x := \inf\{t \geq 0 : X_t < x^* \mid X_0 = x\}$$

denote the time at which a household with initial capital $x \geq x^*$ falls into the area of poverty (the trapping time) and let $\psi(x) = \mathbb{P}(\tau_x < \infty)$ be the infinite-time trapping probability. To study the distribution of the trapping time we apply the expected discounted penalty function at ruin, a concept commonly used in actuarial science (Gerber and Shiu, 1998), such that with a force of interest $\delta \geq 0$ and initial capital $x \geq x^*$, we consider

$$m_\delta(x) = \mathbb{E}\left[w(X_{\tau_x^-} - x^*, \mid X_{\tau_x} - x^* \mid) e^{-\delta\tau_x} \mathbb{1}_{\{\tau_x < \infty\}}\right], \tag{2.3.1}$$

where $\mathbb{1}_{\{A\}}$ is the indicator function of a set $A$ and $w(x_1, x_2)$ for $0 \leq x_1, x_2 < \infty$, is a non-negative penalty function of $x_1$, the capital surplus prior to the trapping time, and $x_2$, the capital deficit at the trapping time. For more details on the so-called Gerber-Shiu risk theory, interested readers may wish to consult Kyprianou (2013).

The probabilistic properties of the trapping time are contained in its distribution function. However, it is sometimes much easier to work with a transformation rather than with the distribution function of a random variable itself. Here, we focus on the Laplace transform, which is particularly useful for nonnegative, absolutely continuous random variables such as the trapping time and is a powerful tool in probability theory. Moreover, the Laplace transform characterises the probability distribution[1]. For a continuous random variable $X$, with probability density function $f_X$, the Laplace transform of $f_X$ is given by the expected value $\mathcal{L}\{f_X\}(s) = \mathbb{E}\left[e^{-sX}\right]$. Note that, specifying the penalty function such that $w(x_1, x_2) = 1$ in (2.3.1), $m_\delta(x)$ becomes the Laplace transform of the trapping time, also interpreted as the expected present value of a unit payment due at the trapping time.

For simplicity, throughout the rest of the chapter we will assume that capital losses are exponentially distributed ($Z_i \sim Exp(\alpha)$).

**Proposition 2.3.1.** *Consider a household capital process (as proposed in Definition 2.2.1) with initial capital $x \geq x^*$, capital growth rate $r$, intensity $\lambda > 0$ and exponentially distributed capital losses with parameter $\alpha > 0$. The Laplace transform of the trapping time is given by*

$$m_\delta(x) = \frac{\lambda}{(\lambda+\delta) U\left(1 - \frac{\lambda}{r}, 1 - \frac{\lambda+\delta}{r}; 0\right)} e^{y(x)} U\left(1 - \frac{\lambda}{r}, 1 - \frac{\lambda+\delta}{r}; -y(x)\right), \tag{2.3.2}$$

*where $\delta \geq 0$ is the force of interest for valuation, $y(x) = -\alpha(x - x^*)$ and $U(\cdot)$ is Tricomi's Confluent Hypergeometric Function as defined in (2.A.6).*

See Appendix 2.A.1 for proof of Proposition 2.3.1.

*Remark* 2.3.1. Figure 2.1a shows that the Laplace transform of the trapping time (2.3.2) approaches the trapping probability as $\delta$ tends to zero, since

$$\lim_{\delta \downarrow 0} m_\delta(x) = \mathbb{P}(\tau_x < \infty) \equiv \psi(x).$$

[1] Only if it exists and is finite in a neighborhood of zero.

As $\delta \to 0$, (2.3.2) yields

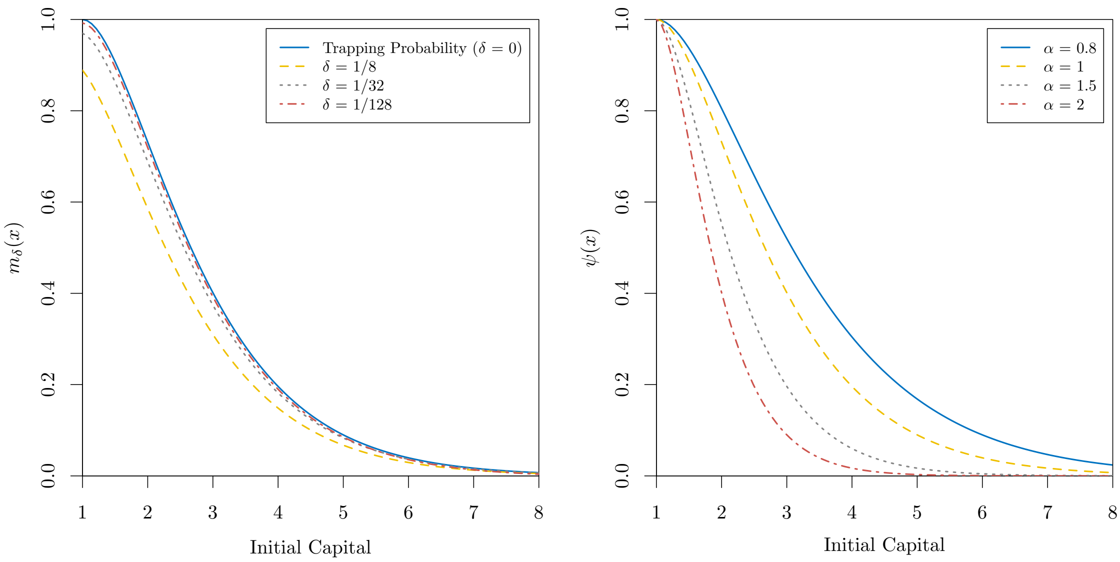
$$\psi(x) = \frac{1}{U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda}{r}; 0\right)} e^{y(x)} U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda}{r}; -y(x)\right).$$

We can further simplify the expression for the trapping probability using the upper incomplete gamma function $\Gamma(a;z) = \int_z^\infty e^{-t} t^{a-1} dt$. Applying the relation

$$\Gamma(a;z) = e^{-z} U(1-a, 1-a; z)$$

(see Equation (13.6.28) of Abramowitz and Stegun (1972)) and the fact that $\Gamma(a;0) = \Gamma(a)$ for $\mathbb{R}(a) > 0$, we have

$$\psi(x) = \frac{\Gamma\left(\frac{\lambda}{r}; -y(x)\right)}{\Gamma\left(\frac{\lambda}{r}\right)}. \tag{2.3.3}$$




(a) (b)

Figure 2.1: (a) Laplace transform $m_\delta(x)$ of the trapping time when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^* = 1$ for $\delta = 0, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}$ (b) Trapping probability $\psi(x)$ when $Z_i \sim Exp(\alpha)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^* = 1$ for $\alpha = 0.8, 1, 1.5, 2$.

Figure 2.1b displays the trapping probability $\psi(x)$ for the stochastic capital process $X_t$. Clearly, increasing the value of the parameter $\alpha$ of the exponential distribution of the capital losses reduces the trapping probability for all households, since losses will more likely take values close to zero and will therefore have less impact on households' capital.

*Remark* 2.3.2. As an application of the Laplace transform of the trapping time, one quantity of interest is the expected trapping time, i.e. the expected time at which a household will fall into the area of poverty. Reducing a household's trapping probability is central to poverty alleviation. However, knowledge of the time at which a household is expected to fall below the poverty line would allow insurers and governments to better prepare for the potential need to lift a household out of poverty.

It provides an alternative comparative measure for the performance analysis of different schemes, helping to inform insurance product design and financial education for consumers. For example, a household with a low expected trapping time may be encouraged to adopt certain risk mitigating behaviours to reduce the impact of shock events and hence the likelihood of them falling below the poverty line. This quantity can be obtained by taking the derivative of $m_\delta(x)$, such that

$$\mathbb{E}\left[\tau_x;\tau_x<\infty\right]=-\left.\frac{d}{d\delta}m_\delta(x)\right|_{\delta=0}, \tag{2.3.4}$$

where $\mathbb{E}\left[\tau_x;\tau_x<\infty\right]$ is analogous to $\mathbb{E}\left[\tau_x\mathbb{1}_{\{\tau_x<\infty\}}\right]$.

**Corollary 2.3.1.** *The expected trapping time under the household capital model proposed in Definition 2.2.1 with initial capital $x\geq x^*$, capital growth rate $r$, intensity $\lambda>0$ and exponentially distributed capital losses with parameter $\alpha>0$ is given by*

$$\begin{aligned}\mathbb{E}\left[\tau_x;\tau_x<\infty\right]=&\frac{\Gamma\left(\frac{\lambda}{r};-y(x)\right)}{\lambda U\left(1-\frac{\lambda}{r},1-\frac{\lambda}{r};0\right)}-\frac{\Gamma\left(\frac{\lambda}{r};-y(x)\right)U^{(c)}\left(1-\frac{\lambda}{r},1-\frac{\lambda}{r};0\right)}{r\left[U\left(1-\frac{\lambda}{r},1-\frac{\lambda}{r};0\right)\right]^2}\\&+e^{y(x)}\frac{U^{(c)}\left(1-\frac{\lambda}{r},1-\frac{\lambda}{r};-y(x)\right)}{rU\left(1-\frac{\lambda}{r},1-\frac{\lambda}{r};0\right)},\end{aligned} \tag{2.3.5}$$

*where $y(x)=-\alpha(x-x^*)$, $U(\cdot)$ is Tricomi's Confluent Hypergeometric Function as defined in (2.A.6) and $U^{(c)}(\cdot)$ its derivative with respect to the second parameter as presented in (2.A.10).*

The mathematical proof of Corollary 2.3.1 is presented in Appendix 2.A.2. Note that, the expected trapping time given that trapping occurs can be calculated by taking the following ratio (see for example, Equation (4.37) of Gerber and Shiu (1998)),

$$\mathbb{E}\left[\tau_x|\tau_x<\infty\right]=\frac{\mathbb{E}\left[\tau_x;\tau_x<\infty\right]}{\psi(x)}.$$

In line with intuition, the expected trapping time is an increasing function of both the capital growth rate $r$ and initial capital $x$. A number of expected trapping times for varying values of $r$ are displayed in Figure 2.2.

*Remark* 2.3.3. The ruin probability for the insurance model (2.2.4) given by

$$\xi(u)=\mathbb{P}(U_t<0\text{ for some }t>0\mid U_0=u),$$

is found by Sundt and Teugels (1995) to satisfy the Integro-Differential Equation (IDE)

$$(\nu u+p)\xi'(u)-\lambda\xi(u)+\lambda\int_0^u\xi(u-z)\,dG_Z(z)+\lambda(1-G_Z(u))=0,\qquad u\geq 0. \tag{2.3.6}$$

Note that when $p=0$, (2.3.6) coincides with the special case of (2.A.1) when $x^*=0$, $w(x_1,x_2)=1$ and $\delta=0$. Thus, the household trapping time can be thought of as the insurer's ruin time. Indeed, the ruin probability in the case of exponential claims when $p=0$, as shown in Section 6 of Sundt and Teugels (1995), is exactly the same as the trapping probability (2.3.3) when $x^*=0$.

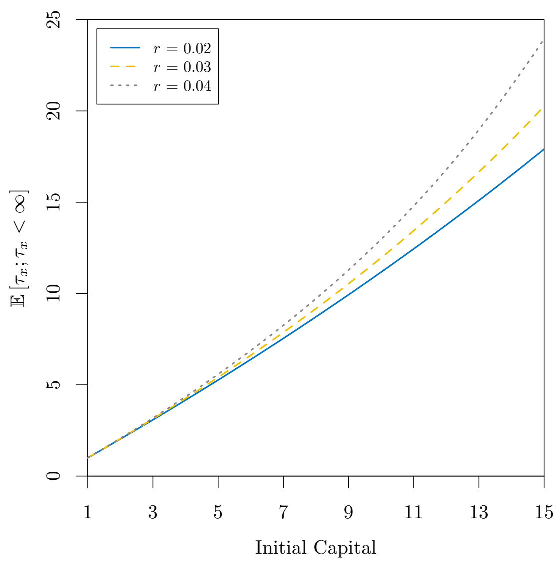



Figure 2.2: Expected trapping time when $Z_i \sim Exp(1)$, $\lambda = 1$ and $x^* = 1$ for $r = 0.02, 0.03, 0.04$.

## 2.4 Introducing Microinsurance

As in Kovacevic and Pflug (2011), we assume households have the option of enrolling in a microinsurance scheme that covers a certain proportion of the capital losses they encounter. The microinsurance policy has proportionality factor $1 - \kappa$, where $\kappa \in [0, 1]$, such that $100 \cdot (1 - \kappa)$ percent of the damage is covered by the microinsurance provider. The premium rate paid by households, calculated according to the expected value principle, is given by

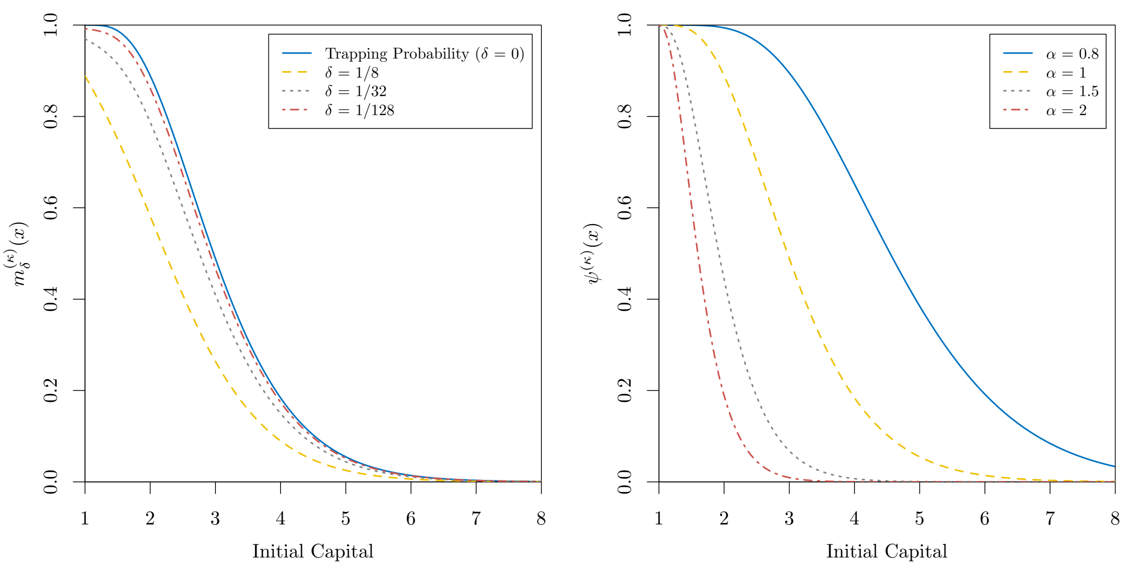
$$\pi(\kappa, \theta) = (1 + \theta) \cdot (1 - \kappa) \cdot \lambda \cdot \mathbb{E}[Z_i], \tag{2.4.1}$$

where $\theta$ is some loading factor. The expected value principle is popular due to its simplicity and transparency. When $\theta = 0$, one can consider $\pi(\kappa, \theta)$ to be the pure risk premium (Albrecher et al., 2017).

The stochastic capital process of a household covered by a microinsurance policy is denoted by $X_t^{(\kappa)}$. We differentiate between all variables and parameters relating to the original uninsured and the insured processes through use of the superscript $(\kappa)$ in the latter case. We assume the basic model parameters are unchanged by the introduction of microinsurance coverage (parameters $a, b$ and $c$ of Kovacevic and Pflug (2011), previously introduced in Section 2.2). Here, we only allow households to select a fixed retention rate, while other studies look for an optimal retention rate process that maximises the expected discounted capital by admitting adjustments in the retention rate after each capital loss throughout the lifetime of the insurance contract (see, for example, Kovacevic and Semmler (2021)).

Since premiums are paid from a household's income, the capital growth rate $r$ is adjusted such that it reflects the lower rate of income generation resulting from the need for premium payment. The premium rate is restricted to prevent certain poverty, which would occur should it exceed the rate of income generation. The capital growth rate of the insured household $r^{(\kappa)} = (1-a)\cdot(b-\pi)\cdot c > 0$ is lower than that of the uninsured household, while the critical capital is higher (see Kovacevic and Pflug (2011) for further discussion). Note that, previous work such as that of Janzen et al. (2021) allow households to choose optimal levels of consumption and insurance coverage over time based on asset holdings and the probability distribution of future assets. Here, all households who can afford to buy insurance enrol in a scheme; that is, as mentioned above, households whose income generation rate is greater than the insurance premium are able to choose any affordable insurance coverage, therefore admitting both optimal and suboptimal choices with respect to the trapping probability. Although this feature aligns with the low levels of financial literacy that characterise the microinsurance environment (Churchill and Matul, 2012), it could initially be considered as a limitation of our model. However, one of the core objectives of the subsidised schemes introduced in Sections 2.5 and 2.6 is to diminish the adverse effects that arise with suboptimal choices and as such any limitation is accounted for.

In between jumps, the insured stochastic growth process $X_t^{(\kappa)}$ behaves in the same manner as (2.2.2), with parameters corresponding to the proportional insurance case of this section. By enrolling in a microinsurance scheme a household's capital losses are reduced to $Y_i := \kappa \cdot Z_i$. Considering the case in which losses follow an exponential distribution with parameter $\alpha > 0$, the structure of the IDE (2.A.1) remains the same. However, acquisition of a proportional microinsurance policy changes the parameter of the distribution $G_Y$ of the random losses $\{Y_i\}$. Namely, we have that $Y_i \sim Exp\left(\alpha^{(\kappa)}\right)$ for $\kappa \in (0, 1]$, where $\alpha^{(\kappa)} := \alpha/\kappa$.




(a) (b)

Figure 2.3: (a) Laplace transform $m_\delta^{(\kappa)}(x)$ of the trapping time when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^{(\kappa)*} = 1$, $\kappa = 0.5$ and $\theta = 0.5$ for $\delta = 0, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}$ (b) Trapping probability $\psi^{(\kappa)}(x)$ when $Z_i \sim Exp(\alpha)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^{(\kappa)*} = 1$, $\kappa = 0.5$ and $\theta = 0.5$ for $\alpha = 0.8, 1, 1.5, 2$.

Following a similar procedure to that in Proposition 2.3.1 (as presented in Appendix 2.A), one easily obtains the Laplace transform of the trapping time and thus the trapping probability for the insured process.

**Proposition 2.4.1.** *Consider the capital process of a household enrolled in a microinsurance scheme with proportionality factor* $1-\kappa \in [0,1]$ *(as introduced in this section). Assume the household has initial capital* $x \geq x^{(\kappa)*}$*, capital growth rate* $r^{(\kappa)}$*, intensity* $\lambda > 0$ *and exponentially distributed capital losses with parameter* $\alpha^{(\kappa)} > 0$*. The Laplace transform of the trapping time is given by*

$$m_\delta^{(\kappa)}(x) = \frac{\lambda}{(\lambda+\delta)U\left(1-\frac{\lambda}{r^{(\kappa)}}, 1-\frac{\lambda+\delta}{r^{(\kappa)}}; 0\right)} e^{y^{(\kappa)}(x)} U\left(1-\frac{\lambda}{r^{(\kappa)}}, 1-\frac{\lambda+\delta}{r^{(\kappa)}}; -y^{(\kappa)}(x)\right), \tag{2.4.2}$$

*where* $\delta \geq 0$ *is the force of interest for valuation and* $y^{(\kappa)}(x) = -\alpha^{(\kappa)}(x - x^{(\kappa)*})$*.*

Figure 2.3a displays the Laplace transform $m_\delta^{(\kappa)}(x)$ for varying values of $\delta$. As mentioned earlier, as $\delta \to 0$, the Laplace transform $m_\delta^{(\kappa)}(x)$ converges to the trapping probability $\psi^{(\kappa)}(x)$.

*Remark* 2.4.1. The trapping probability of the insured process $\psi^{(\kappa)}(x)$, displayed in Figure 2.3b, is given by

$$\psi^{(\kappa)}(x) = \frac{\Gamma\left(\frac{\lambda}{r^{(\kappa)}}; -y^{(\kappa)}(x)\right)}{\Gamma\left(\frac{\lambda}{r^{(\kappa)}}\right)}.$$

As mentioned previously, increasing the value of the parameter $\alpha$ of the exponential distribution of the capital losses reduces the trapping probability since capital losses are likely to have a low impact on household capital. Moreover, note that as the proportionality factor $\kappa \to 0$, the parameter $\alpha^{(\kappa)} := \alpha/\kappa$ of the new capital losses $Y_i$ of the insured capital process increases, leading households to experience low impact losses with a higher probability ($Y_i$ will likely have values close to zero), but to pay higher premiums (2.4.1).

*Remark* 2.4.2. When $\kappa = 0$ the household has full microinsurance coverage, as the microinsurance provider covers the total capital loss experienced by the household. On the other hand, when $\kappa = 1$, no coverage is provided by the insurer, i.e. $X_t = X_t^{(\kappa)}$.

Figure 2.4 presents a comparison between the trapping probabilities of the insured and uninsured processes. As in Kovacevic and Pflug (2011), households with initial capital close to the critical capital (here, the critical capital $x^*$ is 1), i.e. the most vulnerable households, may not receive a real benefit from enrolling in a microinsurance scheme. Although subscribing to a proportional microinsurance scheme reduces capital losses, premium payments make such households more prone to falling into the area of poverty. The intersection point of the two probabilities in Figure 2.4 corresponds to the boundary between households that benefit from the uptake of microinsurance and those who are adversely affected.

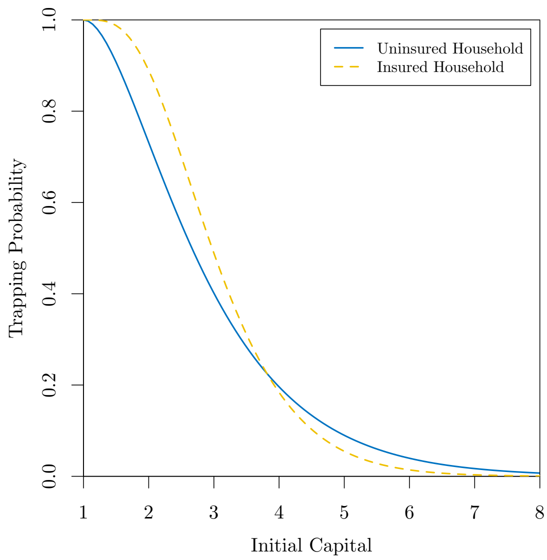



Figure 2.4: Trapping probabilities for the uninsured and insured capital processes when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $\kappa = 0.5$, $\theta = 0.5$ and $x^* = x^{(\kappa)*} = 1$.

## 2.5 Microinsurance with Subsidised Constant Premiums

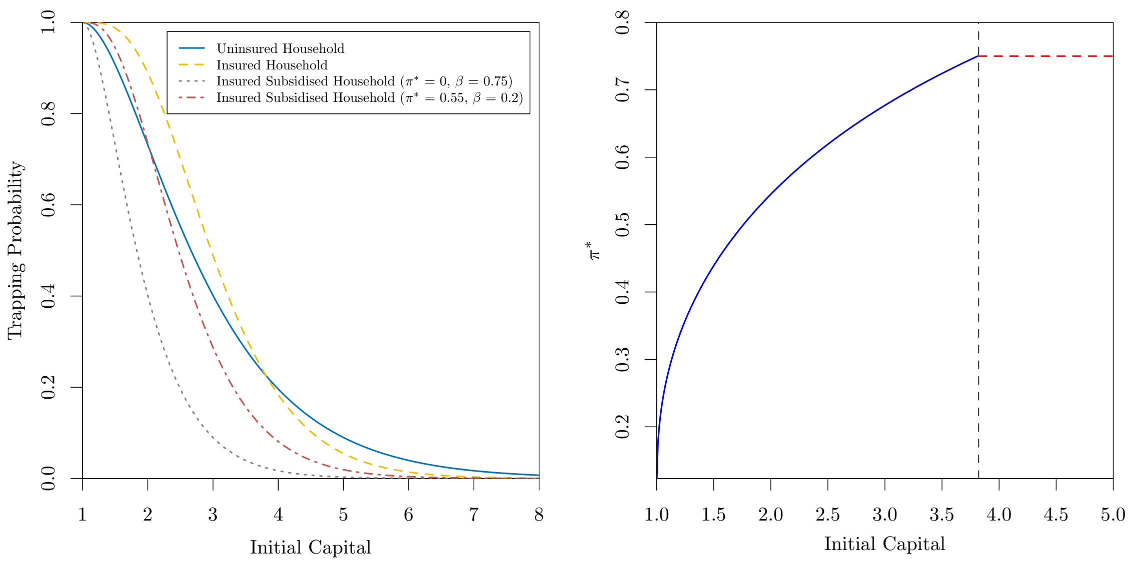
### 2.5.1 General Setting

The preliminary results suggest that microinsurance alone may not be enough to reduce the likelihood of impoverishment for those closest to the poverty line, and so additional aid is required. In this section, we study the cost-effectiveness of government subsidised premiums, considering the case in which the government subsidises an amount $\beta = \pi - \pi^*$, where $\pi^*$ is the premium after subsidisation, such that $\pi \geq \pi^* \geq 0$. As such, lower values of $\pi^*$ correspond to greater government support. When $\pi^* = 0$ the premium is fully subsidised, whereas when $\pi = \pi^*$ households do not receive any subsidies. Note that, in contrast to previous work such as that of Kovacevic and Pflug (2011), where the subsidy is limited to the loading factor, and the self-targeted subsidy strategy of Janzen et al. (2021), where fixed subsidies are provided uniformly to poor households who would anyway purchase insurance, here we extend the possibility of households benefiting from greater subsidisation in line with existing government supported microinsurance schemes, while adjusting for the governmental cost. Some examples for varying levels of subsidisation are presented in Figure 2.5a.

Naturally, we assume that governments are interested in optimising the subsidy provided to households. Governments should provide subsidies to microinsurance providers such that they enhance households' benefits of enrolling in microinsurance schemes, however, they also need to gauge the cost-effectiveness of subsidy provision. Households with capital very close to the critical capital will require additional aid, while government support is not necessarily essential for more privileged households. Since all non-zero values of $\pi^*$ below the optimal value will induce a trapping probability lower than that of the uninsured process through a reduction in premium, one approach to determining the optimal subsidy for households that require government aid is to find the solution of the equation

$$\psi^{\pi^*(\kappa,\theta)}(x) = \psi(x),$$

where $\psi^{\pi^*(\kappa,\theta)}(x)$ and $\psi(x)$ denote the trapping probabilities of the insured subsidised and uninsured processes, respectively. The behaviours of these trapping probabilities can be seen in Figure 2.5a, while the most privileged households do not need help from the government since the non-subsidised insurance lowers their trapping probability below the uninsured, the poorest individuals may require further support. Figure 2.5b illustrates the optimal value of $\pi^*$ for varying initial capital, verifying that, from the point at which the yellow dashed line (insured household) intersects the blue solid line (uninsured household) in Figure 2.5a, payment of the entire premium is affordable for the most privileged households, with the optimal premium remaining constant at $\pi^* = \pi = 0.75$ after this point (red dashed line in Figure 2.5b).




(a) (b)

Figure 2.5: (a) Trapping probabilities for the uninsured, insured and insured subsidised capital processes when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^* = x^{(\kappa)*} = x^{\pi(\kappa,\theta)*} = 1$, $\kappa = 0.5$, $\theta = 0.5$ and $\pi = 0.75$ for $\pi^* = 0, 0.55$ (b) Optimal $\pi^*$ for varying initial capital when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^{\pi(\kappa,\theta)*} = 1$, $\kappa = 0.5$, $\theta = 0.5$ and $\pi = 0.75$.

### 2.5.2 Cost of Social Protection

Next, we assess government cost-effectiveness of the provision of microinsurance premium subsidies to households. Let $\tau_x^{\pi^*(\kappa,\theta)}$ denote the trapping time of a household covered by a subsidised microinsurance policy. Moreover, let $\delta \geq 0$ be the force of interest for valuation and let $S$ denote the present value of all subsidies provided by government until the trapping time such that

$$S = \beta \int_0^{\tau_x^{\pi^*(\kappa,\theta)}} e^{-\delta t} dt = \beta \bar{a}_{\overline{\tau_x^{\pi^*(\kappa,\theta)}}|}.$$

We assume governments provide subsidies according to the strategy introduced in Section 2.5.1, i.e. the government subsidises an amount $\beta = \pi - \pi^*$.

For $x \geq x^{\pi^*(\kappa,\theta)*}$, where $x^{\pi^*(\kappa,\theta)*}$ denotes the critical capital of the insured subsidised process, let $V^{\pi^*(\kappa,\theta)}(x)$ be the expected discounted premium subsidies provided by the government to a household with initial capital $x$ until the trapping time, that is,

$$V^{\pi^*(\kappa,\theta)}(x) = \mathbb{E}\left[S \mid X_0^{\pi^*(\kappa,\theta)} = x\right].$$

**Proposition 2.5.1.** *Consider a household enrolled in a microinsurance scheme with subsidised constant premiums in which the government subsidises an amount $\beta = \pi - \pi^*$, where $\pi \geq \pi^* \geq 0$ (as discussed in Section 2.5.1), with proportionality factor $1 - \kappa \in [0, 1]$. Assume an initial capital $x \geq x^{\pi^*(\kappa,\theta)*}$, capital growth rate $r^{\pi^*(\kappa,\theta)}$, intensity $\lambda > 0$ and exponentially distributed capital losses with parameter $\alpha^{(\kappa)} > 0$. The expected discounted premium subsidies provided by the government to the household until the trapping time is given by*

$$V^{\pi^*(\kappa,\theta)}(x) = \frac{\beta}{\delta}\left[1 - m_\delta^{\pi^*(\kappa,\theta)}(x)\right], \tag{2.5.1}$$

*where $m_\delta^{\pi^*(\kappa,\theta)}(x)$ is the Laplace transform of the trapping time with rate $r^{\pi^*(\kappa,\theta)}$ and critical capital $x^{\pi^*(\kappa,\theta)*}$.*

See Appendix 2.A.3 for proof of Proposition 2.5.1. We now formally define the government's cost of social protection.

**Definition 2.5.1.** Consider (2.3.1), the expected discounted penalty function at trapping of a household enrolled in a subsidised microinsurance scheme with initial capital $x$. Let $w(x_1, x_2) = x_2 + M^{(\kappa)} - x^{\pi^*(\kappa,\theta)*}$ be the penalty function, where $M^{(\kappa)} \geq x^{\pi^*(\kappa,\theta)*}$. Here, $M^{(\kappa)} - x^{\pi^*(\kappa,\theta)*}$ is a constant representing the cost to lift households further away from the area of poverty, while $x_2$ accounts for the cost to lift households up to the critical level $x^{\pi^*(\kappa,\theta)*}$. Thus, the expected discounted penalty function at trapping $m_{\delta,w}^{\pi^*(\kappa,\theta)}(x)$ is the expected present value of the capital deficit at trapping plus a cost $M^{(\kappa)} - x^{\pi^*(\kappa,\theta)*}$ due at the trapping time. We therefore define a government's cost of social protection as the expected discounted premium subsidies, given by (2.5.1), plus the expected present value of the capital deficit and the amount $M^{(\kappa)} - x^{\pi^*(\kappa,\theta)*}$.

*Remark* 2.5.1. The government does not provide subsidies for uninsured households. We consider their expected discounted penalty function at trapping to be $m_{\delta,w}(x)$ with $w(x_1, x_2) = x_2 + M - x^*$. The choice of this particular penalty function is based on the idea that the government, in order to lift a household out of poverty, incurs a cost equal to the household's capital deficit at the moment they fall into poverty plus a fixed cost $M - x^*$ that ensures, with a certain level of confidence, that the household will not return to poverty. This differs from other approaches taken in previous research in which the cost of social protection considers only the present value of the transfers needed to close the poverty gap (see, for example, Ikegami et al. (2018) and Janzen et al. (2021)). In this way, the likelihood of re-incurring these costs for the same household is reduced. Thus, the constant $M$ could be defined in such a way that the government ensures with some probability that households will not fall into the area of poverty again. For instance, let us consider $\epsilon$ to be the most admissible trapping probability for an uninsured household. We can therefore define the statistic

$$M := \inf \{x \geq x^* : \psi(x) < \epsilon\}, \tag{2.5.2}$$

where $M$ is the Minimum Initial Capital (MIC) required to ensure a trapping probability of less than $\epsilon$. This statistic has also been studied from the point of view of an insurance company, where $\epsilon$ represents the most admissible probability that the insurance company will become insolvent (Sattayatham et al., 2013; Constantinescu et al., 2019). As such, the government determines an appropriate amount $M$ such that a household's probability of re-entering the area of poverty is less than $\epsilon \in (0, 1)$. Clearly, higher values of $M$ will increase the certainty that households will not return to poverty. Also note that the value of $M$ will differ between uninsured, insured and insured subsidised households due to the fact that their trapping probabilities are distinct. However, in this study we assume that governments will consider an amount $M^{(\kappa)}$ under all microinsurance schemes (with or without subsidies). That is, we assume that households who are initially enrolled in a microinsurance scheme (with or without subsidies) will be enrolled in a scheme without subsidies just after being lifted away from the area of poverty. Figure 2.6 displays the cost incurred at the trapping time when employing the penalty function $w(x_1, x_2) = x_2 + M - x^*$ for an uninsured household.

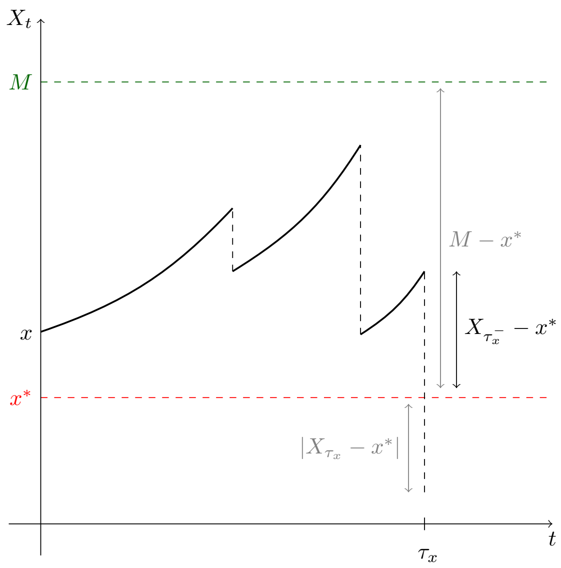



Figure 2.6: The cost incurred by the government at the trapping time is given by $| X_{\tau_x} - x^* |$, the capital deficit at the trapping time, plus $M - x^*$, the cost to lift households further away from the area of poverty.

*Remark* 2.5.2. The government manages the selection of an appropriate force of interest $\delta \geq 0$. For lower force of interest the government discounts future subsidies more heavily, while for higher interest future subsidies almost vanish.

*Remark* 2.5.3. When losses are exponentially distributed with parameter $\alpha^{(\kappa)} > 0$, one can obtain a closed form expression for the cost of social protection. Given our derivation of $V^{\pi^*(\kappa,\theta)}(x)$ in (2.5.1), we determine an expression for $m_{\delta,w}^{\pi^*(\kappa,\theta)}(x)$.

**Proposition 2.5.2.** *Consider a household enrolled in a microinsurance scheme with subsidised constant premiums in which the government subsidises an amount* $\beta = \pi - \pi^*$*, where* $\pi \geq \pi^* \geq 0$*, with proportionality factor* $1-\kappa \in [0,1]$*. Assume an initial capital* $x \geq x^{\pi^*(\kappa,\theta)*}$*, capital growth rate* $r^{\pi^*(\kappa,\theta)}$*, intensity* $\lambda > 0$ *and exponentially distributed capital losses with parameter* $\alpha^{(\kappa)} > 0$*. Furthermore, let* $M^{(\kappa)} - x^{\pi^*(\kappa,\theta)*}$*, with* $M^{(\kappa)} \geq x^{\pi^*(\kappa,\theta)*}$*, be the cost to lift households further away from the area of poverty. The expected discounted cost incurred by the government at the trapping time is given by*

$$m_{\delta,w}^{\pi^*(\kappa,\theta)}(x) = \left[\frac{1}{\alpha^{(\kappa)}} + M^{(\kappa)} - x^{\pi^*(\kappa,\theta)*}\right] m_{\delta}^{\pi^*(\kappa,\theta)}(x), \tag{2.5.3}$$

*where* $m_{\delta}^{\pi^*(\kappa,\theta)}(x)$ *is the Laplace transform of the trapping time with rate* $r^{\pi^*(\kappa,\theta)}$ *and critical capital* $x^{\pi^*(\kappa,\theta)*}$*, and* $\delta \geq 0$ *is the force of interest for valuation.*

See Appendix 2.A.4 for proof of Proposition 2.5.2.

*Remark* 2.5.4. Due to the lack-of-memory property of the exponential distribution the deficit at trapping, given that trapping occurs, is exponentially distributed. One can easily verify this by specifying the penalty function such that for any fixed $u$, $w(x_1, x_2) = \mathbb{1}_{\{x_2 \leq u\}}$ and $\delta = 0$. Similar results to that of Proposition 2.5.2 have been obtained for other risk processes (see, for instance, Example 3.2 of Albrecher et al. (2005)).

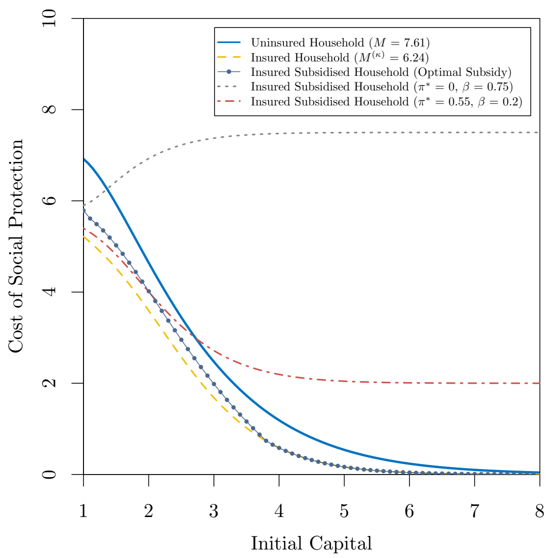



Figure 2.7: Cost of social protection for the uninsured, insured and insured subsidised with $\pi^* = \pi^*_{\texttt{Optimal}}, 0, 0.55$ capital processes when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^* = x^{(\kappa)*} = x^{\pi(\kappa,\theta)*} = 1$, $\kappa = 0.5$, $\theta = 0.5$, $\delta = 0.1$, $\epsilon = 0.01$ and $\pi = 0.75$.

Figure 2.7 displays the governmental cost of social protection for varying initial capital. Here, the value of $M$ is given by (2.5.2). Note that, as mentioned previously, high values of $\delta$ hand a lower weight to future government subsidies, whereas high values of $M$ grant greater certainty that a household will not return to the area of poverty once lifted out of it.

Governments do not benefit from subsidising insurance for the most privileged households since they will subsidise premiums indefinitely, almost surely. Hence, as also illustrated in Figure 2.5b, it is favourable for governments to remove subsidies for this particular household group since their cost of social protection in Figure 2.7 (red dashed-dotted and gray dotted lines for highest values of initial capital) is higher than when uninsured (blue solid line for highest values of initial capital). This is largely due to the fact that governments are still obliged to subsidise a given amount of the premium even though greater initial capital leads to lower trapping probabilities and therefore a reduction in the likelihood of the government needing to lift these households away from the area of poverty.

In addition, Figure 2.7 shows that when providing optimal subsidies, governments can reduce the cost of social protection incurred. Here, the fully subsidised scheme (when $\pi^* = 0$) has a higher cost for all households relative to the scheme that provides optimal subsidies (blue circular-marked line below the gray dotted line), and the difference between the two increases as the initial capital increases, until the moment at which the cost of social protection for the fully subsidised scheme and the scheme that provides optimal subsidies converge to $\beta/\delta$ and zero, respectively. Similarly, for more privileged households, a subsidised scheme (with $\pi^* = 0.55$) has a higher cost relative to the optimal case (blue circular-marked line below the red dashed-dotted line for more privileged households). On the contrary, for the most vulnerable, Figure 2.7 shows the cost of social protection of the scheme that provides optimal subsidies is above the subsidised scheme (blue circular-marked line above the red dashed-dotted line for most vulnerable households) as optimal subsidies for this group provide greater support (i.e. the optimal values for $\pi^*$ are lower). Note that the cost of social protection for the insured (yellow dashed line), non-optimal insured subsidised (gray dotted and red dashed-dotted lines) for the most vulnerable (for those with initial capital lower than $x = 1.362$ when $\pi^* = 0$ and $x = 2.719$ when $\pi^* = 0.55$) and optimal insured subsidised households (blue circular-marked line) is below that of the uninsured, thus highlighting the significance of insurance as a tool for reducing the governmental cost of social protection. Moreover, it is not surprising that the cost of social protection for the insured (yellow dashed line), is lower than the cost of social protection for the rest of the microinsurance schemes, since governments will not provide subsidies, but only an injection of capital in the event of entering poverty.

## 2.6 Microinsurance with Subsidised Flexible Premiums

### 2.6.1 General Setting

Since microinsurance premiums are generally paid as soon as coverage is purchased, a household's capital growth could be constrained after joining a scheme, as observed in the results of Sections 2.4 and 2.5. It is therefore interesting to consider alternative premium payment mechanisms. From the point of view of microinsurance providers, advance premium payments are preferred so that additional income can be generated through investment, naturally leading to lower premium rates. Consumers on the other hand may find it difficult to pay premiums up front. This is a common problem in low-income populations, with research suggesting that consumers prefer to pay smaller installments over time (Churchill and Matul, 2012). Collecting premiums at a time that is inconvenient for households can be futile. Alternative insurance designs in which premium payments are delayed until the insured's income is realised and any indemnities are paid have also been studied. Under such designs, insurance take-up increases, since liquidity constraints are relaxed and concerns regarding insurer default, also prevalent in low-income classes, reduce (Liu and Myers, 2016).

In this section, we introduce an alternative microinsurance subsidy scheme with flexible premium payments. We denote the capital process of a household enrolled in the alternative microinsurance subsidy scheme by $X_t^{(\mathcal{A})}$. As in Section 2.4, we differentiate between variables and parameters relating to the uninsured, insured and alternative insured processes using the superscript $(\mathcal{A})$. Under such an alternative microinsurance subsidy scheme households pay premiums when their capital is above some capital barrier $B \geq x^{(\mathcal{A})*}$, with the premium otherwise paid by the government. In other words, whenever the insured capital process is below the capital level $B$ premiums are entirely subsidised by the government, however, when a household's capital is above $B$ the premium $\pi$ is paid continuously by the household itself. This method of premium collection may motivate households to maintain a level of capital below $B$ in order to avoid premium payments. Consequently, for the purpose of this study, we assume that households always pursue capital growth. Our aim is to study how this alternative microinsurance subsidy scheme could help households reduce their probability of falling into the area of poverty. In addition, we measure the cost-effectiveness of such a scheme from the point of view of the government.

The intangibility of microinsurance makes it difficult to attract potential consumers. Most policyholders will never experience a claim and so cannot perceive the real value of microinsurance, paying more to the scheme (in terms of premium payments) than what they actually receive from it. It is only when claims are settled that microinsurance becomes tangible. The alternative microinsurance subsidy scheme described here could increase client value, since, for example, individuals below the barrier $B$ may submit claims, receive a payout and therefore perceive the value of microinsurance when they suffer a loss, regardless of whether they have ever paid a single premium. Further ways of increasing microinsurance client value include bundling microinsurance with other products and introducing Value Added Services (VAS), which (for health schemes) are services such as telephone hotlines for consultation with doctors or remote diagnosis services, offered to clients outside of the microinsurance contract (Madhur and Saha, 2019).

**Proposition 2.6.1.** *Consider a household enrolled in an alternative microinsurance scheme with subsidised flexible premiums, capital barrier $B \geq x^{(\mathcal{A})*}$ and proportionality factor $1-\kappa \in [0,1]$. Assume an initial capital $x \geq x^{(\mathcal{A})*}$, capital growth rates $r^{(\kappa)}$ and $r$ above and below the barrier, respectively, intensity $\lambda > 0$ and exponentially distributed capital losses with parameter $\alpha^{(\kappa)} > 0$. The Laplace transform of the trapping time is given by*

$$m_\delta^{(\mathcal{A})}(x) = \begin{cases} C_1 M\left(-\frac{\delta}{r}, 1-\frac{\lambda+\delta}{r}; y^{(\mathcal{A})}(x)\right) + C_2 e^{y^{(\mathcal{A})}(x)} U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda+\delta}{r}; -y^{(\mathcal{A})}(x)\right) & \text{for } x^{(\mathcal{A})*} \leq x \leq B, \\ C_3 M\left(-\frac{\delta}{r^{(\kappa)}}, 1-\frac{\lambda+\delta}{r^{(\kappa)}}; y^{(\mathcal{A})}(x)\right) + C_4 e^{y^{(\mathcal{A})}(x)} U\left(1-\frac{\lambda}{r^{(\kappa)}}, 1-\frac{\lambda+\delta}{r^{(\kappa)}}; -y^{(\mathcal{A})}(x)\right) & \text{for } x \geq B, \end{cases} \tag{2.6.1}$$

*where $y^{(\mathcal{A})}(x) = -\alpha^{(\kappa)}(x - x^{(\mathcal{A})*})$ and the constants $C_i$ for $i = 1,2,3,4$ are given by (2.A.13), (2.A.15), (2.A.12) and (2.A.14), respectively.*

A detailed mathematical proof of Proposition 2.6.1 is provided in Appendix 2.A.5.

*Remark* 2.6.1. The trapping probability $\psi^{(\mathcal{A})}(x)$ for the alternative microinsurance subsidy scheme is given by

$$\psi^{(\mathcal{A})}(x) = \begin{cases} 1 - \dfrac{\Gamma\left(\frac{\lambda}{r}\right) - \Gamma\left(\frac{\lambda}{r}; -y^{(\mathcal{A})}(x)\right)}{\left(-y^{(\mathcal{A})}(B)\right)^{\lambda\left(\frac{1}{r}-\frac{1}{r^{(\kappa)}}\right)} \Gamma\left(\frac{\lambda}{r^{(\kappa)}}; -y^{(\mathcal{A})}(B)\right) + \Gamma\left(\frac{\lambda}{r}\right) - \Gamma\left(\frac{\lambda}{r}; -y^{(\mathcal{A})}(B)\right)} & \text{for } x^{(\mathcal{A})*} \leq x \leq B, \\ \dfrac{\left(-y^{(\mathcal{A})}(B)\right)^{\lambda\left(\frac{1}{r}-\frac{1}{r^{(\kappa)}}\right)} \Gamma\left(\frac{\lambda}{r^{(\kappa)}}; -y^{(\mathcal{A})}(x)\right)}{\left(-y^{(\mathcal{A})}(B)\right)^{\lambda\left(\frac{1}{r}-\frac{1}{r^{(\kappa)}}\right)} \Gamma\left(\frac{\lambda}{r^{(\kappa)}}; -y^{(\mathcal{A})}(B)\right) + \Gamma\left(\frac{\lambda}{r}\right) - \Gamma\left(\frac{\lambda}{r}; -y^{(\mathcal{A})}(B)\right)} & \text{for } x \geq B. \end{cases}$$

Similar to the subsidised case, the optimal barrier $B$ can be found by determining the solution of the equation

$$\psi^{(\mathcal{A})}(x) = \psi(x),$$

where $\psi^{(\mathcal{A})}(x)$ and $\psi(x)$ denote the trapping probabilities of the capital process under the alternative microinsurance subsidy scheme and the uninsured capital process, respectively. Some examples for varying initial capital are presented at the end of this section.

*Remark* 2.6.2. When $B \to x^{(\mathcal{A})*}$, the trapping probability for the alternative microinsurance subsidy scheme is equal to the trapping probability obtained for the insured case $\psi^{(\kappa)}(x)$:

$$\lim_{B \to x^{(\mathcal{A})*}} \psi^{(\mathcal{A})}(x) = \frac{\Gamma\left(\frac{\lambda}{r^{(\kappa)}}; -y^{(\kappa)}(x)\right)}{\Gamma\left(\frac{\lambda}{r^{(\kappa)}}\right)}.$$

Moreover, when $B \to \infty$, the trapping probability is given by

$$\lim_{B \to \infty} \psi^{(\mathcal{A})}(x) = \frac{\Gamma\left(\frac{\lambda}{r}; -y^{(\kappa)}(x)\right)}{\Gamma\left(\frac{\lambda}{r}\right)},$$

which is exactly the trapping probability of the insured subsidised process $\psi^{\pi^*(\kappa,\theta)}(x)$ with $\pi^* = 0$.

*Remark* 2.6.3. Figure 2.8 displays the expected trapping time under the alternative microinsurance subsidy scheme for varying initial capital. Again, in line with intuition, the expected trapping time is an increasing function of both the capital level $B$ and initial capital $x$. Steps for obtaining the expected trapping time under the alternative microinsurance subsidy scheme are very similar to those used to derive Equation (2.3.5).

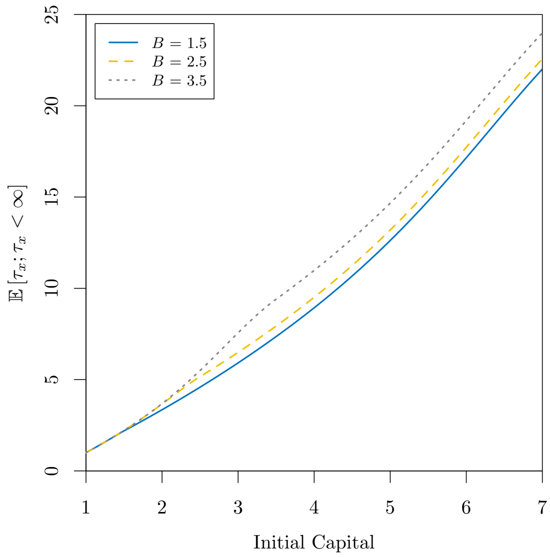
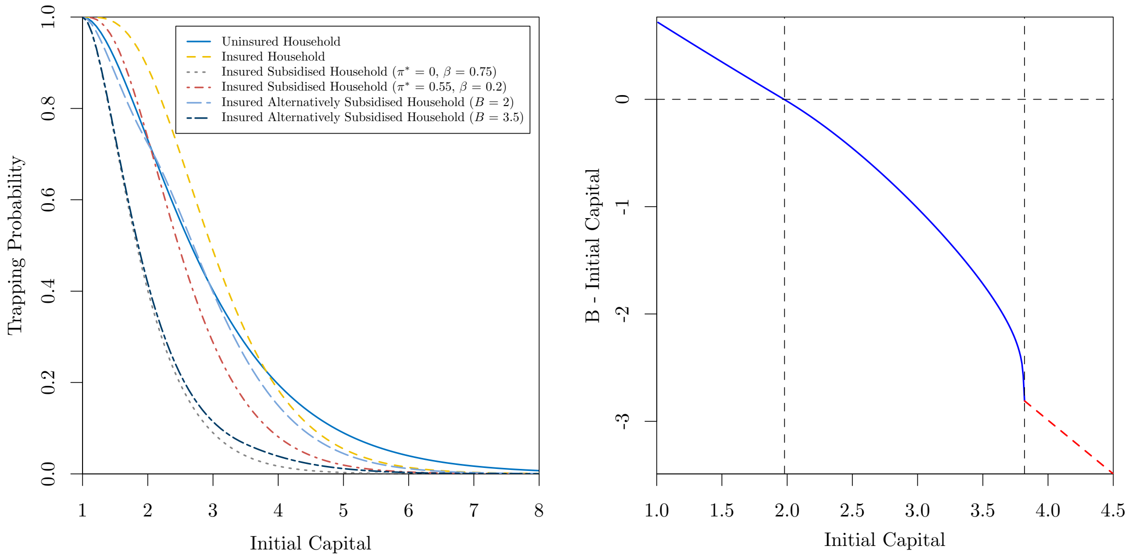



Figure 2.8: Expected trapping time when $Z_i \sim Exp(1)$, $a = 0.8$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^{(\mathcal{A})*} = 1$, $\kappa = 0.5$, and $\theta = 0.5$ for $B = 1.5, 2.5, 3.5$.

Figure 2.9a presents the trapping probabilities for varying initial capital under the uninsured, insured, insured subsidised and insured alternatively subsidised schemes. As expected, increasing the value of the capital barrier $B$ helps households to reduce their probability of falling into the area of poverty, since support from the government is received when their capital resides in the region between the critical capital $x^{(\mathcal{A})*}$ and the capital level $B$. Furthermore, as in the previous section, households with higher levels of initial capital do not need government support, insurance without subsidies decreases their trapping probability to a level below the uninsured (households with initial capital greater than or equal to the point at which the yellow short-dashed line intersects the blue solid line). The optimal barrier for these households is in fact the critical capital, i.e. $B = x^{(\mathcal{A})*}$, this household group can therefore afford to cover the costs of microinsurance coverage themselves.

Figure 2.9b shows that for the most vulnerable, governments should set the barrier level $B$ above their initial capital to remove capital growth constraints associated with premium payments. This level should be selected until the household reaches a capital level that is adequate in ensuring their trapping probability is equal to that of an uninsured household. Conversely, for more privileged households (middle area of Figure 2.9b), the government should establish barriers below their initial capital, with households paying premiums as soon as they enrol in the microinsurance scheme. This behaviour of the optimal barrier is mainly due to the fact that the capital level of such households is distant from the critical capital $x^{(\mathcal{A})*}$. These households are unlikely to fall into the area of poverty after suffering one (non-catastrophe) capital loss, they are instead likely to fall into the region between the critical capital $x^{(\mathcal{A})*}$ and the barrier level $B$ (the area within which the government pays microinsurance premiums), before entering the area of poverty. Thus, the aforementioned region acts as a "buffer", with households in this region benefiting from coverage without the need for premium payments. Increasing initial capital leads to a decrease in the size of the "buffer" region until it disappears when the optimal barrier $B = x^{(\mathcal{A})*}$, as shown by the red dashed line in the right area of Figure 2.9b.




(a) (b)

Figure 2.9: (a) Trapping probabilities for the uninsured, insured, insured subsidised with $\pi^* = 0, 0.55$ and insured alternatively subsidised with $B = 2, 3.5$ capital processes when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^* = x^{(\kappa)*} = x^{\pi(\kappa,\theta)*} = x^{(\mathcal{A})*} = 1$, $\kappa = 0.5$, $\theta = 0.5$ and $\pi = 0.75$ (b) Difference between the optimal barrier and the initial capital, i.e. $B - x$, for varying initial capital, when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^{(\mathcal{A})*} = 1$, $\kappa = 0.5$, $\theta = 0.5$ and $\pi = 0.75$.

### 2.6.2 Cost of Social Protection

Similar to the previous section, it is reasonable to measure the governmental cost-effectiveness of providing microinsurance premium subsidies to households under the alternative microinsurance subsidy scheme. For this reason, we define $\tau_x^{(\mathcal{A})}$ as the trapping time of a household covered by the alternative subsidised microinsurance scheme and $V^{(\mathcal{A})}(x)$ as the expectation of the present value of all subsidies provided by the government to the household until the trapping time, that is

$$V^{(\mathcal{A})}(x) := \mathbb{E}\left[\int_0^{\tau_x^{(\mathcal{A})}} \pi e^{-\delta t} \mathbb{1}_{\left\{X_t^{(\mathcal{A})} < B\right\}} dt \middle| X_0^{(\mathcal{A})} = x\right]. \tag{2.6.2}$$

In this chapter, estimates for $V^{(\mathcal{A})}(x)$ are produced via Monte Carlo simulation. See Appendix 2.B.1 for an efficient algorithm that produces Monte Carlo estimates of $V^{(\mathcal{A})}(x)$.

*Remark* 2.6.4. As in Section 2.5.2, we can easily derive an expression for $m^{(\mathcal{A})}_{\delta,w}(x)$, the expected discounted cost incurred by the government at the trapping time under the alternative microinsurance scheme.

Following a similar procedure to that in the proof of Proposition 2.5.2 (details of which are shown in Appendix 2.A), but for the alternative microinsurance subsidy scheme, one obtains the expected discounted cost incurred by the government at the trapping time.

**Proposition 2.6.2.** *Consider a household enrolled in an alternative microinsurance scheme with subsidised flexible premiums, capital barrier* $B \geq x^{(\mathcal{A})*}$ *and proportionality factor* $1-\kappa \in [0,1]$. *Assume an initial capital* $x \geq x^{(\mathcal{A})*}$, *capital growth rates* $r^{(\kappa)}$ *and* $r$ *above and below the barrier, respectively, intensity* $\lambda > 0$, *exponentially distributed capital losses with parameter* $\alpha^{(\kappa)} > 0$ *and a cost to lift households further away from the area of poverty* $M^{(\kappa)} - x^{(\mathcal{A})*}$, *with* $M^{(\kappa)} \geq x^{(\mathcal{A})*}$. *The expected discounted cost incurred by the government at the trapping time is*

$$m^{(\mathcal{A})}_{\delta,w}(x) = \left[\frac{1}{\alpha^{(\kappa)}} + M^{(\kappa)} - x^{(\mathcal{A})*}\right] m^{(\mathcal{A})}_{\delta}(x), \tag{2.6.3}$$

*where* $m^{(\mathcal{A})}_{\delta}(x)$ *is given by* (2.6.1).

As for the subsidised scheme, under the alternative scheme, we consider the cost of social protection incurred by the government to be equal to the expected discounted subsidies provided until trapping plus the expected discounted cost incurred at trapping, here given by (2.6.2) and (2.6.3), respectively.

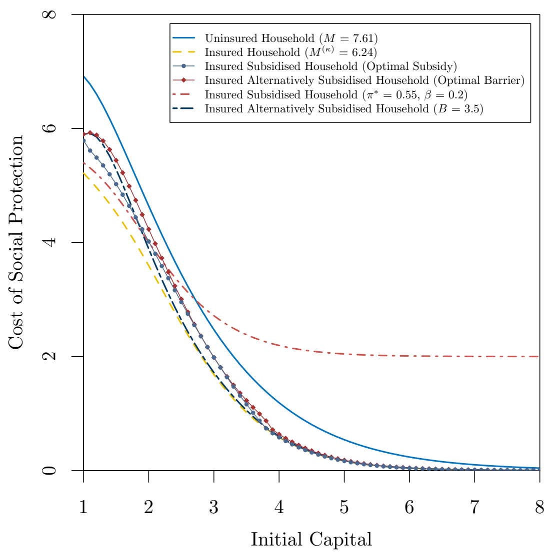



Figure 2.10: Cost of social protection for the uninsured, insured, insured subsidised with $\pi^* = \pi^*_{\texttt{Optimal}}, 0.55$ and insured alternatively subsidised capital processes with $B = B_{\texttt{Optimal}}, 3.5$ when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^* = x^{(\kappa)*} = x^{\pi(\kappa,\theta)*} = x^{(\mathcal{A})*} = 1$, $\kappa = 0.5$, $\theta = 0.5$, $\delta = 0.1$, $\epsilon = 0.01$ and $\pi = 0.75$.

Figure 2.10 compares the cost of social protection for the uninsured, insured, insured subsidised and insured alternatively subsidised households. Cost of social protection for the most vulnerable is reduced with all forms of microinsurance coverage (yellow dashed, red dashed-dotted, blue dashed-dashed, blue circular-marked and red diamond-marked lines are all below the blue solid line for initial capitals close to the critical capital $x^*$). This aligns with the high trapping probability associated with this portion of the population when uninsured, with governments almost surely needing to lift households out of the area of poverty. Although already eliminated when providing optimal subsidies under the insured subsidised scheme (blue circular-marked line below the red dashed-dotted line for the most privileged), the aforementioned drawback of governments subsidising premiums indefinitely almost surely under non-optimal subsidised schemes is also eliminated under both optimal and non-optimal alternative subsidy schemes due to the ceasing of subsidies on households reaching sufficient capital (red diamond-marked and blue dashed-dashed lines below red dashed-dotted line for households with higher levels of capital). Furthermore, as seen in Figure 2.9a, when the barrier level is sufficiently high all households observe a decrease in their trapping probability, almost reaching the trapping probability of a household enrolled in a fully subsidised insurance scheme with $\pi^* = 0$. However, even when high barrier levels are considered, under the alternatively subsidised scheme governments are not required to subsidise premiums indefinitely, since households will pay the entire premium once their capital reaches a sufficient level. This scheme thus makes it possible to reduce the trapping probability for any household, while reducing the cost of social protection incurred by the government, which highlights the cost-efficiency of this alternative scheme.

## 2.7 Conclusion

Comparing the impact of three microinsurance frameworks on the trapping probabilities of low-income households, we provide evidence for the importance of governmentally supported inclusive insurance in the strive towards poverty alleviation. The results of Sections 2.4 and 2.5 support those of Kovacevic and Pflug (2011), highlighting a threshold below which insurance could increase the probability of trapping. Motivated by the recent increased involvement of governments in the support of insurance programmes and maintaining the idea of "smart" subsidies, we have introduced a transparent method with a mathematical foundation for calculating "optimal subsidies" that can strengthen government social protection programmes while lowering the associated costs.

Numerical examples indicate that while the proposed insurance mechanisms (with or without subsidies) reduce the cost of social protection for the most vulnerable, they do not reduce their probability of trapping. This undermines the faculty of inclusive insurance as a cost-effective social protection strategy for poverty alleviation and brings to light questions as to its capability in reducing both the probability of households falling below the poverty line and the associated social protection costs. However, our analysis of a subsidised microinsurance scheme with a premium payment barrier suggests that in general, the trapping probability of a household covered by such a scheme is reduced in comparison to when covered by unsubsidised and (for the most vulnerable) partially subsidised microinsurance schemes, in addition to when uninsured, alleviating this limitation.

More significant influence is observed in relation to the governmental cost of social protection, with the cease of subsidy payments when household capital is sufficient facilitating government savings and therefore increasing social protection efficiency, thus evidencing the relevance of the alternative scheme as a cost-effective social protection strategy for poverty reduction. The cost of social protection for those closest to the area of poverty remains lower than the corresponding uninsured cost in both subsidised frameworks considered, achieving similar results to those obtained with the targeted-subsidisy scheme proposed by Janzen et al. (2021). In our analysis, for such households, governments must account for their support of premium payments, the likely need for household removal from poverty and an extra capital injection to ensure they will not return to poverty with some level of confidence. Nevertheless, total subsidies paid by the government have a small weight within the cost of social protection due to the fact that those closest to the poverty line will fall into the area of poverty almost surely. The capital injection on trapping is also much lower in comparison to that of uninsured households. Each of these factors enhances the reduction in the cost of social protection for the most vulnerable.

Given the decrease in trapping probability and governmental cost of social protection under the barrier strategy scheme considered, this chapter advocates for the development of public-private partnerships (PPPs) for the provision of affordable insurance. Through well-designed subsidy schemes PPPs can reduce vulnerability to poverty in a cost-effective manner. A key motivator for supporting the development of insurance mechanisms is their ability to improve productivity and access to resources. Insurance can improve access to new technologies, credit and hospital services, for example, smoothing the movement of low-income individuals along the economic cycle and providing them with the capacity to move out of poverty and to stay there. Insurance is therefore both productive and protective in preventing poverty. By making insurance more accessible, subsidies improve access to productive resources while providing protective cover.

In presenting the results of the analysis it is important to note the limitations of the adopted approach. All types of insurance are captured in our "insurance" coverage, we therefore do not consider the susceptibility of households to losses of varying severity which would align with the presence of different lines of business, i.e. health, life and agricultural insurance. In addition, the subsidy schemes are assumed to be continuous, with households receiving government subsidies forever in the constant case and while below the barrier in the flexible case. This raises questions in regard to the sustainability of such schemes. An alternative approach would be to phase out subsidisation over time, enabling households to experience and therefore understand the benefit of insurance such that they go on to purchase coverage once no longer subsidised. The barrier strategy would be difficult to implement in practice, requiring continuous tracking of a household's capital over time. Future research will involve adjusting the assumption of random-valued losses to consider random-proportional losses, as in Kovacevic and Pflug (2011). Through the assumption of random-valued losses, a household's level of accumulated capital could fall below zero. In this case, the household would lose more than what it has and would likely continue to lose capital even after surpassing the capital level of zero. Under the random-proportional losses assumption, the concept of trapping is better captured, as once within the area of poverty it is impossible to escape from either side. Moreover, this assumption fits the idea that the amount of capital lost by a household on experiencing an adverse event should depend on the amount of capital that the household currently possesses, e.g. when a household has little capital, then we would expect them to have less to lose.

The main takeaway from this analysis is the importance of the "missing middle", i.e. those close to but above the poverty line, for whom paying the premium is a risk in itself. This takeaway is also applicable in traditional insurance markets. In comparison to large international enterprises, small and medium-sized enterprises (SMEs) with limited liquidity are more likely to require government support in the event of a severe loss. For example, in the pandemic context, the COVID-19 pandemic saw governments stepping in to cover the loss of jobs, wages and, in some cases, hospitalisation costs. Without stakeholder capital injections or such governmental support, SMEs would be extremely susceptible to insolvency. In addition, reinsurance premiums for such an extreme loss would be more severe for smaller enterprises, while larger enterprises are likely to be better posed to protect themselves against such severe risks.

The "missing middle" exists in all fields of insurance, those closest to insolvency have less capacity to protect themselves against the occurrence of a catastrophic loss. Smart solutions supported by large organisations, including governments and intergovernmental organisations, should therefore be designed to mitigate the increased risk faced by the most vulnerable.

# 2.A Appendix A: Mathematical Proofs

## 2.A.1 Proof of Proposition 2.3.1

Using standard arguments based on the infinitesimal generator, the expected discounted penalty function at the trapping time $m_\delta(x)$ as defined in (2.3.1), can be characterised as the solution of the IDE

$$r(x-x^*)m_\delta'(x) - (\lambda+\delta)m_\delta(x) + \lambda\int_0^{x-x^*} m_\delta(x-z)dG_Z(z) = -\lambda A(x), \quad x \geq x^*, \tag{2.A.1}$$

where

$$A(x) := \int_{x-x^*}^{\infty} w(x-x^*, z-(x-x^*))dG_Z(z).$$

When $Z_i \sim Exp(\alpha)$ and $w(x_1,x_2)=1$, Equation (2.A.1) can be written such that

$$r(x-x^*)m_\delta'(x) - (\lambda+\delta)m_\delta(x) + \lambda\int_0^{x-x^*} m_\delta(x-z)\alpha e^{-\alpha z}dz = -\lambda e^{-\alpha(x-x^*)}, \quad x \geq x^*. \tag{2.A.2}$$

Applying the operator $\left(\frac{d}{dx}+\alpha\right)$ to both sides of (2.A.2), together with a number of algebraic manipulations, yields the second order homogeneous differential equation

$$-\frac{(x-x^*)}{\alpha}m_\delta''(x) + \left[\frac{(\lambda+\delta-r)}{\alpha r} - (x-x^*)\right]m_\delta'(x) + \frac{\delta}{r}m_\delta(x) = 0, \qquad x \geq x^*. \tag{2.A.3}$$

Letting $f(y) := m_\delta(x)$, such that $y$ is associated with the change of variable $y := y(x) = -\alpha(x-x^*)$, (2.A.3) reduces to Kummer's Confluent Hypergeometric Equation (Slater, 1960)

$$y\cdot f''(y) + (c-y)f'(y) - af(y) = 0, \qquad y<0, \tag{2.A.4}$$

for $a=-\delta/r$ and $c = 1-(\lambda+\delta)/r$, with regular singular point at $y=0$ and irregular singular point at $y=-\infty$ (corresponding to $x=x^*$ and $x=\infty$, respectively). A general solution of (2.A.4) is given by

$$m_\delta(x) = f(y) = \begin{cases} 1 & x<x^*, \\ A_1 M\left(-\frac{\delta}{r}, 1-\frac{\lambda+\delta}{r}; y(x)\right) + A_2 e^{y(x)} U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda+\delta}{r}; -y(x)\right) & x \geq x^*, \end{cases} \tag{2.A.5}$$

for arbitrary constants $A_1, A_2 \in \mathbb{R}$. Here,

$$M(a,c;z) = {}_1F_1(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\frac{z^n}{n!}$$

is Kummer's Confluent Hypergeometric Function (Kummer, 1837) and $(a)_n = \Gamma(a+n)/\Gamma(n)$ denotes the Pochhammer symbol (Seaborn, 1991). In a similar manner,

$$U(a,c;z) = \begin{cases} \frac{\Gamma(1-c)}{\Gamma(1+a-c)} M(a,c;z) + \frac{\Gamma(c-1)}{\Gamma(a)} z^{1-c} M(1+a-c, 2-c; z) & c \notin \mathbb{Z}, \\ \lim\limits_{\theta \to c} U(a,\theta;z) & c \in \mathbb{Z}, \end{cases} \tag{2.A.6}$$

is Tricomi's Confluent Hypergeometric Function (Tricomi, 1947). This function is generally complex-valued when its argument $z$ is negative, i.e. when $x \geq x^*$ in the case of interest. We seek a real-valued solution of $m_\delta(x)$ over the entire domain, therefore an alternative independent pair of solutions, here, $M(a,c;z)$ and $e^z U(c-a, c; -z)$, to (2.A.4) are chosen for $x \geq x^*$.

To determine the constants $A_1$ and $A_2$ we consider the boundary conditions for $m_\delta(x)$ at $x^*$ and at infinity. Applying Equation (13.1.27) of Abramowitz and Stegun (1972), also known as Kummer's Transformation $M(a,c;z) = e^z M(c-a, c; -z)$, we write (2.A.5) such that

$$m_\delta(x) = e^{y(x)} \left[ A_1 M\left(1 - \frac{\lambda}{r}, 1 - \frac{\lambda+\delta}{r}; -y(x)\right) + A_2 U\left(1 - \frac{\lambda}{r}, 1 - \frac{\lambda+\delta}{r}; -y(x)\right)\right], \tag{2.A.7}$$

for $x \geq x^*$. For $z \to \infty$, it is well-known that

$$M(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)} e^z z^{a-c} \left[1 + O\left(|z|^{-1}\right)\right]$$

and

$$U(a,c;z) = z^{-a} \left[1 + O\left(|z|^{-1}\right)\right]$$

(see for example, Equations (13.1.4) and (13.1.8) of Abramowitz and Stegun (1972)). Asymptotic behaviours of the first and second terms of (2.A.7) as $y(x) \to -\infty$ are therefore given by

$$\frac{\Gamma\left(1 - \frac{\lambda+\delta}{r}\right)}{\Gamma\left(1 - \frac{\lambda}{r}\right)} (-y(x))^{\frac{\delta}{r}} \left(1 + O\left(|-y(x)|^{-1}\right)\right) \tag{2.A.8}$$

and

$$e^{y(x)} (-y(x))^{\frac{\lambda}{r}-1} \left(1 + O\left(|-y(x)|^{-1}\right)\right), \tag{2.A.9}$$

respectively. For $x \to \infty$, (2.A.8) is unbounded, while (2.A.9) tends to zero. The boundary condition $\lim_{x\to\infty} m_\delta(x) = 0$, by definition of $m_\delta(x)$ in (2.3.1), thus implies that $A_1 = 0$. Letting $x = x^*$ in (2.A.2) and (2.A.5) yields

$$\frac{\lambda}{(\lambda+\delta)} = A_2 U\left(1 - \frac{\lambda}{r}, 1 - \frac{\lambda+\delta}{r}; 0\right).$$

Hence, $A_2 = \lambda / \left[(\lambda+\delta) U\left(1 - \lambda/r, 1 - (\lambda+\delta)/r; 0\right)\right]$ and the Laplace transform of the trapping time for $x \geq x^*$ is given by (2.3.2).

### 2.A.2 Proof of Corollary 2.3.1

We differentiate Tricomi's Confluent Hypergeometric Function (2.A.6) with respect to its second parameter. Denote

$$U^{(c)}(a,c;z) \equiv \frac{d}{dc}U(a,c;z).$$

A closed form expression of the aforementioned derivative can be given in terms of series expansions, such that

$$\begin{aligned} U^{(c)}(a,c;z) &= (\eta(a-c+1) - \pi\cot(c\pi))U(a,c;z) \\ &- \frac{\Gamma(c-1)z^{1-c}\log(z)}{\Gamma(a)}M(a-c+1,2-c;z) \\ &- \frac{\Gamma(c-1)z^{1-c}}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{(a-c+1)_k(\eta(a-c+k+1)-\eta(2-c+k))z^k}{(2-c)_k k!} \\ &- \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\sum_{k=0}^{\infty}\frac{\eta(c+k)(a)_k z^k}{(c)_k k!}, \qquad c \notin \mathbb{Z}, \end{aligned} \tag{2.A.10}$$

where $\eta(z) = \frac{d\ln[\Gamma(z)]}{dz} = \Gamma'(z)/\Gamma(z)$ corresponds to Equation (6.3.1) of Abramowitz and Stegun (1972), also known as the digamma function. Calculating (2.3.4) and using (2.A.10), one can derive the expected trapping time (2.3.5).

### 2.A.3 Proof of Proposition 2.5.1

Since

$$S = \frac{\beta}{\delta}\left[1 - e^{-\delta\tau_x^{\pi^*(\kappa,\theta)}}\right],$$

then we consider $m_\delta^{\pi^*(\kappa,\theta)}(x)$, the Laplace transform for the insured process obtained in (2.4.2) with capital growth $r^{\pi^*(\kappa,\theta)}$ to compute $V^{\pi^*(\kappa,\theta)}(x)$ when capital losses are exponentially distributed with parameter $\alpha^{(\kappa)} > 0$. This yields (2.5.1).

### 2.A.4 Proof of Proposition 2.5.2

Following a similar procedure to that in Proposition 2.3.1, consider the integral

$$A(x) := \int_{x-x^{\pi^*(\kappa,\theta)*}}^{\infty} w(x - x^{\pi^*(\kappa,\theta)*}, z - (x - x^{\pi^*(\kappa,\theta)*}))dG_Z(z)$$

$$= \int_{x-x^{\pi^*(\kappa,\theta)*}}^{\infty} \left[z - (x - x^{\pi^*(\kappa,\theta)*}) + M^{(\kappa)} - x^{\pi^*(\kappa,\theta)*}\right]\alpha^{(\kappa)}e^{-\alpha^{(\kappa)}z}dz$$

$$= \left( \frac{1}{\alpha^{(\kappa)}} + M^{(\kappa)} - x^{\pi^*(\kappa,\theta)*} \right) e^{-\alpha^{(\kappa)}(x - x^{\pi^*(\kappa,\theta)*})},$$

which under the assumption $w(x_1, x_2) = x_2 + M^{(\kappa)} - x^{\pi^*(\kappa,\theta)*}$ yields a modified version of the IDE (2.A.1) given by

$$r^{\pi^*(\kappa,\theta)}(x - x^{\pi^*(\kappa,\theta)*}) m_{\delta,w}^{\pi^*(\kappa,\theta)\prime}(x) - (\lambda + \delta) m_{\delta,w}^{\pi^*(\kappa,\theta)}(x) + \lambda \int_0^{x - x^{\pi^*(\kappa,\theta)*}} m_{\delta,w}^{\pi^*(\kappa,\theta)}(x - z) \alpha^{(\kappa)} e^{-\alpha^{(\kappa)} z} dz$$

$$= -\lambda \left( \frac{1}{\alpha^{(\kappa)}} + M^{(\kappa)} - x^{\pi^*(\kappa,\theta)*} \right) e^{-\alpha^{(\kappa)}(x - x^{\pi^*(\kappa,\theta)*})}, \ x \geq x^{\pi^*(\kappa,\theta)*}. \tag{2.A.11}$$

Solving (2.A.11) in the same manner as (2.A.2) gives (2.5.3).

### 2.A.5 Proof of Proposition 2.6.1

Under the alternative microinsurance subsidy scheme, the Laplace transform of the trapping time satisfies the following differential equations:

$$0 = \begin{cases} -\frac{\left(x - x^{(\mathcal{A})*}\right)}{\alpha^{(\kappa)}} m_\delta^{(\mathcal{A})\prime\prime}(x) + \left[ \frac{(\lambda + \delta - r)}{\alpha^{(\kappa)} r} - (x - x^{(\mathcal{A})*}) \right] m_\delta^{(\mathcal{A})\prime}(x) + \frac{\delta}{r} m_\delta^{(\mathcal{A})}(x) & \textit{for } x^{(\mathcal{A})*} \leq x \leq B, \\ -\frac{\left(x - x^{(\mathcal{A})*}\right)}{\alpha^{(\kappa)}} m_\delta^{(\mathcal{A})\prime\prime}(x) + \left[ \frac{(\lambda + \delta - r^{(\kappa)})}{\alpha^{(\kappa)} r^{(\kappa)}} - (x - x^{(\mathcal{A})*}) \right] m_\delta^{(\mathcal{A})\prime}(x) + \frac{\delta}{r^{(\kappa)}} m_\delta^{(\mathcal{A})}(x) & \textit{for } x \geq B. \end{cases}$$

As in Proposition 2.3.1, use of the change of variable $y^{(\mathcal{A})} := y^{(\mathcal{A})}(x) = -\alpha^{(\kappa)}(x - x^{(\mathcal{A})*})$ leads to Kummer's Confluent Hypergeometric Equation, thus Equation (2.6.1) is obtained for arbitrary constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$. Under the boundary condition $\lim_{x \to \infty} m_\delta^{(\mathcal{A})}(x) = 0$ with asymptotic behaviour of the Kummer function $M(a, c; z)$ as presented in Proposition 2.3.1, we deduce that

$$C_3 = 0. \tag{2.A.12}$$

Then, since $m_\delta^{(\mathcal{A})}(x^{(\mathcal{A})*}) = \lambda / (\lambda + \delta)$, we obtain

$$C_1 = \frac{\lambda}{\lambda + \delta} - C_2 U \left( 1 - \frac{\lambda}{r}, 1 - \frac{\lambda + \delta}{r}; 0 \right). \tag{2.A.13}$$

Due to the continuity of the functions $m_\delta^{(\mathcal{A})}(x)$ and $m_\delta^{(\mathcal{A})\prime}(x)$ at $x = B$ and the differential properties of the Confluent Hypergeometric Functions:

$$\frac{d}{dz} M(a, c; z) = \frac{a}{c} M(a + 1, c + 1; z),$$

$$\frac{d}{dz} U(a, c; z) = -a U(a + 1, c + 1; z),$$

upon simplification,

$$C_4 = \frac{\left[ \frac{\lambda}{\lambda + \delta} - C_2 U \left( 1 - \frac{\lambda}{r}, 1 - \frac{\lambda + \delta}{r}; 0 \right) \right] M \left( -\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}; y^{(\mathcal{A})}(B) \right) + C_2 e^{y^{(\mathcal{A})}(B)} U \left( 1 - \frac{\lambda}{r}, 1 - \frac{\lambda + \delta}{r}; -y^{(\mathcal{A})}(B) \right)}{e^{y^{(\mathcal{A})}(B)} U \left( 1 - \frac{\lambda}{r^{(\kappa)}}, 1 - \frac{\lambda + \delta}{r^{(\kappa)}}; -y^{(\mathcal{A})}(B) \right)}$$

(2.A.14)

and

$$C_2 = \frac{\frac{\lambda}{\lambda+\delta}\left[\frac{\delta\alpha^{(\kappa)}}{(r-\lambda-\delta)}M\left(1-\frac{\delta}{r},2-\frac{\lambda+\delta}{r};y^{(\mathcal{A})}(B)\right)+M\left(-\frac{\delta}{r},1-\frac{\lambda+\delta}{r};y^{(\mathcal{A})}(B)\right)\left(\alpha^{(\kappa)}-D\right)\right]}{K}, \tag{2.A.15}$$

where

$$D := \frac{\alpha^{(\kappa)}\left(\frac{\lambda}{r^{(\kappa)}}-1\right)U\left(2-\frac{\lambda}{r^{(\kappa)}},2-\frac{\lambda+\delta}{r^{(\kappa)}};-y^{(\mathcal{A})}(B)\right)}{U\left(1-\frac{\lambda}{r^{(\kappa)}},1-\frac{\lambda+\delta}{r^{(\kappa)}};-y^{(\mathcal{A})}(B)\right)}$$

and

$$\begin{aligned}K :&= M\left(-\frac{\delta}{r},1-\frac{\lambda+\delta}{r};y^{(\mathcal{A})}(B)\right)U\left(1-\frac{\lambda}{r},1-\frac{\lambda+\delta}{r};0\right)\left(\alpha^{(\kappa)}-D\right)\\
&+De^{y^{(\mathcal{A})}(B)}U\left(1-\frac{\lambda}{r},1-\frac{\lambda+\delta}{r};-y^{(\mathcal{A})}(B)\right)\\
&+\frac{\delta\alpha^{(\kappa)}}{(r-\lambda-\delta)}M\left(1-\frac{\delta}{r},2-\frac{\lambda+\delta}{r};y^{(\mathcal{A})}(B)\right)U\left(1-\frac{\lambda}{r},1-\frac{\lambda+\delta}{r};0\right)\\
&-\alpha^{(\kappa)}e^{y^{(\mathcal{A})}(B)}\left(\frac{\lambda}{r}-1\right)U\left(2-\frac{\lambda}{r},2-\frac{\lambda+\delta}{r};-y^{(\mathcal{A})}(B)\right).\end{aligned}$$

# 2.B Appendix B: Monte Carlo Algorithms

## 2.B.1 Efficient Algorithm for Producing Monte Carlo Estimates of $V^{(\mathcal{A})}(x)$

To set up an efficient algorithm for producing Monte Carlo estimates of $V^{(\mathcal{A})}(x)$, we observe the following:

- $X_t^{(\mathcal{A})}$ can only drop below the critical capital $x^{(\mathcal{A})*}$ when a capital loss occurs, hence we only have to check the capital at the times when capital losses occur to see when to stop the process.
- In between any two arrival times of capital losses, the capital process grows exponentially at rates $r$ and $r^{(\kappa)}$, depending if the capital process is below or above the barrier level $B$, respectively.

In the implementation, we generate jump times and jump sizes for this process. The pair $Y_i = (T_i, \tilde{Z}_i)$ reflects the arrival times $T_i$ of the capital losses and the corresponding sizes of the capital losses $\tilde{Z}_i$.

We realise that, over the time interval in between two arrival times of capital losses $(T_{i-1}, T_i]$ either no subsidies are paid when the capital lies above the barrier level $B$ or, otherwise, subsidies are paid from $T_{i-1}$ to a certain time $T_i^{\text{in}}$. Furthermore, we note that for $T_i^{\text{in}} \leq T_i$, one can write

$$\int_{T_{i-1}}^{T_i^{\text{in}}} \pi e^{-\delta s} ds = \frac{\pi}{\delta}\left[e^{-\delta T_{i-1}} - e^{-\delta T_i^{\text{in}}}\right].$$

For $x \leq B$, let $\tau_B = \tau_B(x)$ be the solution to $h_r(t, x) = (x - x^{(\mathcal{A})*})e^{rt} + x^{(\mathcal{A})*} = B$. Namely, $\tau_B = \tau_B(x) = \ln\left[(B - x^{(\mathcal{A})*}) / (x - x^{(\mathcal{A})*})\right]/r$, which is the time when the capital returns to the barrier level $B$ if no loss occurs prior to time $\tau_B$. Thus, conditioning on the arrival times and the sizes of the capital losses leads to

$$\begin{aligned} V^{(\mathcal{A})}(x) &= \mathbb{E}\left[\int_0^{\tau_x^{(\mathcal{A})}} \pi e^{-\delta t} \mathbb{1}_{\left\{X_t^{(\mathcal{A})} < B\right\}} dt \,\middle|\, X_0^{(\mathcal{A})} = x\right] \\ &= \mathbb{E}\left[\sum_{i=1}^{n} \frac{\pi}{\delta}\left(e^{-\delta T_{i-1}} - e^{-\delta T_i^{\text{in}}}\right) \,\middle|\, T_0 = 0, T_n = \tau_x^{(\mathcal{A})}, \{Y_i\}_{i \geq 1}\right] \end{aligned}$$

with subsidy payments finalising at times

$$T_i^{\text{in}} = \min\left\{\tau_B\left(X_{T_{i-1}}^{(\mathcal{A})}\right) + T_{i-1}, T_i\right\} \cdot \mathbb{1}_{\left\{x^* \leq X_{T_{i-1}}^{(\mathcal{A})} < B\right\}} + T_{i-1} \cdot \mathbb{1}_{\left\{X_{T_{i-1}}^{(\mathcal{A})} \geq B\right\}}$$

and

$$X_{T_i}^{(\mathcal{A})} = \left(X_{T_{i-1}}^{(\mathcal{A})} - x^*\right) e^{r\left(T_i^{\text{in}} - T_{i-1}\right)} \cdot \mathbb{1}_{\left\{T_i^{\text{in}} = T_i\right\}}$$

$$+ \left(B - x^*\right) e^{r^{(\kappa)}\left(T_i - T_i^{\text{in}}\right)} \cdot \mathbb{1}_{\left\{T_i^{\text{in}} = \tau_B\left(X^{(\mathcal{A})}_{T_{i-1}}\right) + T_{i-1}\right\}}$$
$$+ \left(X^{(\mathcal{A})}_{T_{i-1}} - x^*\right) e^{r^{(\kappa)}\left(T_i - T_i^{\text{in}}\right)} \cdot \mathbb{1}_{\left\{T_i^{\text{in}} = T_{i-1}\right\}} + x^* - \kappa \cdot \tilde{Z}_i.$$

For a set of samples $\left\{y_i^{(k)}\right\}_{k\geq 1}$ of $\{Y_i\}_{i\geq 1}$, $1 \leq k \leq N$, we compute the Monte Carlo estimate of $V^{(\mathcal{A})}(x)$ as

$$V^{(\mathcal{A})}(x) = \frac{1}{N}\sum_{k=1}^{N}\left[\sum_{i=1}^{n}\frac{\pi}{\delta}\left(e^{-\delta T_{i-1}} - e^{-\delta T_i^{\text{in}}}\right)\middle|\{Y_i\}_{i\geq 1} = \left\{y_i^{(k)}\right\}_{i\geq 1}\right].$$

## 2.C Appendix C: Effects of Underlying Factors on the Trapping Probability

**Reference setup:** $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^* = x^{(\kappa)*} = x^{\pi(\kappa,\theta)*} = x^{(\mathcal{A})*} = 1$, $\kappa = 0.5$ and $\theta = 0.5$.

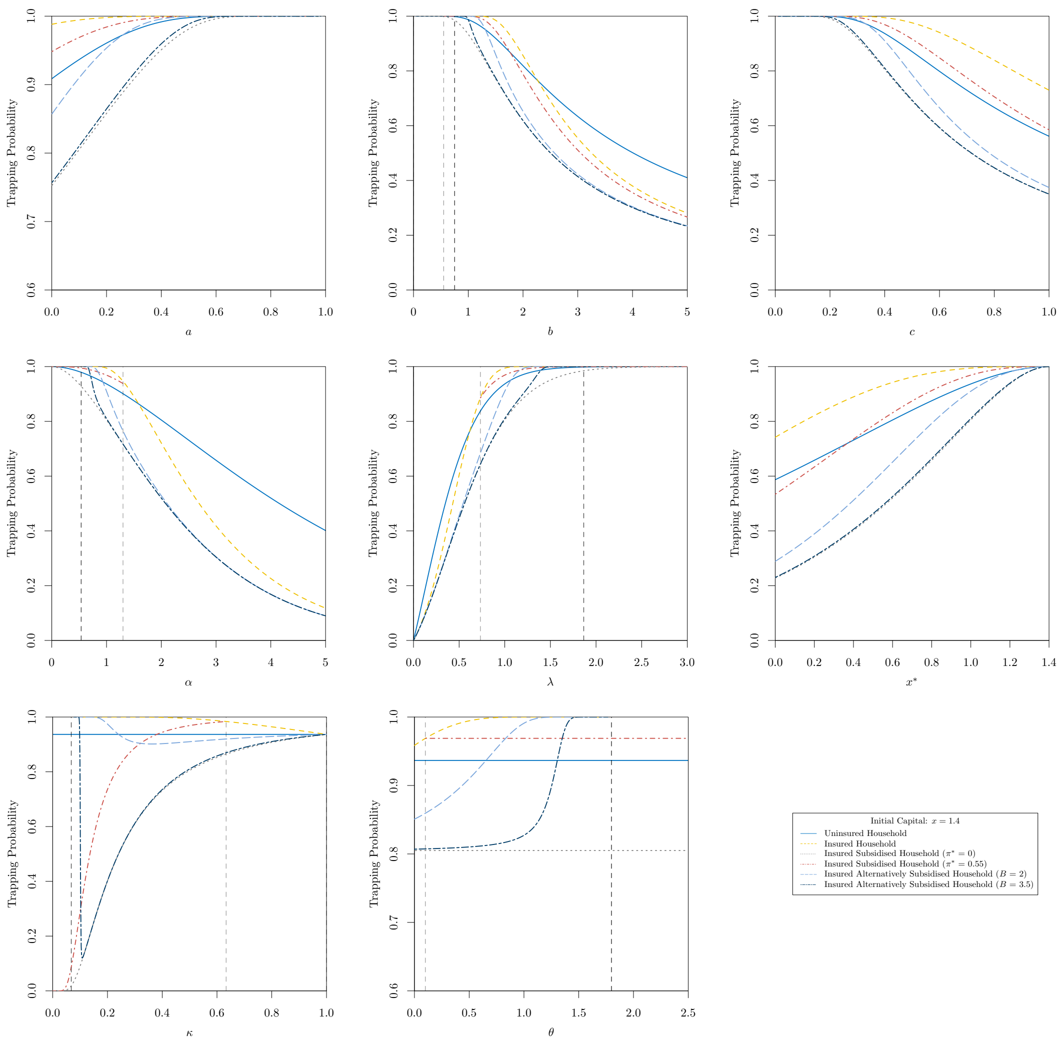



Figure 2.11: Effects of the rate of consumption ($0 < a < 1$), income generation ($0 < b$), investment or savings ($0 < c < 1$), the parameter of the exponential distribution ($\alpha > 0$) (i.e., expected capital loss size), the expected capital loss frequency ($\lambda > 0$), the critical capital ($x \geq x^*$), the proportionality factor $\kappa \in [0, 1]$ and the loading factor ($\theta \geq 0$) for initial capital $x = 1.4$.

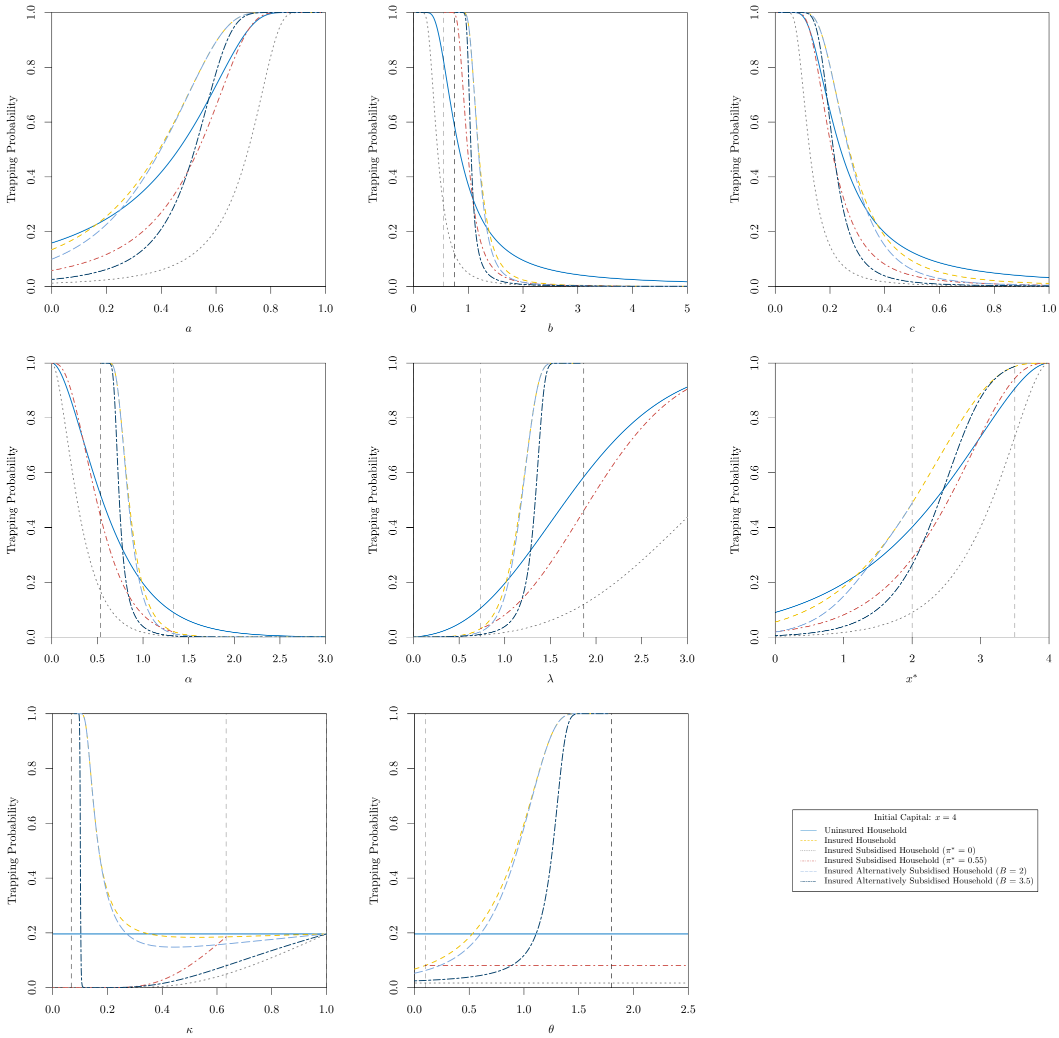



Figure 2.12: Effects of the rate of consumption ($0 < a < 1$), income generation ($0 < b$), investment or savings ($0 < c < 1$), the parameter of the exponential distribution ($\alpha > 0$) (i.e., expected capital loss size), the expected capital loss frequency ($\lambda > 0$), the critical capital ($x \geq x^*$), the proportionality factor $\kappa \in [0, 1]$ and the loading factor ($\theta \geq 0$) for initial capital $x = 4$.

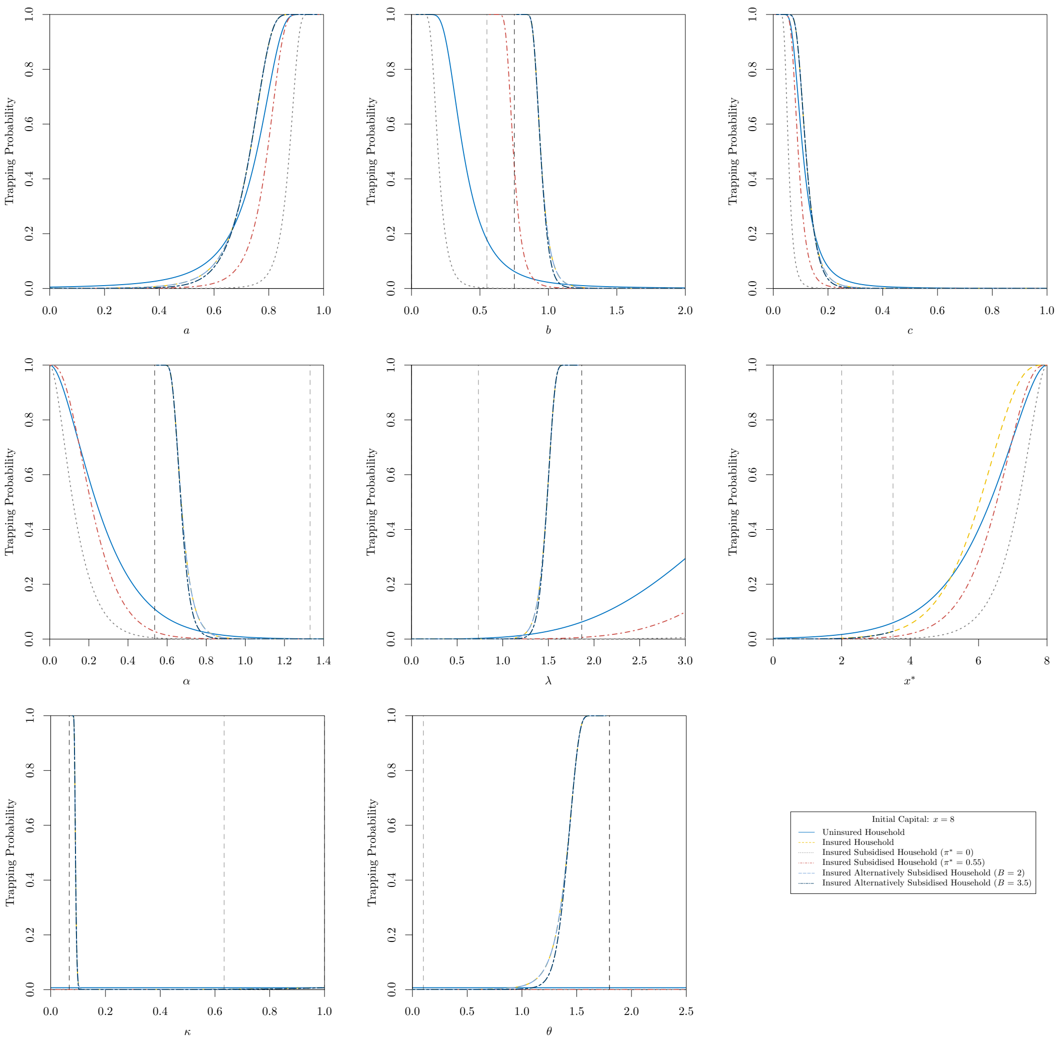



Figure 2.13: Effects of the rate of consumption ($0 < a < 1$), income generation ($0 < b$), investment or savings ($0 < c < 1$), the parameter of the exponential distribution ($\alpha > 0$) (i.e., expected capital loss size), the expected capital loss frequency ($\lambda > 0$), the critical capital ($x \geq x^*$), the proportionality factor $\kappa \in [0, 1]$ and the loading factor ($\theta \geq 0$) for initial capital $x = 8$.

## 2.D Appendix D: Effects of Underlying Factors on the Cost of Social Protection

**Reference setup:** $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^* = x^{(\kappa)*} = x^{\pi(\kappa,\theta)*} = x^{(\mathcal{A})*} = 1$, $\kappa = 0.5$, $\theta = 0.5$, $\delta = 0.1$ and $\epsilon = 0.01$.

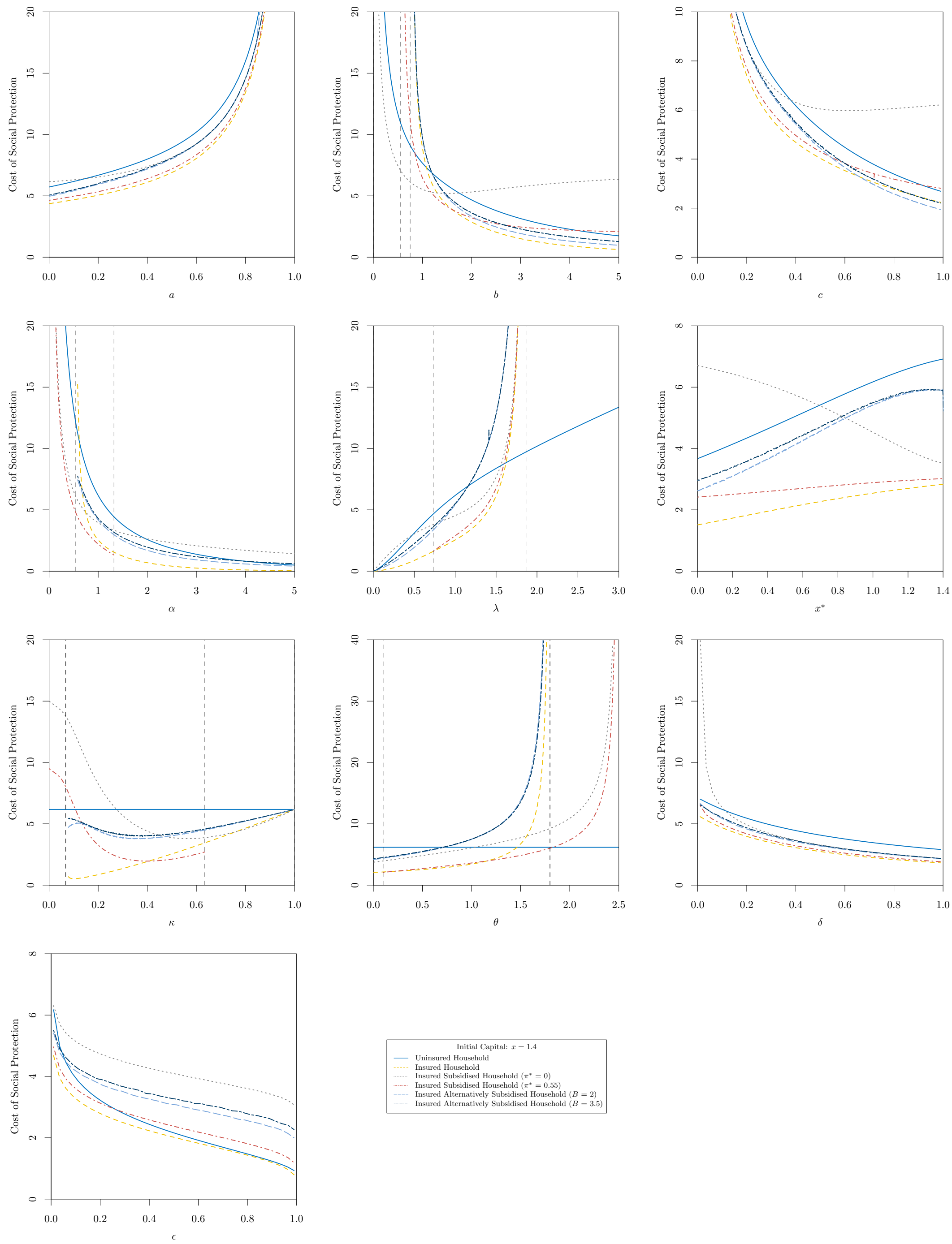



Figure 2.14: Effects of the rate of consumption ($0 < a < 1$), income generation ($0 < b$), investment or savings ($0 < c < 1$), the parameter of the exponential distribution ($\alpha > 0$) (i.e., expected capital loss size), the expected capital loss frequency ($\lambda > 0$), the critical capital ($x \geq x^*$), the proportionality factor $\kappa \in [0, 1]$, the loading factor ($\theta \geq 0$), the force of interest ($\delta \geq 0$) and the most admissible trapping probability $\epsilon \in [0, 1]$ for initial capital $x = 1.4$.

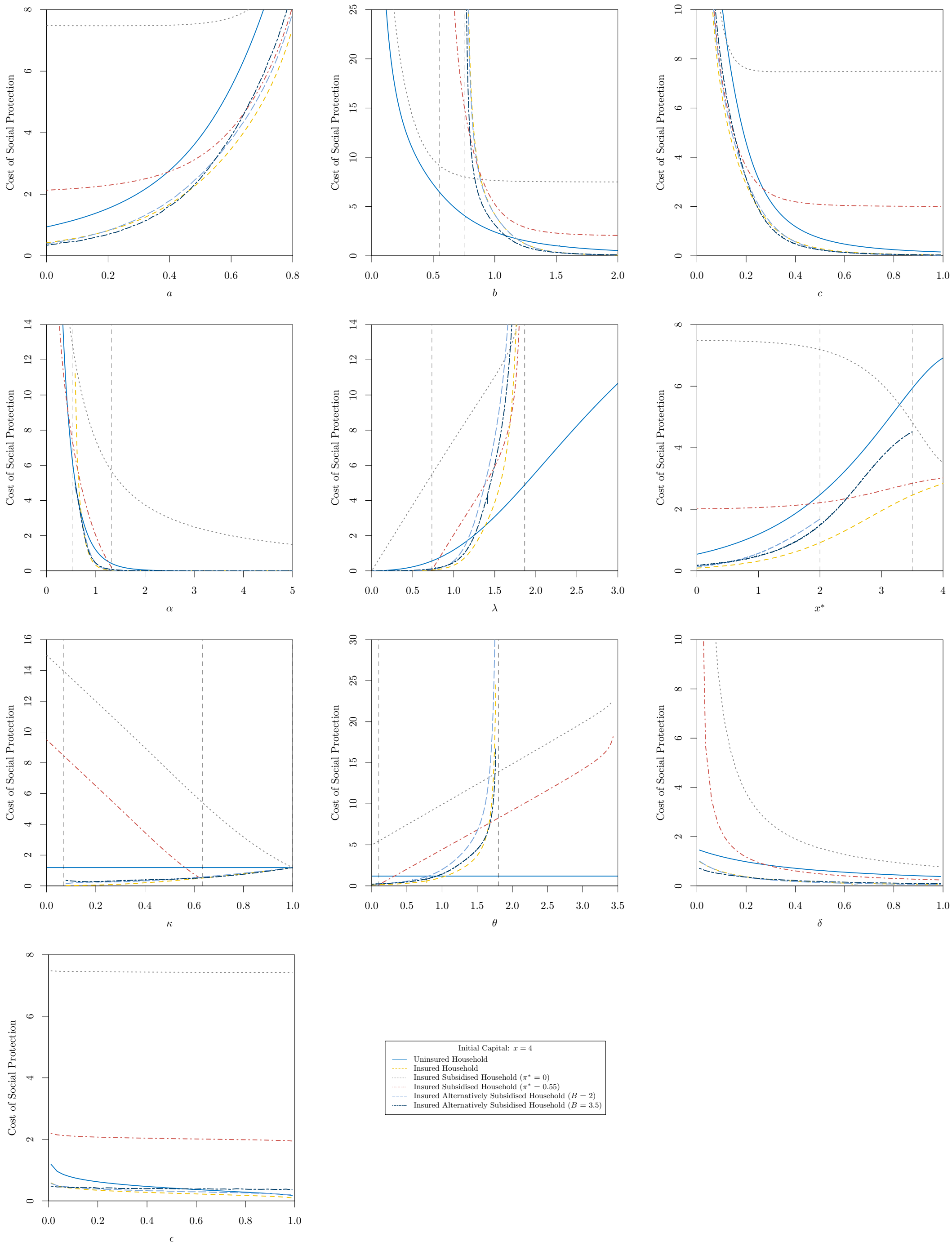



Figure 2.15: Effects of the rate of consumption ($0 < a < 1$), income generation ($0 < b$), investment or savings ($0 < c < 1$), the parameter of the exponential distribution ($\alpha > 0$) (i.e., expected capital loss size), the expected capital loss frequency ($\lambda > 0$), the critical capital ($x \geq x^*$), the proportionality factor $\kappa \in [0, 1]$, the loading factor ($\theta \geq 0$), the force of interest ($\delta \geq 0$) and the most admissible trapping probability $\epsilon \in [0, 1]$ for initial capital $x = 4$.

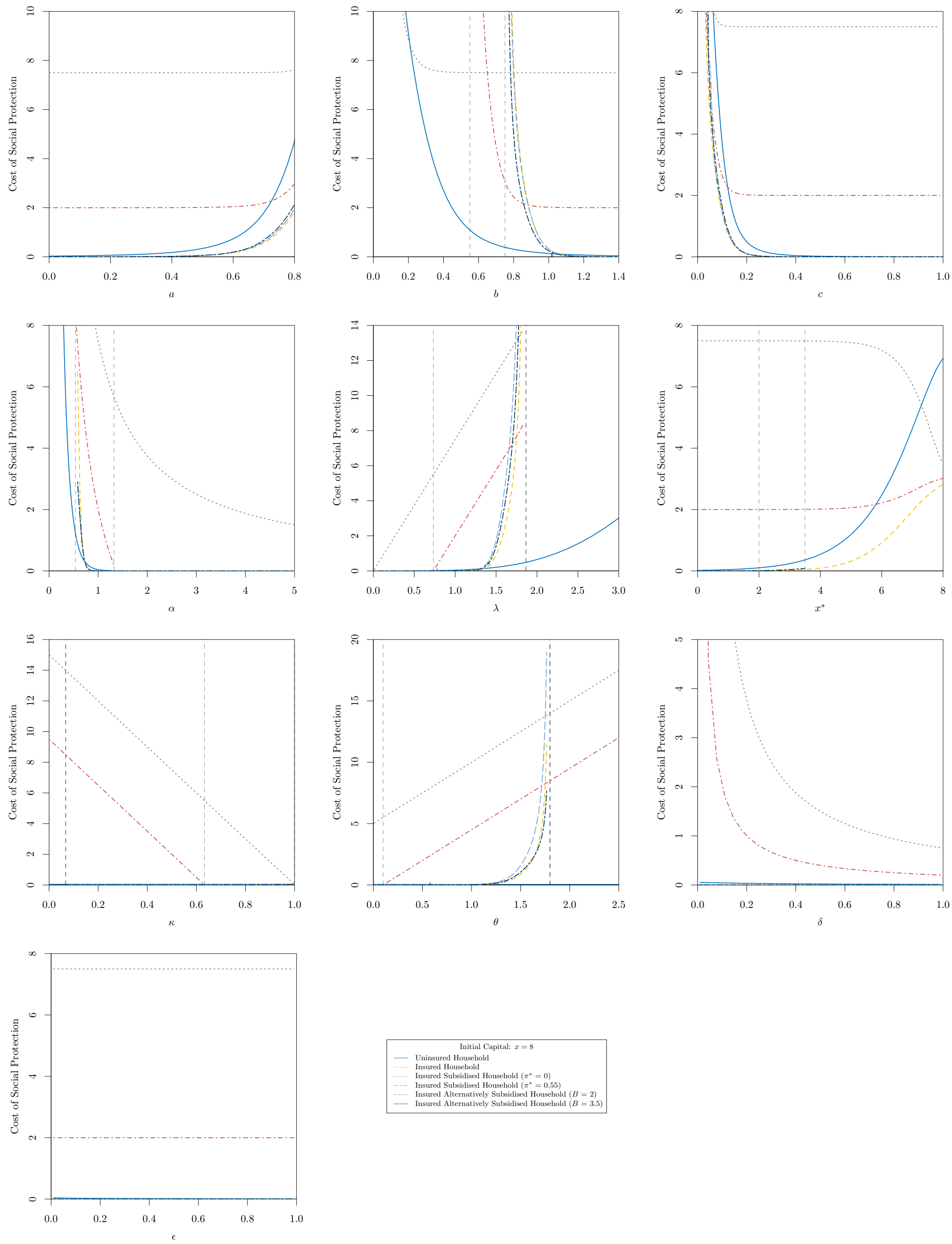



Figure 2.16: Effects of the rate of consumption ($0 < a < 1$), income generation ($0 < b$), investment or savings ($0 < c < 1$), the parameter of the exponential distribution ($\alpha > 0$) (i.e., expected capital loss size), the expected capital loss frequency ($\lambda > 0$), the critical capital ($x \geq x^*$), the proportionality factor $\kappa \in [0, 1]$, the loading factor ($\theta \geq 0$), the force of interest ($\delta \geq 0$) and the most admissible trapping probability $\epsilon \in [0, 1]$ for initial capital $x = 8$.

# Chapter 3

# On the Impact of Insurance on Households Susceptible to Random Proportional Losses: An Analysis of Poverty Trapping

This chapter is based on the following article:

Henshaw, K., J. M. Ramirez, J. M. Flores-Contró, E. A. Thomann, S. H. Loke, and C. D. Constantinescu (2023). On the Impact of Insurance on Households Susceptible to Random Proportional Losses: An Analysis of Poverty Trapping. *Submitted.*

**Abstract.** In this chapter, we consider a risk process with deterministic growth and multiplicative jumps to model the capital of a household. Unlike in other well-studied risk processes, capital losses are assumed to be proportional to the level of accumulated capital at the time of a capital loss event. We seek to derive the probability that a household falls below the poverty line, i.e. the trapping probability, where "trapping" occurs when the level of capital of a household falls below the poverty line, to an area from which it is difficult to escape without external help. Considering the remaining proportion of capital after a capital loss event to be distributed as a particular case of the beta distribution, closed-form expressions for the trapping probability are obtained via analysis of the Laplace transform of the infinitesimal generator of the risk process. To study the impact of insurance on this probability, introduction of an insurance product offering proportional coverage is presented. The infinitesimal generator of the insured risk process gives rise to non-local differential equations that are not easy to solve using techniques commonly used in risk theory. To overcome this, we propose a recursive method for deriving a closed-form solution of the integro-differential equation associated with the infinitesimal generator of the insured risk process and provide a numerical method for estimating the trapping probability. Constraints on parameters of the risk process that prevent certain trapping are also derived in both the uninsured and insured cases using classical results from risk theory.

## 3.1 Introduction

Vulnerable non-poor households (those living just above the poverty line) are extremely susceptible to entering poverty, particularly in the event of a financial loss. This problem, and the true nature of the loss experience of vulnerable non-poor households, must be studied in order to attain poverty reduction. An indicator that can be used to assess financial stability is capital, which in the vulnerable non-poor household environment, where monetary wealth is often limited, the concept of capital should reflect all forms of capital that enable production, whether for trade or self-sustaining purposes. This may include land, property, physical and human capital, with health a form of capital in extreme cases where sufficient health services and food accessibility are not guaranteed (Dasgupta, 1997). The threat of catastrophic loss events is of great concern, particularly under this broad definition of capital. For example, vulnerable non-poor households are predominantly engaged in agricultural work and are exposed, among many other things, to natural disasters in the form of floods and droughts. In contrast to losses relating to health, life or death, agricultural losses can immediately eliminate a high proportion of a household's ability to produce through loss of land and livestock, irrespective of their level of capital.

In this chapter, we study the behaviour of households' capital under the assumption of proportional capital loss experience. Proportionality in loss experience captures the exposure of households of all capital levels to both catastrophic and low severity loss events. This is particularly relevant in the vulnerable non-poor household setting, where, in addition to infrequent but serious events such as natural disasters, more common events such as hospital admissions and household deaths, can be detrimental. To do this, we adopt the ruin-theoretic approach proposed in Kovacevic and Pflug (2011), by using a risk process with deterministic growth and multiplicative losses to model the capital of a household. At capital loss events, accumulated capital is reduced by a random proportion of itself, rather than by an amount of random value, as in Flores-Contró et al. (2021). Processes of this structure are typically referred to as a growth-fragmentation or growth-collapse processes, characterised by their growth in between the random collapse times at which downwards jumps occur. In these models, the randomly occurring jumps have a random size which is dependent on the state of the process immediately before the jump.

Our aim is to derive the probability that a household falls below the poverty line, where this probability mimics an insurer's ruin probability. To the best of our knowledge, only Kovacevic and Pflug (2011), and Flores-Contró et al. (2021) have, so far, studied this problem in the ruin-theoretic setting. As in this earlier work, in this chapter, we consider the probability under two frameworks, one in which the household has no insurance coverage, and the other in which they are proportionally insured. We introduce insurance to assess its effectiveness as a measure of poverty reduction. Aligning with the low-income environment, proportional coverage is assumed to be provided through an inclusive insurance product, specifically designed to cater for those excluded from traditional insurance services or without access to alternative effective risk management strategies. This type of product, targeted towards low-income populations, is commonly referred to as microinsurance. In Flores-Contró et al. (2021), the risk process with deterministic growth and random-value losses is instead used to assess the impact of government premium subsidy schemes on the probability of falling below the poverty line.

Although important, we do not consider the behaviour of a household's capital below the poverty line. Households that live or fall below the poverty line are said to be in a poverty trap, where a poverty trap is a state of poverty from which it is difficult to escape without external help. Poverty trapping is a well-studied topic in development economics (the interested reader may refer to Azariadis and Stachurski (2005), Bowles et al. (2006), Kraay and McKenzie (2014), Barrett et al. (2016) and references therein for further discussion; see Matsuyama (2008) for a detailed description of the mechanics of poverty traps), however, for the purpose of this study, we use the term "trapping" only to describe the event that a household falls into poverty, focusing our interest on vulnerable non-poor households with capital levels above the poverty line.

In Kovacevic and Pflug (2011), estimates of the infinite-time trapping probability of a discretised version of the capital process adopted in this chapter are obtained through numerical simulation. Azaïs and Genadot (2015) perform further numerical analysis on the same model, discussing applications to the capital setting of Kovacevic and Pflug (2011) and to population dynamics, where the poverty line denotes extinction. In both cases, derivation of analytical solutions of infinitesimal generator equations is not attempted. Our main contribution is therefore in the derivation of closed-form solutions of the infinitesimal generator equations associated with risk processes of this type and, in the case of proportional insurance, in the proposition of a novel approach to estimate the trapping probability recursively.

Due to the proportionality of the capital losses, generators of the capital process no longer directly align with those of classical models used to describe the surplus process of an insurer. Obtaining the solution of the infinitesimal generator equation is therefore non-trivial. Indeed, random absolute losses are serially correlated with one another and with the inter-arrival times of capital loss events, in contrast to the random losses considered in traditional risk models. In addition, only the surplus of a household's capital above the poverty line grows exponentially. To ensure that the Lundberg equation is well-defined, and thus prevent certain trapping, constraints on the parameters of the capital growth processes are derived. Laplace transform and derivative operators are then used to obtain the associated trapping probabilities, under no insurance coverage and proportional insurance coverage, respectively.

Research on growth-collapse processes with applications outside the field of actuarial science includes Altman et al. (2002) and Löpker and Van Leeuwaarden (2008) for congestion control in data networks, Eliazar and Klafter (2004) and Eliazar and Klafter (2006) for phenomena in physical systems, Derfel et al. (2012) for cell growth and division, and Peckham et al. (2018) in a model of persistence of populations subject to random shocks. Aligning with the Laplace transform approach adopted in the case of no insurance, Löpker and Van Leeuwaarden (2008) obtain the Laplace transform of the transient moments of a growth-collapse process, while Eliazar and Klafter (2004) consider the state of a growth-collapse process at equilibrium, computing Laplace transforms of the system and of the high- and low-levels of the growth-collapse cycle.

Previous research on the impact of microinsurance mechanisms on the probability of falling below the poverty line from a non-ruin perspective has been undertaken through application of multi-equilibrium models and dynamic stochastic programming (Chantarat et al., 2017; Ikegami et al., 2018; Carter and Janzen, 2018; Liao et al., 2020; Janzen et al., 2021; Kovacevic and Semmler, 2021). With the exception of the latter, each of these studies considers the impact of subsidisation and the associated cost to the subsidy provider. Will et al. (2021) and Henshaw et al. (2023) extend the problem to the group-setting, assessing the impact of risk-sharing on the trapping probability. Will et al. (2021) undertake a simulation-based study and Henshaw et al. (2023) propose a Markov modulated stochastic dissemination model of group wealth interactions, using a bivariate normal approximation to calculate the trapping probability.

Notably, Kovacevic and Pflug (2011), Liao et al. (2020) and Flores-Contró et al. (2021) suggest that purchase of insurance and the associated need for premium payment increases the risk of falling below the poverty line for the most vulnerable. Barriers to microinsurance penetration that exist due to constraints on product affordability resulting from fundamental features of the microinsurance environment likely contribute to such observations. Limited consumer financial literacy and experience, product accessibility and data availability, are examples of the unique characteristics that must be accounted for when designing effective and affordable microinsurance products. Through our analysis, we further investigate the case of proportional loss experience to assess the associated implications on the affordability of insurance.

Janzen et al. (2021) optimise the level of insurance coverage across the population, observing that those in the neighbourhood of the poverty line do not optimally purchase insurance (without subsidies), instead suppressing their consumption and mitigating the probability of falling into poverty. This aligns with the increase in the trapping probability observed in the aforementioned studies, when those closest to the poverty line purchase insurance. Similarly, Kovacevic and Semmler (2021) derive the retention rate process that maximises the expected discounted capital, by allowing adjustments in the retention rate of the policyholder after each capital loss throughout the lifetime of the insurance contract. In this chapter, however, the proportion of insurance coverage and the choice to insure is fixed across the population, as in Kovacevic and Pflug (2011), Chantarat et al. (2017) and Flores-Contró et al. (2021).

The remainder of this chapter is structured as follows. Section 3.2 introduces the capital growth model and its alignment with the classical Crámer-Lundberg model. This connection enables derivation of constraints on the parameters of the risk process that ensure the Lundberg equation is well-defined, thus preventing certain trapping. Derivation of the trapping probability for uninsured losses and $Beta(\alpha, 1)-$distributed remaining proportions of capital is presented in Section 3.3. The trapping probability for households covered by proportional insurance coverage is derived in Section 3.4 for $Beta(1, 1)-$distributed remaining proportions of capital. The non-locality of the differential equations associated with the infinitesimal generator of the insured process is highlighted and a recursive method for estimating the trapping probability is proposed. Uninsured and insured trapping probabilities are compared in Section 3.5 and are presented alongside additional findings of interest. Concluding remarks are provided in Section 5.6.

Throughout the chapter, we use the term "insurance" to refer to any form of microinsurance product. Our analysis does not consider a specific type of product but can be tailored through the selection of appropriate parameters.

## 3.2 The Capital Model

Construction of the capital model follows that of Kovacevic and Pflug (2011). Consider a household with accumulated capital $\{X_t\}_{t\geq 0}$. Under the basic assumption that the household has no loss experience, their growth in accumulated capital is given by

$$\frac{dX_t}{dt} = r \cdot [X_t - x^*]^+ , \tag{3.2.1}$$

where $[x]^+ = \max(x, 0)$. The dynamics in (3.2.1) are built on the assumption that a household's income $I_t$ is split into consumption $C_t$ and savings or investments $S_t$, such that at time $t$,

$$I_t = C_t + S_t, \tag{3.2.2}$$

where consumption is an increasing function of income, given by

$$C_t = \begin{cases} I_t & \text{if } I_t \leq I^*, \\ I^* + a\,(I_t - I^*) & \text{if } I_t > I^*, \end{cases} \tag{3.2.3}$$

for $0 < a < 1$. The critical point below which a household consumes all of its income, with no facility for savings or investment, is denoted by $I^*$. Accumulated capital is assumed to grow proportionally to the level of savings, such that

$$\frac{dX_t}{dt} = cS_t, \tag{3.2.4}$$

for $0 < c < 1$, and income is generated through the accumulated capital, such that

$$I_t = bX_t, \tag{3.2.5}$$

for $b > 0$.

Combining (3.2.2), (3.2.3), (3.2.4) and (3.2.5) gives exactly the dynamics in (3.2.1), where the capital growth rate $r = (1 - a) \cdot b \cdot c > 0$ incorporates household rates of consumption ($0 < a < 1$), income generation ($b > 0$) and investment or savings ($0 < c < 1$), while $x^* = I^*/b > 0$ denotes the threshold below which a household lives in poverty. The notion of a household in this model setting may be extended for consideration of poverty trapping within economic units such as community groups, villages and tribes, in addition to the traditional household structure.

Reflecting the ability of a household to produce, the level of accumulated capital of a household $X_t$ is composed of land, property, physical and human capital. The poverty threshold $x^*$ represents the amount of capital required to forever attain a critical level of income below which a household would not be able to sustain its basic needs, facing elementary problems relating to health and food security. We refer to this threshold as the critical capital or the poverty line. Since (3.2.1) is positive for all levels of capital greater than the critical capital, all points less than or equal to $x^*$ are stationary, the level of capital remains constant if the critical capital is not met. In this basic model, stationary points below the critical capital are not attractors of the system if the initial capital exceeds $x^*$, in which case the capital process grows exponentially with rate $r$.

In line with Kovacevic and Pflug (2011), we expand the dynamics of (3.2.1) under the assumption that households are susceptible to the occurrence of capital losses such as those highlighted in Section 3.1, including severe illness, the death of a household member or breadwinner and catastrophic events such as droughts, floods and earthquakes. The occurrence of loss events is assumed to follow a Poisson process with intensity $\lambda$, where the capital process follows the dynamics of (3.2.1) in between events. On the occurrence of the $i$th capital loss, the capital process experiences a downwards jump to $X_{T_i} \cdot Z_i$, where $Z_i \in [0, 1]$ is the random proportion determining the remaining capital after loss $i$ and $X_{T_i}$ the level of accumulated capital at the loss time. The sequence $\{Z_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables with common distribution function $G_Z$, independent of the Poisson process. In this chapter, it will be assumed that the random proportion determining the remaining capital after each loss event is $Beta(\alpha, \beta)-$distributed ($Z_i \sim Beta(\alpha, \beta)$).

A household reaches the area of poverty if it suffers a loss large enough that the remaining capital is attracted into the poverty trap. Since a household's capital does not grow below the critical capital $x^*$, households that fall into the area of poverty will never escape without external help. Once below the critical capital, households are exposed to the risk of falling deeper into poverty. However, in contrast to Flores-Contró et al. (2021) where random-valued losses are considered, the dynamics of the model do not allow for the possibility of negative capital due to the proportionality of loss experience.

The structure of the process in-between loss events is derived through solution of the Ordinary Differential Equation (ODE) in (3.2.1). The stochastic capital process with deterministic exponential growth and multiplicative losses is then formally defined as follows:

**Definition 3.2.1.** Let $T_i$ be the $i$th event time of a Poisson process $\{N_t\}_{t\geq 0}$ with parameter $\lambda$, where $T_0 = 0$. Let $0 \leq Z_i \leq 1$ be a sequence of i.i.d. random variables with distribution function $G_Z$, independent of the process $\{N_t\}$. For $T_{i-1} \leq t < T_i$, the stochastic growth process of the accumulated capital $X_t$ is defined as

$$X_t = \begin{cases} \left(X_{T_{i-1}} - x^*\right) e^{r(t-T_{i-1})} + x^* & \text{if } X_{T_{i-1}} > x^*, \\ X_{T_{i-1}} & \text{otherwise.} \end{cases} \tag{3.2.6}$$

At the jump times $t = T_i$, the capital process is given by

$$X_{T_i} = \begin{cases} \left[\left(X_{T_{i-1}} - x^*\right) e^{r(T_i - T_{i-1})} + x^*\right] \cdot Z_i & \textit{if } X_{T_{i-1}} > x^*, \\ X_{T_{i-1}} \cdot Z_i & \textit{otherwise.} \end{cases}$$

As in Kovacevic and Pflug (2011) and Flores-Contró et al. (2021), the aim of this chapter is to study the probability that a household falls below the poverty line, i.e. the trapping probability. By Definition 3.2.1, the capital level of the household follows a piecewise deterministic Markov process (Davis, 1984, 1993) of compound Poisson-type, which is deterministic in-between the randomly occurring jump times at which capital losses occur.

The infinite-time trapping probability describes the distribution of the time at which a household becomes trapped, referred to as the trapping time. Given a household has initial capital $x$, their trapping time, denoted $\tau_x$, is given by

$$\tau_x := \inf\left\{t \geq 0 : X_t < x^* | X_0 = x\right\},$$

where $\tau_x$ is fixed at infinity if $X_t \geq x^*$ for all $t$. It then follows that the trapping probability, denoted as $\psi(x)$, is given by

$$\psi\left(x\right) = \mathbb{P}\left(\tau_x < \infty\right).$$

Analysis of the trapping probability can be undertaken through study of the infinitesimal generator. The infinitesimal generator $\mathcal{A}$ of the stochastic process $X_t$ as in Definition 3.2.1 is given by

$$\left(\mathcal{A}\psi\right)(x) = r(x - x^*)\psi'(x) + \lambda \int_0^1 [\psi(x \cdot z) - \psi(x)] dG_Z(z), \tag{3.2.7}$$

for $x \geq x^*$. The remainder of the chapter works towards solving $\left(\mathcal{A}\psi\right)(x) = 0$, in line with the classical theorem of Paulsen and Gjessing (1997). Intuitively, the boundary conditions of the trapping probability are

$$\lim_{x \to x^{*+}} \psi(x) = 1 \quad \text{and} \quad \lim_{x \to \infty} \psi(x) = 0, \tag{3.2.8}$$

such that under the assumption that $\psi(x)$ is a bounded and twice continuously differentiable function on $x \geq x^*$, with a bounded first derivative, and since we consider only what happens above the critical capital $x^*$, the theorem of Paulsen and Gjessing (1997) is applicable.

Closed-form expressions for Laplace transforms of ruin (trapping) probabilities are often more easily obtained than for the probability itself. However, multiplication of the initial capital by the random proportion in the integral function makes Laplace transform methods typically used in risk theory no longer straightforward. Solution of the Integro-Differential Equation (IDE) in (3.2.7) has so far only been undertaken numerically (see, for example, Kovacevic and Pflug (2011)). In this chapter, closed-form trapping probabilities are obtained through solution of $\left(\mathcal{A}\psi\right)(x) = 0$ for particular cases of $G_Z$, the distribution function of the remaining proportions of capital.

First, note that there exists a relationship between the capital model of Definition 3.2.1 and the classical Crámer-Lundberg model. This enables specification of an upper bound on the trapping probability of the capital growth process $X_t$ through Lundberg's inequality, derived in Lundberg (1926). Consider an adjustment of the capital process that is discretised at loss event times such that $\tilde{X}_i = X_{T_i}$, i.e. the capital process studied in Kovacevic and Pflug (2011). Defining $L_i := \log\left(\tilde{X}_i\right)$ and setting $x^* = 0$ yields

$$L_i = L_{i-1} + r(T_i - T_{i-1}) + \log(Z_i) = \log x + rT_i + \sum_{k=1}^{i} \log(Z_k), \qquad (3.2.9)$$

where $\log(Z_k) < 0$. The model on the right-hand side of (3.2.9) is a version of the classical Crámer-Lundberg model introduced by Lundberg (1903) and later studied by Cramér (1930), which assumes an insurance company collects premiums continuously and pays claims of random size at random times. The corresponding surplus process is given by

$$U_t = u + pt - \sum_{i=1}^{P_t} Y_i,$$

where $u = U_0 \geq 0$ is the insurer's initial capital, $p$ is the constant premium rate, $\{P_t\}_{t\geq 0}$ is a Poisson process with intensity $\mathcal{I}$ which counts the number of claims in the time interval $[0, t]$ and $\{Y_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. claim sizes with distribution function $G_Y$.

The net profit condition is a constraint that ensures, on average, that the capital gains of a household are superior to their losses. If this condition is not satisfied then trapping is certain. It is well-known in risk theory that if the net profit condition holds, the process $U_t$ converges to infinity almost surely as $t \to \infty$ and there is a positive probability that $U_t \geq 0$ for all $t$. As a consequence of the net profit condition, it also holds that $\lim_{u\to\infty} \psi^{\text{RUIN}}(u) = 0$, where $\psi^{\text{RUIN}}(u)$ is the ruin probability under the classical Crámer-Lundberg model. However, derivation of the net profit condition from the drift of $U_t$ to infinity is not always straightforward. The Lundberg equation provides an alternative method for deriving the net profit condition. Assume that there exists a constant $R > 0$ such that the process $\left(e^{-RL_i}\right)_{i\geq 0}$ is a martingale. The resulting equation is the Lundberg equation, and is given by

$$\mathbb{E}[e^{-R\log(Z_i)}]\mathbb{E}[e^{-Rr\tilde{T}_i}] = \mathbb{E}[e^{-R(\log(Z_i)+r\tilde{T}_i)}] = 1,$$

where $\tilde{T}_i = T_i - T_{i-1}$ denotes the inter-arrival time and the unique solution $R$ is the adjustment coefficient. Thus, for $R$ to exist, it must hold that $\mathbb{E}[\log(Z_i) + r\tilde{T}_i] > 0$. In fact, for $R$ to exist the net profit condition must hold. As such, the existence of $R$ ensures that $\lim_{u\to\infty} \psi^{\text{RUIN}}(u) = 0$.

Then, if $\mathbb{E}[\log(Z_i) + r\tilde{T}_i] > 0$, the logarithmic process in (3.2.9) converges to infinity almost surely, and

$$\lim_{\log x\to\infty} \mathbb{P}(L_i < 0 | L_0 = \log x) = 0.$$

Since $\log x \to \infty$ implies $x \to \infty$ it holds that

$$\lim_{x\to\infty} \psi(x) \sim \lim_{x\to\infty} \psi(x|x^* = 0) \leq \lim_{x\to\infty} \mathbb{P}(X_t < 1 | X_0 = x)$$

$$= \lim_{\log x \to \infty} \mathbb{P}(L_i < 0 | L_0 = \log x) = 0,$$

where we have applied the equivalence of $\tilde{X}_i$ and $X_t$ at loss event times and the fact that asymptotically, the behaviour of the trapping probability $\psi(x)$ remains unchanged for any $x^*$. The upper boundary condition in (3.2.8) therefore holds if $\mathbb{E}[\log(Z_i) + r\tilde{T}_i] > 0$.

In Sections 3.3 and 3.4, we use the net profit condition to derive constraints on the parameters of the capital process for uninsured and proportionally insured households, respectively. The closed-form trapping probabilities are then derived through consideration of the associated infinitesimal generators for uninsured losses with $Beta(\alpha, 1)-$distributed remaining proportions of capital (Section 3.3) and proportionally insured losses with $Beta(1, 1)-$distributed remaining proportions of capital (Section 3.4). Laplace transform methods are applied in Section 3.3 and a derivative approach in Section 3.4, where a solution of the infinitesimal generator equation is derived recursively.

## 3.3 Derivation of Trapping Probability Under No Insurance Coverage

Under the assumption of remaining proportions of capital with distribution $Z_i \sim Beta(\alpha, 1)$, letting $u = x \cdot z$ reduces the infinitesimal generator of the capital growth process in (3.2.7) to

$$\left(\mathcal{A}\psi\right)(x) = r(x - x^*)\psi'(x) - \lambda\psi(x) + \frac{\alpha\lambda}{x^\alpha}\int_0^x \psi(u)u^{\alpha-1}du, \tag{3.3.1}$$

for $x \geq x^*$.

**Proposition 3.3.1.** *Consider a household capital process as proposed in Definition 3.2.1 with initial capital $x \geq x^*$, capital growth rate $r$, loss intensity $\lambda > 0$ and remaining proportions of capital with distribution $Beta(\alpha, 1)$. The adjustment coefficient of the corresponding Lundberg equation exists if*

$$\frac{\lambda}{r} < \alpha. \tag{3.3.2}$$

*Proof.* For $Beta(\alpha, 1)-$distributed remaining proportions of capital, given that $Z_i$ and $\tilde{T}_i$ are independent and since $\mathbb{E}[\log(Z_i)] = \alpha \int_0^1 \log(z)z^{\alpha-1}dz$, $\mathbb{E}[\log(Z_i) + r\tilde{T}_i]$ holds if and only if (3.3.2) is satisfied, as required.

□

We now derive the trapping probability through solution of $\left(\mathcal{A}\psi\right)(x) = 0$ in line with the discussion of Section 3.2. Since households face certain trapping if the net profit condition is violated, our analysis focuses only on the region for which (3.3.2) holds.

**Proposition 3.3.2.** *Consider a household capital process as proposed in Definition 3.2.1 with initial capital $x \geq x^*$, capital growth rate $r$, loss intensity $\lambda > 0$ and remaining proportions of capital with distribution Beta$(\alpha, 1)$. The closed-form trapping probability is given by*

$$\psi(x) = \frac{\Gamma(\alpha) \cdot {}_2F_1(\alpha - \frac{\lambda}{r}, 1 - \frac{\lambda}{r}; 1 + \alpha - \frac{\lambda}{r}; y(x)^{-1})}{\left(\alpha - \frac{\lambda}{r}\right) \Gamma\left(\alpha - \frac{\lambda}{r}\right) \Gamma\left(\frac{\lambda}{r}\right)} y(x)^{\frac{\lambda}{r} - \alpha}, \qquad (3.3.3)$$

*where $y(x) = \frac{x}{x^*}$ and ${}_2F_1(\cdot)$ is Gauss's Hypergeometric Function, for $\alpha > \frac{\lambda}{r}$.*

*Proof.* Setting $H_1(x) = \psi(x) \cdot x^{\alpha-1}$ and $H_2(x) = 1$ in (3.3.1), we have

$$r(x - x^*)\psi'(x) - \lambda\psi(x) + \frac{\alpha\lambda}{x^\alpha}(H_1 * H_2)(x) = 0, \qquad x \geq x^*, \qquad (3.3.4)$$

where $f * g$ denotes the convolution of the functions $f$ and $g$, i.e.

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau \quad \textit{for} \quad f, g : [0, \infty) \to \mathbb{R}.$$

Let $\mathcal{L}\{\psi(x)\} = F(s)$ denote the Laplace transform of the trapping probability $\psi(x)$, i.e.

$$\mathcal{L}\{\psi(x)\} = F(s) = \int_0^\infty \psi(x)e^{-sx}dx, \qquad x \geq 0.$$

Applying the Laplace transform to (3.3.4) yields

$$s^2F^{(\alpha+1)}(s) + s\left[\alpha + 1 + \frac{\lambda}{r} + x^*s\right]F^{(\alpha)}(s) + \alpha\left[\frac{\lambda}{r} + x^*s\right]F^{(\alpha-1)}(s) = 0, \qquad (3.3.5)$$

where the Convolution Theorem (see, for example, Theorem 3.4 of Dyke (2001)) and the following elementary properties of Laplace transforms:

$$\begin{aligned}
&\mathcal{L}\{x^n\psi(x)\} = (-1)^nF^{(n)}(s),\\
&\mathcal{L}\{\psi'(x)\} = sF(s) - \psi(0) \quad \text{and}\\
&\mathcal{L}\{x^n\psi'(x)\} = (-1)^nsF^{(n)}(s) - n(-1)^{n-1}F^{(n-1)}(s),
\end{aligned}$$

were used. To address the higher order derivatives, we let $y(s) := F^{(\alpha-1)}(s)$, such that $y'(s) := F^{(\alpha)}(s)$ and $y''(s) := F^{(\alpha+1)}(s)$. Then, (3.3.5) is equivalent to

$$s^2y''(s) + s\left[\alpha + 1 + \frac{\lambda}{r} + x^*s\right]y'(s) + \alpha\left[\frac{\lambda}{r} + x^*s\right]y(s) = 0. \qquad (3.3.6)$$

The above is a second order ODE of the kind (139) from Zaitsev and Polyanin (2003). Hence, letting $y = s^kw$, where $k$ is a root of the quadratic equation $k^2 + (\alpha + \lambda/r)\,k + (\alpha\lambda)\,/r = 0$, i.e. $k_1 = -\lambda/r$ or $k_2 = -\alpha$ (here, we take $k_2 = -\alpha$ and omit the case $k_1 = -\lambda/r$ in which a similar procedure must be followed), (3.3.6) reduces to

$$sw''(s) + \left[1 + \frac{\lambda}{r} - \alpha + x^*s\right]w'(s) = 0,$$

which has as solution

$$w(s) = C_1 \int_0^s e^{-x^* t} t^{-\left(1+\frac{\lambda}{r}-\alpha\right)} dt + C_2.$$

Then, under the substitution $u = x^* \cdot t$,

$$y(s) = F^{(\alpha-1)}(s) = C_1 x^{*\left(\frac{\lambda}{r}-\alpha\right)} s^{-\alpha} \gamma\left(\alpha - \frac{\lambda}{r}; x^* s\right) + C_2 s^{-\alpha} \qquad \textit{for} \quad \frac{\lambda}{r} < \alpha, \tag{3.3.7}$$

where $\gamma(a; x) = \int_0^x e^{-t} t^{a-1} dt$ for $\mathbb{R}(a) > 0$ is the lower incomplete gamma function (see, for example, Equation (6.5.2) of Abramowitz and Stegun (1972)). From Equation (2) of Section 3.10.1 of Prudnikov et al. (1992), which states that the Laplace transform of the piecewise function

$$f(t) = \begin{cases} \frac{\Gamma(v)}{\Gamma(\mu)} t^{\mu-1} & 0 < t < a, \\ \\ \frac{a^v t^{\mu-v-1}}{v\Gamma(\mu-v)} {}_2F_1\left(v, v-\mu+1; v+1; \frac{a}{t}\right) & a < t, \end{cases}$$

is $\mathcal{L}\{f(t)\} = s^{-\mu}\gamma(v, as)$ for $\mathbb{R}(v-\mu) < 1$, $\mathbb{R}(\mu, a)$, and $\mathbb{R}(s) > 0$, inverting (3.3.7) yields

$$\psi(x) = \begin{cases} C_2 \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} + C_1 x^{*\left(\frac{\lambda}{r}-\alpha\right)} \frac{\Gamma(\alpha-\frac{\lambda}{r})}{\Gamma(\alpha)} (-1)^{1-\alpha} & 0 < x < x^*, \\ C_2 \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} + C_1 \frac{(-1)^{1-\alpha}}{(\alpha-\frac{\lambda}{r})\Gamma(\frac{\lambda}{r})} {}_2F_1\left(\alpha - \frac{\lambda}{r}, 1 - \frac{\lambda}{r}; \alpha - \frac{\lambda}{r} + 1; \frac{x^*}{x}\right) x^{\frac{\lambda}{r}-\alpha} & x^* < x, \end{cases}$$

where we have used the fact that $\frac{d}{ds}F(s) = -\mathcal{L}\{x\psi(x)\}$ to prove by induction that $\frac{d^n}{ds^n}F(s) = (-1)^n \mathcal{L}\{x^n \psi(x)\}$.

From the boundary conditions for $\psi(x)$ in (3.2.8), we have

$$C_2 = 0 \quad \text{and} \quad C_1 = \frac{\Gamma(\alpha)}{(-1)^{1-\alpha}\Gamma(\alpha - \frac{\lambda}{r})} x^{*(\alpha-\frac{\lambda}{r})}$$

such that (3.3.3) holds.

□

**Corollary 3.3.1.** *The closed-form trapping probability in* (3.3.3) *is equivalent to*

$$\psi(x) = 1 - \frac{\Gamma(\alpha)}{\Gamma\left(1+\frac{\lambda}{r}\right)\Gamma\left(\alpha-\frac{\lambda}{r}\right)} \left(1 - y(x)^{-1}\right)^{\frac{\lambda}{r}} {}_2F_1\left(\frac{\lambda}{r}, 1+\frac{\lambda}{r}-\alpha; 1+\frac{\lambda}{r}; 1-y(x)^{-1}\right), \tag{3.3.8}$$

*where* $y(x) = \frac{x}{x^*}$ *and* ${}_2F_1(\cdot)$ *is Gauss's Hypergeometric Function, for* $\alpha > \frac{\lambda}{r}$.

*Proof.* First, we apply the following hypergeometric transform

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} z^{-a} {}_2F_1\left(a, a-c+1; a+b-c+1; 1-\frac{1}{z}\right)$$

$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}z^{a-c}{}_2F_1\left(c-a,1-a;c-a-b+1;1-\frac{1}{z}\right),$$

which holds for $|\arg z| < \pi$ and $|\arg(1-z)| < \pi$ (see, for instance, Equation (15.3.9) from Abramowitz and Stegun (1972)), to (3.3.3), where we extend the gamma function to negative non-integer values by the relation

$$\Gamma(x+1) = x\Gamma(x),$$

(see, for instance, Equation (6.1.15) from Abramowitz and Stegun (1972)) for $x < 0, x \notin \mathbb{Z}$. Then, applying the relation

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z)$$

(see, for example, Equation (15.3.3) from Abramowitz and Stegun (1972)) and transforming via the formula

$${}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right) \tag{3.3.9}$$

(see, for example, Equation (15.3.4) from Abramowitz and Stegun (1972)) yields (3.3.8).

□

*Remark* 3.3.1. Substitution of $\alpha = 1$ into (3.3.3), or equivalently (3.3.8), yields the closed-form trapping probability under uniformly distributed remaining proportions of capital, i.e. the case $Z_i \sim Beta(1,1)$.

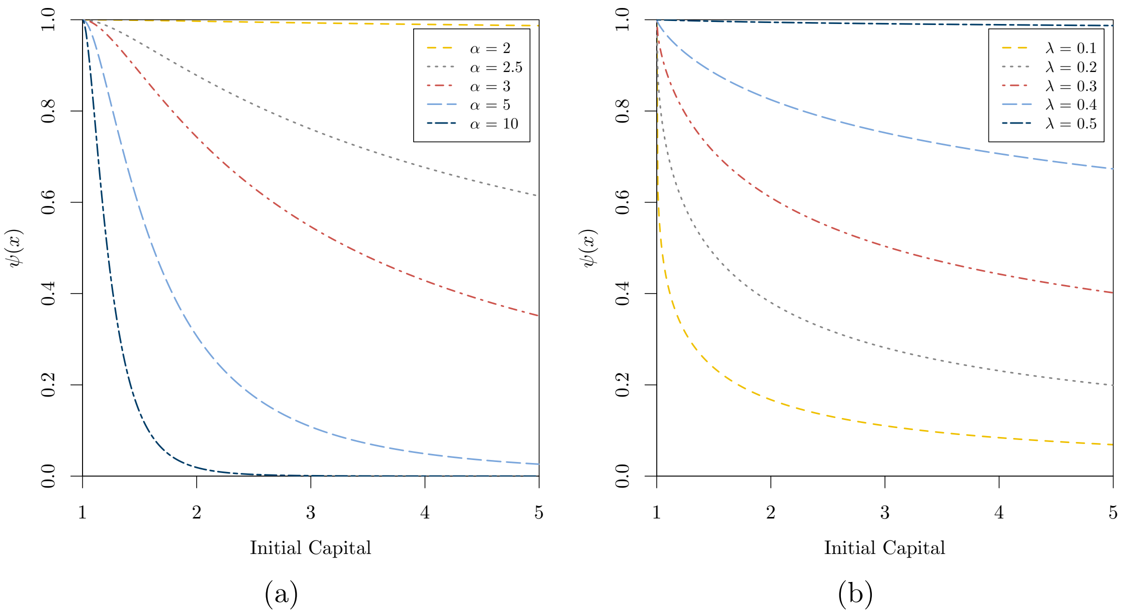



Figure 3.1: (a) Trapping probability $\psi(x)$ when $Z_i \sim Beta(\alpha, 1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$ and $x^* = 1$ for $\alpha = 2, 2.5, 3, 5, 10$ (b) Trapping probability $\psi(x)$ when $Z_i \sim Beta(1,1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$ and $x^* = 1$ for $\lambda = 0.1, 0.2, 0.3, 0.4, 0.5$.

The closed-form trapping probability for households susceptible to proportional losses with $Beta(\alpha, 1)$ distributed remaining proportions of capital, as derived in

Proposition 3.3.2, is presented in Figure 3.1a for varying initial capital $x$ and shape parameter $\alpha$. Note that the trapping probability tends to one as $\lambda/r$ tends to $\alpha$ in line with the constraint of Proposition 3.3.1. The low value of the rate parameter $\lambda$ reflects the vulnerability of low-income households to both high and low frequency loss events, while aligning with the constraint in Proposition 3.3.1. Increasing $\alpha$ increases the mean of the distribution of the remaining proportion of capital. Observation of a decreasing trapping probability with increasing $\alpha$ is therefore intuitive and aligns with the reduction in loss. Figure 3.1b presents the same trapping probability for varying loss frequency $\lambda$ and fixed $\alpha = 1$. In this case, remaining proportions of capital are uniformly distributed as in Section 3.4. Increasing the frequency of loss events increases the trapping probability, as is to be expected. Parameters $a, b$ and $c$ are selected to correspond with those in Flores-Contró et al. (2021).

Particularly, high levels of accumulated capital are not relevant in the microinsurance and poverty trapping context. However, the asymptotic behaviour of the analytic trapping probability at infinity is interesting for understanding the behaviour of the function. Since $\lim_{z\to 0} {}_2F_1(a, b; c; z) = 1$ (see, for example, Kristensson (2010)), (3.3.3) behaves asymptotically like the power function

$$\frac{\Gamma(\alpha)}{\left(\alpha - \frac{\lambda}{r}\right)\Gamma\left(\alpha - \frac{\lambda}{r}\right)\Gamma\left(\frac{\lambda}{r}\right)} y(x)^{\frac{\lambda}{r} - \alpha}, \tag{3.3.10}$$

such that the uninsured trapping probability has power-law asymptotic decay as $x \to \infty$.

We now compare the decay of the household-level trapping probability under proportional losses and no insurance coverage with that of the exponentially distributed random-valued loss case of Flores-Contró et al. (2021). The equivalent uninsured trapping probability under random-valued losses for $x \geq x^*$ is given by

$$\psi^{\text{EXPONENTIAL}}(x) = \frac{\Gamma\left(\frac{\lambda}{r}; g(x)\right)}{\Gamma\left(\frac{\lambda}{r}\right)}, \tag{3.3.11}$$

where $g(x) = \mu(x - x^*)$, $\mu$ is the exponential loss parameter and $\Gamma(a; z)$ is the upper incomplete gamma function defined as $\Gamma(a; z) := \int_z^\infty e^{-t} t^{a-1} dt$. The probability in (3.3.11) follows

$$g(x)^{\frac{\lambda}{r} - 1} e^{-g(x)} (1 + O(|g(x)|^{-1})), \tag{3.3.12}$$

asymptotically. The limiting behaviour of the ratio of (3.3.12) to (3.3.10) is

$$C x^{\alpha - \frac{\lambda}{r}} (x - x^*)^{\frac{\lambda}{r} - 1} e^{-g(x)} (1 + O(|g(x)|^{-1})),$$

for a constant $C = \left[\mu^{\frac{\lambda}{r} - 1}\left(\alpha - \frac{\lambda}{r}\right)\Gamma\left(\alpha - \frac{\lambda}{r}\right)\Gamma\left(\frac{\lambda}{r}\right) x^{*\frac{\lambda}{r} - \alpha}\right] / \Gamma(\alpha)$. The trapping probability in the random-valued case therefore decays at a faster rate than when a household experiences proportional losses, with the severity of this difference dependent on the parameters of the loss distributions. This result is intuitive, since proportional losses are riskier than random-valued losses at high capital levels due to the non-zero probability of a household losing all (or a high proportion) of its capital. This is particularly severe in the uniform case of Section 3.4, where high and low levels of proportional losses are equally likely. When $\alpha = 1$, the trapping probability in the random-valued case decays exponentially faster than in the proportional case. A comparison of the decay of the trapping probability under proportional losses against that of random-valued losses is provided in Figure 3.2b, where the probabilities are plotted on the logarithmic scale. Here, the slower rate of decay under proportional losses is clearly observable.

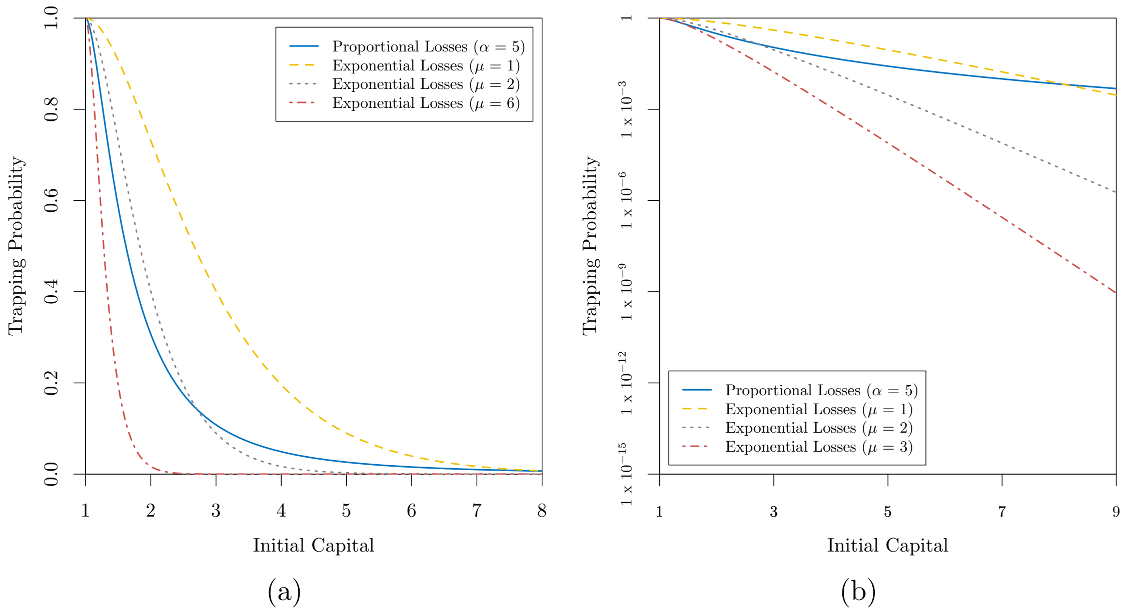



Figure 3.2: (a) Comparison between the trapping probability $\psi^{\text{EXPONENTIAL}}(x)$ in (3.3.11) with $Exp(\mu)-$distributed random-valued losses for $\mu = 1, 2, 6$ and the trapping probability $\psi(x)$ in (3.3.3) with $Beta(5, 1)-$distributed proportional losses where $a = 0.1$, $b = 1.4$, $c = 0.4$, $x^* = 1$ and $\lambda = 1$ (b) The same curves as in Figure 3.2a for $\alpha = 5$ and $\mu = 1, 2, 3$ on the logarithmic scale.

Figure 3.2 compares trapping probabilities under proportional (3.3.3) and random-valued (3.3.11) losses for a given set of parameters. Trapping probabilities for a number of exponential claim size distributions are compared with the trapping probability under proportional losses with an expected value of approximately 16.7% of accumulated capital. For random-valued claim sizes with an expected value of 0.5 ($\mu = 2$) the trapping probability is greater than for proportional losses for the most vulnerable, however, as capital increases the trapping probability under proportional losses exceeds the random-valued case. If the expected claim size increases to one ($\mu = 1$) the trapping probability for proportional losses is significantly lower than in the random-valued case at all levels of initial capital. Compared to the mean loss associated with $Beta(5, 1)-$distributed remaining proportions, an expected claim size of one is low with respect to high levels of initial capital. For $x = 6$ the two loss rates coincide. This therefore suggests that for equivalent loss size, the trapping probability for proportional losses is reduced in comparison to random-valued losses. However, for capital levels below this point random-valued losses account for a greater proportion of capital than the proportional loss case selected for comparison and thus the increased trapping probability is intuitive. Further analysis would be needed to validate the consistency in the reduction of the probability for equivalent losses.

## 3.4 Derivation of Trapping Probability Under Proportional Insurance Coverage

In line with Kovacevic and Pflug (2011) and Flores-Contró et al. (2021), in this section, we extend the model under the assumption that capital losses are covered by a proportional insurance product. Consider the presence of a fixed premium insurance product that covers $100 \cdot (1-\kappa)$ percent of all household losses, where $1-\kappa$ for $\kappa \in (0,1]$ is the proportionality factor. Assume that coverage is purchased by all households. Under proportional insurance coverage, the critical capital (or poverty line) and capital growth rate associated with an insured household must account for the need for premium payments. As such, define

$$r^{(\kappa)} = (1-a) \cdot (b - \pi(\kappa, \theta)) \cdot c \quad \text{and} \quad x^{(\kappa)*} = \frac{I^*}{b - \pi(\kappa, \theta)}, \tag{3.4.1}$$

where $\pi(\kappa, \theta)$ is the premium rate and is calculated according to the expected value principle,

$$\pi(\kappa, \theta) = (1+\theta) \cdot (1-\kappa) \cdot \lambda \cdot \mathbb{E}\left[1 - Z_i\right].$$

Parameters $a, b$ and $c$ are household rates of consumption, income generation and investment or savings as defined in Section 3.2 and the parameter $\theta$ is the loading factor specified by the insurer. We assume that these parameters, and the critical income $I^*$, are not changed by the introduction of insurance. However, due to the need for premium payments, the critical capital in the insured case is greater than that of an uninsured household, while the capital growth rate is reduced.

The associated capital risk process covered by an insurance policy is denoted by $X_t^{(\kappa)}$ and has an analogous structure to that of Definition 3.2.1, with the remaining proportion of capital after each loss event instead denoted $Y_i$, where $Y_i = 1 - \kappa(1 - Z_i) \in [1-\kappa, 1]$. As such, in between loss events, where $T_{i-1} \leq t < T_i$, the capital growth process follows (3.2.6). At event times $t = T_i$, the process is given by

$$X_t^{(\kappa)} = \begin{cases} \left[\left(X_{T_{i-1}}^{(\kappa)} - x^{(\kappa)*}\right) e^{r^{(\kappa)}(T_i - T_{i-1})} + x^{(\kappa)*}\right] \cdot Y_i & \text{if } X_{T_{i-1}}^{(\kappa)} > x^{(\kappa)*}, \\ X_{T_{i-1}}^{(\kappa)} \cdot Y_i & \text{otherwise.} \end{cases} \tag{3.4.2}$$

Note that for $\kappa = 1$, the capital model in (3.4.2) and the parameters $r^{(\kappa)}$ and $x^{(\kappa)*}$ exactly correspond to those of an uninsured household, as discussed in Section 3.3.

**Proposition 3.4.1.** *Consider a household capital process defined by (3.2.6) in between loss events and by (3.4.2) at loss event times, with coverage proportionality factor $1-\kappa \in (0,1]$. For initial capital $x \geq x^{(\kappa)*}$, capital growth rate $r^{(\kappa)}$, loss intensity $\lambda > 0$ and remaining proportions of capital $Z_i$ with distribution Beta$(\alpha, 1)$, the adjustment coefficient of the corresponding Lundberg equation exists if*

$$\frac{r^{(\kappa)}}{\lambda} > \frac{\kappa}{(\alpha+1)(1-\kappa)} {}_2F_1\left(1, \alpha+1; \alpha+2; -\frac{\kappa}{1-\kappa}\right), \tag{3.4.3}$$

*where ${}_2F_1(\cdot)$ is Gauss's Hypergeometric Function.*

*Proof.* The condition that must hold for the adjustment coefficient $R$ to exist under proportional insurance coverage, and thus for the net profit condition to be satisfied, is

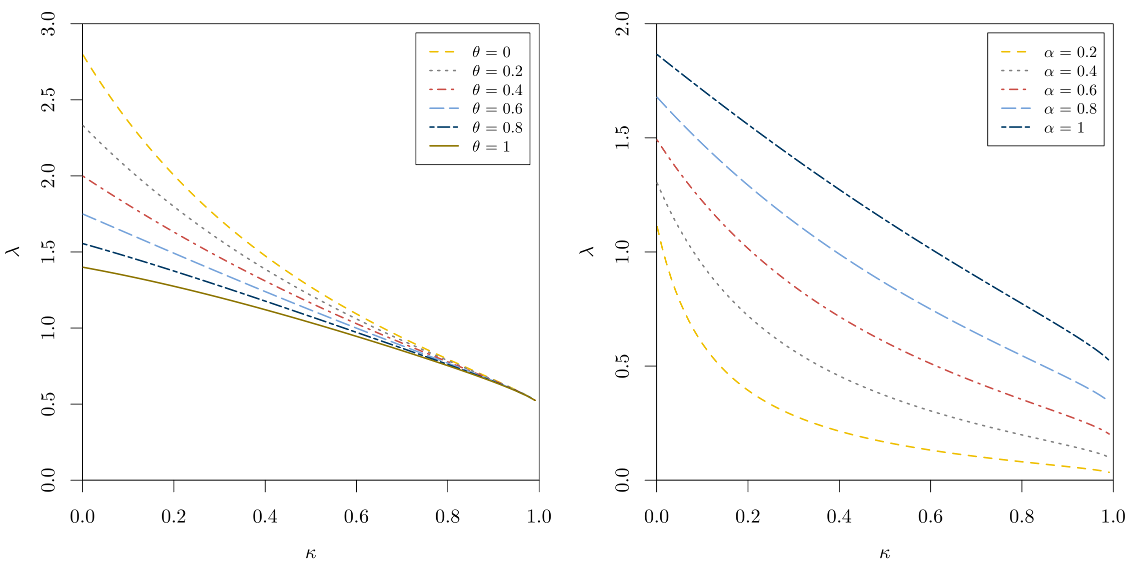
$$\mathbb{E}[r^{(\kappa)}\tilde{T}_i + \log(1-\kappa(1-Z_i))] > 0 \iff \mathbb{E}[\log(1-\kappa(1-Z_i))] > -\frac{r^{(\kappa)}}{\lambda}.$$

For $Z_i \sim Beta(\alpha, 1)$, using integration by parts,

$$\mathbb{E}[\log(1-\kappa(1-Z_i))] = -\kappa \int_0^1 (1-\kappa+\kappa z)^{-1} z^{\alpha} dz,$$

the right-hand side of which is the integral representation of Gauss's Hypergeometric Function (see, for instance, Equation (15.3.1) from Abramowitz and Stegun (1972)), giving exactly (3.4.3), as required.

$\square$




(a) (b)

Figure 3.3: Upper boundary of the region defined by the constraint on $\frac{\lambda}{r^{(\kappa)}}$ in (3.4.4) for $a = 0.1$, $b = 1.4$, $c = 0.4$ with (a) fixed $\alpha = 1$ and different values of $\theta$ and (b) fixed $\theta = 0.5$ and different values of $\alpha$.

*Remark* 3.4.1. For $Z_i \sim Beta(1, 1)$, the constraint for existence of the adjustment coefficient reduces to

$$\frac{r^{(\kappa)}}{\lambda} > 1 + \frac{1-\kappa}{\kappa}\log(1-\kappa). \tag{3.4.4}$$

The constraint on $\lambda$ in (3.4.4) is presented in Figure 3.3a for varying $\theta$ and Figure 3.3b for varying $\alpha$. Note that the sensitivity of the constraint to the loading factor $\theta$ increases for decreasing $\kappa$ and thus increasing insurance coverage. In the experiments considered in Figure 3.3b, the constraint is bounded above by the uniform case, where $\alpha = 1$. This indicates that the parameter region in which certain trapping is prevented is greater for uniformly distributed remaining proportions of capital. In a similar manner, Figure 3.3a implies that lowering the loading factor $\theta$ increases the region in which certain trapping is prevented when remaining proportions are uniformly distributed.

*Remark* 3.4.2. For $\kappa = 1$, since

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad (c \neq 0,-1,-2,..., \mathbb{R}\,(c-a-b)>0)\,, $$

(see, for example, Equation (15.1.20) of Abramowitz and Stegun (1972)), applying the identity (3.3.9) reduces (3.4.3) to the uninsured constraint in (3.3.2).

We approach the derivation of the trapping probability of the insured process in a manner analogous to that described in Section 3.3, noting the adjustment in the domain of the random variable capturing the remaining proportion of capital. The infinitesimal generator corresponding to the capital process in (3.2.6) and (3.4.2) is given by

$$ \left(\mathcal{A}\psi^{(\kappa)}\right)(x) = r^{(\kappa)}(x - x^{(\kappa)*})\psi'^{(\kappa)}(x) + \lambda \int_{1-\kappa}^{1} [\psi^{(\kappa)}(x \cdot y) - \psi^{(\kappa)}(x)] dG_Y(y), \quad (3.4.5) $$

where $\psi^{(\kappa)}(x)$ is the trapping probability under proportional insurance coverage and $G_Y(y) = G_Z\left(1 - (1-y)/\kappa\right)$ is the distribution function of $Y_i$. To derive $\psi^{(\kappa)}(x)$ we consider only the case $\alpha = 1$, i.e. $Z_i \sim Beta(1,1)$, where remaining proportions of capital are uniformly distributed and $dG_Y(y) = \frac{dy}{\kappa}$.

Solution of $\left(\mathcal{A}\psi^{(\kappa)}\right)(x) = 0$ is again sought to obtain the trapping probability of the insured process, where $\psi^{(\kappa)}(x)$ is assumed to be a bounded and twice continuously differentiable function on $x \geq x^{(\kappa)*}$ with a bounded first derivative and boundary conditions as in (3.2.8). Using equivalent arguments to those presented in the discussion of the net profit condition in Section 3.2, if (3.4.4) is satisfied the boundary condition $\lim_{x\to\infty} \psi^{(\kappa)}(x) = 0$ holds. Households face certain trapping if the net profit condition is violated, therefore our analysis focuses only on the region in which (3.4.4) holds.

Taking the derivative of the infinitesimal generator (3.4.5) with $dG_Y(y) = \frac{dy}{\kappa}$, yields

$$ \begin{aligned} r^{(\kappa)}\left(x^2 - x^{(\kappa)*}x\right)\psi''^{(\kappa)}(x) &+ \left[\left(2r^{(\kappa)} - \lambda\right)x - rx^{(\kappa)*}\right]\psi'^{(\kappa)}(x) \\ &+ \frac{\lambda\,(1-\kappa)}{\kappa}\psi^{(\kappa)}(x) - \frac{\lambda(1-\kappa)}{\kappa}\psi^{(\kappa)}\left(\left(1-\kappa\right)x\right) = 0. \end{aligned} \quad (3.4.6) $$

As such, even in the simple case of uniformly distributed remaining proportions of capital, application of the differential operator induces a non-local term in the resulting differential equation (3.4.6). When taking the Laplace transform of $\left(\mathcal{A}\psi^{(\kappa)}\right)(x) = 0$, as in Section 3.3, a non-local differential equation is also obtained. Derivation of the trapping probability is therefore highly intractable when adopting classical approaches.

The non-locality is caused by the lower integral limit in (3.4.5). To overcome this, consider the following. If $y$ is such that $x \cdot y \leq x^{(\kappa)*}$ then $\psi^{(\kappa)}(x \cdot y)$ is known.

In fact, for all $y \in [1-\kappa, x^{(\kappa)*}/x]$ trapping occurs with the first loss, such that $\psi^{(\kappa)}(x \cdot y) = 1$. For $y$ in this interval, the integral in (3.4.5) is trivial. Exploiting this observation, we redefine the infinitesimal generator as a piecewise function with boundary at $x = x^{(\kappa)*}/\left(1-\kappa\right)$, where $1-\kappa$ is the lower bound of $Y_i$. In this way, for $x(1-\kappa) > x^{(\kappa)*}$ a household cannot become trapped by the first loss for any realisation of $Y_i$. We therefore obtain a piecewise IDE that can be solved in a standard manner for $x(1-\kappa) < x^{(\kappa)*}$, but for $x(1-\kappa) > x^{(\kappa)*}$ the problem of non-locality remains. Our approach, as described below, partitions the domain of $\psi^{(\kappa)}(x)$ into subintervals such that the solution of $\left(\mathcal{A}\psi^{(\kappa)}\right)(x) = 0$ for $x$ in any given subinterval is informed by the solution in the previous subinterval. We begin by considering the two fundamental subintervals, divided where $x = x^{(\kappa)*}/\left(1-\kappa\right)$.

The behaviour of the capital process above the critical capital $x^{(\kappa)*}$ determines a household's trapping probability, with only surplus capital above the critical capital growing exponentially. Thus, additionally consider the change of variable $h(x) = \psi^{(\kappa)}(x + x^{(\kappa)*})$ for $x > 0$. Then, for $\tilde{x} = x - x^{(\kappa)*} > 0$, the piecewise infinitesimal generator $\left(\mathcal{A}h\right)(\tilde{x})$ is given by

$$\begin{cases} r^{(\kappa)}(\tilde{x} + x^{(\kappa)*})\tilde{x}h'(\tilde{x}) - \lambda(\tilde{x} + x^{(\kappa)*})h(\tilde{x}) + \frac{\lambda}{\kappa}\int_{(\tilde{x}+x^{(\kappa)*})(1-\kappa)}^{\tilde{x}+x^{(\kappa)*}} h(u - x^{(\kappa)*})du & \tilde{x} > \frac{x^{(\kappa)*}\kappa}{1-\kappa}, \\ r^{(\kappa)}(\tilde{x} + x^{(\kappa)*})\tilde{x}h'(\tilde{x}) - \lambda(\tilde{x} + x^{(\kappa)*})h(\tilde{x}) + \frac{\lambda}{\kappa}\int_{x^{(\kappa)*}}^{\tilde{x}+x^{(\kappa)*}} h(u - x^{(\kappa)*})du + \lambda x^{(\kappa)*} - \frac{\lambda\tilde{x}(1-\kappa)}{\kappa} & \tilde{x} < \frac{x^{(\kappa)*}\kappa}{1-\kappa}, \end{cases} \tag{3.4.7}$$

where the subintervals on the domain of $\tilde{x}$ have interface at $x = x^{(\kappa)*}/\left(1-\kappa\right)$. Under this change of variable and assuming $r^{(\kappa)}/\lambda$ satisfies (3.4.4), the trapping probability satisfies $\left(\mathcal{A}h\right)(\tilde{x}) = 0$, with boundary conditions,

$$\lim_{\tilde{x}\to 0} h(\tilde{x}) = 1 \quad \text{and} \quad \lim_{\tilde{x}\to\infty} h(\tilde{x}) = 0.$$

For this purpose, we consider the derivative of the piecewise IDE in (3.4.7). Fixing $\left(\mathcal{A}h\right)(\tilde{x}) = 0$ and taking the derivative with respect to $\tilde{x}$ yields

$$\begin{aligned} \tilde{x}(\tilde{x} + x^{(\kappa)*})h''(\tilde{x}) + \left[\left(2 - \frac{\lambda}{r^{(\kappa)}}\right)\tilde{x} + x^{(\kappa)*}\left(1 - \frac{\lambda}{r^{(\kappa)}}\right)\right]h'(\tilde{x}) \\ + \frac{\lambda(1-\kappa)}{r^{(\kappa)}\kappa}h(\tilde{x}) = \frac{\lambda(1-\kappa)}{r^{(\kappa)}\kappa}h((1-\kappa)\tilde{x} - x^{(\kappa)*}\kappa), \end{aligned} \tag{3.4.8}$$

for $\tilde{x} > \left(x^{(\kappa)*}\kappa\right)/\left(1-\kappa\right)$ and

$$\tilde{x}(\tilde{x} + x^{(\kappa)*})h''(\tilde{x}) + \left[\left(2 - \frac{\lambda}{r^{(\kappa)}}\right)\tilde{x} + x^{(\kappa)*}\left(1 - \frac{\lambda}{r^{(\kappa)}}\right)\right]h'(\tilde{x}) + \frac{\lambda(1-\kappa)}{r^{(\kappa)}\kappa}h(\tilde{x}) = \frac{\lambda(1-\kappa)}{r^{(\kappa)}\kappa} \tag{3.4.9}$$

for $\tilde{x} < \left(x^{(\kappa)*}\kappa\right)/\left(1-\kappa\right)$, where, as mentioned, we observe the non-local term $h((1-\kappa)\tilde{x} - x^{(\kappa)*}\kappa)$ for $\tilde{x} > \left(x^{(\kappa)*}\kappa\right)/\left(1-\kappa\right)$.

First consider the homogeneous parts of (3.4.8) and (3.4.9), noting their equivalence. Letting $f(z) := h(\tilde{x})$, such that $z$ is associated with the change of variable

$z := z(\tilde{x}) = -\tilde{x}/x^{(\kappa)*}$, the homogeneous differential equation reduces to Gauss's Hypergeometric Differential Equation (Slater, 1960)

$$z(1-z)\cdot f''(z) + [c_1 - (1+a_1+b_1)z]f'(z) - a_1 b_1 f(z) = 0, \tag{3.4.10}$$

for $a_1 = \frac{1}{2}\left(1 - \frac{\lambda}{r^{(\kappa)}}\right) - \frac{1}{2}\sqrt{\left(1+\frac{\lambda}{r^{(\kappa)}}\right)^2 - \frac{4\lambda}{r^{(\kappa)}\kappa}}$, $b_1 = \frac{1}{2}\left(1 - \frac{\lambda}{r^{(\kappa)}}\right) + \frac{1}{2}\sqrt{\left(1+\frac{\lambda}{r^{(\kappa)}}\right)^2 - \frac{4\lambda}{r^{(\kappa)}\kappa}}$ and $c_1 = 1 - \lambda/r^{(\kappa)}$, with regular singular points at $z = 0, 1, \infty$ (corresponding to $\tilde{x} = 0, -x^{(\kappa)*}, -\infty$, respectively). A general solution of (3.4.10) in the neighborhood of the singular point $z = 0$ is given by

$$\begin{aligned} f(z) := h(\tilde{x}) = &C_1 (1-z)^{-a_1} {}_2F_1\left(a_1, c_1 - b_1; c_1; \frac{z}{z-1}\right) \\ &+ C_2 z^{1-c_1} (1-z)^{c_1 - a_1 - 1} {}_2F_1\left(1 + a_1 - c_1, 1 - b_1; 2 - c_1; \frac{z}{z-1}\right), \end{aligned}$$

for arbitrary constants $C_1, C_2 \in \mathbb{R}$ (see for example, Equations (15.5.9) and (15.5.10) of Abramowitz and Stegun (1972)), where ${}_2F_1(\cdot)$ is Gauss's Hypergeometric Function.

Returning to the inhomogeneous differential equations in (3.4.8) and (3.4.9), let

$$\mathscr{L} = r(\tilde{x})\frac{d^2}{d\tilde{x}^2} + p(\tilde{x})\frac{d}{d\tilde{x}} + q(\tilde{x}),$$

where $r(\tilde{x}) = \tilde{x}\left(\tilde{x} + x^{(\kappa)*}\right)$, $p(\tilde{x}) = \left(2 - \lambda/r^{(\kappa)}\right)\tilde{x} + x^{(\kappa)*}\left(1 - \lambda/r^{(\kappa)}\right)$ and $q(\tilde{x}) = [\lambda(1-\kappa)] / \left(r^{(\kappa)}\kappa\right)$, denote the linear, second order operator for which

$$\begin{aligned} u(\tilde{x}) &= \left(1 + \frac{\tilde{x}}{x^{(\kappa)*}}\right)^{-a_1} {}_2F_1\left(a_1, a_1; 1 - \frac{\lambda}{r^{(\kappa)}}; \frac{\tilde{x}}{\tilde{x} + x^{(\kappa)*}}\right), \\ v(\tilde{x}) &= \left(\frac{\tilde{x}}{\tilde{x} + x^{(\kappa)*}}\right)^{\frac{\lambda}{r^{(\kappa)}}} \left(1 + \frac{\tilde{x}}{x^{(\kappa)*}}\right)^{-a_1} {}_2F_1\left(\frac{\lambda}{r^{(\kappa)}} + a_1, \frac{\lambda}{r^{(\kappa)}} + a_1; 1 + \frac{\lambda}{r^{(\kappa)}}; \frac{\tilde{x}}{\tilde{x} + x^{(\kappa)*}}\right), \end{aligned} \tag{3.4.11}$$

forms the fundamental solution set, and let

$$G(\tilde{x}, x') = \frac{u(x')v(\tilde{x}) - u(\tilde{x})v(x')}{r(x')W(x')}, \tag{3.4.12}$$

be the Green's function corresponding to $\mathscr{L}$, where $W(x) = u(x)v'(x) - u'(x)v(x)$ is the Wronskian of $u$ and $v$. The system in (3.4.7) that is to be solved can therefore be characterised as follows

$$(\mathscr{L}h)(\tilde{x}) = \begin{cases} \frac{\lambda(1-\kappa)}{r^{(\kappa)}\kappa} h((1-\kappa)\tilde{x} - x^{(\kappa)*}\kappa) & \tilde{x} > \frac{x^{(\kappa)*}\kappa}{1-\kappa}, \\ \frac{\lambda(1-\kappa)}{r^{(\kappa)}\kappa} & \tilde{x} < \frac{x^{(\kappa)*}\kappa}{1-\kappa}. \end{cases} \tag{3.4.13}$$

Now, let the surplus of capital above the critical capital $\tilde{x} \in [0, \infty)$ be separated into subintervals $I_j = [\tilde{x}_j, \tilde{x}_{j+1}]$, where $\{\tilde{x}_j\}_{j \in \mathbb{N}_0}$ is an increasing sequence and $\tilde{x}_0 = 0$. Moreover, defining a set of recursive kernels by

$$\begin{cases} g_1(x, s_1) = G(x, s_1), \\ g_{j+1}(x, s_1, ..., s_{j+1}) = G(x, s_{j+1}) g_j(\ell(s_{j+1}), s_1, ..., s_j), \end{cases}$$

for $j \geq 1$. Then, the following theorem holds, where the proposition of a solution of the type (3.4.14) is informed by the solution of (3.4.9).

**Theorem 3.4.1.** *Consider a household capital process defined by (3.2.6) in between loss events and by (3.4.2) at loss event times, with coverage proportionality factor $1-\kappa \in (0,1]$. Assume initial capital $x$ such that $\tilde{x} \geq 0$, capital growth rate $r^{(\kappa)}$ and loss intensity $\lambda > 0$ such that $\lambda / r^{(\kappa)}$ satisfies (3.4.4), and remaining proportions of capital with distribution Beta$(1,1)$. Then, a solution of $(\mathcal{A}h)(\tilde{x}) = 0$ for the infinitesimal generator $(\mathcal{A}h)(\tilde{x})$ in (3.4.7), that satisfies $\lim_{\tilde{x}\to 0} h(\tilde{x}) = 1$, is given by the piecewise function*

$$h(\tilde{x}) = 1 + Ay_j(\tilde{x}), \quad \tilde{x} \in I_j, \tag{3.4.14}$$

*for any constant $A$, where the functions $y_j(\tilde{x})$ are defined for $\tilde{x} \geq \tilde{x}_j$ and are given by the recursion*

$$\begin{cases} y_0(\tilde{x}) = v(\tilde{x}), \\ \\ y_{j+1}(\tilde{x}) = y_j(\tilde{x}) + c^{j+1} \int_{\tilde{x}_{j+1}}^{\tilde{x}} \int_{\tilde{x}_j}^{\ell(s_{j+1})} \cdots \int_{\tilde{x}_1}^{\ell(s_2)} g_{j+1}(\tilde{x}, s_1, .., s_{j+1}) v(\ell(s_1)) ds_1 \cdots ds_{j+1}, \end{cases} \tag{3.4.15}$$

*where $c = [\lambda(1-\kappa)] / (r^{(\kappa)}\kappa)$, $\tilde{x}_{j+1} = (\tilde{x}_j + x^{(\kappa)*}\kappa) / (1-\kappa)$ and $\ell(x) = (1-\kappa)x - x^{(\kappa)*}\kappa$.*

*Proof.* First consider the IDE for the solution in the first interval $I_0 = [\tilde{x}_0, \tilde{x}_1]$ given in (3.4.7) for $\tilde{x} < (x^{(\kappa)*}\kappa) / (1-\kappa)$, where we define $\tilde{x}_0$ and $\tilde{x}_1$ to be the lower and upper limits of the first interval, namely 0 and $(x^{(\kappa)*}\kappa) / (1-\kappa)$, respectively. Proposing an Ansatz $h_p(\tilde{x}) = C$ for the particular solution yields $C = 1$, such that the general solution of $h(\tilde{x})$ for $\tilde{x} \in I_0$ is exactly

$$h(\tilde{x}) = C_1 u(\tilde{x}) + C_2 v(\tilde{x}) + 1.$$

The lower boundary condition for $h(\tilde{x})$ in this interval, given by $\lim_{\tilde{x}\to 0} h(\tilde{x}) = 1$, then holds if and only if $C_1 = 0$. Letting $A = C_2$ and $y_0(\tilde{x}) = v(\tilde{x})$, $h(\tilde{x}) = 1 + Ay_0(\tilde{x})$ for $\tilde{x} \in I_0$, as required.

To solve in the upper part of the infinitesimal generator IDE, i.e. for intervals $I_j = [\tilde{x}_j, \tilde{x}_{j+1}]$ where $j \geq 1$, consider (3.4.7) for $\tilde{x} > (x^{(\kappa)*}\kappa) / (1-\kappa)$. By the solution in the interval $I_0$, $h((1-\kappa)\tilde{x} - x^{(\kappa)*}\kappa)$ is known where

$$\tilde{x}_0 < (1-\kappa)\tilde{x} - x^{(\kappa)*}\kappa < \tilde{x}_1 \iff \tilde{x}_1 < \tilde{x} < \frac{\tilde{x}_1 + x^{(\kappa)*}\kappa}{1-\kappa}.$$

As such, letting $\tilde{x}_2 := (\tilde{x}_1 + x^{(\kappa)*}\kappa) / (1-\kappa)$, a solution for (3.4.7) for $\tilde{x} > (x^{(\kappa)*}\kappa) / (1-\kappa)$ can be obtained in the interval $I_1 = [\tilde{x}_1, \tilde{x}_2]$. In fact, for any interval $I_{j+1}$, a solution can be determined by observing the value of the function in the previous interval, since $h((1-\kappa)\tilde{x} - x^{(\kappa)*}\kappa)$ for $\tilde{x} > \tilde{x}_{j+1}$ is known, up to a point, by the solution in $I_j$. It is simple to prove by induction that the upper limit of the $j$-th interval is given by

$$\tilde{x}_{j+1} = \frac{\tilde{x}_j + x^{(\kappa)*}\kappa}{1-\kappa}. \tag{3.4.16}$$

Suppose that for all $\tilde{x} \in I_j$ for $j \geq 1$, $\tilde{y}_j(\tilde{x}) = h(\tilde{x}) = 1 + Ay_j(\tilde{x})$. Then, by (3.4.13) for $\tilde{x} > (x^{(\kappa)*}\kappa) / (1-\kappa)$, it must hold that

$$\left(\mathscr{L}\tilde{y}_{j+1}\right)(\tilde{x}) = \frac{\lambda(1-\kappa)}{r^{(\kappa)}\kappa}\tilde{y}_j((1-\kappa)\tilde{x} - x^{(\kappa)*}\kappa) \iff \left(\mathscr{L}y_{j+1}\right)(\tilde{x}) = cy_j(\ell(\tilde{x})), \tag{3.4.17}$$

for all $\tilde{x} \geq \tilde{x}_{j+1}$, denoting $c = [\lambda(1-\kappa)] / (r^{(\kappa)}\kappa)$ and $\ell(x) = (1-\kappa)x - x^{(\kappa)*}\kappa$. It therefore remains to prove that (3.4.17) holds when $y_{j+1}(\tilde{x})$ is given by the recursion in (3.4.15). To prove by induction, consider the case $j = 0$,

$$\left(\mathscr{L}y_1\right)(\tilde{x}) = \mathscr{L}\left[y_0(\tilde{x}) + c\int_{\tilde{x}_1}^{\tilde{x}} G(\tilde{x}, s_1)v(\ell(s_1))ds_1\right].$$

By definition, $\left(\mathscr{L}y_0\right)(\tilde{x}) = 0$ when $y_0$ is in the solution set and $\mathscr{L}\left[\int^{\tilde{x}} G(\tilde{x}, s)\phi(s)ds\right] = \phi(\tilde{x})$. As such,

$$\left(\mathscr{L}y_1\right)(\tilde{x}) = cv(\ell(\tilde{x})) = cy_0(\ell(\tilde{x})),$$

as required. Assume (3.4.17) holds for $j = k-1$. Then, $\left(\mathscr{L}y_k\right)(\tilde{x}) = cy_{k-1}(\ell(\tilde{x}))$ for $\tilde{x} \geq \tilde{x}_k$. Finally, consider the case $j = k$. By (3.4.15),

$$\begin{aligned}&\left(\mathscr{L}y_{k+1}\right)(\tilde{x}) = cy_{k-1}(\ell(\tilde{x})) \\ &+ c^{k+1}\mathscr{L}\left[\int_{\tilde{x}_{k+1}}^{\tilde{x}} G(\tilde{x}, s_{k+1})\int_{\tilde{x}_k}^{\ell(s_{k+1})}\cdots\int_{\tilde{x}_1}^{\ell(s_2)} g_k(\ell(s_{k+1}), s_1, .., s_k)v(\ell(s_1))ds_1\cdots ds_{k+1}\right],\end{aligned}$$

which, by definition of the Green's function, is equivalent to

$$\begin{aligned}\left(\mathscr{L}y_{k+1}\right)(\tilde{x}) &= cy_{k-1}(\ell(\tilde{x})) \\ &+ c^{k+1}\int_{\tilde{x}_k}^{\ell(\tilde{x})}\cdots\int_{\tilde{x}_1}^{\ell(s_2)} g_k(\ell(\tilde{x}), s_1, .., s_k)v(\ell(s_1))ds_1\cdots ds_k = cy_k(\ell(\tilde{x})),\end{aligned}$$

as required.

□

*Remark* 3.4.3. For $\kappa = 1$, since $\lim_{\kappa\to 1}(x^{(\kappa)*}\kappa) / (1-\kappa) = \infty$, the upper limit of the first subinterval $\tilde{x}_1 = \infty$. The IDE in (3.4.7) for $\tilde{x} < (x^{(\kappa)*}\kappa) / (1-\kappa)$ therefore holds over the whole domain $\tilde{x} > 0$ and the solution in Theorem 3.4.1 reduces to $h(\tilde{x}) = 1 + Av(\tilde{x})$, the solution in the first interval $I_0$. In this case, the constant $A$ can be derived analytically such that the upper boundary condition on the trapping probability, given by $\lim_{\tilde{x}\to\infty} h(\tilde{x}) = 0$, holds. The resulting trapping probability is exactly that of the uninsured case in (3.3.3) of Proposition 3.3.2 (or equivalently, in (3.3.8) of Corollary 3.3.1).

The characterisation of the trapping probability $\psi^{(\kappa)}(x)$ satisfying (3.4.5) in the case of $Beta(1,1)-$ distributed proportional losses will follow from Theorem 3.4.1 if it can be shown that a solution of the form (3.4.14) tends to zero as $\tilde{x} \to \infty$, in line with the upper boundary condition. Specifically, we define the piecewise function

$$y(x) = y_j(x - x^{(\kappa)*}), \qquad \textit{where} \quad x - x^{(\kappa)*} \in I_j, \tag{3.4.18}$$

with $y_j$ and $I_j$ as in Theorem 3.4.1, and pose the following conjecture

**Conjecture 3.4.1.** *The limit* $L := \lim_{x\to\infty} y(x)$ *exists and is different from zero.*

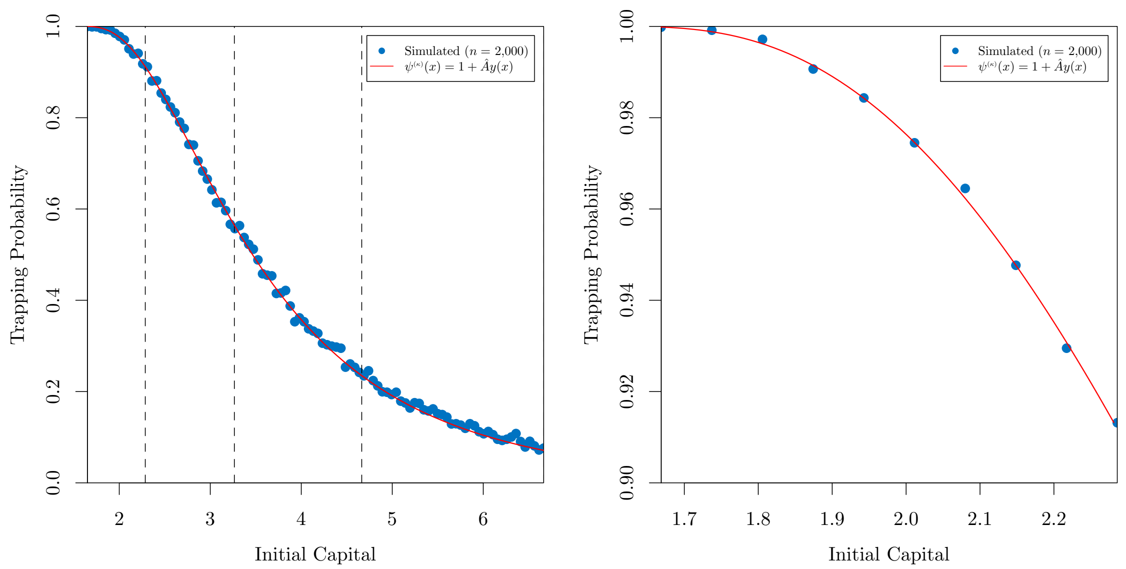



Figure 3.4: (a) Comparison between the trapping probability estimated via $\psi^{(\kappa)}(x)$ in (3.4.19) and simulations of the capital process $X_t$. Each simulation point is obtained from an ensemble of 2,000 realisations of $\{X_t : 0 \le t \le 500\}$ for different values of the initial capital $X_0 = x$. The vertical lines mark the subintervals $x^{(\kappa)*} + I_j$, $0 \le j \le 3$ used in the construction of $y$ in Theorem 3.4.1 (b) The estimate $\hat{A} = -3.556$ is obtained by fitting $1 + Av(x + x^{(\kappa)*})$ to the simulated data for $x$ in the first subinterval. Parameters used are $\lambda = 1$ and $\kappa = 0.3$. The values of $r^{(\kappa)}$ and $x^{(\kappa)*}$ are computed via (3.4.1) with $a = 0.1$, $b = 1.4$, $c = 0.4$, $x^* = 1$ and $\theta = 0.5$.

If Conjecture 3.4.1 holds, then (3.4.14) yields that

$$\psi^{(\kappa)}(x) = 1 - \frac{y(x)}{L}, \tag{3.4.19}$$

is the unique solution to $\left(\mathcal{A}\psi^{(\kappa)}\right)(x) = 0$, $\psi^{(\kappa)}(x^{(\kappa)*+}) = 1$, and $\lim_{x\to\infty} \psi^{(\kappa)}(x) = 0$, as desired. Numerical computation of $y(x)$ in (3.4.18) for large $x$ is not a trivial matter, as the functions $v$ and $G$ in (3.4.11) and (3.4.12), respectively, are highly oscillatory for large values of $\tilde{x}$. Nevertheless, our numerical experiments appear to indicate that Conjecture 3.4.1 holds. Moreover, if Conjecture 3.4.1 is assumed to hold, there exists a practical method for estimating the true value of $A$ in (3.4.14) and for obtaining a very good approximation to $\psi^{(\kappa)}(x)$.

Note that, by (3.4.15)

$$\psi^{(\kappa)}(x) = 1 + Av(x + x^{(\kappa)*}), \quad x \in \left[x^{(\kappa)*}, \frac{x^{(\kappa)*}}{1-\kappa}\right], \tag{3.4.20}$$

which is easily computed for any value of $A$. In addition, the process $X_t$ can be simulated to obtain estimates of the trapping probability for any initial capital

$x \in [x^{(\kappa)*}, x^{(\kappa)*}/(1-\kappa)]$. As such, an estimate $\hat{A}$ for the conjectured value of $A$ can be estimated by fitting $\psi^{(\kappa)}(x)$ to the simulated data. A comparison between the trapping probability estimated via $\psi^{(\kappa)}(x)$ in (3.4.19) and simulated data is presented in Figure 3.4 for a given set of parameters.

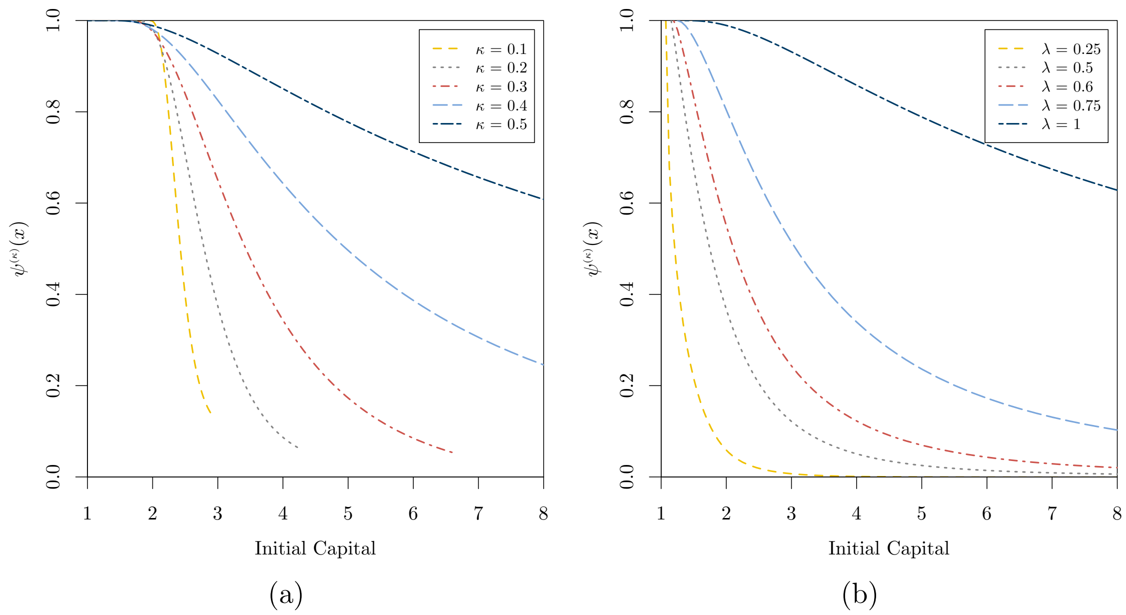



Figure 3.5: Estimation of the trapping probability $\psi^{(\kappa)}(x)$ via (3.4.19) assuming Conjecture 3.4.1 for (a) $\lambda = 1$ for different values of $\kappa$ and (b) $\kappa = 0.5$ for different values of $\lambda$. Each curve is computed with the first three iterates of (3.4.15) via numerical integration, the value of $A$ is then estimated as explained in Figure 3.4. For each case, the values of $r^{(\kappa)}$ and $x^{(\kappa)*}$ are computed via (3.4.1) with $a = 0.1$, $b = 1.4$, $c = 0.4$, $x^* = 1$ and $\theta = 0.5$ and $\lambda$ is selected such that (3.4.4) holds.

The trapping probability for proportionally insured households susceptible to proportional losses with $Beta(1, 1)-$distributed remaining proportions of capital, estimated via (3.4.19), is presented in Figure 3.5a for varying initial capital $x$ and proportionality factor $\kappa$. For small values of $\kappa$ and at higher subintervals, calculation of the trapping probability is highly computationally intensive. In Figures 3.5a and 3.5b, trapping probabilities are estimated for the first four subintervals, i.e. $I_j$ for $0 \le j \le 3$. The limits of $I_j$ in (3.4.16) are functions of $\kappa$. As such, changing the value of $\kappa$ causes the trapping probability curves to terminate at different points, determined by the upper limit of $I_3$, as can be observed in Figure 3.5a.

Note that, in Figure 3.5a, as $\kappa$ tends to zero the trapping probability tends towards a step function. This is indicative of the fact that for $\kappa = 0$ households have full insurance coverage and do not experience loss events, inducing a trapping probability that is zero-valued for all levels of capital above the critical capital due to the restriction on the premium that ensures positive capital growth. Increasing $\kappa$ and thus decreasing the level of insurance coverage intuitively causes an increase in the trapping probability. Figure 3.5b presents the same trapping probability for varying loss frequency $\lambda$ and fixed $\kappa$, where half of every loss is insured. Increasing the frequency of loss events increases the trapping probability. For $\lambda = 0.5$, under the parameter set considered in this figure, $\lambda/r$ is extremely close to one. Therefore, in the case of no insurance, households exhibiting this loss behaviour would be close to certain ruin. As presented in Figure 3.3, purchase of insurance eases this constraint, significantly reducing the probability of trapping. The fact that both figures presenting the estimated trapping probability are intuitive, provides further evidence for Conjecture 3.4.1.

## 3.5 Discussion

Figure 3.6 presents a comparison of trapping probabilities for the uninsured and insured capital processes as derived in (3.3.3) and (3.4.19), respectively, for two values of the parameter $\lambda$. For $\lambda = 0.25$, the insured trapping probability lies below the uninsured at almost all levels of initial capital, decaying at a much faster rate. Only for initial capital extremely close to the critical capital does the uninsured probability lie below the insured. At the higher loss frequency of $\lambda = 0.5$, the uninsured trapping probability lies close to one throughout the range of initial capital considered, significantly higher than the equivalent probability for insured losses at all capital levels. Note that in this case, $\lambda/r$ lies close to the uninsured constraint preventing certain trapping in (3.3.2).

Sensitivity analysis on the trapping probabilities in (3.3.3) and (3.4.19) is presented in Figure 3.7 for low levels of initial capital and varying $\kappa$ and $\lambda$. Specifically, trapping probabilities for households with capital between $x = x^*$, the uninsured poverty line, and $x = x^{(\kappa)*}/(1-\kappa)$, the upper limit of the first subinterval $I_0$, corresponding to the trapping probability in $I_0$ given in (3.4.20), are presented. At this more granular level, the intersection point of the curves can be observed more clearly. This intersection point indicates when proportional insurance coverage is beneficial for reducing poverty trapping. In the estimation of the insured trapping probability, the increase in critical capital associated with the need for premium payment is accounted for through specification of $x^{(\kappa)*}$, where an insured household is deemed to be trapped when their capital falls below $I^*/(b-\pi(\kappa,\theta))$, where the critical income $I^* = b$ under the assumption of no change in the basic model parameters due to the purchase of insurance. Thus, in the insured case, households with initial capital slightly above $x^*$ have already become trapped.

As in Kovacevic and Pflug (2011) and Flores-Contró et al. (2021) the increase in the trapping probabilities of the most vulnerable households when proportionally insured is observed in all cases considered. However, importantly, this increase occurs for a much smaller proportion of the low-income sample. Denoting the intersection point of the uninsured and insured trapping probabilities by $x_c$, the significance of the distance between the intersection point and the critical capital $x^*$ is presented in Figure 3.8 for varying $\kappa$ and $\lambda$. Considering four levels of the loading factor $\theta$, the distance is positive under all sets of parameters tested. The depiction of $x_c - x^*$ in these figures highlights that the level of capital at which insurance becomes beneficial lies much closer to the poverty line than for more extreme (Kovacevic and Pflug, 2011) and random-valued losses (Flores-Contró et al., 2021), with only small distances between the intersection point and the critical capital observed. These results suggest that purchase of proportional insurance for proportional losses is beneficial for a larger proportion of those closest to the poverty line. In particular, proportional coverage appears to be more affordable than classical coverage for random-valued losses.

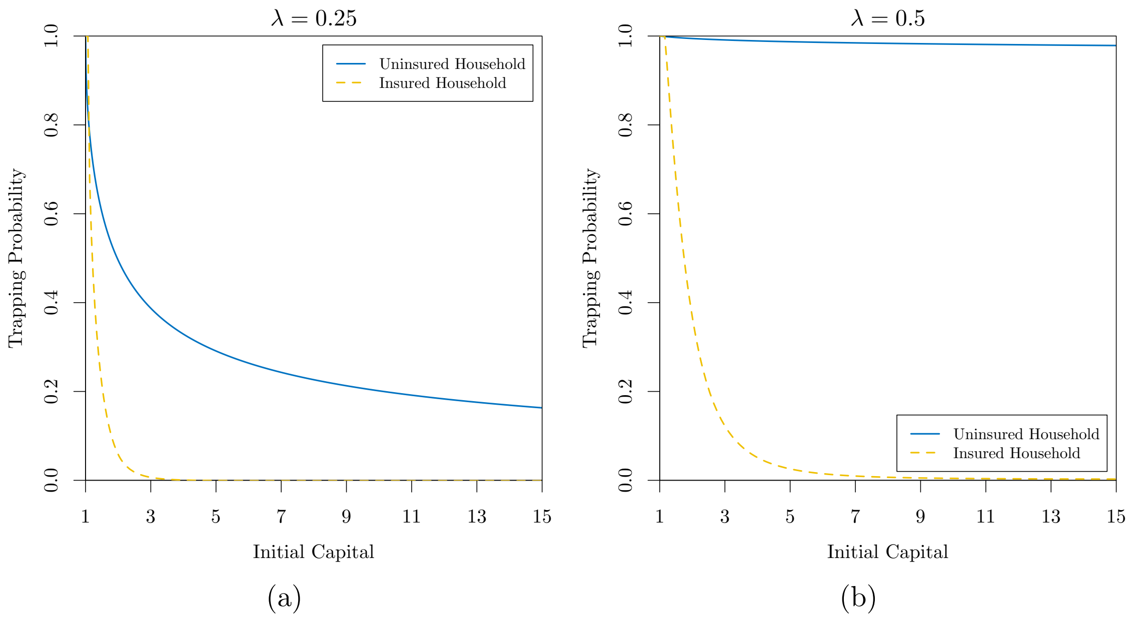



Figure 3.6: Comparison between the trapping probabilities of uninsured and insured households for $\kappa = 0.5$ and two different values of $\lambda$. Dashed curves are computed via (3.4.19) assuming Conjecture 3.4.1 and solid curves via (3.3.3). For each case, the values of $r^{(\kappa)}$ and $x^{(\kappa)*}$ are computed via (3.4.1) with $a = 0.1$, $b = 1.4$, $c = 0.4$, $x^* = 1$ and $\theta = 0.5$. Recall that for uninsured losses, by (3.3.2) it must hold that $\frac{\lambda}{r} < 1$.

Our consideration of a poverty line that varies with the level of insurance coverage accounts for the fact that premium payments limit a household's level of capital. We therefore consider "extreme poverty" at an individualised level. In Kovacevic and Pflug (2011) and Flores-Contró et al. (2021) the uninsured trapping probability is instead compared with the insured trapping probability for a fixed critical capital $x^*$, irrespective of the parameters $\kappa$, $\lambda$ and $\theta$. Such a specification could be used to consider trapping with respect to an international poverty line, which is fixed for all households. Under this alternative assumption, the trapping probability under proportional insurance coverage of Section 3.4 lies below the uninsured probability of Section 3.3 at all capital levels. In this case, the purchase of insurance therefore does not increase the probability of trapping for any household above the poverty line.

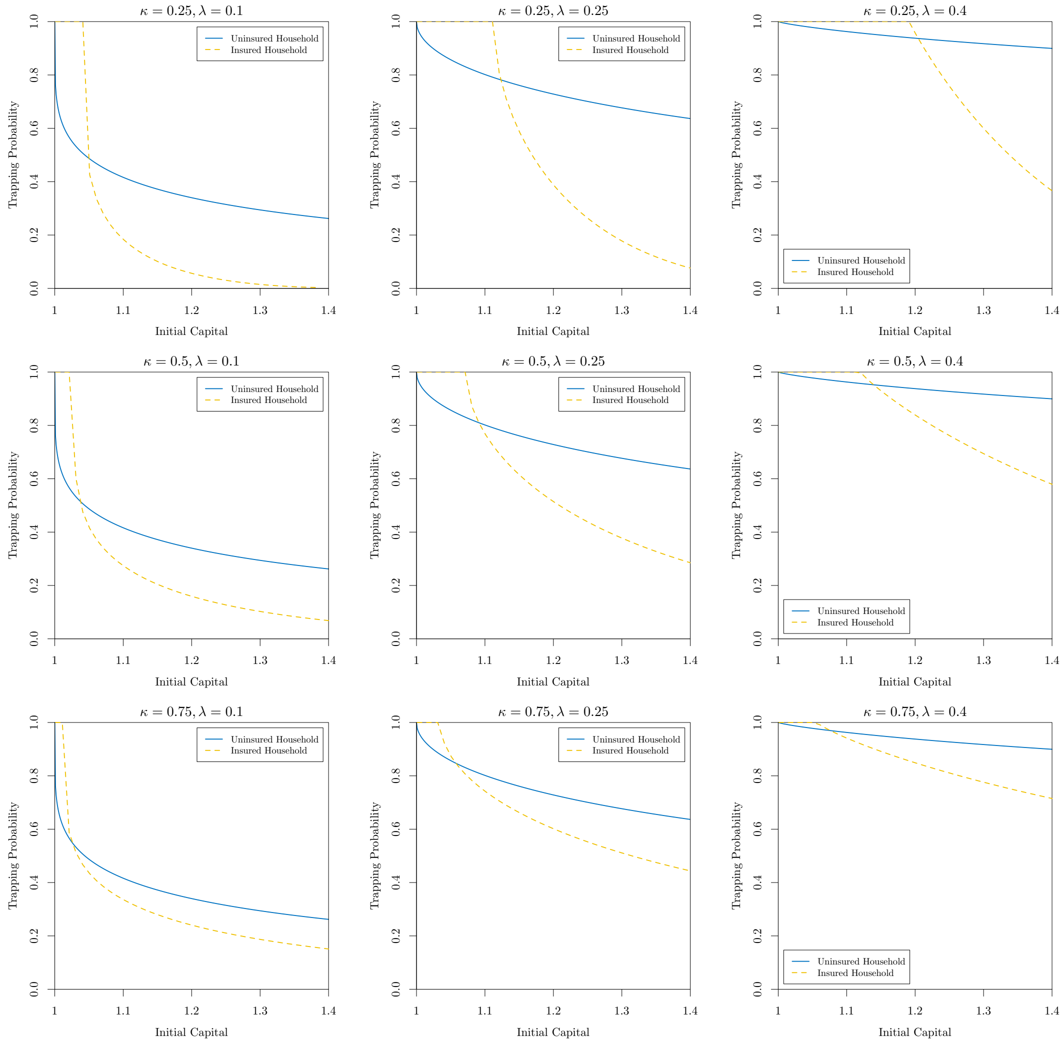



Figure 3.7: Comparison of the trapping probabilities of uninsured and insured households for small values of initial capital, $x \in [1, x^{(\kappa)*}/(1-\kappa)]$ and different values of $\kappa$ and $\lambda$, showing the existence of a level $x_c > x^*$ such that for $1 < x < x_c$ it is better for households not to insure. Dashed curves are computed as in Figure 3.5b and solid curves using expression (3.3.3). For each case, the values of $r^{(\kappa)}$ and $x^{(\kappa)*}$ are computed via (3.4.1) with $a = 0.1$, $b = 1.4$, $c = 0.4$, $x^* = 1$ and $\theta = 0.5$.

Mathematical differences between the uninsured and insured capital processes and the associated parameter constraints may also provide indications of the impact of insurance. In Figure 3.3, the constraint that ensures existence of the Lundberg equation is presented. For uninsured losses with uniformly distributed remaining proportions of capital ($Z_i \sim Beta(1,1)$), by (3.3.2), an equivalent figure would display a horizontal line at $\lambda = r$. For the case considered in Figure 3.3, $r = 0.504$. As such, for all levels of $\theta$, there exists a region in which the uninsured constraint in (3.3.2) is violated, while the insured constraint in (3.4.4) is not. This indicates that for households without insurance, the Lundberg equation fails to be well-defined in more cases. Increasing the level of insurance coverage therefore increases the loss frequency for which the net profit condition is satisfied. As a result, certain trapping is avoided in more cases.

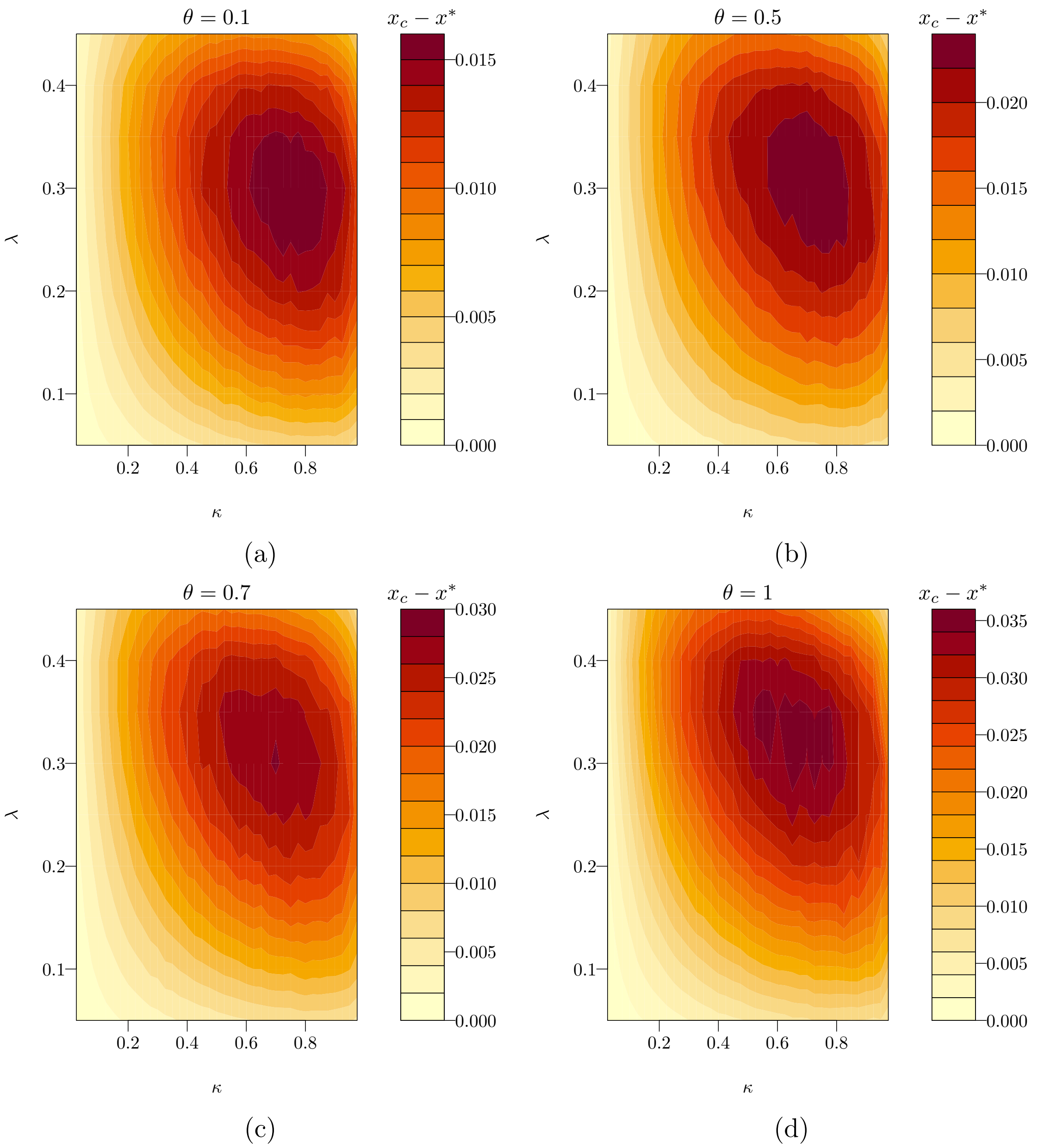



Figure 3.8: Estimated distance between $x_c$ and $x^*$, i.e. $x_c - x^*$, for different values of $\lambda$ and $\kappa$ for which $\frac{\lambda}{r} < 1$ and (a) $\theta = 0.1$, (b) $\theta = 0.5$, (c) $\theta = 0.7$ and (d) $\theta = 1$, where $x_c$ is the intersection point of the uninsured and insured trapping probabilities. For each case, the values of $r^{(\kappa)}$ and $x^{(\kappa)*}$ are computed via (3.4.1) with $a = 0.1$, $b = 1.4$, $c = 0.4$ and $x^* = 1$.

Due to the increasing complexity of (3.4.15) the constant $A$ appears in an increasingly convoluted manner throughout the subintervals $I_j$. As we move through $I_j$ for increasing $\tilde{x}$, estimation of the trapping probability under proportional insurance coverage becomes computationally intensive, particularly for small values of $\kappa$. However, analysis of the algebraic decay of the trapping probability can provide further insight into the behaviour of the function at high capital levels. Solution of the transcendental equation

$$r^{(\kappa)}\gamma - \lambda + \frac{\lambda\alpha}{\kappa}\int_{1-\kappa}^{1} y^{\gamma}\left(1 - \frac{1-y}{\kappa}\right)^{\alpha-1} dy = 0, \tag{3.5.1}$$

derived from $\left(\mathcal{A}\psi^{(\kappa)}\right)(x) = 0$ for $\left(\mathcal{A}\psi^{(\kappa)}\right)(x)$ in (3.4.5) for $Z_i \sim Beta(\alpha, 1)$ under the assumption of polynomial asymptotic decay to zero at infinity: $\psi^{(\kappa)}(x) \sim (x - x^*)^{\gamma}$ as $x \to \infty$ for constant $\gamma$, highlights that for $Beta(1,1)$ distributed remaining proportions of capital, as in Section 3.4, as $\kappa$ increases and households maintain a higher risk level the trapping probability decays more slowly as initial capital $x$ approaches infinity. The same observation can be found with less significance for fixed $\kappa$ and decreasing $\lambda/r^{(\kappa)}$. Solution of the transcendental equation in (3.5.1) for $\alpha > 0$ and $\kappa = 1$ yields that the trapping probability decays only if $\lambda/r < \alpha$, providing exactly the Lundberg condition in the case of no insurance coverage.

## 3.6 Conclusion

We have considered an adjustment of the capital process of Flores-Contró et al. (2021) in which low-income households are susceptible to losses proportional to their accumulated capital level, as in Kovacevic and Pflug (2011). Under the assumption of proportional losses we capture the exposure of households of all capital levels to both catastrophic and low severity loss events, a feature particularly significant in the low-income setting. Typically considered to be protected from capital losses, households with higher levels of capital are still susceptible to large proportional losses on the occurrence of extreme events, particularly in agriculturally rich areas. In addition to high severity loss events, low-income households closest to the poverty line experience large proportional losses due to events typically considered less severe in the high-income setting, such as hospital admissions and household deaths.

Focusing on the probability that a household falls below the poverty line, referred to as the trapping probability, in the analysis of this chapter we have solved, for the first time analytically, infinitesimal generator equations associated with a capital process with exponential growth and multiplicative jumps. We have considered two cases: (i) households with no insurance coverage and (ii) households with proportional insurance coverage. In both cases, closed-form solutions of the infinitesimal generator equations associated with the trapping probability were derived alongside constraints on the parameters of the model that prevent certain trapping. Through the derivation of these probabilities we provide insights into the impact of proportional insurance for proportional losses. Comparison between the proportional assumption of this chapter and the random-valued assumption of Flores-Contró et al. (2021) was additionally presented.

For households with no insurance coverage, explicit trapping probabilities for $Beta(\alpha, 1)-$distributed remaining proportions of capital were obtained using Laplace transform methods. In comparison to the corresponding trapping probability for random-valued losses, the proportional trapping probability exhibits a slower rate of decay, in line with the non-zero probability of high-income households losing a large proportion of their wealth.

Consideration of proportional insurance coverage requires redefinition of the infinitesimal generator of the process. Even under the assumption of uniformly distributed remaining proportions of capital the structure of the proportional insurance product induces non-local functional terms in the derivative and Laplace transform of the infinitesimal generator. Classical methods for solving the infinitesimal generator to derive the trapping probability are therefore not applicable. To overcome this, we propose a recursive method for deriving a solution of the IDE and estimate the unique solution numerically through the conjecture of the existence of a limit. Although only analytic up to a constant, the estimated trapping probability performs well when compared with simulations of the capital process and provides intuitive results under sensitivity analysis. Future work will involve deriving a mathematical proof that this conjecture holds.

Comparing trapping probabilities under no insurance coverage and proportional insurance coverage suggests that the increase in trapping probability observed under random-valued losses is less severe in this proportional case. This finding is in contrast to that of Kovacevic and Pflug (2011), where an increase in trapping probability similar to that of Flores-Contró et al. (2021) is observed under the same proportional model. However, this result is likely highly dependent on the specification of parameters. It should be noted that the distribution of the remaining proportion of capital considered in the numerical example of Kovacevic and Pflug (2011) is such that losses have an expected value of 88%, an extremely high proportion given a loss frequency parameter of $\lambda = 1$. In turn, the associated premium rates are high and will constrain capital growth more significantly. The lower rate associated with the distribution selected for presentation in the analysis of this chapter captures losses of varying severity, as is the experience of a low-income population, and will necessitate reduced premiums. Furthermore, when considering a critical capital that is fixed as in Kovacevic and Pflug (2011), irrespective of a household's insured status, the increase in trapping probability associated with purchase of insurance is not observed at any level of capital.

Ultimately, the findings of this chapter suggest that insurance for proportional losses is more affordable than coverage for losses of random value. This aligns with the idea that premiums are normalised to wealth under the proportional loss structure, thus improving the variability in the affordability of premiums characteristic of insurance for random-valued losses. As such, if the assumption of proportionality is correct, in the context of subsidisation, the proportion of the low-income population requiring full government support may be narrower than anticipated. Under consideration of a universal poverty line, such as the international poverty line, insurance is beneficial at all capital levels. However, when considering the impact of insurance at a more granular level, where the critical level increases with increasing coverage, for those with capital just above the critical capital, as in the findings of existing studies, insurance and the associated need for premium payments increases their probability of falling below the poverty line.

# Chapter 4

# The Gerber-Shiu Expected Discounted Penalty Function: An Application to Poverty Trapping

This chapter is based on the following article:

Flores-Contró, J. M. (2024). The Gerber-Shiu Expected Discounted Penalty Function: An Application to Poverty Trapping. *Submitted.*

**Abstract.** In this chapter, we consider a risk process with deterministic growth and prorated losses to model the capital of a household. Our work focuses on the analysis of the trapping time of such a process, where trapping occurs when a household's capital level falls into the poverty area, a region from which it is difficult to escape without external help. A function analogous to the classical Gerber-Shiu expected discounted penalty function is introduced, which incorporates information on the trapping time, the capital surplus immediately before trapping and the capital deficit at trapping. Given that the remaining proportion of capital upon experiencing a capital loss is $Beta(\alpha, 1)-$distributed, closed-form expressions are obtained for quantities typically studied in classical risk theory, including the Laplace transform of the trapping time and the distribution of the capital deficit at trapping. In particular, we derive a model belonging to the generalised beta (GB) distribution family that describes the distribution of the capital deficit at trapping given that trapping occurs. Affinities between the capital deficit at trapping and a class of poverty measures, known as the Foster-Greer-Thorbecke (FGT) index, are presented. The versatility of this model to estimate FGT indices is assessed using household microdata from Burkina Faso's *Enquête Multisectorielle Continue (EMC) 2014*.

## 4.1 Introduction

Recently, risk theory has proven to be a powerful tool to analyse a household's infinite-time trapping probability (the probability of a household's capital falling into the area of poverty at some point in time) (see, for instance, Kovacevic and Pflug (2011), Flores-Contró et al. (2021) and Henshaw et al. (2023)). The classical risk process, also known as the Cramér-Lundberg model, which was introduced by Cramér and Lundberg at the beginning of the last century (Lundberg, 1903, 1926; Cramér, 1930), has been adapted to better portray the capital of a household. For example, in Kovacevic and Pflug (2011), Flores-Contró et al. (2021) and Henshaw et al. (2023), only the surplus of a household's current capital above a critical capital level (or poverty line) grows exponentially, unlike the linear premium income for an insurer's surplus in the Cramér-Lundberg model. Moreover, Kovacevic and Pflug (2011) and Henshaw et al. (2023) consider household capital losses as a proportion of the accumulated capital, yielding absolute losses that are serially correlated with each other and with the inter-arrival times of loss events. In contrast, losses in the Cramér-Lundberg model are given by a sequence of i.i.d. claim sizes and are subtracted from the insurer's surplus rather than prorated. Similar models with prorated jumps have been studied outside the actuarial science domain (see Altman et al. (2002), Altman et al. (2005) and Löpker and Van Leeuwaarden (2008), for an application of this type of model on data transmission over the internet; Eliazar and Klafter (2004) and Eliazar and Klafter (2006) for their use in representing the behaviour of physical systems with a growth-collapse pattern; and Derfel et al. (2012) for the adoption of these processes to modelling the division and growth of cell-populations).

This chapter examines the household capital process with proportional losses originally introduced in Kovacevic and Pflug (2011) and subsequently studied in Azaïs and Genadot (2015) and Henshaw et al. (2023). Previous work on this capital process focuses solely on studying the infinite-time trapping probability. Indeed, Kovacevic and Pflug (2011) and Azaïs and Genadot (2015) use numerical methods to estimate the trapping probability, without aiming to find an analytical solution for the probability. However, as stated by Asmussen and Albrecher (2010), the ideal situation in risk theory is to derive closed-form solutions for trapping probabilities. To this end, Henshaw et al. (2023) apply Laplace transform techniques to solve the infinitesimal generator of the household's capital risk process and obtain a closed-form expression for the infinite-time trapping probability under the assumption of $Beta(\alpha, 1)-$distributed remaining proportions of capital.

Although the infinite-time trapping probability is a very important indicator for studying poverty dynamics, policy makers and other stakeholders may need additional information on other quantities to fully understand a household's transition into poverty. A clear example of a quantity of interest is the income short-fall (or income gap), which is defined as the absolute value of the difference between a poor household's income (or consumption) and some poverty line. A household's income short-fall serves as key component in a number of poverty measures (see, for instance, Sen (1976), where a simple poverty measure, the income-gap ratio, assesses the percentage of household's mean income short-fall from the poverty line and Foster et al. (1984), where the well-known Foster-Greer-Thorbecke (FGT) index weights the income gaps of the poor to estimate the aggregate poverty of an economic entity). The primary objective of incorporating household levels of income short-fall in poverty measures is the elimination of certain measurement issues. That is, numerous poverty measures, such as the head-count index, which calculates the proportion of the population living below the poverty line and has been considered as one of the most common indices for measuring poverty since the first studies of poverty were conducted (see, Booth (1889) and Rowntree (1901)), ignore the depth of poverty and the distribution of income among the poor, making them deficient as poverty indicators (Sen, 1976). Consequently, this underlines the importance of exploring additional quantities such as a household's income short-fall.

Apart from facilitating the study of the infinite-time trapping probability, classical risk theory provides additional tools that allow the examination of other quantities of interest, such as a household's income short-fall at the trapping time (the time at which a household's capital falls into the area of poverty), thus granting a much deeper understanding of a household's transition into poverty. In particular, the Gerber-Shiu expected discounted penalty function, which was originally introduced by Gerber and Shiu (1998), gives information about three quantities: the time of ruin, the deficit at ruin, and the surplus prior to ruin, corresponding to the first time an insurer's surplus becomes negative, the undershoot and the overshoot of the insurer's surplus at ruin, respectively. These three random variables play an important role within the risk management strategy of an insurance company. For instance, risk measures such as the Value-at-Risk and the Tail-Value-at-Risk have a close link with the deficit at ruin, while from a monitoring perspective, the surplus prior to ruin could be thought of as an early warning signal for the insurance company. The (ruin) time at which any such event takes place is then of critical importance (Landriault and Willmot, 2009). Extensive literature on these variables exists for the Cramér-Lundberg model and its variations (see, for example, Gerber and Shiu (1997a), Gerber and Shiu (1998), Lin and Willmot (1999), Lin and Willmot (2000), Chiu and Yin (2003), Landriault and Willmot (2009) and references therein).

Certainly, a household's trapping time can be thought of as the ruin time of an insurer, while the capital surplus prior to trapping and the capital deficit at trapping are analogous to the insurer's surplus prior to ruin and the deficit at ruin, respectively. Therefore, the Gerber-Shiu expected discounted penalty function can be applied to study these quantities. Recently, for example, Flores-Contró et al. (2021) employed the Gerber-Shiu expected discounted penalty function to study the distribution of the trapping time of a household's capital risk process with deterministic growth and $Exp(\alpha)-$distributed losses. Using classical risk theory techniques, Flores-Contró et al. (2021) also assess how the introduction of an insurance policy alters the distribution of the trapping time. Kovacevic and Semmler (2021) have also recently highlighted the importance of studying such trapping times to optimise the retention rates of insurance policies purchased by households. In this chapter, for the household capital process with proportional losses, we obtain closed-form expressions for the Gerber-Shiu expected discounted penalty function under the assumption of $Beta(\alpha, 1)-$distributed remaining proportions of capital. Thus, the first contribution of this chapter lies in the derivation of analytical equations for the Gerber-Shiu expected discounted penalty function, which to the best of our knowledge, have not been previously obtained for this particular risk process.

Given the importance of the income short-fall and its key role in widely used poverty measures, the second contribution of this chapter lies in obtaining a compelling microeconomic foundation, which emerges from the derivation of the Gerber-Shiu expected discounted penalty function for the household capital process, to model the distribution of the income short-fall. This is particularly important as parametric estimation of income distributions has long been used to model income since the introduction of the Pareto (1967) law. One of the main advantages of parametric estimation of income distributions is that explicit formulas, as functions of the parameters of the theoretical income distribution, are available to measure poverty and inequality. This allows, for example, to further interpret the shape parameters of the theoretical income distribution, as well as to carry out sensitivity analyses of poverty measures to variations in the shape parameters (Graf and Nedyalkova, 2014). In economics, it is well-known that the processes of income generation and distribution must be connected, underpinned by a microeconomic foundation, to the functional form of any model that adequately represents the distribution of personal income (Callealta Barroso et al., 2020). Our results reveal that the distribution of a household's income short-fall belongs to the generalised beta (GB) distribution family, a group of models that have been widely used in economics for modelling income.

To assess the validity of our results, we fit the derived GB model to household microdata from Burkina Faso's *Enquête Multisectorielle Continue (EMC) 2014*. Poverty measures are estimated using both the observed income short-fall data and the fitted theoretical income short-fall distribution. Goodness-of-fit tests and comparisons between theoretical and empirical poverty measures suggest that risk theory is a promising theoretical framework for studying poverty dynamics. That is, by appropriately adapting the classical Cramér-Lundberg model to better portray a household's capital, risk theory provides a vast framework with a diverse set of tools to explore. The application of risk theory techniques to study poverty dynamics is just beginning and its potential is yet to be discovered.

The remainder of the chapter is organised as follows. In Section 4.2, we introduce the capital of a household and its connection with the Cramér-Lundberg model. Section 4.3 provides a brief discussion on the GB distribution family and its application in economics for modelling income. In Section 4.4, the trapping time and the Gerber-Shiu expected discounted penalty function are defined. Moreover, an Integro-Differential Equation (IDE) for the Gerber-Shiu expected discounted penalty function is also derived. We obtain in Section 4.4.1 a closed-form expression for the Laplace transform of the trapping time when the remaining proportion of capital is $Beta(\alpha, 1)-$distributed. Apart from characterising uniquely the probability distribution of the trapping time, Section 4.4.1 also shows how the Laplace transform of the trapping time can be applied to estimate other quantities of interest such as the expected trapping time. Likewise, Section 4.4.2 studies the capital deficit at trapping by means of the Gerber-Shiu expected discounted penalty function for $Beta(\alpha, 1)-$distributed remaining proportions of capital and shows that the distribution of the capital deficit at trapping given that trapping occurs is described by a model belonging to the GB distribution family. Section 4.5 introduces the FGT index in more detail and discusses affinities between the index and the capital deficit at trapping. Built on Sections 4.4.2 and 4.5, a GB distribution is fitted to household microdata from Burkina Faso's *Enquête Multisectorielle Continue (EMC) 2014* in Section 4.6. In addition, FGT indices are estimated using the fitted distribution. To evaluate the adequacy of the model, empirical values of the poverty measures are compared with theoretical estimates and goodness-of-fit tests are assessed. Lastly, concluding remarks are discussed in Section 4.7.

## 4.2 The Capital of a Household

In classical risk theory, the insurance risk process with deterministic investment $\{U_t\}_{t\geq 0}$ is given by

$$U_t = u + pt + \nu \int_0^t U_s \, ds - \sum_{i=1}^{P_t} Y_i, \tag{4.2.1}$$

where $u = U_0 \geq 0$ is the insurer's initial surplus, $p$ is the incoming premium rate per unit time, $\nu$ is the risk-free interest rate, $\{P_t\}_{t\geq 0}$ is a Poisson process with intensity $\mathcal{I}$ counting the number of claims in the time interval $[0, t]$ and $\{Y_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. claim sizes with distribution function $G_Y$. Initially introduced by Segerdahl (1942), this model was subsequently studied by Harrison (1977) and Sundt and Teugels (1995). Readers may wish to consult Paulsen (1998) for a detailed literature review on this model.

Adopting traditional risk theory techniques, this chapter examines ideas proposed in Kovacevic and Pflug (2011). In particular, we study a household's capital process $\{X_t\}_{t\geq 0}$ with a deterministic exponential growth and multiplicative capital loss (collapse) structure. The process grows exponentially with a rate $r > 0$, which incorporates household rates of consumption $(0 < a < 1)$, income generation $(0 < b)$ and investment or savings $(0 < c < 1)$, above a critical capital (or poverty line) $x^* > 0$ whereas below this critical threshold it remains constant. At time $T_i$, the $i$th capital loss event time of a Poisson process $\{N_t\}_{t\geq 0}$ with parameter $\lambda$, the capital process jumps (downwards) to $Z_i \cdot X_{T_i}$, where $\{Z_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables with distribution function $G_Z$ supported in $[0, 1]$, independent of the process $N_t$, representing the proportions of remaining capital after each loss event. Therefore, a household's capital process in between jumps is given by

$$X_t = \begin{cases} \left(X_{T_{i-1}} - x^*\right) e^{r(t - T_{i-1})} + x^* & \textit{if } X_{T_{i-1}} > x^*, \\ X_{T_{i-1}} & \textit{otherwise}, \end{cases} \tag{4.2.2}$$

for $T_{i-1} \leq t < T_i$ and $T_0 = 0$. On the other hand, at the jump times $t = T_i$, the capital process is given by

$$X_{T_i} = \begin{cases} \left[\left(X_{T_{i-1}} - x^*\right) e^{r(T_i - T_{i-1})} + x^*\right] \cdot Z_i & \textit{if } X_{T_{i-1}} > x^*, \\ X_{T_{i-1}} \cdot Z_i & \textit{otherwise.} \end{cases} \tag{4.2.3}$$

The stochastic process $\{X_t\}_{t\geq 0}$ is a piecewise-determinsitic Markov process (Davis, 1984, 1993) and its infinitesimal generator is given by

$$(\mathcal{A}f)(x) = r(x - x^*)f'(x) + \lambda \int_0^1 [f(x \cdot z) - f(x)] \, \mathrm{d}G_Z(z), \qquad x \geq x^*.$$

There exist many similarities between the household capital process and other well-known risk processes. For instance, observe that when $p = 0$, the insurance risk process (4.2.1) is equivalent to the household capital process above the critical capital $x^* = 0$ with claim losses subtracted from the insurer's surplus rather than prorated. Furthermore, taking the logarithm of a discretised version of the household capital process, that is, setting the critical capital $x^* = 0$ and taking the logarithm of (4.2.3), yields a version of the classical risk process (see, for instance, Kovacevic and Pflug (2011) and Henshaw et al. (2023)), also known as the Cramér-Lundberg model, introduced by Cramér and Lundberg at the beginning of the last century (Lundberg, 1903, 1926; Cramér, 1930). This model considers linear premium income for the surplus of an insurance company with losses given by a sequence of i.i.d. claim sizes. Clearly, the Cramér-Lundberg model could also be seen as a particular case of the risk process (4.2.1) with $\nu = 0$. Despite these resemblances, there are also a number of discrepancies between the household capital process and those commonly studied in the actuarial science literature. Firstly, only the surplus of a household's current capital above the critical capital grows exponentially. Secondly, household losses are defined as a proportion of the accumulated capital, yielding absolute losses that are serially correlated with each other and with the inter-arrival times of loss events (Kovacevic and Pflug, 2011; Henshaw et al., 2023).

## 4.3 The Generalised Beta Distribution Family

The probability density function (p.d.f.) of the generalised beta (GB) distribution family is given by

$$GB(y; a, b, c, p, q) = \frac{|a| y^{ap-1} \left(1 - (1-c)(y/b)^a\right)^{q-1}}{b^{ap} \, \mathrm{B}(p,q) \left(1 + c(y/b)^a\right)^{p+q}} \qquad \textit{for} \qquad 0 < y^a < \frac{b^a}{1-c}, \tag{4.3.1}$$

and zero otherwise, where $a \neq 0$; $0 \leq c \leq 1$; $b, p, q > 0$; and $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt$ denotes the beta function (see, for instance, Equation (6.2.1) from Abramowitz and Stegun (1972)). The GB includes other distributions as special or limiting cases (see, for example, McDonald and Xu (1995)). In particular, the beta of the first kind (B1), with p.d.f.

$$B1(y; b, p, q) := GB(y; a = 1, b, c = 0, p, q) = \frac{y^{p-1}(b-y)^{q-1}}{b^{p+q-1} B(p,q)} \qquad \textit{for} \qquad 0 < y < b, \tag{4.3.2}$$

arises as the model that describes the distribution of a household's income shortfall, for the particular case in which the remaining proportions of capital $Z_i$ are

$Beta(\alpha,1)-$distributed. Indeed, the results obtained in Section 4.4.2 validate the adequacy of the B1 distribution as a model of income distribution and, in particular, as a model for the distribution of the income short-fall. Thurow (1970) was the first to adopt the standard beta distribution ($Beta(p,q) := B1(y;b=1,p,q)$) to analyse factors contributing to income inequality among whites and blacks. One of the main advantages of the beta distribution is that it includes the gamma distribution as a limiting case and therefore provides at least as good a fit as the gamma. This is an important feature, especially since the gamma distribution has also been considered to model income distribution (Salem and Mount, 1974). In the 1980s, seeking to improve the goodness of fit of the two-parameter standard beta distribution, McDonald (1984) introduced the generalized beta of the first and second kind ($GB1 := GB(y;a,b,c=0,p,q)$ and $GB2 := GB(y;a,b,c=1,p,q)$), two four-parameter distributions that nest most of the previously used models of two and three parameters as special cases or limit distributions (e.g. the Singh-Maddala distribution (Singh and Maddala, 1976)). Subsequently, McDonald and Xu (1995) introduced (4.3.1), a five-parameter distribution that has clearly played an important role for modelling income. In fact, many distributions (belonging or not to the GB distribution family) with a varying number of parameters have been used in the literature to model income (see Hlasny (2021) for a detailed survey).

## 4.4 When and How Households Become Poor?

Let

$$\tau_x := \inf\{t \geq 0 : X_t < x^* \mid X_0 = x\}$$

denote the time at which a household with initial capital $x \geq x^*$ falls into the area of poverty (the trapping time), where $\psi(x) = \mathbb{P}(\tau_x < \infty)$ is the infinite-time trapping probability. To study the distribution of the trapping time, we apply the Gerber-Shiu expected discounted penalty function at ruin, a concept commonly used in actuarial science (Gerber and Shiu, 1998), such that with a force of interest $\delta \geq 0$ and initial capital $x \geq x^*$, we consider

$$m_\delta(x) = \mathbb{E}\left[w(X_{\tau_x^-} - x^*, \mid X_{\tau_x} - x^* \mid) e^{-\delta\tau_x} \mathbb{1}_{\{\tau_x < \infty\}}\right], \qquad (4.4.1)$$

where $\mathbb{1}_{\{A\}}$ is the indicator function of a set $A$, and $w(x_1, x_2)$, for $0 \leq x_1 < \infty$ and $0 < x_2 \leq x^*$, is a non-negative penalty function of $x_1$, the capital surplus prior to the trapping time, and $x_2$, the capital deficit at the trapping time. For more details on the so-called Gerber-Shiu risk theory, interested readers may wish to consult Kyprianou (2013). The function $m_\delta(x)$ is useful for deriving results in connection with joint and marginal distributions of $\tau_x$, $X_{\tau_x^-} - x^*$ and $\mid X_{\tau_x} - x^* \mid$. For example, when $\delta$ is considered as the argument, (4.4.1) can be viewed in terms of a Laplace transform. That is, (4.4.1) is the Laplace transform of the trapping time $\tau_x$ if one sets $w(x_1, x_2) = 1$[1]. Another choice, for any fixed $y$, is $w(x_1, x_2) = \mathbb{1}_{\{x_2 \leq y\}}$ for $\delta = 0$, for which (4.4.1) leads to the distribution function of the capital deficit

[1]Recall that, for a continuous random variable $Y$, with p.d.f. $f_Y$, the Laplace transform of $f_Y$ is given by the expected value $\mathcal{L}\{f_Y\}(s) = \mathbb{E}\left[e^{-sY}\right]$.

at trapping. It is not difficult to realise that, by appropriately choosing a penalty function $w(x_1, x_2)$ and force of interest $\delta$, various risk quantities can be modelled. He et al. (2023) provide a non-exhaustive list of such risk quantities. In this chapter, we are mainly interested in studying the Laplace transform of the trapping time and the distribution of the capital deficit at trapping. Thus, we will focus our analysis on the choices mentioned above. Following Gerber and Shiu (1998), our goal is to derive a functional equation for $m_\delta(x)$ by applying the law of iterated expectations to the right-hand side of (4.4.1).

**Theorem 4.4.1.** *The Gerber-Shiu expected discounted penalty function at trapping, $m_\delta(x)$, for $x \geq x^*$, satisfies the following Integro-Differential Equation (IDE)*

$$r(x - x^*)m'_\delta(x) - (\delta + \lambda)m_\delta(x) + \lambda \int_{x^*/x}^{1} m_\delta(x \cdot z)dG_Z(z) = -\lambda A(x), \quad (4.4.2)$$

*where $A(x) := \int_0^{x^*/x} w(x - x^*, x^* - x \cdot z)dG_Z(z)$, with boundary conditions*

$$m_\delta(x^*) = \frac{\lambda}{\delta + \lambda} A(x^*) \quad \text{and} \quad \lim_{x\to\infty} m_\delta(x) = 0.$$

*Proof.* For $h > 0$, consider the time interval $(0, h)$, and condition on the time $t$ and the proportion $z$ of remaining capital after the first capital loss in this time interval. Since the inter-arrival times of losses are exponentially distributed, the probability that there is no loss up to time $h$ is $e^{-\lambda h}$, and the probability that the first capital loss occurs between time $t$ and time $t + dt$ is $e^{-\lambda t}\lambda dt$. If

$$z < \frac{x^*}{(x - x^*)e^{rt} + x^*},$$

where $0 < x^*/\left[(x - x^*)e^{rt} + x^*\right] \leq 1$, trapping has occurred with the first loss. Hence,

$$\begin{aligned} m_\delta(x) &= e^{-(\delta+\lambda)h} m_\delta((x - x^*)e^{rh} + x^*) \\ &+ \int_0^h \left[ \int_0^{\frac{x^*}{(x-x^*)e^{rt}+x^*}} w\left((x - x^*)e^{rt}, x^* - ((x - x^*)e^{rt} + x^*) \cdot z\right) dG_Z(z) \right] e^{-(\delta+\lambda)t}\lambda dt \\ &+ \int_0^h \left[ \int_{\frac{x^*}{(x-x^*)e^{rt}+x^*}}^{1} m_\delta(((x - x^*)e^{rt} + x^*) \cdot z)dG_Z(z) \right] e^{-(\delta+\lambda)t}\lambda dt. \end{aligned} \quad (4.4.3)$$

Note that every part of the above integral equation (IE) is differentiable with respect to $h$. Thus, by symmetry one can also establish the differentiability of $m_\delta(x)$ with respect to $x$ (see, for example, Remark 1.11 in Asmussen and Albrecher (2010) where a similar argument is presented for the ruin probability of risk processes with non-proportional random-valued losses). Differentiating (4.4.3) with respect to $h$ and setting $h = 0$, (4.4.2) is obtained.

□

### 4.4.1 The Trapping Time

As noted previously, specifying the penalty function such that $w(x_1, x_2) = 1$, (4.4.1) becomes the Laplace transform of the trapping time, also interpreted as the expected present value of a unit payment due at the trapping time. Thus, Equation (4.4.2) can then be written such that

$$0 = r(x - x^*)m'_\delta(x) - (\delta + \lambda)m_\delta(x) + \lambda G_Z\left(\frac{x^*}{x}\right) + \lambda \int_{x^*/x}^{1} m_\delta(x \cdot z)dG_Z(z). \tag{4.4.4}$$

*Remark* 4.4.1. In general, it is not straightforward to obtain the solution of (4.4.4) for general distribution functions $G_Z$. Hence, throughout this chapter, it will be assumed that $Z_i \sim Beta(\alpha, 1)$, case for which the distribution function is $G_Z(z) = z^\alpha$ and the p.d.f. is $g_Z(z) = \alpha z^{\alpha-1}$ for $0 < z < 1$, where $\alpha > 0$. Under this assumption, one can derive a closed-form expression for the Laplace transform of the trapping time.

**Proposition 4.4.1.** *Consider a household capital process defined as in (4.2.2) and (4.2.3), with initial capital $x \geq x^*$, capital growth rate $r$, intensity $\lambda > 0$ and remaining proportions of capital with distribution $Beta(\alpha, 1)$ where $\alpha > 0$; that is, $Z_i \sim Beta(\alpha, 1)$. The Laplace transform of the trapping time is given by*

$$m_\delta(x) = \frac{\lambda \cdot {}_2F_1\left(b, b - c + 1; b - a + 1; y(x)^{-1}\right)}{(\lambda + \delta){}_2F_1\left(b, b - c + 1; b - a + 1; 1\right)} y(x)^{-b}, \tag{4.4.5}$$

*where $\delta \geq 0$ is the force of interest for valuation, ${}_2F_1(\cdot)$ is Gauss's Hypergeometric Function as defined in (4.4.10), $y(x) = \frac{x}{x^*}$, $a = \frac{-(\delta+\lambda-\alpha r)-\sqrt{(\delta+\lambda-\alpha r)^2+4r\alpha\delta}}{2r}$, $b = \frac{-(\delta+\lambda-\alpha r)+\sqrt{(\delta+\lambda-\alpha r)^2+4r\alpha\delta}}{2r}$ and $c = \alpha$.*

*Proof.* Under the assumption $Z_i \sim Beta(\alpha, 1)$, the IDE (4.4.4) can be written such that

$$0 = r(x - x^*)m'_\delta(x) - (\delta + \lambda)m_\delta(x) + \lambda \left(\frac{x^*}{x}\right)^\alpha + \lambda \int_{x^*/x}^{1} m_\delta(x \cdot z)\alpha z^{\alpha-1}dz. \tag{4.4.6}$$

Applying the operator $\frac{d}{dx}$ to both sides of (4.4.6), together with a number of algebraic manipulations, yields to the following second order Ordinary Differential Equation (ODE)

$$0 = r(x^2 - xx^*)m''_\delta(x) + \left[(r(1 + \alpha) - \delta - \lambda)x - r\alpha x^*\right] m'_\delta(x) - \alpha\delta m_\delta(x) \tag{4.4.7}$$

Letting $f(y) := m_\delta(x)$, such that $y$ is associated with the change of variable $y := y(x) = x/x^*$, Equation (4.4.7) reduces to Gauss's Hypergeometric Differential Equation (Slater, 1960)

$$y(1 - y) \cdot f''(y) + [c - (1 + a + b)y]f'(y) - abf(y) = 0, \tag{4.4.8}$$

for $a = \frac{-(\delta+\lambda-\alpha r)-\sqrt{(\delta+\lambda-\alpha r)^2+4r\alpha\delta}}{2r}$, $b = \frac{-(\delta+\lambda-\alpha r)+\sqrt{(\delta+\lambda-\alpha r)^2+4r\alpha\delta}}{2r}$ and $c = \alpha$, with regular singular points at $y = 0, 1, \infty$ (corresponding to $x = 0, x^*, \infty$, respectively). A general solution of (4.4.8) in the neighborhood of the singular point $y = \infty$ is given by

$$\begin{aligned} f(y) :=& m_\delta(x) = A_1 y(x)^{-a} {}_2F_1\left(a, a-c+1; a-b+1; y(x)^{-1}\right) \\ &+ A_2 y(x)^{-b} {}_2F_1\left(b, b-c+1; b-a+1; y(x)^{-1}\right), \end{aligned} \tag{4.4.9}$$

for arbitrary constants $A_1, A_2 \in \mathbb{R}$ (see for example, Equations (15.5.7) and (15.5.8) of Abramowitz and Stegun (1972)). Here,

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{4.4.10}$$

is Gauss's Hypergeometric Function (Gauss, 1866) and $(a)_n = \Gamma(a+n)/\Gamma(n)$ denotes the Pochhammer symbol (Seaborn, 1991).

To determine the constants $A_1$ and $A_2$, we use the boundary conditions at $x^*$ and at infinity. The boundary condition $\lim_{x\to\infty} m_\delta(x) = 0$, thus implies that $A_1 = 0$. Letting $x = x^*$ in (4.4.6) and (4.4.9) yields

$$\frac{\lambda}{\lambda+\delta} = A_2 \cdot {}_2F_1\left(b, b-c+1; b-a+1; 1\right).$$

Hence, $A_2 = \lambda / \left[(\lambda+\delta) {}_2F_1\left(b, b-c+1; b-a+1; 1\right)\right]$ and the Laplace transform of the trapping time is given by (4.4.5).

□

*Remark* 4.4.2. Figure 4.1a shows that the Laplace transform of the trapping time approaches the trapping probability as $\delta$ tends to zero, i.e.

$$\lim_{\delta\downarrow 0} m_\delta(x) = \mathbb{P}(\tau_x < \infty) \equiv \psi(x).$$

As $\delta \to 0$, (4.4.5) yields

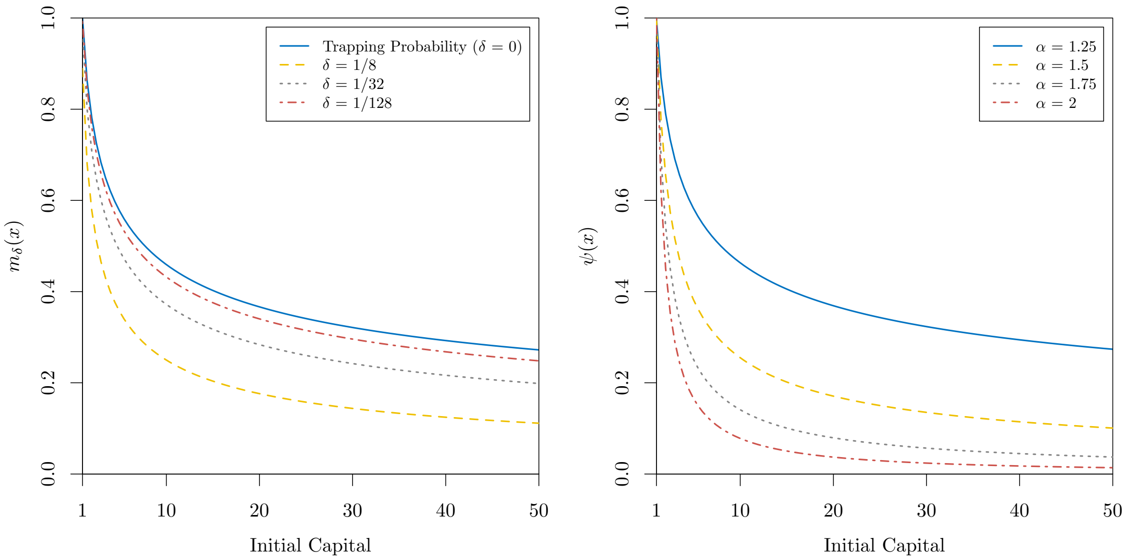
$$\psi(x) = \frac{{}_2F_1\left(\alpha - \frac{\lambda}{r}, 1 - \frac{\lambda}{r}; 1 + \alpha - \frac{\lambda}{r}; y(x)^{-1}\right)}{{}_2F_1\left(\alpha - \frac{\lambda}{r}, 1 - \frac{\lambda}{r}; 1 + \alpha - \frac{\lambda}{r}; 1\right)} y(x)^{\frac{\lambda}{r}-\alpha}, \tag{4.4.11}$$

for $\alpha > \lambda/r$. Indeed, (4.4.11) was recently derived in Henshaw et al. (2023) using Laplace transform techniques. Figure 4.1b displays the trapping probability $\psi(x)$ for the capital process $X_t$. Note that, as mentioned in Henshaw et al. (2023), we can further simplify the expression for the trapping probability using some properties of Gauss's Hypergeometric Function. Namely,

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad (c \neq 0, -1, -2, ..., \mathbb{R}\left(c-a-b\right) > 0)$$

(see, for example, Equation (15.1.20) of Abramowitz and Stegun (1972)). Applying this relation, we obtain

$$\psi(x) = \frac{\Gamma(\alpha) \cdot {}_2F_1\left(\alpha - \frac{\lambda}{r}, 1 - \frac{\lambda}{r}; 1 + \alpha - \frac{\lambda}{r}; y(x)^{-1}\right)}{\left(\alpha - \frac{\lambda}{r}\right) \Gamma\left(\alpha - \frac{\lambda}{r}\right) \Gamma\left(\frac{\lambda}{r}\right)} y(x)^{\frac{\lambda}{r} - \alpha}.$$




(a) (b)

Figure 4.1: (a) Laplace transform $m_\delta(x)$ of the trapping time when $Z_i \sim Beta(1.25, 1)$, $a = 0.1$, $b = 3$, $c = 0.4$, $\lambda = 1$, $x^* = 1$ for $\delta = 0, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}$ (b) Trapping probability $\psi(x)$ when $Z_i \sim Beta(\alpha, 1)$, $a = 0.1$, $b = 3$, $c = 0.4$, $\lambda = 1$, $x^* = 1$ for $\alpha = 1.25, 1.5, 1.75, 2$.

*Remark* 4.4.3. As an application of the Laplace transform of the trapping time, one particular quantity of interest is the expected trapping time; i.e. the expected time at which a household will fall into the area of poverty. This can be obtained by taking the derivative of $m_\delta(x)$:

$$\mathbb{E}\left[\tau_x; \tau_x < \infty\right] = -\left.\frac{d}{d\delta} m_\delta(x)\right|_{\delta=0}, \tag{4.4.12}$$

where $\mathbb{E}\left[\tau_x; \tau_x < \infty\right]$ is equivalent to $\mathbb{E}\left[\tau_x \mathbb{1}_{\{\tau_x < \infty\}}\right]$. As such, we differentiate Gauss's Hypergeometric Function with respect to its first, second and third parameters. Denote

$$\begin{aligned}
{}_2F_1^{(a)}(a, b; c; z) &\equiv \frac{d}{da} {}_2F_1(a, b; c; z), \\
{}_2F_1^{(b)}(a, b; c; z) &\equiv \frac{d}{db} {}_2F_1(a, b; c; z), \text{ and} \\
{}_2F_1^{(c)}(a, b; c; z) &\equiv \frac{d}{dc} {}_2F_1(a, b; c; z).
\end{aligned}$$

A closed-form expression of the aforementioned derivatives is given in terms of the Kampé de Fériet function (Appell and Kampé De Fériet, 1926):

$$F_{R,S,U}^{A,B,D}\left(\begin{array}{l} a_1,\ldots,a_A;b_1,\ldots,b_B;d_1,\ldots,d_D; \\ r_1,\ldots,r_R;s_1,\ldots,s_S;u_1,\ldots,u_U; \end{array} x,y\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m+n}\prod_{j=1}^{B}(b_j)_m\prod_{j=1}^{D}(d_j)_n}{\prod_{j=1}^{R}(r_j)_{m+n}\prod_{j=1}^{S}(s_j)_m\prod_{j=1}^{U}(u_j)_n}\frac{x^m}{m!}\frac{y^n}{n!},$$

such that (see, for example, Equations (9a) and (9b) of Ancarani and Gasaneo (2009)),

$$_2F_1^{(a)}(a,b;c;z)=\frac{zb}{c}F_{2,1,0}^{2,2,1}\left(\begin{array}{c} a+1,b+1;1,a;1; \\ 2,c+1;a+1;; \end{array} z,z\right),$$

$$_2F_1^{(b)}(a,b;c;z)=\frac{za}{c}F_{2,1,0}^{2,2,1}\left(\begin{array}{c} a+1,b+1;1,b;1; \\ 2,c+1;b+1;; \end{array} z,z\right) \text{ and } \tag{4.4.13}$$

$$_2F_1^{(c)}(a,b;c;z)=-\frac{zab}{c^2}F_{2,1,0}^{2,2,1}\left(\begin{array}{c} a+1,b+1;1,c;1; \\ 2,c+1;c+1;; \end{array} z,z\right).$$

This is not the first time that the Kampé de Fériet function appears in ruin theory, as it arises in the study of some risk processes that consider the payment of dividends provided by the insurer (see, for example, Albrecher and Cani (2017)).

**Corollary 4.4.1.** *The expected trapping time under the household capital process defined as in (4.2.2) and (4.2.3), with initial capital $x \geq x^*$, capital growth rate $r$, intensity $\lambda > 0$ and remaining proportions of capital with distribution $Beta(\alpha, 1)$ where $\alpha > 0$; that is, $Z_i \sim Beta(\alpha, 1)$ is given by*

$$\begin{aligned}
&\mathbb{E}\left[\tau_x;\tau_x<\infty\right]=\frac{1}{r(\alpha r-\lambda)\Gamma\left(\frac{\lambda}{r}\right)^2\Gamma\left(\alpha-\frac{\lambda}{r}+1\right)^2}\Gamma(\alpha)y(x)^{\frac{\lambda}{r}-\alpha}\\
&\Bigg[\Gamma(\alpha)_2F_1\left(1-\frac{\lambda}{r},\alpha-\frac{\lambda}{r};1+\alpha-\frac{\lambda}{r};y(x)^{-1}\right)\left((\alpha r+\lambda)_2F_1^{(c)}\left(\alpha-\frac{\lambda}{r},1-\frac{\lambda}{r};1+\alpha-\frac{\lambda}{r};1\right)\right.\\
&\left.+\lambda\left({}_2F_1^{(a)}\left(\alpha-\frac{\lambda}{r},1-\frac{\lambda}{r};1+\alpha-\frac{\lambda}{r};1\right)+{}_2F_1^{(b)}\left(\alpha-\frac{\lambda}{r},1-\frac{\lambda}{r};1+\alpha-\frac{\lambda}{r};1\right)\right)\right)\\
&+\left(\frac{1}{\lambda}\right)\Gamma\left(\frac{\lambda}{r}\right)\Gamma\left(1+\alpha-\frac{\lambda}{r}\right)\left({}_2F_1\left(1-\frac{\lambda}{r},\alpha-\frac{\lambda}{r};1+\alpha-\frac{\lambda}{r};y(x)^{-1}\right)\left(r(\alpha r-\lambda)+\lambda^2\ln\left[y(x)\right]\right)\right.\\
&-\lambda\left((\alpha r+\lambda)\,{}_2F_1^{(c)}\left(\alpha-\frac{\lambda}{r},1-\frac{\lambda}{r};1+\alpha-\frac{\lambda}{r};y(x)^{-1}\right)\right.\\
&\left.\left.+\lambda\left({}_2F_1^{(a)}\left(\alpha-\frac{\lambda}{r},1-\frac{\lambda}{r};1+\alpha-\frac{\lambda}{r};y(x)^{-1}\right)+{}_2F_1^{(b)}\left(\alpha-\frac{\lambda}{r},1-\frac{\lambda}{r};1+\alpha-\frac{\lambda}{r};y(x)^{-1}\right)\right)\right)\right)\Bigg],
\end{aligned} \tag{4.4.14}$$

*where $y(x) = \frac{x}{x^*}$, ${}_2F_1(\cdot)$ is Gauss's Hypergeometric Function as defined in (4.4.10) and ${}_2F_1^{(a)}(\cdot)$, ${}_2F_1^{(b)}(\cdot)$ and ${}_2F_1^{(c)}(\cdot)$ its derivatives with respect to the first, second and third parameters, respectively, as introduced in (4.4.13).*

*Proof.* Calculating (4.4.12) and using (4.4.13), one can derive the expected trapping time (4.4.14). □

Moreover, we can calculate the expected trapping time given that trapping occurs by taking the following ratio (see for example, Equation (4.37) of Gerber and Shiu (1998)),

$$\mathbb{E}\left[\tau_x | \tau_x < \infty\right] = \frac{\mathbb{E}\left[\tau_x; \tau_x < \infty\right]}{\psi(x)}.$$

A number of expected trapping times for varying values of the capital growth rate $r$ are displayed in Figure 4.2. One observes that the expected trapping time is, for a fixed initial capital, typically higher when considering a lower capital growth rate $r$, which at first sight may look counter-intuitive, as a higher capital growth rate $r$ means a faster exponential growth. Nevertheless, this indicates that for high capital growth rates $r$, those trajectories that do not lead to trapping quickly, will very likely avoid it later.

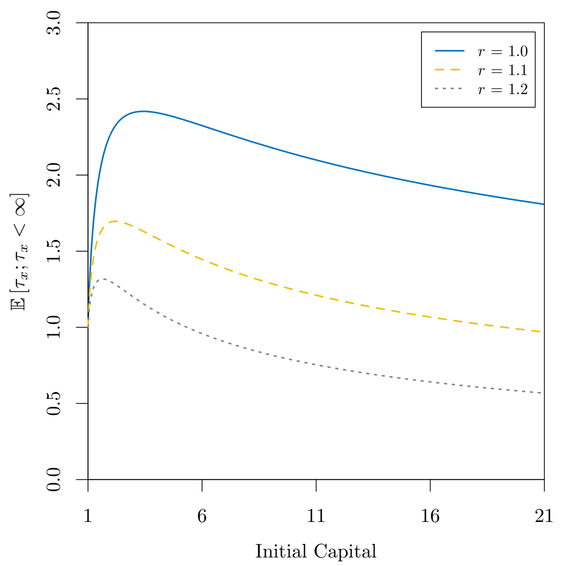



Figure 4.2: Expected trapping time $\mathbb{E}\left[\tau_x; \tau_x < \infty\right]$ when $Z_i \sim Beta(1.5, 1)$, $\lambda = 1$ and $x^* = 1$ for $r = 1.0, 1.1, 1.2$.

### 4.4.2 The Capital Deficit at Trapping

The capital deficit at trapping is the absolute value of the difference between a household's level of capital at the trapping time and the critical capital, i.e. the amount $\mid X_{\tau_x} - x^* \mid$. Specifying the penalty function such that for any fixed $y$, $w(x_1, x_2) = \mathbb{1}_{\{x_2 \leq y\}}$, (4.4.1) becomes the distribution function of the capital deficit at the trapping time discounted at a force of interest $\delta \geq 0$. This choice leads to the following proposition

**Proposition 4.4.2.** *Consider a household capital process defined as in* (4.2.2) *and* (4.2.3)*, with initial capital $x \geq x^*$, capital growth rate $r$, intensity $\lambda > 0$ and remaining proportions of capital with distribution $Beta(\alpha, 1)$ where $\alpha > 0$; that is, $Z_i \sim Beta(\alpha, 1)$. The distribution function of the discounted capital deficit at the trapping time is given by*

$$F_\delta(y; \tau_x < \infty | x) = m_\delta(x) \cdot \left[1 - \left(1 - \frac{y}{x^*}\right)^\alpha\right] \quad \text{for} \quad 0 \leq y \leq x^*, \qquad (4.4.15)$$

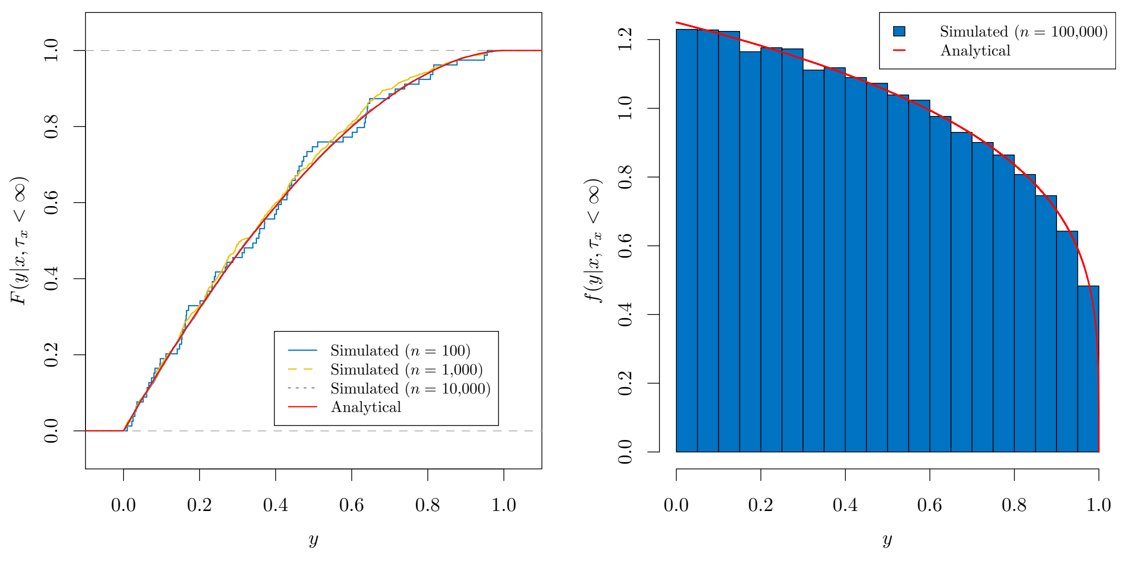
*where $m_\delta(x)$ is the Laplace transform of the trapping time given by (4.4.5) and $\delta \geq 0$ is the force of interest for valuation.*

*Proof.* The choice $w(x_1, x_2) = \mathbb{1}_{\{x_2 \leq y\}}$ yields a modified version of the IDE (4.4.2), with $A(x) = y\,(x)^{-\alpha} - [(x^* - y)\,/x]^{\alpha}$ for $y(x) = x/x^*$. Following a similar procedure to that of Proposition 4.4.1 leads to (4.4.15). □

*Remark* 4.4.4. One can easily obtain $f_\delta(y; \tau_x < \infty|x)$, the p.d.f. of the discounted capital deficit at the trapping time, by differentiating $F_\delta(y; \tau_x < \infty|x)$ w.r.t. $y$. That is,

$$f_\delta(y; \tau_x < \infty|x) := \frac{d}{dy} F_\delta(y; \tau_x < \infty|x) = m_\delta(x) \cdot \frac{\alpha}{x^*}\left(1 - \frac{y}{x^*}\right)^{\alpha-1} \quad \textit{for} \quad 0 < y < x^*,$$

where $m_\delta(x)$ is the Laplace transform of the trapping time given by (4.4.5) and $\delta \geq 0$ is the force of interest for valuation.




(a) (b)

Figure 4.3: (a) $F(y|x, \tau_x < \infty)$ when $Z_i \sim Beta(1.75, 1)$, $r = 1.08$, $\lambda = 1$, $x = 1.25$, $x^* = 1$ (b) $f(y|x, \tau_x < \infty)$ when $Z_i \sim Beta(1.25, 1)$, $r = 1.08$, $\lambda = 1$, $x = 1.25$, $x^* = 1$.

*Remark* 4.4.5. Note that, setting $\delta = 0$ yields $F(y; \tau_x < \infty|x)$, the distribution of the capital deficit at trapping. Furthermore, we can calculate the distribution of the capital deficit at trapping given that trapping has occurred. This is given by

$$F(y|x, \tau_x < \infty) := \frac{F(y; \tau_x < \infty|x)}{\psi(x)} = 1 - \left(1 - \frac{y}{x^*}\right)^{\alpha} \quad \textit{for} \quad 0 \leq y \leq x^*. \tag{4.4.16}$$

Moreover, differentiating $F(y|x, \tau_x < \infty)$ w.r.t. $y$ leads to the p.d.f. of the capital deficit at trapping given that trapping has occurred,

$$f(y|x, \tau_x < \infty) := \frac{d}{dy} F(y|x, \tau_x < \infty) = \frac{\alpha}{x^*}\left(1 - \frac{y}{x^*}\right)^{\alpha-1} \quad \textit{for} \quad 0 < y < x^*.$$

(4.4.17)

Figure 4.3 compares both, analytical and simulated, distribution and p.d.f. of the capital deficit at trapping given that trapping occurs. Simulated quantities were generated using the Euler-Maruyama method, a well-known technique mainly used to approximate numerical solutions of Stochastic Differential Equations (SDEs) (see, for example, Kloeden and Eckhard (1995)). Not surprisingly, Figure 4.3a clearly shows that the simulated quantities converge to the theoretical distribution (4.4.16) as the number of simulations $n$ increases, while Figure 4.3b displays how the theoretical p.d.f. given by (4.4.17) perfectly fits the simulated observations.

By comparing (4.3.2) with (4.4.17) one concludes that the capital deficit at trapping given that trapping occurs follows the beta distribution of the first kind (B1). Indeed, if we denote the random variable $Y := \mid X_{\tau_x} - x^* \mid \Big| \tau_x < \infty$, we have that $Y \sim B1(y; b = x^*, p = 1, q = \alpha)$. Similarly, one can write $Y \overset{d}{=} x^* \cdot (1 - Z_i)$, where $\overset{d}{=}$ denotes equality in distribution.

## 4.5 A Class of Poverty Measures and its Connection with the Capital Deficit at Trapping

Poverty measures serve as the main tool for the evaluation of anti-poverty policies (e.g. cash transfer programmes) and poverty itself. Since Sen (1976), following his axiomatic approach, researchers have formulated numerous poverty measures over the years. The Foster-Greer-Thorbecke (FGT) index (Foster et al., 1984) is undoubtedly one of the most important of these poverty measures and has been widely applied in empirical works. In fact, the FGT index has become the standard measure for international poverty assessments and is regularly reported on by individual countries and international organisations such as the World Bank (for a detailed review of the contributions of the FGT index over the 25 years since its publication, see Foster et al. (2010)). The FGT index emerged as an alternative to the "rank weighting" approach, which was originally applied in the "Sen measure" (see Theorem 1 from Sen (1976)), and accounts for the normalised gap and the rank order of a person in the group of the poor. The FGT index contemplates instead a "short-fall weighting" method, which considers the income short-fall expressed as a share of the poverty line.

Let $F_X(x)$ be the distribution function of the income variable $X$ from a population with continuous p.d.f. $f_X(x)$ at a given point $x$. The FGT class of poverty measures indexed by $\gamma \geq 0$ is defined as follows

$$FGT_\gamma = \int_0^z \left( \frac{z - x}{z} \right)^\gamma f_X(x)\, dx, \tag{4.5.1}$$

where $z$ is the poverty line. Particular cases of the FGT class of poverty measures include $FGT_0$, which is simply the head-count index and as mentioned in Section 4.1, calculates the proportion of households living below the poverty line. Another common measure is $FGT_1$, a normalisation of the income-gap ratio originally introduced by Sen (1976). This poverty measure is commonly referred to as the poverty gap index. In contrast, the poverty severity index, $FGT_2$, is a weighted sum of income short-falls (as a proportion of the poverty line), where the weights are the proportionate income short-falls themselves. Note that, a larger $\gamma$ in (4.5.1) gives greater emphasis to the poorest poor. Hence, this parameter is viewed as a measure of poverty aversion (Foster et al., 1984).

From (4.5.1), one can write

$$FGT_\gamma = \frac{H(z)}{z^\gamma}\mathbb{E}_x\left[D(z,x)|x<z\right],$$

where $H(z) = F_X(z)$ is the head-count index and $D(z,x) = (z-x)^\gamma$ is a function that describes the level of deprivation suffered by an individual whose income $x$ is less than the poverty line $z$. Clearly, $D(z,x)$ is in terms of an individual's income short-fall $y := z - x$.

We now consider a household's capital process as defined in Section 4.1. Under this model, a household's income is generated through capital: $I_t = bX_t$, where $b > 0$ holds (see Equation (4) in Kovacevic and Pflug (2011)). Taking $b = 1$ leads to the case for which a household's income is equal to its capital. Thus, the results obtained in Section 4.4 also apply to a household's income. On this basis, from Section 4.4.2 yields that $Y \sim B1(y; b = x^*, p = 1, q = \alpha)$, where the random variable $Y$ denotes the income short-fall (or income deficit) at trapping given that trapping occurs. In this case, the $FGT_\gamma$ index is given in terms of the $\gamma$th moment of $Y$,

$$FGT_\gamma = \frac{H(z)}{z^\gamma}\mathbb{E}\left[Y^\gamma\right] = H(z)\left(\frac{x^*}{z}\right)^\gamma\frac{B\left(1+\alpha,\gamma\right)}{B\left(1,\gamma\right)},$$

where we used the fact that the $h$th moment of a random variable $W \sim B1(w; b, p, q)$ is given by

$$\mathbb{E}\left[W^h\right] = \frac{b^h B(p+q,h)}{B(p,h)}, \tag{4.5.2}$$

(see, for instance, Table 1 from McDonald (1984)).

*Remark* 4.5.1. One can also compute the $h$th moment of the capital deficit at trapping given that trapping occurs by means of the Gerber-Shiu expected discounted penalty function. Indeed, choosing $w(x_1,x_2) = x_2^h$ yields a modified version of the IDE (4.4.2), with $A(x) = \alpha\cdot x^{*h}\cdot B(\alpha,h+1)\cdot y\left(x\right)^{-\alpha}$ for $y(x) = x/x^*$. Thus, solving (4.4.2) as in Proposition 4.4.1 yields to the $h$th moment of the discounted capital deficit at trapping,

$$\mathbb{E}\left[\mid X_{\tau_x} - x^*\mid^h e^{-\delta\tau_x}; \tau_x < \infty\right] = \alpha\cdot x^{*h}\cdot B\left(\alpha, h+1\right)\cdot m_\delta(x).$$

Setting $\delta = 0$ yields $\mathbb{E}\left[\mid X_{\tau_x} - x^*\mid^h; \tau_x < \infty\right]$, the $h$th moment of the capital deficit at trapping. Consequently, the $h$th moment of the capital deficit at trapping given that trapping occurs is given by

$$\mathbb{E}\left[Y^h\right] := \frac{\mathbb{E}\left[\mid X_{\tau_x} - x^*\mid^h; \tau_x < \infty\right]}{\psi(x)} = \alpha\cdot x^{*h}\cdot B\left(\alpha,h+1\right) = \frac{\alpha\cdot x^{*h}\cdot h\cdot B(\alpha,h)}{h+\alpha}, \tag{4.5.3}$$

where we applied the property $B(p, q+1) = \frac{q \cdot B(p,q)}{p+q}$ of the beta function. Clearly, (4.5.3) is equal to (4.5.2) for the case $b = x^*$, $p = 1$ and $q = \alpha$.

## 4.6 An Application to Burkina Faso's Household Microdata

### 4.6.1 Context and Data

Burkina Faso is located in West Africa with an area of $274,200$ km$^2$. In 2021, the population was estimated at just over 20.3 million, with the capital Ouagadougou being the country's largest city. Historically, its economy has been largely based on agriculture, which provides a living for more than 80% of the population. Burkina Faso's main subsistence crops are sorghum, millet, maize and rice, while the country has been one of Africa's leading producers of cotton and gold (Brugger and Zanetti, 2020; Engels, 2023).

The country's climate is characterised by a dry tropical climate that alternates a short rainy season with a long dry season. Due to its geographical location, bordering the Sahara Desert, Burkina Faso's climate is subject to seasonal and annual variations. Furthermore, the country is divided into three different climatic zones, the Sahelian zone in the north, the North-Sudanian zone in the centre and the South-Sudanian zone in the south, which receive an average annual rainfall of less than 600 mm, between 600 and 900 mm and more than 900 mm, respectively (Alvar-Beltrán et al., 2020).

Household microdata from Burkina Faso's Continuous Multisector Survey (*Enquête Multisectorielle Continue (EMC)) 2014*[2] is used to evaluate the adequacy of the $B1(y; b = x^*, p = 1, q = \alpha)$ model to describe income short-fall distribution. The survey was conducted from 17 January 2014 to 24 November 2014 by the National Institute of Statistics and Demography (*Institut National de la Statistique et de la Démographie* (INSD)). The EMC had as main objective the generation of sound data for poverty analyses. A total of $10,411$ households were interviewed, with a 96.4% of interviews accepted.

The main variable of interest generated in the survey is consumption, which in the EMC is given in units of the West African CFA (*Communauté financière en Afrique*) franc per person per day in average prices in Ouagadougou during the EMC field work. To identify the poor, a minimum food basket of around thirty products was defined. Determining the cost of this food basket and other basic needs, the absolute poverty line was estimated at $153,530$ CFA. A person is poor if he/she lives in a poor household and a household is poor if the annual per capita consumption is below the absolute poverty line which is equivalent to 421 CFA per capita consumption per day.

[2]For a detailed overview of the survey, interested readers may wish to consult the survey's official report (in French): Institut National de la Statistique et de la Démographie (INSD) (2015).

### 4.6.2 Estimators for Parameters of the B1 Model

In this chapter, we use the method-of-moments (MoM) to estimate the parameters $\alpha$ and $x^*$ of the B1 model. Assume that $y_1, y_2, ..., y_n$ is a random sample of income short-fall of size $n$. Letting $M_k = \frac{1}{n}\sum_{i=1}^{n} y_i^k$ denote the $k$th sample moment yields to the method-of-moments estimators (MMEs) for $\alpha$ and $x^*$, given by

$$\hat{\alpha}_{MME} = \frac{2\left[M_2 - M_1^2\right]}{2M_1^2 - M_2} \qquad \text{and} \qquad \hat{x^*}_{MME} = \frac{M_1 \cdot M_2}{2M_1^2 - M_2},$$

respectively. These estimators are derived by equating the first two sample moments ($M_1$ and $M_2$) with the theoretical moments (Equation (4.5.3) for $h = 1, 2$) and by subsequently solving for the two parameters, $\alpha$ and $x^*$. Tables 4.1, 4.2 and 4.3 show the MMEs for $\alpha$ and $x^*$ at a national level, by area of residence and by region, respectively. In addition, the maps in Figure 4.4 display these estimates by region, giving a comprehensive geographical overview of the parameters. These results will be discussed more in detail in Section 4.6.4.

### 4.6.3 Evaluating the Goodness-of-Fit of the B1 Model

The non-parametric one-sample Kolmogorov-Smirnov (KS) test and the $R^2$ coefficient are used to assess the goodness-of-fit of the B1 model. To conduct the KS test, we calculate the KS statistic, which is given by

$$D = \max_{y} \left| F_n(y) - F(y) \right|, \tag{4.6.1}$$

where $F_n(y)$ is the empirical distribution function defined as $F_n(y) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{y_i \le y\}}$ and $F(y)$ is (4.4.16), the distribution function of the B1 model. The null ($H_0$) and alternative ($H_1$) hypotheses of the KS goodness-of-fit test are:

$H_0$: *the household income short-fall data follows the B1 model* and

$H_1$: *the household income short-fall data does not follows the B1 model.*

The null hypothesis $H_0$ is rejected at a significance level $\alpha_{KS}$ if the p-value of the KS statistic is less than $\alpha_{KS}$. The p-value is computed based on the limiting distribution of the KS statistic (4.6.1) (Marsaglia et al., 2003),

$$\lim_{n\to\infty} \mathbb{P}(\sqrt{n}D \le y) = \frac{\sqrt{2\pi}}{y} \sum_{i=1}^{\infty} e^{-(2i-1)^2\pi^2/\left(8y^2\right)}.$$

We further support the KS test by considering the $R^2$ coefficient, which quantifies the degree of correlation between the observed and predicted probabilities under an assumed distribution. Here, a value of $R^2$ that is close to one indicates that the B1 model is a good fit for the household income short-fall data. The $R^2$ coefficient is computed as follows:

$$R^2 = \frac{\sum_{i=1}^{n}\left[\hat{F}\left(y_i\right) - \bar{F}(y)\right]^2}{\sum_{i=1}^{n}\left[\hat{F}\left(y_i\right) - \bar{F}(y)\right]^2 + \sum_{i=1}^{n}\left[F_n\left(y_i\right) - \hat{F}\left(y_i\right)\right]^2},$$

where $F_n(y_i)$ is the empirical distribution function for the $i$th household income short-fall, $\hat{F}(y_i)$ is the estimated distribution function for the $i$th household income short-fall under the B1 model and $\bar{F}(y)$ is the average of $\hat{F}(y_i)$.

To verify our assumptions and model specifications, we also consider graphical methods. We plot the distribution function (4.4.16) and the p.d.f. (4.4.17) against the empirical distribution $F_n(y)$ and the histogram of the observed income short-fall data, respectively. In addition, we use the B1 model quantile-quantile (Q-Q) and probability-probability (P-P) plots to support the assumption of a $B1(y; b = x^*, p = 1, q = \alpha)$ distribution. Appendices 4.A, 4.B and 4.C show these graphical methods at a national level, by area of residence and by region, respectively.

### 4.6.4 Results and Discussion

The assumption of the B1 distribution can be investigated based on the p-value of the Kolmogorov-Smirnov (KS) statistic and the $R^2$ coefficient, which are shown in Tables 4.1, 4.2 and 4.3. As shown in these tables, with the exception of the estimates at a national level, all p-values of the KS test are higher than the significance level of $\alpha_{KS} = 0.05$. This indicates that the B1 model significantly describes the household income short-fall data by both area of residence and by region. This is borne out by the estimated values for the $R^2$ coefficient, which are found higher than 0.99 for almost all the cases, suggesting that the B1 model explains more than 99% of the variation in the data, but the remaining (less than 1%) variation is attributed to errors and cannot be explained by the model. The Cascades region is the only case attaining a lower value for the $R^2$ coefficient: 0.9870, which is nevertheless still very close to one, so that the model can still be considered to describe a large part of the variation in the data.

| Country | Type | $\alpha$ | $x^*$ | p-value (KS test) | $R^2$ | Poverty Gap Index ($FGT_1$) | Poverty Severity Index ($FGT_2$) |
|---|---|---|---|---|---|---|---|
| Burkina Faso | Direct | - | - | - | - | 0.096 | 0.032 |
| Burkina Faso | MME | 1.50 | 87,209.01 | 0.02379 | 0.9983 | 0.091 | 0.029 |

*p-value> $\alpha_{KS} = 0.05$.

Table 4.1: B1 distribution fitted parameters and poverty measures for Burkina Faso.

Graphical methods displayed in Appendices 4.A, 4.B and 4.C provide an additional tool to evaluate the assumption of the B1 distribution. Plot (a) in the appendices shows a density plot in which are plotted the p.d.f. (4.4.17) of a B1 model and the histogram of the observed income short-fall data. On the other hand, Plot (c) in the appendices displays a distribution plot in which the B1 distribution function (4.4.16) is plotted against the empirical distribution function $F_n(y)$. From these plots, one can observe that the B1 model fits the histogram and the empirical distribution well, respectively. Moreover, if the household income short-fall data is found to follow the B1 model, observations on both the quantile-quantile (Q-Q) and probability-probability (P-P) plots will appear to form almost a straight diagonal line. In general, household income short-fall data is positioned in the diagonal by area of residence and by region, suggesting the appropriateness of the B1 distribution[3]. In particular, one cannot expect the observed income short-fall data points to follow the reference diagonal line in the upper quantiles as income short-fall data is sparse in the tails.

| Area of Residence | Type | $\alpha$ | $x^*$ | p-value (KS test) | $R^2$ | Poverty Gap Index ($FGT_1$) | Poverty Severity Index ($FGT_2$) |
|---|---|---|---|---|---|---|---|
| Urban | Direct | - | - | - | - | 0.052 | 0.016 |
| Urban | MME | 1.53 | 82,925.53 | 0.4103* | 0.9961 | 0.051 | 0.016 |
| Rural | Direct | - | - | - | - | 0.118 | 0.040 |
| Rural | MME | 1.48 | 88,002.48 | 0.0754* | 0.9985 | 0.111 | 0.036 |

*p-value$> \alpha_{KS} = 0.05$.

Table 4.2: B1 distribution fitted parameters and poverty measures by area of residence.

As underlined in Section 4.6.2, parameter estimates are based on the Method of Moments (MoM) and are shown in Tables 4.1, 4.2 and 4.3. From Section 4.1, one can realise that under the assumption of $Beta(\alpha, 1)-$distributed remaining proportions of capital, higher values of $\alpha$ yield to a greater expected remaining proportion of capital upon experiencing a capital loss (i.e. the distribution is left-skewed or equivalently, the remaining proportions of capital $Z_i \in [0, 1]$ are more likely to have values close to one). On this basis, it is possible to assess the magnitude of capital losses experienced by households in Burkina Faso. For instance, Figure 4.4a displays how households experience capital losses of varying magnitude depending on the geographical area in which they reside. In fact, Figure 4.4a shows interestingly that these magnitudes appear to be linked to or dependent on the different climatic zones[4]. These findings are in line with previous research, which highlights the country's economy dependence on rain-fed agriculture and livestock husbandry, which in turn makes it vulnerable to climate risks such as droughts and floods (see, for example, Zampaligré et al. (2014)). In particular, Table 4.2 and Figure 4.4b show that the estimates of the critical capital $x^*$ for Hauts-Bassins, Centre-Est, Centre and Boucle du Mouhoun are higher (greater than 90,000 CFA), compared to those for the rest of regions, suggesting that the country's potentially poorest households live in these regions. However, it is important to note that the poverty gap index ($FGT_1$) and the poverty severity index ($FGT_2$) shown in Table 4.2 for Boucle du Mouhoun attain higher values due to the fact that the head-count index is higher (56%) compared to that of Hauts-Bassins (35%), Centre-Est (34%) and Centre (17%). Similarly, households residing in the Nord region seem to experience the most adverse capital losses, as its estimated value for $\alpha$ shown in Table 4.2 is the lowest among all regions. Moreover, Nord's high head-count index (65%) and its uniform distribution shape are major contributors to the high poverty gap ($FGT_1$) and poverty severity ($FGT_2$) index.

[3]This is also true at a national level. However, recall that the null hypothesis $H_0$ for the KS test was rejected.

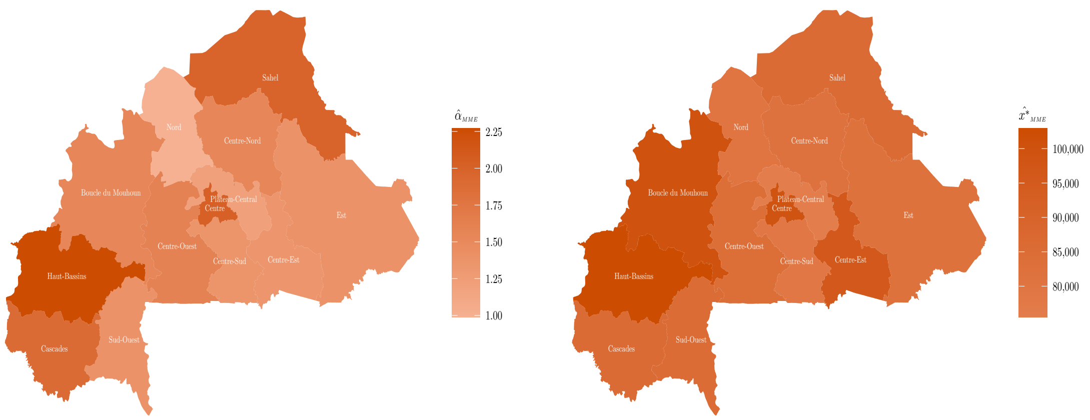
[4]See Alvar-Beltrán et al. (2020) for a detailed map of Burkina Faso with the different climatic zones.

| Region | Type | $\alpha$ | $x^*$ | p-value (KS test) | $R^2$ | Poverty Gap Index ($FGT_1$) | Poverty Severity Index ($FGT_2$) |
|---|---|---|---|---|---|---|---|
| Boucle du Mouhoun | Direct | - | - | - | - | 0.143 | 0.052 |
| Boucle du Mouhoun | MME | 1.54 | 91,848.66 | 0.8662* | 0.9986 | 0.131 | 0.044 |
| Cascades | Direct | - | - | - | - | 0.038 | 0.011 |
| Cascades | MME | 1.89 | 86,053.24 | 0.7089* | 0.9870 | 0.038 | 0.011 |
| Centre | Direct | - | - | - | - | 0.037 | 0.012 |
| Centre | MME | 2.03 | 99,257.41 | 0.9487* | 0.9974 | 0.037 | 0.012 |
| Centre-Est | Direct | - | - | - | - | 0.096 | 0.036 |
| Centre-Est | MME | 1.34 | 95,997.68 | 0.493* | 0.9931 | 0.092 | 0.035 |
| Centre-Nord | Direct | - | - | - | - | 0.082 | 0.026 |
| Centre-Nord | MME | 1.55 | 81,599.65 | 0.8051* | 0.9937 | 0.075 | 0.023 |
| Centre-Ouest | Direct | - | - | - | - | 0.107 | 0.034 |
| Centre-Ouest | MME | 1.60 | 84,363.89 | 0.3283* | 0.9930 | 0.099 | 0.030 |
| Centre-Sud | Direct | - | - | - | - | 0.095 | 0.030 |
| Centre-Sud | MME | 1.37 | 79,048.83 | 0.9878* | 0.9978 | 0.090 | 0.027 |
| Est | Direct | - | - | - | - | 0.109 | 0.035 |
| Est | MME | 1.39 | 82,056.03 | 0.9768* | 0.9980 | 0.104 | 0.033 |
| Hauts-Bassins | Direct | - | - | - | - | 0.076 | 0.025 |
| Hauts-Bassins | MME | 2.27 | 102,924.19 | 0.7671* | 0.9974 | 0.072 | 0.023 |
| Nord | Direct | - | - | - | - | 0.176 | 0.063 |
| Nord | MME | 0.99 | 79,977.12 | 0.1357* | 0.9928 | 0.170 | 0.059 |
| Plateau Central | Direct | - | - | - | - | 0.104 | 0.034 |
| Plateau Central | MME | 1.22 | 75,536.12 | 0.7121* | 0.9963 | 0.097 | 0.030 |
| Sahel | Direct | - | - | - | - | 0.050 | 0.015 |
| Sahel | MME | 1.98 | 86,059.27 | 0.9505* | 0.9973 | 0.048 | 0.013 |
| Sud-Ouest | Direct | - | - | - | - | 0.081 | 0.027 |
| Sud-Ouest | MME | 1.39 | 85,579.68 | 0.4962* | 0.9941 | 0.082 | 0.027 |

*p-value> $\alpha_{_{KS}} = 0.05$.

Table 4.3: B1 distribution fitted parameters and poverty measures by region.

The robustness of the poverty gap index ($FGT_1$) and the poverty severity index ($FGT_2$) at a national level, by area of residence and by region, when specifying the B1 model as the income short-fall distribution can be evaluated in Tables 4.1, 4.2 and 4.3, respectively. Comparing the estimates using the B1 distribution assumption with the direct (empirical) values of the poverty measures, one can see how close the estimates are to the direct values of the FGT indices, thus reinforcing the assumption of the B1 distribution for income short-fall data.




(a) (b)

Figure 4.4: Method of moment estimators (MMEs) by region: (a) $\hat{\alpha}_{_{MME}}$ (b) $\hat{x^*}_{_{MME}}$.

In Figure 4.5, we contrast the level of poverty and the changes that would have occurred in the poverty level of selected regions. Both the poverty gap and the poverty severity index show an enormous progress in poverty reduction for greater expected remaining proportion of capital upon experiencing a capital loss (i.e. higher values of $\alpha$). Higher values for $\alpha$ can be attained with risk mitigation strategies such as subsidised insurance programmes (Flores-Contró et al., 2021). Similarly, as expected, Figure 4.5 shows that for higher values of critical capital $x^*$, poverty level increases.

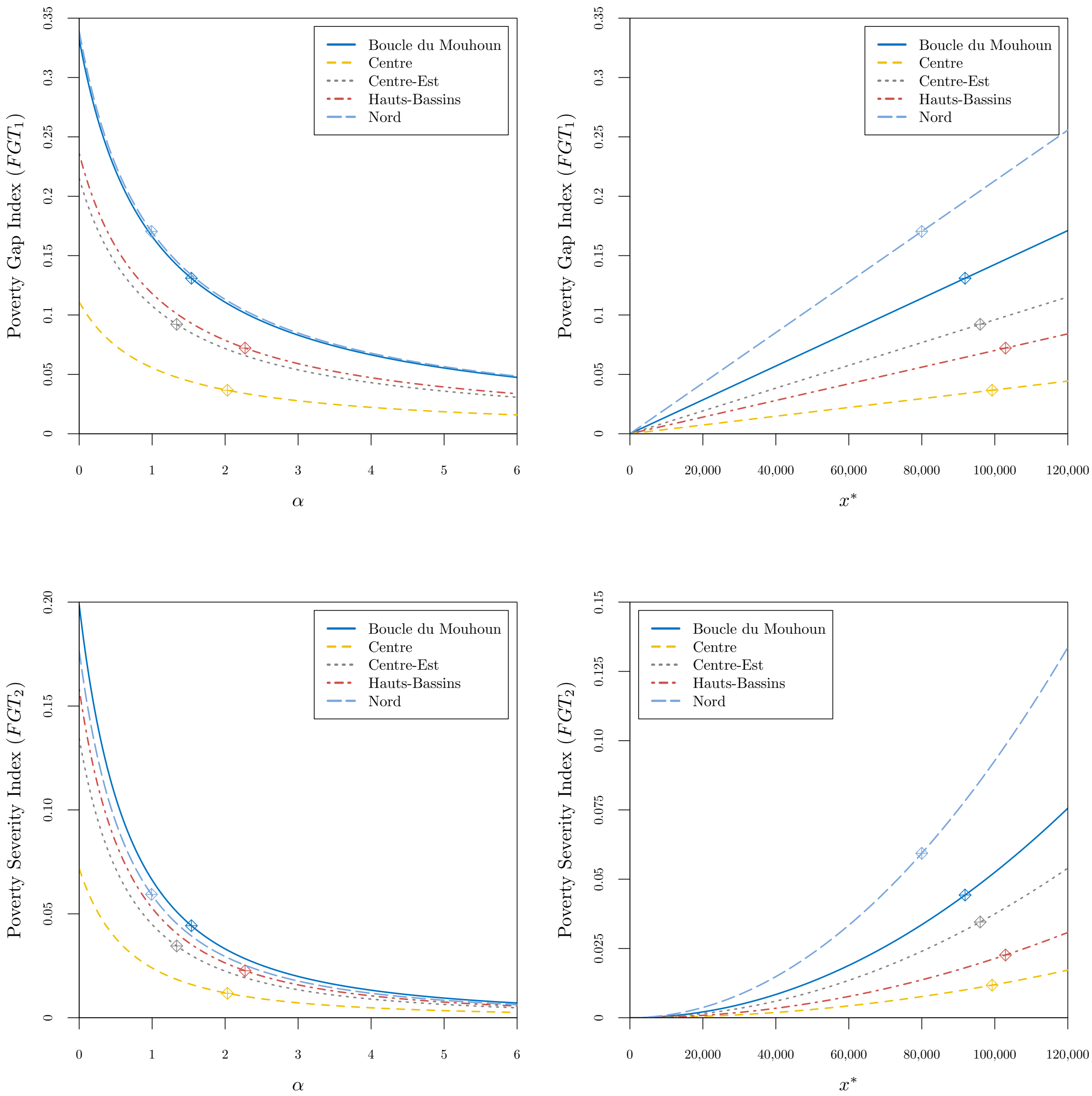



Figure 4.5: Sensitivity of the poverty gap and the poverty severity index to $\alpha$ and $x^*$ for selected regions. Markers show the actual values of the poverty measures.

## 4.7 Conclusion

This chapter studies the Gerber-Shiu expected discounted penalty function for the household capital process introduced in Kovacevic and Pflug (2011). The Gerber-Shiu function incorporates information on the trapping time, the capital surplus immediately before trapping and the capital deficit at trapping. Recent work focuses on only analysing the infinite-time trapping probability (Henshaw et al., 2023), therefore overlooking quantities of particular interest such as the undershoot and the overshoot of a household's capital at trapping. To the best of our knowledge, we derive for the first time a functional equation for the Gerber-Shiu function and we solve it for the particular case in which the remaining proportions of capital upon experiencing a capital loss are $Beta(\alpha, 1)-$distributed. As a result, we obtain closed-form expressions for important quantities such as the Laplace transform of the trapping time and the distribution of the capital deficit at trapping. These quantities are particularly important as they provide crucial information towards understanding a household's transition into poverty.

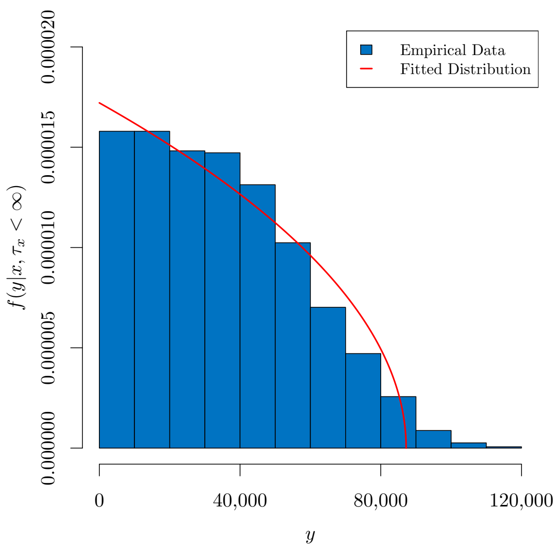
Using risk theory techniques, we derive a microeconomic foundation for the beta of the first kind (B1) as a suitable model to represent the distribution of personal income deficit (or income short-fall). It is indeed interesting that our findings are in line with previous research in development economics, where the generalised beta (GB) distribution family and its derivatives (including the B1 model) have shown to be appropriate models to describe the distribution of personal income.

Affinities between the capital deficit at trapping and a class of poverty measures, known as the Foster-Greer-Thorbecke (FGT) index, are also presented. In addition, we provide empirical evidence of the suitability of the B1 distribution for modelling Burkina Faso's household income short-fall data from the Continuous Multisector Survey (*Enquête Multisectorielle Continue (EMC)) 2014*. Indeed, in this chapter, the B1 model is fitted to Burkina Faso's household income short-fall data, and it is found that the B1 distribution fitted to the data well, suggesting that this model is appropriate for describing the income short-fall distribution. Moreover, we show how the poverty gap index and the poverty severity index can be calculated from the estimated B1 income short-fall distribution. One of the main advantages of parametric distributions such as the B1 distribution is that (poverty) indicators can be presented as functions of the parameters of the chosen distribution. Thus parametric modelling allows to gain insight into the relationship between (poverty) indicators and the distribution of the parameters.

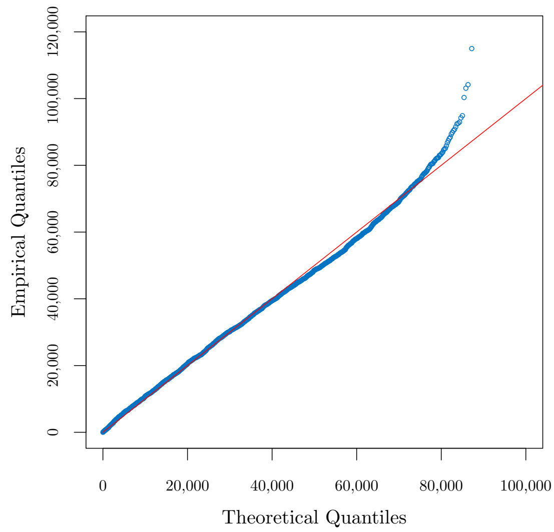
Future research can consider other distributions supported in $[0, 1]$ for the remaining proportions of capital. In this way, one could arrive at other distributions for the capital deficit at trapping that have also been used previously to model personal income (e.g. the lognormal distribution and the power-law distribution). However, this is not straightforward, as finding a closed-form solution for the Integro-Differential Equation (IDE) derived in Theorem 4.4.1 when considering more general distributions for the remaining proportion of capital is challenging. In addition, it might also be interesting to carry out the same analysis with household microdata from other countries in order to verify the results obtained with Burkina Faso's EMC.

## 4.A Appendix A: Goodness-of-Fit Plots for Burkina Faso

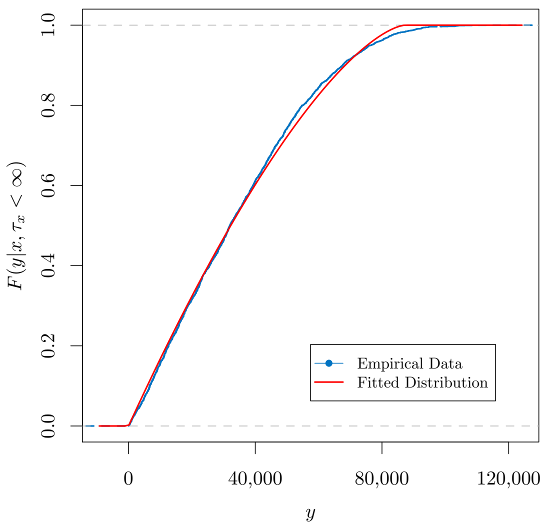
Burkina Faso



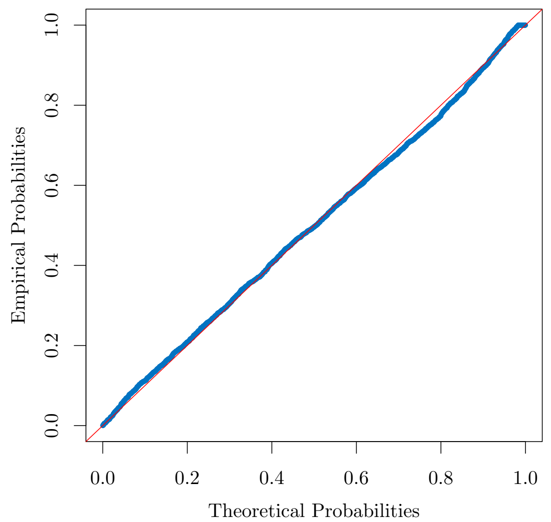

(a) Comparison of Probability Density




(b) Quantile-Quantile (Q-Q) Plot



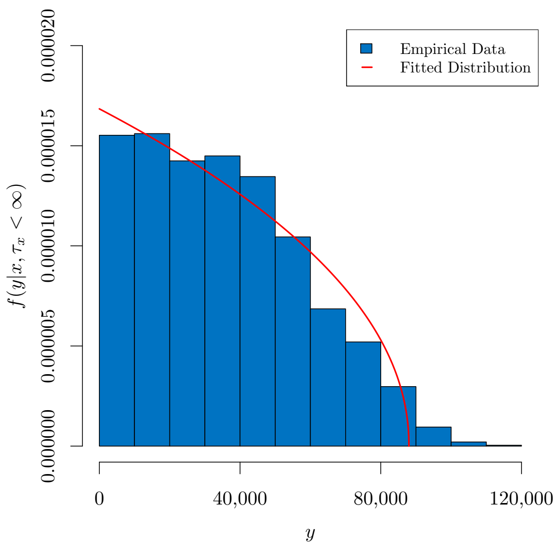

(c) Comparison of Probability Distributions



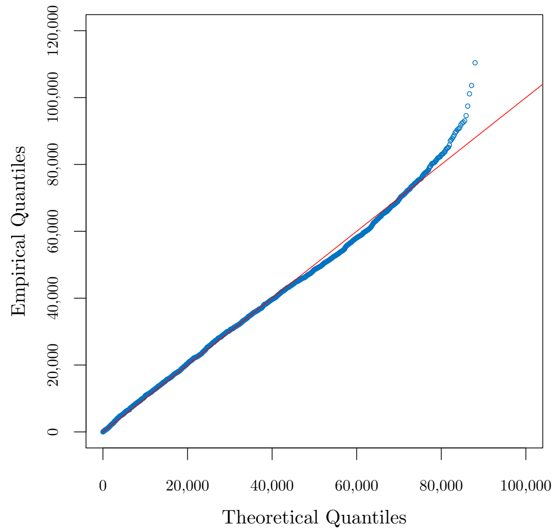

(d) Probability-Probability (P-P) Plot

## 4.B Appendix B: Goodness-of-Fit Plots by Area of Residence

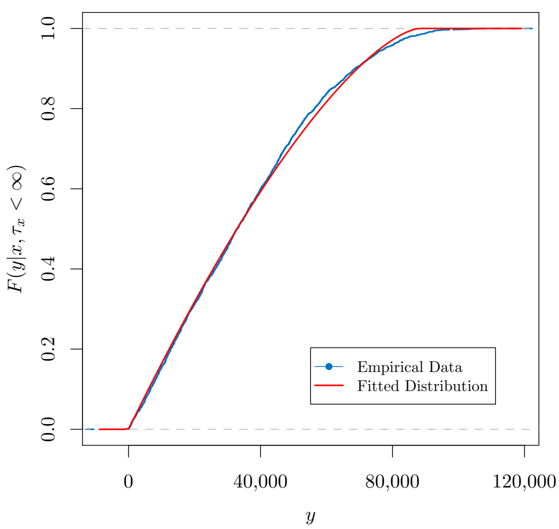
Rural



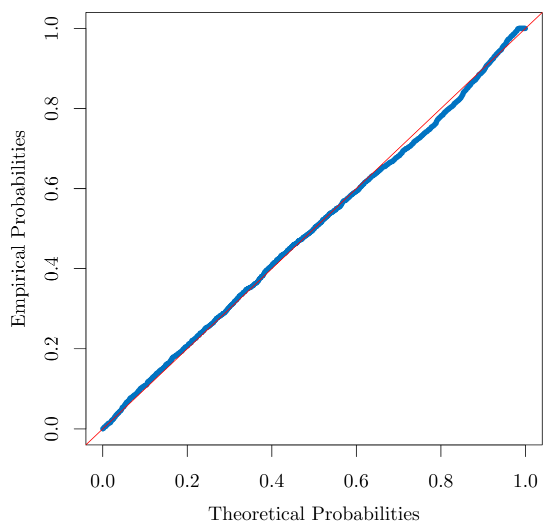

(a) Comparison of Probability Density




(b) Quantile-Quantile (Q-Q) Plot

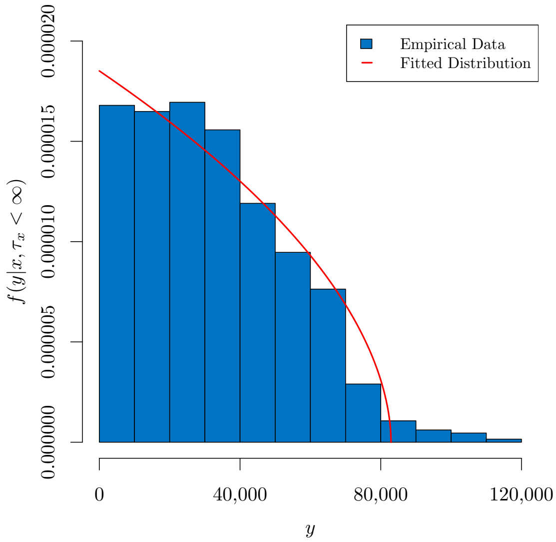



(c) Comparison of Probability Distributions

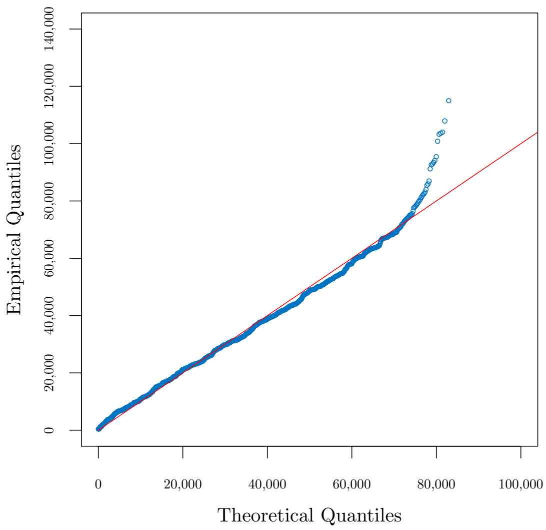



(d) Probability-Probability (P-P) Plot

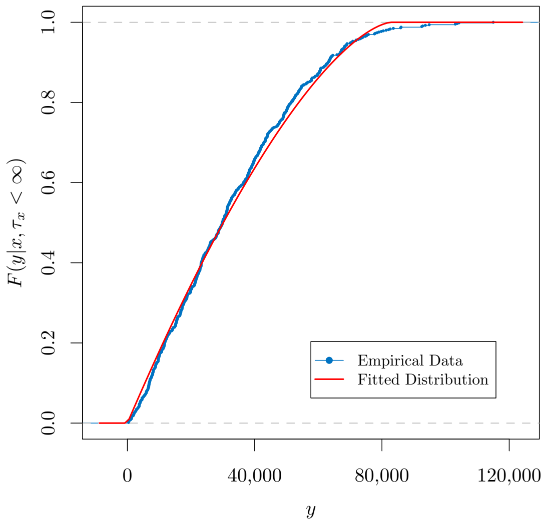
Urban



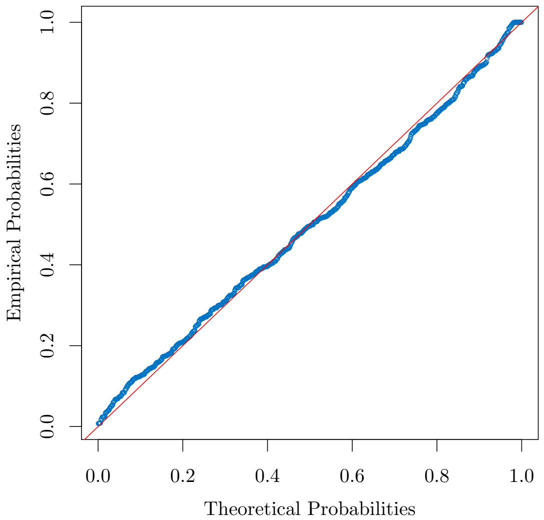

(a) Comparison of Probability Density




(b) Quantile-Quantile (Q-Q) Plot



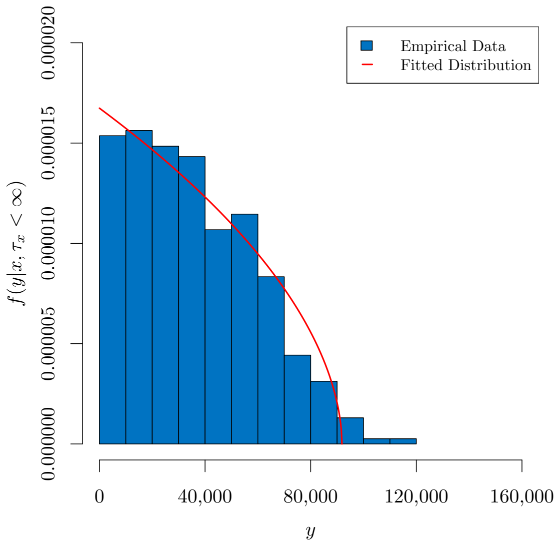

(c) Comparison of Probability Distributions



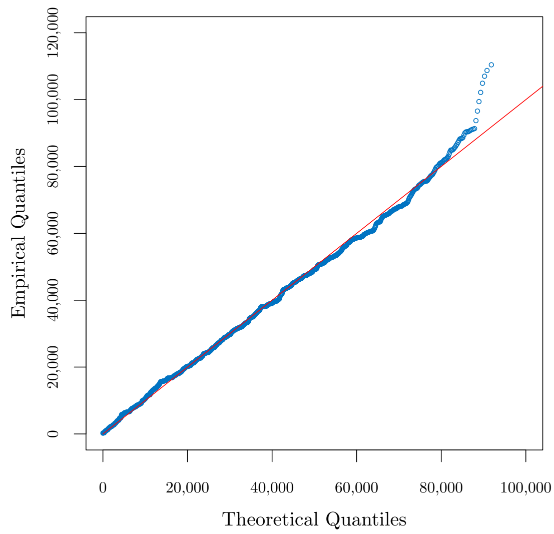

(d) Probability-Probability (P-P) Plot

## 4.C Appendix C: Goodness-of-Fit Plots by Region

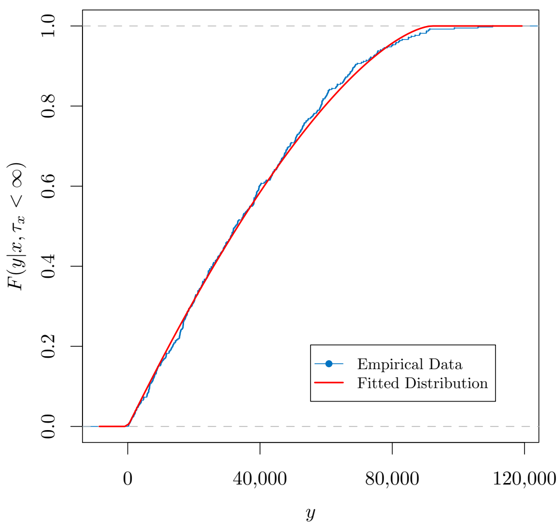
Boucle du Mouhoun



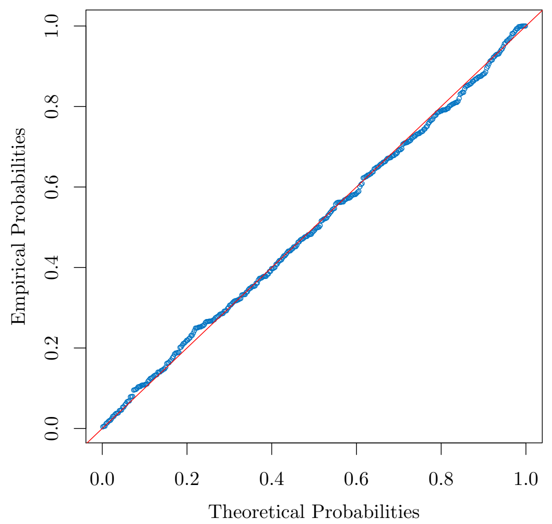

(a) Comparison of Probability Density




(b) Quantile-Quantile (Q-Q) Plot

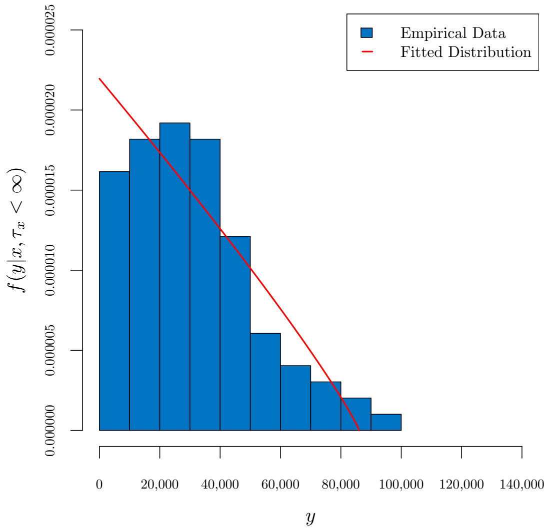



(c) Comparison of Probability Distributions

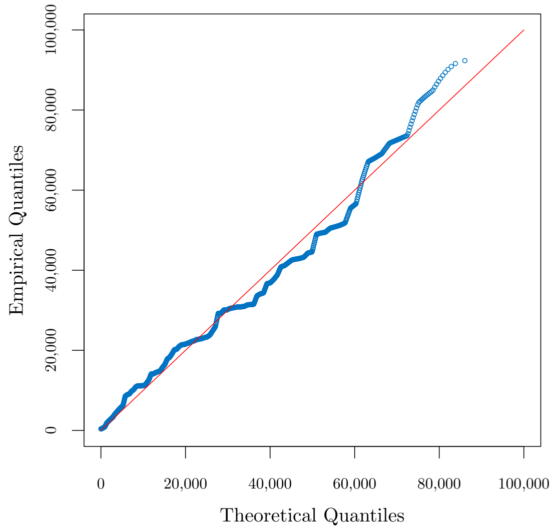



(d) Probability-Probability (P-P) Plot

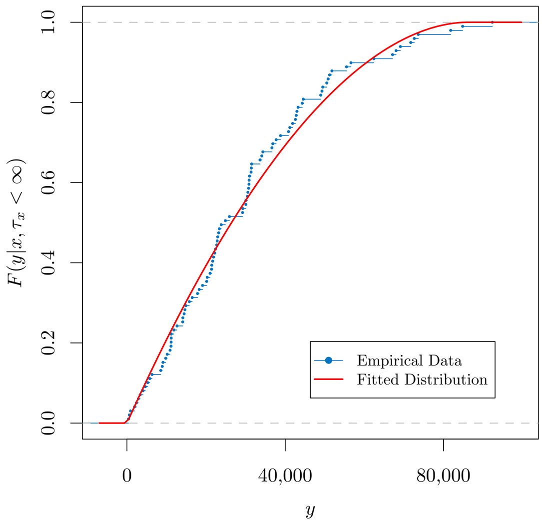
## Cascades



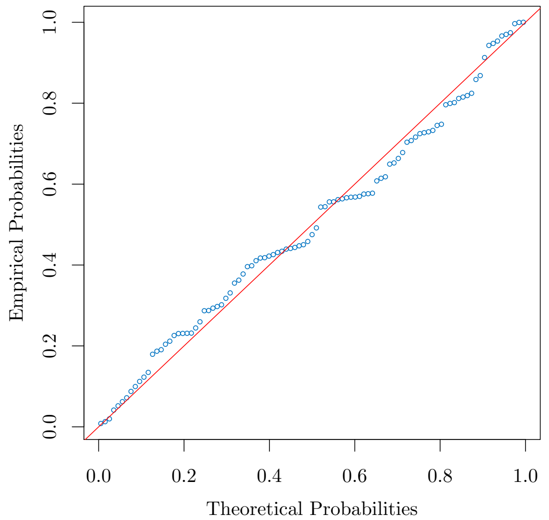

(a) Comparison of Probability Density




(b) Quantile-Quantile (Q-Q) Plot

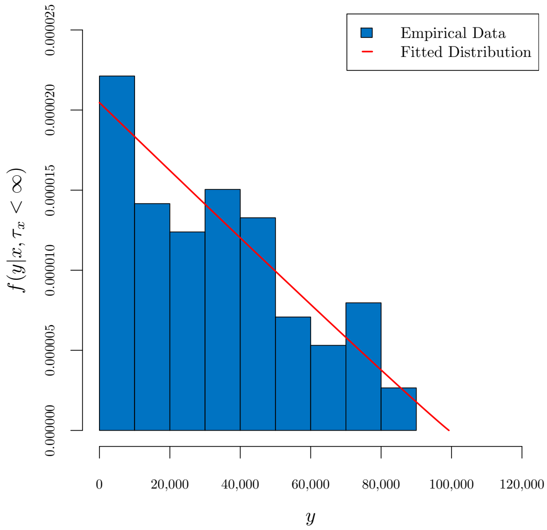



(c) Comparison of Probability Distributions

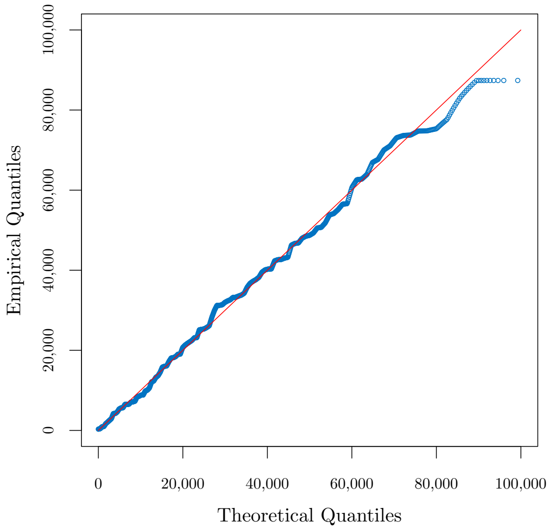



(d) Probability-Probability (P-P) Plot

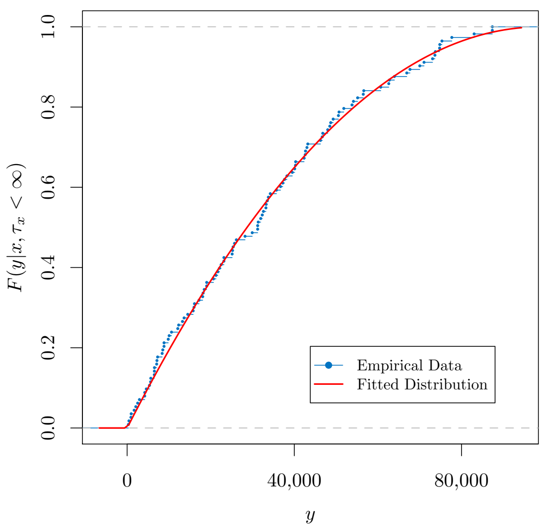
Centre



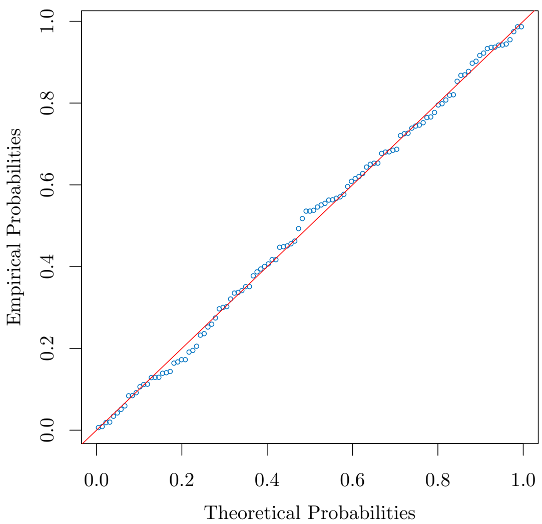

(a) Comparison of Probability Density




(b) Quantile-Quantile (Q-Q) Plot

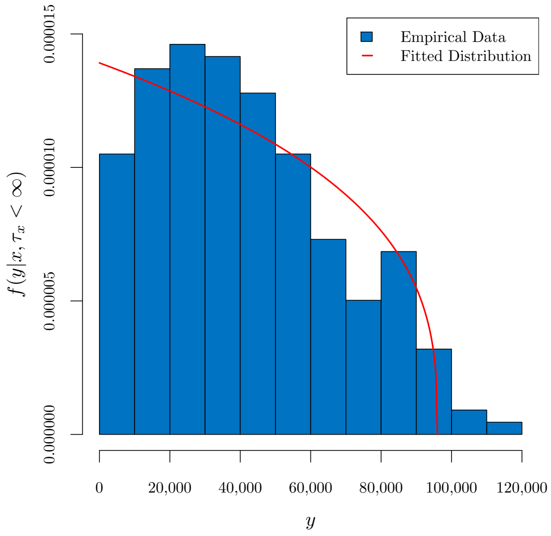



(c) Comparison of Probability Distributions

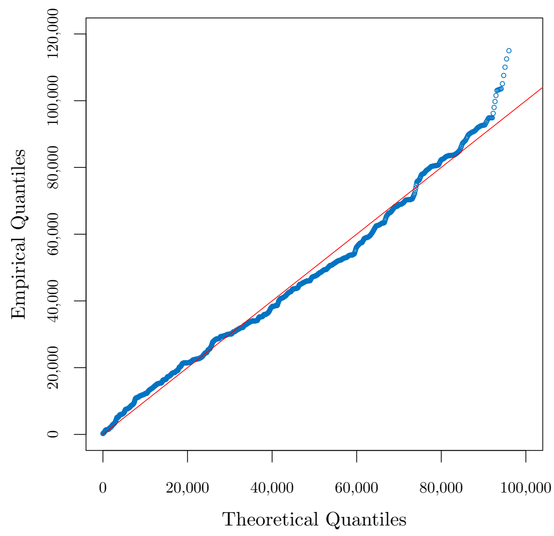



(d) Probability-Probability (P-P) Plot

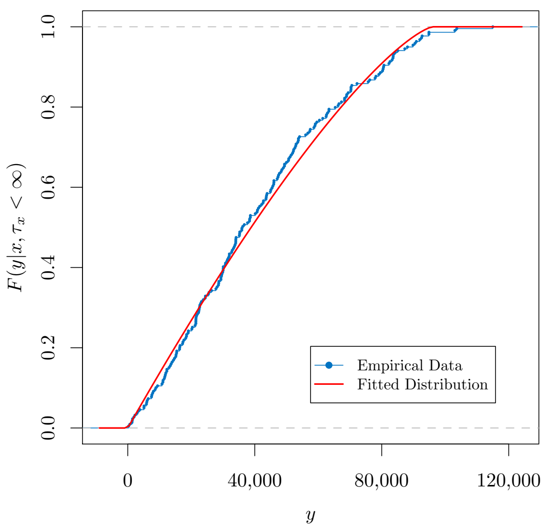
## Centre-Est



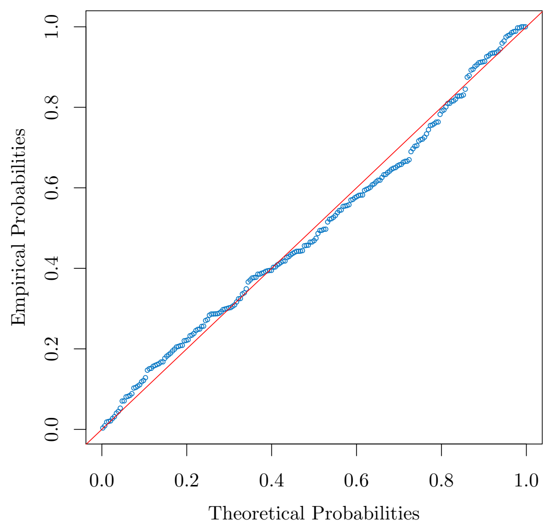

(a) Comparison of Probability Density




(b) Quantile-Quantile (Q-Q) Plot

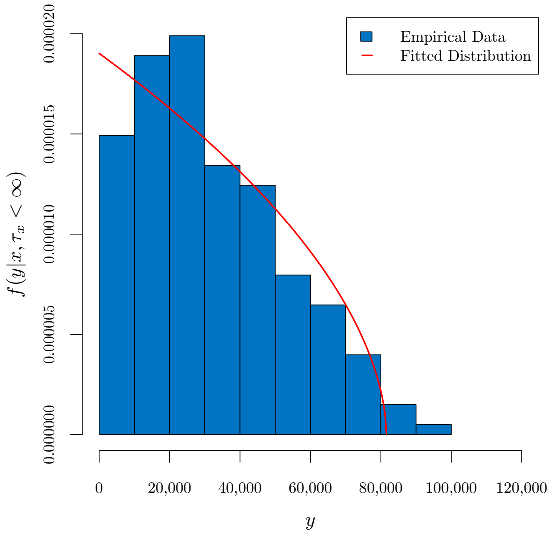



(c) Comparison of Probability Distributions




(d) Probability-Probability (P-P) Plot

Centre-Nord

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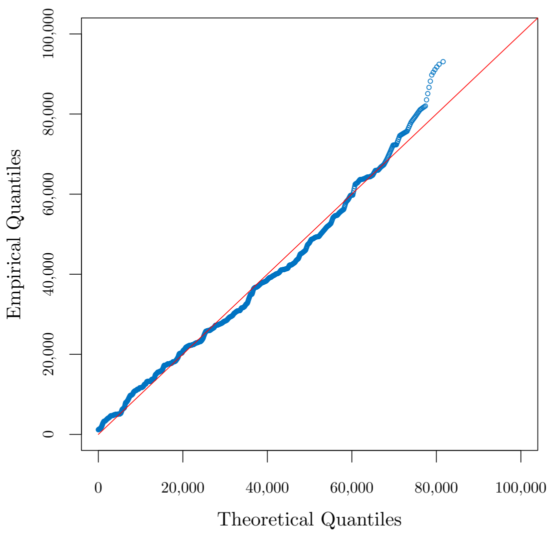


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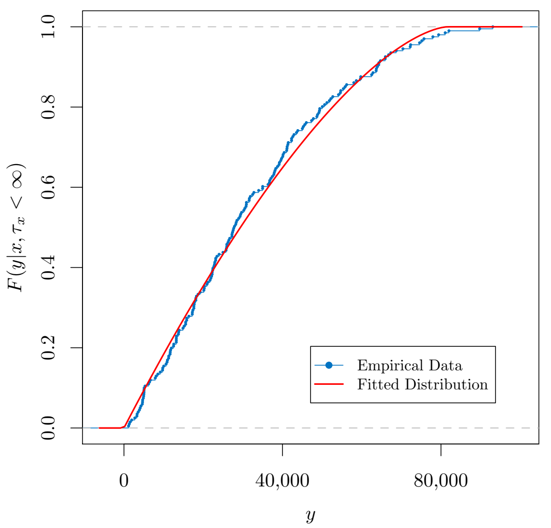


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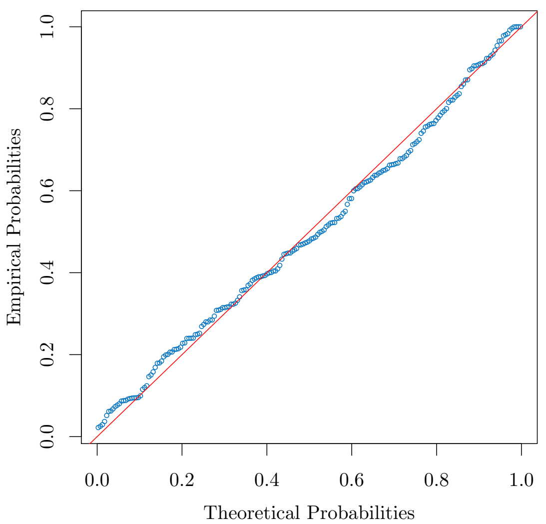
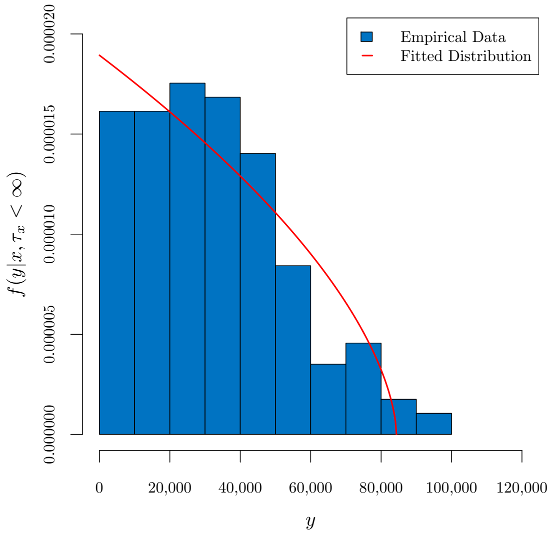



(d) Probability-Probability (P-P) Plot

## Centre-Ouest

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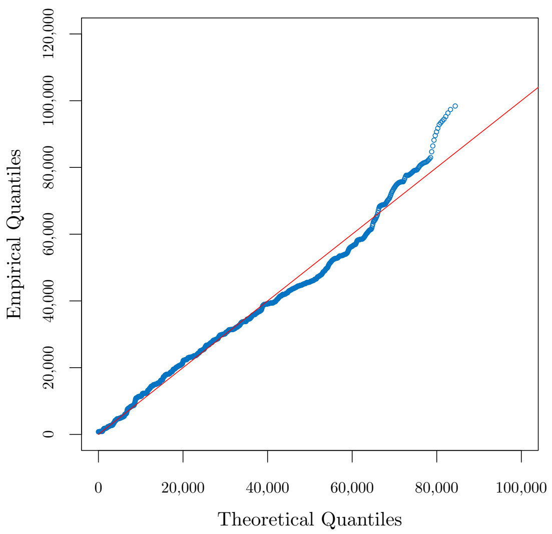


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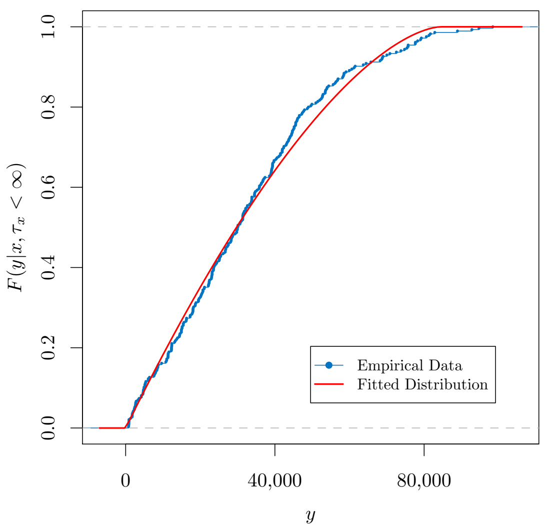


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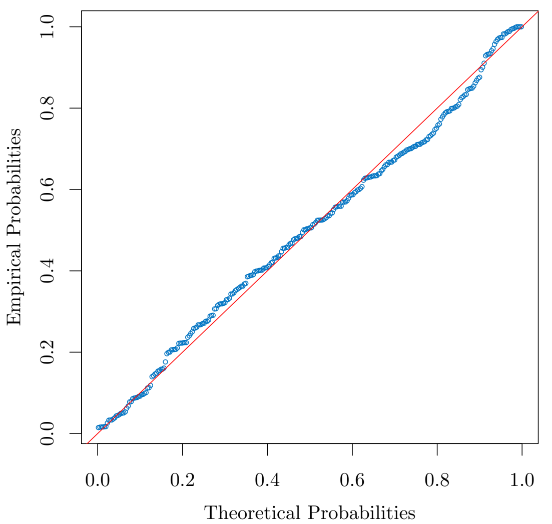
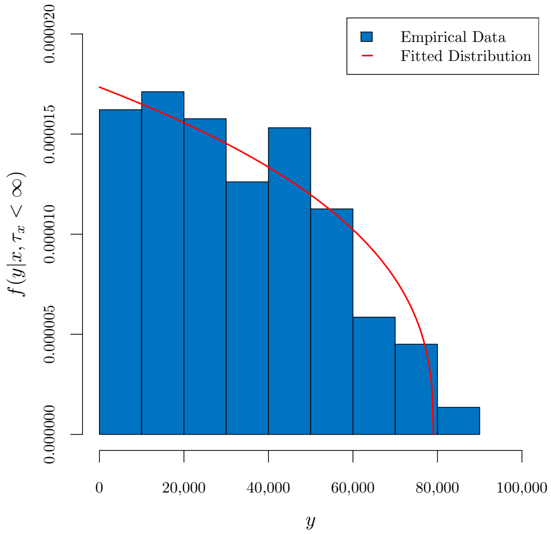



(d) Probability-Probability (P-P) Plot

Centre-Sud

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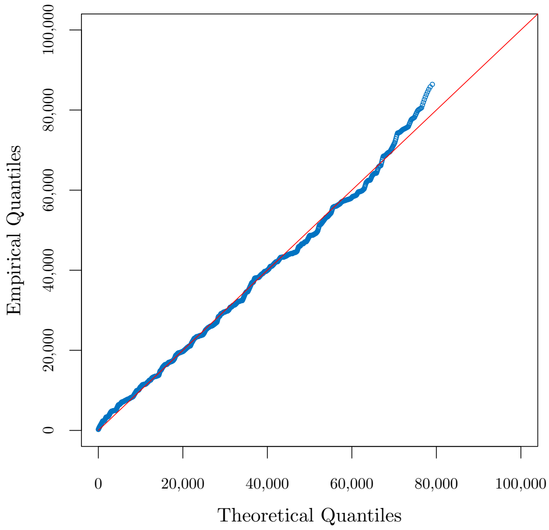


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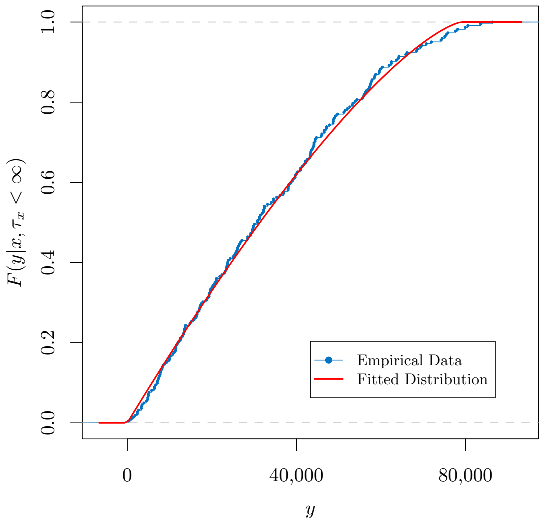


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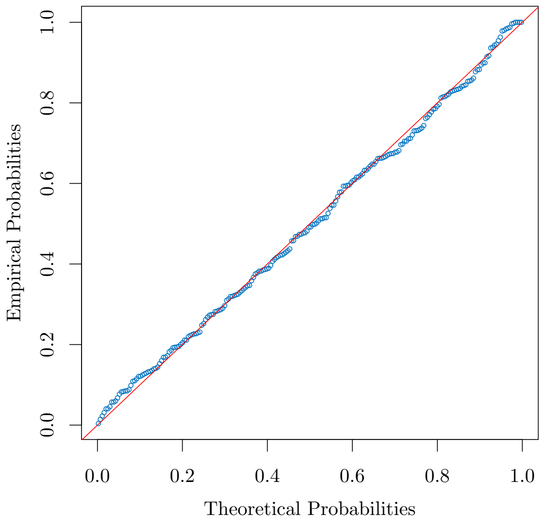


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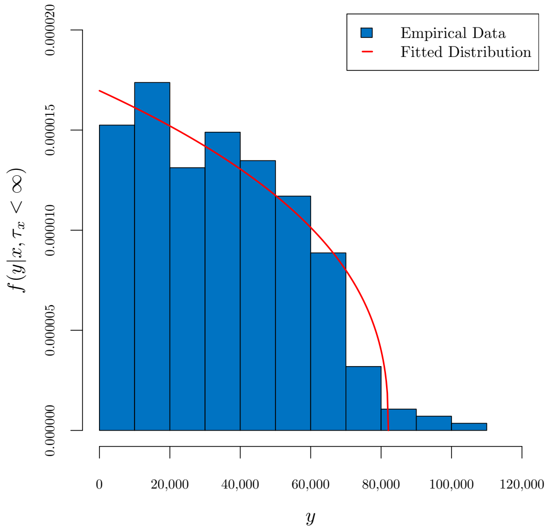
Est

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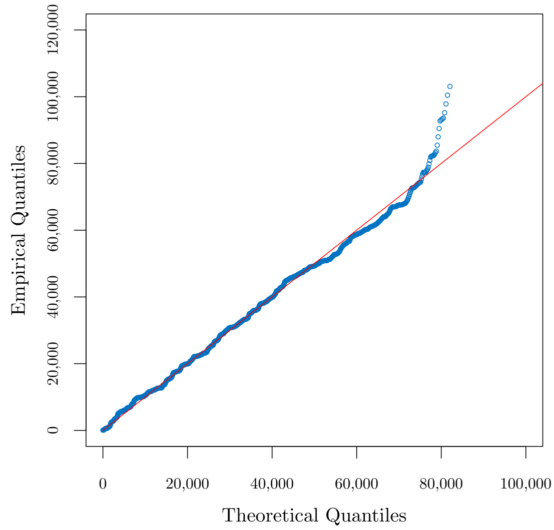


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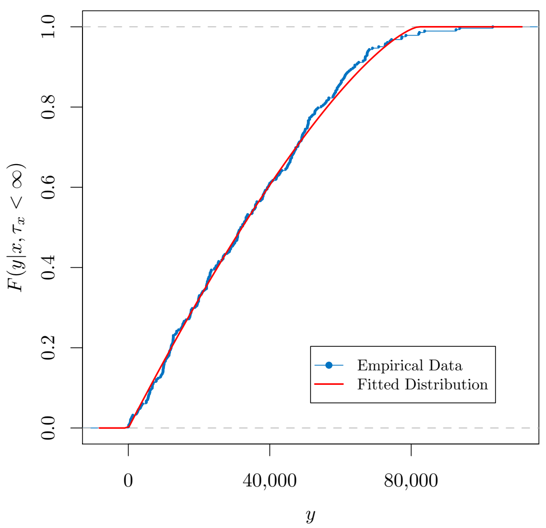


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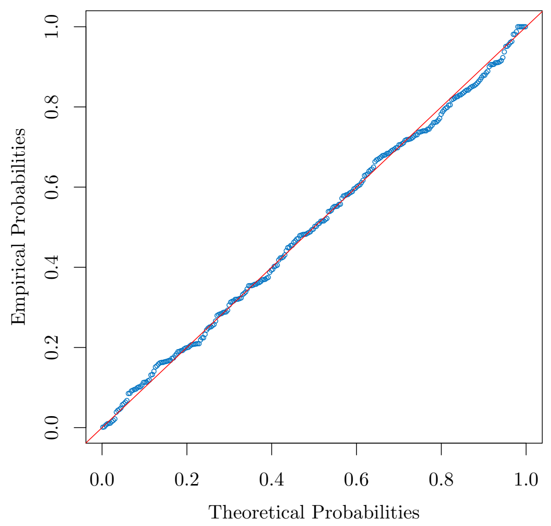
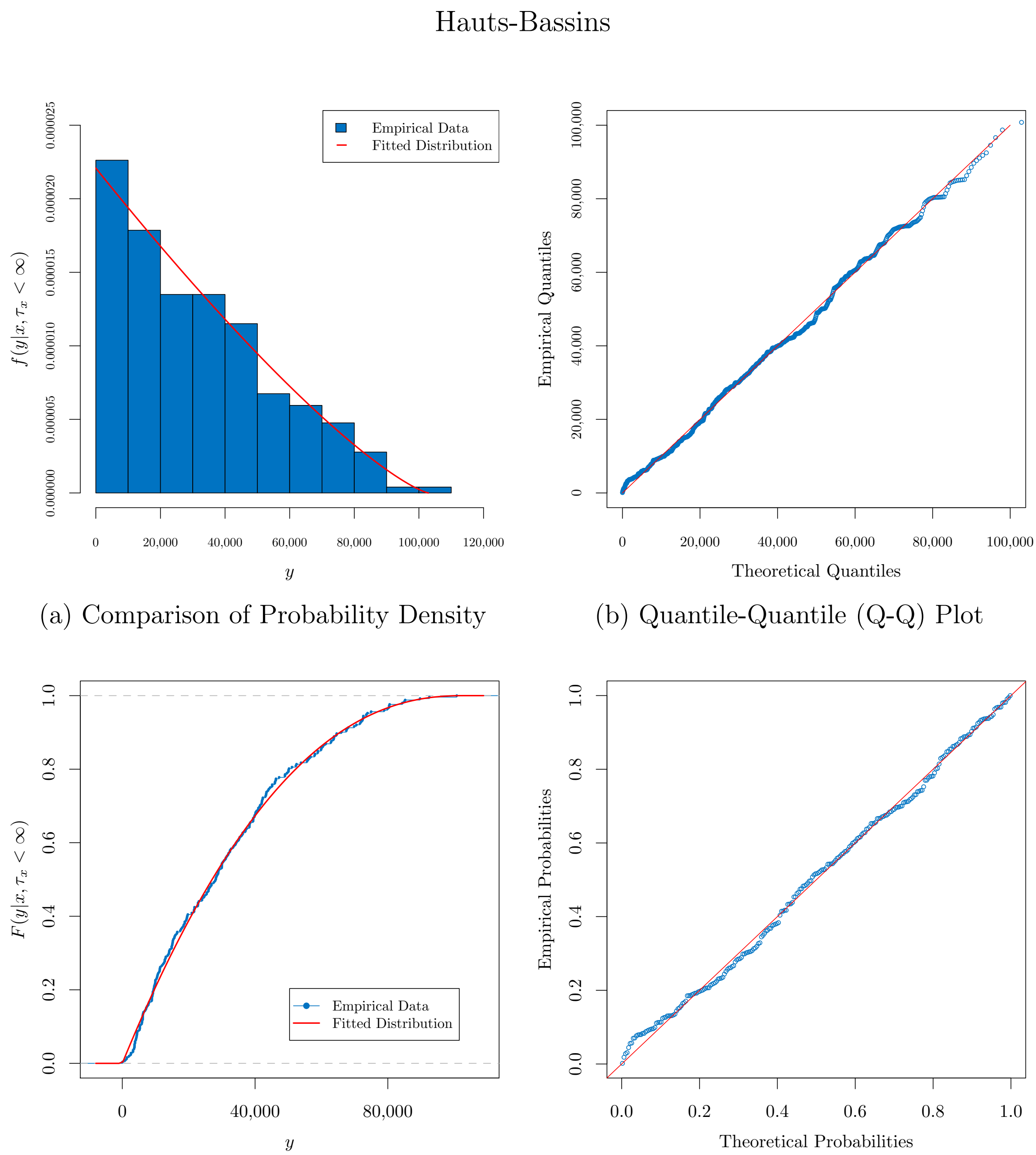



(d) Probability-Probability (P-P) Plot



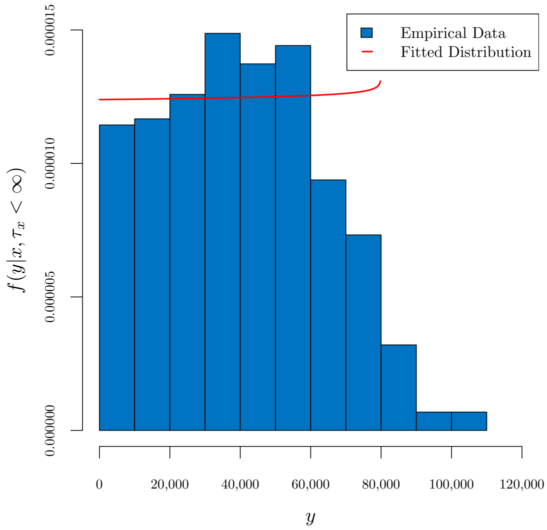

(a) Comparison of Probability Density

(b) Quantile-Quantile (Q-Q) Plot

(c) Comparison of Probability Distributions

(d) Probability-Probability (P-P) Plot

Nord

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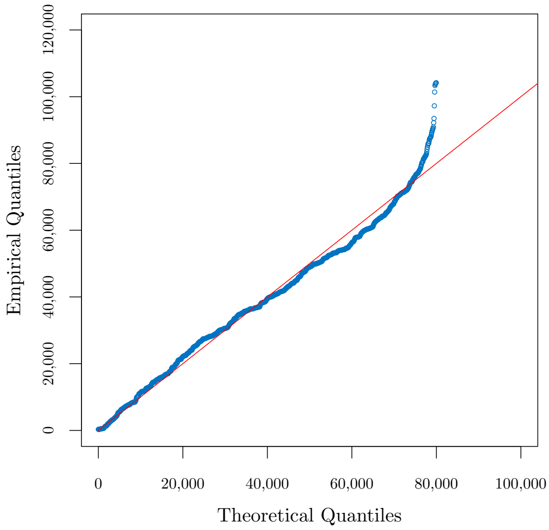


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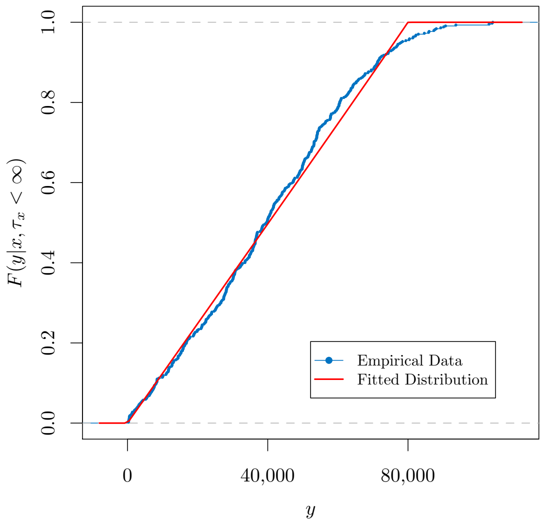
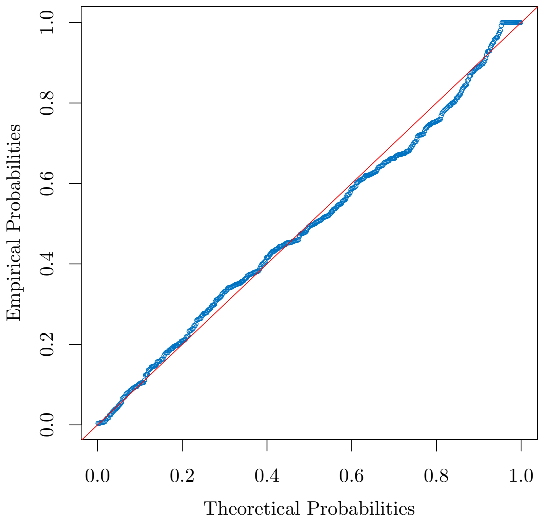
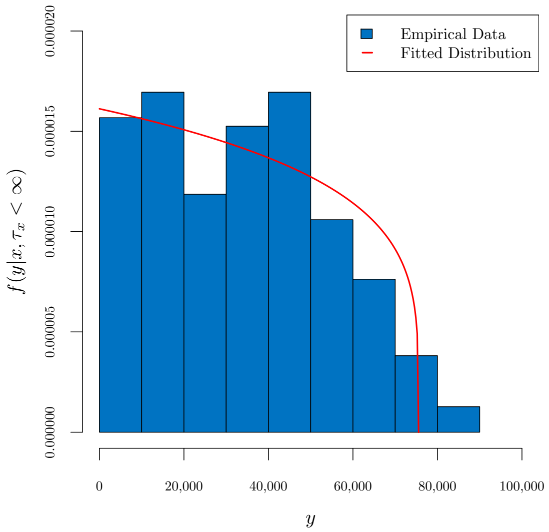



(c) Comparison of Probability Distributions




(d) Probability-Probability (P-P) Plot

Plateau Central

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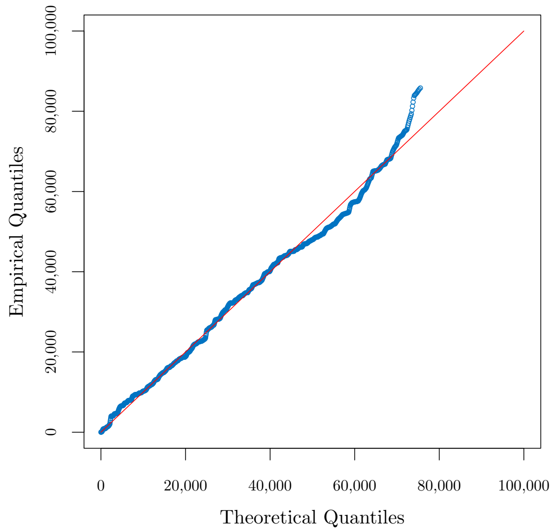


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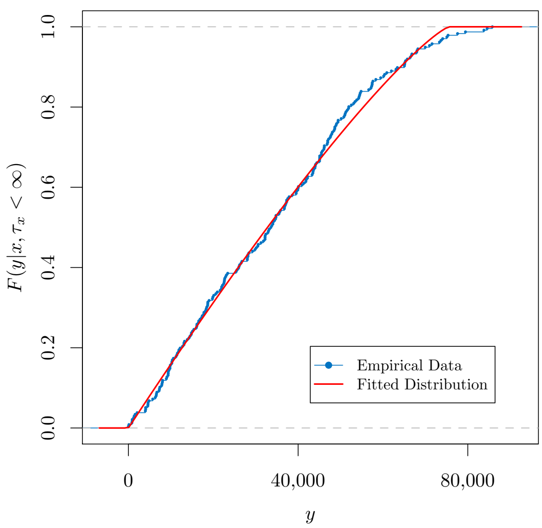


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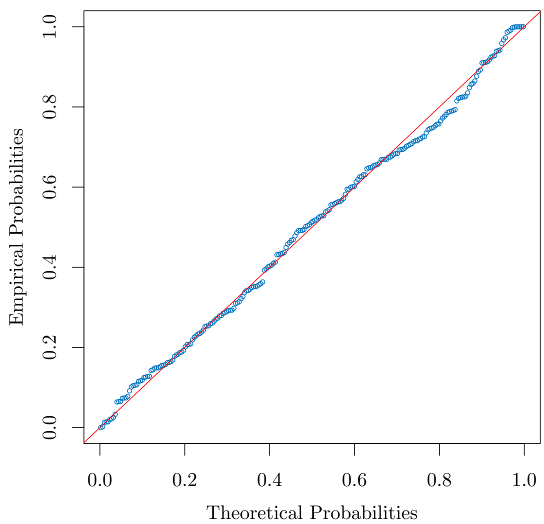
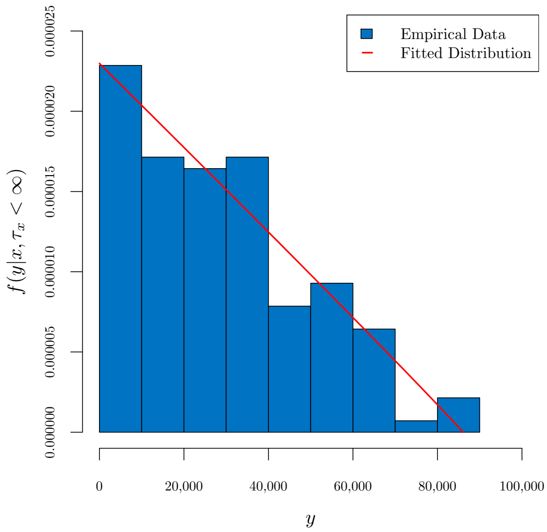



(d) Probability-Probability (P-P) Plot

Sahel

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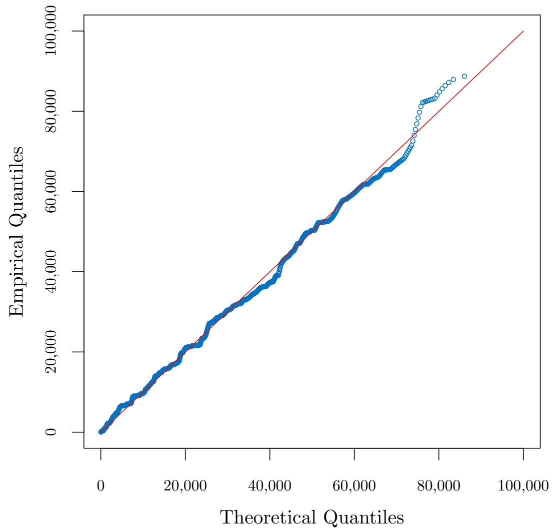
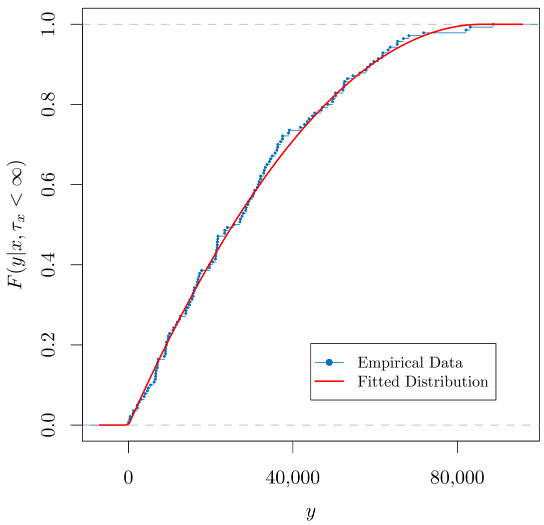
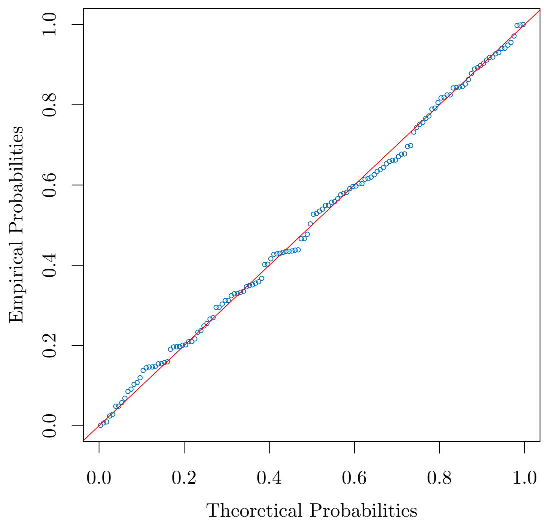



(b) Quantile-Quantile (Q-Q) Plot

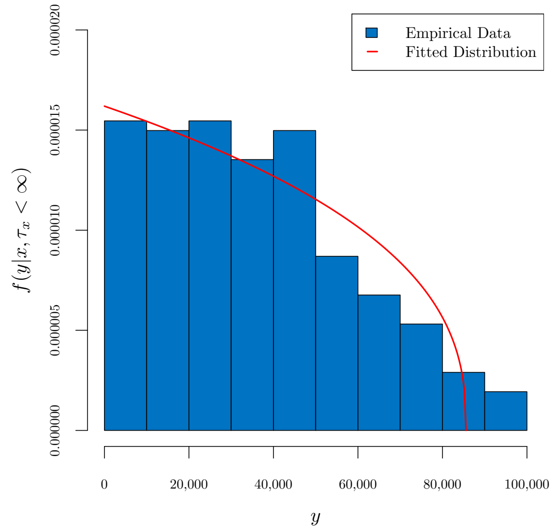



(c) Comparison of Probability Distributions




(d) Probability-Probability (P-P) Plot

Sud-Ouest

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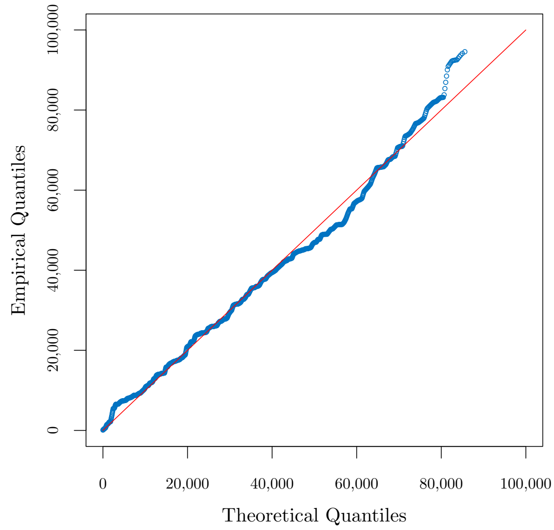


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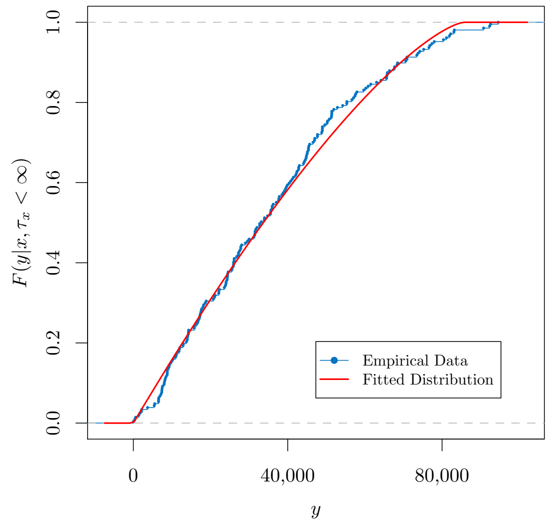
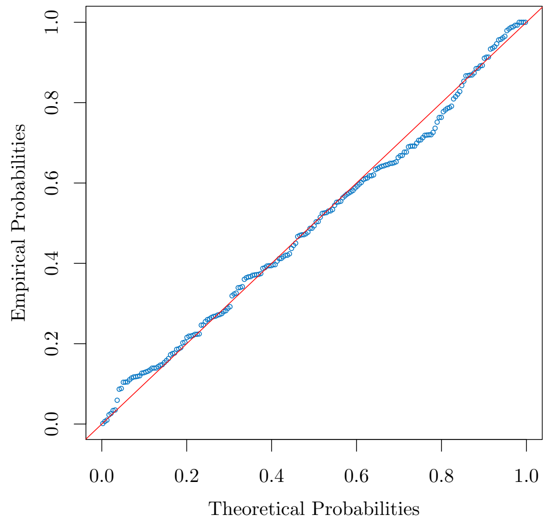



(c) Comparison of Probability Distributions




(d) Probability-Probability (P-P) Plot

# Chapter 5

# The Role of Direct Capital Cash Transfers Towards Poverty and Extreme Poverty Alleviation - An Omega Risk Process

This chapter is based on the following article:

Flores-Contró, J. M. and S. Arnold (2024). The Role of Direct Capital Cash Transfers Towards Poverty and Extreme Poverty Alleviation - An Omega Risk Process. Published in *Scandinavian Actuarial Journal*. https://doi.org/10.1080/03461238.2024.2321574.

**Abstract.** Trapping refers to the event when a household falls into the area of poverty. Households that live or fall into the area of poverty are said to be in a poverty trap, where a poverty trap is a state of poverty from which it is difficult to escape without external help. Similarly, extreme poverty is considered as the most severe type of poverty, in which households experience severe deprivation of basic human needs. In this chapter, we consider an Omega risk process with deterministic growth and a multiplicative jump (collapse) structure to model the capital of a household. It is assumed that, when a household's capital level is above a certain capital barrier level that determines a household's eligibility for a capital cash transfer programme, its capital grows exponentially. As soon as its capital falls below the capital barrier level, the capital dynamics incorporate external support in the form of direct transfers (capital cash transfers) provided by donors or governments. Otherwise, once trapped, the capital grows only due to the capital cash transfers. Under this model, we first derive closed-form expressions for the trapping probability and then do the same for the probability of extreme poverty, which only depends on the current value of the capital given by some extreme poverty rate function. Numerical examples illustrate the role of capital cash transfers on poverty and extreme poverty dynamics.

# 5.1 Introduction

In development economics, households that live or fall below the poverty line are said to be in a poverty trap, where a poverty trap is a state of poverty from which it is difficult to escape without external help. Similarly, extreme poverty refers to the most severe type of poverty, characterised by severe deprivation of basic human needs, including food, safe drinking water, sanitation facilities, health, shelter, education and information (United Nations and World Summit for Social Development, 1996).

According to the World Bank (2018), the number of people living in extreme poverty declined from 36% in 1990 to 10% of the world's population in 2015. However, this downward trend has been decelerating throughout the years. Indeed, recent research published by the United Nations University World Institute for Development Economics Research (UNU-WIDER) shows that, due to the COVID-19 crisis, global poverty could increase for the first time since 1990 (Sumner et al., 2020), therefore threatening one of the global public's priorities: ending poverty. In 2015, owing to the importance of the topic, world leaders agreed on seventeen Sustainable Development Goals (SDGs) which engage not only public and private sectors but also society in attaining a better and more sustainable future for all. Among these goals, eradicating extreme poverty by 2030 is at the top of the list of priorities, followed by other targets among which, the reduction of at least by half of the proportion of people living in poverty and the implementation of appropriate social protection programmes, stand out (SDG 1: End poverty in all its forms everywhere) (United Nations, 2015).

Poverty is not an individualised condition, as it does not affect only those who are poor. That is, poverty causes enormous economic, social and psychological costs to both the poor and the non-poor. Crime, access to and affordability of health care and economic productivity are just a few examples of common global concerns that are exacerbated by poverty (Rank et al., 2021). Child poverty is a clear example of how poverty affects us all. For instance, children who grow up in poverty are much more likely to commit crime as adults (Bjerk, 2007). More crime means higher correction costs and a rise in private spending on crime prevention (e.g. in buying alarms and locks). Similarly, growing up in poverty can have harmful effects on a person's health (Brooks-Gunn and Duncan, 1997; Case et al., 2002; Ravallion, 2016). This causes hospitals and health insurers to spend more on the treatment of preventable diseases (Children's Defense Fund (U.S.), 1994), jeopardising access to and affordability of health care. Lastly, poor children are often less exposed to education (Rank et al., 2014) and they may therefore have fewer qualifications, which in turn translates into lower paid and more unstable jobs. This results in lower economic productivity in adulthood for poor children. Specifically, for the United States of America, McLaughlin and Rank (2018) indicate the aggregate annual cost of child poverty amounts to USD 1.0298 trillion, representing 5.4% of the country's gross domestic product (GDP). Moreover, McLaughlin and Rank (2018) also estimate that, for every dollar spent on reducing childhood poverty, the country would save at least seven dollars with respect the economic costs of poverty.

Cash transfer programmes are one of the main social protection strategies to reduce poverty and are therefore considered important mechanisms to help achieve SDG 1. In their simplest form, these programmes transfer cash, whether in small, regular amounts, or as lump sums, to people living below the poverty line and are generally funded by governments, international organisations, donors or nongovernmental organisations (NGOs) (Garcia and Moore, 2012). Moreover, cash transfers are usually classified as unconditional (UCTs) or conditional (CCTs), with the former not requiring beneficiaries to undertake any specific actions nor meet any conditions whereas the latter needs them to have some specific behavioural conditions in exchange of the cash transfer (Baird et al., 2014), such as enrolling children in school or taking them to regular health check-ups (Handa and Davis, 2006).

Adopting a ruin-theoretic approach, this chapter studies the impact of regular UCTs on poverty and extreme poverty dynamics and, particularly, their effectiveness in reducing the likelihood of a household living in poverty and extreme poverty. Previous research has addressed the role of UCTs as a pathway out of extreme poverty for households. Handa et al. (2016) study two programmes, the Child Grant Programme (CGP) and the Multiple Category Targeted Programme (MCP), which were implemented in 2010 by the Ministry of Community Development, Mother and Child Health (MCDMCH) of the Government of Zambia. The authors find that both of these UCTs go far beyond their primary objective of protecting food security and consumption, as they also have an enormous impact on households' productive capacity. Although a flat transfer of USD 12 per month may not permanently lift households out of the poverty trap, their results suggest these programmes can help raise the standards of living of the country's population. In the same way, Ambler and De Brauw (2017) show that the Benazir Income Support Program (BISP), an UCT initiative introduced in 2008 by the Government of Pakistan, has increased women empowerment in the country, frequently associated with economic growth (Duflo, 2012), which at the same time has been linked with poverty reduction (Adams, 2003). As a matter of fact, UCTs have recently gained popularity as a cost-effective social protection strategy to attain some public policy objectives, including poverty alleviation (Aker, 2013; Baird et al., 2014; Blattman and Niehaus, 2014; Haushofer and Shapiro, 2016; Jensen et al., 2017; Pega et al., 2022).

Despite the growing interest in studying the impact of UCTs on poverty dynamics over the years, most studies have adopted an empirical approach. This chapter is an attempt to attach a mathematically based theoretical framework to the vast empirical literature. In this chapter, we extend the model proposed by Kovacevic and Pflug (2011). Here, a household's capital process $X = \{X_t\}_{t\geq 0}$ grows exponentially at a rate $r > 0$, which incorporates household rates of consumption, income generation and investment or savings, above a critical capital level (or poverty line) $x^* > 0$, whereas below a capital barrier level $B > x^*$, the capital also integrates external support in the form of direct transfers (capital cash transfers) provided by donors or governments at a rate $c_T > 0$. At time $T_i$, the $i$th capital loss event time, the capital process jumps (downwards) to $Z_i \cdot X_{T_i}$, where $\{Z_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables with distribution function $G_Z$ supported in $(0, 1]$, representing the proportions of remaining capital after each loss event (in the present chapter, it will be regularly assumed the random variables are $Beta(\alpha, 1)-$distributed). A more comprehensive picture of this model is introduced in Section 5.2.

The probability of falling (trapping probability) and the moment at which a household falls (trapping time) into the poverty trap have recently attracted the interest of some researchers (see, for example, Kovacevic and Pflug (2011), Azaïs and Genadot (2015), Flores-Contró et al. (2021), Henshaw et al. (2023) and Flores-Contró (2024)). These studies focus on analysing the behaviour of a household's capital above the critical capital, hence overlooking its evolution below this threshold. That is, under this set up, a household's capital process is killed at the trapping time $\tau_x^{\text{P}} := \inf\{t \geq 0 : X_t < x^* \mid X_0 = x\}$. In this chapter, we assume households may escape from the poverty trap only due to external support received in the form of capital cash transfers. Therefore, we define the random variable $\tau_x^{\text{EP}}$ for $x \in (0, \infty)$ as the time of extreme poverty i.e. the moment at which a household becomes extremely poor and $\psi^{\text{EP}}(x) = \mathbb{P}(\tau_x^{\text{EP}} < \infty)$ as the probability of extreme poverty. Hence, under this new set up, a household's capital process is killed at the time of extreme poverty. The approach taken here differs from the aforementioned studies, where the area of poverty $\Lambda = [0, x^*]$ was considered as an absorbing state from which it was not possible to escape. To explore these ideas, we consider an Omega risk process, which in classical risk theory, distinguishes between ruin (negative surplus) and bankruptcy (going out of business). Thus, it is assumed that, even with negative surplus levels, an insurance company can do business as usual and continue until bankruptcy occurs.

The Omega model was first introduced in Albrecher et al. (2011), where closed-form formulas for the expected discounted dividends until bankruptcy under a dividend barrier strategy are obtained for the case in which the surplus of an insurance company is modelled as a Brownian motion. Similarly, Gerber et al. (2012), Albrecher and Lautscham (2013) and Wang et al. (2016) derive explicit expressions for the expected discounted penalty function at bankruptcy and the probability of bankruptcy when the surplus of an insurance company is modelled as a Brownian motion, a compound Poisson risk model with exponential claim sizes and an Ornstein-Uhlenbeck process, respectively. Certainly, the Omega model has been extensively studied during the last decade in the actuarial science literature, with researchers incorporating the bankruptcy concept into traditional ruin models. A particular clear example of this is in Cui and Nguyen (2016), where an Omega model with surplus-dependent tax payments and capital injections in a time-homogeneous diffusion setting is studied. This work not only incorporates features from the Omega model (Albrecher et al., 2011) but also from traditionally well-studied ruin models such as the risk model with tax (Albrecher and Hipp, 2007) and the risk model with capital injections (Albrecher and Ivanovs, 2014). More recently, Gao and He (2019) and He et al. (2019) obtain analytical results for the expected discounted penalty function and the probability of bankruptcy for surplus processes under three- and two-step premium rate settings, respectively. In like manner, Gao et al. (2022) also derive results for the expected discounted dividends until bankruptcy for a jump-diffusion surplus process with a two-step premium rate under a dividend barrier strategy. Besides, alternative versions of the Omega model have also been considered. For instance, Kaszubowski (2019) allows the surplus process to evolve below zero but assumes it is killed once it falls below some fixed level $-d < 0$.

Under the classical risk theory set up, the probability of bankruptcy is quantified by a bankruptcy rate function $\omega(x)$, where $x$ represents the value of the negative surplus.

The bankruptcy rate function is defined in such a way in which the probability of bankruptcy increases when the deficit grows. Consequently, for the household capital process, the bankruptcy event is swaped for the extreme poverty one and an extreme poverty rate function $\omega(X_s)$, which is assumed to be locally bounded and dependent on the capital level below the critical capital $x^*$, is defined on $(0, x^*]$. Namely, for some capital $X_s < x^*$ and no prior extreme poverty event, the probability of extreme poverty on the time interval $[s, s + dt)$ is given by $\omega(X_s)\,dt$. Moreover, we assume that $\omega(\cdot) \geq 0$ and $\omega(x) \geq \omega(y)$ for $0 < x \leq y$ to reflect the likelihood of extreme poverty does not decrease as the capital approaches zero. Clearly, when $\omega(y) \equiv \infty$ for all $y < x^*$, the probability of extreme poverty is equal to the trapping probability $\psi^{\text{P}}(x) = \mathbb{P}(\tau_x^{\text{P}} < \infty)$.

In general, UCTs target the poor. However, in recent years, cash transfer programmes have reached unprecedented levels of coverage. For example, in 2020, in response to the COVID-19 pandemic, one out of six people in the world received at least one cash transfer payment (Gentilini, 2022). As a consequence of this expansion, it is now more common to encounter UCTs targeting other population groups, such as the vulnerable non-poor (those living just above the poverty line). One example is Ingreso Solidario, an UCT programme in Colombia that was implemented in April 2020 as a response to the COVID-19 pandemic. Ingreso Solidario provided monthly transfers of approximately USD 40 to eligible households: poor households not covered by pre-existing social programmes and non-poor households deemed vulnerable based on an assessment of their living conditions (Vera-Cossio et al., 2023). The capital model considered in this chapter allows for the assessment of targeted UCTs, either to the poor only (letting $B \to x^{*+}$) or to both poor and vulnerable non-poor households (when $B > x^*$), on poverty dynamics. Moreover, the capital model is in line with the idea that spending on poverty reduction and prevention can help save on the economic costs of poverty. As such, when capital cash transfers target only the poor, the essential aim of the UCT programme is to lift households out of poverty. On the other hand, when the UCT programme targets both the poor and the vulnerable non-poor, the programme hopes to prevent the vulnerable non-poor from falling into poverty, apart from lifting the poor out of poverty. Nevertheless, both settings pursue one same objective: poverty reduction.

Particular attention should be paid to the fact that the targeted UCTs considered in this chapter, either to the poor only or to both poor and vulnerable non-poor households, prevent households from becoming extremely poor, as extreme poverty implies poverty (recall that a household is at risk of becoming extremely poor only when its capital lies below the critical capital $x^*$ or, in other words, a household can become extremely poor only when it is already poor). This is consistent with how extreme poverty is currently measured. For instance, the World Bank uses the International Poverty Line (IPL), set at USD 2.15 per person per day, to measure extreme poverty (Jolliffe et al., 2022). The IPL is also the most relevant poverty line to measure poverty in low-income countries, whereas in other countries, other poverty lines are used to measure poverty. For example, the poverty line is set at USD 3.65 and USD 6.85 per person per day, in lower and upper middle-income countries, respectively (Jolliffe et al., 2022). According to the World Bank's definition of extreme poverty, it is clear how extreme poverty implies poverty. In general, extreme poverty differs from conventional poverty in that it has greater depth (degree of deprivation), larger length (duration over time) and greater breadth (the number of dimensions such as illiteracy and malnutrition, among others) (Emran et al., 2014). Because of these characteristics, the economic costs of extreme poverty are also expected to be higher than those of conventional poverty. Hence, extreme poverty should be avoided by all means, and should be considered and studied separately.

The remainder of the chapter is structured as follows. In Section 5.2, we introduce the capital model, with special emphasis on its behavior inside and outside the poverty area. Explicit equations, their solutions and numerical illustrations for the trapping probability are given in Section 5.3 for the particular case in which the remaining proportions of capital are $Beta(\alpha, 1)-$distributed. In particular, a comparison between the trapping probability of the original capital model introduced by Kovacevic and Pflug (2011) and the one proposed in this chapter is presented in Appendix 5.B. The event of extreme poverty and the time when it occurs are discussed in Section 5.4. In addition, closed-form solutions and numerical illustrations for the probability of extreme poverty are derived in Section 5.4, assuming constant and exponential extreme poverty rate functions for the particular case in which the remaining proportions of capital are $Beta(\alpha, 1)-$distributed. Following Albrecher and Lautscham (2013), Section 5.5 illustrates how to approximate the probability of extreme poverty for more general cases by making use of an efficient Monte Carlo simulation method. Finally, concluding remarks are discussed in Section 5.6.

## 5.2 The Capital Model

This chapter extends the capital process originally proposed in Kovacevic and Pflug (2011), where an individual household's income $I_t$ at time $t$ comprises consumption $C_t$ and savings (investments) $S_t$. Hence, as in the original capital process, income dynamics are given by

$$I_t = C_t + S_t.$$

Moreover, consumption is an increasing function of income and its dynamics are given by

$$C_t = \begin{cases} I_t & \text{if } I_t \leq I^*, \\ I^* + a\,(I_t - I^*) & \text{if } I_t > I^*, \end{cases}$$

where $0 < a < 1$. It is assumed that permanent consumption below $I^*$ might result in severe adverse effects on health (Kovacevic and Pflug, 2011). Figure 5.1a shows the dynamics of consumption and savings. Consider the accumulated capital $X_t$ up to time $t$ follows the dynamics

$$\frac{dX_t}{dt} = c_s S_t,$$

with $0 < c_s < 1$, and income is generated through capital

$$I_t = bX_t,$$

where $0 < b$ holds.

Putting all these pieces together and defining $x^* \cdot b = I^*$, one gets the dynamical system

$$\frac{dX_t}{dt} = r \cdot [X_t - x^*]^+ ,$$

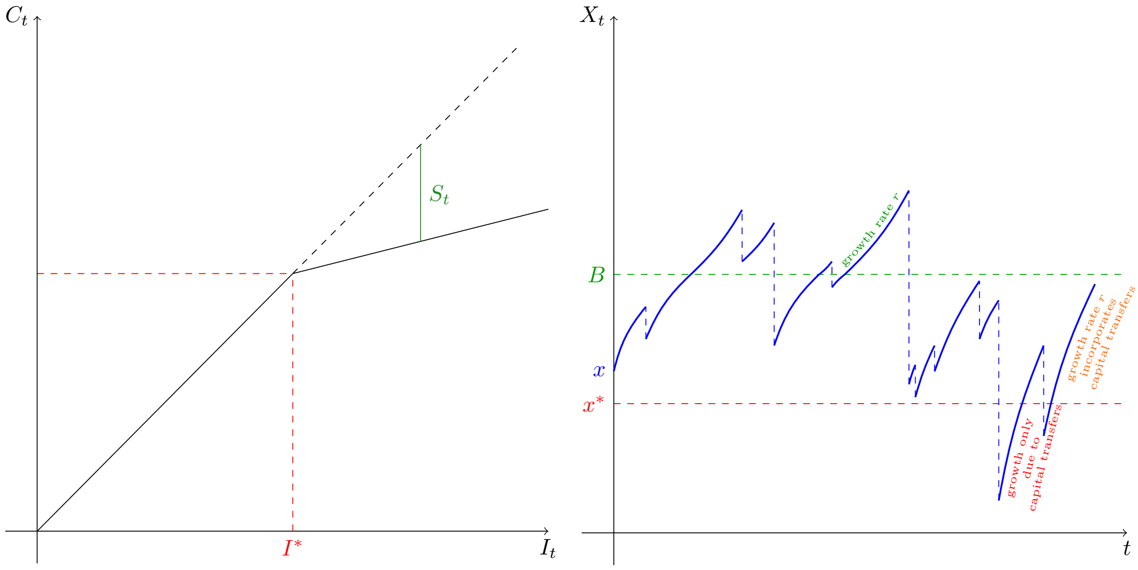
where $r = (1 - a) \cdot b \cdot c_S > 0$ and $x^* > 0$ represents the threshold below which a household lives in poverty, also interpreted as the amount of capital needed to acquire the critical income $I^*$ as a perpetuity (Kovacevic and Pflug, 2011).

We now also consider direct transfers (capital cash transfers) provided by donors or governments only to those deemed eligible. Assume a household qualifies to be a beneficiary of the unconditional capital cash transfer programme when its accumulated capital $X_t$ up to time $t$ is below some capital barrier level $B > x^*$ and that the external support will be provided at a rate $c_T > 0$. The main objective of the proposed UCTs is to reduce the gap between the capital barrier level and the accumulated capital $X_t$ up to time $t$ for those households with capital levels below the capital barrier level $B > x^*$. Under this framework, one gets the dynamical system

$$\frac{dX_t}{dt} = r \cdot [X_t - x^*]^+ + c_T \cdot [B - X_t]^+. \tag{5.2.1}$$

In line with the ideology that households are susceptible to the occurrence of capital losses, including severe illness, the death of a household member or breadwinner and catastrophic events such as floods and earthquakes, we model the occurrence of these events with a Poisson process with intensity $\lambda$ and consider the capital process follows the dynamics of (5.2.1) in between events. On the occurrence of a loss, the household's capital at the event time is reduced by a random proportion $0 \leq 1 - Z_i \leq 1$. Hence, the fraction of the capital not destroyed at the event time is given by $Z_i$. The sequence $\{Z_i\}_{i=1}^{\infty}$ is independent of the Poisson process and i.i.d. with common distribution function $G_Z$. A trajectory of the capital process $X_t$ is shown in Figure 5.1b.

Here, the trajectories of the piecewise-deterministic process (Davis, 1984) behave as follows: if the capital lies above the capital barrier level $B > x^*$, then the capital grows exponentially at a rate $r$, whereas if the capital lies above the critical capital $x^*$ but below the capital barrier level $B > x^*$, then the capital growth is composed by both the individual household rate $r$ and the external support rate $c_T$; otherwise, the capital growth only incorporates the external support rate $c_T$. Note that, both the critical capital $x^*$ and the capital barrier level $B > x^*$ act as equilibrium levels for the process. That is, the further above the current value of the process is from the critical capital $x^*$, the faster the capital will depart from the critical capital $x^*$ at the individual household rate $r$. Similarly, the further below the current value of the process is from the capital barrier level $B > x^*$, the faster the capital will grow to the capital barrier level $B > x^*$ at the external support rate $c_T$ (there is a "$B-$reverting" effect where $c_T$ is the rate of reversion).




(a) (b)

Figure 5.1: (a) Consumption and savings (b) Trajectory of the stochastic process $X_t$.

## 5.3 When and How Households Become Poor?

In this section, we will study the trapping time $\tau_x^{P}$, which is defined as the time at which a household with initial capital $x \geq x^*$ falls into the area of poverty. That is,

$$\tau_x^{P} := \inf\left\{t \geq 0 : X_t < x^* \mid X_0 = x\right\}.$$

Note that, we use the superscript "$^{P}$" to distinguish trapping-related variables and functions. Our analysis will involve the expected discounted penalty function, a concept commonly used in actuarial science (Gerber and Shiu, 1998). The expected discounted penalty function contains information on the trapping time $\tau_x^{P}$ itself and two related random variables, the capital surplus prior to the trapping time $X_{\tau_x^{P}-} - x^*$ and the capital deficit at the trapping time $\mid X_{\tau_x^{P}} - x^* \mid$.

For a force of interest $\delta \geq 0$ and initial capital $x \geq x^*$, the expected discounted penalty function is defined as

$$m_\delta^{P}(x) = \mathbb{E}\left[w^{P}(X_{\tau_x^{P}-} - x^*, \mid X_{\tau_x^{P}} - x^* \mid) e^{-\delta \tau_x^{P}} \mathbb{1}_{\{\tau_x^{P} < \infty\}}\right], \tag{5.3.1}$$

where $\mathbb{1}_{\{A\}}$ is the indicator function of a set $A$, and $w^{P}(x_1, x_2)$, for $0 \leq x_1 < \infty$ and $0 < x_2 \leq x^*$, is a non-negative penalty function of $x_1$, the capital surplus prior to the trapping time, and $x_2$, the capital deficit at the trapping time. For more details on the so-called Gerber-Shiu risk theory, interested readers may wish to consult Kyprianou (2013). The function $m_\delta^{P}(x)$ is useful for deriving results in connection with joint and marginal distributions of $\tau_x^{P}$, $X_{\tau_x^{P}-} - x^*$ and $\mid X_{\tau_x^{P}} - x^* \mid$. For instance, (5.3.1) could also be viewed in terms of a Laplace transform when $\delta$ is serving as the argument. Indeed, if we let $w^{P}(x_1, x_2) = 1$, (5.3.1) is the Laplace transform of the trapping time $\tau_x^{\scriptscriptstyle P}$ [1]. Another choice is $w^{\scriptscriptstyle P}(x_1, x_2) = \mathbb{1}_{\{x_1 \leq x, x_2 \leq y\}}$ for $\delta = 0$, for which (5.3.1) leads to the joint distribution function of the capital surplus prior to the trapping time and the capital deficit at the trapping time. It is not difficult to realise that, by appropriately choosing a penalty function $w^{\scriptscriptstyle P}(x_1, x_2)$ and force of interest $\delta$, various risk quantities can be modelled. A non-exhaustive list of such risk quantities is given in He et al. (2023). In this chapter, we are mainly interested in studying the impact of capital cash transfers on the probability of falling into the poverty trap. Thus, we will focus our analysis on the risk quantity $\psi^{\scriptscriptstyle P}(x) = \mathbb{P}(\tau_x^{\scriptscriptstyle P} < \infty)$, which can be derived by choosing $w^{\scriptscriptstyle P}(x_1, x_2) = 1$ and $\delta = 0$ in (5.3.1).

Following Gerber and Shiu (1998), our goal is to derive a functional equation for $m_\delta^{\scriptscriptstyle P}(x)$ by applying the law of iterated expectations to the right-hand side of (5.3.1).

We point out that $m_\delta^{\scriptscriptstyle P}(x)$ has different sample paths for $x \geq B$ and $x^* \leq x < B$. Hence, we distinguish the two situations by writing $m_\delta^{\scriptscriptstyle P}(x) = m_{\delta,u}^{\scriptscriptstyle P}(x)$ for $x \geq B$ and $m_\delta^{\scriptscriptstyle P}(x) = m_{\delta,l}^{\scriptscriptstyle P}(x)$ for $x^* \leq x < B$. Similarly, we write $\psi_u^{\scriptscriptstyle P}(x) = \mathbb{P}(\tau_x^{\scriptscriptstyle P} < \infty)$ for $x \geq B$ and $\psi_l^{\scriptscriptstyle P}(x) = \mathbb{P}(\tau_x^{\scriptscriptstyle P} < \infty)$ for $x^* \leq x < B$.

*Remark* 5.3.1. Note that, when $B = x^*$, above the critical capital $x^*$, the dynamics of the capital process follow those of the original process (Kovacevic and Pflug, 2011). Thus, the trapping probability $\psi^{\scriptscriptstyle P}(x)$ and the expected discounted penalty function at the trapping time $m_\delta^{\scriptscriptstyle P}(x)$, are equivalent to those studied in Henshaw et al. (2023) and Flores-Contró (2024), respectively. Clearly, this also holds true when $c_T = 0$. Appendix 5.B evidences this behaviour for a set of selected parameters.

**Theorem 5.3.1.** *When $x \geq B$, we have*

$$m_{\delta,u}^{\scriptscriptstyle P}(x) = \frac{\lambda}{r}(x - x^*)^{\frac{\lambda+\delta}{r}} \int_x^\infty \frac{1}{(u - x^*)^{\frac{\lambda+\delta}{r}+1}} \left[\int_{B/u}^1 m_{\delta,u}^{\scriptscriptstyle P}(u \cdot z) dG_Z(z) \right.$$
$$\left. + \int_{x^*/u}^{B/u} m_{\delta,l}^{\scriptscriptstyle P}(u \cdot z) dG_Z(z) + A^{\scriptscriptstyle P}(u)\right] du, \tag{5.3.2}$$

*and when $x^* \leq x < B$, we have*

$$m_{\delta,l}^{\scriptscriptstyle P}(x) = \frac{\lambda}{r - c_T}(x + x^{**})^{\frac{\lambda+\delta}{r-c_T}} \int_x^B \frac{1}{(u + x^{**})^{\frac{\lambda+\delta}{r-c_T}+1}} \left[\int_{x^*/u}^1 m_{\delta,l}^{\scriptscriptstyle P}(u \cdot z) dG_Z(z) + A^{\scriptscriptstyle P}(u)\right]$$

$$du + \frac{\lambda}{r}(B - x^*)^{\frac{\lambda+\delta}{r}} \left(\frac{x + x^{**}}{B + x^{**}}\right)^{\frac{\lambda+\delta}{r-c_T}} \int_B^\infty \frac{1}{(v - x^*)^{\frac{\lambda+\delta}{r}+1}} \left[\int_{B/v}^1 m_{\delta,u}^{\scriptscriptstyle P}(v \cdot z) dG_Z(z) \right.$$
$$\left. + \int_{x^*/v}^{B/v} m_{\delta,l}^{\scriptscriptstyle P}(v \cdot z) dG_Z(z) + A^{\scriptscriptstyle P}(v)\right] dv, \tag{5.3.3}$$

*where the function $A^{\scriptscriptstyle P}(u)$ is given by*

$$A^{\scriptscriptstyle P}(u) := \int_0^{x^*/u} w^{\scriptscriptstyle P}(u - x^*, x^* - u \cdot z) dG_Z(z).$$

[1] We know from probability theory that, for a continuous random variable $Y$, with probability density function $f_Y$, the Laplace transform of $f_Y$ is given by the expected value $\mathcal{L}\{f_Y\}(s) = \mathbb{E}\left[e^{-sY}\right]$.

See Appendix 5.A.1 for proof of Theorem 5.3.1.

*Remark* 5.3.2. We point out that the Integral Equations (IEs) (5.3.2) and (5.3.3) allow us to consider the differentiability of the functions $m^{P}_{\delta,u}(x)$ and $m^{P}_{\delta,l}(x)$. For instance, it is easy to see from (5.3.2) and (5.3.3) that $m^{P}_{\delta,u}(x)$ and $m^{P}_{\delta,l}(x)$ are differentiable in $(B,\infty)$ and $(x^*,B)$, respectively. Furthermore, they satisfy the following condition

$$m^{P}_{\delta,u}(B) = m^{P}_{\delta,l}(B^-). \tag{5.3.4}$$

The existence and uniqueness of the required solution to the IEs derived in Theorem 5.3.1 should be justified in each case (see, for example, Mihálykó and Mihálykó (2011) for an analysis of the existence and uniqueness of the solution of an IE for the expected discounted penalty function of a risk process with dependent inter-arrival times and claim sizes). Now, by differentiating the IEs (5.3.2) and (5.3.3), we obtain the Integro-Differential Equations (IDEs) for $m^{P}_{\delta,u}(x)$ and $m^{P}_{\delta,l}(x)$ in the following theorem

**Theorem 5.3.2.** *When $x \geq B$, we have*

$$\begin{aligned} r(x-x^*)m'^{P}_{\delta,u}(x) - (\lambda+\delta)m^{P}_{\delta,u}(x) + \lambda\Bigg[\int_{B/x}^{1} m^{P}_{\delta,u}(x\cdot z)dG_Z(z) \\ + \int_{x^*/x}^{B/x} m^{P}_{\delta,l}(x\cdot z)dG_Z(z) + A^{P}(x)\Bigg] = 0, \end{aligned} \tag{5.3.5}$$

*and when $x^* \leq x < B$, we have*

$$(r-c_T)(x+x^{**})m'^{P}_{\delta,l}(x) - (\lambda+\delta)m^{P}_{\delta,l}(x) + \lambda\left[\int_{x^*/x}^{1} m^{P}_{\delta,l}(x\cdot z)dG_Z(z) + A^{P}(x)\right] = 0. \tag{5.3.6}$$

*In addition, the boundary conditions for $m^{P}_{\delta,u}(x)$ and $m^{P}_{\delta,l}(x)$ are given by (5.3.4), $\lim_{x\to\infty} m^{P}_{\delta,u}(x) = 0$ and $m^{P}_{\delta,l}(x^*) = \frac{1}{\lambda+\delta}\left[c_T(B-x^*)m'^{P}_{\delta,l}(x^*) + \lambda A^{P}(x^*)\right]$.*

*Remark* 5.3.3. Equation (5.3.6) for $m^{P}_{\delta,l}(x)$ is independent of $m^{P}_{\delta,u}(x)$. However, $m^{P}_{\delta,l}(x)$ is subject to the boundary condition (5.3.4) which is involved with $m^{P}_{\delta,u}(x)$. Furthermore, it is easy to see from (5.3.4), (5.3.5) and (5.3.6) that $m^{P}_{\delta,u}(x)$ and $m^{P}_{\delta,l}(x)$ satisfy

$$m'^{P}_{\delta,u}(B) = m'^{P}_{\delta,l}(B^-). \tag{5.3.7}$$

### 5.3.1 The Trapping Time

Sometimes it is easier to work with a transformation rather than with the original distribution function of a random variable. In this chapter, we focus on studying the Laplace transform of the random variables of interest. The Laplace transform of a random variable characterises the probability distribution uniquely and is known to be a powerful tool in probability theory and, in particular, quite useful when studying nonnegative random variables. Recall that, by specifying the penalty function such that $w^{P}(x_1, x_2) = 1$, $m^{P}_{\delta}(x)$ becomes the Laplace transform of the trapping time, also interpreted as the expected present value of a unit payment due at the trapping time.

Thus, with $w^{P}(x_1, x_2) = 1$, Equations (5.3.5) and (5.3.6) can then be written such that when $x \geq B$,

$$
\begin{aligned}
0 = r(x - x^*)m'^{P}_{\delta,u}(x) - (\lambda + \delta)m^{P}_{\delta,u}(x) + \lambda \left[ \int_{B/x}^{1} m^{P}_{\delta,u}(x \cdot z)dG_Z(z) \right. \\
\left. + \int_{x^*/x}^{B/x} m^{P}_{\delta,l}(x \cdot z)dG_Z(z) + G_Z\left(\frac{x^*}{x}\right) \right], \qquad (5.3.8)
\end{aligned}
$$

and when $x^* \leq x < B$,

$$
\begin{aligned}
0 = (r - c_{T})(x + x^{**})m'^{P}_{\delta,l}(x) - (\lambda + \delta)m^{P}_{\delta,l}(x) \\
+ \lambda \left[ \int_{x^*/x}^{1} m^{P}_{\delta,l}(x \cdot z)dG_Z(z) + G_Z\left(\frac{x^*}{x}\right) \right]. \qquad (5.3.9)
\end{aligned}
$$

**Proposition 5.3.1.** *Consider a household capital process with initial capital $x \geq x^*$, capital growth rate $r$, capital barrier level $B$, capital transfer rate $c_{T}$, intensity $\lambda > 0$ and remaining proportions of capital with distribution Beta$(\alpha, 1)$ where $\alpha > 0$; that is, $Z_i \sim$ Beta$(\alpha, 1)$. The Laplace transform of the trapping time is given by*

$$
m^{P}_{\delta}(x) = \begin{cases} A^{P}_{2,u} y^{P}_{u}(x)^{-b^{P}_{u}} {}_2F_1\left(b^{P}_{u}, b^{P}_{u} - c^{P}_{u} + 1; b^{P}_{u} - a^{P}_{u} + 1; y^{P}_{u}(x)^{-1}\right) & \text{for } x \geq B, \\[2ex] A^{P}_{1,l} y^{P}_{l}(x)^{-a^{P}_{l}} {}_2F_1\left(a^{P}_{l}, a^{P}_{l} - c^{P}_{l} + 1; a^{P}_{l} - b^{P}_{l} + 1; y^{P}_{l}(x)^{-1}\right) & \\ \quad + A^{P}_{2,l} y^{P}_{l}(x)^{-b^{P}_{l}} {}_2F_1\left(b^{P}_{l}, b^{P}_{l} - c^{P}_{l} + 1; b^{P}_{l} - a^{P}_{l} + 1; y^{P}_{l}(x)^{-1}\right) & \text{for } x^* \leq x < B, \end{cases} \qquad (5.3.10)
$$

*where $\delta \geq 0$ is the force of interest for valuation, ${}_2F_1(\cdot)$ is Gauss's Hypergeometric Function as defined in (5.A.8), $y^{P}_{u}(x) = \frac{x}{x^*}$, $a^{P}_{u} = \frac{-(\delta+\lambda-\alpha r)-\sqrt{(\delta+\lambda-\alpha r)^2+4r\alpha\delta}}{2r}$, $b^{P}_{u} = \frac{-(\delta+\lambda-\alpha r)+\sqrt{(\delta+\lambda-\alpha r)^2+4r\alpha\delta}}{2r}$, $c^{P}_{u} = c^{P}_{l} = \alpha$, $y^{P}_{l}(x) = -\frac{x}{x^{**}}$ with $x^{**} = \frac{c_{T}B - rx^*}{r - c_{T}}$, $a^{P}_{l} = \frac{-(\delta+\lambda-\alpha(r-c_{T}))-\sqrt{(\delta+\lambda-\alpha(r-c_{T}))^2+4(r-c_{T})\alpha\delta}}{2(r-c_{T})}$, $b^{P}_{l} = \frac{-(\delta+\lambda-\alpha(r-c_{T}))+\sqrt{(\delta+\lambda-\alpha(r-c_{T}))^2+4(r-c_{T})\alpha\delta}}{2(r-c_{T})}$ and the constants $A^{P}_{2,u}$, $A^{P}_{1,l}$ and $A^{P}_{2,l}$ are given by (5.A.9), (5.A.10), and (5.A.11), respectively.*

The mathematical proof of Proposition 5.3.1 is presented in Appendix 5.A.2.

*Remark* 5.3.4. As mentioned previously, the Laplace transform of the trapping time approaches the trapping probability as $\delta$ tends to zero, i.e.

$$
\lim_{\delta \downarrow 0} m^{P}_{\delta}(x) = \mathbb{P}(\tau^{P}_{x} < \infty) \equiv \psi^{P}(x),
$$

for $\alpha > \frac{\lambda}{r}$. If the net profit condition $\alpha > \frac{\lambda}{r}$ does not hold, then trapping would be certain (Henshaw et al., 2023).

Figures 5.2a[2] and 5.2b display the trapping probability $\psi^{P}(x)$ for the capital process $X_t$. Not surprisingly, Figure 5.2a shows the trapping probability is a decreasing function of both the capital transfer rate $c_T$ and the initial capital. In particular, it is worth noting the important role the capital transfer rate $c_T$ can play in attaining lower trapping probabilities for households with initial capital below the capital barrier level $B$ as very high capital transfer rates $c_T$ have the potential to level the likelihood of becoming poor for this particular group. However, high capital transfer rates $c_T$ seem to be less efficient for attaining lower trapping probabilities for households with initial capital above the capital barrier level $B$. This is due to the fact that households with initial capital above the capital barrier level $B$ are exposed to never receiving a capital cash transfer. Indeed, if they experience a large loss, they are likely to fall directly into the poverty trap without ever receiving a capital cash transfer. From Figure 5.2b, we can also highlight the importance of the capital barrier level $B$ to reach lower values for the trapping probability. Although a higher capital transfer rate $c_T$ and a higher capital barrier level $B$ may lead to lower trapping probabilities, the sensitivity analyses shown in Appendices 5.B and 5.C suggest the trapping probability is less sensitive to these parameters compared to the probability of extreme poverty.

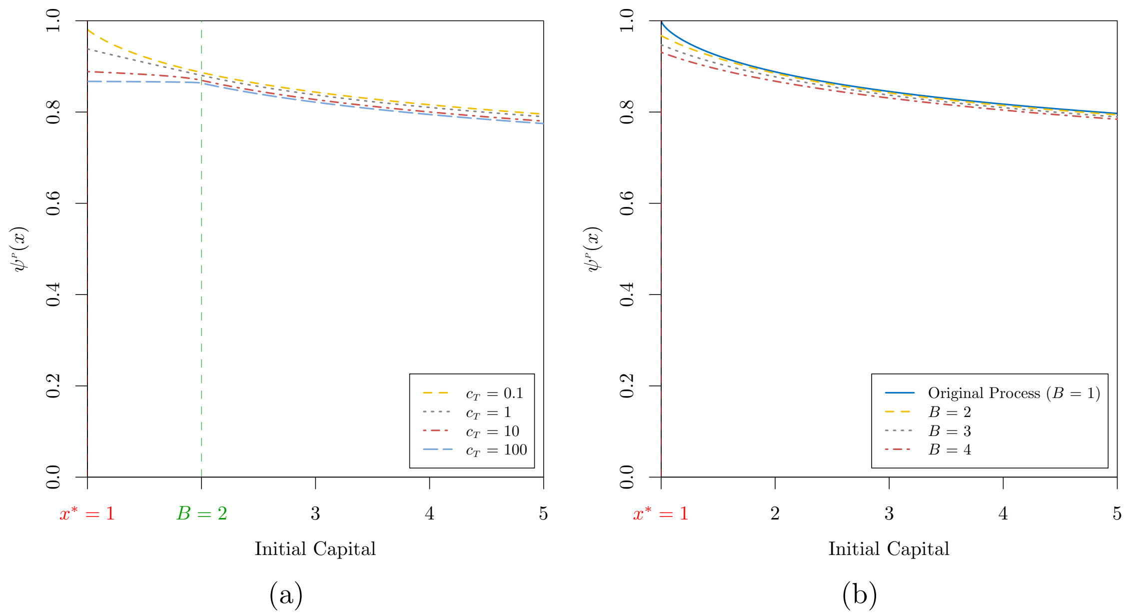



Figure 5.2: (a) Trapping probability $\psi^{P}(x)$ when $Z_i \sim Beta(0.8, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $B = 2$, $\lambda = 1$ and $x^* = 1$ for $c_T = 0.1, 1, 10, 100$ (b) Trapping probability $\psi^{P}(x)$ when $Z_i \sim Beta(0.8, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $c_T = 0.25$, $\lambda = 1$ and $x^* = 1$ for $B = 1, 2, 3, 4$.

[2] A GitHub repository with some code used in this chapter is available at https://github.com/josemiguelflores/TheRoleofDirectCapitalCashTransfers.git

## 5.4 When and How Households Become Extremely Poor?

We define the random variable $\tau_x^{\text{EP}}$ for $x \in (0, \infty)$ as the time of extreme poverty i.e. the moment at which a household with initial capital $x$ becomes extremely poor and $\psi^{\text{EP}}(x) = \mathbb{P}(\tau_x^{\text{EP}} < \infty)$ as the probability of extreme poverty. Note that, we use the superscript "EP" to distinguish extreme poverty-related variables and functions. The probability of extreme poverty is quantified by an extreme poverty rate function $\omega(x)$, where $x$ represents the value of capital below the critical capital $x^*$. The extreme poverty rate function is defined in such a way in which the probability of extreme poverty increases when the capital deficit $\mid X_s - x^* \mid$ grows. Namely, for some capital $X_s < x^*$ and no prior extreme poverty event, the probability of extreme poverty on the time interval $[s, s+dt)$ is given by $\omega(X_s)\, dt$. The expected discounted penalty function at extreme poverty is therefore given by

$$m_\delta^{\text{EP}}(x) = \mathbb{E}\left[w^{\text{EP}}(X_{\tau_x^{\text{EP}}-}, \mid X_{\tau_x^{\text{EP}}} - x^* \mid) e^{-\delta \tau_x^{\text{EP}}} \mathbb{1}_{\{\tau_x^{\text{EP}} < \infty\}}\right],$$

where $w^{\text{EP}}(x_1, x_2)$, for $0 \leq x_1 < x^*$ and $0 < x_2 \leq x^*$, is a non-negative penalty function of $x_1$, the accumulated capital prior to the time of extreme poverty, and $x_2$, the capital deficit at the time of extreme poverty. Note that, for the case of the expected discounted penalty function at extreme poverty, it is reasonable to consider the accumulated capital immediately before extreme poverty instead of the capital surplus, which was considered in (5.3.1) for the expected discounted penalty function at the trapping time.

We point out that $m_\delta^{\text{EP}}(x)$ has different sample paths for $x \geq B$, $x^* \leq x < B$ and $0 < x < x^*$. Hence, we distinguish the three situations by writing $m_\delta^{\text{EP}}(x) = m_{\delta,u}^{\text{EP}}(x)$ for $x \geq B$, $m_\delta^{\text{EP}}(x) = m_{\delta,m}^{\text{EP}}(x)$ for $x^* \leq x < B$ and $m_\delta^{\text{EP}}(x) = m_{\delta,l}^{\text{EP}}(x)$ for $0 < x < x^*$. Similarly, we write $\psi_u^{\text{EP}}(x) = \mathbb{P}(\tau_x^{\text{EP}} < \infty)$ for $x \geq B$, $\psi_m^{\text{EP}}(x) = \mathbb{P}(\tau_x^{\text{EP}} < \infty)$ for $x^* \leq x < B$ and $\psi_l^{\text{EP}}(x) = \mathbb{P}(\tau_x^{\text{EP}} < \infty)$ for $0 < x < x^*$.

Proceeding as in Section 5.3, one derives the following IEs for the expected discounted penalty function at extreme poverty in the following theorem

**Theorem 5.4.1.** *When $x \geq B$, we have*

$$m_{\delta,u}^{\text{EP}}(x) = \frac{\lambda}{r}(x - x^*)^{\frac{\lambda+\delta}{r}} \int_x^\infty \frac{1}{(v_u - x^*)^{\frac{\lambda+\delta}{r}+1}} \left[\int_0^{x^*/v_u} m_{\delta,l}^{\text{EP}}(v_u \cdot z) dG_Z(z) + \int_{x^*/v_u}^{B/v_u} m_{\delta,m}^{\text{EP}}(v_u \cdot z) dG_Z(z) + \int_{B/v_u}^{1} m_{\delta,u}^{\text{EP}}(v_u \cdot z) dG_Z(z)\right] dv_u, \quad (5.4.1)$$

*when $x^* \leq x < B$, we have*

$$m_{\delta,m}^{\text{EP}}(x) = \frac{\lambda}{r - c_T}(x + x^{**})^{\frac{\lambda+\delta}{r-c_T}} \int_x^B \frac{1}{(v_m + x^{**})^{\frac{\lambda+\delta}{r-c_T}+1}} \left[\int_0^{x^*/v_m} m_{\delta,l}^{\text{EP}}(v_m \cdot z) dG_Z(z) + \int_{x^*/v_m}^{1} m_{\delta,m}^{\text{EP}}(v_m \cdot z) dG_Z(z)\right] dv_m$$

$$+\frac{\lambda}{r}(B-x^*)^{\frac{\lambda+\delta}{r}}\left(\frac{x+x^{**}}{B+x^{**}}\right)^{\frac{\lambda+\delta}{r-c_T}}\int_B^\infty \frac{1}{(v_u-x^*)^{\frac{\lambda+\delta}{r}+1}}\left[\int_0^{x^*/v_u} m^{EP}_{\delta,l}(v_u\cdot z)dG_Z(z)\right.$$
$$\left.+\int_{x^*/v_u}^{B/v_u} m^{EP}_{\delta,m}(v_u\cdot z)dG_Z(z)+\int_{B/v_u}^{1} m^{EP}_{\delta,u}(v_u\cdot z)dG_Z(z)\right]dv_u, \tag{5.4.2}$$

*and when* $0<x<x^*$*, we have*

$$m^{EP}_{\delta,l}(x)=-\frac{1}{c_T(x-B)^{\frac{\lambda+\delta}{c_T}}}\int_x^{x^*}\frac{1}{(v_l-B)^{1-\frac{\lambda+\delta}{c_T}}}e^{\frac{1}{c_T}\int_x^{v_l}\frac{\omega(u_l)}{u_l-B}du_l}\omega(v_l)w^{EP}(v_l,x^*-v_l)dv_l$$
$$-\frac{\lambda}{c_T(x-B)^{\frac{\lambda+\delta}{c_T}}}\int_x^{x^*}\frac{1}{(v_l-B)^{1-\frac{\lambda+\delta}{c_T}}}e^{\frac{1}{c_T}\int_x^{v_l}\frac{\omega(u_l)}{u_l-B}du_l}\int_0^1 m^{EP}_{\delta,l}(v_l\cdot z)dG_z(z)dv_l$$
$$+\frac{\lambda}{r-c_T}\left(\frac{x^*-B}{x-B}\right)^{\frac{\lambda+\delta}{c_T}}(x^*+x^{**})^{\frac{\lambda+\delta}{r-c_T}}\int_{x^*}^{B}\frac{1}{(v_m+x^{**})^{\frac{\lambda+\delta}{r-c_T}+1}}e^{\frac{1}{c_T}\int_x^{x^*}\frac{\omega(u_l)}{u_l-B}du_l}$$
$$\left[\int_0^{x^*/v_m} m^{EP}_{\delta,l}(v_m\cdot z)dG_Z(z)+\int_{x^*/v_m}^{1} m^{EP}_{\delta,m}(v_m\cdot z)dG_Z(z)\right]dv_m$$
$$+\frac{\lambda}{r}\left(\frac{x^*-B}{x-B}\right)^{\frac{\lambda+\delta}{c_T}}\left(\frac{x^*+x^{**}}{B+x^{**}}\right)^{\frac{\lambda+\delta}{r-c_T}}(B-x^*)^{\frac{\lambda+\delta}{r}}$$
$$\int_B^\infty\frac{1}{(v_u-x^*)^{\frac{\lambda+\delta}{r}+1}}e^{\frac{1}{c_T}\int_x^{x^*}\frac{\omega(u_l)}{u_l-B}du_l}\left[\int_0^{x^*/v_u} m^{EP}_{\delta,l}(v_u\cdot z)dG_Z(z)\right.$$
$$\left.+\int_{x^*/v_u}^{B/v_u} m^{EP}_{\delta,m}(v_u\cdot z)dG_Z(z)+\int_{B/v_u}^{1} m^{EP}_{\delta,u}(v_u\cdot z)dG_Z(z)\right]dv_u. \tag{5.4.3}$$

See Appendix 5.A.3 for proof of Theorem 5.4.1.

*Remark* 5.4.1. We point out that the IEs (5.4.1), (5.4.2) and (5.4.3) allow us to consider the differentiability of the functions $m^{EP}_{\delta,u}(x)$, $m^{EP}_{\delta,m}(x)$ and $m^{EP}_{\delta,l}(x)$. For instance, it is easy to see from (5.4.1), (5.4.2) and (5.4.3) that $m^{EP}_{\delta,u}(x)$, $m^{EP}_{\delta,m}(x)$ and $m^{EP}_{\delta,l}(x)$ are differentiable in $(B,\infty)$, $(x^*,B)$ and $(0,x^*)$, respectively. Furthermore, they satisfy the following condition

$$m^{EP}_{\delta,u}(B)=m^{EP}_{\delta,m}(B^-) \tag{5.4.4}$$

and

$$m^{EP}_{\delta,m}(x^*)=m^{EP}_{\delta,l}(x^{*-}). \tag{5.4.5}$$

As for Theorem 5.3.1, the existence and uniqueness of the required solution to the IEs derived in Theorem 5.4.1 should be justified in each case. Now, by differentiating the IEs (5.4.1), (5.4.2) and (5.4.3), we obtain the IDEs for $m^{EP}_{\delta,u}(x)$, $m^{EP}_{\delta,m}(x)$ and $m^{EP}_{\delta,l}(x)$ in the following theorem

**Theorem 5.4.2.** *When* $x\geq B$*, we have*

$$r(x-x^*)m'^{EP}_{\delta,u}(x)-(\lambda+\delta)m^{EP}_{\delta,u}(x)+\lambda\left[\int_0^{x^*/x} m^{EP}_{\delta,l}(x\cdot z)dG_Z(z)\right.$$

$$+\int_{x^*/x}^{B/x} m^{\text{EP}}_{\delta,m}(x\cdot z)dG_Z(z)+\int_{B/x}^{1} m^{\text{EP}}_{\delta,u}(x\cdot z)dG_Z(z)\Bigg]=0, \tag{5.4.6}$$

*when $x^*\le x<B$, we have*

$$(r-c_T)\left(x+x^{**}\right)m'^{\text{EP}}_{\delta,m}(x)-(\lambda+\delta)m^{\text{EP}}_{\delta,m}(x)+\lambda\Bigg[\int_0^{x^*/x} m^{\text{EP}}_{\delta,l}(x\cdot z)dG_Z(z)$$

$$+\int_{x^*/x}^{1} m^{\text{EP}}_{\delta,m}(x\cdot z)dG_Z(z)\Bigg]=0, \tag{5.4.7}$$

*and when $0<x<x^*$, we have*

$$c_T(x-B)m'^{\text{EP}}_{\delta,l}(x)+[\lambda+\delta+\omega(x)]m^{\text{EP}}_{\delta,l}(x)$$
$$-\omega(x)w^{\text{EP}}(x,x^*-x)-\lambda\int_0^1 m^{\text{EP}}_{\delta,l}(x\cdot z)dG_Z(z)=0. \tag{5.4.8}$$

*In addition, the boundary conditions for $m^{\text{EP}}_{\delta,u}(x)$, $m^{\text{EP}}_{\delta,m}(x)$ and $m^{\text{EP}}_{\delta,l}(x)$ are given by (5.4.4), (5.4.5) and $\lim_{x\to\infty} m^{\text{EP}}_{\delta,u}(x)=0$.*

*Remark* 5.4.2. Equation (5.4.8) for $m^{\text{EP}}_{\delta,l}(x)$ is independent of $m^{\text{EP}}_{\delta,u}(x)$ and $m^{\text{EP}}_{\delta,m}(x)$. However, $m^{\text{EP}}_{\delta,l}(x)$ is subject to the boundary condition (5.4.5) which is involved with $m^{\text{EP}}_{\delta,m}(x)$. At the same time, $m^{\text{EP}}_{\delta,m}(x)$ is subject to the boundary condition (5.4.4) which is involved with $m^{\text{EP}}_{\delta,u}(x)$. Furthermore, it is easy to see from (5.4.6), (5.4.7) and (5.4.8) that $m^{\text{EP}}_{\delta,u}(x)$, $m^{\text{EP}}_{\delta,m}(x)$ and $m^{\text{EP}}_{\delta,l}(x)$ satisfy

$$m'^{\text{EP}}_{\delta,u}(B)=m'^{\text{EP}}_{\delta,m}(B^-) \tag{5.4.9}$$

and

$$m'^{\text{EP}}_{\delta,m}(x^*)=m'^{\text{EP}}_{\delta,l}(x^{*-}). \tag{5.4.10}$$

### 5.4.1 The Time of Extreme Poverty

Focusing again in studying the Laplace transform of the random variable of interest (the time of extreme poverty) we note that by specifying the penalty function such that $w^{\text{EP}}(x_1,x_2)=1$, $m^{\text{EP}}_{\delta}(x)$ becomes the Laplace transform of the time of extreme poverty, also interpreted as the expected present value of a unit payment due at the time of extreme poverty. Thus, Equation (5.4.8) can then be written such that when $0<x<x^*$,

$$0=c_T(x-B)m'^{\text{EP}}_{\delta,l}(x)+[\lambda+\delta+\omega(x)]m^{\text{EP}}_{\delta,l}(x)-\omega(x)-\lambda\int_0^1 m^{\text{EP}}_{\delta,l}(x\cdot z)dG_Z(z). \tag{5.4.11}$$

#### Examples of Extreme Poverty Rate Functions

**Constant Extreme Poverty Rate Functions.** Let $\omega_1(x)\equiv\omega_c\cdot\mathbb{1}_{\{x<x^*\}}$ with $\omega_c>0$. This is the simplest case of extreme poverty rate function and it could be interpreted as the situation in which the events of extreme poverty occur at discrete times. For instance, let $\xi_1, \xi_2, ...$ be i.i.d. exponential random variables with mean $\frac{1}{\omega_c}$ and $\Xi_k = \xi_1 + \xi_2 + ... + \xi_k$ denote the $kth$ event of extreme poverty (e.g. unexpected loss of assets or health), with $k = 1, 2, ....$ In this context, extreme poverty occurs when at such an event of extreme poverty the capital level is below $x^*$ (Albrecher et al., 2013).

**Proposition 5.4.1.** *Consider a household capital process with initial capital $x \geq x^*$, capital growth rate $r$, capital barrier level $B$, capital transfer rate $c_{\scriptscriptstyle T}$, intensity $\lambda > 0$ and remaining proportions of capital with distribution Beta$(\alpha, 1)$ where $\alpha > 0$; that is, $Z_i \sim$ Beta$(\alpha, 1)$. For a constant extreme poverty rate function $\omega_1(x) \equiv \omega_c \cdot \mathbb{1}_{\{x<x^*\}}$, with $\omega_c > 0$, the Laplace transform of the time of extreme poverty is given by*

$$m_\delta^{\text{EP}}(x) = \begin{cases} A_{2,u}^{\text{EP}} y_u^{\text{EP}}(x)^{-b_u^{\text{EP}}} {}_2F_1\left(b_u^{\text{EP}}, b_u^{\text{EP}} - c_u^{\text{EP}} + 1; b_u^{\text{EP}} - a_u^{\text{EP}} + 1; y_u^{\text{EP}}(x)^{-1}\right) & \text{for } \; x \geq B, \\ \\ A_{1,m}^{\text{EP}} y_m^{\text{EP}}(x)^{-a_m^{\text{EP}}} {}_2F_1\left(a_m^{\text{EP}}, a_m^{\text{EP}} - c_m^{\text{EP}} + 1; a_m^{\text{EP}} - b_m^{\text{EP}} + 1; y_m^{\text{EP}}(x)^{-1}\right) & \\ + A_{2,m}^{\text{EP}} y_m^{\text{EP}}(x)^{-b_m^{\text{EP}}} {}_2F_1\left(b_m^{\text{EP}}, b_m^{\text{EP}} - c_m^{\text{EP}} + 1; b_m^{\text{EP}} - a_m^{\text{EP}} + 1; y_m^{\text{EP}}(x)^{-1}\right) & \text{for } \; x^* \leq x < B, \\ \\ \dfrac{\omega_c}{\delta + \omega_c} + A_{1,l}^{\text{EP}} {}_2F_1\left(a_l^{\text{EP}}, b_l^{\text{EP}}; c_l^{\text{EP}}; y_l^{\text{EP}}(x)\right) & \text{for } \; 0 < x < x^*, \end{cases}$$

*where $\delta \geq 0$ is the force of interest for valuation, ${}_2F_1(\cdot)$ is Gauss's Hypergeometric Function as defined in (5.A.8), $y_u^{\text{EP}}(x) = \frac{x}{x^*}$, $a_u^{\text{EP}} = \frac{-(\delta+\lambda-\alpha r)-\sqrt{(\delta+\lambda-\alpha r)^2+4r\alpha\delta}}{2r}$, $b_u^{\text{EP}} = \frac{-(\delta+\lambda-\alpha r)+\sqrt{(\delta+\lambda-\alpha r)^2+4r\alpha\delta}}{2r}$, $y_m^{\text{EP}}(x) = -\frac{x}{x^{**}}$, with $x^{**} = \frac{c_{\scriptscriptstyle T} B - r x^*}{r - c_{\scriptscriptstyle T}}$, $c_u^{\text{EP}} = c_m^{\text{EP}} = c_l^{\text{EP}} = \alpha$, $a_m^{\text{EP}} = \frac{-(\delta+\lambda-\alpha(r-c_{\scriptscriptstyle T}))-\sqrt{(\delta+\lambda-\alpha(r-c_{\scriptscriptstyle T}))^2+4(r-c_{\scriptscriptstyle T})\alpha\delta}}{2(r-c_{\scriptscriptstyle T})}$, $b_m^{\text{EP}} = \frac{-(\delta+\lambda-\alpha(r-c_{\scriptscriptstyle T}))+\sqrt{(\delta+\lambda-\alpha(r-c_{\scriptscriptstyle T}))^2+4(r-c_{\scriptscriptstyle T})\alpha\delta}}{2(r-c_{\scriptscriptstyle T})}$, $y_l^{\text{EP}}(x) = \frac{x}{B}$, $a_l^{\text{EP}} = \frac{\alpha c_{\scriptscriptstyle T}+\lambda+\delta+\omega_c-\sqrt{(\alpha c_{\scriptscriptstyle T}+\lambda+\delta+\omega_c)^2-4\alpha c_{\scriptscriptstyle T}(\delta+\omega_c)}}{2c_{\scriptscriptstyle T}}$, $b_l^{\text{EP}} = \frac{\alpha c_{\scriptscriptstyle T}+\lambda+\delta+\omega_c+\sqrt{(\alpha c_{\scriptscriptstyle T}+\lambda+\delta+\omega_c)^2-4\alpha c_{\scriptscriptstyle T}(\delta+\omega_c)}}{2c_{\scriptscriptstyle T}}$ and the constants $A_{2,u}^{\text{EP}}$, $A_{1,m}^{\text{EP}}$, $A_{2,m}^{\text{EP}}$ and $A_{1,l}^{\text{EP}}$ are obtained as explained in Appendix 5.A.4.*

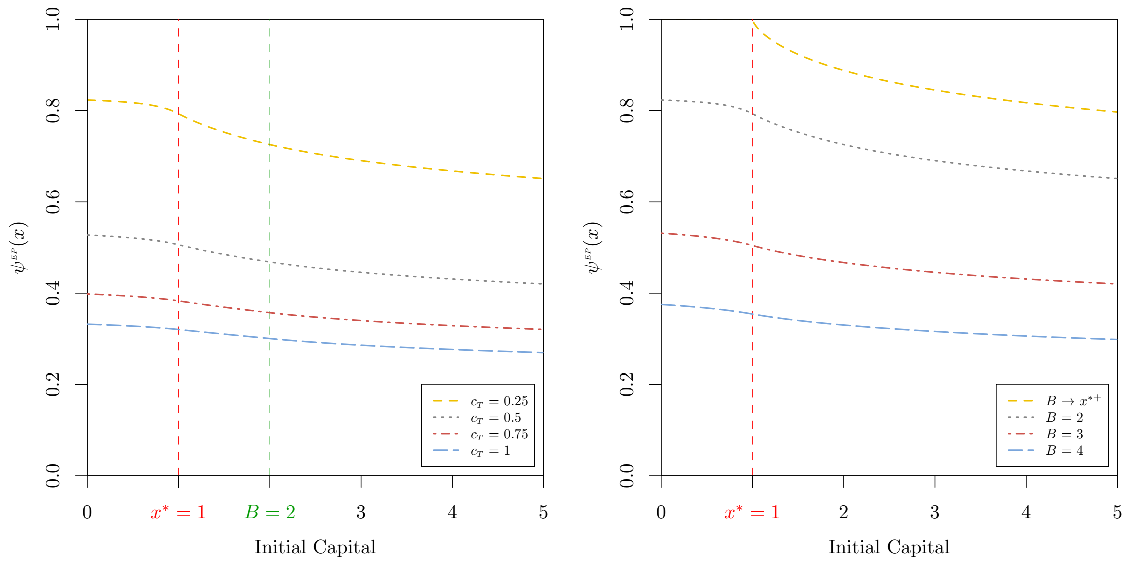
See Appendix 5.A.4 for proof of Proposition 5.4.1.

*Remark* 5.4.3. As for the trapping time, the Laplace transform of the time of extreme poverty approaches the probability of extreme poverty as $\delta$ tends to zero, i.e.

$$\lim_{\delta \downarrow 0} m_\delta^{\text{EP}}(x) = \mathbb{P}(\tau_x^{\text{EP}} < \infty) \equiv \psi^{\text{EP}}(x),$$

for $\frac{\lambda}{r} < \alpha$.

Figure 5.3 shows the probability of extreme poverty $\psi^{\text{EP}}(x)$ for the capital process $X_t$ for a constant extreme poverty rate function. As shown in Figure 5.2 for the case of the trapping probability, Figure 5.3a and 5.3b reveal the probability of extreme poverty is also a decreasing function of the cash transfer rate $c_{\scriptscriptstyle T}$, the capital barrier level $B$ and the initial capital. In addition, in line with the definition of the extreme poverty rate function, Figure 5.4 demonstrates the probability of extreme poverty is an increasing function of the extreme poverty rate function. Furthermore, Figure 5.4 also plots the trapping probability obtained in Section 5.3 for reference, which is given by the particular case when $\omega_c \equiv \infty$ and therefore represents an upper bound for the probability of extreme poverty. Appendix 5.C provides a sensitivity analysis for the probability of extreme poverty with a constant extreme poverty rate function.




(a) (b)

Figure 5.3: (a) Probability of extreme poverty $\psi^{EP}(x)$ when $Z_i \sim Beta(0.8, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $B = 2$, $\lambda = 1$, $x^* = 1$ and $\omega_1(x) = 0.02$ for $c_T = 0.25, 0.5, 0.75, 1$ (b) Probability of extreme poverty $\psi^{EP}(x)$ when $Z_i \sim Beta(0.8, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $c_T = 0.25$, $\lambda = 1$, $x^* = 1$ and $\omega_1(x) = 0.02$ for $B \to x^{*+}$ and $B = 2, 3, 4$.

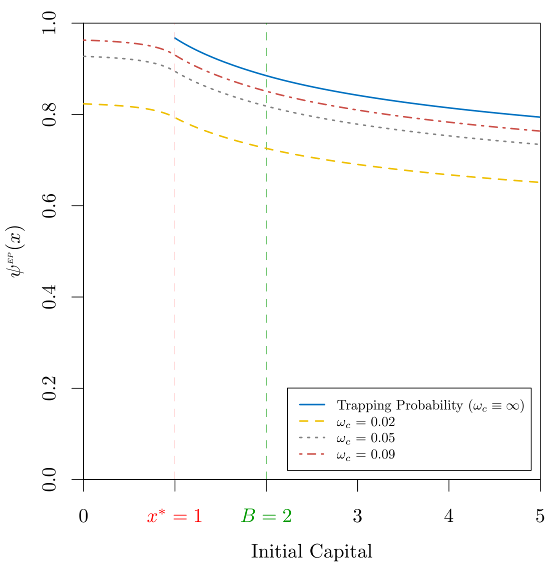



Figure 5.4: Probability of extreme poverty $\psi^{EP}(x)$ when $Z_i \sim Beta(0.8, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $c_T = 0.25$, $B = 2$, $\lambda = 1$, $x^* = 1$ and $\omega_1(x) = \omega_c$ for $\omega_c = 0.02, 0.05, 0.09$.

**Exponential Extreme Poverty Rate Functions.** Let now $\omega_2(x) = \frac{\beta}{x} \cdot \mathbb{1}_{\{x<x^*\}}$, for some $\beta > 0$. In this case, the extreme poverty rates take fairly flat values for lower deficit levels and reach higher values when the capital level gets close to zero. Such a function could be considered as the analogous version of the exponential bankruptcy rate function studied in Albrecher and Lautscham (2013).

*Remark* 5.4.4. In general, it is not straightforward to obtain the solution of (5.A.18) for more general extreme poverty rates $\omega(x)$, as functions of $\omega(x)$ appear both in the coefficients of the homogeneous equation and in the inhomogeneous term. Thus, for the particular case of exponential extreme poverty rate functions $\omega_2(x) = \frac{\beta}{x} \cdot \mathbb{1}_{\{x<x^*\}}$, we will only discuss the probability of extreme poverty.

**Proposition 5.4.2.** *Consider a household capital process with initial capital $x \geq x^*$, capital growth rate $r$, capital barrier level $B$, capital transfer rate $c_T$, intensity $\lambda > 0$ and remaining proportions of capital with distribution $Beta(\alpha, 1)$ where $\alpha > 0$; that is, $Z_i \sim Beta(\alpha, 1)$. For an exponential extreme poverty rate function $\omega_2(x) = \frac{\beta}{x} \cdot \mathbb{1}_{\{x<x^*\}}$, with $\beta > 0$, the probability of extreme poverty is given by*

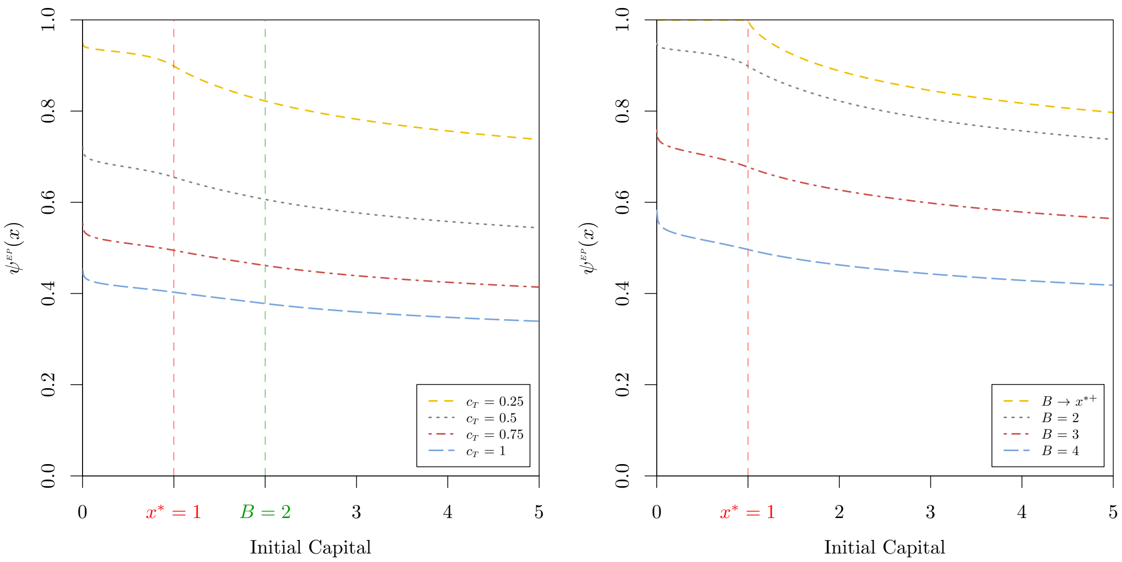
$$\psi^{EP}(x) = \begin{cases} A^{EP}_{2,u} \left(\frac{x}{x^*}\right)^{\frac{\lambda}{r}-\alpha} {}_2F_1\left(\alpha - \frac{\lambda}{r}, 1 - \frac{\lambda}{r}; 1 + \alpha - \frac{\lambda}{r}; \frac{x^*}{x}\right) & \textit{for} \quad x \geq B, \\ A^{EP}_{1,m} y^{EP}_m(x)^{-a^{EP}_m} {}_2F_1\left(a^{EP}_m, a^{EP}_m - c^{EP}_m + 1; a^{EP}_m - b^{EP}_m + 1; y^{EP}_m(x)^{-1}\right) & \\ \quad + A^{EP}_{2,m} y^{EP}_m(x)^{-b^{EP}_m} {}_2F_1\left(b^{EP}_m, b^{EP}_m - c^{EP}_m + 1; b^{EP}_m - a^{EP}_m + 1; y^{EP}_m(x)^{-1}\right) & \textit{for} \quad x^* \leq x < B, \\ 1 + A^{EP}_{2,l} y^{EP}_l(x)^{2-\alpha-c^{EP}_l} {}_2F_1\left(a^{EP}_l - c^{EP}_l + 1, b^{EP}_l - c^{EP}_l + 1; 2 - c^{EP}_l; y^{EP}_l(x)\right) & \textit{for} \quad 0 < x < x^*, \end{cases}$$

*where ${}_2F_1(\cdot)$ is Gauss's Hypergeometric Function as defined in (5.A.8), $y^{EP}_m(x) = -\frac{x}{x^{**}}$, with $x^{**} = \frac{c_T B - r x^*}{r - c_T}$, $a^{EP}_m = \frac{-(\lambda-\alpha(r-c_T))-\sqrt{(\lambda-\alpha(r-c_T))^2}}{2(r-c_T)}$, $b^{EP}_m = \frac{-(\lambda-\alpha(r-c_T))+\sqrt{(\lambda-\alpha(r-c_T))^2}}{2(r-c_T)}$, $c^{EP}_m = \alpha$, $y^{EP}_l(x) = \frac{x}{B}$, $a^{EP}_l = 1 - \alpha$, $b^{EP}_l = \frac{c_T+\lambda}{c_T}$ and $c^{EP}_l = -\frac{Bc_T(\alpha-2)+\beta}{Bc_T}$ for $\alpha > \frac{\lambda}{r}$ and the constants $A^{EP}_{2,u}$, $A^{EP}_{1,m}$, $A^{EP}_{2,m}$ and $A^{EP}_{2,l}$ are obtained as explained in Appendix 5.A.5.*

The mathematical proof of Proposition 5.4.2 is given in Appendix 5.A.5.

Figures 5.5 and 5.6 display the probability of extreme poverty when dealing with an exponential extreme poverty rate function. Evidently, under this setup, the probability of extreme poverty attains higher values compared to that under which a constant extreme poverty rate is considered. This can be verified by comparing Figures 5.3a and 5.5a, for several cash transfer rates $c_T$, Figures 5.3b and 5.5b, for different capital barrier levels $B$, and Figures 5.4 and 5.6, for different values of the extreme poverty rate, respectively. This result is not particularly surprising because of the fact that the exponential extreme poverty rate takes higher values for higher capital deficits while the constant extreme poverty rate remains flat for all capital levels. Appendix 5.C also presents a sensitivity analysis of the probability of extreme poverty for an exponential extreme poverty rate function.

As mentioned previously, Appendix 5.C shows how sensitive the probability of extreme poverty is with respect to all the underlying parameters (for both constant and exponential extreme poverty rate functions). In particular, the sensitivity with respect to the cash transfer rate $c_T$ and the capital barrier level $B$ is worth noting. These results accentuate the importance of selecting an appropriate cash transfer rate $c_T$ (i.e. an adequate frequency or intensity of the capital cash transfers) and a suitable capital barrier level $B$ (i.e. an opportune targeting), when designing the social protection strategy for achieving extreme poverty reduction.




(a) (b)

Figure 5.5: (a) Probability of extreme poverty $\psi^{EP}(x)$ when $Z_i \sim Beta(0.8, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $B = 2$, $\lambda = 1$, $x^* = 1$ and $\omega_2(x) = \frac{0.02}{x}$ for $c_T = 0.25, 0.5, 0.75, 1$ (b) Probability of extreme poverty $\psi^{EP}(x)$ when $Z_i \sim Beta(0.8, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $c_T = 0.25$, $\lambda = 1$, $x^* = 1$ and $\omega_2(x) = \frac{0.02}{x}$ for $B \to x^{*+}$ and $B = 2, 3, 4$.

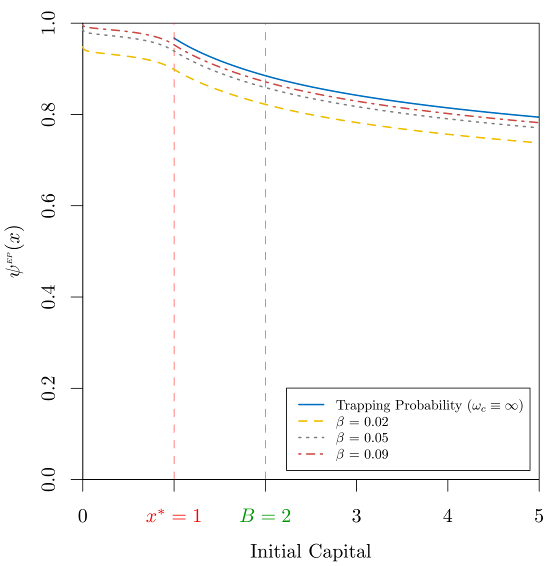
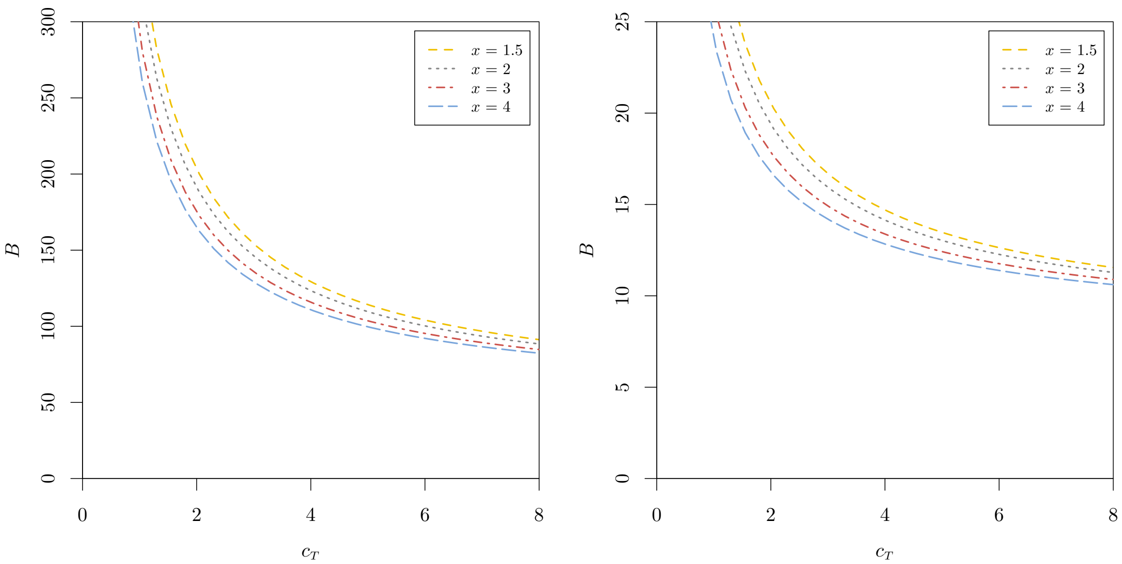



Figure 5.6: Probability of extreme poverty $\psi^{EP}(x)$ when $Z_i \sim Beta(0.8, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $c_T = 0.25$, $B = 2$, $\lambda = 1$, $x^* = 1$ and $\omega_2(x) = \frac{\beta}{x}$ for $\beta = 0.02, 0.05, 0.09$.

Figures 5.7a and 5.7b provide an example of the cash transfer rate $c_T$ and the capital barrier level $B$ required to attain a given target trapping probability and probability of extreme poverty (for a constant extreme poverty rate function), respectively. Clearly, policy makers must decide between reducing the intensity of the capital cash transfers (lowering the capital cash transfer rate $c_T$) to a wider group of households (increasing the capital barrier level $B$) or increasing the intensity of the capital cash transfers (rising the capital cash transfer rate $c_T$) to a narrower group of households (lowering the capital barrier level $B$) in order to achieve the target probabilities, showing an evident trade-off between these two parameters.




(a) (b)

Figure 5.7: (a) Cash transfer rate $c_T$ and capital barrier level $B$ required to attain a given target trapping probability of $\psi^{P}(x) = 0.01$ when $Z_i \sim Beta(1.25, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $\lambda = 1$ and $x^* = 1$ for initial capital $x = 1.5, 2, 3, 4$ (b) Cash transfer rate $c_T$ and capital barrier level $B$ required to attain a given target probability of extreme poverty of $\psi^{EP}(x) = 0.01$ when $Z_i \sim Beta(1.25, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $\lambda = 1$, $x^* = 1$ and $\omega_c = 0.09$ for initial capital $x = 1.5, 2, 3, 4$.

## 5.5 Monte Carlo Simulation

In general, it is not straightforward to derive explicit formulas for both the trapping probability and the probability of extreme poverty when more general cases are considered. Monte Carlo simulation is an alternative way to produce estimates for both quantities and is particularly useful when dealing with cases for which closed-form formulas are not available. In this section, following Albrecher and Lautscham (2013), we introduce a simple and efficient methodology that allows to generate fairly accurate approximations for the probability of extreme poverty.

### 5.5.1 Methodology

Following Albrecher and Lautscham (2013), we note that for any capital level $x \in (0, \infty)$ it holds that

$$\psi^{\text{EP}}(x) = 1 - \mathbb{E}\left[e^{-\int_0^\infty \omega(X_t)\mathbb{1}_{\{X_t<x^*\}}dt} \mid X_0 = x\right],$$

as extreme poverty can only be avoided if there is no event of the Poisson process with level-dependent intensity $\omega(\cdot)$ during the time the capital process spends below the critical capital $x^*$. The above expectation can then be computed by conditioning on the simulated sample path. Concretely, conditioning on the remaining proportions $\Theta_i$, with

$$\begin{aligned}\Psi^{\text{EP}}(\omega, x) \mid (T_1, \Theta_1), (T_2, \Theta_2) \ldots &= \int_0^\infty \omega(X_t) \cdot \mathbb{1}_{\{X_t<x^*\}}dt \\ &= -\sum_{i=0}^{\infty} \mathbb{1}_{\{X_{T_i}<x^*\}} \int_{T_i}^{\min\left(T_{i+1}, T_i+\tau_{x^*}\left(X_{T_i}\right)\right)} \omega(X_s)\, ds\end{aligned} \tag{5.5.1}$$

with $T_0 = 0$, we can write

$$\psi^{\text{EP}}(\omega, x) = \mathbb{E}_{(T_1,\Theta_1),(T_2,\Theta_2)\ldots}\left[1 - e^{\Psi^{\text{EP}}(\omega,x)|(T_1,\Theta_1),(T_2,\Theta_2)\ldots}\right].$$

In particular, for the two choices $\omega_1(x) = \omega_c$, $\omega_c > 0$, and $\omega_2(x) = \frac{\beta}{x}$, $\beta > 0$, (5.5.1) reads

$$\begin{aligned}\Psi^{\text{EP}}(\omega_1, x) \mid (T_1, \Theta_1), (T_2, \Theta_2) \ldots &= -\sum_{i=0}^{\infty} \mathbb{1}_{\{X_{T_i}<x^*\}} \int_{T_i}^{\min\left(T_{i+1}, T_i+\tau_{x^*}\left(X_{T_i}\right)\right)} \omega_c\, ds \\ &= -\omega_c \sum_{i=0}^{\infty} \mathbb{1}_{\{X_{T_i}<x^*\}} \left[\min\left(T_{i+1} - T_i, \tau_{x^*}\left(X_{T_i}\right)\right)\right]\end{aligned} \tag{5.5.2}$$

and

$$\begin{aligned}\Psi^{\text{EP}}(\omega_2, x) \mid (T_1, \Theta_1), (T_2, \Theta_2) \ldots &= -\sum_{i=0}^{\infty} \mathbb{1}_{\{X_{T_i}<x^*\}} \int_{T_i}^{\min\left(T_{i+1}, T_i+\tau_{x^*}\left(X_{T_i}\right)\right)} \frac{\beta}{\left(X_{T_i} - B\right) e^{c_T(T_i-s)} + B} ds \\ &= -\frac{\beta}{c_T B} \sum_{i=0}^{\infty} \mathbb{1}_{\{X_{T_i}<x^*\}} \Big[c_T \min\left(T_{i+1} - T_i, \tau_{x^*}\left(X_{T_i}\right)\right) \\ &+ \ln\left(B + \left(X_{T_i} - B\right) e^{c_T\left[T_i - min\left(T_{i+1}, T_i+\tau_{x^*}\left(X_{T_i}\right)\right)\right]}\right) - \ln\left(X_{T_i}\right)\Big],\end{aligned} \tag{5.5.3}$$

respectively. Figure 5.8 depicts a particular path, and the shaded area refers to $\Psi^{\text{EP}}(\omega, x) \mid (T_1, \Theta_1), (T_2, \Theta_2) \ldots$ as in (5.5.1).

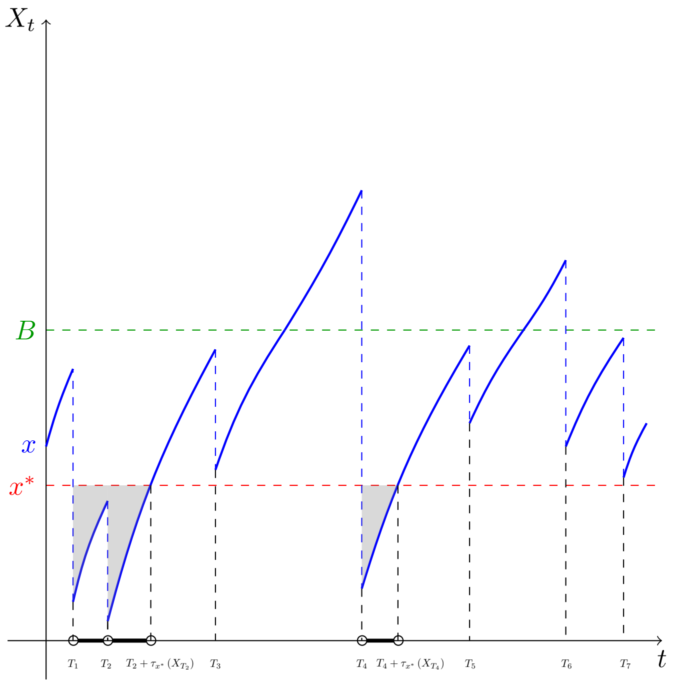
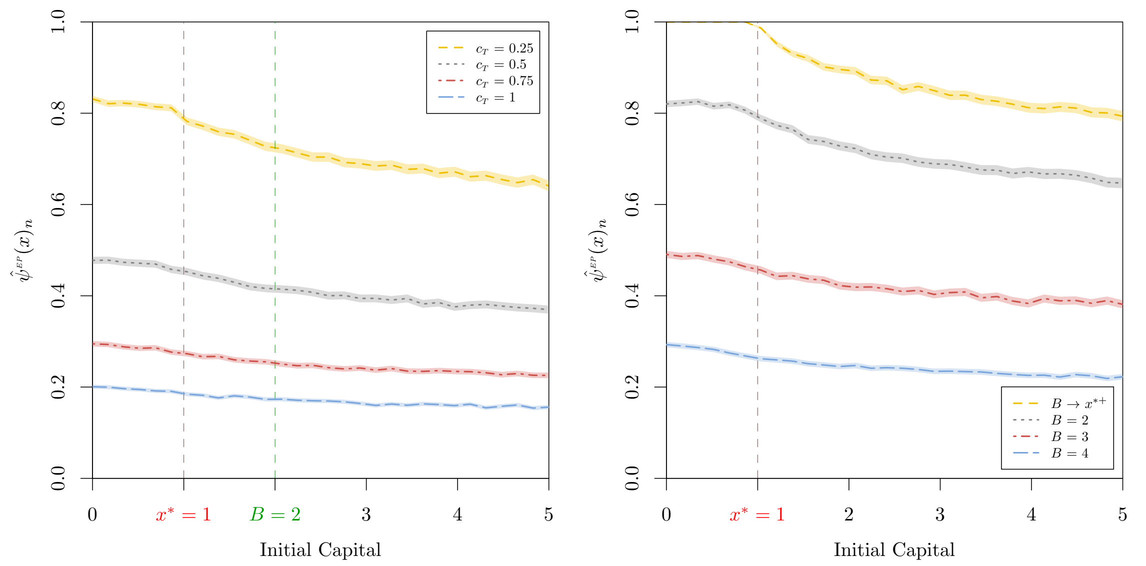



Figure 5.8: Computation of $\Psi^{\text{EP}}(\omega, x)$ conditional on a realised sample path.




(a) (b)

Figure 5.9: (a) Probability of extreme poverty $\hat{\psi}^{\text{EP}}(x)_n$ when $n = 10,000$, $Z_i \sim Beta(0.8, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $B = 2$, $\lambda = 1$, $x^* = 1$ and $\omega_1(x) = 0.02$ for $c_T = 0.25, 0.5, 0.75, 1$ (b) Probability of extreme poverty $\psi^{\text{EP}}(x)_n$ when $n = 10,000$, $Z_i \sim Beta(0.8, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $c_T = 0.25$, $\lambda = 1$, $x^* = 1$ and $\omega_1(x) = 0.02$ for $B \to x^{*+}$ and $B = 2, 3, 4$.

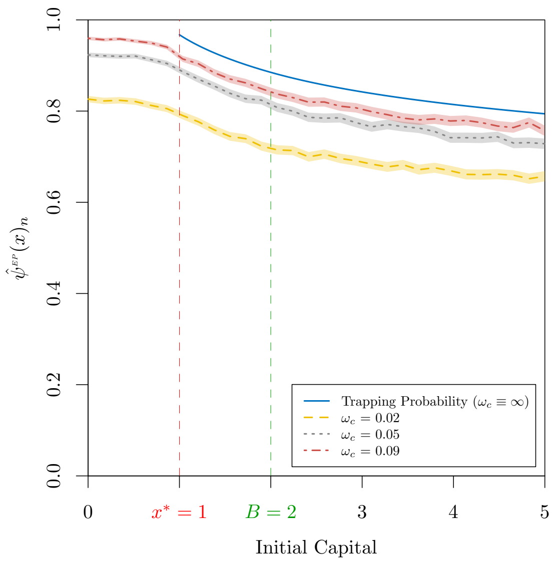
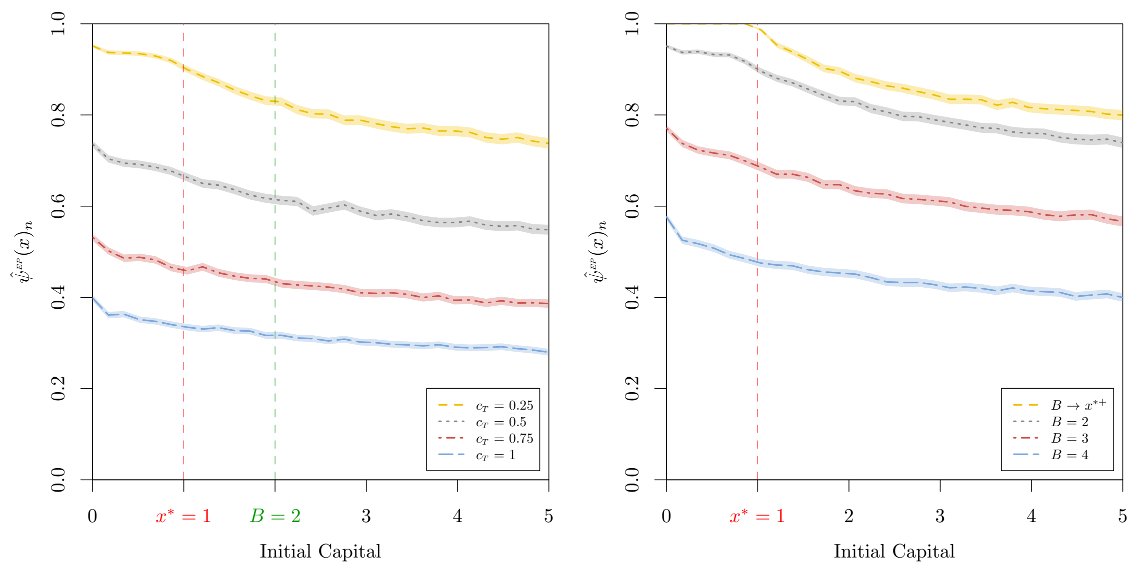



Figure 5.10: Probability of extreme poverty $\hat{\psi}^{\text{EP}}(x)_n$ when $n = 10,000$, $Z_i \sim Beta(0.8, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $c_T = 0.25$, $B = 2$, $\lambda = 1$, $x^* = 1$ and $\omega_1(x) = \omega_c$ for $\omega_c = 0.02, 0.05, 0.09$.




(a) (b)

Figure 5.11: (a) Probability of extreme poverty $\hat{\psi}^{\text{EP}}(x)_n$ when $n = 10,000$, $Z_i \sim Beta(0.8, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $B = 2$, $\lambda = 1$, $x^* = 1$ and $\omega_2(x) = \frac{0.02}{x}$ for $c_T = 0.25, 0.5, 0.75, 1$ (b) Probability of extreme poverty $\psi^{\text{EP}}(x)$ when $Z_i \sim Beta(0.8, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $c_T = 0.25$, $\lambda = 1$, $x^* = 1$ and $\omega_2(x) = \frac{0.02}{x}$ for $B \to x^{*+}$ and $B = 2, 3, 4$.

In the following simulations, $n$ capital process paths are generated and for the $k$th such path, the function $\Psi^{\text{EP}}(\omega, x)_k \mid (T_1, \Theta_1), (T_2, \Theta_2) \ldots$ is computed as per (5.5.2) and (5.5.3). The estimator of the probability of extreme poverty is given by

$$\hat{\psi}^{\text{EP}}(x)_n = \frac{1}{n}\sum_{k=1}^{n}\left(1 - e^{\Psi^{\text{EP}}(\omega,x)_k}\right),$$

and the two sided 99% confidence interval of the estimator can be written as

$$\left(\max\left[\hat{\psi}^{\text{EP}}(x)_n - \frac{2.81}{\sqrt{n}}\sigma_n, 0\right], \min\left[\hat{\psi}^{\text{EP}}(x)_n + \frac{2.81}{\sqrt{n}}\sigma_n, 1\right]\right),$$

with $\sigma_n = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}\left(1 - e^{\Psi^{\text{EP}}(\omega,x)_k} - \hat{\psi}^{\text{EP}}(x)_n\right)^2}$, such that the bounds of the confidence interval converge to $\hat{\psi}^{\text{EP}}(x)_n$ for $n \to \infty$.

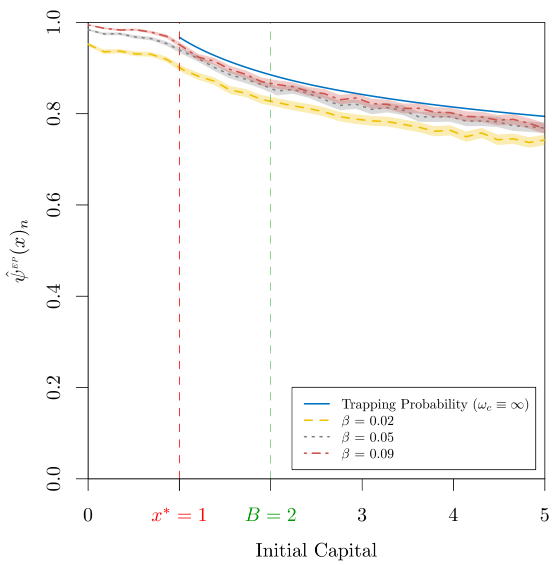



Figure 5.12: Probability of extreme poverty $\hat{\psi}^{\text{EP}}(x)_n$ when $n = 10,000$, $Z_i \sim Beta(0.8, 1)$, $a = 0.1$, $b = 4$, $c_S = 0.4$, $c_T = 0.25$, $B = 2$, $\lambda = 1$, $x^* = 1$ and $\omega_2(x) = \frac{\beta}{x}$ for $\beta = 0.02, 0.05, 0.09$.

Figures 5.9, 5.10, 5.11 and 5.12 provide an example of the Monte Carlo methodology discussed in this section. A comparison of Figure 5.3 with Figure 5.9, Figure 5.4 with Figure 5.10, Figure 5.5 with Figure 5.11 and Figure 5.6 with Figure 5.12, respectively, provides insight into the ability of this method to produce approximations of the probability of extreme poverty when considering more general cases. Although, in general, Monte Carlo simulations produce fairly accurate approximations, it is especially important to note that, for some cases of selected parameters, Monte Carlo simulations may lead to less accurate approximations. Comparing Figures 5.3a and 5.9a, and Figures 5.3b and 5.9b, for higher capital cash transfer rates $c_T$ and higher capital barrier levels $B$, respectively, leads to a clear evidence of this imprecision. In this particular case, the differences between the closed-form formula and the Monte Carlo approximates are mainly due to the fact that for high capital cash transfer rates $c_T$ and capital barrier levels $B$, the capital trajectory will grow rapidly up to the capital barrier level $B$, even in those cases where capital levels close to zero are reached, whereas for the closed-form formula, this would almost certainly be considered as an event of extreme poverty. Nevertheless, it is also worth noting that even though there are the aforementioned discrepancies, Monte Carlo estimates are able to capture the main trend in the probability of extreme poverty.

As mentioned previously, the proposed methodology could be of great advantage when dealing with dynamics for which closed-form formulas are not available. For instance, one could produce approximates of the probability of extreme poverty for situations under which the remaining proportions of capital after experiencing a loss are non $Beta(\alpha, 1)-$distributed; i.e. when the random variables $Z_i$ follow another distribution with support in $(0, 1]$.

## 5.6 Conclusion

Using standard techniques from actuarial science and, in particular, from ruin theory, this study analyses the efficiency of regular unconditional cash transfer (UCT) programmes in achieving one of the global public's priorities: ending poverty in all its forms everywhere. Introducing an alternative version of the household's capital model originally proposed in Kovacevic and Pflug (2011), where we consider a particular group of households are entitled to benefit from capital cash transfers and, adopting ideas from the Omega risk process, first introduced in Albrecher et al. (2011), this chapter focuses on studying two main random variables: the trapping time and the time of extreme poverty. While the trapping time has been previously studied for more common risk processes (see, for example, Flores-Contró et al. (2021) and Flores-Contró (2024)), to the best of our knowledge, this is the first work that considers the trapping time and the time of extreme poverty under the dynamics of a household's capital process that incorporates capital cash transfers. Furthermore, for the particular case of the time of extreme poverty, this work also introduces the concept of the extreme poverty rate function for the first time. This chapter analyses the behavior of two main risk measures associated to these random times: the trapping probability and the probability of extreme poverty.

From a ruin-theoretic perspective, our main contribution is obtaining closed-form solutions for both risk measures, which is considered to be the ideal situation when working with ruin probabilities (Asmussen and Albrecher, 2010). We derive explicit formulas for both the trapping probability and the probability of extreme poverty assuming the proportion of the remaining capital of a household after experiencing a loss is $Beta(\alpha, 1)-$distributed. Moreover, for the particular case of the probability of extreme poverty, we also consider two examples of extreme poverty rate functions for which closed-from solutions for the probability of extreme poverty are available: constant and exponential extreme poverty rate functions. Nevertheless, explicit formulas are generally not straightforward to obtain for more general cases. Hence, following Albrecher and Lautscham (2013), in Section 5.5 we also illustrate how to produce approximations of the probability of extreme poverty via an efficient Monte Carlo simulation method.

Numerical examples presented in Sections 5.3 and 5.4 indicate that regular UCT programmes are an efficient social protection strategy to keep households out of poverty and extreme poverty, as their trapping probability and the probability of extreme poverty, respectively, decrease when they are part of such strategy. In particular, the role played by both the capital cash transfer rate $c_T$ and the capital barrier level $B$ for attaining lower probabilities is outlined. Our findings can provide policy makers with a mathematically sound starting point for designing UCT programmes. That is, our model, for instance, could provide insights during the planning phase of an UCT programme to policy makers about the impact on the probability of (extreme) household impoverishment when targeting a particular group of households (depending on the selection of the capital barrier level $B$). Moreover, the sensitivity of the probability of (extreme) household impoverishment to the frequency or intensity of the capital cash transfers (depending on the choice of the capital cash transfer rate $c_T$) can also be assessed with our results. Furthermore, it is important to note that our analyses show that the probability of extreme poverty appears to be more sensitive to changes in these parameters, compared to the trapping probability, therefore suggesting that policy makers should specially watch out on these parameters when designing social protection strategies aimed at reducing extreme poverty.

From the point of view of development economics, previous empirical studies are in line with our findings. Furthermore, our work presents an alternative approach to analyse cash transfer programmes and may represent a point of departure for applying knowledge of another discipline, such as actuarial science, in development economics.

It is important to highlight some of the limitations of our study. For example, due to the construction of the model, our analysis does not capture the direct effect of an UCT programme on a household's consumption. Recently, Habimana et al. (2021) show how Rwanda's UCT programme (VUP-Direct Support) increases a household's total and food consumption. In the same way, in its current form, the capital model is unable to incorporate the rationale behind conditional cash transfer (CCT) programmes, as it does not track any beneficiary actions such as: enrollment and attendance of children and adolescents in school, use of health services and uptake of food and nutritional supplements (Cruz et al., 2017). Alternative versions of the proposed model should address these issues.

Finally, future research should also consider the cost of an UCT programme. This cost could be estimated, for instance, by computing the total expected discounted value of capital cash transfers made to a household. This concept would be analogous to other well-known quantities previously studied in ruin theory, such as the expected discounted capital injections (Albrecher and Ivanovs, 2014). These quantities could, for example, be useful for estimating the required capital cash transfer rate $c_T$ and capital barrier level $B$ such that, for a given social protection budget, the trapping probability or probability of extreme poverty is minimised.

# 5.A Appendix A: Mathematical Proofs

## 5.A.1 Proof of Theorem 5.3.1

For $x \geq B$, the capital immediately before the first capital loss is $h_r(t,x) = (x - x^*)e^{rt} + x^*$. Hence, by conditioning on the time and the remaining proportion of the first capital loss and discounting the expected values to time 0 at the force of interest $\delta$, when $x \geq B$ we obtain

$$\begin{aligned} m_{\delta,u}^{\mathbb{P}}(x) = \int_0^\infty \lambda e^{-(\lambda+\delta)t} \Bigg[ &\int_{B/h_r(t,x)}^1 m_{\delta,u}^{\mathbb{P}}(h_r(t,x)\cdot z)dG_Z(z) \\ &+ \int_{x^*/h_r(t,x)}^{B/h_r(t,x)} m_{\delta,l}^{\mathbb{P}}(h_r(t,x)\cdot z)dG_Z(z) \\ &+ \int_0^{x^*/h_r(t,x)} w^{\mathbb{P}}(h_r(t,x) - x^*, x^* - h_r(t,x)\cdot z)dG_Z(z) \Bigg] dt. \end{aligned} \tag{5.A.1}$$

The above equation for $m_{\delta,u}^{\mathbb{P}}(x)$ involves $m_{\delta,l}^{\mathbb{P}}(x)$ for $x^* \leq x < B$. When the initial capital is below the capital barrier level $B$, the capital growth is driven by both the capital growth rate $r$ and the capital transfer rate $c_T$ before the capital returns to the capital barrier level $B$. Thus, for $x^* \leq x < B$, let $\tau_B := \tau_B(x)$ be the solution to

$$h_{r-c_T}(t,x) = (x + x^{**})e^{(r-c_T)t} - x^{**} = B,$$

with $x^{**} = (c_T B - r x^*)\,/\,(r - c_T)$. Namely, $\tau_B := \tau_B(x) = \ln\left[(B + x^{**})\,/\,(x + x^{**})\right]/(r - c_T)$, which is the time when the capital returns to the capital barrier level $B$ if no capital loss occurs prior to time $\tau_B$. Furthermore, $h_{r-c_T}(t,x) < B$ for $t < \tau_B$ and $h_{r-c_T}(\tau_B, x) = B$. Moreover, $h_{r-c_T}(t,x)$ is the capital at time $t \leq \tau_B$ if no capital loss occurs prior to time $\tau_B$. Thus, by conditioning on the time and the remaining proportion of the first capital loss and discounting the expected values to time 0 at the force of interest $\delta$, when $x^* \leq x < B$ we obtain

$$\begin{aligned} m_{\delta,l}^{\mathbb{P}}(x) = &\int_0^{\tau_B} \lambda e^{-(\lambda+\delta)t} \Bigg[ \int_{x^*/h_{r-c_T}(t,x)}^1 m_{\delta,l}^{\mathbb{P}}(h_{r-c_T}(t,x)\cdot z)dG_Z(z) \\ &+ \int_0^{x^*/h_{r-c_T}(t,x)} w^{\mathbb{P}}\left(h_{r-c_T}(t,x) - x^*, x^* - h_{r-c_T}(t,x)\cdot z\right) dG_Z(z) \Bigg] dt \\ &+ \int_{\tau_B}^\infty \lambda e^{-(\lambda+\delta)t} \Bigg[ \int_{B/h_r(t-\tau_B,B)}^1 m_{\delta,u}^{\mathbb{P}}(h_r(t-\tau_B,B)\cdot z)dG_Z(z) \\ &+ \int_{x^*/h_r(t-\tau_B,B)}^{B/h_r(t-\tau_B,B)} m_{\delta,l}^{\mathbb{P}}(h_r(t-\tau_B,B)\cdot z)dG_Z(z) \\ &+ \int_0^{x^*/h_r(t-\tau_B,B)} w^{\mathbb{P}}(h_r(t-\tau_B,B) - x^*, x^* - h_r(t-\tau_B,B)\cdot z)dG_Z(z) \Bigg] dt. \end{aligned} \tag{5.A.2}$$

Now, changing variables $u = h_r(t,x)$ in (5.A.1), we obtain (5.3.2). Moreover, first changing variables $u = h_{r-c_T}(t,x)$ in the integrals with respect to $t$ from 0 to $\tau_B$ in (5.A.2), and then changing variables $v = h_r(t-\tau_B, B)$ in the integrals with respect to $t$ from $\tau_B$ to $\infty$ in (5.A.2), we obtain (5.3.3).

### 5.A.2 Proof of Proposition 5.3.1

When $Z_i \sim Beta(\alpha, 1)$, i.e. $g_Z(z) = \alpha z^{\alpha-1}\mathbb{1}_{\{0<z<1\}}$ with $\alpha > 0$, Equations (5.3.8) and (5.3.9) can be written such that when $x \geq B$,

$$0 = r(x-x^*)m'^{P}_{\delta,u}(x) - (\lambda+\delta)m^{P}_{\delta,u}(x) + \lambda\left[\int_{B/x}^{1} m^{P}_{\delta,u}(x\cdot z)\alpha z^{\alpha-1}dz + \int_{x^*/x}^{B/x} m^{P}_{\delta,l}(x\cdot z)\alpha z^{\alpha-1}dz + \left(\frac{x^*}{x}\right)^{\alpha}\right], \tag{5.A.3}$$

and when $x^* \leq x < B$,

$$0 = (r-c_T)(x+x^{**})m'^{P}_{\delta,l}(x) - (\lambda+\delta)m^{P}_{\delta,l}(x) + \lambda\left[\int_{x^*/x}^{1} m^{P}_{\delta,l}(x\cdot z)\alpha z^{\alpha-1}dz + \left(\frac{x^*}{x}\right)^{\alpha}\right]. \tag{5.A.4}$$

Applying the operator $\frac{d}{dx}$ to both sides of (5.A.3) and (5.A.4), together with a number of algebraic manipulations, yields to the following second order Ordinary Differential Equations (ODEs),

$$x \geq B : 0 = r(x^2 - xx^*)m''^{P}_{\delta,u}(x) + \left[(r(1+\alpha)-\delta-\lambda)x - r\alpha x^*\right] m'^{P}_{\delta,u}(x) - \alpha\delta m^{P}_{\delta,u}(x) \tag{5.A.5}$$

and

$$x^* \leq x < B : 0 = (r-c_T)(x^2 + xx^{**})m''^{P}_{\delta,l}(x) + \left[\left((r-c_T)(1+\alpha)-\delta-\lambda\right)x + \alpha(r-c_T)x^{**}\right] m'^{P}_{\delta,l}(x) - \alpha\delta m^{P}_{\delta,l}(x). \tag{5.A.6}$$

Letting $f^{P}_i(y^{P}_i) := m^{P}_{\delta,i}(x)$ for $i = u,l$, such that $y^{P}_u$ and $y^{P}_l$ are associated with the change of variables $y^{P}_u := y^{P}_u(x) = x/x^*$ and $y^{P}_l := y^{P}_l(x) = -x/x^{**}$, respectively, Equations (5.A.5) and (5.A.6) reduce to Gauss's Hypergeometric Differential Equation (Slater, 1960)

$$0 = y^{P}_i(1-y^{P}_i)\cdot f''^{P}_i(y^{P}_i) + [c^{P}_i - (1+a^{P}_i+b^{P}_i)y^{P}_i]f'^{P}_i(y^{P}_i) - a^{P}_i b^{P}_i f^{P}_i(y^{P}_i), \tag{5.A.7}$$

for $a^{P}_l = \frac{-(\delta+\lambda-\alpha(r-c_T)) - \sqrt{(\delta+\lambda-\alpha(r-c_T))^2+4(r-c_T)\alpha\delta}}{2(r-c_T)}$, $b^{P}_l = \frac{-(\delta+\lambda-\alpha(r-c_T)) + \sqrt{(\delta+\lambda-\alpha(r-c_T))^2+4(r-c_T)\alpha\delta}}{2(r-c_T)}$, $a^{P}_u = \frac{-(\delta+\lambda-\alpha r) - \sqrt{(\delta+\lambda-\alpha r)^2+4r\alpha\delta}}{2r}$, $b^{P}_u = \frac{-(\delta+\lambda-\alpha r) + \sqrt{(\delta+\lambda-\alpha r)^2+4r\alpha\delta}}{2r}$ and $c^{P}_u = c^{P}_l = \alpha$ with regular singular points at $y^{P}_i = 0, 1, \infty$ (corresponding to $x = -\infty, -x^{**}, 0, x^*$ and

$\infty$). A general solution of (5.A.7) in the neighborhood of the singular point $y_i^{P} = \infty$ is given by

$$
\begin{aligned}
f_i^{P}(y_i^{P}) := m_{\delta,i}^{P}(x) = A_{1,i}^{P} y_i^{P}(x)^{-a_i^{P}} {}_2F_1\left(a_i^{P}, a_i^{P} - c_i^{P} + 1; a_i^{P} - b_i^{P} + 1; y_i^{P}(x)^{-1}\right) \\
+ A_{2,i}^{P} y_i^{P}(x)^{-b_i^{P}} {}_2F_1\left(b_i^{P}, b_i^{P} - c_i^{P} + 1; b_i^{P} - a_i^{P} + 1; y_i^{P}(x)^{-1}\right),
\end{aligned}
$$

for arbitrary constants $A_{1,i}^{P}, A_{2,i}^{P} \in \mathbb{R}$ (see for example, Equations (15.5.7) and (15.5.8) of Abramowitz and Stegun (1972)). Here,

$$
{}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{5.A.8}
$$

is Gauss's Hypergeometric Function (Gauss, 1866) and $(a)_n = \Gamma(a+n)/\Gamma(n)$ denotes the Pochhammer symbol (Seaborn, 1991).

To determine the constants $A_{1,i}^{P}$ and $A_{2,i}^{P}$ we use the boundary conditions at $x^*$ and at $\infty$. In addition, we use (5.3.4), (5.3.7) and the differential properties of Gauss's Hypergeometric Function

$$
\frac{d}{dz} {}_2F_1(a, b; c; z) = \frac{ab}{c} {}_2F_1(a+1, b+1; c+1; z).
$$

The boundary condition $\lim_{x\to\infty} m_{\delta,u}^{P}(x) = 0$, by definition of $m_\delta^{P}(x)$ in (5.3.1), thus implies that $A_{1,u}^{P} = 0$. Moreover, letting $x = x^*$ in (5.3.9) yields

$$
m_{\delta,l}^{P}(x^*) = \frac{1}{\lambda + \delta}\left[c_T (B - x^*)\, m_{\delta,l}'^{P}(x^*) + \lambda\right].
$$

Hence, this yields to

$$
\begin{aligned}
A_{2,u}^{P} = \Big[ & \lambda y_u^{P}(B)^{b_u^{P}} x^* y_l^{P}(B)^{-(a_l^{P}+b_l^{P})} y_l^{P}(x^*)^{a_l^{P}} \Big( a_l^{P} {}_2\tilde{F}_1\left(a_l^{P}+1, a_l^{P} - c_l^{P} + 1; a_l^{P} - b_l^{P} + 1; y_l^{P}(B)^{-1}\right) \\
& {}_2\tilde{F}_1\left(b_l^{P}, b_l^{P} - c_l^{P} + 1; b_l^{P} - a_l^{P} + 1; y_l^{P}(B)^{-1}\right) - b_l^{P} {}_2\tilde{F}_1\left(a_l^{P}, a_l^{P} - c_l^{P} + 1; a_l^{P} - b_l^{P} + 1; y_l^{P}(B)^{-1}\right) \\
& {}_2\tilde{F}_1\left(b_l^{P}+1, b_l^{P} - c_l^{P} + 1; b_l^{P} - a_l^{P} + 1; y_l^{P}(B)^{-1}\right)\Big)\Big] / \\
& \Big[\Gamma\left(1 - a_u^{P} + b_u^{P}\right) \Big( y_l^{P}(B)^{-b_l^{P}} \Big( (\delta+\lambda)\, x^* {}_2\tilde{F}_1\left(a_l^{P}, a_l^{P} - c_l^{P} + 1; a_l^{P} - b_l^{P} + 1; y_l^{P}(x^*)^{-1}\right) \\
& + c_T a_l^{P} (B - x^*)\, {}_2\tilde{F}_1\left(a_l^{P}+1, a_l^{P} - c_l^{P} + 1; a_l^{P} - b_l^{P} + 1; y_l^{P}(x^*)^{-1}\right)\Big) \\
& \Big( b_u^{P} {}_2\tilde{F}_1\left(b_u^{P}+1, b_u^{P} - c_u^{P} + 1; b_u^{P} - a_u^{P} + 1; y_u^{P}(B)^{-1}\right) {}_2\tilde{F}_1\left(b_l^{P}, b_l^{P} - c_l^{P} + 1; b_l^{P} - a_l^{P} + 1; y_l^{P}(B)^{-1}\right) \\
& - b_l^{P} {}_2\tilde{F}_1\left(b_u^{P}, b_u^{P} - c_u^{P} + 1; b_u^{P} - a_u^{P} + 1; y_u^{P}(B)^{-1}\right) {}_2\tilde{F}_1\left(b_l^{P}+1, b_l^{P} - c_l^{P} + 1; b_l^{P} - a_l^{P} + 1; y_l^{P}(B)^{-1}\right)\Big) \\
& - y_l^{P}(B)^{-a_l^{P}} y_l^{P}(x^*)^{a_l^{P} - b_l^{P}} \Big( b_u^{P} {}_2\tilde{F}_1\left(b_u^{P}+1, b_u^{P} - c_u^{P} + 1; b_u^{P} - a_u^{P} + 1; y_u^{P}(B)^{-1}\right) \\
& {}_2\tilde{F}_1\left(a_l^{P}, a_l^{P} - c_l^{P} + 1; a_l^{P} - b_l^{P} + 1; y_l^{P}(B)^{-1}\right) - a_l^{P} {}_2\tilde{F}_1\left(b_u^{P}, b_u^{P} - c_u^{P} + 1; b_u^{P} - a_u^{P} + 1; y_u^{P}(B)^{-1}\right) \\
& {}_2\tilde{F}_1\left(1 + a_l^{P}, a_l^{P} - c_l^{P} + 1; a_l^{P} - b_l^{P} + 1; y_l^{P}(B)^{-1}\right)\Big) \\
& \Big( (\delta+\lambda)\, x^* {}_2\tilde{F}_1\left(b_l^{P}, b_l^{P} - c_l^{P} + 1; b_l^{P} - a_l^{P} + 1; y_l^{P}(x^*)^{-1}\right) \\
& + c_T b_l^{P} (B - x^*)\, {}_2\tilde{F}_1\left(b_l^{P}+1, b_l^{P} - c_l^{P} + 1; b_l^{P} - a_l^{P} + 1; y_l^{P}(x^*)^{-1}\right)\Big)\Big)\Big],
\end{aligned} \tag{5.A.9}
$$

$$
\begin{aligned}
A_{1,l}^{P} = \Big[ & \lambda x^{*} y_l^{P}\,(x^{*})^{a_l^{P}} \Big(1 + 1/\Big(-1 + \Big(y_l^{P}\,(B)^{a_l^{P}-b_l^{P}}\, y_l^{P}\,(x^{*})^{b_l^{P}-a_l^{P}} \\
& \Big((\delta+\lambda)\,x^{*}\,{}_2\tilde{F}_1\left(a_l^{P}, a_l^{P}-c_l^{P}+1; a_l^{P}-b_l^{P}+1; y_l^{P}(x^{*})^{-1}\right) \\
& + c_{T} a_l^{P}\,(B-x^{*})\,{}_2\tilde{F}_1\left(a_l^{P}+1, a_l^{P}-c_l^{P}+1; a_l^{P}-b_l^{P}+1; y_l^{P}(x^{*})^{-1}\right)\Big) \\
& \Big(b_u^{P}\,{}_2\tilde{F}_1\left(b_u^{P}+1, b_u^{P}-c_u^{P}+1; b_u^{P}-a_u^{P}+1; y_u^{P}(B)^{-1}\right) \\
& {}_2\tilde{F}_1\left(b_l^{P}, b_l^{P}-c_l^{P}+1; b_l^{P}-a_l^{P}+1; y_l^{P}(B)^{-1}\right) - b_l^{P}\,{}_2\tilde{F}_1\left(b_u^{P}, b_u^{P}-c_u^{P}+1; b_u^{P}-a_u^{P}+1; y_u^{P}(B)^{-1}\right) \\
& {}_2\tilde{F}_1\left(b_l^{P}+1, b_l^{P}-c_l^{P}+1; b_l^{P}-a_l^{P}+1; y_l^{P}(B)^{-1}\right)\Big)\Big) \,/ \\
& \Big(\Big(b_u^{P}\,{}_2\tilde{F}_1\left(b_u^{P}+1, b_u^{P}-c_u^{P}+1; b_u^{P}-a_u^{P}+1; y_u^{P}(B)^{-1}\right) \\
& {}_2\tilde{F}_1\left(a_l^{P}, a_l^{P}-c_l^{P}+1; a_l^{P}-b_l^{P}+1; y_l^{P}(B)^{-1}\right) - a_l^{P}\,{}_2\tilde{F}_1\left(b_u^{P}, b_u^{P}-c_u^{P}+1; b_u^{P}-a_u^{P}+1; y_u^{P}(B)^{-1}\right) \\
& {}_2\tilde{F}_1\left(a_l^{P}+1, a_l^{P}-c_l^{P}+1; a_l^{P}-b_l^{P}+1; y_l^{P}(B)^{-1}\right)\Big) \\
& \Big((\delta+\lambda)\,x^{*}\,{}_2\tilde{F}_1\left(b_l^{P}, b_l^{P}-c_l^{P}+1; b_l^{P}-a_l^{P}+1; y_l^{P}(x^{*})^{-1}\right) \\
& + c_{T} b_l^{P}\,(B-x^{*})\,{}_2\tilde{F}_1\left(b_l^{P}+1, b_l^{P}-c_l^{P}+1; b_l^{P}-a_l^{P}+1; y_l^{P}(x^{*})^{-1}\right)\Big)\Big)\Big)\Big)\Big] \,/ \\
& \\
& \Big[(\delta+\lambda)\,x^{*}\,{}_2\tilde{F}_1\left(a_l^{P}, a_l^{P}-c_l^{P}+1; a_l^{P}-b_l^{P}+1; y_l^{P}(x^{*})^{-1}\right) \\
& + c_{T} a_l^{P}\,(B-x^{*})\,{}_2\tilde{F}_1\left(a_l^{P}+1, a_l^{P}-c_l^{P}+1; a_l^{P}-b_l^{P}+1; y_l^{P}(x^{*})^{-1}\right)\Big]
\end{aligned}
\tag{5.A.10}
$$

and

$$
\begin{aligned}
A_{2,l}^{P} = \Big[ & \lambda x^{*} y_l^{P}(B)^{-a_l^{P}} y_l^{P}(x^{*})^{a_l^{P}}\,\Gamma\left(1 + a_l^{P} - b_l^{P}\right)\Big(b_u^{P}\,{}_2\tilde{F}_1\left(b_u^{P}+1, b_u^{P}-c_u^{P}+1; b_u^{P}-a_u^{P}+1; y_u^{P}(B)^{-1}\right) \\
& {}_2\tilde{F}_1\left(a_l^{P}, a_l^{P}-c_l^{P}+1; a_l^{P}-b_l^{P}+1; y_l^{P}(B)^{-1}\right) - a_l^{P}\,{}_2\tilde{F}_1\left(b_u^{P}, b_u^{P}-c_u^{P}+1; b_u^{P}-a_u^{P}+1; y_u^{P}(B)^{-1}\right) \\
& {}_2\tilde{F}_1\left(a_l^{P}+1, a_l^{P}-c_l^{P}+1; a_l^{P}-b_l^{P}+1; y_l^{P}(B)^{-1}\right)\Big)\, sin\left(\left(a_l^{P}-b_l^{P}\right)\pi\right)\Big] \,/ \\
& \Big[\left(a_l^{P}-b_l^{P}\right)\pi\,\Big(-y_l^{P}(B)^{-b_l^{P}}\,\Big((\delta+\lambda)\,x^{*}\,{}_2\tilde{F}_1\left(a_l^{P}, a_l^{P}-c_l^{P}+1; a_l^{P}-b_l^{P}+1; y_l^{P}(x^{*})^{-1}\right) \\
& + c_{T} a_l^{P}\,(B-x^{*})\,{}_2\tilde{F}_1\left(a_l^{P}+1, a_l^{P}-c_l^{P}+1; a_l^{P}-b_l^{P}+1; y_l^{P}(x^{*})^{-1}\right)\Big) \\
& \Big(b_u^{P}\,{}_2\tilde{F}_1\left(b_u^{P}+1, b_u^{P}-c_u^{P}+1; b_u^{P}-a_u^{P}+1; y_u^{P}(B)^{-1}\right){}_2\tilde{F}_1\left(b_l^{P}, b_l^{P}-c_l^{P}+1; b_l^{P}-a_l^{P}+1; y_l^{P}(B)^{-1}\right) \\
& - b_l^{P}\,{}_2\tilde{F}_1\left(b_u^{P}, b_u^{P}-c_u^{P}+1; b_u^{P}-a_u^{P}+1; y_u^{P}(B)^{-1}\right){}_2\tilde{F}_1\left(b_l^{P}+1, b_l^{P}-c_l^{P}+1; b_l^{P}-a_l^{P}+1; y_l^{P}(B)^{-1}\right)\Big) \\
& + y_l^{P}(B)^{-a_l^{P}} y_l^{P}(x^{*})^{a_l^{P}-b_l^{P}}\,\Big(b_u^{P}\,{}_2\tilde{F}_1\left(b_u^{P}+1, b_u^{P}-c_u^{P}+1; b_u^{P}-a_u^{P}+1; y_u^{P}(B)^{-1}\right) \\
& {}_2\tilde{F}_1\left(a_l^{P}, a_l^{P}-c_l^{P}+1; a_l^{P}-b_l^{P}+1; y_l^{P}(B)^{-1}\right) - a_l^{P}\,{}_2\tilde{F}_1\left(b_u^{P}, b_u^{P}-c_u^{P}+1; b_u^{P}-a_u^{P}+1; y_u^{P}(B)^{-1}\right) \\
& {}_2\tilde{F}_1\left(a_l^{P}+1, a_l^{P}-c_l^{P}+1; a_l^{P}-b_l^{P}+1; y_l^{P}(B)^{-1}\right)\Big) \\
& \Big((\delta+\lambda)\,x^{*}\,{}_2\tilde{F}_1\left(b_l^{P}, b_l^{P}-c_l^{P}+1; b_l^{P}-a_l^{P}+1; y_l^{P}(x^{*})^{-1}\right) \\
& c_{T} b_l^{P}\,(B-x^{*})\,{}_2\tilde{F}_1\left(b_l^{P}+1, b_l^{P}-c_l^{P}+1; b_l^{P}-a_l^{P}+1; y_l^{P}(x^{*})^{-1}\right)\Big)\Big)\Big],
\end{aligned}
\tag{5.A.11}
$$

where ${}_2\tilde{F}_1(a,b;c;z) = {}_2F_1(a,b;c;z)/\Gamma(c)$ denotes the Regularised Hypergeometric Function. Therefore, the Laplace transform of the trapping time is given by (5.3.10).

### 5.A.3 Proof of Theorem 5.4.1

Using similar arguments as those for Theorem 5.3.1, we know that for $x \geq B$, the capital immediately before the first capital loss is $h_r(t,x) = (x - x^*)e^{rt} + x^*$ and the capital has three possibilities at time $t$, that it is more than $B$, that it is between $x^*$ and $B$, and that it is between 0 and $x^*$. Thus, by conditioning on the time and the remaining proportion of the first capital loss and discounting the expected values to time 0 at the force of interest $\delta$, when $x \geq B$ we obtain

$$\begin{aligned} m^{\text{EP}}_{\delta,u}(x) = \int_0^\infty \lambda e^{-(\lambda+\delta)t} \Bigg[ \int_0^{x^*/h_r(t,x)} & m^{\text{EP}}_{\delta,l}(h_r(t,x)\cdot z)dG_Z(z) \\ + \int_{x^*/h_r(t,x)}^{B/h_r(t,x)} & m^{\text{EP}}_{\delta,m}(h_r(t,x)\cdot z)dG_Z(z) \\ + \int_{B/h_r(t,x)}^{1} & m^{\text{EP}}_{\delta,u}(h_r(t,x)\cdot z)dG_Z(z)\Bigg] dt \end{aligned} \tag{5.A.12}$$

Then, doing the change of variable $v_u = h_r(t,x)$ in the integrals with respect to $t$ from 0 to $\infty$ in (5.A.12), we obtain (5.4.1).

For $x^* \leq x < B$, there are two possibilities. First, $t < \tau_{\scriptscriptstyle B}$ and the capital has not yet reached the capital barrier level $B$. In this case, we know the capital immediately before time $t$ is $h_{r-c_T}(t,x) = (x+x^{**})e^{(r-c_T)t} - x^{**}$ and the capital has two possibilities at time $t$, that it is between $x^*$ and $B$, and that it is between 0 and $x^*$. Second, for $t > \tau_{\scriptscriptstyle B}$, that is, no capital loss occurs before the capital exceeds the capital barrier $B$. In this case, we also know the capital immediately before time $t$ is $h_r(t-\tau_{\scriptscriptstyle B}, B) = (B - x^*)e^{r(t-\tau_{\scriptscriptstyle B})} + x^*$ and the accumulated capital has three possibilities at time $t$, that it is more than $B$, that it is between $x^*$ and $B$, and that it is between 0 and $x^*$. Hence, by conditioning on the time and the remaining proportion of the first capital loss and discounting the expected values to time 0 at the force of interest $\delta$, when $x^* \leq x < B$ we obtain

$$\begin{aligned} m^{\text{EP}}_{\delta,m}(x) = & \int_0^{\tau_B} \lambda e^{-(\lambda+\delta)t} \Bigg[ \int_0^{x^*/h_{r-c_T}(t,x)} m^{\text{EP}}_{\delta,l}(h_{r-c_T}(t,x)\cdot z)dG_Z(z) \\ & + \int_{x^*/h_{r-c_T}(t,x)}^{1} m^{\text{EP}}_{\delta,m}(h_{r-c_T}(t,x)\cdot z)dG_Z(z)\Bigg] dt \\ & + \int_{\tau_B}^{\infty} \lambda e^{-(\lambda+\delta)t} \Bigg[ \int_0^{x^*/h_r(t-\tau_B,B)} m^{\text{EP}}_{\delta,l}(h_r(t-\tau_B,B)\cdot z)dG_Z(z) \\ & + \int_{x^*/h_r(t-\tau_B,B)}^{B/h_r(t-\tau_B,B)} m^{\text{EP}}_{\delta,m}(h_r(t-\tau_B,B)\cdot z)dG_Z(z) \\ & + \int_{B/h_r(t-\tau_B,B)}^{1} m^{\text{EP}}_{\delta,u}(h_r(t-\tau_B,B)\cdot z)dG_Z(z)\Bigg] dt \end{aligned} \tag{5.A.13}$$

Now, first changing variables $v_m = h_{r-c_T}(t,x)$ in the integrals with respect to $t$ from 0 to $\tau_{\scriptscriptstyle B}$ in (5.A.13), and then changing variables $v_u = h_r(t-\tau_{\scriptscriptstyle B}, B)$ in the integrals with respect to $t$ from $\tau_{\scriptscriptstyle B}$ to $\infty$ in (5.A.13), we obtain (5.4.2).

For $0 < x < x^*$, let $\tau_{x^*} := \tau_{x^*}(x)$ be the solution to

$$h_{c_T}(t,x) = (x-B)e^{-c_T t} + B = x^*.$$

Namely, $\tau_{x^*} := \tau_{x^*}(x) = -\ln\left[(x^*-B)/(x-B)\right]/c_T$, which is the time when the capital returns to the critical capital $x^*$ if no capital loss occurs prior to time $\tau_{x^*}$. Furthermore, $h_{c_T}(t,x) < x^*$ for $t < \tau_{x^*}$ and $h_{c_T}(\tau_{x^*},x) = x^*$. Moreover, $h_{c_T}(t,x)$ is the capital at time $t \le \tau_{x^*}$ if no capital loss occurs prior to time $\tau_{x^*}$.

Thus, for $0 < x < x^*$, there are three possibilities. First, $t < \tau_{x^*}$ and the capital up to time $t$ has not reached the critical capital $x^*$. In this case, the capital immediately before time $t$ is $h_{c_T}(t,x) = (x-B)e^{-c_T t} + B$. Second, $\tau_{x^*} \le t < \tau_{x^*} + \tau_B(x^*)$ and the capital has not yet reached the capital barrier level $B$ and no capital loss occurs before the capital exceeds the critical capital $x^*$. In this case, the capital immediately before time $t$ is $h_{r-c_T}(t-\tau_{x^*},x^*) = (x^*+x^{**})e^{(r-c_T)(t-\tau_{x^*})} - x^{**}$ and the capital up to time $t$ has two possibilities, that it is between $x^*$ and $B$, and that it is between 0 and $x^*$. Third, $t \ge \tau_{x^*} + \tau_B(x^*)$, that is, no capital loss occurs before the capital up to time $t$ exceeds the capital barrier level $B$. In this case, the capital immediately before time $t$ is $h_r(t-\tau_{x^*}-\tau_B(x^*),B) = (B-x^*)e^{r(t-\tau_{x^*}-\tau_B(x^*))} + x^*$ and the capital up to time $t$ has three possibilities, that it is more than $B$, that it is between $x^*$ and $B$, and that it is between 0 and $x^*$. Hence, by conditioning on the time and the remaining proportion of the first capital loss and discounting the expected values to time 0 at the force of interest $\delta$, when $0 < x < x^*$ we obtain

$$\begin{aligned}
m^{\text{EP}}_{\delta,l}(x) &= \int_0^{\tau_{x^*}} e^{-(\lambda+\delta)t} e^{-\int_0^t \omega(h_{c_T}(y,x))dy} \omega\left(h_{c_T}(t,x)\right) w^{\text{EP}}\left(h_{c_T}(t,x), x^* - h_{c_T}(t,x)\right) dt \\
&+ \int_0^{\tau_{x^*}} \lambda e^{-(\lambda+\delta)t} e^{-\int_0^t \omega(h_{c_T}(y,x))dy} \int_0^1 m^{\text{EP}}_{\delta,l}(h_{c_T}(t,x)\cdot z) dG_Z(z) dt \\
&+ \int_{\tau_{x^*}}^{\tau_{x^*}+\tau_B(x^*)} \lambda e^{-(\lambda+\delta)t} e^{-\int_0^{\tau_{x^*}} \omega(h_{c_T}(y,x))dy} \\
&\left[\int_0^{x^*/h_{r-c_T}(t-\tau_{x^*},x^*)} m^{\text{EP}}_{\delta,l}(h_{r-c_T}(t-\tau_{x^*},x^*)\cdot z) dG_Z(z)\right. \\
&\left.+ \int_{x^*/h_{r-c_T}(t-\tau_{x^*},x^*)}^1 m^{\text{EP}}_{\delta,m}(h_{r-c_T}(t-\tau_{x^*},x^*)\cdot z) dG_Z(z)\right] dt \\
&+ \int_{\tau_{x^*}+\tau_B(x^*)}^{\infty} \lambda e^{-(\lambda+\delta)t} e^{-\int_0^{\tau_{x^*}} \omega(h_{c_T}(y,x))dy} \\
&\left[\int_0^{x^*/h_r(t-\tau_{x^*}-\tau_B(x^*),B)} m^{\text{EP}}_{\delta,l}(h_r(t-\tau_{x^*}-\tau_B(x^*),B)\cdot z) dG_Z(z)\right. \\
&+ \int_{x^*/h_r(t-\tau_{x^*}-\tau_B(x^*),B)}^{B/h_r(t-\tau_{x^*}-\tau_B(x^*),B)} m^{\text{EP}}_{\delta,m}(h_r(t-\tau_{x^*}-\tau_B(x^*),B)\cdot z) dG_Z(z) \\
&\left.+ \int_{B/h_r(t-\tau_{x^*}-\tau_B(x^*),B)}^1 m^{\text{EP}}_{\delta,u}(h_r(t-\tau_{x^*}-\tau_B(x^*),B)\cdot z) dG_Z(z)\right] dt
\end{aligned} \tag{5.A.14}$$

Now, first changing variables $v_l = h_{c_T}(t,x)$ and $u_l = h_{c_T}(y,x)$ in the integrals with respect to $t$ from 0 to $\tau_{x^*}$ in (5.A.14), then changing variables $v_m = h_{r-c_T}(t-\tau_{x^*},x^*)$ in the integrals with respect to $t$ from $\tau_{x^*}$ to $\tau_{x^*}+\tau_B(x^*)$ in (5.A.14) and lastly changing variables $v_u = h_r(t - \tau_{x^*} - \tau_{B}(x^*), B)$ in the integrals with respect to $t$ from $\tau_{x^*} + \tau_{B}(x^*)$ to $\infty$ in (5.A.14) we obtain (5.4.3).

### 5.A.4 Proof of Proposition 5.4.1

When $Z_i \sim Beta(\alpha, 1)$, i.e. $g_Z(z) = \alpha z^{\alpha-1}\mathbb{1}_{\{0<z<1\}}$ with $\alpha > 0$, Equations (5.4.6), (5.4.7) and (5.4.11) can be written such that when $x \geq B$,

$$0 = r(x - x^*)m'^{\text{EP}}_{\delta,u}(x) - (\lambda+\delta)m^{\text{EP}}_{\delta,u}(x) + \lambda\left[\int_0^{x^*/x} m^{\text{EP}}_{\delta,l}(x\cdot z)\alpha z^{\alpha-1}dz + \int_{x^*/x}^{B/x} m^{\text{EP}}_{\delta,m}(x\cdot z)\alpha z^{\alpha-1}dz + \int_{B/x}^{1} m^{\text{EP}}_{\delta,u}(x\cdot z)\alpha z^{\alpha-1}dz\right], \tag{5.A.15}$$

when $x^* \leq x < B$,

$$0 = (r - c_T)(x + x^{**})m'^{\text{EP}}_{\delta,m}(x) - (\lambda+\delta)m^{\text{EP}}_{\delta,m}(x) + \lambda\left[\int_0^{x^*/x} m^{\text{EP}}_{\delta,l}(x\cdot z)\alpha z^{\alpha-1}dz + \int_{x^*/x}^{1} m^{\text{EP}}_{\delta,m}(x\cdot z)\alpha z^{\alpha-1}dz\right], \tag{5.A.16}$$

and when $0 < x < x^*$,

$$0 = c_T(x - B)m'^{\text{EP}}_{\delta,l}(x) + [\lambda+\delta+\omega_c]m^{\text{EP}}_{\delta,l}(x) - \omega_c - \lambda\int_0^1 m^{\text{EP}}_{\delta,l}(x\cdot z)\alpha z^{\alpha-1}dz. \tag{5.A.17}$$

Applying the operator $\frac{d}{dx}$ to both sides of (5.A.15), (5.A.16) and (5.A.17), together with a number of algebraic manipulations, yields to the following second order ODEs,

$$x \geq B : 0 = r(x^2 - xx^*)m''^{\text{EP}}_{\delta,u}(x) + [(r(1+\alpha) - \delta - \lambda)x - r\alpha x^*]\, m'^{\text{EP}}_{\delta,u}(x) - \alpha\delta m^{\text{EP}}_{\delta,u}(x),$$

$$x^* \leq x < B : 0 = (r - c_T)(x^2 + xx^{**})m''^{\text{EP}}_{\delta,m}(x) + [((r - c_T)(1+\alpha) - \delta - \lambda)x + \alpha(r - c_T)x^{**}]\, m'^{\text{EP}}_{\delta,m}(x) - \alpha\delta m^{\text{EP}}_{\delta,m}(x)$$

and

$$\begin{aligned} 0 < x < x^* : 0 = {} & c_T(x^2 - Bx)m''^{\text{EP}}_{\delta,l}(x) \\ & + [(c_T(1+\alpha) + \delta + \lambda + \omega_c)x - \alpha c_T B]\, m'^{\text{EP}}_{\delta,l}(x) \\ & + \alpha(\delta + \omega_c)m^{\text{EP}}_{\delta,l}(x) - \alpha w_c. \end{aligned} \tag{5.A.18}$$

Hence, for $0 < x < x^*$, $m^{\text{EP}}_{\delta,l}(x)$ satisfies the nonhomogeneous differential equation (5.A.18), when the extreme poverty rate function $\omega_1(x) = \omega_c$ (constant value) and the penalty function $w^{\text{EP}}(x_1, x_2) = 1$. The particular solution of $m^{\text{EP}}_{\delta,l}(x)$ is

$$m^{*\text{EP}}_{\delta,l}(x) = \frac{\omega_c}{\delta + \omega_c}.$$

Therefore, the general solution of $m^{\text{EP}}_{\delta,l}(x)$ is given by

$$m^{\text{EP}}_{\delta,l}(x) = h^{\text{EP}}_l(x) + \frac{\omega_c}{\delta + \omega_c},$$

where $h^{\text{EP}}_l(x)$ is the homogeneous solution of (5.A.18). Then, following a similar procedure to that of Proposition 5.3.1, letting $f^{\text{EP}}_l(y^{\text{EP}}_l) := h^{\text{EP}}_l(x)$, such that $y^{\text{EP}}_l$ is associated with the change of variable $y^{\text{EP}}_l := y^{\text{EP}}_l(x) = x/B$, the homogeneous part of Equation (5.A.18) reduces to Equation (5.A.7) for $c^{\text{EP}}_l = \alpha$, $a^{\text{EP}}_l = \frac{\alpha c_{\tau}+\lambda+\delta+\omega_c-\sqrt{(\alpha c_{\tau}+\lambda+\delta+\omega_c)^2-4\alpha c_{\tau}(\delta+\omega_c)}}{2c_{\tau}}$ and $b^{\text{EP}}_l = \frac{\alpha c_{\tau}+\lambda+\delta+\omega_c+\sqrt{(\alpha c_{\tau}+\lambda+\delta+\omega_c)^2-4\alpha c_{\tau}(\delta+\omega_c)}}{2c_{\tau}}$, with regular singular points at $y^{\text{EP}}_l = 0, 1, \infty$ (corresponding to $x = 0, B$ and $\infty$). A general solution of (5.A.7) in the neighborhood of the singular point $y^{\text{EP}}_l = 0$ is given by

$$\begin{aligned} f^{\text{EP}}_l(y^{\text{EP}}_l) :=& h^{\text{EP}}_l(x) = A^{\text{EP}}_{1,l}\,{}_2F_1\left(a^{\text{EP}}_l, b^{\text{EP}}_l; c^{\text{EP}}_l; y^{\text{EP}}_l(x)\right) \\ &+ A^{\text{EP}}_{2,l}\, y^{\text{EP}}_l(x)^{1-c^{\text{EP}}_l}\,{}_2F_1\left(a^{\text{EP}}_l - c^{\text{EP}}_l + 1, b^{\text{EP}}_l - c^{\text{EP}}_l + 1; 2 - c^{\text{EP}}_l; y^{\text{EP}}_l(x)\right), \end{aligned} \tag{5.A.19}$$

for arbitrary constants $A^{\text{EP}}_{1,l}, A^{\text{EP}}_{2,l} \in \mathbb{R}$ (see for example, Equations (15.5.3) and (15.5.4) of Abramowitz and Stegun (1972)). Due to the fact that $m^{\text{EP}}_{\delta,l}(x)$ is finite, we can then conclude that $A^{\text{EP}}_{2,l} = 0$, as the second term of (5.A.19) is unbounded when $x \to 0^+$ for $\alpha > 0$. Thus, the solution of $m^{\text{EP}}_{\delta,l}(x)$ is given by

$$m^{\text{EP}}_{\delta,l}(x) = A^{\text{EP}}_{1,l}\,{}_2F_1\left(a^{\text{EP}}_l, b^{\text{EP}}_l; c^{\text{EP}}_l; y^{\text{EP}}_l(x)\right) + \frac{\omega_c}{\delta + \omega_c}.$$

Then, following the proof of Proposition 5.3.1, one can easily obtain the solutions for $m^{\text{EP}}_{\delta,u}(x)$ and $m^{\text{EP}}_{\delta,m}(x)$, when $x \geq B$ and $x^* \leq x < B$, respectively.

Considering the continuity of $m^{\text{EP}}_{\delta}(x)$ and $m'^{\text{EP}}_{\delta}(x)$ at the critical capital $x^*$ and the capital barrier level $B$, that is, using (5.4.4), (5.4.5), (5.4.9) and (5.4.10), one can derive a system of equations from which the unknown coefficients $A^{\text{EP}}_{2,u}$, $A^{\text{EP}}_{1,m}$, $A^{\text{EP}}_{2,m}$ and $A^{\text{EP}}_{1,l}$, can be determined to obtain an explicit solution for $m^{\text{EP}}_{\delta}(x)$.

### 5.A.5 Proof of Proposition 5.4.2

Following a similar procedure to that in Appendix 5.A.4, for $0 < x < x^*$, one can derive from (5.4.11) the following nonhomogeneous second order ODE for $\psi^{\text{EP}}_l(x)$, when the extreme poverty rate function $\omega_2(x) = \frac{\beta}{x}$ (exponential extreme poverty rate), the penalty function $w^{\text{EP}}(x_1, x_2) = 1$ and the force of interest $\delta = 0$,

$$x^2 (x - B)\, \psi''^{\text{EP}}_l(x) + x \left[\frac{c_T (1+\alpha) + \lambda}{c_T} x + \frac{\beta - \alpha B c_T}{c_T}\right] \psi'^{\text{EP}}_l(x) + \frac{\beta(\alpha - 1)}{c_T} \psi^{\text{EP}}_l(x) - \frac{\beta(\alpha-1)}{c_T} = 0. \tag{5.A.20}$$

Clearly, $\psi^{*\text{EP}}_l(x) = 1$ is always a particular solution of Equation (5.A.20), so that one can write

$$\psi^{\text{EP}}_l(x) = h^{\text{EP}}_l(x) + 1,$$

where $h_l^{\text{EP}}(x)$ is the homogeneous solution of (5.A.20). Now, making the substitution $h_l^{\text{EP}}(x) = x^{1-\alpha} g_l^{\text{EP}}(x)$, Equation (5.A.20) yields to the following second order ODE

$$x\,(x-B)\,g_l''^{\text{EP}}(x) + \left[\frac{c_T\,(3-\alpha)+\lambda}{c_T}x + \frac{Bc_T\,(\alpha-2)+\beta}{c_T}\right] g_l'^{\text{EP}}(x) + \frac{(1-\alpha)\,(c_T+\lambda)}{c_T} g_l^{\text{EP}}(x) = 0.$$

A second substitution, $y_l^{\text{EP}} := y_l^{\text{EP}}(x) = x/B$, such that $f_l^{\text{EP}}\left(y_l^{\text{EP}}(x)\right) = g_l^{\text{EP}}(x)$, produces Equation (5.A.7) for $a_l^{\text{EP}} = 1-\alpha$, $b_l^{\text{EP}} = (c_T+\lambda)/c_T$ and $c_l^{\text{EP}} = -\left[Bc_T\,(\alpha-2)+\beta\right]/(Bc_T)$, with regular singular points at $y_l^{\text{EP}} = 0, 1, \infty$ (corresponding to $x = 0, B$ and $\infty$). Thus, knowing that a general solution of (5.A.7) in the neighborhood of the singular point $y_l^{\text{EP}} = 0$ is of the form (5.A.19) and that $h_l^{\text{EP}}(x) = x^{1-\alpha} g_l^{\text{EP}}(x)$ one obtains the homogenous solution

$$\begin{aligned} h_l^{\text{EP}}(x) &= A_{1,l}^{\text{EP}} y_l^{\text{EP}}(x)^{1-\alpha} {}_2F_1\left(a_l^{\text{EP}}, b_l^{\text{EP}}; c_l^{\text{EP}}; y_l^{\text{EP}}(x)\right) \\ &\quad + A_{2,l}^{\text{EP}} y_l^{\text{EP}}(x)^{2-c_l^{\text{EP}}-\alpha} {}_2F_1\left(a_l^{\text{EP}} - c_l^{\text{EP}} + 1, b_l^{\text{EP}} - c_l^{\text{EP}} + 1; 2 - c_l^{\text{EP}}; y_l^{\text{EP}}(x)\right), \end{aligned} \tag{5.A.21}$$

for arbitrary constants $A_{1,l}^{\text{EP}}, A_{2,l}^{\text{EP}} \in \mathbb{R}$. Due to the fact that $\psi_l^{\text{EP}}(x)$ is finite, we can then conclude that $A_{1,l}^{\text{EP}} = 0$, as the first term of (5.A.21) is unbounded when $x \to 0^+$ for $\alpha > 0$. Hence, the solution of $\psi_l^{\text{EP}}(x)$ is given by

$$\psi_l^{\text{EP}}(x) = A_{2,l}^{\text{EP}} y_l^{\text{EP}}(x)^{2-c_l^{\text{EP}}-\alpha} {}_2F_1\left(a_l^{\text{EP}} - c_l^{\text{EP}} + 1, b_l^{\text{EP}} - c_l^{\text{EP}} + 1; 2 - c_l^{\text{EP}}; y_l^{\text{EP}}(x)\right) + 1.$$

As in Appendix 5.A.4, following the proof of Proposition 5.3.1 for $\delta = 0$, one can easily obtain the solutions for $\psi_u^{\text{EP}}(x)$ and $\psi_m^{\text{EP}}(x)$, when $x \geq B$ and $x^* \leq x < B$, respectively.

Finally, due to the continuity of $\psi^{\text{EP}}(x)$ and $\psi'^{\text{EP}}(x)$ at the critical capital $x^*$ and the capital barrier level $B$, that is, using (5.4.4), (5.4.5), (5.4.9) and (5.4.10) for $\delta = 0$ and $w^{\text{EP}}(x_1, x_2) = 1$, one can derive a system of equations from which the unknown coefficients $A_{2,u}^{\text{EP}}$, $A_{1,m}^{\text{EP}}$, $A_{2,m}^{\text{EP}}$ and $A_{2,l}^{\text{EP}}$, can be determined to derive a closed-form expression for $\psi^{\text{EP}}(x)$.

# 5.B Appendix B: Effects of Underlying Factors on the Trapping Probability

We consider the influence of the parameters on the trapping probability by varying them in a reasonable range, keeping all other parameters constant. The reference setup is given below.

**Reference setup:** $a = 0.1$, $b = 4$, $c_S = 0.4$, $Z_i \sim Beta(0.8, 1)$, $\lambda = 1$, $x^* = 1$, $B = 2$ and $c_T = 1$.

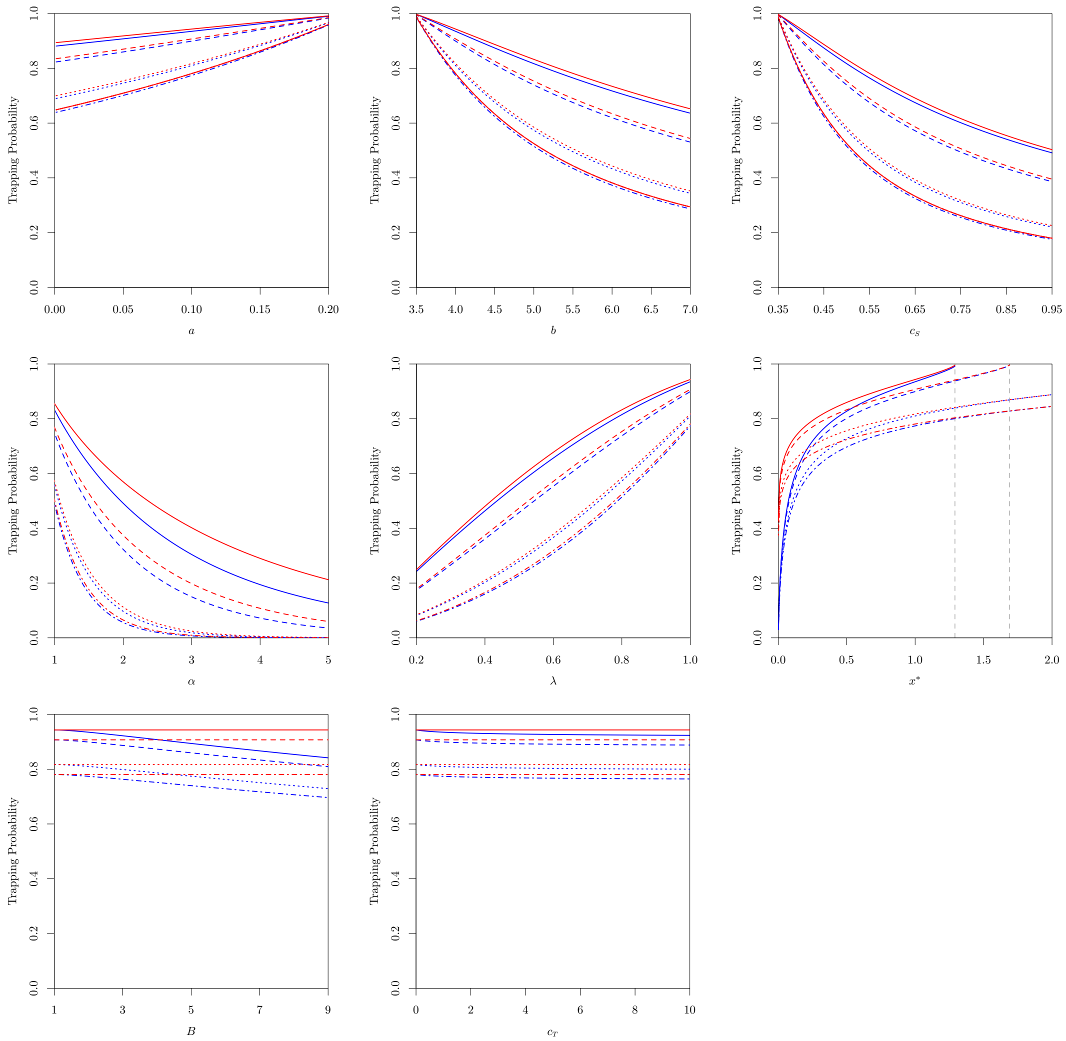



Figure 5.13: Effects of the rate of consumption ($0 < a < 1$), income generation ($b > 0$), investment or savings ($0 < c_S < 1$), the parameter of the Beta distribution ($\alpha > 0$) (i.e., expected remaining proportion of capital), the expected capital loss frequency ($\lambda > 0$), the critical capital ($x \geq x^*$), the capital barrier level ($B > x^*$) and the capital transfer rate ($c_T > 0$) on the trapping probability of the original model obtained in Henshaw et al. (2023) (in red) and on the trapping probability of the model with capital cash transfers (in blue) for initial capital $x = 1.3$ (solid), $1.7$ (dashed), $4.0$ (dotted), $6.0$ (dashed-dotted).

## 5.C Appendix C: Effects of Underlying Factors on the Probability of Extreme Poverty

We consider the influence of the parameters on the probability of extreme poverty by varying them in a reasonable range, keeping all other parameters constant. The reference setup is given below.

**Reference setup:** $a = 0.1$, $b = 4$, $c_S = 0.4$, $Z_i \sim Beta(0.8, 1)$, $\lambda = 1$, $x^* = 1$, $B = 2$, $c_T = 1$, $\omega_1(x) = 0.05$ and $\omega_2(x) = \frac{0.05}{x}$.

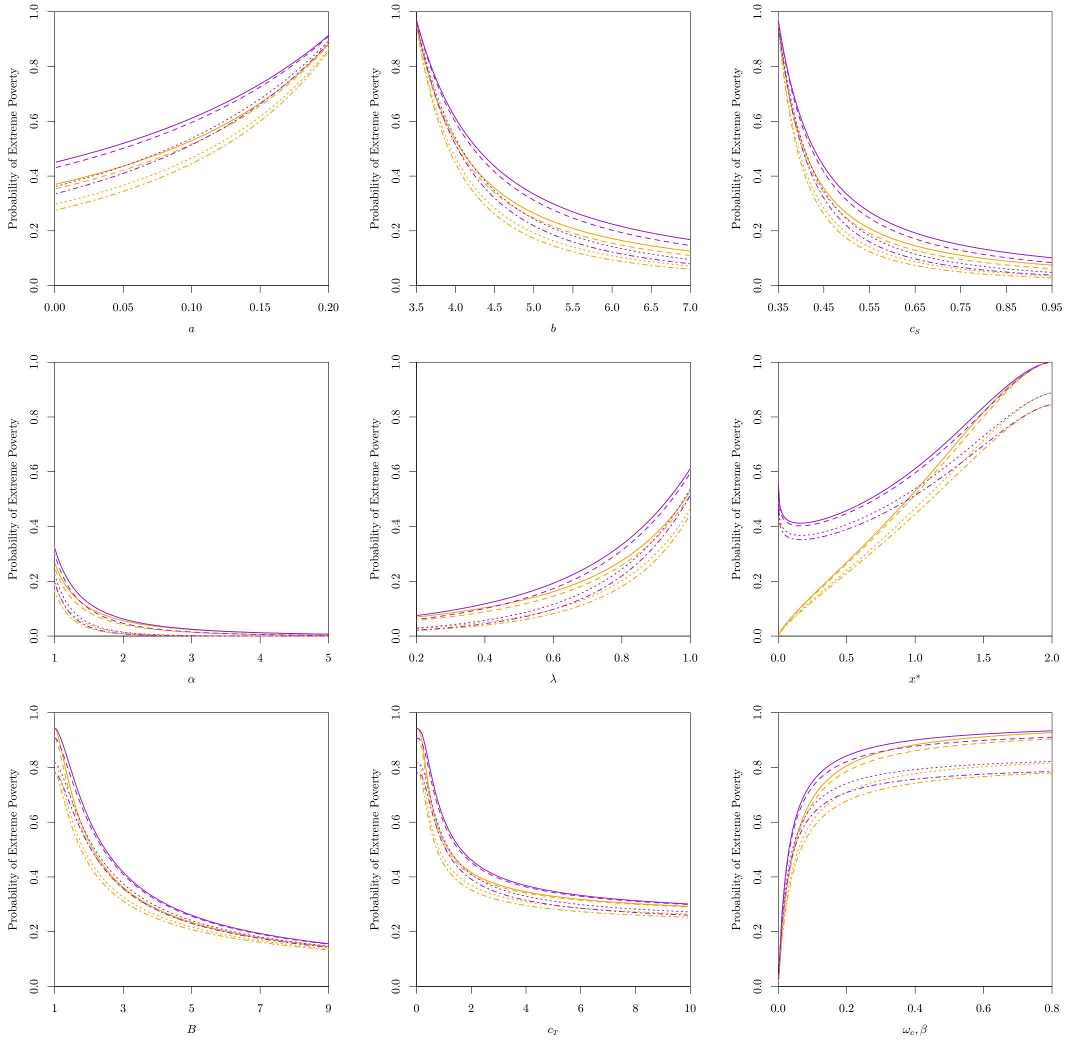



Figure 5.14: Effects of the rate of consumption ($0 < a < 1$), income generation ($b > 0$), investment or savings ($0 < c_S < 1$), the parameter of the Beta distribution ($\alpha > 0$) (i.e., expected remaining proportion of capital), the expected capital loss frequency ($\lambda > 0$), the critical capital ($x \geq x^*$), the capital barrier level ($B > x^*$), the capital transfer rate ($c_T > 0$) and the extreme poverty rate function on the probability of extreme poverty for a constant extreme poverty rate function (in orange) and an exponential extreme poverty rate function (in purple) for initial capital $x = 1.3$ (solid), $1.7$ (dashed), $4.0$ (dotted), $6.0$ (dashed-dotted).

# General Conclusions

The aim of this thesis is to assess the extent to which social protection strategies (and complementary approaches such as insurance) are effective tools to alleviate poverty, one of the most important common goals of humankind in recent history. This is done using techniques from ruin theory, a branch of insurance mathematics that deals with the study of stochastic processes and their fluctuations. Ruin theory, among other things, provides tools to analyse the behaviour of a household's capital over time and thus the possibility of estimating the trapping probability of a household (the probability that a household will ever fall into the poverty trap).

In Chapter 2, we highlight the insufficiency of microinsurance alone as a means of protecting vulnerable non-poor households (those households living with capital levels that are just above the poverty line) from falling into the poverty trap, and thus provide evidence of the importance of government-sponsored microinsurance in the fight against poverty. In particular, we show that government-sponsored schemes (here, we consider one with subsidised constant premiums and another one with subsidised flexible premiums) reduce the trapping probability of households and, at the same time, the cost of social protection incurred by the government, thus suggesting that government-sponsored inclusive insurance schemes are cost-effective instruments for poverty alleviation. The results of Chapter 2 also contribute to the literature on ruin theory by deriving closed-form formulas for both the trapping probability and the cost of social protection incurred by the government (under the scheme with subsidised constant premiums) when considering a particular variation of the Cramér-Lundberg model that aims to portray a household's capital over time.

To better represent the fact that vulnerable non-poor households have less to lose than more privileged households (those with higher levels of capital), and also the fact that the poverty trap represents an absorbing state from which households cannot escape without external help (either downwards or upwards), in Chapter 3 we adjust the capital model studied in Chapter 2 and consider a growth-collapse model in which capital losses are prorated from the household's accumulated capital rather than subtracted. Insurance coverage under the assumption of this model suggests that the increase in trapping probability observed under random-valued losses considered in Chapter 2 is less severe in this proportional case. The main contribution of Chapter 3 to the literature on ruin theory is the derivation of a closed-form formula for the trapping probability of a household. To the best of our knowledge, we present this formula for the very first time for a risk process with proportional jumps. Moreover, a novel recursive approach is introduced to derive the trapping probability under the assumption that the household acquires proportional microinsurance coverage.

Instead of focusing solely on the trapping probability of a household, Chapter 4 further examines the conditions under which the trapping event occurs (for the household capital process with proportional jumps also studied in Chapter 3). The Gerber-Shiu expected discounted penalty function, a concept commonly used in ruin theory, serves as the main tool for such analysis. This function provides information on three key random variables: the trapping time, the capital surplus prior to trapping and the capital deficit at trapping. In Chapter 4, we provide analytic expressions for the Gerber-Shiu expected discounted penalty function of the risk process with proportional jumps. Moreover, by applying this concept, we derive a compelling microeconomic foundation, arising from the derivation of the Gerber-Shiu expected discounted penalty function itself, to model the distribution of a household's income short-fall (the absolute value of the difference between a poor household's income (or consumption) and some poverty line). These results are important, as a household's income short-fall is the main component of a class of poverty indicators and these, in turn, serve as a tool for monitoring and evaluating the performance of social protection strategies.

The role of non-contributory transfers (unconditional cash transfers) in poverty alleviation is analysed in Chapter 5. Numerical experiments conducted in Chapter 5 suggest that regular unconditional cash transfer programmes are efficient social protection strategies to keep households out of poverty and extreme poverty, as their trapping probability and the probability of extreme poverty (the probability that a household will ever become extremely poor), respectively, are reduced when they are part of such strategies. Chapter 5 also stresses the importance of selecting appropriate design and implementation features, such as the frequency of the transfers and the beneficiary selection criteria. The main contribution of Chapter 5 to the literature on ruin theory is the derivation of explicit formulas for both the trapping probability and the probability of extreme poverty. In particular, for the probability of extreme poverty, Chapter 5 also introduces for the first time the concept of extreme poverty rate function. Here, the probability of extreme poverty of a household depends on the current value of the capital given by the extreme poverty rate function.

This thesis represents a starting point for applying knowledge from another discipline, such as insurance mathematics, to development economics. There are many avenues to which this work can be extended. However, as highlighted in Chapter 1, it is true that perhaps the introduction of new risk processes that best describe the phenomenon of interest remains as the main task for future research. For example, in inclusive insurance, the importance of bundling insurance products with other types of products to reach scale (e.g. linking agricultural insurance to agricultural supplies such as seeds and fertiliser) has been highlighted. Accordingly, a next step would be to explore risk processes that allow for the bundling of certain products. Similarly, it would be interesting to consider, for example, the possibility of assessing the role of unconditional cash transfers alongside inclusive insurance as joint instruments. Risk processes that incorporate households' risk preferences over time and the direct effects of inclusive insurance (or unconditional cash transfer programmes) on a household's consumption should also be considered. All these would lead to more complex risk processes and thus to difficulties in applying traditional techniques from ruin theory. Therefore, apart from the fact that these models will better match reality, they will also represent an interesting challenge from a mathematical point of view, as they could give rise to problems never seen before in the insurance mathematics literature.

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