A RUIN MODEL WITH DEPENDENCE BETWEEN CLAIM SIZES AND CLAIM INTERVALS *

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Abstract

We consider a generalization of the classical ruin model to a dependent setting, where the distribution of the time between two claim occurrences depends on the previous claim size. Exact analytical expressions for the Laplace transform of the ruin function are derived. The results are illustrated by several examples.

1 Introduction

The classical Cramer-Lundberg model to describe the surplus process of an insurance portfolio relies on the assumption of independence among claim sizes and between claim sizes and claim inter-occurrence times. However, in practice this assumption is often too restrictive and there is a need for more general models where the independence assumptions can be relaxed. Recently, various results have been obtained concerning the asymptotic behaviour of the probability of ruin for dependent claims. In the case of light-tailed claim sizes, Nyrhinen [12, 13] derived Lundberg-type limiting results using large deviations techniques and Müller and Pflug [11] introduced dependence orderings to relate the limiting ruin probabilities. The behaviour of the Lundberg exponent as a function of a dependence measure has been investigated in Albrecher and Kantor [2]. For heavy-tailed claim size distributions, the asymptotic behaviour of the ruin probability with dependent claims was studied e.g. in Asmussen et al. [5] and Mikosch and Samorodnitsky [9, 10]. However, all these results are of asymptotic nature and it is a challenging problem to obtain results on the probability of ruin in a dependent setting, also for smaller values of the initial capital.

Motivated by a related model in queueing theory (cf. Boxma and Perry [6]), in this paper a generalization of the classical ruin model is considered, where the distribution of the time between two claim occurrences depends on the previous claim size.

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For this specific dependent model, we derive exact solutions for the probability of survival by means of Laplace-Stieltjes transforms. This seems to be the first exact formula for the ruin probability in a continuous-time risk model allowing for dependency and thus should be viewed as a starting point for deriving analytical solutions in more general dependent scenarios. For example, we would like to consider (i) more general claim inter-occurrence distributions, and (ii) situations in which the claim sizes and claim inter-occurrence times depend on a common Markov chain (cf. [1, 8]). The paper is organized in the following way: In Section 2 we introduce the risk model and derive the exact expressions for the probability of survival. In Section 3, several related models that allow for a similar treatment are discussed. Section 4 contains some numerical illustrations and investigates the effect of ignoring the dependence structure.

2 The Model

Let us consider the following risk model for the surplus process $R(t)$ of an insurance portfolio:

$$R(t) = x + ct - \sum_{j=1}^{N(t)} B_j,$$

where x is the initial capital, c is the premium density which is assumed to be constant, B_j is the size of the jth claim and $N(t)$ is the number of claims up to time t. Let B_i be a sequence of i.i.d. random variables with distribution function $B(\cdot)$, mean β and Laplace-Stieltjes transform (LST) $\tilde{b}(\cdot)$. We assume the claim occurrence process to be of the following Markovian type: If a claim B_i is larger than a threshold T_i, then the time until the next claim is exponentially distributed with rate λ_1, otherwise it is exponentially distributed with rate λ_2. The quantities T_i are assumed to be i.i.d. random variables with distribution function $T(\cdot)$. In the sequel, B (T) shall denote a generic claim size (threshold) with distribution $B(\cdot)$ ($T(\cdot)$).

2.1 Exact Solutions

We are interested in the probability of survival $\phi(x)$, i.e. $\mathbb{P}(R(t) \geq 0 \,\forall\, t > 0 \,|\, R(0) = x)$. Let us assume that

$$\beta < c[\frac{\mathbb{P}(B > T)}{\lambda_1} + \frac{\mathbb{P}(B \leq T)}{\lambda_2}], \tag{1}$$

which is the net profit condition, and $\mathbb{P}(B > 0) = \mathbb{P}(T > 0) = 1$.
Let $\phi_i(x)$ $(i = 1, 2)$ denote the probability of survival with initial capital x given that the first claim occurs according to the exponential distribution with rate λ_i. Then we get

$$\phi_1(x) = (1-\lambda_1\,dt)\phi_1(x+c\,dt)+ \\ +\lambda_1\,dt\int_0^{x+c\,dt}\Big[\mathbb{P}(T\le y)\phi_1(x+c\,dt-y)+\mathbb{P}(T>y)\phi_2(x+c\,dt-y)\Big]dB(y).$$

Taylor expansion and rearranging yields

$$c\frac{d\phi_1}{dx}(x)-\lambda_1\phi_1(x)+\lambda_1\int_0^x\mathbb{P}(T\le y)\phi_1(x-y)dB(y)+ \\ +\lambda_1\int_0^x\mathbb{P}(T>y)\phi_2(x-y)dB(y)=0. \quad (2)$$

Similarly we obtain

$$c\frac{d\phi_2}{dx}(x)-\lambda_2\phi_2(x)+\lambda_2\int_0^x\mathbb{P}(T\le y)\phi_1(x-y)dB(y)+ \\ +\lambda_2\int_0^x\mathbb{P}(T>y)\phi_2(x-y)dB(y)=0. \quad (3)$$

Define, for Re $s\ge 0$:

$$\begin{aligned}\chi_1(s) &:= \mathbb{E}[e^{-sB}\,1_{(B>T)}]=\int_{x=0}^{\infty}e^{-sx}\,T(x)dB(x),\\ \chi_2(s) &:= \mathbb{E}[e^{-sB}\,1_{(B\le T)}]=\int_{x=0}^{\infty}e^{-sx}(1-T(x))dB(x),\end{aligned}$$

and denote the Laplace transform of $\phi_i(x)$ by

$$\tilde{\phi}_i(s):=\int_0^{\infty}e^{-sx}\phi_i(x)\,dx.$$

Note that $\chi_1(s)+\chi_2(s)=\tilde{b}(s)$.

From (2) and (3) it follows that for Re $s\ge 0$ we have

$$\begin{aligned}\tilde{\phi}_1(s)\Big[c\,s-\lambda_1+\lambda_1\chi_1(s)\Big]+\lambda_1\tilde{\phi}_2(s)\chi_2(s) &= c\phi_1(0+),\\ \tilde{\phi}_2(s)\Big[c\,s-\lambda_2+\lambda_2\chi_2(s)\Big]+\lambda_2\tilde{\phi}_1(s)\chi_1(s) &= c\phi_2(0+),\end{aligned}$$

which can further be simplified to

$$\tilde{\phi}_1(s) = \frac{c\phi_1(0+)\Big[c\,s - \lambda_2 + \lambda_2\chi_2(s)\Big] - c\lambda_1\chi_2(s)\phi_2(0+)}{\Big[c\,s - \lambda_1 + \lambda_1\chi_1(s)\Big]\cdot\Big[c\,s - \lambda_2 + \lambda_2\chi_2(s)\Big] - \lambda_1\lambda_2\chi_1(s)\chi_2(s)} \tag{4}$$

and

$$\tilde{\phi}_2(s) = \frac{c\phi_2(0+)\Big[c\,s - \lambda_1 + \lambda_1\chi_1(s)\Big] - c\lambda_2\chi_1(s)\phi_1(0+)}{\Big[c\,s - \lambda_1 + \lambda_1\chi_1(s)\Big]\cdot\Big[c\,s - \lambda_2 + \lambda_2\chi_2(s)\Big] - \lambda_1\lambda_2\chi_1(s)\chi_2(s)}. \tag{5}$$

Note that the denominators on the right-hand side of (4) and (5) coincide.

Remark: If we set $\lambda_1 = \lambda_2 := \lambda$ in (4) we obtain

$$\begin{aligned}\tilde{\phi}_1(s) &= \frac{c\phi_1(0+)\Big[c\,s - \lambda + \lambda\chi_2(s)\Big] - c\lambda\chi_2(s)\phi_1(0+)}{(c\,s - \lambda + \lambda\chi_1(s))(c\,s - \lambda + \lambda\chi_2(s)) - \lambda^2\chi_1(s)\chi_2(s)} \\ &= \frac{c\phi_1(0+)}{c\,s - \lambda + \lambda\tilde{b}(s)},\end{aligned}$$

and thus we retain the classical Pollaczek-Khintchine formula for the independent setting.

For complete solution we now need to determine the quantities $\phi_i(0+)$. Since $\lim_{x\to\infty}\phi_i(x) = 1$ we have

$$\lim_{s\to 0} s\tilde{\phi}_i(s) = 1 \quad (i = 1, 2). \tag{6}$$

Using (6) w.l.o.g. in (4) (equation (5) would lead to the same result), we obtain

$$\begin{aligned}1 &= \lim_{s\to 0}\left(s\,\frac{c\phi_1(0+)\Big[c\,s - \lambda_2 + \lambda_2\chi_2(s)\Big] - c\lambda_1\chi_2(s)\phi_2(0+)}{\Big[c\,s - \lambda_1 + \lambda_1\chi_1(s)\Big]\cdot\Big[c\,s - \lambda_2 + \lambda_2\chi_2(s)\Big] - \lambda_1\lambda_2\chi_1(s)\chi_2(s)}\right) \\ &= \frac{c\lambda_2\phi_1(0+)(-1+\chi_2(0)) - c\lambda_1\chi_2(0)\phi_2(0+)}{\lim_{s\to 0}\frac{(c\,s-\lambda_1+\lambda_1\chi_1(s))\cdot(c\,s-\lambda_2+\lambda_2\chi_2(s))-\lambda_1\lambda_2\chi_1(s)\chi_2(s)}{s}} \\ &= \frac{c\lambda_2\phi_1(0+)(-1+\chi_2(0)) - c\lambda_1\chi_2(0)\phi_2(0+)}{c\lambda_1(\chi_1(0)-1) + c\lambda_2(\chi_2(0)-1) - \lambda_1\lambda_2(\chi_1'(0)+\chi_2'(0))\Big]}.\end{aligned} \tag{7}$$

Now we can use the relations $\chi_2(0) = \mathbb{P}(B \le T)$, $\chi_1(0) = \mathbb{P}(B > T)$ and thus $\chi_1(0) + \chi_2(0) = 1$ and also $\mathbb{E}(B\,1_{(B\le T)}) = -\chi_2'(0)$, $\mathbb{E}(B\,1_{(B>T)}) = -\chi_1'(0)$ and $\beta = -\chi_1'(0) - \chi_2'(0)$. In this way (7) can be substantially simplified yielding

$$(1 - \phi_1(0+))\frac{\mathbb{P}(B > T)}{\lambda_1} + (1 - \phi_2(0+))\frac{\mathbb{P}(B \le T)}{\lambda_2} = \frac{\beta}{c}. \tag{8}$$

Remark: For the special case $\lambda_1 = \lambda_2 := \lambda$ we obtain from (8)

$$\phi_1(0+) = \phi_2(0+) = \frac{c - \lambda\beta}{c}, \tag{9}$$

which is the well-known formula for the survival probability with zero initial capital in the classical independent case.

We now need a second equation for $\phi_1(0+)$ and $\phi_2(0+)$. Using Rouché's theorem, one can show the following:

Lemma 1. *The denominator of* (4) *has exactly one zero* σ *with Re* $\sigma > 0$.

Proof. Rewrite the denominator of (4) and (5) as $cs(h_1(s) + h_2(s))$, in which

$$h_1(s) \;:=\; cs - \lambda_1 - \lambda_2,$$

$$h_2(s) \;:=\; \lambda_1\chi_1(s) + \lambda_2\chi_2(s) + \frac{\lambda_1\lambda_2\beta}{c}\frac{1-\tilde{b}(s)}{\beta s}.$$

We wish to show that this denominator has exactly one zero for Re $s > 0$; note that the behaviour of $\phi_i(s)$ at $s = 0$ has already been analysed and exploited in (7). Let us now apply Rouché's theorem to the closed contour C, consisting of the imaginary axis from $-ir$ to $+ir$ and a semi-circle in the right halfplane with radius r and origin O; we shall let $r \to \infty$. $h_1(s)$ and $h_2(s)$ are analytic inside C; notice that $\frac{1-\tilde{b}(s)}{\beta s}$ is the LST of $\int_0^x \frac{1-B(y)}{\beta}\mathrm{d}y$ which is the residual (forward recurrence) claim size distribution. Hence it is analytic and (as will be used below) bounded by one in absolute value in the right halfplane. Furthermore, $h_1(s)$ has exactly one zero inside C for r large enough. For the application of Rouché's theorem it remains to show that $|h_1(s)| > |h_2(s)|$ on C. This is clearly true on the semi-circle. On the imaginary axis, $|h_1(s)| \geq \lambda_1 + \lambda_2$, whereas, under the condition (1),

$$\begin{aligned} |h_2(s)| &\leq \lambda_1\chi_1(0) + \lambda_2\chi_2(0) + \frac{\lambda_1\lambda_2\beta}{c} \\ &< \lambda_1 + \lambda_2. \end{aligned} \tag{10}$$

□

In fact, it is easy to see that σ is real, with $0 < \sigma < \frac{\lambda_1+\lambda_2}{c}$, since $h_1(0) + h_2(0) < 0$ and $h_1(\frac{\lambda_1+\lambda_2}{c}) + h_2(\frac{\lambda_1+\lambda_2}{c}) > 0$.
Since $\tilde{\phi}_i(s)$ is an analytic function for Re $s \geq 0$, σ must also be a zero of the numerators of (4) and (5). In both cases this yields the same relation between $\phi_1(0+)$ and $\phi_2(0+)$, namely

$$\phi_2(0+) = \frac{c\sigma - \lambda_2 + \lambda_2\chi_2(\sigma)}{\lambda_1\chi_2(\sigma)}\phi_1(0+) = \frac{\lambda_2\chi_1(\sigma)}{c\sigma - \lambda_1 + \lambda_1\chi_1(\sigma)}\phi_1(0+). \tag{11}$$

Combined with (4), (5) and (8), this completes the determination of $\tilde{\phi}_i(s)$, $i = 1, 2$.

Remark. Note that whenever $\chi_i(s)$ $(i = 1, 2)$ are rational functions (which is e.g. fulfilled if the corresponding conditional distributions are phase-type), then the Laplace-Stieltjes transforms (4) and (5) can be inverted explicitly to yield exact formulae for $\phi_i(x)$ $(i = 1, 2)$ (see e.g. Spiegel [14]). Since the class of phase-type distributions is dense (in the sense of weak convergence) in the class of all distributions on the positive half-line, one can approximate any given distribution arbitrarily closely by a phase-type distribution and use the exact solutions above (algorithms for phase-type fitting are e.g. discussed in Asmussen [3]).

Example 1. For the special case $T \sim Exp(\mu)$ we obtain

$$\begin{aligned} \chi_2(s) &= \int_{x=0}^{\infty} e^{-sx} e^{-\mu x} dB(x) = \tilde{b}(s+\mu), \\ \chi_1(s) &= \tilde{b}(s) - \tilde{b}(s+\mu). \end{aligned}$$

If in addition $B \sim Exp(\nu)$, with $\nu = 1/\beta$, then we have

$$\chi_2(s) = \frac{\nu}{\nu + s + \mu}, \quad \chi_1(s) = \frac{\nu}{\nu + s} - \frac{\nu}{\nu + s + \mu},$$

and thus σ in (11) is the unique solution s with Re $s > 0$ of

$$\Big(c\,s + \frac{\lambda_1 \mu \nu}{(\nu + s)(\nu + \mu + s)} - \lambda_1\Big)\Big(c\,s + \frac{\lambda_2 \nu}{\nu + \mu + s} - \lambda_2\Big) - \frac{\lambda_1 \lambda_2 \mu \nu^2}{(\nu + \mu + s)^2 (\nu + s)} = 0.$$

Since the Laplace-Stieltjes transforms are rational functions in this case, they can be inverted explicitly for any given parameter values (see Section 4 for a specific numerical example).

Example 2. For a deterministic threshold (i.e. $T_i = T^*$ a.s. for all $i \geq 1$ and some constant $T^* > 0$) and exponential claim sizes ($B_i \sim Exp(\nu)$) we obtain

$$\chi_1(s) = \frac{\nu}{\nu + s} e^{-(\nu+s)T^*} \quad \text{and} \quad \chi_2(s) = \frac{\nu}{\nu + s}(1 - e^{-(\nu+s)T^*}). \tag{12}$$

2.2 Comparison to Model with Independence

The availability of analytical solutions for the survival probability allows one to investigate the error produced by neglecting a dependency structure of the above kind. Indeed, assuming independence when in fact the dependency structure of Model 1 is present, an estimation of distribution of the inter-occurrence time W_i would lead to the mixing density

$$f_{W_i}(x) = \mathbb{P}(B_i > T_i)\lambda_1 e^{-\lambda_1 x} + \mathbb{P}(B_i \leq T_i)\lambda_2 e^{-\lambda_2 x},$$

i.e. one would assume to have a renewal model (also called Sparre Andersen risk model) with a hyper-exponential inter-arrival distribution. For such a model, the Lundberg coefficient R, given it exists, can easily be determined as the unique positive solution of $\tilde{b}(-R)\tilde{w}(cR) = 1$, where $\tilde{w}(\cdot)$ denotes the Laplace transform of $f_{W_i}(x)$ (see e.g. Asmussen [4]). An illustrative example for the difference of the corresponding survival probabilities is given in Section 4.

3 Related Models

In the following, we list a number of related dependency models for which exact solutions for the survival probability can be derived in an analogous way:

3.1 Model 2

Let for every $t > 0$ the risk process be in one of the two states $i = 1, 2$, corresponding to the rate λ_i of the exponential distribution for the time until the next claim occurs. At the time of a claim occurrence the state of the system may change depending on the corresponding claim size. If a claim B_j is smaller than a threshold T_j, then the state of the risk process changes, otherwise it does not. The quantities T_j are again assumed to be i.i.d. random variables with distribution function $T(\cdot)$. The net profit condition in this model is

$$2\beta < c(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}). \tag{13}$$

Then the analysis of $\phi_i(x)$ (which is the survival probability with initial capital x, given that the system starts out in state i) is analogous to the previous section and we obtain

$$c\frac{d\phi_1}{dx}(x) - \lambda_1\phi_1(x) + \lambda_1 \int_0^x \mathbb{P}(T \le y)\phi_1(x-y)dB(y) + \\ + \lambda_1 \int_0^x \mathbb{P}(T > y)\phi_2(x-y)dB(y) = 0 \tag{14}$$

and

$$c\frac{d\phi_2}{dx}(x) - \lambda_2\phi_2(x) + \lambda_2 \int_0^x \mathbb{P}(T \le y)\phi_2(x-y)dB(y) + \\ + \lambda_2 \int_0^x \mathbb{P}(T > y)\phi_1(x-y)dB(y) = 0, \tag{15}$$

from which it follows that for Re $s \ge 0$

$$\tilde{\phi}_1(s) = \frac{c\phi_1(0+)\Big[c\,s - \lambda_2 + \lambda_2\chi_1(s)\Big] - c\lambda_1\chi_2(s)\phi_2(0+)}{\Big[c\,s - \lambda_1 + \lambda_1\chi_1(s)\Big]\cdot\Big[c\,s - \lambda_2 + \lambda_2\chi_1(s)\Big] - \lambda_1\lambda_2\chi_2^2(s)} \tag{16}$$

and

$$\tilde{\phi}_2(s) = \frac{c\phi_2(0+)\Big[c\,s - \lambda_1 + \lambda_1\chi_1(s)\Big] - c\lambda_2\chi_2(s)\phi_1(0+)}{\Big[c\,s - \lambda_1 + \lambda_1\chi_1(s)\Big]\cdot\Big[c\,s - \lambda_2 + \lambda_2\chi_1(s)\Big] - \lambda_1\lambda_2\chi_2^2(s)}, \tag{17}$$

where $\tilde{\phi}_i(s)$ is again the Laplace transform of $\phi_i(x)$. Note that the denominators on the right-hand side of (16) and (17) again coincide.
Let us now determine $\phi_1(0+)$ and $\phi_2(0+)$. As for Model 1, one equation for these two unknowns follows from $\lim_{s\to 0} s\tilde{\phi}_i(s) = 1$, yielding

$$\lambda_2(1-\phi_1(0+)) + \lambda_1(1-\phi_2(0+)) = 2\frac{\lambda_1\lambda_2}{c}\beta. \tag{18}$$

A second equation is obtained by noticing that there is a real number $\tau \in (0, \frac{\lambda_1+\lambda_2}{c})$ that makes the denominator of (16), and similarly (17), zero. Indeed, write the denominator of (16) and (17) as $cs(k_1(s)+k_2(s))$, in which

$$k_1(s) := cs - \lambda_1 - \lambda_2,$$

$$k_2(s) := (\lambda_1+\lambda_2)\chi_1(s) + \frac{\lambda_1\lambda_2}{cs}[(1-\chi_1(s))^2 - \chi_2(s)^2].$$

Now observe that $k_1(0)+k_2(0) < 0$ if the net profit condition (13) holds, whereas $k_1(\frac{\lambda_1+\lambda_2}{c}) + k_2(\frac{\lambda_1+\lambda_2}{c}) > 0$. Since $\tilde{\phi}_i(s)$ is an analytic function for Re $s \geq 0$, τ must also be a zero of the numerators of (16) and (17). In both cases this yields the same relation between $\phi_1(0+)$ and $\phi_2(0+)$, namely

$$\phi_2(0+) = \frac{c\tau - \lambda_2 + \lambda_2\chi_1(\tau)}{\lambda_1\chi_2(\tau)}\phi_1(0+) = \frac{\lambda_2\chi_2(\tau)}{c\tau - \lambda_1 + \lambda_1\chi_1(\tau)}\phi_1(0+). \tag{19}$$

We have not proved that τ is the *only* zero of the denominator of (16) for Re $s > 0$ (application of Rouché's theorem seems much more involved here than in the case of Model 1). However, that is not needed: If (13) holds, then there should be unique solutions $\phi_1(x)$ and $\phi_2(x)$ of the integro-differential equations (14) and (15). $\tilde{\phi}_1(s)$ and $\tilde{\phi}_2(s)$ as given in (16) and (17) with $\phi_1(0+)$ and $\phi_2(0+)$ given by (18) and (19) are the Laplace transforms of functions $\phi_1(x)$ and $\phi_2(x)$ that satisfy those integro-differential equations, so we need not look further. See Cohen and Down [7] for more general ideas about handling queueing systems without taking recourse to Rouché's theorem.

Remark: For the special case $\lambda_1 = \lambda_2 := \lambda$ we again obtain from (18) the survival probability (9) with zero initial capital in the independent case.

If, alternatively, the state of the risk process changes at the time of a claim occurrence, given that B_j is larger than a threshold T_j and remains in its state otherwise, we get instead of (16) and (17):

$$\tilde{\phi}_1(s) = \frac{c\phi_1(0+)\Big[c\,s - \lambda_2 + \lambda_2\chi_2(s)\Big] - c\lambda_1\chi_1(s)\phi_2(0+)}{\Big[c\,s - \lambda_1 + \lambda_1\chi_2(s)\Big]\cdot\Big[c\,s - \lambda_2 + \lambda_2\chi_2(s)\Big] - \lambda_1\lambda_2\chi_1^2(s)}, \tag{20}$$

and

$$\tilde{\phi}_2(s) = \frac{c\phi_2(0+)\Big[c\,s - \lambda_1 + \lambda_1\chi_2(s)\Big] - c\lambda_2\chi_1(s)\phi_1(0+)}{\Big[c\,s - \lambda_1 + \lambda_1\chi_2(s)\Big]\cdot\Big[c\,s - \lambda_2 + \lambda_2\chi_2(s)\Big] - \lambda_1\lambda_2\chi_1^2(s)}, \tag{21}$$

and $\phi_1(0+)$ and $\phi_2(0+)$ follow from (18) and

$$\phi_2(0+) = \frac{c\zeta - \lambda_2 + \lambda_2\chi_2(\zeta)}{\lambda_1\chi_1(\zeta)}\phi_1(0+) = \frac{\lambda_2\chi_1(\zeta)}{c\zeta - \lambda_1 + \lambda_1\chi_2(\zeta)}\phi_1(0+), \tag{22}$$

where, similar to τ above, ζ is the real zero of the denominator of (20) in $(0, \frac{\lambda_1+\lambda_2}{c})$.

3.2 Another Variant of Model 1

Let us now look at the following variant of Model 1 with applications in reinsurance: As in Model 1, we take the claim intervals $W_{i+1} \sim Exp(\lambda_1)$ if $B_i > T_i$, and $W_{i+1} \sim Exp(\lambda_2)$ if $B_i \leq T_i$ for all $i \geq 1$, where T_i are again i.i.d. threshold variables. However, now the actual claim payment is $\min(B_i, T_i)$. Thus the threshold T_i can be interpreted as the retention level of an XL-type reinsurance on the claim size (note that a deterministic threshold is a special case of this model). For the analysis of this model, we have to introduce the Laplace-Stieltjes transform

$$\psi(s) := \mathbb{E}[e^{-sT}\, 1_{(T<B)}] = \int_{x=0}^{\infty} e^{-sx}(1 - B(x))dT(x).$$

Note that $\chi_2(s) + \psi(s) = \mathbb{E}[e^{-s\min(B,T)}]$ and thus $\mathbb{E}[\min(B,T)] = -\chi_2'(0) - \psi'(0)$. A similar derivation along the lines of Section 2.1 leads to

$$\tilde{\phi}_1(s) = \frac{c\phi_1(0+)\Big[c\,s - \lambda_2 + \lambda_2\chi_2(s)\Big] - c\lambda_1\chi_2(s)\phi_2(0+)}{\Big[c\,s - \lambda_1 + \lambda_1\psi(s)\Big]\cdot\Big[c\,s - \lambda_2 + \lambda_2\chi_2(s)\Big] - \lambda_1\lambda_2\psi(s)\chi_2(s)} \tag{23}$$

and

$$\tilde{\phi}_2(s) = \frac{c\phi_2(0+)\Big[c\,s - \lambda_1 + \lambda_1\psi(s)\Big] - c\lambda_2\psi(s)\phi_1(0+)}{\Big[c\,s - \lambda_1 + \lambda_1\psi(s)\Big]\cdot\Big[c\,s - \lambda_2 + \lambda_2\chi_2(s)\Big] - \lambda_1\lambda_2\psi(s)\chi_2(s)}, \tag{24}$$

where $\phi_i(0+)$ (i=1,2) are the solutions of the two equations

$$\lambda_2\mathbb{P}(B > T)(1 - \phi_1(0+)) + \lambda_1\mathbb{P}(B \leq T)(1 - \phi_2(0+)) = \frac{\lambda_1\lambda_2}{c}\mathbb{E}[\min(B,T)] \tag{25}$$

and

$$\phi_2(0+) = \frac{c\gamma - \lambda_2 + \lambda_2\chi_2(\gamma)}{\lambda_1\chi_2(\gamma)}\phi_1(0+) = \frac{\lambda_2\psi(\gamma)}{c\gamma - \lambda_1 + \lambda_1\psi(\gamma)}\phi_1(0+),$$

where here γ is the unique positive zero of the denominator of (23). Note that the existence and uniqueness of γ can, as in Model 1, be easily shown by Rouché type arguments.

Remark. In Boxma and Perry [6] a queueing model with the above dependence structure between service and subsequent interarrival times has been investigated. However, sample path duality between the corresponding workload process and our risk process does not hold for this particular dependence structure, as can also be seen from the difference between (23) and (24) and the formulae (3.8) and (3.9) of [6].

4 Numerical Illustrations

Example 1. Let $T \sim Exp(2)$, $B \sim Exp(1)$, $c = 2$, $\lambda_1 = 3$, $\lambda_2 = 1$. The net profit condition (1) is obviously fulfilled. Then the inversion of the Laplace transforms (4) and (5) yields

$$\begin{aligned}\phi_1(x) &= 1 - 0.007\, e^{-3.161\, x} - 0.938\, e^{-0.065\, x},\\ \phi_2(x) &= 1 - 0.003\, e^{-3.161\, x} - 0.867\, e^{-0.065\, x},\end{aligned} \tag{26}$$

where here and in the sequel all numerical values are rounded to their last digit (cf. Figure 1a).

Let us now compare (26) to $\phi(x)$ in a model with the assumption of independence as described in Section 2.2. The inter-arrival density in the independent model is then given by $f_{W_i}(x) = 2\, e^{-3x} + \frac{1}{3} e^{-x}$. The Lundberg exponent in this renewal risk model is the positive solution of

$$\frac{1}{1-R}\left(\frac{1}{3+3cR} + \frac{2}{3+cR}\right) = 1,$$

i.e. $R = 0.077$. In this specific example, there is even an analytical solution for the survival probability in the corresponding renewal model available, since the claim size distribution is exponential. This solution can be derived utilizing a sample path duality to a related queuing process (see e.g. Asmussen [4]) and we obtain

$$\phi_{ind}(x) = 1 - 0.923 e^{-0.077x}.$$

This should be compared with the stationary version of the dependent setting

$$\phi_{dep}(x) = \frac{2}{3}\phi_1(x) + \frac{1}{3}\phi_2(x) = 1 - 0.006 e^{-3.161x} - 0.915 e^{-0.065x}.$$

Note that concerning the asymptotic behavior, the Lundberg exponent of $\phi_{dep}(x)$ is smaller than the one of $\phi_{ind}(x)$, i.e. ignoring the dependence structure underestimates the inherent risk, especially for larger values of initial capital x (cf. Figure 1b).

Example 2. Let $T \sim Exp(1)$, $B \sim Exp(1)$, $c = 2$, $\lambda_1 = 1$, $\lambda_2 = 2$. Then the inversion of the Laplace transforms (4) and (5) yields

$$\begin{aligned}\phi_1(x) &= 1 - 0.632\, e^{-0.355\, x} + 0.017\, e^{-1.889\, x},\\ \phi_2(x) &= 1 - 0.798\, e^{-0.355\, x} + 0.028\, e^{-1.889\, x}.\end{aligned} \tag{27}$$

If we again compare (27) to $\phi(x)$ in a model with the assumption of independence, then the inter-arrival density is now given by $f_{W_i}(x) = e^{-2x} + \frac{1}{2} e^{-x}$. The Lundberg exponent in this renewal risk model is the positive solution of

$$\frac{1}{1-R}\left(\frac{1}{2(1+cR)} + \frac{1}{2+cR}\right) = 1,$$

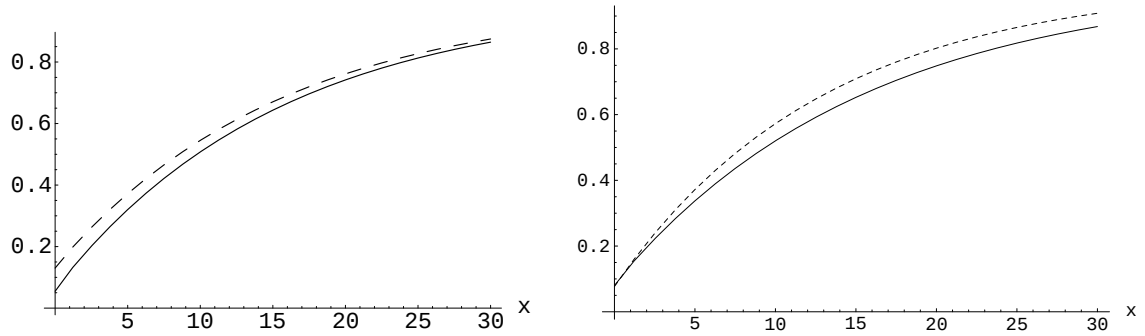

Figure 1: Survival probabilities in Example 1. Left: $\phi_1(x)$ (solid line) and $\phi_2(x)$ (dashed line). Right: $\phi_{dep}(x)$ (solid line) and $\phi_{ind}(x)$ (dotted line)

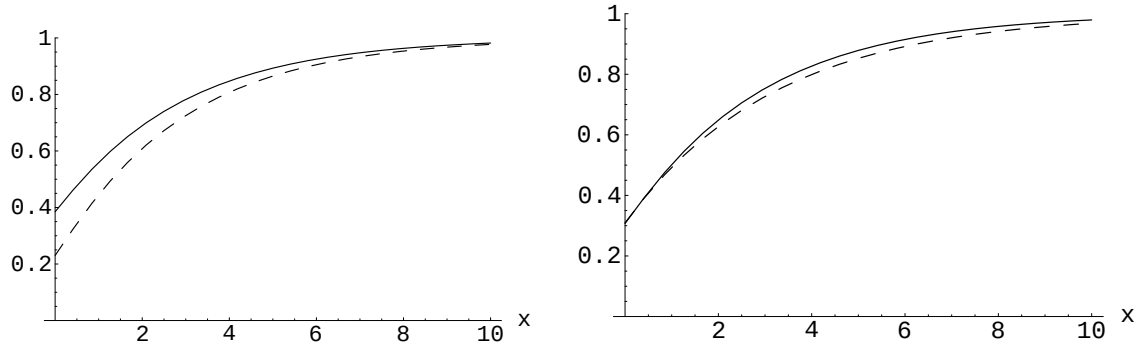

Figure 2: Survival probabilities in Example 2. Left: $\phi_1(x)$ (solid line) and $\phi_2(x)$ (dashed line). Right: $\phi_{dep}(x)$ (solid line) and $\phi_{ind}(x)$ (dotted line)

i.e. $R = 0.309$. Again, we even have an analytical solution for $\phi_{ind}(x)$ in the corresponding renewal model available:

$$\phi_{ind}(x) = 1 - 0.691e^{-0.309x}.$$

The stationary version of the dependent setting yields

$$\phi_{dep}(x) = \frac{1}{2}\phi_1(x) + \frac{1}{2}\phi_2(x) = 1 - 0.715e^{-0.355x} + 0.023e^{-1.889x}.$$

Note that in this case, the Lundberg exponent of $\phi_{ind}(x)$ is smaller than the one of $\phi_{dep}(x)$, i.e. the independent setting is "more dangerous". This is, heuristically, due to the fact that for this choice of parameters a larger claim is likely to be followed by a longer inter-occurrence time (see also Figure 2b).

Example 3. Let us again consider the setting of Example 2, but now with a deterministic threshold $T_i = 1$ a.s. for all $i \geq 1$ (so the value of T_i equals the expected value of the threshold variable of Example 2). According to (12) we have $\chi_1(s) = \frac{1}{1+s}e^{-s-1}$ and $\chi_2(s) = \frac{1}{1+s}(1 - e^{-s-1})$ and we obtain $\phi_1(0+) = 0.337$ and $\phi_2(0+) = 0.190$. The resulting Laplace transforms (4) and (5) can easily be inverted numerically by a Bromwich contour integration. Table 1 illustrates the fact that the distribution of the threshold has a significant effect on the survival probabilities.

	$T = 1$		$T \sim Exp(1)$	
x	$\phi_1(x)$	$\phi_2(x)$	$\phi_1(x)$	$\phi_2(x)$
0	0.337	0.190	0.384	0.230
0.5	0.419	0.285	0.477	0.343
1	0.499	0.380	0.559	0.445
1.5	0.570	0.469	0.630	0.533
2	0.632	0.545	0.690	0.609
2.5	0.684	0.610	0.740	0.672
3	0.730	0.666	0.782	0.725
3.5	0.768	0.714	0.818	0.770
4	0.802	0.755	0.847	0.807
4.5	0.830	0.790	0.872	0.839
5	0.854	0.820	0.893	0.865

Table 1: Comparison of $\phi_i(x)$ for Examples 2 and 3

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