Resource Allocation When Projects Have Ranges of Increasing Returns*

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Abstract

We examine the problem of optimally allocating a fixed budget to a finite number of different investment projects whose returns are uncertain and depend on the budget allocated to each of them. The marginal productivity of capital in a project is first increasing then decreasing with the amount of capital invested in it. Such a shape is particularly prevalent when the output is a probability such as the chance of escaping infection or succeeding with a R&D project. When the total budget is below some lower cutoff value, the entire budget is invested in a single project. Above this cutoff, the share invested in a project can be discontinuous and non-monotone in the total budget. Above an upper cutoff budget, all projects receive more capital as the budget increases. If the projects are identical, each will get the same budget. Moreover, under some conditions, the aggregate benefit function also presents first increasing and then decreasing returns to scale as the amount invested increases.

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1 Introduction

We address a common capital allocation problem in which an agent can implement various investment projects at different levels. In the public realm, this could be a federal agency that has to determine a vaccination strategy for a contagious disease, such as avian flu, with a limited national vaccination budget. Given that herd immunity develops, it is well known that the social benefit curve from a vaccination campaign is S-shaped in the proportion of the vaccination inoculated in a locale. A similar problem arises if a very limited amount of a drug is to be distributed to reduce new infections in a nation heavily afflicted with HIV. An egalitarian allocation might have little beneficial effect. This application to public health policy operates in an uncertain environment as far as the probability of been contaminated is not known. Therefore, we consider the expected social benefit of such campaigns.

Our general model also applies to a range of private-sector allocation problems. Thus, a credit-rationed entrepreneur may have the potential to invest in various independent projects, each of which offers increasing and then decreasing expected returns to invested capital. Many efforts to market a new product in different areas under uncertainty have this character. In fact, a number of uncertain events presents a cumulative distribution that is S-shaped (the logistic distribution for instance). Such a function is used to model resource-limited exponential growth, for example. Another branch of the literature related to our topic concerns the analysis of the financing of R&D projects. Weitzman [20] considered the case where there is a finite number of different opportunities, each yielding an unknown reward. He proposed an algorithm that tells at each stage whether or not to continue searching and if so, which project to finance. This could apply to the optimal sequential search strategy for developing various uncertain technologies that meet the same or similar purpose. This model has been extended by Roberts and Weitzman [18] in a more general framework. Our analysis approaches this one except that the expected benefit function generated by each project is assumed to be known in our case. Note that we work with expectations; S-shaped productivities are particularly prevalent in uncertainty situations.

The central ingredient of the general model is that the marginal benefit of each possible action is hump-shaped, i.e., the marginal benefit of an action is maximum at some intermediate intensity of that action. The objective function, which is the sum of the benefits extracted from the different actions, is therefore not concave in the vector of decision variables. In fact, under this assumption, it is tempting to give up the idea of distributing resources to all projects, because projects funded at a modest scale offer low productivity.

The question of increasing returns in an economy has been widely studied, in a theoretical framework, notably in the seventies. Indeed, with increasing returns, a competitive market may lack an equilibrium. Many authors proposed solutions to avoid this problem (see for instance Rader [17], Aoki [1], Crémer [6], Brown and Heal [4] or Heal [10]). The optimal financing of a finite number of projects, each presenting an S-shape has been addressed in particular in an important early paper by Ginsberg [9]. He characterizes the solution in the general case and explains how the budget is usually shared among an increasing number of projects when the benefit functions are identical. He typically uses the average benefit function to solve the problem. We adopt a different approach. We focus on plausible shapes for the benefit function and we find that they yield sensible solutions. To do so, we introduce some families of functions defined by interesting and fairly general properties for the marginal benefit function. We also highlight the features of the aggregate benefit function.

Section 2 motivates the paper by giving examples where the benefit function is S-shaped. Section 3 states some general properties in the case of a low budget level. Section 4 analyzes higher budget levels for the case of identical benefit functions. Section 5 tackles the case of heterogeneous benefit functions in the context of a high budget level. Section 6 describes the properties of the aggregate benefit function, and section 7 concludes.

2 S-Shaped Productivity in Various Domains

This section argues that a total productivity curve that is S-shaped is found across a broad array of areas. The shape usually arises because two conflicting forces are at work: (1) Small investments accomplish little. Thus, \$100,000 will not produce a sophisticated new invention, nor dent the national consciousness in a media campaign for a new product. (2) Beyond a certain level of investment decreasing returns set in. Thus, the invention is likely to have been developed if it will ever be developed at affordable cost, and the product will likely be widely known. Hence, productivity first rises with expenditure at an increasing rate, and then the rate decreases. The S-shaped curve, as is shown in Figure 1, emerges.

The concept of herd immunity is well known in epidemiology. Each individual who gets immunized against a communicable disease within a closed population conveys a positive externality. Since he can no longer get the disease, he can no longer communicate it to others. The first few immunizations yield little external benefit, since there remain so many other individuals who can still convey infection. However, once a significant proportion of individuals has been vaccinated, the whole population is substantially protected, which leads to the label herd immunity. Beyond a certain point, additional vaccinations therefore yield little additional

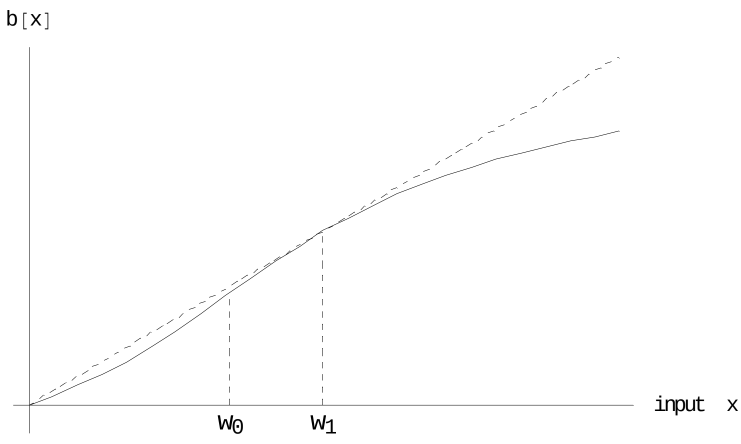

Figure 1: The benefit function as a function of the amount invested in the specific project.

protection (see for instance Fine [7]). Once more, all these considerations apply in an uncertain environment: contamination only arises with a certain probability.

Efforts to produce inventions have long been recognized to exhibit an S-shaped in the function relating probability of success to level of investment. Such a shape is clearly delineated in the empirical analysis of patenting as a function of R&D expenditures by a firm in Scherer [16] Figure 1.With low expenditures, the probability of getting a patent is very low but since a sufficiently high level of expenditures has been devoted to research, an increase in the patenting probability is more difficult to achieve. S-shaped curves product performance are a driving concept behind Utterback's [19] (pp. 158-160) analysis of radical innovations, and Christensen's [5] (pp. 39-41) model of disruptive technologies. Successor (radical or disruptive) technologies come along when the first technology is operating beyond its inflection point. Kuznets [11] (pp. 31-33) noted the same S-shape phenomenon for an industry as a whole, which might be relevant say for government R&D and tax policies that seeks to push various industries forward.

Little [12] provides an overview look at the returns to aggregate advertising of various products, drawing on the work of others. He identifies S-shaped responses, e.g., of sales/capita in response to advertising/capita, though he also alerts readers to more complex patterns. He concludes (p. 639) "that advertising models should accommodate S-shaped curves."

In general, in any investment arena where there is a range of increasing returns, we should expect to find S-shaped response curves. That is because we know that decreasing returns set in, since except where natural resources are involved, we do not see one product, or one firm, or one industry dominating a major economy. When two or more entities must compete for investment, and where those entities each experience S-shaped returns, the lessons of this paper apply.

As we already mentioned, this S-shape applies to a wide number of uncertain events. There exist random variables whose cumulative distribution function is S-shaped as for instance the logistic distribution. Balakrishnan [2] in his handbook of the logistic distribution proposes different applications for this distribution function. It was first applied to model population growth. But Oliver [14] used the logistic distribution to model the spread of innovation, and more precisely the thousands of agricultural tractors in Great Britain from 1950 to 1965. In economics, logistic distribution has been used to model income distribution (Fisk [8]) or agricultural production (Oliver [13]). It is also widely employed in public health. For instance, the ratio disease incidence among those exposed versus those not exposed to the risk factor of interest (it is called the relative risk) may be estimated with a logistic function. Moreover, Plackett [15] was the first to use the logistic function in the analysis of survival data. He developed a model that applies to operations on cancer patients, but also to labour turnover, business failures or animal experiments. Therefore, this article applies to a large number of risky events whose distribution is S-shaped. However, in the rest of the article, we are going to speak from an investor who has a given budget to invest in different projects without specifying their nature.

3 Low budget levels: general case

We first consider the problem in the general case where a finite number of projects presenting different benefit functions b_1,..., b_n are available to the investor. We assume that $b_i(0) = 0\ \forall i$, and that each function b_i is increasing in its argument. Finally, we assume, as shown in Figure 1 that there exists a critical investment level w_0 such that each function b_i is locally convex in $[0, w_0[$, and that it is locally concave in $]w_0, +\infty[$. The investor is endowed with a budget w to finance these projects and he finances each project b_i with an amount x_i. In this case, the problem he has to solve reads

$$B(w) = \max_{x_1 \dots x_n} b_1(x_1) + \dots + b_{n-1}(x_{n-1}) + b_n(x_n)$$

subject to

$$\begin{array}{l} x_i \geq 0 \quad \forall i = 1 \dots n, \\ \sum_{i=1}^{n} x_i \leq w. \end{array}$$

The budget constraint is binding since each function b_i is increasing. As Ginsberg [9], we focus on the average benefit function to obtain the following result.

Lemma 1 *Suppose there are n benefit functions $b_1, \cdots, b_n$ each characterized by $w_i^* = \arg\max b_i(x)/x$. If $w \leq \min_i w_i^*$, then the entire budget w goes to the project with the highest benefit $b_i(w)$.*

Proof: Suppose without loss of generality that $w_1^* = \min_i w_i^*$ and that $w \leq w_1^*$. Suppose moreover that $x_1, ..., x_n$ belong to $E = \{x_i \geq 0 \mid x_1 + ... x_n = w\}$. Consider the integer j such that $\frac{b_j(w)}{w} \geq \frac{b_i(w)}{w} \forall i \neq j$. For all i, $x_i \mapsto b(x_i)/x_i$ is increasing over the interval $[0, w]$. This implies in turn that $\forall i$

$$\frac{b_i(x_i)}{x_i} \leq \frac{b_i(w)}{w} \leq \frac{b_j(w)}{w}.$$

These n inequalities lead to

$$b_1(x_1) + ... + b_n(x_n) \leq \frac{x_1}{w} b_j(w) + ... + \frac{x_n}{w} b_j(w) = b_j(w).$$

Because this is true for all x_i in E, the above inequality means that any allocation $(x_1, ..., x_n)$ is dominated by an allocation in which the entire budget is allocated to one project.■

When the total budget is less than $\min_i w_i^*$, it is optimal to invest the entire budget in the project with the highest benefit because of the increasing returns to scale at low intensities. The following parts will be devoted to the study of the optimal allocation in the case of higher budget levels. We are going to solve this problem in two steps, focusing first on the case of identical benefit functions, and studying then different benefit functions.

4 When the budget level increases: the case of identical benefit functions

To have an intuition of the results, we first analyze the case of two identical benefit functions. We then extend the results to a finite number of benefit functions using induction arguments.

4.1 Two identical projects

4.1.1 General properties

In the general case, we rapidly characterize the solution as Ginsberg [9] already did. If x and $w - x$ denote the budget invested in each project, the choice problem is

$$B(w) = \max_{0 \leq x \leq w} \quad b(x) + b(w - x). \tag{1}$$

There exist three types of solutions to this maximization program as the following proposition tells us.

Lemma 2 *The optimal solution of problem (1) belongs to one of the following three types*

1. *the full-specialization type* $x^* \in \{0, w\}$*,*

2. *the symmetric -or egalitarian- type* $x^* = w/2$*,*

3. *the asymmetric interior type* $x^* = \widehat{x}(w)$*, where* $\widehat{x}(w) < w_0 < w - \widehat{x}(w)$ *and* $b'(\widehat{x}(w)) = b'(w - \widehat{x}(w))$*.*

Proof: See the Appendix.

Notice that program (1) is symmetric relative to $w/2$. Therefore, in the rest of our discussion of the 2 identical-projects case, we are going to focus on solutions that are greater or equal to $w/2$. The full-specialization solution will designate w, the equal solution $w/2$ and the asymmetric interior solution $w - \widehat{x}(w)$. In the case of the full-specialization solution, the entire budget is devoted to only one project. In the case of the symmetric solution, both projects get exactly the same amount, and in the case of the asymmetric interior solution, the two projects get a different positive amount.

According to Lemma 1, when w is less than $w_1 = \arg\max b(x)/x$, all the budget is devoted to a unique project. Moreover, because of the convexity of function b on $[0, w_0]$, when the budget level is less than $2w_0$, the objective function in (1) is convex in the decision variable x in its domain, and the symmetric strategy can never be a maximum. In the next lemma, we determine the condition under which the specialized solution dominates the diversified one.

Lemma 3 *There exists a single* $w_2 \in R \cup \{+\infty\}$ *such that* $b(w) \geq 2b(w/2)$ *for all* $w \leq w_2$ *and* $b(w) \leq 2b(w/2)$ *for all* $w > w_2$*.*

Proof: See the Appendix.■

In words, w_2 is the unique critical wealth level below which full - specialization dominates the symmetric solution, and above which the symmetric allocation is preferred to full-specialization. This implies that, when the budget is increased, it is never optimal to switch from the diversified solution to the specialized one. The following lemma provides more insights about how the optimal strategy evolves as the budget level w is increased.

Lemma 4 *Consider the case of two identical projects. The optimal investment strategy has the following characteristics:*

1. *For low budget levels, the full-specialization strategy is optimal;*

2. *Then, as the total budget level* w *increases, the optimal strategy can switch to an asymmetric interior solution, or directly to the symmetric allocation ;*

3. Once the symmetric allocation is selected, it remains optimal for all larger w.

Proof: See the Appendix. ■

According to Lemma 4, once a strategy (an asymmetric interior solution or the equal strategy) dominates the full-specialization strategy for a given budget level w, the full-specialization strategy will not be optimal for any budget level that is higher than w. Moreover, if the symmetric strategy is optimal for a given budget level, it will remain optimal for any higher budget levels. This particular result has already been proved by Ginsberg [9]. Intuitively, when the budget level is low, the investor prefers to favour one project by investing the whole budget in it because of the low productivity at low budget levels. On the contrary, when the total budget level is high enough, the investor prefers to share the budget equally between both projects because of the projects' decreasing productivity from w_0 on. In between, the investor wants to invest a strictly positive amount in each project but he still favours one project to the detriment of the other. It is not worth to invest everything in a single project since, from the inflection point on, marginal benefit of investing in a project is decreasing. However, the investor prefers to take advantage of the higher productivity for one project.

To illustrate this result, we consider an example where the benefit function is given by

$$b(x) = \frac{x^\gamma}{x^\gamma + k(1-x)^\gamma},$$

with $\gamma = 2$ and $k = 2$. Observe that $w_0 = 0.613$ and $w_1 = 0.816$ in this numerical example. We have drawn the optimal strategy as a function of the total budget w in Figure 2. When w is smaller than 1, it is optimal to invest everything in one project. When w is between 1 and 1.225, the asymmetric interior solution is optimal. Finally, for larger w, the symmetric strategy is optimal.

Concerning asymmetric interior solutions, let us observe that, as wealth w increases, one of the two projects will get a *smaller* budget, as seen in Figure 2.[1] This comes from the full differentiation of the first-order condition[2], which yields

$$\frac{d\widehat{x}}{dw} = \frac{b''(w-\widehat{x})}{b''(\widehat{x}) + b''(w-\widehat{x})} \text{ and } \frac{d\,(w-\widehat{x})}{dw} = \frac{b''(\widehat{x})}{b''(\widehat{x}) + b''(w-\widehat{x})}.$$

As $\widehat{x} < w_0 < w - \widehat{x}$, $b''(\widehat{x})$ is positive and $b''(w-\widehat{x})$ is negative, $d\widehat{x}/dw$ and $d(w-\widehat{x})/dw$ must have opposite signs.

[1] Observe that there is another reason for why the project-specific budgets do not increase monotonically the total budget. When the optimal strategy switches from full specialization to full diversification, the previously financed project gets a 50% reduction in its budget.

[2] $b'(x^*) - b'(w-x^*) \begin{cases} = 0 & \text{if} \quad x^* < w, \\ \geq 0 & \text{if} \quad x^* = w. \end{cases}$

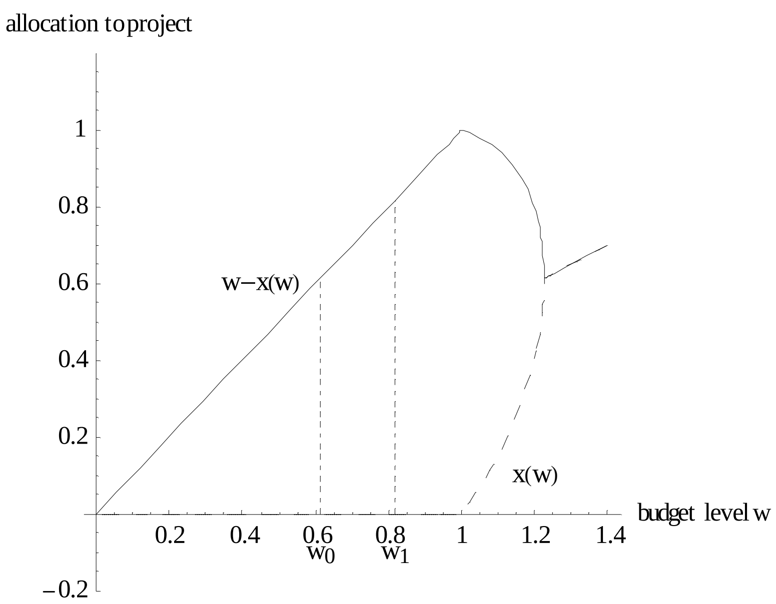

Figure 2: The optimal investment in projects 1 and 2 as a function of the total budget.

To get more information on the evolution of the solution to the maximization program (1) when the total budget level increases, we hereafter study three particular classes of functions: symmetric benefit functions, benefit functions that are "pulled down", and benefit functions that are "lifted up". This analysis necessitates to focus on the shape of the marginal benefit function and this process that is new relative to Ginsberg [9] allows to characterize the cases for which the outcome is very simple.

4.1.2 Symmetric Benefit Functions

A symmetric benefit function can be seen as either having symmetric first order derivatives with respect to the inflection point w_0, or as a 180° rotated function from the part below w_0 to the part above w_0. In this case, the marginal benefit function is symmetric relative to the axis $w = w_0$. With such a benefit function, any asymmetric interior allocation $\widehat{x}$ is excluded and the unique switching wealth level w_2 equals $2w_0$, as stated by the following proposition.

Proposition 1 *Suppose that b' is symmetric in the sense that $b'(w_0 + \delta) = b'(w_0 - \delta)$ for all $\delta \in [0, w_0]$. Then, the fully specialized strategy is optimal if w is smaller than $2w_0$, whereas the symmetric strategy is optimal if w is larger than $2w_0$.*

Proof: See the Appendix. ■

When the benefit function is symmetric, the optimal strategy is full-specialization when $w \leq 2w_0$, and is egalitarian otherwise. In other words, for any budget w below $2w_0$, one project gets all the budget w, otherwise the two projects get exactly the same amount $w/2$. This special case serves as a benchmark for the next two cases, where the benefit function is not symmetric.

The analysis is thus more simple if it is conducted using the marginal benefit function. We consider two cases. In the first, beyond the inflection point both the total and marginal functions lie below their equivalent function for the hypothetical symmetric case[3]. We refer to this as having the benefit functions (both total and marginal) "pulled down". In the second case, both the total and marginal benefit functions lie above their symmetric counterparts. We call this the "lifted up" case.

4.1.3 "Pulled Down" (PD) Benefit Functions

Let us first give the definition of a benefit function that is stretched to the left.

Definition 1 *A benefit function is said to be pulled down (PD) if* $b'(w_0 + \delta) \leq b'(w_0 - \delta)$ *for all* $\delta \in [0, w_0]$.

If a benefit function is PD, beyond the inflection point w_0 the marginal benefit curve is pulled to the left so that it lies everywhere below the symmetric case curve. In Figure 3, the marginal benefit function in the PD case is represented.

Given PD, once the maximal productivity $b'(w_0)$ has been reached, the increase in productivity is less than in the symmetric case. The following lemma allows us to characterize the solution.

Proposition 2 *Suppose that the benefit function* b *is PD, then* $w_2 < 2w_0$*. Moreover, the symmetric strategy is optimal whenever* $w > 2w_0$.

Proof: See the Appendix. ■

When the benefit function is PD and the budget level is larger than $2w_0$, it is optimal to share equally the budget between the two projects. Remember that we obtained the same result in the symmetric case. Because the increase in productivity from w_0 on is less rapid than in the symmetric case, the attractiveness of the specialized solution is weakened. Since it was already inferior to the egalitarian solution in the symmetric case, this result is reinforced in the PD case.

[3] When the total benefit function is symmetric, the shape of the marginal benefit function to the right of the inflection point is a mirror reflection of what is to the left. For the asymmetric case, if the marginal benefit function lies below its symmetric counterpart, then the corresponding total benefit function will also lie below its counterpart, but the reverse is not true.

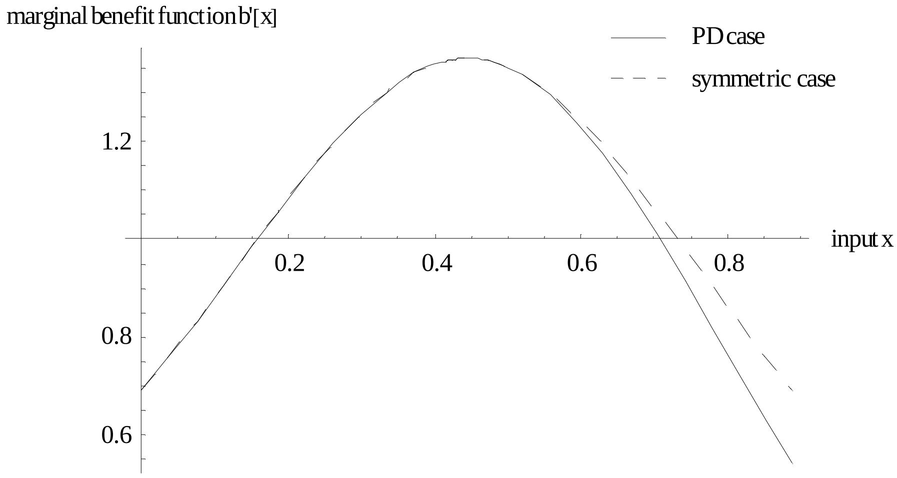

Figure 3: Marginal benefit function in the PD case.

As an illustration, we focus on the case of a logistic distribution function on $[0,1]$ whose cumulative distribution function is equal to $\frac{1}{2\tanh(1)}\left(\tanh\left(\frac{8x-4}{4}\right)+\tanh(1)\right)$. In order to introduce the concept of PD and LU benefit functions, we add a transformation and the benefit function equals

$$b(x)=\frac{1+\alpha x}{1+\alpha}\frac{1}{2\tanh(1)}\left(\tanh\left(\frac{8x-4}{4}\right)+\tanh(1)\right).$$

If $\alpha=0$, b is a symmetric function with respect to w_0. If α is negative, function b is PD, and if α is positive, function b is lifted up. In Figure 4, we take α to be equal to -0.2. In this case, $w_0=0.445$ and $w_2=0.793<2w_0$.

For $w>2w_0=0.890$, the symmetric allocation is optimal as we see on the graph. An asymmetric interior solution exists when $w\leq 2w_0$.

4.1.4 "Lifted Up" (LU) Benefit Functions

A second important case arises when both the total and marginal benefit functions are "lifted up", so they lie above their hypothetical symmetric functions. We define this term more formally as:

Definition 2 *A benefit function is said to be lifted up (LU) if* $b'(w_0+\delta)\geq b'(w_0-\delta)$ *for all* $\delta\in[0,w_0]$.

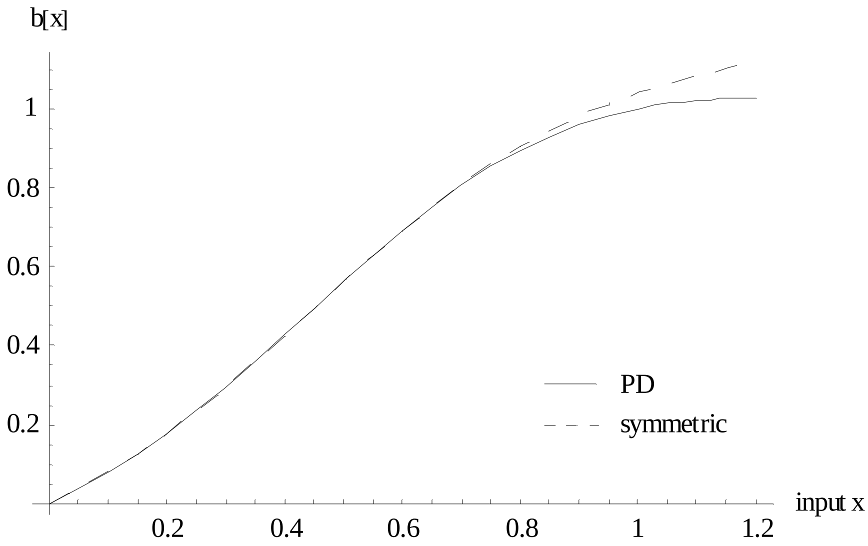

Figure 4: A PD benefit function.

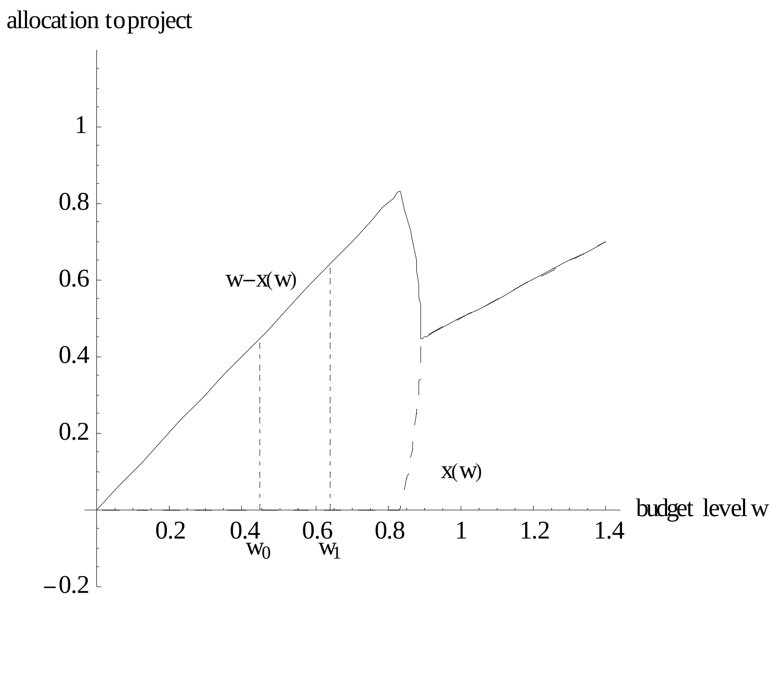

Figure 5: Optimal strategies for a PD benefit function.

If a benefit function is LU[4], beyond the inflection point w_0 the marginal benefit curve is pulled to the right so that it lies everywhere above the symmetric case curve. Therefore, the increase in productivity is more rapid than in the symmetric case beyond w_0. Intuitively, this reinforces the attractiveness of the more specialized strategies. In other words, it should be more likely that one project has a greater share of the total budget than the other even when $w > 2w_0$. This contrasts with the case where the benefit function is PD and where as soon as $2w_0$ is reached, both projects get the same amount. Let us introduce the quantity $x(\delta)$ defined by $b'(w_0 - \delta) = b'(x(\delta))$, with $x(\delta) > w_0$. It is defined for all $\delta \in [0, w_0]$. In fact, for each δ, there exists a $w(\delta)$ such that $x(\delta) = w(\delta) - (w_0 - \delta)$ and $w_0 - \delta$ is an asymmetric interior solution. We are interested in the quantity $z(\delta) = x(\delta) - (w_0 + \delta)$ (see Figure 6). It corresponds to the horizontal distance between both curves. As b is LU, we know that $z(\delta) \geq 0$ for all $\delta \in [0, w_0]$. A condition on this function $z(.)$ allows us to characterize the shape of the optimal solution in the case of a LU benefit function.

Proposition 3 *Suppose that the benefit function b is LU, then $w_2 > 2w_0$. Moreover*

1. *If $w \leq 2w_0$, then the full-specialization strategy is optimal.*
2. *If $w > 2w_0$ and if $\delta \longmapsto z(\delta)$ is increasing, then the optimal strategy cannot be an asymmetric interior one.*

Proof: See the Appendix. ■

We see that, in the LU case, the egalitarian strategy will not in general be optimal even when $w > 2w_0$. If function z is increasing, we have a complete characterization of the solution: it employs the full-specialization strategy for $w \leq w_2$ and then switches to the equal-allocation strategy for $w > w_2$. Because $w_2 > 2w_0$, even when the total budget is higher than $2w_0$, it is still optimal to favour one project. This is due to the more rapid increase in the productivity beyond w_0 relative to the symmetric case. Given that z is an increasing function, the marginal productivity decreases less rapidly than in the symmetric case. A LU benefit function is thus all the more attractive. Thanks to this condition, an interior allocation satisfying the first order condition is a local minimum and should therefore not be taken into account for the search of the optimal solution (the second order condition is not satisfied). If the condition stated in Proposition 3 is not satisfied, the asymmetric solution may be a local maximum. Therefore, the optimal strategy may be to begin with the full-specialization strategy when w is very low,

[4]Note that if a benefit function is LU, it cannot be PD, therefore LU and PD are mutually exclusive. However, these two notions are not mutually inclusive since a benefit function might be neither PD nor LU.

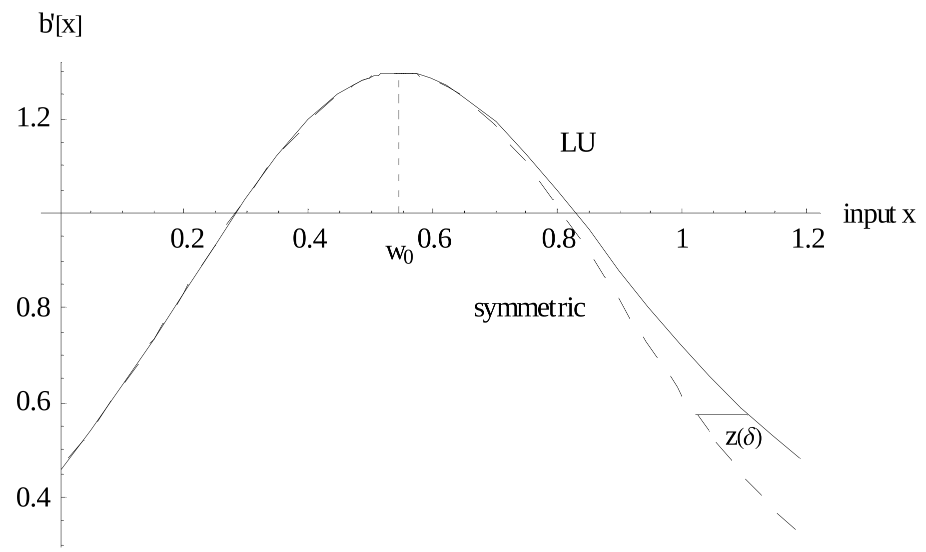

Figure 6: A LU marginal benefit function together with function z.

and as w increases to switch successively to an asymmetric interior solution and then to the equal-allocation strategy.

To illustrate this case, we consider the example of the previous paragraph with a positive α, which implies that the benefit function is LU. In this case, the optimal allocation belongs to $\{w, w/2\}$ according to the ranking of w with respect to w_2 (if it exists). For this numerical example, we take α to be equal to 0.2. The marginal benefit function of each project is represented on Figure 6 together with function $z(.)$.

Since function $z(.)$ is increasing, no asymmetric interior solution will be adopted. In this case, $w_0 = 0.545$ and $w_2 = 1.126$. The full-specialization strategy is optimal for $w \leq 1.126$ and the equal strategy is optimal for $w > 1.126$ as we can see on Figure 7.

Before we turn to the case of a finite number of identical benefit functions, we look briefly at the continuity of the optimal investment strategy.

4.1.5 Some remarks on the continuity of the optimal investment strategy

Up to now, when optimal investment strategies have had interior asymmetric solutions, they have always been continuous. But that may not always be the case, as we now illustrate. Suppose that there exists a range of the budget level such that the asymmetric interior solution exists. Defining two thresholds $\underline{w}$ and $\overline{w}$, the optimal allocation equals

$$x(w) = \begin{cases} w \text{ if } w < \underline{w} \\ w - \widehat{x}(w) \text{ if } \underline{w} \leq w < \overline{w} \\ \frac{w}{2} \text{ if } w \geq \overline{w} \end{cases}.$$

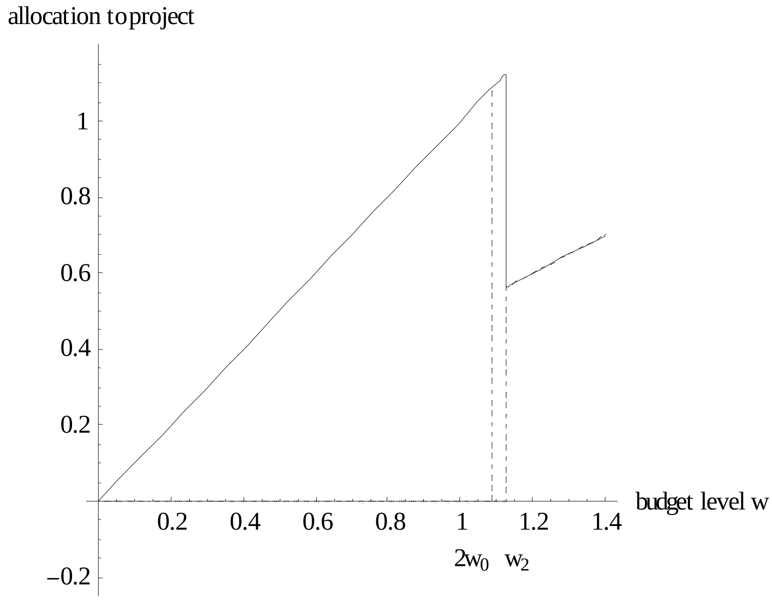

Figure 7: Optimal strategy for a LU benefit funtion.

The proof of Lemma 4 showed that function $b(w) - [b(\widehat{x}(w)) + b(w - \widehat{x}(w))]$ had a unique zero, namely, $\underline{w}$. Let us define, if it exists, $w_s > w_0$ by $b'(w_s) = b'(0)$. Thus $\widehat{x}(w_s) = 0$ is a solution of the first order condition $b'(\widehat{x}(w_s)) = b'(w_s - \widehat{x}(w_s))$ and it is the solution of the equation

$$b(w) - [b(\widehat{x}(w)) + b(w - \widehat{x}(w))] = 0. \tag{2}$$

The left hand side of equation 2 is strictly positive when $w < w_s$ and strictly negative when $w < w_s$. Therefore, $w_s = \underline{w}$ and the allocation is continuous at $\underline{w}$. However, w_s may not exist. Then the allocation may not be continuous as the following example shows. Let us consider the benefit function defined by its first derivative:

$$b'(x) = \begin{cases} kx & \text{if} \quad x \leq w_0, \\ \frac{a}{x^6} + d & \text{if} \quad x > w_0. \end{cases}$$

With $x_0 = 0.5$, $k = 0.1$ and $d = 0.01$, and a and c such that functions b and b' are continuous at x_0, the derivative of the benefit function is represented in Figure 8.

As $d > 0$, w_s does not exist and the optimal allocation will not be continuous.

At the point where the project ceases receiving all funding, its optimal input drops discontinuously.

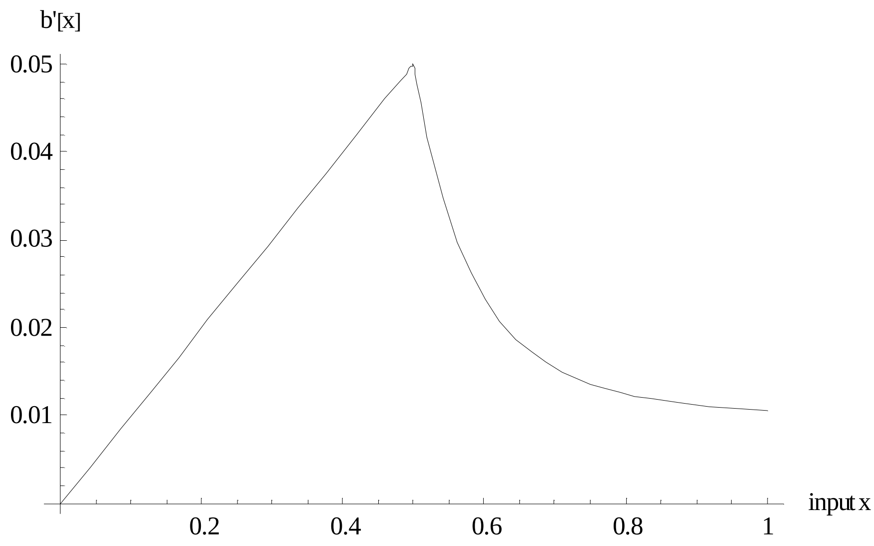

Figure 8: A marginal benefit function.

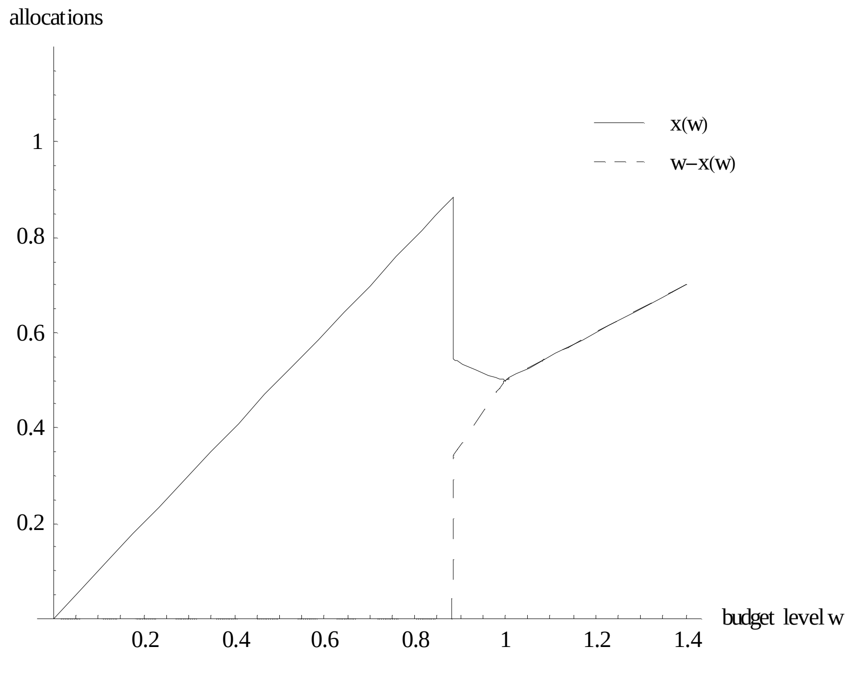

Figure 9: A non continuous allocation.

We studied the optimal allocation in the case of two identical benefit functions. We now turn to the case of a finite number of identical benefit functions.

4.2 n identical projects

As already stated in section 3, with n projects, the investor has to solve the maximization program

$$B(w) = \max_{x_1,\dots,x_{n-1}} b(x_1) + \dots + b(x_{n-1}) + b(w - x_1 - \dots - x_{n-1}) \tag{3}$$

subject to

$$\begin{array}{l} x_i \geq 0, \forall i = 1 \dots n-1, \\ \sum_{i=1}^{n-1} x_i \leq w. \end{array}$$

It is convenient to maximize this benefit function in two steps.

1. A first maximization

$$\max_{x_1,\dots,x_{n-2}} b(x_1) + \dots + b(x_{n-1}) + b(w - x_1 - \dots - x_{n-1}) \tag{4}$$

is equivalent to finding the optimal allocation between $n-1$ projects when the total budget level that is available is equal to $w - x_{n-1}$. The solutions to this maximization are denoted $x_1^*(x_{n-1}, w), \dots, x_{n-2}^*(x_{n-1}, w)$.

2. Then, there remains to solve

$$\begin{aligned} &\max_{x_{n-1}} b(x_1^*(x_{n-1}, w)) + \dots + b\left(x_{n-2}^*(x_{n-1}, w)\right) + b(x_{n-1}) + \\ &+ b\left(w - x_1^*(x_{n-1}, w) - \dots - x_{n-2}^*(x_{n-1}, w) - x_{n-1}\right). \end{aligned} \tag{5}$$

Therefore, this kind of problem has to be solved using induction arguments. Before doing this, let us first introduce the thresholds $w_2, \dots, w_i, \dots, w_n$ defined by

$$(i-1)\, b\left(\frac{w_i}{i-1}\right) = ib\left(\frac{w_i}{i}\right), \forall i = 2 \dots n-1. \tag{6}$$

Note that we recover the definition of w_2. In the following lemma, we prove the uniqueness of these thresholds, and we rank them.

Lemma 5 *The thresholds $w_2, \dots, w_n$ are uniquely defined by equation (6) and satisfy*

$$w_0 < w_2 < \dots < w_i < \dots < w_n.$$

Moreover, $\forall i = 2, \dots n-1$,

$$\begin{array}{ll} (i-1)\, b\left(\frac{w}{i-1}\right) > ib\left(\frac{w}{i}\right) & \forall w < w_i, \\ (i-1)\, b\left(\frac{w}{i-1}\right) < ib\left(\frac{w}{i}\right) & \forall w > w_i. \end{array}$$

Proof: See the Appendix. ■

If you only consider equal allocations among the financed projects, Lemma 5 tells us that as the total budget w increases, it is optimal to share it equally between an increasing number of projects (1, then 2, to finally end up with the financing of all n projects).

4.2.1 The Case of a Symmetric Benefit Function

In the case of a symmetric benefit function, the optimal allocation is quite natural as stated in the following proposition.

Proposition 4 *Suppose b is a symmetric benefit function. Then, the optimal strategy is to share equally the budget between all financed projects. Moreover,*

- *If $w < w_2$, only one project gets the entire budget w;*
- $\forall w \in [w_i, w_{i+1}[$, *$i$ projects are financed* , $\forall i = 2...n-1$;
- *If $w \geq w_n$, then all the n projects are financed.*

Proof: See the Appendix.■

We have generalized the result of the previous section with two identical benefit functions. As wealth increases, the optimal strategy is to share equally the total amount between an increasing number of projects. We now turn to the study of other families of functions.

4.2.2 The Case of a LU Benefit Function

It is difficult to extend the results concerning PD benefit functions in the case of n different projects. Indeed, recall that in the two projects case, we did not find a condition under which no asymmetric interior solution exists. Therefore, in this study of the n projects case, we generalize the 2 projects case we focus on to the case of lifted up benefit functions.

Proposition 5 *Suppose b is a LU benefit function and $\delta \mapsto z(\delta)$ is an increasing function. Then, the optimal strategy is to share equally the budget between all financed projects. Moreover,*

- *If $w < w_2$, only one project gets the entire budget w;*
- $\forall w \in [w_i, w_{i+1}[$, *$i$ projects are financed* $\forall i = 2...n-1$;
- *If $w \geq w_n$, then all the n projects are financed.*

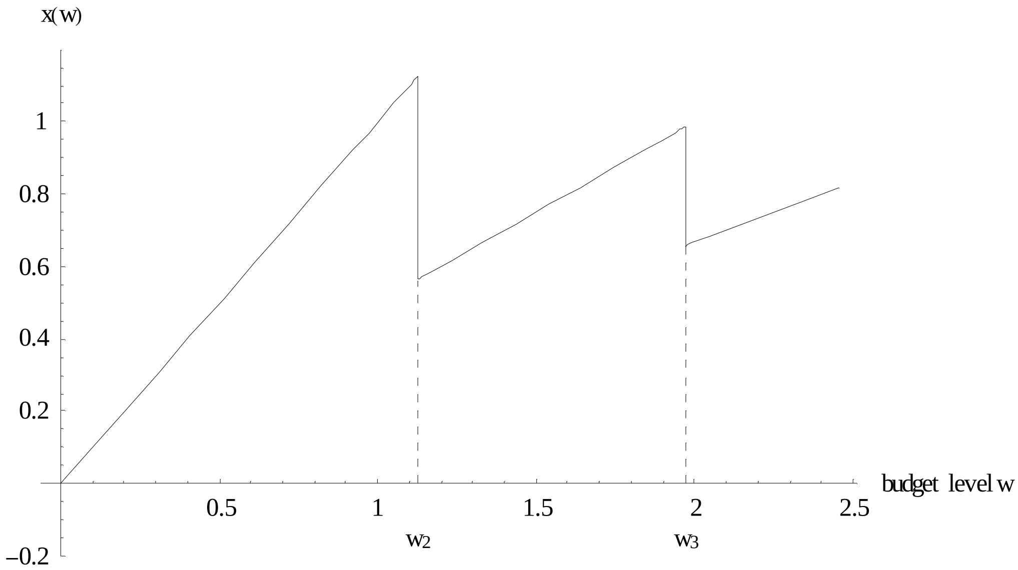

Figure 10: Optimal strategy for a LU benefit function with three identical projects.

Proof: See the Appendix.■

If the benefit function is LU, the decrease in productivity for wealth levels higher or equal than w_0 is less rapid than in the symmetric case. Therefore, it is optimal to increase the number of financed projects as the total budget w increases and to share it equally between all financed projects. We present the result in the case of a LU benefit function and of 3 projects on Figure 10 for the functions' family studied in the previous section

$$b(x) = \frac{1+\alpha x}{1+\alpha}\frac{1}{2\tanh(1)}\left(\tanh\left(\frac{8x-4}{4}\right)+\tanh(1)\right).$$

and $\alpha = 0.2$.

In this part, we managed to give the shape of the optimal allocation when the budget level increases in the case of a finite number of identical benefit functions. Now, we determine how these results generalize to the case of heterogeneous benefit functions.

5 When the budget level increases: the case of different benefit functions

Heterogeneity makes the problem much trickier. We consider thus a special case of two different projects, the second operating at a much larger scale than the first one. Their benefit functions are

$$b(x) \text{ and } c(x) = kb(x/j) \text{ with } 1 < k < j. \tag{7}$$

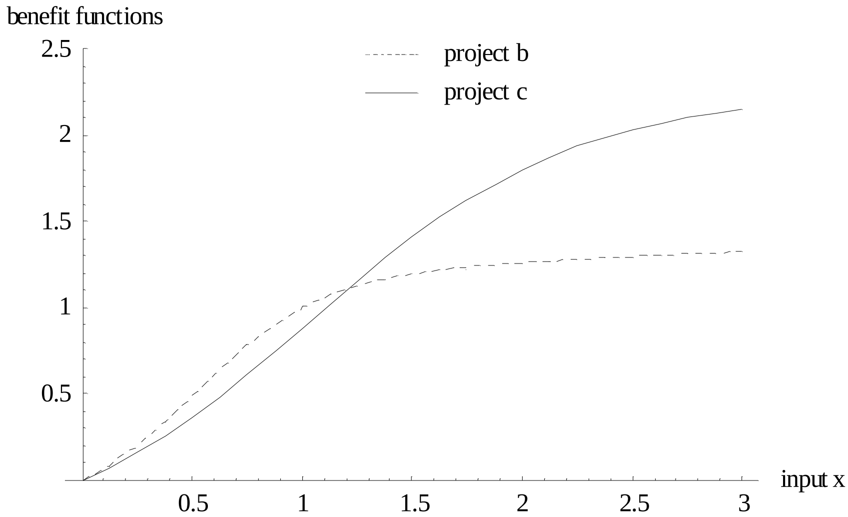

Figure 11: Benefit functions b and c.

They are represented on Figure 11.

Denote $w_1^* = \arg\max b(x)/x$ (resp. $w_2^* = \arg\max c(x)/x$). w_2^* is such that $c(w_2^*)/w_2^* = kb(w_2^*/j)/w_2^* = c'(w_2^*) = (k/j)\,b'(w_2^*/j)$. Thus, $w_2^* = jx_1^* > w_1^*$. Moreover, $x \mapsto b(x)/x$ is increasing on $[0, w_1^*]$ and $x/j < x$. Therefore, $\forall x \leq w_1^*$:

$$\begin{aligned} \tfrac{1}{x}b(x) &> \tfrac{j}{x}b\left(\tfrac{x}{j}\right), \\ b(x) &> jb\left(\tfrac{x}{j}\right), \\ &> kb\left(\tfrac{x}{j}\right), \end{aligned}$$

and $\forall x \leq w_1^*, b(x) > c(x)$. According to Lemma 1, if the total budget level w is less than w_1^*, then project b gets the entire budget w and project c gets nothing. Indeed, for low budget levels, project b is more profitable than project c and gets the entire financing. Before going further into the study of the optimal allocation, we describe the potential solutions to the investor's maximization program

$$\max_{0 \leq x \leq w} b(w-x) + c(x). \tag{8}$$

Lemma 6 *Suppose b is a benefit function and $c(x) = kb(x/j)$ with $1 < k < j$. The optimal solution of program (8) belongs to one of the following five types:*

1. $x(w) = 0$: *the whole budget goes to project b,*
2. $x(w) = w$: *the whole budget goes to project c,*
3. $\widehat{x}^1(w)$ *with* $b'\left(w - \widehat{x}^1(w)\right) = \frac{k}{j}b'\left(\frac{\widehat{x}^1(w)}{j}\right)$ *and* $w_0 < \frac{\widehat{x}^1(w)}{j} < w - \widehat{x}^1(w)$. *This solution will be called "interior solution 1",*

4. $\widehat{x}^2(w)$ *with* $b'\left(w-\widehat{x}^2(w)\right)=\frac{k}{j}b'\left(\frac{\widehat{x}^2(w)}{j}\right)$, $\frac{\widehat{x}^2(w)}{j}<w_0<w-\widehat{x}^2(w)$, *and* $\frac{k}{j^2}b''\left(\frac{x}{j}\right)+b''(w-x)\leq 0$. *This solution will be called "interior solution 2",*

5. $\widehat{x}^3(w)$ *with* $b'\left(w-\widehat{x}^3(w)\right)=\frac{k}{j}b'\left(\frac{\widehat{x}^3(w)}{j}\right)$, $w-\widehat{x}^3(w)<w_0<\frac{\widehat{x}^3(w)}{j}$, *and* $\frac{k}{j^2}b''\left(\frac{x}{j}\right)+b''(w-x)\leq 0$. *This solution will be called "interior solution 3".*

Proof: See the Appendix.■

In this case, there exist three interior solutions. Taking the derivative of the first order condition, $b'(w-x(w))=\frac{k}{j}b'\left(\frac{x(w)}{j}\right)$, with respect to w leads to

$$\frac{d\widehat{x}^i(w)}{dw}=\frac{b''\left(w-\widehat{x}^i(w)\right)}{\frac{j}{k^2}b''\left(\frac{\widehat{x}^i(w)}{j}\right)+b''\left(w-\widehat{x}^i(w)\right)}\text{ and} \tag{9}$$

$$\frac{d\left(w-\widehat{x}^i(w)\right)}{dw}=\frac{\frac{j}{k^2}b''\left(\frac{\widehat{x}^i(w)}{j}\right)}{\frac{j}{k^2}b''\left(\frac{\widehat{x}^i(w)}{j}\right)+b''\left(w-\widehat{x}^i(w)\right)}. \tag{10}$$

The ranking of $\frac{\widehat{x}^i(w)}{j}$ and $w-\widehat{x}^i(w), i=1,2,3$ relative to w_0 allows us to note that $\widehat{x}^1(w)$ and $\widehat{x}^2(w)$ are increasing functions of w, $\widehat{x}^3(w)$ is a decreasing function of w. $w-\widehat{x}^1(w)$ and $w-\widehat{x}^3(w)$ are increasing functions of w, but $w-\widehat{x}^2(w)$ is a decreasing function of w. When the interior solution 2 solves the maximization program (8), project c gets an increasing amount of the total budget whereas project b gets a decreasing amount of the total budget. This result is similar than in the case of homogenous benefit function. Quite the opposite happens with the interior solution 3: project b gets an increasing amount of the total budget whereas project c gets a decreasing amount of the total budget. This is not the case anymore with the interior solution 1, where the two optimal solutions are increasing functions of the total budget w: as the budget increases, each project gets more financing. The following lemma[5] characterizes the optimal allocation.

Lemma 7 *Suppose b is a benefit function and $c(x)=kb(x/j)$ with $1<k<j$. The optimal solution to the maximization program (8) has the following characteristics:*

1. *When $x\leq w_1^*$, project b gets the whole budget,*

2. *When interior solution 1, $\widehat{x}^1(w)$, is the optimal allocation, it will remain so for any higher budget level.*

Proof: See the Appendix.■

[5] Edward Shpiz helped with this lemma and with Lemma 6.

The path of the optimal allocation between the two projects as a function of the budget is quite different from the case where the benefit functions were identical. No general result indeed holds on the way the different allocations link together. However, there are two similarities. First, when the budget level is very low, only one project, project b, is financed. Second, once the interior solution 1 is reached, the funding of each project increases with w. But contrary to the identical benefit functions case, the two projects are not financed at the same level. Between the allocation that gives all the budget to project b and the interior solution 1, anything may happen. In particular, each project can have an allocation that is an increasing function of the total budget whereas the allocation of the other project is a decreasing function of the total budget (interior solutions 2 and 3). Moreover, it can be the case that one project stops being financed (when project c gets all the budget). It can also happen that the three allocations mix together. In order to illustrate this discussion, we consider the family of benefit function studied in section 3

$$b\left(x\right)=\frac{1+\alpha x}{1+\alpha}\frac{1}{2\tanh\left(1\right)}\left(\tanh\left(\frac{7x-3.5}{4}\right)+\tanh\left(\frac{3.5}{4}\right)\right).$$

with $\alpha=0.05$, $k=1.8$ and $j=2$. With these parameters, the inflection point, w_0, is equal to 0.5159, and $w_1^*=0.7601$. The numerical resolution of this example gives the following results (that are depicted on Figure 12):

- If $w<1.0987$, then project b gets the entire budget,
- If $1.0987<w<1.2632$, then the optimal solution is the interior solution 2. The two projects are financed, but as w increases, project b is less financed whereas project c is more financed,
- If $1.2632<w<1.6224$, then project c gets the whole budget w,
- If $1.6224<w<1.7381$, the interior solution 2 is once again the optimal solution,
- If $w>1.7381$, the interior solution 1 is optimal, meaning that the funding of each project increases with w.

When the budget level is low, only project b gets financing. But as soon as the budget level w increases and once the inflexion point is crossed, it becomes less profitable whereas project c still presents an increasing marginal productivity. Therefore, project c begins to be funded and project b gets a lower share of the total budget before being totally abandoned. There is a range of budget levels for which project c gets the whole budget: indeed, for these values of w, both marginal productivities are decreasing, but project c still presents a higher marginal

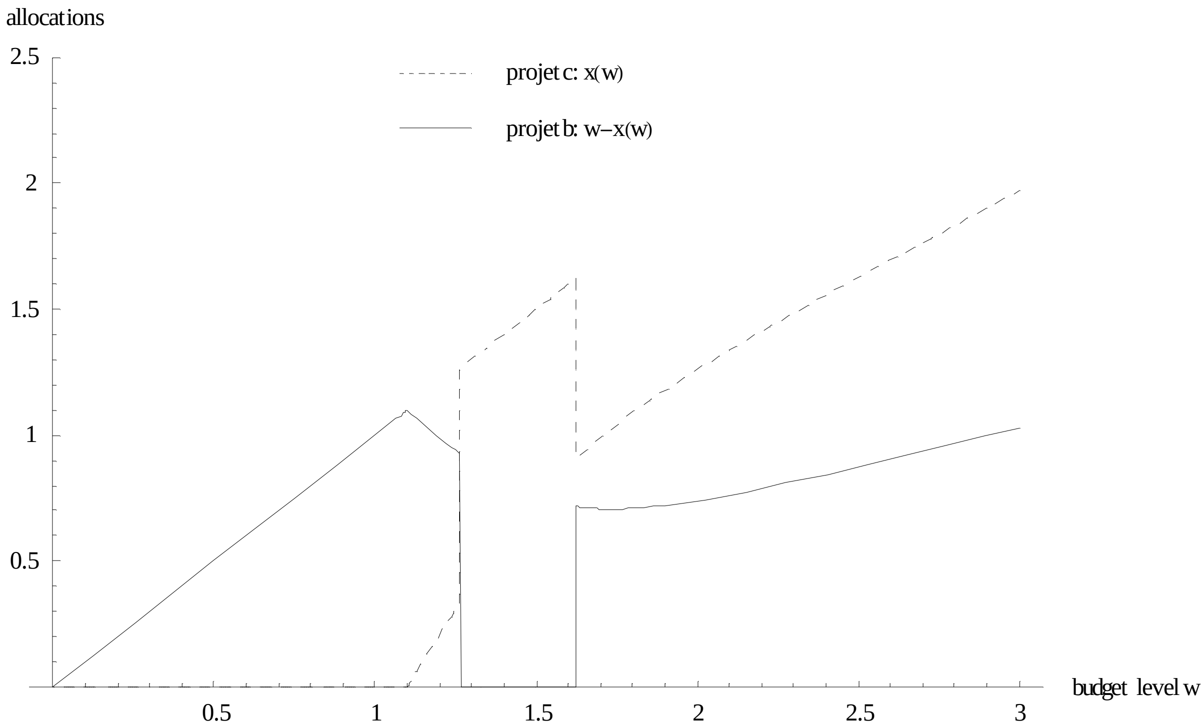

Figure 12: Optimal allocations with heterogenous benefit functions.

productivity. But on $[1.6224, 1.7381]$, project b comes back to life. The two decreasing marginal productivities get closer, but the dominance of project c relative to project b makes b lose funding as the budget increases. Once the last threshold 1.7381 is crossed, both projects get an increasing funding as w gets larger. Indeed, the two marginal productivities, while decreasing, converge and the two projects are worth being financed with a strictly positive budget share. The optimal allocation is thus much more complex than in the homogenous case; indeed the financing of each project may not be monotone with the budget level w. Moreover, the number of financed projects is not increasing with the total budget that is available. It is difficult to get a more precise description of the optimal allocation, but the main result is that after a succession of financing and non-financing of the different projects, they both end up being financed in an increasing way as the total funding increases.

6 Properties of the aggregate benefit function

The object of this section is to study function B defined by

$$B(w) = \max_{x_1 \ldots x_{n-1}} b_1(x_1) + \ldots + b_{n-1}(x_{n-1}) + b_n(w - x_1 - \ldots - x_{n-1})$$

subject to

$$x_i \geq 0 \quad \forall i = 1...n-1,$$
$$\sum_{i=1}^{n-1} x_i \leq w.$$

We can first state a simple property of function B.

Lemma 8 *Function B is increasing in the budget level w.*

Proof: This is an application of the envelope theorem.■

From now on, our analysis will narrow to the case of identical benefit functions. Indeed, we have noted in the previous sections that it is very difficult to obtain general properties in the case of different benefit functions. We begin with the case where the characterization of the optimal solution is straightforward, i.e. when the benefit function is LU.

6.1 The case of LU benefit functions

In section 4, we proved that under some condition, the optimal strategy is to equally finance an increasing number of projects as the budget increases. What is the impact of this result on the aggregate benefit function?

Proposition 6 *Suppose that b is a LU benefit function and that function z is increasing. Then, function B is convex on $[0, w_0]$ and concave on $[w_0, +\infty[$.*

Proof: See the Appendix.■

On Figure 13, we illustrate this result with the LU benefit functions' family we already used. Note that it is not correct to say that the aggregate benefit function is S-shaped although it is successively convex and concave. Indeed, as the first derivative of function B is not continuous[6], there is a kink around w_2 and B can be seen as locally convex around this threshold (the derivative $b'(w)$ is lower than $b'(w/2)$). It is straightforward to extend this result to the case of a symmetric benefit function.

6.2 The general case

Recall that in the general case the characterization of the optimal strategy is tricky. Therefore, we concentrate on the case of two identical benefit functions. According to Proposition 4, we know how the solution evolves as the budget increases and we can state the following proposition.

Proposition 7 *In the case of two identical benefit functions, if there exist w for which the interior solution is optimal, then function B is successively convex, concave, convex to end up concave as the budget w increases.*

[6] There is indeed no reason that $b'(w) = b'\left(\frac{w}{2}\right)$ at $w = w_2$.

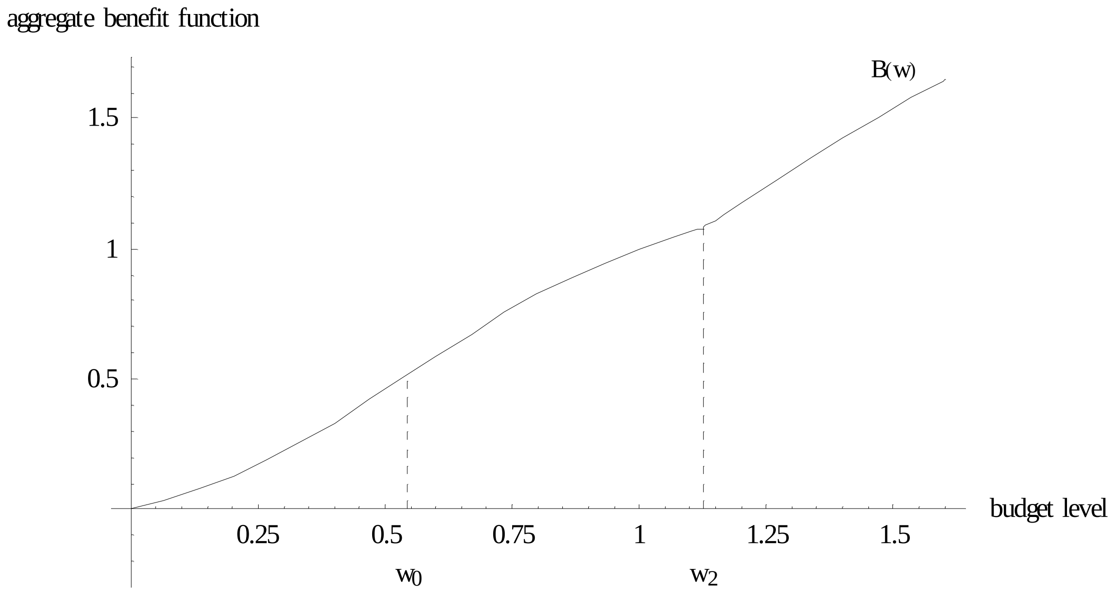

Figure 13: Aggregate benefit function when the functions are LU.

Proof: If an asymmetric interior solution exists, B has the following shape:

$$B(w) = \begin{cases} b(w) & \text{if} \quad w < \underline{w} \\ b(\widehat{x}(w)) + b(w - \widehat{x}(w)) & \text{if} \quad \underline{w} \leq w < \overline{w} \\ 2b\left(\frac{w}{2}\right) & \text{if} \quad w \geq \overline{w} \end{cases},$$

where $\underline{w} \geq w_1$ and $\overline{w} \geq w_2$. Let us focus on what happens on $[\underline{w}, \overline{w}]$. According to the envelope theorem, $B'(w) = b'(w - \widehat{x}(w))$ and thus $B''(w) = \left(1 - \frac{d\widehat{x}(w)}{dw}\right) b''(w - \widehat{x}(w))$. Recall that in this case $w - \widehat{x}(w) > w_0$, therefore $b''(w - \widehat{x}(w)) \leq 0$ and $\left(1 - \frac{d\widehat{x}(w)}{dw}\right) \leq 0$: B is thus convex in this case. It follows that B is convex on $[0, w_0]$, concave on $[w_0, \underline{w}]$, convex on $[\underline{w}, \overline{w}]$, concave on $[\overline{w}, +\infty[$. ■

We illustrate this proposition with the benefit function $b(x) = \frac{x^2}{x^2+2(1-x)^2}$ on Figure 11. B is convex on $[0, w_0]$, concave on $[w_0, \underline{w}]$, convex on $[\underline{w}, \overline{w}]$ and concave on $[\overline{w}, +\infty[$. In fact, when the asymmetric interior solution is optimal, B is convex meaning that the marginal aggregate benefit function is increasing. Indeed, in this case, the project that begins to be financed has a greater weight than the other one in terms of the second derivatives of the aggregate benefit function (the increase in the marginal benefit function is more important than the decrease). But once this property is not satisfied anymore, we turn to the equal allocation and the aggregate marginal benefit function is concave. Note that in this example, the first derivative of function B is continuous. This comes from the continuity of function x in the special example we treated.

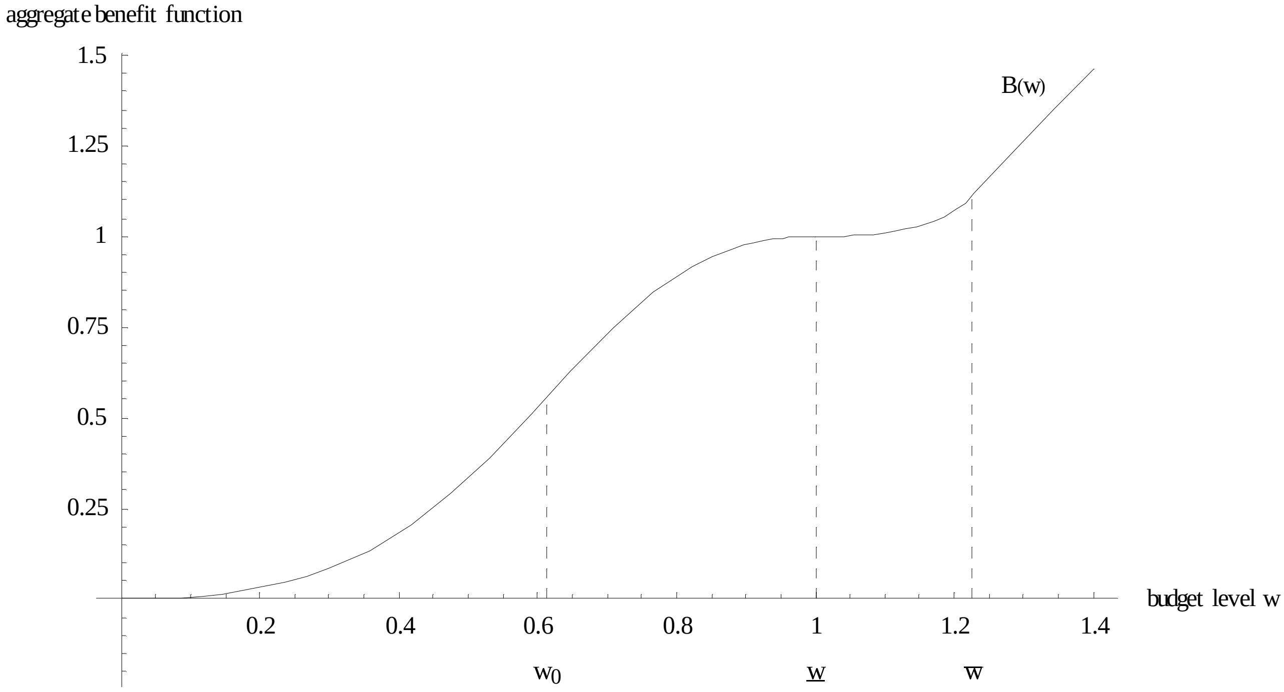

Figure 14: Aggregate benefit function in the general case with two projects.

However, as we discussed in paragraph 4.1.5, this is not always the case and function B may present a kink as in the case of LU benefit functions.

7 Concluding remarks

We study the investment decision of an investor with multiple available projects, each presenting a range of increasing returns before returns decline. Such decisions are common across a great range of fields, such as allocating R&D investment, advertising budgets, or inoculations for communicable diseases, and are particularly prevalent when outcomes are uncertain or indexed probabilities. With n identical projects, when budget levels are low, the investor favours one project by investing the whole budget in it. Once he decides to invest a strictly positive amount in each project for a given budget level, he will keep on investing strictly positive amounts in each project.

The properties of the optimal allocation are most easily seen with just two projects. As the budget increases, allocations may be unequal though positive, and a project may actually experience a reduction in budget over some range. When the total budget level is high enough, the investor shares the budget equally between the two projects, and this equal strategy remains optimal for any higher budget level. When the benefit function has a plausible shape, what we label lifted up, the optimal investment strategy goes from full specialization to equal division without passing through a range with positive but unequal division. These results extend immediately to the case of a finite number of projects.

Matters are more complex when the benefit for the projects may differ. Qualitatively, however, the same local and global marginal efficiency requirements must be satisfied, and the prime features of efficient allocations are maintained. Thus, first one project gets all resources. Then there is a range where multiple projects get funding, and the funding for some way be non-monotonic with the total budget. Finally, when the budget is large, all projects get funded, and the funding for each increases as the budget grows further.

The aggregate benefit function for the lifted up case is first convex and then concave. More generally, the aggregate benefit function is successively convex, concave, convex.... to end up concave in the budget level. In short, an apparently straightforward and commonly encountered resource allocation problem turns out to have an intriguingly complex solution, despite perfectly intuitive efficiency conditions.

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Appendix

A Proof of Lemma 2

If we denote by λ the Lagrangian multiplier associated with the constraint $x \leq w$, the Lagrangian of program (1) reads:

$$L = b(x) + b(w-x) + \lambda (w-x)$$

Therefore, the first order condition of this program, $\partial L/\partial x = 0$, leads to:

$$b'(x^*) - b'(w-x^*) \begin{cases} = 0 & \text{if} \quad x^* < w, \\ \geq 0 & \text{if} \quad x^* = w. \end{cases} \tag{11}$$

So, if $\lambda > 0$, the constraint binds and $x = w$ is the solution of the program.

On the contrary, if $\lambda = 0$, then $b'(x^*) - b'(w-x^*) = 0$. This solution has two possible solutions: either $x^* = w/2$ or $x^* = \widehat{x}$. Because of the properties of the benefit function (b' is increasing if $x \leq w_0$ and b' is decreasing if $x \geq w_0$), this last possibility is possible iff $\widehat{x} < w_0 < w - \widehat{x}$. Indeed, suppose by contradiction that $w_0 < x^* < w - x^*$. Then, consider an alternative allocation $(x^* + \varepsilon, w - x^* - \varepsilon)$ with $0 < \varepsilon < (w - 2x^*)/2$. Because b is locally concave above w_0, it implies that

$$b(x^*) + b(w-x^*) < b(x^* + \varepsilon) + b(w - x^* - \varepsilon).$$

This is a contradiction.■

B Proof of Lemma 3

We study function $x \longmapsto f(x) = b(x) - 2b(x/2)$. The local convexity of the function b on the interval $[0, w_0]$ implies that $f'(x) \geq 0$ for all $x < w_0$. In the same way, the local concavity of b for $x > w_0$, implies that $f'(x) \leq 0 \quad \forall x > 2w_0$. We hereafter show that there exists a unique a such that $f'(x) = 0$. Suppose by contradiction that there exist a_1 and a_2, $0 < a_1 < a_2$, such that

$$f'(a_1) = f'(a_2) = 0.$$

Note that $b'(a_i/2) = b'(a_i)$ is possible only if $a_i/2 < w_0 < a_i$. Thus, we must have that $w_0 < a_1 < a_2$, which implies in turn that $b'(a_1) > b'(a_2)$. Similarly, we must have that $a_1/2 < a_2/2 < w_0$, which implies that $b'(a_1/2) < b'(a_2/2)$. Combining this last result with the initial assumption that $b'(a_i/2) = b'(a_i)$ implies that $b'(a_1) = b'(a_1/2) < b'(a_2/2) = b'(a_2)$, or equivalently $b'(a_1) < b'(a_2)$, a contradiction. It follows that there exists a unique a such that

$$\begin{array}{ll} \forall x \leq a & f'(x) \geq 0, \\ \forall x \geq a & f'(x) \leq 0. \end{array}.$$

Therefore, f is increasing on $[0,a]$ and decreasing on $[a,+\infty[$. As $f(0)=0$, if a positive zero exists to f, it is unique. It implies that f changes sign only once, from positive to negative.$\blacksquare$

C Proof of Lemma 4

We are going to show the following three assertions:

1. As w increases, one can never switch from the fully diversified solution to the full-specialization one.
2. As w increases, one can never switch from an asymmetric interior solution to the full-specialization one.
3. As w increases, one can never switch from the fully diversified solution to an asymmetric interior one.

The first property is a direct consequence of Lemma 3. Now, we prove the second result. Consider a range of w for which an asymmetric interior solution $(\widehat{x}, w-\widehat{x})$ exists, where $\widehat{x}(w)$ is defined by the asymmetric solution to equation $b'(\widehat{x}) = b'(w-\widehat{x})$. We know from Proposition 2 that $\widehat{x} < w_0 < w-\widehat{x}$. Let us study the function $w \longmapsto g(w) = b(w) - [b(\widehat{x}(w)) + b(w-\widehat{x}(w))]$. Consider any solution $w=\overline{w}$ of equation $g(w)=0$. We show that this implies that $g'(\overline{w})$ be nonpositive. Indeed, by the envelope theorem, we have that

$$g'(\overline{w}) = b'(\overline{w}) - b'(\overline{w}-\widehat{x}(\overline{w})).$$

Because $w_0 < \overline{w}-\widehat{x}(\overline{w}) \leq \overline{w}$, we have that b' is decreasing between $\overline{w}-\widehat{x}(\overline{w})$ and $\overline{w}$. It implies that $b'(\overline{w}-\widehat{x}(\overline{w}))$ is larger than $b'(\overline{w})$, or equivalently, that $g'(\overline{w})$ is nonpositive. It implies that if one switches between the fully specialized solution and the asymmetric interior solution when wealth increases, it can only be from the former to the latter.

To prove the third result, consider a range of w for which an asymmetric interior solution $(\widehat{x}, w-\widehat{x})$ exists, where $\widehat{x}(w)$ is defined by the asymmetric solution to equation $b'(\widehat{x}) = b'(w-\widehat{x})$. Let us study the function $w \longmapsto h(w) = b(\widehat{x}(w)) + b(w-\widehat{x}(w)) - 2b(w/2)$. Consider any solution $w=\overline{w}$ of equation $h(w)=0$. We show that this implies that $h'(\overline{w})$ be nonpositive. Indeed, by the envelope theorem, we have that

$$h'(\overline{w}) = b'(\overline{w}-\widehat{x}(\overline{w})) - b'(\overline{w}/2).$$

We know from Proposition 2 that $\widehat{x} < w_0 < w-\widehat{x}$. We also know that b' is increasing and then decreasing in interval $[\widehat{x}(\overline{w}), \overline{w}-\widehat{x}(\overline{w})]$, and that the values of b' are the same at the boundaries of this interval. Because $\overline{w}/2$ belongs to this interval, we have that $b'(\overline{w}/2)$ is larger than $b'(\overline{w}-\widehat{x}(\overline{w}))$, or equivalently, that $h'(\overline{w})$ is nonpositive. It implies that if one switches between the asymmetric interior solution and the equal solution when wealth increases, it can only be from the former to the latter.■

D Proof of Proposition 1

We first prove that the first-order condition $b'(x)=b'(w-x)$ may have only one root at $x=w/2$ when $w\neq 2w_0$. Suppose by contradiction that there exists $\widehat{x}\neq w/2$ such that $b'(\widehat{x})=b'(w-\widehat{x})$. By symmetry, this can be true only if $\widehat{x}=w-\widehat{x}$, or if $w_0-(\widehat{x}-w_0)=w-\widehat{x}$. The first case is equivalent to $\widehat{x}=w/2$, a contradiction. The second case is equivalent to $w=2w_0$, also a contradiction. Thus, $x=w/2$ is the only candidate for an interior optimum.

We then show that $x=w/2$ is a minimum of the objective function when w is smaller than $2w_0$. To show this, we prove that $b'(x)\leq b'(w-x)$ for all x smaller than $w/2$. Two case must be considered depending upon whether $w-x$ is smaller or larger than w_0. When $w-x<w_0$, both x and $w-x$ are smaller than w_0. Because b' is increasing in this range, we indeed obtain that $b'(x)\leq b'(w-x)$ if $x\leq w-x$, which is true. When $w-x>w_0$, x and $w-x$ are on opposite sides of w_0. But $b'(w-x)=b'(w_0+(w-x-w_0))$ is by symmetry equal to $b'(w_0-(w-x-w_0))$, whose argument is smaller than w_0. Because b' is increasing in this range, it implies that $b'(x)\leq b'(x+(2w_0-w))$ if $w\leq 2w_0$, which is also true.

A parallel proof can be written when w is larger than $2w_0$.■

E Proof of the Proposition 2

Suppose b is PD. We are going to prove that $b(2w_0)<2b(w_0)$.

$$\begin{array}{rcl} b(2w_0) & = & \int_0^{w_0} b'(x)\,dx+\int_{w_0}^{2w_0} b'(x)\,dx \\ & = & \int_0^{w_0} b'(w_0-\delta)\,d\delta+\int_0^{w_0} b'(w_0+\delta)\,d\delta \\ & < & 2\int_0^{w_0} b'(w_0-\delta)\,d\delta \\ & = & 2b(w_0). \end{array}$$

Therefore, $b(2w_0)<2b(w_0)$ and $w_2<2w_0$.

We turn to the second part of the Proposition and suppose that there exist $\overline{w}$ and $\widehat{x}$ such that $b'(\widehat{x})=b'(\overline{w}-\widehat{x})$ with $\widehat{x}<w_0<\overline{w}-\widehat{x}$. Define $\widehat{\delta}\in[0,w_0]$ such that $\widehat{x}=w_0-\widehat{\delta}$. We have that

$$\begin{array}{rcl} b'(\widehat{x}) & = & b'\left(w_0-\widehat{\delta}\right), \\ & = & b'(\overline{w}-\widehat{x}), \\ & = & b'\left(\overline{w}-\left(w_0-\widehat{\delta}\right)\right), \\ & \geq & b'\left(w_0+\widehat{\delta}\right). \end{array}$$

As b' is decreasing for $x \geq x_{0,}$ the above equality thus implies that $\overline{w} - \left(w_0 - \widehat{\delta}\right) \leq w_0 + \widehat{\delta}$, hence $\overline{w} \leq 2w_0$. Therefore, if $\overline{w}$ and $\widehat{x}$ exist, we must have that $\overline{w} \leq 2w_0$. Thus, if $w > 2w_0$, no interior asymmetric interior solution exists and the solution belongs to $\{w, w/2\}$. But we know that for a PD benefit function $2w_0 > w_2$. Therefore, if $w > 2w_0$, $2b\,(w/2) > b\,(w)$ and the allocation $\{w/2\}$ is the solution for $w > 2w_0$. ■

F Proof of Proposition 3

Suppose b is LU. We are going to prove that $b\,(2w_0) > 2b\,(w_0)$.

$$\begin{array}{rcl} b\,(2w_0) & = & \int_0^{w_0} b'\,(x)\,dx + \int_{w_0}^{2w_0} b'\,(x)\,dx, \\ & = & \int_0^{w_0} b'\,(w_0 - \delta)\,d\delta + \int_0^{w_0} b'\,(w_0 + \delta)\,d\delta, \\ & > & 2\int_0^{w_0} b'\,(w_0 - \delta)\,d\delta, \\ & = & 2b\,(w_0)\,. \end{array}$$

Therefore, $b\,(2w_0) > 2b\,(w_0)$ and $w_2 > 2w_0$.

Suppose now there exist $\overline{w}$ and $\widehat{x}$ such that $b'\,(\widehat{x}) = b'\,(\overline{w} - \widehat{x})$ with $\widehat{x} < w_0 < \overline{w} - \widehat{x}$. Define $\delta \in [0, w_0]$ such that $\widehat{x} = w_0 - \delta$. We have that

$$\begin{array}{rcl} b'\,(\widehat{x}) & = & b'\,(w_0 - \delta)\,, \\ & = & b'\,(\overline{w} - \widehat{x})\,, \\ & = & b'\,(\overline{w} - (w_0 - \delta))\,, \\ & \leq & b'\,(w_0 + \delta)\,. \end{array}$$

As b' is decreasing for $x \geq x_{0,}$ the above equality thus implies that $\overline{w} - (w_0 - \delta) \geq w_0 + \delta$, hence $\overline{w} \geq 2w_0$. Therefore, if $\overline{w}$ and $\widehat{x}$ exist, we must have that $\overline{w} \geq 2w_0$. Thus, if $w \leq 2w_0$, no interior asymmetric interior solution exists and the solution belongs to $\{w, w/2\}$. However, in this case, $w/2$ is a local minimum Therefore, if $w \leq 2w_0$, the solution is the full-specialization strategy $\{0, w\}$.

We now focus on the case where $w > 2w_0$, as $\widehat{x} = w_0 - \delta$, the condition on function z implies that $w\,(\delta) - \widehat{x}\,(\delta) - (w_0 + \delta)$ is an increasing function, where $w\,(\delta)$ is defined by $b'\,(\widehat{x}\,(\delta)) = b'\,(w\,(\delta) - \widehat{x}\,(\delta))$. Therefore, $w'\,(\delta) \geq 0$. Differentiating $b'\,(\widehat{x}\,(\delta)) = b'\,(w\,(\delta) - \widehat{x}\,(\delta))$ with respect to δ and recalling that $\widehat{x}\,(\delta) = w_0 - \delta$ lead to $(w'\,(\delta) + 1)\,b''\,(w\,(\delta) - \widehat{x}\,(\delta)) = -b''\,(\widehat{x}\,(\delta))$. As $w'\,(\delta) \geq 0$, this implies that $b''\,(w\,(\delta) - \widehat{x}\,(\delta)) + b''\,(\widehat{x}\,(\delta)) \geq 0$ and the asymmetric interior solution is a local minimum and is not a potential candidate for the optimal allocation. The solution to (1) belongs thus to $\{w, w/2\}$.■

G Proof of Lemma 5

We are going to prove this lemma in three steps. First, since b is a convex function on $[0, w_0]$, $b\left(\frac{w_0}{2}\right) < \frac{1}{2}b\,(w_0)$, meaning that $w_0 < w_2$.

Concerning the uniqueness of the thresholds defined by equation (6), let us consider function $f_i(x) = (i-1)\,b\left(\frac{x}{i-1}\right) - ib\left(\frac{x}{i}\right)$. The first derivative, $f_i'(x) = b'\left(\frac{x}{i-1}\right) - b'\left(\frac{x}{i}\right)$ is strictly negative when $x > iw_0$ (since b' is decreasing on $[w_0, +\infty[$) and is strictly positive when $x < (i-1)\,w_0$ (since b' is increasing on $[0, w_0]$). We hereafter show that there exists a unique a such that $f_i'(a) = 0$. Suppose by contradiction that there exist a and c, with $(i-1)\,w_0 < a < c < iw_0$ such that $b'\left(\frac{a}{i-1}\right) = b'\left(\frac{a}{i}\right)$ and $b'\left(\frac{c}{i-1}\right) = b'\left(\frac{c}{i}\right)$. $w_0 < \frac{a}{i-1} < \frac{c}{i-1} < \frac{iw_0}{i-1}$ implies that $b'(w_0) > b'\left(\frac{a}{i-1}\right) > b'\left(\frac{c}{i-1}\right) > b'\left(\frac{iw_0}{i-1}\right)$, and $\frac{(i-1)w_0}{i} < \frac{a}{i} < \frac{c}{i} < w_0$ implies that $b'\left(\frac{(i-1)w_0}{i}\right) < b'\left(\frac{a}{i}\right) < b'\left(\frac{c}{i}\right) < b'(w_0)$. This leads to a contradiction since $b'\left(\frac{a}{i-1}\right) = b'\left(\frac{a}{i}\right)$ and $b'\left(\frac{c}{i-1}\right) = b'\left(\frac{c}{i}\right)$. Therefore, a is unique and f_i is increasing on $[0, a]$ and decreasing on $[a, +\infty[$. As $f_i(0) = 0$, if a positive zero w_i exists to f_i, it is unique. f_i changes sign only once, from positive to negative.

Now, we prove that $w_i < w_{i+1}, \forall i = 2...n-1$. According to equation (6), w_{i+1} is such that $ib\left(\frac{w_{i+1}}{i}\right) = (i+1)\,b\left(\frac{w_{i+1}}{i+1}\right)$, or by dividing each member by x, $\frac{i}{w_{i+1}}b\left(\frac{w_{i+1}}{i}\right) = \frac{i+1}{w_{i+1}}b\left(\frac{w_{i+1}}{i+1}\right)$. We have already proved that function $x \mapsto b(x)/x$ is single peaked, increasing on $[0, w_1]$ and then decreasing on $[w_1, +\infty[$. Therefore, in order $\frac{i}{w_{i+1}}b\left(\frac{w_{i+1}}{i}\right) = \frac{i+1}{w_{i+1}}b\left(\frac{w_{i+1}}{i+1}\right)$ to hold, it must be the case that $\frac{w_{i+1}}{i+1} < w_1$ and $\frac{w_{i+1}}{i} > w_1$, or equivalently $iw_1 < w_{i+1} < (i+1)\,w_1$. To compare w_i and w_{i+1}, let us compute $ib\left(\frac{w_{i+1}}{i}\right) - (i-1)\,b\left(\frac{w_{i+1}}{i-1}\right)$. As $w_{i+1} > iw_1$, $w_1 < \frac{w_{i+1}}{i} < \frac{w_{i+1}}{i-1}$, $x \mapsto b(x)/x$ is decreasing and therefore $\frac{i}{w_{i+1}}b\left(\frac{w_{i+1}}{i}\right) > \frac{i-1}{w_{i+1}}b\left(\frac{w_{i+1}}{i-1}\right)$, meaning that $w_{i+1} > w_i$.■

H Proof of Proposition 4

We are going to prove this result using an induction argument.

First with two benefit functions, we know according to Proposition 1 that when $w < w_2$, then the optimal allocation is $x^*(w) = \{w, 0\}$ and when $w \geq w_2$, then the optimal allocation is $x^*(w) = \{w/2, w/2\}$.

Now, we suppose that the result holds when the investor has the choice between $n-1$ projects. Let us prove that it holds when the investor has n projects. According to the previous discussion, we maximize the investor's program in two steps. First, we solve

$$\max_{x_1,\ldots,x_{n-2}} b(x_1) + ... + b(x_{n-1}) + b(w - x_1 - ... - x_{n-1}).$$

As the result holds when the investor has the choice between $n-1$ projects, the solution to this program is known.

$$x_1^*(x_{n-1}, w) = \begin{cases} w - x_{n-1} & \text{if } w - x_{n-1} \leq w_2, \\ \frac{w - x_{n-1}}{2} & \text{if } w_2 < w - x_{n-1} \leq w_3, \\ ... & \\ \frac{w - x_{n-1}}{n-1} & \text{if } w - x_{n-1} > w_{n-1} \end{cases},$$

$$x_i^*(x_{n-1},w)=\begin{cases}0 & \text{if } w-x_{n-1}\le w_2,\\ \ldots & \\ \frac{w-x_{n-1}}{i} & \text{if } w_i<w-x_{n-1}\le w_{i+1},\quad \forall i=2,\ldots n-2.\\ \ldots & \\ \frac{w-x_{n-1}}{n-1} & \text{if } w-x_{n-1}>w_{n-1},\end{cases}$$

There remains to solve the second step. Suppose that $w_i < w - x_{n-1} \le w_{i+1}$. Therefore, the maximization comes down to

$$\max_{x_{n-1}} b(x_{n-1}) + ib\left(\frac{w-x_{n-1}}{i}\right),$$

subject to

$$\begin{array}{l} x_{n-1} < w - w_i, \\ x_{n-1} \ge w - w_{i+1}. \end{array} \tag{12}$$

The first order conditions, $b'(x_{n-1}) = b'((w-x_{n-1})/i)$, lead to the following candidate solutions

1. $x_{n-1} = (w-x_{n-1})/i$, then $x_{n-1} = \frac{w}{i+1}$ and condition (12) implies $w > \frac{i+1}{i}w_i$,
2. x_{n-1}^1 such that $b'\left(x_{n-1}^1\right) = b'\left(\left(w - x_{n-1}^1\right)/i\right)$ and $x_{n-1}^1 < w_0 < \left(w - x_{n-1}^1\right)/i$. We call x_{n-1}^1 the asymmetric interior solution 1,
3. x_{n-1}^2 such that $b'\left(x_{n-1}^2\right) = b'\left(\left(w - x_{n-1}^2\right)/i\right)$ and $\left(w - x_{n-1}^2\right)/i < w_0 < x_{n-1}^2$ (maximization problem (5) is not symmetric anymore). In this case, condition (12) leads to $w > w_0 + w_i$. We call x_{n-1}^2 the asymmetric interior solution 1. There are also the two corner solutions,
4. $x_{n-1} = w$ but this can be eliminated because condition (12) leads to $w_i < 0$,
5. $x_{n-1} = 0$.

We are going to prove that the two asymmetric interior solutions x_{n-1}^1 and x_{n-1}^2 do not exist. We first focus on x_{n-1}^1. To do so, we study the second order conditions (SOC) of the maximization program (5): $b''(x_{n-1}) + \frac{1}{i}b''\left(\frac{w-x_{n-1}}{i}\right) \le 0$. As b is a symmetric benefit function, $b''(x_{n-1}) + b''\left(\frac{w-x_{n-1}}{i}\right) = 0$. Therefore,

$$\begin{array}{rcl} b''(x_{n-1}) + \frac{1}{i}b''\left(\frac{w-x_{n-1}}{i}\right) & = & \frac{1}{i}\left(b''(x_{n-1}) + b''\left(\frac{w-x_{n-1}}{i}\right)\right) + \frac{i-1}{i}b''(x_{n-1}), \\ & = & \frac{i-1}{i}b''(x_{n-1}), \\ & > & 0 \text{ since } x_{n-1}^1 < w_0. \end{array}$$

Thus, the asymmetric interior solution, if it exists is a local minimum.

To eliminate the other asymmetric interior solution, we first prove an intermediate result, that is, when the benefit function is symmetric, then $w_i \ge iw_0, \forall i = 2, ..., n-1$. To do so, let us compute $(i-1)\,b\left(\frac{i}{i-1}w_0\right)-ib\left(w_0\right)$. Let us define δ such that $b\left(\frac{i}{i-1}w_0\right)=b\left(w_0+\delta\right)$, implying that $\delta=\frac{w_0}{i-1}$ and that $b\left(w_0-\delta\right)=b\left(\frac{i-2}{i-1}w_0\right)$. Since b is a symmetric function, $b\left(w_0+\delta\right)=2b\left(w_0\right)-b\left(w_0-\delta\right)$, and

$$\begin{aligned}
(i-1)\,b\left(\tfrac{i}{i-1}w_0\right)-ib\left(w_0\right) &= (i-1)\,b\left(w_0+\delta\right)-ib\left(w_0\right), \\
&= (i-1)\left(2b\left(w_0\right)-b\left(w_0-\delta\right)\right)-ib\left(w_0\right), \\
&= (i-2)\,b\left(w_0\right)-(i-1)\,b\left(\tfrac{i-2}{i-1}w_0\right), \\
&= (i-2)\left(b\left(w_0\right)-\tfrac{i-1}{i-2}b\left(\tfrac{i-2}{i-1}w_0\right)\right), \\
&\geq 0 \text{ since } b \text{ is convex on } [0,w_0].
\end{aligned}$$

Therefore, $w_i\geq iw_0, \forall i=2,...,n-1$. Let us now turn to the study of x_{n-1}^2. As b is symmetric, $x_{n-1}^2-\frac{w-x_{n-1}^2}{i}=2\left(x_{n-1}^2-w_0\right)$, $x_{n-1}^2=\frac{2ix_0-w}{i-1}$ has to be strictly greater than w_0 implying that $w<(i+1)\,w_0$. Moreover, condition (12), $w-x_{n-1}^2>w_i$, or equivalently $iw>(i-1)\,w_i+2ix_0$ has to be satisfied. In order these two inequalities to be compatible, we need $\frac{i-1}{i}w_i+2w_0<(i+1)\,w_0$, that is equivalent to $w_i<iw_0$. This not true, therefore, the asymmetric interior solution x_{n-1}^2 does not exist and the optimal strategy x_{n-1} is either equal to $\frac{w}{i+1}$ or to 0.■

I Proof of Proposition 5

As in the symmetric case, we are going to prove this result using an induction argument.

First of all with two benefit functions, we know according to Proposition 3 that when $w<w_2$, the optimal allocation is $x^*(w)=\{w,0\}$ and when $w\geq w_2$, the optimal allocation is $x^*(w)=\{w/2,w/2\}$.

Now, we suppose that the result holds when the investor has the choice between $n-1$ projects. Let us prove that it then holds when the investor has n projects. According to the previous discussion, we maximize the investor's program in two steps. First of all, we solve

$$\max_{x_1,\ldots,x_{n-2}} b\left(x_1\right)+...+b\left(x_{n-1}\right)+b\left(w-x_1-...-x_{n-1}\right).$$

As the result holds when the investor has the choice between $n-1$ projects, we know how to solve this program.

$$x_1^*\left(x_{n-1},w\right)=\begin{cases} w-x_{n-1} & \text{if } w-x_{n-1}\leq w_2, \\ \frac{w-x_{n-1}}{2} & \text{if } w_2<w-x_{n-1}\leq w_3, \\ \ldots & \\ \frac{w-x_{n-1}}{n-1} & \text{if } w-x_{n-1}>w_{n-1} \end{cases},$$

$$x_i^*\left(x_{n-1},w\right)=\begin{cases} 0 & \text{if } w-x_{n-1}\leq w_2, \\ \ldots & \\ \frac{w-x_{n-1}}{i} & \text{if } w_i<w-x_{n-1}\leq x_{i+1}, \quad \forall i=2,...n-2. \\ \ldots & \\ \frac{w-x_{n-1}}{n-1} & \text{if } w-x_{n-1}>w_{n-1}, \end{cases}$$

There remains to solve the second step. Suppose that $w_i < w - x_{n-1} \leq w_{i+1}$. Therefore, the maximization comes down to

$$\max_{x_{n-1}} b(x_{n-1}) + ib\left(\frac{w - x_{n-1}}{i}\right),$$

subject to

$$\begin{array}{l} x_{n-1} < w - w_i, \\ x_{n-1} \geq w - w_{i+1}. \end{array} \tag{13}$$

The first order conditions, $b'(x_{n-1}) = b'((w - x_{n-1})/i)$, lead to the following candidate solutions

1. $x_{n-1} = (w - x_{n-1})/i$, then $x_{n-1} = \frac{w}{i+1}$ and condition (13) leads to $w > \frac{i+1}{i} w_i$,
2. x_{n-1}^1 such that $b'\left(x_{n-1}^1\right) = b'\left(\left(w - x_{n-1}^1\right)/i\right)$ and $x_{n-1}^1 < w_0 < \left(w - x_{n-1}^1\right)/i$. We call x_{n-1}^1 the asymmetric interior solution 1,
3. x_{n-1}^2 such that $b'\left(x_{n-1}^2\right) = b'\left(\left(w - x_{n-1}^2\right)/i\right)$ and $\left(w - x_{n-1}^2\right)/i < w_0 < x_{n-1}^2$ (maximization problem (5) is not symmetric anymore). In this case, condition (13) leads to $w > w_0 + w_i$. We call x_{n-1}^2 the asymmetric interior solution 1. There are also the two corner solutions,
4. $x_{n-1} = w$ but this can be eliminated because condition (13) leads to $w_i < 0$,
5. $x_{n-1} = 0$.

We are going to prove that the two asymmetric interior solutions x_{n-1}^1 and x_{n-1}^2 do not exist. We first focus on x_{n-1}^1. If we define $\delta_1 \in [0, w_0]$ such that $x_{n-1}^1 = w_0 - \delta_1$, $z(\delta_1) = (w(\delta_1) - (i+1) w_0 - (i-1)\delta_1)/i$. As it is increasing by assumption, $w'(\delta_1) \geq i - 1$. But $w(\delta_1)$ is defined by $b'(w_0 - \delta_1) = b'((w(\delta_1) - w_0 + \delta_1)/i)$. Differentiating this expression with respect to δ_1 leads to

$$-b''(w_0 - \delta_1) = \frac{w'(\delta_1) + 1}{i} b''((w - w_0 + \delta_1)/i).$$

As $w'(\delta_1) \geq i - 1$, $(w'(\delta_1) + 1)/i \geq 1$, and the following inequalities hold:

$$\begin{array}{rcl} b''(w_0 - \delta_1) & = & -\frac{w'(\delta_1)+1}{i} b''((w - w_0 + \delta_1)/i), \\ & \geq & -b''((w - w_0 + \delta_1)/i), \\ & \geq & -(1/i) b''((w - w_0 + \delta_1)/i). \end{array}$$

Therefore $b''(w_0 - \delta_1) + \frac{1}{i} b''((w - w_0 + \delta_1)/i) \geq 0$ and the asymmetric interior solution 1, if it exists, is unique and is a local minimum.

Before studying x_{n-1}^2, let us prove an intermediate result, that is $w_i > iw_0$, $\forall i \geq 2$. To do so, we compute $ib(w_0) - (i-1)b\left(\frac{i}{i-1}w_0\right)$.

$$\begin{aligned} ib(w_0) - (i-1)b\left(\tfrac{i}{i-1}w_0\right) &= \int_0^{w_0} b'(w_0 - \delta)\,d\delta - (i-1)\int_0^{\frac{w_0}{i-1}} b'(w_0+\delta)\,d\delta, \\ &\leq \int_0^{w_0} b'(w_0-\delta)\,d\delta - \int_0^{\frac{w_0}{i-1}} b'(w_0-\delta)\,d\delta, \\ &\quad -(i-2)\int_0^{\frac{w_0}{i-1}} b'(w_0+\delta)\,d\delta, \\ &= \int_{\frac{w_0}{i-1}}^{w_0} b'(w_0-\delta)\,d\delta - (i-2)\int_0^{\frac{w_0}{i-1}} b'(w_0+\delta)\,d\delta, \\ &= -b\left(w_0 - \tfrac{w_0}{i-1}\right) - (i-2)\left(b\left(w_0+\tfrac{w_0}{i-1}\right) - b(w_0)\right), \\ &< 0. \end{aligned}$$

Therefore, $w_i > iw_0$, $\forall i \geq 2$. We thereafter focus on the solution x_{n-1}^2 such that $b'\left(x_{n-1}^2\right) = b'\left(\left(w - x_{n-1}^2\right)/2\right)$ and $\left(w - x_{n-1}^2\right)/2 < w_0 < x_{n-1}^2$. There exists $\delta_2 > 0$ such that $x_{n-1}^2 = w_0 + \delta_2$. In this case,

$$z(\delta_2) = (w(\delta_2) + (i-1)\delta_2 - (i+1)w_0)/i.$$

Condition (13) implies that $w - x_{n-1}^2 > w_i$. Let us define $F(\delta_2) = w(\delta_2) - w_0 - \delta_2 - w_i$ Condition (13) implies that $F(\delta_2) > 0, \forall \delta_2 > 0$. $F'(\delta_2) = w'(\delta_2) - 1$. Recall that $w(\delta_2)$ is defined by $b'(w_0 + \delta_2) = b'((w(\delta_2) - w_0 - \delta_2)/i)$. Taking the derivative of this expression with respect to δ_2, this leads to $b''(w_0+\delta_2) = 1/i\,(w'(\delta_2) - 1)\,b''((w(\delta_2) - w_0 - \delta_2)/i)$.

This last equality holds if and only if $w'(\delta_2) - 1 < 0$. Therefore $F(\delta_2)$ is strictly decreasing. $F(0) = w(0) - w_0 - w_i$ and $w(0) = (i+1)w_0$ for a LU benefit function. Therefore, $F(0) < 0$ and condition (13)is violated and this asymmetric interior solution 2 cannot exist for a LU benefit function.

The two asymmetric interior solutions have been eliminated, thus $x_{n-1} = \frac{w}{i+1}$ or $x_{n-1} = 0$ and the proposition is proved.■

J Proof of Lemma 6

The first two candidate solutions are the two corner solutions. Let us now focus on interior solutions characterized by the first order conditions $b'(w - \widehat{x}) = \frac{k}{j}b'\left(\frac{\widehat{x}}{j}\right)$. As $1 < k < j$, it follows that $b'\left(\frac{\widehat{x}}{j}\right) > b'(w - \widehat{x})$. There are four candidate solutions to this inequality:

1. $b'\left(\frac{\widehat{x}}{j}\right) > b'(w-\widehat{x})$ with $\frac{\widehat{x}}{j} > w_0$ and $w - \widehat{x} > w_0$. As b' is decreasing $\forall x > w_0$, it is the case if and only if $w_0 < \frac{\widehat{x}}{j} < w - \widehat{x}$. The second order condition, $\frac{k}{j^2}b''\left(\frac{\widehat{x}}{j}\right) + b''(w-\widehat{x}) \leq 0$, is satisfied because of the concavity of function b on $[w_0, +\infty[$. This candidate solution is therefore called the "interior solution 1".

2. $b'\left(\frac{\widehat{x}}{j}\right) > b'(w-\widehat{x})$ with $\frac{\widehat{x}}{j} < w_0$ and $w-\widehat{x} < w_0$. In this case, $\frac{k}{j^2}b''\left(\frac{\widehat{x}}{j}\right) + b''(w-\widehat{x}) \geq 0$ and this solution is a local minimum. It can therefore be skipped.

3. $b'\left(\frac{\widehat{x}}{j}\right) > b'(w-\widehat{x})$ with $\frac{\widehat{x}}{j} < w_0$ and $w-\widehat{x} > w_0$. This candidate turns out to be a potential solution if and only if the second order condition is satisfied, $\frac{k}{j^2}b''\left(\frac{\widehat{x}}{j}\right) + b''(w-\widehat{x}) \leq 0$.

4. $b'\left(\frac{\widehat{x}}{j}\right) > b'(w-\widehat{x})$ with $\frac{\widehat{x}}{j} > w_0$ and $w-\widehat{x} < w_0$. This candidate turns out to be a potential solution if and only if the second order condition is satisfied, $\frac{k}{j^2}b''\left(\frac{\widehat{x}}{j}\right) + b''(w-\widehat{x}) \leq 0$.■

K Proof of Lemma 7

The first result is an application of Lemma 1. Concerning the other results, they are similar to the results of Proposition 4. We are going to prove the following five results:

1. As w increases, one can never switch from the interior solution 2 to the allocation that gives the whole budget to project b,

2. As w increases, one can never switch from the interior solution 1 to the interior solution 2,

3. As w increases, one can never switch from the interior solution 1 to the allocation that gives the whole budget to project c,

4. As w increases, one can never switch from the interior solution 1 to the allocation that gives the whole budget to project b,

5. As w increases, one can never switch from the interior solution 1 to the interior solution 3.

We successively prove the five assertions.

1. Consider a range of w for which an interior solution 2 $(\widehat{x}^2, w-\widehat{x}^2)$ exists, where $\widehat{x}^2(w)$ is defined by $b'\left(w-\widehat{x}^2(w)\right) = \frac{k}{j}b'\left(\frac{\widehat{x}^2(w)}{j}\right)$ with $\frac{\widehat{x}^2(w)}{j} < w_0 < w-\widehat{x}^2(w)$. Let us study the function $w \longmapsto g_1(w) = b(w) - \left[kb(\frac{\widehat{x}^2(w)}{j}) + b(w-\widehat{x}^2(w))\right]$. Consider any solution $w=\overline{w}$ of equation $g_1(w)=0$. We show that this implies that $g_1'(\overline{w})$ be nonpositive. Indeed, by the envelope theorem, we have that

$$g_1'(\overline{w}) = b'(\overline{w}) - b'(\overline{w}-\widehat{x}^2(\overline{w})).$$

As $\overline{w} > w - \widehat{x}^2(w) > w_0$, b' is decreasing and $b'(\overline{w}) < b'\left(w - \widehat{x}^2(w)\right)$. Therefore, $g_1'(\overline{w})$ is nonpositive. It implies that if one switches between the allocation that gives the whole budget to project b to the interior solution 2, it can only be from the former to the latter.

2. Consider a range of w for which an interior solution 1 $(\widehat{x}^1, w - \widehat{x}^1)$ and an interior solution 2 $(\widehat{x}^2, w - \widehat{x}^2)$ exist, where $\widehat{x}^2(w)$ is defined by $b'\left(w - \widehat{x}^2(w)\right) = \frac{k}{j} b'\left(\frac{\widehat{x}^2(w)}{j}\right)$ with $\frac{\widehat{x}^2(w)}{j} < w_0 < w - \widehat{x}^2(w)$ and where $\widehat{x}^1(w)$ is defined by $b'\left(w - \widehat{x}^1(w)\right) = \frac{k}{j} b'\left(\frac{\widehat{x}^1(w)}{j}\right)$ and $w_0 < \frac{\widehat{x}^1(w)}{j} < w - \widehat{x}^1(w)$. Let us study the function $w \longmapsto g_2(w) = kb(\frac{\widehat{x}^2(w)}{j}) + b(w - \widehat{x}^2(w)) - \left[kb(\frac{\widehat{x}^1(w)}{j}) + b(w - \widehat{x}^1(w))\right]$. Consider any solution $w = \overline{w}$ of equation $g_2(w) = 0$. We show that this implies that $g_2'(\overline{w})$ be nonpositive. Indeed, by the envelope theorem, we have that

$$g_2'(\overline{w}) = b'(\overline{w} - \widehat{x}^2(\overline{w})) - b'(\overline{w} - \widehat{x}^1(\overline{w})).$$

As $\widehat{x}^2(\overline{w}) < jx_0 < \widehat{x}^1(\overline{w})$, $\overline{w} - \widehat{x}^2(\overline{w}) > \overline{w} - \widehat{x}^1(\overline{w}) > w_0$, by assumption. Therefore, b' is decreasing and $b'\left(\overline{w} - \widehat{x}^2(\overline{w})\right) < b'\left(\overline{w} - \widehat{x}^1(\overline{w})\right)$. Therefore, $g_2'(\overline{w})$ is nonpositive. It implies that if one switches between the interior solution 2 to the interior solution 1, it can only be from the former to the latter.

3. Consider a range of w for which an interior solution 1 $(\widehat{x}^1, w - \widehat{x}^1)$ exists, where $\widehat{x}^1(w)$ is defined by $b'\left(w - \widehat{x}^1(w)\right) = \frac{k}{j} b'\left(\frac{\widehat{x}^1(w)}{j}\right)$ and $w_0 < \frac{\widehat{x}^1(w)}{j} < w - \widehat{x}^1(w)$. Let us study the function $w \longmapsto g_3(w) = kb\left(\frac{w}{j}\right) - \left[kb(\frac{\widehat{x}^1(w)}{j}) + b(w - \widehat{x}^1(w))\right]$. Consider any solution $w = \overline{w}$ of equation $g_3(w) = 0$. We show that this implies that $g_3'(\overline{w})$ be nonpositive. Indeed, by the envelope theorem, we have that

$$g_3'(\overline{w}) = \frac{k}{j} b'(\frac{\overline{w}}{j}) - b'(\overline{w} - \widehat{x}^1(\overline{w})).$$

As $w_0 < \frac{\widehat{x}^1(\overline{w})}{j}$ and $\widehat{x}^1(\overline{w}) < w$, $b'\left(\frac{\overline{w}}{j}\right) < b'\left(\frac{\widehat{x}^1(\overline{w})}{j}\right)$ and $\frac{k}{j} b'\left(\frac{\overline{w}}{j}\right) < \frac{k}{j} b'\left(\frac{\widehat{x}^1(\overline{w})}{j}\right) = b'(\overline{w} - \widehat{x}^1(\overline{w}))$. Therefore, $g_3'(\overline{w})$ is nonpositive. It implies that if one switches between the allocation that gives the whole budget to project c to the interior solution 1, it can only be from the former to the latter.

4. Consider a range of w for which an interior solution 1 $(\widehat{x}^1, w - \widehat{x}^1)$ exists, where $\widehat{x}^1(w)$ is defined by $b'\left(w - \widehat{x}^1(w)\right) = \frac{k}{j} b'\left(\frac{\widehat{x}^1(w)}{j}\right)$ and $w_0 < \frac{\widehat{x}^1(w)}{j} < w - \widehat{x}^1(w)$. Let us study the function $w \longmapsto g_4(w) = b(w) - \left[kb(\frac{\widehat{x}^1(w)}{j}) + b(w - \widehat{x}^1(w))\right]$. Consider any solution $w = \overline{w}$ of equation $g_4(w) = 0$. We show that this implies that $g_4'(\overline{w})$ be nonpositive. Indeed, by the envelope theorem, we have that

$$g_4'(\overline{w}) = b'(\overline{w}) - b'(\overline{w} - \widehat{x}^1(\overline{w})).$$

As $\overline{w} > w - \widehat{x}^1(w) > w_0$, b' is decreasing and $b'(\overline{w}) < b'\left(w - \widehat{x}^1(w)\right)$. Therefore, $g_4'(\overline{w})$ is nonpositive. It implies that if one switches between the allocation that gives the whole budget to project b to the interior solution 1, it can only be from the former to the latter.

5. Consider a range of w for which an interior solution 1 $(\widehat{x}^1, w - \widehat{x}^1)$ and an interior solution 3 $(\widehat{x}^3, w - \widehat{x}^3)$ exist, where $\widehat{x}^3(w)$ is defined by $b'\left(w - \widehat{x}^3(w)\right) = \frac{k}{j} b'\left(\frac{\widehat{x}^3(w)}{j}\right)$ with $w - \widehat{x}^3(w) < w_0 < \frac{\widehat{x}^3(w)}{j}$ and where $\widehat{x}^1(w)$ is defined by $b'\left(w - \widehat{x}^1(w)\right) = \frac{k}{j} b'\left(\frac{\widehat{x}^1(w)}{j}\right)$ and $w_0 < \frac{\widehat{x}^1(w)}{j} < w - \widehat{x}^1(w)$. Let us study the function $w \longmapsto g_5(w) = kb(\frac{\widehat{x}^3(w)}{j}) + b(w - \widehat{x}^3(w)) - \left[kb(\frac{\widehat{x}^1(w)}{j}) + b(w - \widehat{x}^1(w))\right]$. Consider any solution $w = \overline{w}$ of equation $g_5(w) = 0$. We show that this implies that $g_5'(\overline{w})$ be nonpositive. Indeed, by the envelope theorem, we have that

$$g_5'(\overline{w}) = b'(\overline{w} - \widehat{x}^3(\overline{w})) - b'(\overline{w} - \widehat{x}^1(\overline{w})).$$

As $w - \widehat{x}^3(w) < w_0 < \frac{\widehat{x}^3(w)}{j}$ and $w_0 < \frac{\widehat{x}^1(w)}{j} < w - \widehat{x}^1(w)$, it follows that $w - \widehat{x}^3(w) < w_0 < \frac{\widehat{x}^1(w)}{j} < \frac{\widehat{x}^3(w)}{j} < w - \widehat{x}^1(w)$. $g_5'(\overline{w}) = b'(\overline{w} - \widehat{x}^3(\overline{w})) - b'(\overline{w} - \widehat{x}^1(\overline{w})) = \frac{k}{j} b'\left(\frac{\widehat{x}^3(\overline{w})}{j}\right) - \frac{k}{j} b'\left(\frac{\widehat{x}^1(\overline{w})}{j}\right)$. As $\frac{\widehat{x}^3(\overline{w})}{j} > \frac{\widehat{x}^1(\overline{w})}{j} > w_0$, b' is decreasing and $\frac{k}{j} b'\left(\frac{\widehat{x}^3(w)}{j}\right) < \frac{k}{j} b'\left(\frac{\widehat{x}^1(w)}{j}\right)$. Therefore, $g_5'(\overline{w})$ is nonpositive. It implies that if one switches between the interior solution 3 to the interior solution 1, it can only be from the former to the latter.■

L Proof of Proposition 6:

The first step is to prove that in this case, $w_i > iw_0$ $\forall i = 2...n$. The comes down to proving that

$$(i-1)\, b\left(\frac{iw_0}{i-1}\right) \geq ib(w_0).$$

We have:

$$\begin{aligned}
(i-1)\, b\left(\tfrac{iw_0}{i-1}\right) &= (i-1)\int_0^{w_0} b'(w_0 - \delta)\, d\delta + (i-1)\int_0^{\frac{w_0}{i-1}} b'(w_0 + \delta)\, d\delta \\
&\geq (i-1)\int_0^{w_0} b'(w_0 - \delta)\, d\delta + (i-1)\int_0^{\frac{w_0}{i-1}} b'(w_0 - \delta)\, d\delta \\
&= ib(w_0) + (i-1)\left(b(w_0) - b\left(\tfrac{i-2}{i-1} w_0\right)\right) - b(w_0) \\
&\geq ib(w_0)
\end{aligned}$$

The last inequality holds since $(i-1)\left(b(w_0) - b\left(\frac{i-2}{i-1} w_0\right)\right) - b(w_0) \geq 0$. Indeed, as $\frac{i-2}{i-1} w_0 < w_0 < w_1$, it follows that $\frac{b\left(\frac{i-2}{i-1} w_0\right)}{\frac{i-2}{i-1} w_0} \leq \frac{b(w_0)}{w_0}$.

Recall now that under the conditions stated in the proposition, the aggregate benefit function has the following expression:

$$B\left(w\right)=\begin{cases} b\left(w\right) & \text{if} \quad w<w_2 \\ 2b\left(\frac{w}{2}\right) & \text{if} \quad w_2\leq w<w_3 \\ ... & \\ ib\left(\frac{w}{i}\right) & \text{if} \quad w_i\leq w<w_{i+1} \\ ... & \\ nb\left(\frac{w}{n}\right) & \text{if} \quad w\geq w_n \end{cases}.$$

Thanks to the result we just proved, we find that B is convex on $[0, w_0]$ and concave on $[w_0, +\infty[$.■