Year : 2016

ASSET LIABILITY MANAGEMENT AND JOINT MORTALITY MODELLING IN OLD-AGE INSURANCE

TOUKOUROU Youssouf

UNIL | Université de Lausanne

FACULTÉ DES HAUTES ÉTUDES COMMERCIALES

DÉPARTEMENT DE SCIENCES ACTUARIELLES

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THÈSE DE DOCTORAT

présentée à la

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pour l'obtention du grade de
Docteur en Sciences Actuarielles

par

Youssouf TOUKOUROU

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La thèse est intitulée :

ASSET LIABILITY MANAGEMENT AND JOINT MORTALITY MODELLING IN OLD-AGE INSURANCE

Lausanne, le 13 juin 2016

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Université de Lausanne
Faculté des Hautes Études Commerciales

Doctorat en Sciences Actuarielles

Par la présente, je certifie avoir examiné la thèse de doctorat de

Youssouf TOUKOUROU

Sa thèse remplit les exigences liées à un travail de doctorat.
Toutes les révisions que les membres du jury et le soussigné ont demandées durant le colloque de thèse ont été prises en considération et reçoivent ici mon approbation.

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Université de Lausanne
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Doctorat en Sciences Actuarielles

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Declaration of Authorship

I, Youssouf A. F. TOUKOUROU, declare that this thesis titled, “ASSET LIABILITY MANAGEMENT AND JOINT MORTALITY MODELLING IN OLD-AGE INSURANCE” and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:

"It's not because things are difficult that we dare not venture. It's because we dare not venture that they are difficult."

Seneca

"Celui qui trouve sans chercher est celui qui a longtemps cherché sans trouver."

Gaston Bachelard

"In the end, it's not the years in your life that count. It's the life in your years."

Abraham Lincoln

Acknowledgements

The completion of this thesis would not have been possible without the valuable inputs, support and information provided by many people to whom I am indebted.

First and foremost, I am deeply grateful to my supervisor Prof. François Dufresne for giving me the opportunity to work on these interesting topics and for all his extensive guidance and support during my PhD study.

I would also like to express my gratitude to Prof. Enkelejd Hashorva for his collaboration and advices on a part of my thesis. Also, a special thanks to Gildas Ratovomirija, Prof. Joël Wagner and Dr. Stefan Thonhauser for all the fruitful discussions we had about this thesis.

Many thanks to my PhD committee: Prof. Frédéric Planchet, Prof. Joël Wagner and Prof. Dufresne who took the time to read my thesis and provided some constructive feedbacks which improved its quality.

I would also like to express my gratitude to all the members of the Department of Actuarial Science, especially Héloise, Arian, Gildas, Peiman, Lanpeng, Jevgenijs, Maissa, Michel, William, etc.

Finally, I would like to thank my family and friends for the support they provided me through these five years. I would particularly like to aknowledge my father and my mother, whom without their love, encouragement and support, I would not have completed my thesis.

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[1] www.coopers.ch

To my mother and my father…

Chapter 0

Synthesis Report

Studying old-age population is an active field of research in actuarial science. Especially in the current context of aging population in OECD countries, the actuary has a leading role in managing the related risks. In this thesis, the old-age challenges are adressed considering the pension fund asset liability management (ALM) study as well as the modelling of joint mortality. The thesis contains three chapters.

Chapter 1: Asset Liability Management for Pension Funds: A Survey

Before examining a specific model of ALM for pension funds, it is of interest to review the different methodologies discussed in the literature. In this chapter, the analysis is conducted in two steps. At first, the reader is introduced to the different features of a pension fund. The Swiss system is discussed as an example. In particular, we describe the Swiss three pillars system, emphasize the future reforms and discuss its implications. A brief comparison with some selected OECD countries shows that the Swiss pension funds perform quite well. Secondly, we identify two types of risks in a pension fund: the financial risks and the demographic risks. The ALM framework provides the theoretical background for managing these risks. Considering some key ALM methods, the chapter analyses the advantages and disadvantages of the models.

Chapter 2: On Integrated Chance Constraints in ALM for Pension Funds

The goal of this chapter is to discuss a concrete ALM model using the stochastic programming framework. In this respect, we discuss the role of *integrated chance constraints* (ICC) as quantitative risk constraints in ALM for pension funds. We define two types of ICC: the *one period* integrated chance constraint (OICC) and the *multiperiod* integrated chance constraint (MICC). As their names suggest, the OICC covers only one period whereas several periods are taken into account with the MICC. A multistage stochastic linear programming model is therefore developed for this purpose and a special mention is paid to the modeling of the MICC.

Based on a numerical example, we firstly analyse the effects of the OICC and the MICC on the optimal decisions (asset allocation and contribution rate) of a pension fund. By definition, the MICC is more restrictive and safer compared to the OICC. Secondly, we quantify this MICC safety increase. The results show that although the optimal decisions from the OICC and the MICC differ, the total costs are very close, showing that the MICC might represent a good alternative.

Chapter 3: On Bivariate Lifetime Modeling in Life Insurance Applications

Mortality has an important impact on the pension fund population. Chapter 3 proposes a model that describes the lifetimes within a married couple. Insurance and annuity products covering several lives require the modelling of the joint distribution of future lifetimes. In the interest of simplifying calculations, it is common in practice to assume that the future lifetimes among a group of people are independent. However, extensive research over the past decades suggests otherwise. In this chapter, a copula approach is used to model the dependence between lifetimes within a married couple using data from a large Canadian insurance company. As a novelty, the age difference and the gender of the elder partner are introduced as an argument of the dependence parameter. Maximum likelihood techniques are thus implemented for the parameter estimation. Not only do the results make clear that the correlation decreases with age difference, but also the dependence between the lifetimes is higher when husband is older than wife. A goodness-of-fit procedure is applied in order to assess the validity of the model. Finally, considering several products available on the life insurance market, the paper concludes with practical illustrations.

Chapter 1

Asset Liability Management for Pension Funds: A Survey

1.1 Introduction

A pension scheme[1] can be defined as a financial institution that collects discretionary or mandatory contibutions from its members during their working period in order to provide, to themselves or their dependants, regular incomes in some specific situations. These different situations may include retirement, disability and death. The amount of the benefit is defined by an objective rule. That is, the same pension amount is given to beneficiaries having exactly the same characteristics. Prescribed by law (e.g. national social security system) or by a convention (e.g. occupational pension by the employer), the fund is affiliated to a sponsor (state, private company or employer, etc.) that is the guarantor of the continuity of the system. In addition, the members have the possibility to combine different kinds of securities namely social security, occupational pension and individual saving in order to maintain a certain standard of living at retirement or when death or disability occurs. That is the basic concept of the multi-pillar approach, see the report World Bank, 1994.

Pension funds are exposed to two main types of risks namely (1) demographic risk, and (2) financial risks. The first type includes all kind of uncertainties that affect the structure or the size of the population of pension fund members. For instance, a high proportion of retirees associated with a low proportion of working population has a negative effect on the pension system. The financial risks regroup the uncertainties related to investment and economical trends. These risks are essentially related to financial assets, interest rate, inflation rate and credit. Devolder et al., 2013 analyses the different types of pension fund risks. In this respect, the main

[1] In this chapter, the term *pension* includes public social security systems as well as private pensions.

challenge of the fund is to manage the pension contributions in order to secure the pension payments, taking into account the financial market uncertainties. That is the fundamental objective of an asset liability management (ALM) study.

The world of pension funds has been facing several changes over the last decade. The life expectancy is continuously increasing as opposed to the working population that is decreasing. The financial returns are particularly low with an abnormal volatility of the market. In such context, the risk management is getting highly complex, requiring the expertise of different specialists. In this respect, the aim of this chapter is to firstly introduce the reader to the different characteristics of a pension fund and secondly, provide a literature review on the well known ALM methods for pension funds.

The rest of the chapter is organized as follows. Section 1.2 highlights the different features of a pension fund. In Section 1.3, the Swiss pension fund system is discussed and a comparison is made with some selected OECD countries. In the following part, we perform a survey on the main risk management tools that are discussed in the literature. In particular, Section 1.4 is dedicated to the advantages and disadvantages of well known ALM methods for pension funds whereas Section 1.5 is devoted to longevity risks. Section 1.6 concludes.

1.2 Pension funds features

Pension funds may differ in many aspects. Firstly, a pension plan can be classified in three broad types: the defined contribution (DC), the defined benefit (DB) and the hybrid pension scheme which is a combination of the two firsts. In a DC fund, the active members and the employer pay a fixed contribution (percentage of salary) into an individual account. The contributions are invested on the financial markets and the accumulated amount credited with interest (positive or negative) serves to pay the benefits at retirement. It is important to notice here that the level of benefit is based on the accumulated amount. Under the DB framework, the pension fund promises a specified amount of benefit at retirement that is predetermined by a formula based on the participant's wage history and years of service as well as the plan parameters rather than the accumulated amount as in the DC. Thus, the current contribution rate is calculated basing on the projected benefits. In a practical manner, the main difference between DB and DC lies in who is carrying the risks. The retirement wealth accumulation under the DC system depends on the allocation policy of the employee who takes the full risk whereas in a DB plan, the risk is partially or totally transferred to the sponsor. Bodie et al., 1988 describe the different characteristics of DB and DC with respect to the risks faced by employers and employees. The authors examine the trade-offs involved in the choice between DB and DC plans. DC plans are by definition fully funded whereas solvency is one of the most important issues in DB plans. Nowadays, DC plans are gaining more and more ground due to the complexity in the management of DB pension funds and Bodie et al., 1988 believe that DC plans would necessarily dominate DB plans because of the flexibility of DC plans design. Broadbent et al., 2006 analyse the factors leading to the shift from DB to DC and its implications regarding asset allocation and risk management. The authors exhibit the flexibility of DC plans and explain that an employee prefering a DB framework can always buy the appropriate life annuity products. That said, from an employee point of view, the DB is nowadays preferable to the DC for the two following reasons: the pension benefits are somehow guaranteed and the return on the financial markets are low.

Secondly, pension funds (including state pension systems) can be characterized by the way they are financed. We distinguish the pay as you go (PAYG) system and the funded system. In the PAYG mechanism, the contributions paid one year by the active members are directly transferred to the beneficiaries. Thus, there is no retirement capital accumulation in such a plan and this approach is based on the fact that the benefits for retired people are paid by the current working population. In such mechanism, the continuity of the sponsor is of great importance because if the system must stop at any time for any reason, there is clearly a big problem for affiliates. Under a funded system, the retirement capital is built up by contributions during the working period and that capital will serve to pay the retirement benefits. As a remark, in the funded system, the available capital can be invested on the financial market and probably generate higher rate of return whereas no actual capital is accumulated under the PAYG system. Kuné, 2001 explains how the future demographic and economic developments is anticipated under funded system, and concludes that it provides the best way of securing pension liabilities. However, literature is controversial regarding which one is the best. A priori, there are substancial transition costs in switching from PAYG to funded scheme. According to the Aaron-condition, we can say that a funded system is better than a PAYG system when the real rate of interest exceeds the real rate of growth of wages and salary, see Aaron, 1966, Siebert, 1997 and Kuné, 2001. The strength of the PAYG is that a minimum income is guaranted independently of the capital market perfomance while financial and inflation risks have to be considered as uninsurable under the funded system. From Brown, 1997, any fund that pretends to improve the pension benefits should fullfill the following three criteria:

- it must increase gross national savings,
- those savings must be used in a manner that increases worker productivity,
- there cannot exist a better method of achieving the first two stated goals,

and these conditions can be fulfilled by any system, i.e. PAYG or funded system.

Thirdly, depending on the sponsor, pension plans are also specified by their legal aspect. It can be either public, in which case it is regulated under public sector law (e.g. state, region or public company), or private where it is regulated under private sector law or convention. Based on the notional contribution rate, Colin et al., 1999 analyse the differences between the two types. As the first pension funds were PAYG and public, there is a tendancy in the literature to associate public plan with PAYG plan. But in principle, a pension fund may indeed be public funded or private PAYG. The discussions in favor of the privatisation of pension funds turn around two points (see, e.g. Banks and Emmerson, 2000 and Thompson, 1998):

- private funded funds provide a higher return than public sector funded schemes, and
- they are less sensitive to the political risk.

Finally, as it can be the case in some countries (e.g. Australia), the employee may have the freedom to decide which pension fund he/she wants to be affiliated with. Thus, based on its current situation and its personal retirement objectives, the employee can choose the fund that suits him best. In that case, the risk is essentially carried by himself. A typical example is Australia which has one of the best pension systems in the world. In Europe, the application of the free choice of the pension fund has been subject to a number of criticisms. The opponents argue that such freedom would destroy the principle of solidarity between retirees and contributing employees, which is the foundation of the pension fund system in most European countries. Hence, we may end up with some funds which only contain disabled and retired people and no active members would join them. Such freedom to pension members would increase the retirement costs. That is, pension funds would spend additional fees in trying to win new members whereas the employees would pay some fiduciaries who can advise them for choosing the best fund. That said, in European countries, instead of being allowed to choose the pension fund, more and more DC retirement funds are giving to their members the possibility to choose within several risk profiles (i.e. investment strategies).

1.3 Swiss pension system: present and future

As we are in Switzerland, we are interested in the particular case of the Swiss retirement plan. For this reason, in this section, we firstly introduce the Swiss three pillars system and its features. Secondly, attention is paid to future challenges, especially by considering the project "*Prévoyance 2020*" and analysing its implications.

1.3.1 The Swiss three pillars

The Swiss retirement pension system comprises three parts, which together make up the total pension system, see Figure 1.1. Its goal is to guarantee a financial soundness to the individual survivors in old age and its survivals. We distinguish:

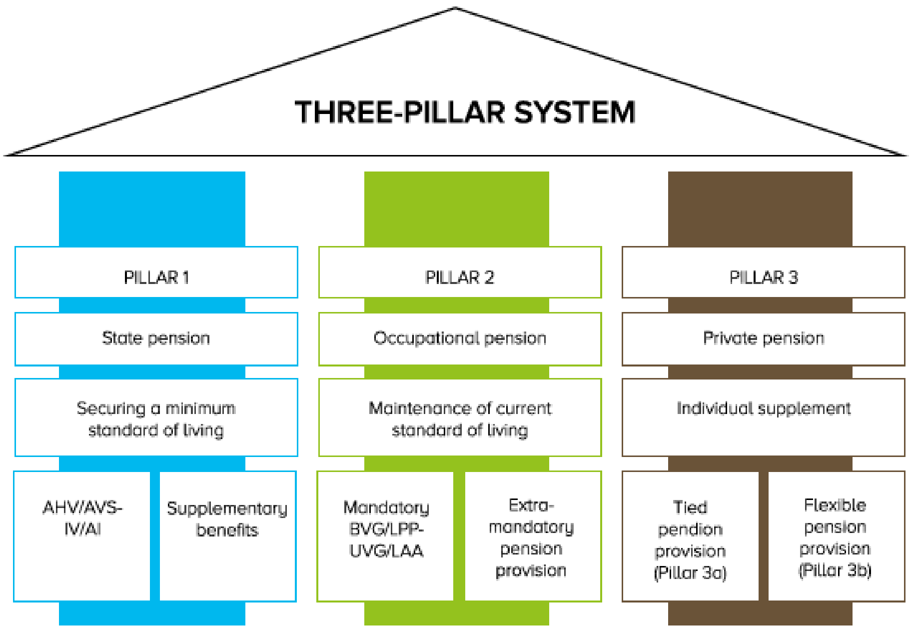


FIGURE 1.1: The Swiss three pillars system[2]

- **Federal Old Age and Survivor Insurance (AVS) and Federal Disability Insurance (AI):** With a total benefit of about 41 billions in 2014, the Swiss first pillar is composed of the old-age retirement income, the survival pension in case of death and of the disability insurance. It aims to ensure a basic standard of living. All persons who have resided or worked in Switzerland for a sufficient period of time are entitled to the first pillar. It works on a PAYG basis under which the current contributions are used to pay the current old-age pensions. Any person living or working in Switzerland is insured and has to pay contributions from the year in which she/he turns 21 or from age 17 if already working. Contributions are evenly split between the employer and the employee and are directly deducted from employees' salaries. The benefit incomes from the first pillar are also subject to income tax.

- **Occupational pension scheme:** The Federal Act on Occupational Retirement, Survivors' and Disability Pension Plans (LPP) entered into force on January 1, 1985. As the second pillar of the Swiss retirement plan, the occupational pension complements the first pillar in order to reach, on average, an income of 60% of the last salary, allowing the retired to maintain its anterior standard of living. Old age, death and disability are covered. It is mandatory for all workers earning a yearly salary higher than CHF $21'150$ (as of 2015), and optional for self-employed persons. The financial system is based on a fully funded approach where the employee's monthly contributions are accumulated up to retirement. The employer pays at least half of the yearly contribution. In principle, the contribution is a certain proportion of the annual salary; the insured salary is limited at CHF $84'600$ (as of 2015). The contributions are yearly accumulated at a yearly minimum rate of 1.75%, also called the minimum interest rate. At retirement, the accumulated amount is converted into pension at an annual rate called the conversion rate. The minimum level of this conversion rate is set by the regulation and makes the Swiss second pillar system so particular. Swiss pension funds usually provide an income which is larger than the required minimum conversion rate. As for the first pillar, pension incomes are subject to income tax whereas the contributions are tax free. Entirely set up and managed by employers and employees, the pension funds can be DB, DC (mostly), or hybrid. Notice that the Swiss defined contribution plans are somehow special and are often considered as cash balance plans (hybrid plan) according to international classification, see OECD, 2011.

- **Individual or private pension:** Based on a fully funded mechanism, it constitutes the third pillar of the Swiss pension system and is totally optional. The individual pension is managed by the person (employee or self employed) itself. It may serve to close any gaps that exist in a person's pension coverage and allows to cover any supplementary individual needs. It can be split into tied pension provisions (Pillar 3a) and flexible pension provisions (Pillar 3b). The main difference is that Pillar 3a benefits from some fiscal incentives whereas Pillar 3b does not. As one could expect, the amount of contributions into the Pillar 3a each year is limited.

1.3.2 Comparison with other countries

Based on both arrangements and performances, the Swiss retirement system is known as being one of the most developed systems in the world. So is its second pillar. As explained above, the first pillar operates under the PAYG whereas second and third pillars are mostly fully funded. The total assets of the occupational pension represent almost 119% of the GDP, showing that Switzerland has a mature pension system. The occupational pension funds operate under an hybrid regime where the plan sponsor shares the investment risk and all the assets are pooled (actually a cash balance plan). Almost all the occupational pension assets are managed by autonomous funds. The life expectancy at 65 is 19.4 years for men and 22.4 for women (OECD, 2013). In addition, employees are automatically enrolled to their employer's pension fund concerning the second pillar. Only the third pillar allows for a free choice of the insurance company or bank.

The goal of this section is to analyse how the Swiss system perfoms compared to other countries. A cross-country comparison of pension systems is likely to be controversial as each country has its particular economic, social, political, cultural and historical circumstances. However, there are certain indicators that, across the range of systems, allow to assess the regime. In order to ease the comparison, a sample of seven countries will be considered namely Canada, China, Denmark, Germany, Netherland, UK and USA in addition to Switzerland. They are essentially OECD countries to which we have added China as it is currently one of the major economies in the world. The evaluation of the Swiss system is made from three perspectives namely adequacy, sustainability and integrity. In principle, based on those three pespectives, the Melbourne Mercer pension report (Ralston et al., 2012) identifies more than 40 indicators in order to calculate a pension index for each country. Based on their importance, only some of them will be discussed here.

Firstly, the replacement rate analyses to what extent the provided benefits ensure a certain quality of life and represents the major measurable outcome from the system. It contributes to measure the adequacy of the system. Figure 1.2 compares the minimum replacement rate and the net replacement rate for the different countries (OECD, 2013). The net replacement rate is computed as the ratio of the retirement income from the two first pillars (mandatory) over the net average lifetime wage. The middle red bar in Figure 1.2 represents the targeted rate suggested by the World Bank (World Bank, 1994). Switzerland meets that condition right after the Netherlands, Denmark and China. The targeted minimum replacement rate is

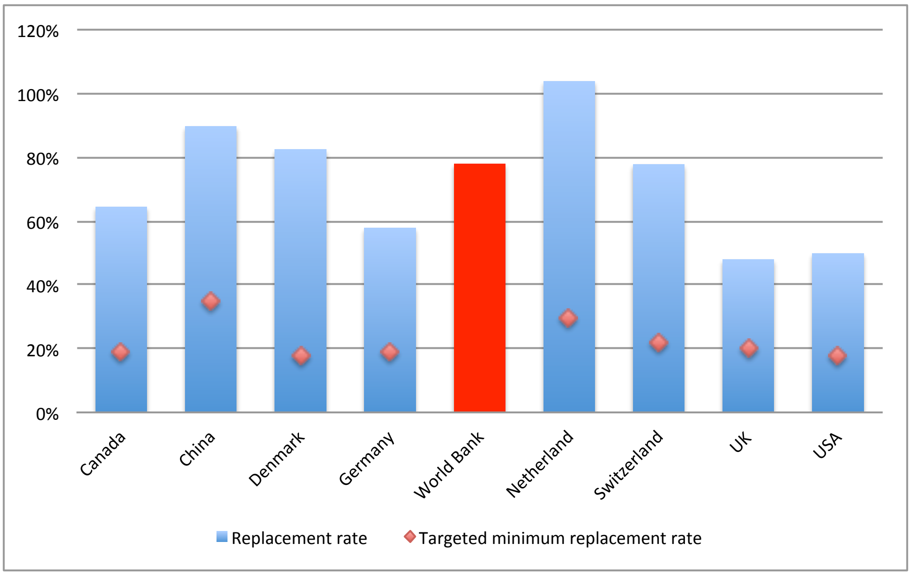


FIGURE 1.2: Net pension replacement rate by earnings for a median earner (OECD, 2013)

based on the intended basic pension of each country. From the figure, it can be seen that even though Denmark has a higher replacement rate than Switzerland, the minimum pension is more important in Switzerland. In addition, the home ownership rate (European Commission, 2013), displayed in Figure 1.3, is an important factor in affecting the financial security during retirement. It has a positive impact on the retirement income in the sense that the retiree pays no rent (if fully owned). The figure clearly shows that Switzerland has the lowest rate.

Secondly, some indicators influence the likelihood that the current system will be able to provide the benefits into the future, i.e. the sustainability of the system. The total amount of pension assets gives an idea on the development of a pension system. Figure 1.4 compares the total asset for private occupational pensions. The amount are calculated as percentage of GDP. Switzerland appears second with a percentage of 119% of the GDP. In Germany, Denmark and China, the total assets are particularly low, certainly due to the importance of the public funds and/or pension insurance contracts. The private pension coverage rate represents an other way for assessing the maturity of pension system. It is computed as the proportion of the working age population who are members of private pension plans. Figure 1.5 displays the value of the coverage rate for (quasi[3]) mandatory private pension in selected OECD countries. As mentioned in the Melbourne Mercer pension report (Ralston et al., 2012), a higher proportion of coverage amongst the workforce

[3]It is mandatory only for eligible employees.

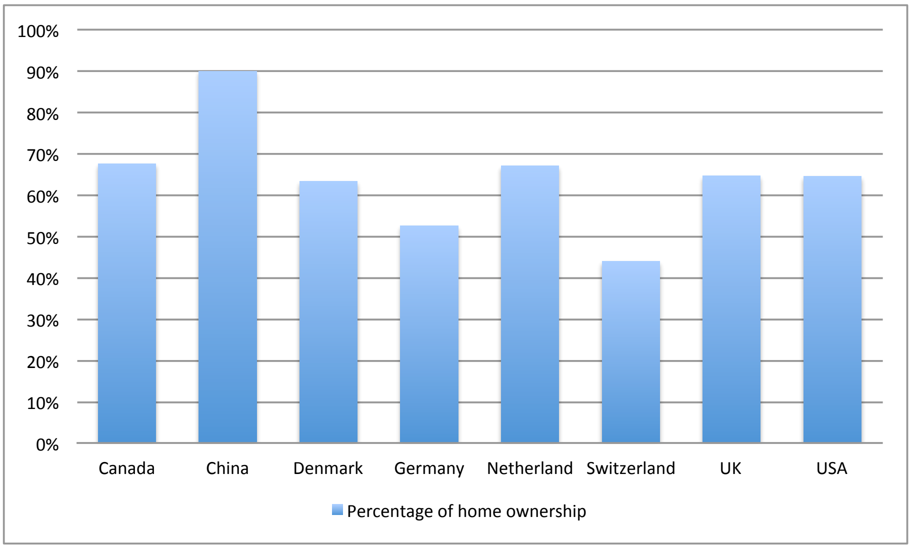


FIGURE 1.3: Home ownership rate by countries (European Commission, 2013)

increases the likelihood that the overall retirement income system will be sustainable in the future as it reduces pressure on government expenditure. Some country key indicators for retirement sustainability are reported in Table 1.1.

Finally, the integrity of system can be rated by the way of indicators related to regulation and governance, protection and communication for members, and costs. More specifically, those indicators affect the level of confidence that the citizens of each country have in their system. In this respect, the Melbourne Mercer pension report clearly shows that Switzerland is among the world leaders in terms of integrity of the retirement system. However, there are also some areas for improvements for the Swiss retirement system in general that have been pointed out in the literature. The expert suggestions are essentially inciting the increase of the retirement pension age over time, the increase of the home ownership and the reduction of the access to funds before retirement.

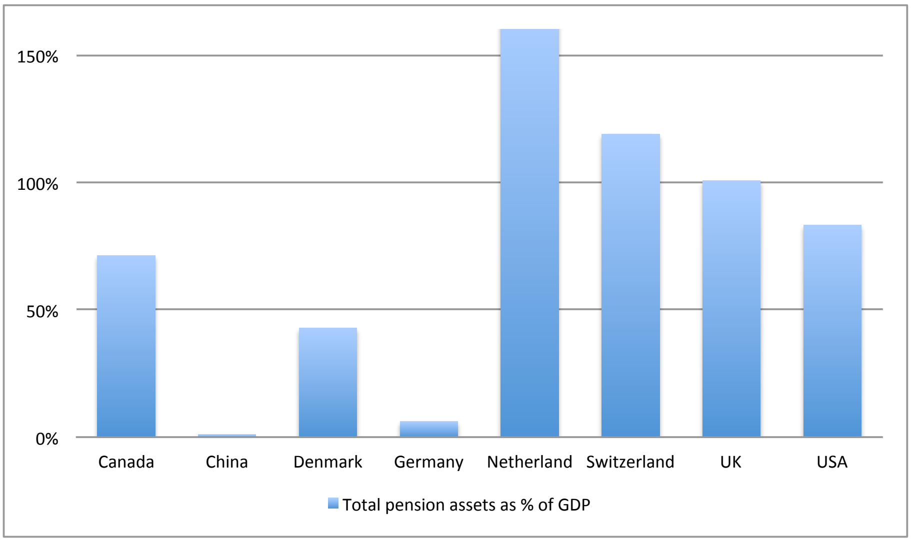


FIGURE 1.4: Total pension assets as % of GDP (OECD, 2013)

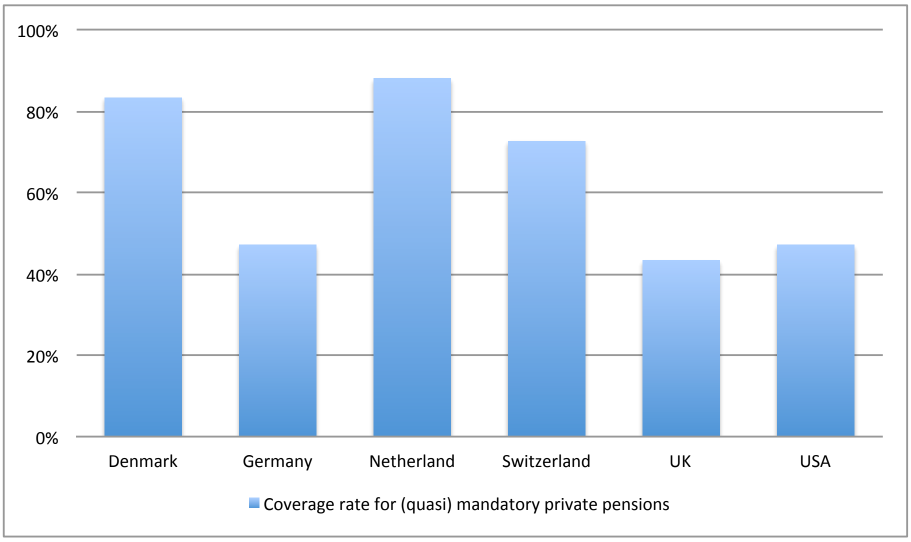


FIGURE 1.5: Coverage rate for (quasi) mandatory private pensions (OECD, 2012)

Retirement indicators	Countries								
	CAN	CHN	DNK	DEU	NLD	CHE	UK	USA	OECD
Fertility	1.61	1.67	1.67	1.41	1.68	1.52	1.83	1.86	1.6
Life expectancy at birth	81.4	75.2	79.3	80.7	80.9	82.5	80.4	78.9	80
Life expectancy at 65	20.3	15.6	18.5	19.4	19.3	20.8	19.4	19.3	19.3
% of pop. over 65 (%)	16.0	9.5	18.6	21.4	18.1	18.2	18.1	14.7	16.2
Old-age dependency ratio (%)	22.0	12.0	28.0	32.0	26.0	26.0	26.0	21.0	23.5

TABLE 1.1: Some key indicators for retirement system sustainability (OCDE, 2013)

1.3.3 The planned pension reforms

The retirement 2020 reform project

The Swiss population of elders is growing at a faster rate, the life expectancy is increasing and the capital market offers return rates which are increasingly low. These are the reasons why the Federal Council, in collaboration with some Swiss social institutions, has adopted the "Prévoyance 2020" reform project. As its name suggests, the retirement 2020 reform project sets new guidelines for the Swiss retirement system in 2020. The reform aims to:

- maintain the standard of benefits in the Swiss retirement system,
- ensure the financial stability of the two first pillars and
- adapt the system to the society changes.

The suggestions are of many kinds but, we focus on those which are relevant from an actuarial point of view. Firstly, the notion of "retirement age" (which is currently 64 for women and 65 for men) is replaced by a "reference age" of 65 for both women and men. In that case, the insured (man or woman) can retire at any age between 62 and 70, increasing the flexibility of the system and indirectly, encouraging people who want to work later than 65. The reader should notice that Switzerland has one of the highest occupational rates for people aged over 60. The reference age is applicable to the first and the second pillar of the Swiss retirement system. Under some conditions, it is even possible to combine a pension income with a remunerative activity. If a member decides to retire earlier (resp. later) than 65, the annuity is reduced (resp. increased) relatively. Secondly, the minimum conversion rate (which is currently of 6.8%) will be reduced by 0.2 percentage point each year in order to reach a rate of 6% after four years. In order to maintain the level of benefits, an increase in the second pillar contribution rate is also proposed. Thirdly, the part of the salary which is subject to the second pillar is enlarged, extending the number of employees which are insured under occupational insurance. This latter is particularly interesting for workers with low salary as the minimum salary required to be part of the second pillar is reduced. Last but not least, the first pillar is underfunded and requires the support of the Confederation; the situation is going to be worse in the next years due to the demographic trends. In order to fund this shortfall and to reduce the participation of the Confederation, the project proposes to increase the VAT by two percentage points.

Implications of the reform on risks

The project "Prévoyance 2020" aims to address the current and future problems of the Swiss retirement pension system. The matters are of many kinds. In the following paragraphs, we discuss the reform viability with respect to the first and second pillars. This is without loss of generality as the influence on the third pillar is negligible.

Firstly, the first pillar, based on a PAYG, is mainly influenced by demographic changes. Moreover, the longevity of the population and the imbalance in the age pyramid tends to increase the old-aged dependency ratio (ADR). The ADR is the ratio of the number of persons older than 65 to the active population (ages ranging from 20 to 64). Figure 1.8 in Section 1.5 gives an overview of the ADR in selected countries. The continuous increase of ADR means that the trend of increasing expenses of the AVS must be supported by an ever smaller number of actives. For example in Switzerland, in 2010, 3.64 actives were providing one pension benefit; it will be 3 actives in 2020 and less than 2 for 2050; thus requiring an increase in the contribution of active people to the first pillar. To avoid the rise of AVS contribution rate, two solutions are proposed. As a first step, decrease the total amount of pension expenses by giving the possibility for people to work longer than 65 in order to reduce the proportion of inactive people, i.e. the ADR. Besides, define additional contribution funding through a rise in VAT, for example. The advantage of an increase in VAT is that it spreads the additional costs of the first pillar over the entire population, and not only over the working population.

Secondly, the problems encountered in the occupational pension scheme are mostly related to longevity and low yields on the capital markets. Longevity is the fact that, on average, people live longer than expected. Ceteris paribus, the increase in life expectancy at age 65 leads to a greater retirement savings needs. On average, an increase of one year of the lifetime would lead to a 5% increase in the total expenses. To address the problem of longevity, a reduction of the conversion rate is initially proposed.

The conversion rate is the rate at which the retirement saving is converted into pension annuity. It mainly depends on two parameters: technical interest rate and life expectancy. The technical interest rate helps to evaluate the present value of future pension payments and is yearly fixed by politics. In general, its value is highly correlated to the low risk interest rate. In the literature, it is common to assume a fix margin between the expected low risk interest rate and the technical interest

	i							
n	0.5%	1.0%.	1.5%	2.0%	2.5%	3.0%.	3.5%	4.0%
15	0.067	0.070	0.073	0.076	0.079	0.082	0.085	0.088
16	0.063	0.066	0.069	0.072	0.075	0.078	0.081	0.084
17	0.060	0.062	0.065	0.068	0.071	0.074	0.077	0.080
18	0.056	0.059	0.062	0.065	0.068	0.071	0.074	0.077
19	0.053	0.056	0.059	0.062	0.065	0.068	0.071	0.074
20	0.051	0.053	0.056	0.059	0.062	0.065	0.068	0.072
21	0.048	0.051	0.054	0.057	0.060	0.063	0.066	0.069
22	0.046	0.049	0.052	0.055	0.058	0.061	0.064	0.067

TABLE 1.2: Calculated conversion rate as function of life expectancy and technical interest rate

rate, especially due to uncertainty. For ease of understanding, the conversion rate is approached by:

$$Conv_{rate}\left(i,n\right) := \frac{1}{a_{\bar{n}|i}} - c = \frac{i}{1-\left(1+i\right)^{-n}} - c \tag{1.1}$$

where n is the life expectancy at retirement, $i := y - r$ the technical interest rate, y the expected low risk interest rate and r the fix margin. The overall computed value is often reduced by c in order to take, dependants as well as administration fees, into account. Table 1.2 displays the evolution of the conversion rate with respect to n and i assuming $c := 0.2\%$. For example, on average, either an increase of two years in life expectancy or a decrease of one percentage point of technical interest rate would lead to a decrease of 0.6 percentage point of the calculated conversion rate.

Nowadays, as the interest rates are getting low and the life expectancy is increasing, it is reasonnable to decrease the conversion rate. A conversion rate of 6%, corresponding to a 12% decrease from 6.8% and leading to a similar decrease in total expenses, is recommended. Assuming a life expectancy at retirement $n = 21$ and considering a technical interest rate $i \leq 2.5\%$, Table 1.2 strengthens this evidence. However, the problem of maintaining the pensions to, at least, the current level arises. For this purpose, the reform recommends to increase the level of second pillar contribution rates. That, added to the increase in average retirement age would contribute to guarantee the financial stability of Swiss occupational pension funds.

1.4 Asset Liability Management

In this subsection, we will focus our attention on an occupational fund evolving under a fully funded mechanism. Employees pay contributions to the fund during their working period. These contributions represent the assets. At retirement, they receive their pension incomes which constitute the liabilities of the pension fund. Thus, the goal of an asset liability management (ALM) study is to determine the best ways to manage the current assets and contribution rates in order to guarantee, to a certain extent, the payments of future retirement pensions. Among the different ALM methods available in the literature, there are deteministic ALMs, asset only, the surplus optimization, stochastic control, stochastic programming and Monte Carlo simulation models.

1.4.1 Deterministic ALM

The use of ALM methods has a long tradition in pension funds. Even though the cash outflows were estimated in a stochastic way, the methods were originally deterministic. The asset are mainly allocated into bonds, which are assumed to be less risky. Moreover, the bonds could be either corporate or government and they guarantee the payment of a certain amount of coupons at the end of each period. Based on economical and actuarial assumptions, the level of benefits to be paid are estimated. The goal of those models is to determine the adequate allocation that protects the fund from any unexpected movements in cash flows and/or interest rates, i.e. immunization. There are mainly two types of models.

Firstly, according to *cash flow matching*, bonds are chosen in such a way that, at the end of each period of study, the total amount of coupons to be received equals the total amount of pensions to be paid. When the forecasts of liabilities are good, the impact of the interest rate fluctuations are reduced. However, time value of money, reinvestment and liquidity (market) risks can cause serious issues, especially when the projection of cash outflows appears to be far from reality. It is also difficult (if not impossible) to find bonds that enables a perfect match for a realistic pension fund problem (e.g. long duration).

Secondly, in the *duration matching* approach, the asset allocation is defined such that the duration of asset equals the duration of liability. The duration of a financial product that consists of fixed cash flows, for example a bond, is the weighted average of the times until those fixed cash flows are received. For a portfolio, it is the weighted average of the individual durations. Roughly speaking, it is a measure in determining a bond's sensitivity to interest rates. One can show that the duration is equivalent to minus the elasticity of the bond price with respect to one plus its yield. In this respect, the duration matching approach is based on the inverse relationship between the price of a coupon bond and reinvestment return on the coupon. In principle, the variation in interest rate makes the values of assets and liabilities change by (approximately) the same amount; that is its main advantage. We refer the reader to Macaulay, 1938, Redington, 1952 and Koopmans, 1942 for more details.

As a limit to this model, the approach assumes flat term structure of interest rate and some arbitrary changes in the interest rate − not parallel to the yield curve as it is often the case in real world − leads to a mismatch between assets and liabilities and becomes difficult to manage. Large changes in interest rates are not taken into account neither. For adaptation to pension funds, the asset portfolio should be rebalanced dynamically and the present value of assets should equal the present value of liabilities. A more advanced version of this method also suggests that assets and liabilities are arranged so that the total convexity of the assets exceeds the convexity of the liabilities. The convexity of a financial product measures its curvature as a function of interest rate. It represents the second derivative of the bond price whereas the duration is related to the first derivative (slope). The asset portfolio is thus constructed by solving the three following equations:

$$\begin{aligned} Asset\ market\ value &= Liability\ market\ value \\ Asset\ duration &= Liability\ duration \\ Asset\ convexity &\geq Liability\ convexity \end{aligned}$$

In some cases, it is also possible to combine the two methods defined above and that is called the *combination matching*. Hiller and Schaack, 1990 examined the drawbacks of the different deterministic ALM methods. Linear programming theory provides the necessary framework for implementing the deterministic approaches. As an example, the paper by Kocherlakota et al., 1990 applies the duality theory of linear programming to provide insights for generalizing and solving the cash-flow matching problem.

A major problem of immunization methods is that they could be costly since the yields on bonds are low. So, the participation of financial market in the payment of a pension is relatively small. To illustrate the effects of the asset rate of return, consider a defined benefit pension scheme whose objective is to reach a pension

	$r \rightarrow$						
$g \downarrow$	0.00	0.01	0.02	0.04	0.08	0.12.	0.16
0.01	0.22	0.18	0.14	0.09	0.03	0.01	0.00
0.02	0.26	0.21	0.16	0.10	0.03	0.01	0.00
0.04	0.35	0.27	0.21	0.12	0.04	0.01	0.00
0.08	0.56	0.42	0.31	0.17	0.05	0.02	0.00

TABLE 1.3: Fair contribution rate variations with respect to r and g

amount of 60% of the last salary. We consider a single person who started working at 25 and retires at 65, and whose pension account is fully funded. A certain proportion c_r of the annual salary W_t at the end of year $t,\ 1 \leq t \leq 40$, is paid into the fund at the end of each year and represents the yearly contribution. Salary increases yearly by a fixed rate g and r is the average return on assets. We have $W_{t+1} = W_t\,(1+g)$ and at retirement age 65,

Total amount of assets = Present value of future pensions,

that is

$$c_r W_1 \sum_{t=0}^{39} (1+g)^t\,(1+r)^{40-t} = 0.60 \cdot W_{65} \cdot a_{65}$$

where over the first year of work, $W_1 = 5000$ and a_{65} is the single premium of a life annuity of one at age 65. Assuming $a_{65} := 12$, Table 1.3 describes the evolution of the fair contribution rate with respect to the salary growth g and the average return on assets r. It can be seen that when g decreases or r increases, the fair contribution rate tends to decrease. A pension fund wants to attain its objectives at the lowest cost. Investing in riskier assets permits to increase the average return on assets and therefore, allows to reduce the contribution rate. The difficulty in modeling such assets and the growing uncertainty of economical, financial and actuarial parameters in the pension fund make the deterministic approaches inefficient.

1.4.2 The Asset Only Framework

The asset only method is one of the oldest approaches in ALM; it is also well known in practice. Theoretically, it should not be considered as an ALM method, but in practice, it is one of the most widely used methods. As its name suggest, the optimal asset allocation is obtained by modeling assets and liabilities separately. First of all, based on economical and actuarial assumptions, the liability and the future obligations are projected over the period of study. Secondly, considering the initial level of total assets, one can determine the yearly average rate of return on assets required in order to meet a certain proportion of the projected liability. Once this target is set, the pension fund hires asset managers who attempt to beat the targeted rate of return at the lowest risk. The maximum level of risk has to be well defined by the fund. Assuming the risk measure is the variance of the asset portfolio and under standard conditions, the ALM problem therefore consists in a mean-variance optimization problem whose aim is to provide, at least, the yearly rate of return while minimizing the risk exposition, i.e. the variance. The modern portfolio theory (see Markowitz, 1952, Ingersoll, 1987, Huang and Litzenberger, 1988) proposes some powerful tools to solve such problem.

As an example, consider a pension fund with a total liability $L_0 = 100$, which is expecting to reach $L_1 = 101$ at time 1. We assume that the total asset is currently at $A_0 = 95$. At a given time t, the funding ratio F_t is the ratio of the total assets over the total liabilities; $F_0 = 0.95$ for $t = 0$. The value $F_t < 1$ means that the institution is underfunded at time t. Our goal is to reach a funding ratio of $F_1 = 1$ at time 1, i.e.

$$\frac{A_1}{L_1} = 1 \Leftrightarrow A_1 = L_1 = 101 \Rightarrow r_1 = \frac{A_1}{A_0} - 1 = 6.32\%,$$

where r_1 is the rate of return on asset required in order to meet the one-year projected liability. Thus, the optimization problem is

$$\begin{aligned}
&\min_{x_i} \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j \sigma_{ij}, \\
&\sum_{j=1}^{m} x_j = 1, \\
&\sum_{j=1}^{m} x_j \mu_j \geq r_1 = 6.32\%, \\
&x_j \geq 0 \quad \forall j \in \{1, \cdots, m\},
\end{aligned}$$

where m is the number of asset j, x_j the proportion of A_0 invested in asset j, r_1^j the rate of return on asset j over the year, $\mu_j = \mathbf{E}\left[r_1^j\right]$ the one-year expected value of the return on asset j and σ_{ij} measures the covariance between assets i and j. This process eliminates the sources of portfolio risks that do not provide an expected return premium for investors. The randomness of the liability is obviously not taken into account in this model. For $m = 2$ classes of assets, let's assume

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0.028 \\ 0.073 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} 0.029^2 & 0.00176 \\ 0.00176 & 0.169^2 \end{pmatrix}.$$

Thus, the asset allocation is optimal at

$$x_1 = 21.87\% \qquad \text{and} \qquad x_2 = 78.13\%$$

for a variance level of 1.81%. This example is simplified as in practice, the number of assets is much larger than two. The mean variance portfolio theory was initially set up for single-period optimization problems. In pension funds, the ALM problem is a long term problem. Applying the mean variance approach to pension fund ALM requires a multi-period approach. Under several sets of reasonable assumptions, many authors (see Fama, 1970, Hakansson, 1970 and Hakansson, 1974 and Merton, 1969) have discussed this issue. As a result, the multi-period problem can be solved as a sequence of single-period problems.

In practice, other constraints on the asset weights can be added. They are often risk, legal, budget, regulatory and operating constraints. These constraints can significantly alter the efficient frontier. It is also natural to define a set of feasible portfolios and to choose the best one based on simulations. The asset only method has the advantage of being relatively simple to implement; the optimal allocation problem can be solved by any financial actor able to afford the return needed to cover liabilities. However, the dependence between total asset and total liability is obviously not taken into account in the portfolio determination. More specifically, a significant difference between actual and projected liabilities might cause severe difficulties. The surplus optimization approach, discussed in the next subsection, addresses that issue.

1.4.3 Surplus Optimization

The total liability is the discounted expected value of future payments. At a given time, it represents the amount the fund has to own if it has to close at that time. As future payments are random, the total liability is a random variable influenced by many factors. The factors can be actuarial or economical. The economical risks of the liabilities are often related to inflation, salary growth and discount rates. As these risks also have an effect on the total amount of assets, it is important to include the random liability as part of the ALM model and not only as the target wealth for the pension fund portfolio optimization. More clearly, the correlations between asset and liability have to be considered in the determination of the optimal portfolio. That is the main problem of the asset only approach. For example, the assets, with higher covariance to liability, tend to reduce the risk exposition of the pension fund, see Sharpe and Tint, 1990 and Keel and Müller, 1995. Leibowitz and Henriksson, 1988 showed that an asset such as cash, which should typically reduce the riskiness of an all-asset portfolio, may actually increase the riskiness of a portfolio that includes liability.

Including liability in the optimal asset allocation decision has a long story in finance theory. We refer the readers to Sharpe and Tint, 1990, Elton and Gruber, 1992, Leibowitz et al., 1992 among others. In what follows, we present the surplus optimization model proposed by Sharpe and Tint, 1990 because, although liability is taken into account, the ALM model is put in a very close relationship to the traditional mean-variance portfolio theory. As the asset only is one of the most used in practice, it would be easy to switch from the asset only to the surplus optimization.

The surplus is the excess value of the total asset compared to the total liability. The surplus optimization model focuses on the surplus instead of the total asset as in the asset only model. We denote by S_t the surplus at time t, $t \in \{0; 1\}$ such that

$$S_t = A_t - kL_t,$$

where A_t and L_t are, respectively, the total amount of assets and liabilities. The parameter k denotes the importance attached to the liability. For example, $k = 1$ for a full surplus optimization whereas $k = 0$ for an asset only optimization. The variable t denotes the time index and especially in this model, $t = 0$ is the current time whereas $t = 1$ is the end of the period of study (one period approach). Let R_A and R_L denote, respectively, the return on asset and liability:

$$A_1 = A_0\,(1 + R_A) \text{ and } L_1 = L_0\,(1 + R_L)\,. \tag{1.2}$$

The return on liability (known as the liability appreciation rate) is calculated from value changes, just as it would be for any asset. Over the period of study, the variation of the surplus is

$$S_1 - S_0 = (A_1 - kL_1) - (A_0 - kL_0) = A_0R_A - kL_0R_L.$$

Following Sharpe and Tint, 1990 and Keel and Müller, 1995, this amount can be expressed relative to the level of assets at time 0. That is

$$Z = \frac{S_1 - S_0}{A_0} = R_A - k\frac{1}{F_0}R_L; \quad \text{where } F_0 = \frac{A_0}{L_0},$$

and F_0 is the funding ratio at time 0. The variable Z is influenced by many factors. They are:

- random factors such as returns on assets and liabilities,
- the current situation of the fund, i.e. the initial funding ratio F_0, and
- the asset allocation and the contribution rate which are the only factors that the decision maker can influence. They are also called decision variables.

As in the mean variance approach, the surplus ALM problem consists in determining x_j such that

$$\begin{aligned} &\min_{x_j} \quad Var\left[Z\right], \\ \text{subject to} \quad &\sum_{j=1}^{m} x_j = 1, \\ &\mathbf{E}\left[Z\right] \geq R_s, \\ &x_j \geq 0 \quad \forall j \in \{1, \cdots, m\}, \end{aligned} \tag{1.3}$$

where R_s is the targeted minimum return on Z and x_j denotes the proportion of A_0 invested in asset class j. The symbol $Var\left[X\right]$ denotes the variance of the random variable X. Applying the properties of expectation and variance, we obtain:

$$\mathbf{E}\left[Z\right] = \mathbf{E}\left[R_A - \frac{k}{F_0}R_L\right] = \mathbf{E}\left[R_A\right] - \frac{k}{F_0}\mathbf{E}\left[R_L\right] \quad \text{and} \tag{1.4}$$

$$\begin{aligned} Var\left[Z\right] &= Var\left[R_A - \frac{k}{F_0}R_L\right] \\ &= Var\left[R_A\right] + \frac{k^2}{F_0^2}Var\left[R_L\right] - 2\frac{k}{F_0}coVar\left(R_A, R_L\right) \end{aligned} \tag{1.5}$$

where $coVar\left(R_A, R_L\right)$ is the covariance between R_A and R_L. We recall that $R_A = \sum x_i r_i \Rightarrow \mathbf{E}\left[R_A\right] = \sum x_i \mu_i$ and $Var\left[R_A\right] = \sum_i \sum_j x_i x_j \sigma_{ij}$ and $\sigma_{ij},\ r_i$ and μ_i are defined in the previous section. In equation 1.4, controlling $\mathbf{E}\left[Z\right]$ is equivalent to controlling $\mathbf{E}\left[R_A\right]$ as the second part of the expression does not contain any decision variable, i.e. the decision maker has no influence on it. Similarly, minimizing $Var\left[Z\right]$ is equivalent to minimizing $Var\left[R_A\right] - 2\frac{k}{F_0}coVar\left(R_A, R_L\right)$. The surplus problem is thus

$$\begin{aligned} &\min_{x_j} \quad \sum_i \sum_j x_i x_j \sigma_{ij} - 2\frac{k}{F_0}coVar\left(R_A, R_L\right), \\ \text{subject to} \quad &\sum_{j=1}^{m} x_j = 1, \\ &\sum_j x_j \mu_j i \geq R_s, \\ &x_j \geq 0 \quad \forall j \in \{1, \cdots, m\}. \end{aligned} \tag{1.6}$$

The optimization problem formulated in (1.6) is similar to the one introduced in the asset only approach. The only difference is the second term in the objective function:

$$-2\frac{k}{F_0}coVar\left(R_A, R_L\right). \tag{1.7}$$

The latter expression is function of the importance to be attached to liability k, the initial funding ratio F_0 and the covariance between assets and liabilities. It represents the part of the model that is dedicated to the hedge of liability. This is one of the reasons why the expression (1.7) is often called *liability hedging credit* in the literature, see e.g. Sharpe and Tint, 1990. When the expression (1.7) is equal to zero, surplus optimization is equivalent to asset only. That is, using the asset only approach in an ALM study for pension fund is equivalent to making one of the following assumptions:

- liability is independent of asset or
- liability is non-existent in our problem.

From the system of equations (1.6), we can see that a positive covariance between assets and liability tends to reduce the objective and thus, improves the risk exposition of the pension fund.

Based on the model of Sharpe and Tint, 1990 and the Merton's Capital Asset Pricing Model (Merton, 1973), Keel and Müller, 1995 show that the efficient portfolios can be decomposed into a minimum risk component, liability component and a return generating component. It is the essence of the *liability driven investment* (LDI) method. A brief introduction is presented in Section 1.4.4. The surplus optimization model has the advantage of being simple to implement and provides an analytical solution to the pension fund ALM problem. More specifically, this approach by Sharpe and Tint, 1990 allows a full or partial consideration of the liability. However, it is a one period approach and as the pension fund ALM problem is a long term problem, a dynamic model would be more appropriate. In practice, as mentionned in Wane et al., 2011, a serious issue may arise from the determination of the liability return and of its covariance with the asset classes. What is the real probability distribution of liability? The papers by Kritzman, 1990 and Meder and Staub, 2006 could shed some light on this issue.

1.4.4 Stochastic Control

The stochastic control challenge is to cast the ALM problem in a continuous time framework, attempting to take into account the issues of surplus optimization due to long term and dynamicity in pension funds. This method is not really accepted in practice where it is often considered as an academic model. It is an extension of the surplus optimization approach. The stochastic control approach is based on optimal portfolio selection theory introduced in the intertemporal asset allocation models of Merton, 1969, Merton, 1971 and Merton, 1973. A first application to pension funds problem can be found in Merton, 1993 who discusses the lifetime portfolio selection problem for a university endowment fund. Since then, the literature of stochastic control has flourished and we can cite Taylor, 2002, Brennan et al., 1997, Brennan et al., 1998, Cairns, 2000, Cairns et al., 2006, Blake et al., 2013, Rudolf and Ziemba, 2004, Haberman and Sung, 1994 and Menoncin and Scaillet, 2006 among others.

To start off the procedure, the analysis utilizes a system of stochastic differential equations for modelling the stochastic parameters over time of assets and liabilities models. The dynamics of each process has, firstly, to be established explicitly. Secondly, the dynamics and the objective of the ALM problem will serve to define the objective function. The value function can be considered as the value of the objective function corresponding to the optimal decision. We recall that our goal is to determine the asset allocation (and eventually the contribution rate) leading to the value function. To solve the problem, we use the dynamic programming approach; the idea being to connect the optimization problem with a certain differential equation known as Hamilton-Jacobi-Bellman (HJB) equation. Under some regularity conditions, the solution of the HJB equation is the value function. The last step consists in recovering the optimal strategy by plugging-in the value function into the HJB equation. We refer the reader to Merton, 2010 to learn more about the resolution of general stochastic control problems. The main challenge of this approach lies in determining the solution of the HJB equation. Two main algorithms are often used to solve a stochastic control problem: the dynamic programming algorithm and the finite element algorithm, see Ziemba and Mulvey, 1998 and Merton, 2010. The main difference is that the dynamic programming algorithm results in an exact analytical solution whereas the finite element algorithm uses numerical techniques to find an approximate solution to the HJB equation. The latter is particularly useful when an analytical solution is not available. Appendix 1 proposes an example of implementation using the first resolution approach.

The ALM problem considered in the Appendix 1 leads to a stochastic control framework with a state variable F_t and one decision variable w. Considering a power utility function, the use of the dynamic programming algorithm enables to find the exact optimal asset allocation. When it is not possible to determine an exact solution (e.g. Taylor, 2002), finite element algorithms (e.g. finite difference technique) may provide a good approximation of the exact solution, see Kushner and Dupuis, 2013. The dynamics of the model allow to include the process of information being revealed progressively through time. The portfolio allocation can be rebalanced at any time. The stochastic process of each variable is accurately incorporated into the ALM framework. The assets and liabilities are influenced by many sources − common or not − of risk and the risk aversion is accommodated. Under some theoretical assumptions, all these together are integrated in a system that results in a simple and tractable solution. These arguments explain the research interests for the stochastic control.

Nonetheless, the stochastic control approach may pose some issues. Firstly, as pointed out in Mulvey and Ziemba, 1998, it applies to problems in which the state space can be kept manageable, i.e. with at most three or four state variables. In the Appendix 1, the only state variable considered is the funding ratio F_t. That reduces considerably the problem. In a more realistic problem, stochastic processes can be driven by more than one variable, e.g. six variables in the financial scenario model of Ahlgrim et al., 2005 and four variables in the strategic asset allocation model of (Brennan et al., 1998 and Brennan et al., 1997). Moreover, the generation of confidence limits is difficult to calculate and modeling errors may arise due to the state space approximation. Secondly, estimating the parameters of the stochastic differential equation is not so simple. Especially, in contrast to the asset random variables, the stochastic model estimation of the liability process can be difficult as the evaluation methods often vary with regulation. Also, finding a random process that regroups all sources of uncertainty specific to liability − economic, financial and actuarial − as assumed in Appendix 1 can be difficult. To finish, from a practical point of view, the stochastic control seems very technical in the sense that it requires some advanced and specific mathematical knowledge.

1.4.5 Stochastic Programming

The stochastic programming (SP) approach offers an alternative to dynamic stochastic control for setting dynamic investment strategies. Often based on Monte-Carlo simulations due to its complexity, SP gives a flexible and powerful tool for ALMs. Even though its use in modeling complex portfolio optimization problems dates back the early seventies, SP is relatively new as its real application to pension fund practices only traces back to the work of Kallberg et al., 1982 and Kusy and Ziemba, 1986. Its importance lies in its ability to bring together many kinds of features in a

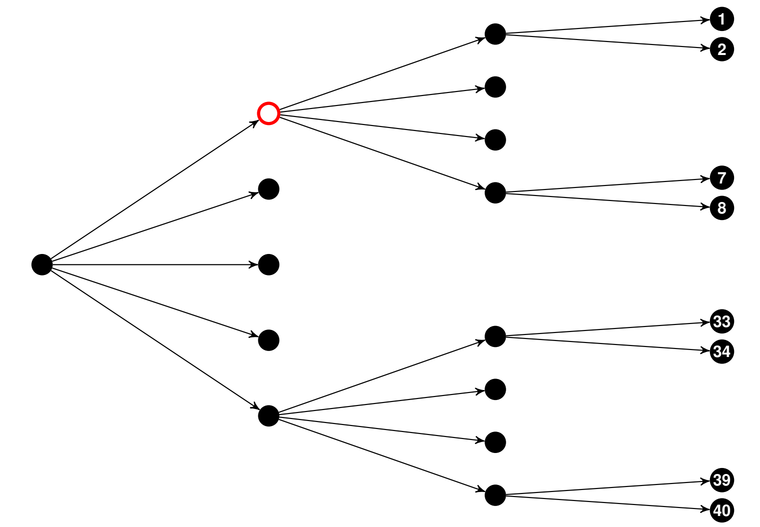


FIGURE 1.6: A scenario tree with 40 scenarios and 66 nodes.

common framework. The key idea is to approach randomness via a tree structure called *scenario tree* (see Figure 2.1) and optimal decision is taken at each node of the tree considering the information available at that point. The ALM optimization is done in several steps.

Firstly, as in the SC approach, the dynamic of each process has to be clearly defined and its parameters accurately estimated. Secondly, the resulting process equation will be discretized and serves to construct the scenario tree; the ALM model is formulated accordingly. To illustrate this, let us consider a random vector X which may take five possible values between time 0 and time 1, then four possibilities between 1 and 2 and finally two outcomes between 2 and 3 as displayed in Figure 2.1. From time 0 to time 3, the branching structure is $1-5-4-2-1$ leading to a total of $S = 5 \times 4 \times 2 = 40$ paths or outcomes. We have a time horizon of $T = 3$ years which has been divided into three periods of one year each. Each path from $t = 0$ to $t = 3$ is called a *scenario*. Different scenarios may share a common path up to some stage and then diverge; that is the case of scenarios 1 and 2. A *node* (t, s) is a possible outcome of the stochastic random variable at a given time t in a scenario s. Each node represents an opportunity to take a decision. The total number of nodes in the tree is determinant of the size of the stochastic program. Figure 2.1 describes a scenario tree with 66 nodes and 40 scenarios. Let $\mathcal{S} = \{1, \cdots, S\}$ be the set of scenarios. Each scenario s has a probability p^s, where $p^s > 0$ and $\sum_{s=1}^{S} p^s = 1$. We denote by X_t^s the random variable X at the node (t, s) and by w_t^s the vector of decision. For an ALM study for pension fund, the vector w_t may contain asset allocation, contribution rate, remedial contribution (see Toukourou and Dufresne, 2015 that is also presented in Chapter 2), indexation rate of future payments, etc. Once this event tree is built, decisions are taken at the beginning of each period (at each node) with the full knowledge of the past and considering the remaining future represented by the subtree rooted at (t, s). But, only time 0 values of the decision variables are crucial to the decision maker, since, almost surely, a true realization of the random data will be different from the set of generated scenarios. Readers must know that the branching structure, the time horizon, the duration of each period and the number of scenarios may vary from a model to another.

In addition, one should avoid anticipativity in the model by making one single decision at each time t for all paths by adding explicit constraints. That is, for any two different scenarios s_1 and s_2 ($s_1, s_2 \in \mathcal{S}$ and $s_1 \neq s_2$) having the same history up to time t, we enforce $w_t^{s_1} = w_t^{s_2}$. For example, at the empty red circle of Figure 2.1, $w_1^1 = w_1^2 = \cdots = w_1^8$.

Constructing a scenario tree close enough to the initial stochastic model can be challenging. There is an entire literature on this topic and authors often use simulations. Kouwenberg, 2001 discusses different approaches for the scenario tree generation whereas the model proposed by Høyland and Wallace, 2001 tries to fit the conditional moments at each node. Finally, once the variables, objective and constraints are established explicitely in a SP framework, many algorithms solve the problem. Optimality is defined in terms of current costs plus expected future costs. We refer the reader to Birge and Louveaux, 2011 and Shapiro et al., 2009 for a general introduction to SP. As stated in Mulvey and Ziemba, 1998, the primary algorithms for solving stochastic programs fall into three categories: direct solution by interior point methods, augmented Lagrangian decomposition and nested Bender's or regularized decomposition. The paper by Pflug and Świętanowski, 1999 has considered all these methods. Several models and advanced algorithms for stochastic linear programs can be found in Kall and Mayer, 2011 and Kali and Wallace, 1994.

The SP approach was firstly adopted by Bradley and Crane, 1972 with their application to a fixed income security portfolio problem. Kusy and Ziemba, 1986 show, based on a five-year period application to the Vancouver City Saving Credit Union, that SLP is theoretically and operationally superior to a corresponding deterministic linear programming model. The authors have proved that the effort required for the implementation of ALM and its computational requirements are comparable to those of the deterministic model. For a Japanese insurance company, Carino et al., 1994 developed a model that enables the decision makers to include risk management tools as well as including the complex regulations imposed by the Japanese insurance laws and practices. Over the two years of experiment, the resulting investment strategy has been fruitful as it has yielded extra income of 42 basis points (US$ 79 million). More recently, considering a Finnish pension company, Hilli et al., 2007 focus on the modeling of the stochastic factors and analyse the obtained numerical solution. Dert, 1995 pioneered the inclusion of chance constraints in multistage recourse models for pension funds. Chance constrained programs often lead to integer programming for which, it may be difficult to determine a tractable solution. As an alternative to chance constraints, Haneveld et al., 2010 proposed the integrated chance constraints, whose feasibility set is more handleable as it does not require integers. Toukourou and Dufresne, 2015 analyse the effect of integrated chance constraints as quantitative risk constraints in ALM for pension funds and introduce the multiperiod integrated chance constraints. The literature of SP in ALM also includes Consigli and Dempster, 1998, Bogentoft et al., 2001, Drijver, 2005, Faleh, 2011, Aro and Pennanen, 2013 among others.

On a practical point of view, the SP approach is among the best ALM approaches as it includes more of the essential elements of the real problem faced by the pension plan than the alternative approaches cited above. In general, most of the practical features of a realistic pension fund can be taken into account; that is, operational or regulatory restrictions and policy requirements can be modeled as a set of constraints in an optimization program, whose objective describes the pension fund goal.

A practical application of the SP approach is discussed in Chapter 2. The success of the model for institutional application also demonstrates the practicability of the approach. Many reasons justify its interest. The uncertainty of assets and liabilities is modeled through a discrete distribution which integrates the dependence between financial and demographic risks. The framework has a long time horizon split into subperiods allowing for dynamic optimal decisions. Uncertainty is incorporated in the decision process and the revealed informations are included dynamically. All these can be incorporated in a single and consistent structure; a simple linear program is sometime sufficient. As an example of current application of stochastic linear programming, Geyer and Ziemba, 2008 have developped a linear ALM model − InnoALM − for the Siemens pension fund. In use since 2000, this model has become the only consistently implemented and fully integrated proprietary tool for assessing pension allocation issues within Siemens AG worldwide.

Despite the undeniable applicability of SP, some difficulties are still to raise. Firstly, according to SP framework, the future is described by the scenario tree and determining a discrete distribution that represents the continuous uncertainty – well enough – is challenging. Secondly, implementing some realistic constraints in the model may cause infeasibility or untractability when the feasibility set is non-convex. For example, the value-at-risk constraint (VaR) also known as *chance constraint* requires the use of integer variables and may lead to untractability; see Dert, 1995 for a successful application. Binaries are also used in Drijver, 2005 where remedial actions are taken only when the funding ratio falls short in two consecutive years. In many cases, the SP solution algorithm requires advanced optimization knowledge that could make the model hardly implementable.

1.4.6 Simulation methods

Actuaries have to analyse the long-term viability of the pension fund with respect to future contributions, pay-outs to beneficiaries and other uncertainties. As its name suggests, this approach uses Monte-Carlo simulations to evaluate the feasible decisions from which, it draws the best ones. That is done in three steps. In order to ease understanding, we consider a pension fund that wants to determine the asset allocation that maximizes its terminal funding ratio over a certain period of study.

As for the other approaches, the first step is inevitably to define clearly the probability distribution followed by the stochastic variables of the model: financial and demographic. Secondly, based on these distributions, the simulation model or framework is defined. The simulation model serves to forecast the possible future paths of the variables of interest: assets, liabilities and then, funding ratio in our case. The future paths of assets and liabilities are obviously influenced by the current decisions taken on the strategic asset allocation. This last one is set depending on the fund's objectives. Finally, the procedure for choosing the best allocations is simple. For each possible asset allocation, a certain number of possible outcomes (simulations) of the terminal funding ratio will be generated. Considering the performance of each allocation, the three or five best decisions are submitted to the pension fund board of trustees. The performance is measured in terms of average return and risk measures, e.g.: VaR, CVaR, volatility, probability of underfunding, etc.

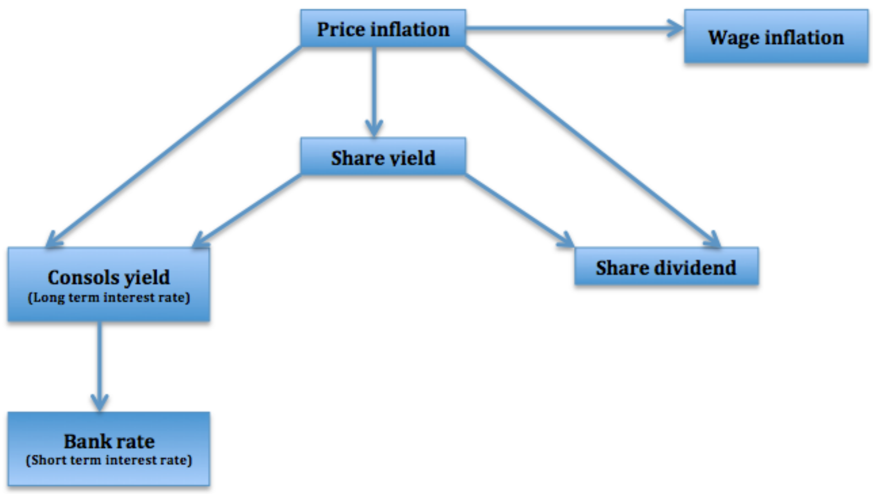


FIGURE 1.7: The cascade structure of the Wilkie model

The simulation approach is the most used in pension fund consulting nowadays, and research is still progressing. However, there are models that have particularly marked the literature. We can cite Wilkie, 1984, revisited in Wilkie, 1995 and Cairns et al., 2008, Ahlgrim et al., 2005 for modelling financial economic scenarios and the models of Møller and Steffensen, 2007 for the liability simulations.

The Wilkie investment model – from the name of its author Professor A. David Wilkie, 1984 – is described in Appendix 2. It is a financial asset model that seeks to describe the behavior of inflation and various economics factors over time. Originally based on data for the United Kingdom from 1919 to 1982, the Wilkie model is formulated as a system of autoregressive processes where each one characterizes an economic variable. Basically, the variables are the retail price index (inflation), the share yield, the share dividends and the consols yield (long term interest rate), interconnected by a cascade structure as shown in Figure 1.7. The wage inflation and the short term interest rate has been added in a later version of the model, see Wilkie, 1995. The main factor of the model which influences all asset prices is the retail price index (price inflation); it essentially drives the composite model.

As one would expect, there have been a lot of discussions in the literature. Daykin and Hey, 1990 have pointed out that the model tends to generate negative inflation with a probability that is higher than what observations actually show. From the Financial Management Group (FIMAG, UK) working group on the Wilkie model, Geoghegan et al., 1992 concluded that the model should be estimated using post-1945 data, including as recent data as are available, this in order to integrate the fundamental changes that had affected the process generating the data after the

Second World War. The constant variance of the model has also been questioned. Ludvik, 1993 has noticed that the model tends to underestimate the correlation between the UK stocks and the fixed interest rate security. The author also summarises the reasons why the short term behaviour of stochastic asset models is relevant in asset/liability studies for pension funds. An updated version of the Wilkie model is published in Wilkie, 1995 where the variables wages index, short term interest rate and property rentals are added to the model, and the framework exploits the cointegrated models and ARCH models. Huber, 1997 noted that certain economic theories and the constancy of the model's parameter values did not appear to have been specifically considered. In Whitten and Thomas, 1999, the application of non-linear modelling to investment series is explored, considering both ARCH techniques and threshold modelling. The authors suggest a threshold autoregressive system as a useful progression of the Wilkie model. In Cairns et al., 2008, the model parameters are updated to 2007 and the authors conclude that the model is still satisfactory even though the estimated parameter are not stable. The stability of the model parameters is discussed in Wilkie et al., 2011. Based on the Wilkie model, Şahin, 2010 introduces the yield-macro model which is also an investment model for actuarial use in the UK. More recently in Şahin et al., 2014, the yield-macro model is compared to Wilkie based on updated UK data (up to the year 2009).

Many applications of the hierarchy structural modeling method are done late on. Thomson, 1996 has applied an adjusted version of the Wilkie model to the case of South Africa. The ALM investment model introduced in Yakoubov et al., 1999 uses earnings rather than dividends to generate price returns and the equity return is divided into three components: dividend yield, earnings growth, and change in market rating. An overview of eight stochastic asset models is provided in Lee and Wilkie, 2000. The literature on asset simulation models is very generous. We can also cite the financial scenario simulation model of Ahlgrim et al., 2005, the Towers Perin (Mulvey and Thorlacius, 1998) model with a cascade structure using stochastic differential equations, the Falcon Asset model (Dempster et al., 1998) developped by Falcon Asset Management, among others.

1.5 Demographic risks

By definition, the pension scheme is an institution covering a specific population. Therefore, the demographic risks regroup all the uncertainties that arise from the possible future changes in the pension fund population, i.e. active and inactive

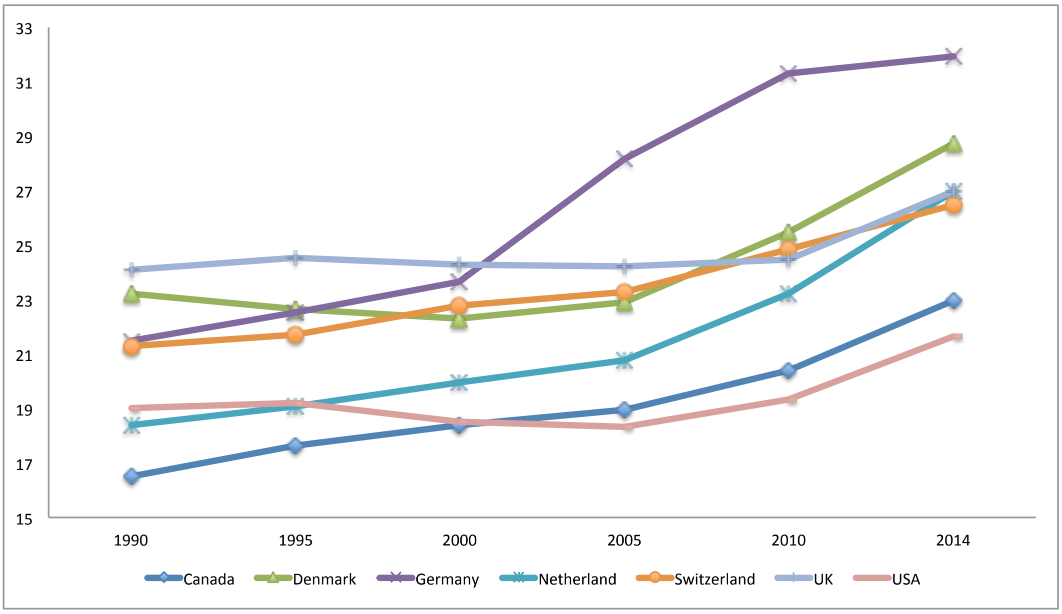


FIGURE 1.8: Evolution of the old age dependency ratio in selected OECD countries (OECD, 2015)

members of the fund. The changes can be either quantitative – related to the size of the population – or qualitative concerning the structure of the population, e.g. proportion of actives or non-actives. Among the risks affecting the population of a pension fund, we have the longevity risk, the renewal risk and the lapse risk, see Devolder et al., 2013.

As mentioned above, longevity leads to an (unexpected) increase of the proportion of retirees in the pension population. The increase is essentially due to the developments in mortality and life expectancy. The Figure 1.8 displays the evolution since 1990 of the old age dependency ratio in selected OECD countries. It can be seen that the trend line is up for all the selected countries showing that the longevity is expected to continue increasing in the coming years. Since 2000, Germany is having the highest ADR with an increase of more than seven years over the fourteen-year period. In Switzerland, the change from 2000 to 2014 is about four years for a value of 26% in 2014. More details on the situation in Switzerland are given in Subsection 1.3.3.

The augmentation of the life expectancy has a serious drawback on a pension fund liabilities. A pension fund can either include mortality changes in its actuarial models and/or cede the risk to a third party, i.e. insurer, reinsurer or an other financial institution. For the first approach, there are several models that include the mortality improvements in the literature. These models use the prospective mortality table that follows different cohorts over time. The starting point in this field is the Lee and Carter, 1992 method in which mortality is determined by two components namely the age and the time. The theory is based on the time series analysis for which the mortality rate μ_a is given by

$$\ln \mu_a (t) = \alpha_a + \beta_a \kappa_t + \epsilon_{at}$$

where α_a, β_a and κ_t are parameters to be estimated and ϵ_{at} is a set of random disturbances. The index a indicates the age group (or the age) whereas t is the time index. For estimation purposes, the following contraints have been added to the model

$$\sum_a \beta_a = 1 \qquad and \qquad \sum_t \kappa_t = 0.$$

The ordinary least squares provide the theoretical background for estimating the parameters α, β and κ, i.e. by minimizing the squared errors. That is

$$\left(\hat{\alpha}, \hat{\beta}, \hat{\kappa}\right) = \arg \min_{(\alpha,\beta,\kappa)} \sum_{a=a_{\min}}^{a_{\max}} \sum_{t=t_{\min}}^{t_{\max}} \left(\ln \hat{\mu}_a (t) - \alpha_a - \beta_a \kappa_t\right)^2 .$$

We refer the reader to Lee and Carter, 1992 and Bell, 1997 for a complete description of the model. Lee, 2000, Lee, 2003 and Delwarde and Denuit, 2005 introduce actuaries to the concept with several extensions and applications. It must however be noted that the model does not allow for the inclusion of expert judgments concerning the future advances in mortality. Thus, no information other than previous history can be introduced. The reader is referred to Gutterman and Vanderhoof, 1998 for a complete discussion on the question.

For a pension fund, an other way for managing the longevity risk consists in transfering this risk to insurers or reinsurers, e.g. by buying life annuities from insurance companies that then pay the pensions. In this respect, longevity linked securities are getting popular nowadays on the financial market, see e.g. Barrieu and Albertini, 2010 (Sections 20 and 21).

In addition, the renewal risk is related to the decrease in the new members entering the fund, leading to a reduction of the total contribution while the lapse risk arises from members who leave the pension fund before retirement (e.g. disability, dismissal, etc.).

1.6 Conclusion

This chapter provides a literature review of the asset liability management (ALM) methodologies in pension funds. Before describing the different models, it is of interest to explain the particular features of pension schemes. In order to emphasize those caracteristics, the Swiss pension system is described and compared to some other OECD countries. It can be seen from the observations that the Swiss system performs quite well.

Two types of risks have been identified for pension funds: the demographic risks and the financial risks. The ALM framework provides the theoretical background for managing the financial risks. Several approaches including deterministic ALM, asset only, surplus optimization, stochastic control, stochastic programming and Monte Carlo simulations are discussed. The pros and cons of each model are also analysed. It can be seen that the deterministic ALM does not allow to take advantage of the financial market. The liabilities of the pension fund are not taken into account in the asset only framework whereas the surplus optimization and the stochastic control models are difficult to implement in practice. Among the different ALM approaches discussed, only the stochastic programming and the simulation methods allow for the integration of the main features of a pension fund in a single model. Chapter 2 discusses the role of integrated chance constraints as part of a stochastic program in ALM for pension funds.

Concerning the demographic risks, longevity is the most important and a brief introduction to the Lee and Carter model is given. Another source of demographic risk is the mortality. Chapter 3 combines the copula models and the Gompertz mortality law to model the bivariate lifetime of married couples.

1.7 Appendices

1.7.1 Appendix 1: Stochastic control: a case study

As an example, we discuss the case of a pension fund whose goal is to determine the best investment policy $\mathbf{w} := (w_t)_{t \geq 0}$ that maximizes the expected utility of the terminal funding ratio. We consider the Merton portfolio allocation problem with liability such as presented in Martinelli, 2007. We consider a financial market in which $n+1$ assets are traded continuously within the time horizon $[0, T]$. There are n risky assets and one risk-free asset. The process of the risk-free asset value P_t^0 satisfies the following differential equation:

$$\mathrm{d}P_t^0 = P_t^0 r \mathrm{d}t \tag{1.8}$$

with $P_0^0 = p_0 > 0$ and r is the risk free rate. The price process of each risky asset i is $P_t^i, \ i \in \{1, \cdots, n\}$ and satisfies the following differential equation:

$$\mathrm{d}P_t^i = P_t^i \left\{ \mu_i \mathrm{d}t + \sum_{j=1}^{n} \sigma_{ij} \mathrm{d}W_t^j \right\} \tag{1.9}$$

where $W_t = \left(W_t^1, \cdots, W_t^n\right)$ is a standard $\mathcal{F}_{t \geq 0}-$adapted $n-$dimensional Wiener process which is described through a standard probability space (Ω, A, P); $\mathcal{F}_\infty \subset A$ and $\mathcal{F}_0$ is trivial. We set $P_0^i = p^i, \ i \in \{1, \cdots, n\}$. The vector $\mu = (\mu_i)'_{i=1,\cdots,n}$ and the matrix $\sigma = (\sigma_{ij})_{i,j=1,\cdots,n}$ are, respectively, the expected return vector and the asset return variance-covariance matrix. Besides, σ, μ and r are progressively measurable and uniformly bounded processes in $[0, T]$ with σ a non singular matrix. In principle, the variables σ, μ and r are function of time and a state variable X which regroups various sources of uncertainty influencing the value of assets and liability. Here, for sake of simplicity, we assume they are constant. The liability process is

$$\mathrm{d}L_t = L_t \left\{ \mu_L \mathrm{d}t + \sum_{j=1}^{n} \sigma_{L,j} \mathrm{d}W_t^j + \sigma_{L,\epsilon} \mathrm{d}W_t^\epsilon \right\} \tag{1.10}$$

where W_t^ϵ is a standard brownian motion uncorrelated to W and μ_L the appreciation rate of the value of total liability. The variable W_t^ϵ incorporates all the sources of uncertainty specific to liability risk such as salary growth, inflation and actuarial risks.

Now, we consider an investor that invests her initial wealth A_0 into the financial market subject to liability with an initial value L_0. At any time t, the total asset A_t

is invested into the n risky assets and the risk-free asset. Let $\mathcal{W}$ be the set of admissible strategies. The investment policy $\mathbf{w} = \left(w_t^{'} = (w_{1t}, \cdots, w_{nt})\right)_{t\geq 0}$, $\mathbf{w} \in \mathcal{W}$, is predictable and describes, at each time t, which proportion to allocate into each asset. Thus, the total asset process $A_t^{\mathbf{w}}$, describing the wealth at time t under the strategy $\mathbf{w}$, satisfies the following differential equation:

$$\begin{aligned} \frac{\mathrm{d}A_t^{\mathbf{w}}}{A_t^{\mathbf{w}}} &= \left(1 - w_t^{'}\mathbf{1}\right)\frac{\mathrm{d}P_t^0}{P_t^0} + w_t^{'}\frac{\mathrm{d}P_t}{P_t} \\ &= \left(r + w_t^{'}(\mu - r\mathbf{1})\right)\mathrm{d}t + w_t^{'}\sigma \mathrm{d}W_t \\ \Rightarrow \mathrm{d}A_t^{\mathbf{w}} &= A_t^{\mathbf{w}}\left[\left(r + w_t^{'}(\mu - r\mathbf{1})\right)\mathrm{d}t + w_t^{'}\sigma \mathrm{d}W_t\right] \end{aligned}$$

where $\mathbf{1}$ is the $n-$dimensional vector of ones. The process $A_t^{\mathbf{w}}$ is therefore a geometric brownian motion with a drift $\left(r + w_t^{'}(\mu - r\mathbf{1})\right)$ and a volatility $\left(w_t^{'}\sigma\right)$. We recall the objective of the pension fund which is to maximize the expected utility of the terminal funding ratio:

$$\max_{\mathbf{w}\in W} \mathbf{E}_{0,F_0}\left[U\left(F_T^{\mathbf{w}}\right)\right] \tag{1.11}$$

with $F_T^{\mathbf{w}} = \frac{A_T^{\mathbf{w}}}{L_T}$, $U(\bullet)$ the utility function and $\mathbf{E}_{t,F_t}[\bullet]$ the conditional expectation at time t for an initial funding ratio F_0. A simple application of Ito's Lemma implies that the instantaneous funding ratio stochastic process under strategy $\mathbf{w}$ is given by

$$\mathrm{d}F_t^{\mathbf{w}} = \mathrm{d}\left(\frac{A_t^{\mathbf{w}}}{L_t}\right) = \frac{1}{L_t}\mathrm{d}A_t^{\mathbf{w}} - \frac{A_t^{\mathbf{w}}}{L_t^2}\mathrm{d}L_t - \frac{1}{L_t^2}\mathrm{d}A_t^{\mathbf{w}}\mathrm{d}L_t + \frac{A_t^{\mathbf{w}}}{L_t^3}\left(\mathrm{d}L_t\right)^2$$

and can be rewritten as

$$\begin{aligned} \frac{\mathrm{d}F_t^{\mathbf{w}}}{F_t^{\mathbf{w}}} &= \left(\left(r - \mu_L + \sigma_L^{'}\sigma_L + \sigma_{L,\epsilon}^2\right) + w_t^{'}\left((\mu - r\mathbf{1}) - \sigma\sigma_L\right)\right)\mathrm{d}t \\ &\quad + \left(w_t^{'}\sigma - \sigma_L^{'}\right)\mathrm{d}W_t - \sigma_{L,\epsilon}\mathrm{d}W_t^{\epsilon}. \end{aligned} \tag{1.12}$$

Let us define the mean return and the volatility of the funding ratio portfolio such that

$$\begin{aligned} \mu_F(w_t) &\equiv \left(r - \mu_L + \sigma_L^{'}\sigma_L + \sigma_{L,\epsilon}^2\right) + w_t^{'}\left((\mu - r\mathbf{1}) - \sigma\sigma_L\right) \\ \sigma_F(w_t) &\equiv \left(\left(w_t^{'}\sigma - \sigma_L^{'}\right)^{'}\left(w_t^{'}\sigma - \sigma_L^{'}\right) + \sigma_{L,\epsilon}^2\right)^{\frac{1}{2}} \end{aligned}$$

and we consider the derived associated value function:

$$V(t, F_t) = \max_{\mathbf{w}\in W} V^{\mathbf{w}}(t, F_t)$$

where the objective $V^{\mathbf{w}}(t, F_t)$ is defined by:

$$V^{\mathbf{w}}(t, F_t) = \mathbf{E}_{t,F_t}\left[U\left(F_T^{\mathbf{w}}\right)\right].$$

The process F_t known, at time t, plays the role of state variable (see Merton, 2010 and Chapter 3 of Touzi, 2012). We define by $C^{1,2}$ a set of function one time differentiable in t and twice differentiable in F_t. For a function $\varphi(t, F_t) \in C^{1,2}$ and assuming a constant strategy w, the infinitesimal generator $\mathcal{L}$ applied to function ψ leads to

$$\mathcal{L}^w \varphi(t, F_t) = \varphi_t + F\varphi_F \mu_F^w + \frac{1}{2} F^2 \varphi_{FF} \left(\sigma_F^w\right)^2$$

where φ_t and φ_F are respectively the first derivative with respect to t and F respectively. The symbol φ_{FF} denotes the second derivative of function φ with respect to F. This equation is derived using the Ito lemma. In our problem, the HJB equation associated is given by

$$\sup_{w \in W} \left\{\mathcal{L}^w V(t, F_t)\right\} = 0 \qquad \text{subject to } V^w(T, F_T) = U(F_T). \tag{1.13}$$

Assuming that there exists a maximizer w^* and the function $V \in C^{1,2}$, the optimization of (1.13) with respect to w leads to the following differential equation

$$\begin{aligned} & FV_F \frac{\partial \mu_F^w}{\partial w}(w^*) + \frac{1}{2} V_{FF} \frac{\partial \left(\mu_F^w\right)^2}{\partial w}(w^*) = 0 \\ \Leftrightarrow \quad & FV_F\left((\mu - r\mathbf{1}) - \sigma\sigma_L\right) + \frac{1}{2} V_{FF}\left(w^{*'}\sigma\sigma' - \sigma\sigma_L\right) = 0. \end{aligned}$$

The optimal asset allocation obtained from this equation is

$$\begin{aligned} w^* = w^*(t, F_t) &= -\left(\sigma\sigma'\right)^{-1}\left((\mu - r\mathbf{1}) - \sigma\sigma_L\right)\frac{V_F(t, F_t)}{FV_{FF}(t, F_t)} + \left(\sigma\sigma'\right)^{-1}\sigma\sigma_L \\ &= -\left(\sigma\sigma'\right)^{-1}(\mu - r\mathbf{1})\frac{V_F(t, F_t)}{FV_{FF}(t, F_t)} + \left(1 + \frac{V_F(t, F_t)}{FV_{FF}(t, F_t)}\right)\left(\sigma'\right)^{-1}\sigma_L, \end{aligned}$$

highlighting the three funds separation theorem. That is, the proportion $w_M = \frac{\left(\sigma\sigma'\right)^{-1}(\mu - r\mathbf{1})}{\mathbf{1}\left(\sigma\sigma'\right)^{-1}(\mu - r\mathbf{1})}$ called the *performance seeking portfolio* (PSP) serves, in a first hand, to generate high returns on the market and is similar to the standard mean variance efficient portfolio. Secondly, the proportion $w_L = \frac{\left(\sigma'\right)^{-1}\sigma_L}{\mathbf{1}\left(\sigma'\right)^{-1}\sigma_L}$ called the *liability hedging portfolio* (LHP) helps to control the variability of the pension fund liabilities. Finally, the rest is invested in risk-free asset. We refer the reader to Martinelli, 2007 for further details concerning this example.

The obtained solution is subject to the following assumptions:

- the value function $V \in C^{1,2}$ (not allways fullfilled) and
- a maximizer exists.

For the special case of a risk averse power utility function of the type:

$$U\left(F_T\right) = \frac{\left(F_T\right)^{1-\gamma}}{1-\gamma} \tag{1.14}$$

where γ is the risk aversion parameter, $U \in C^{1,2}$, the ratio $\frac{V_F}{FV_{FF}} = \frac{1}{\gamma}$ (also called the Arrow-Pratt coefficient of risk tolerance) and the optimal strategy w^* is given by

$$w^* = \left(\sigma\sigma^{'}\right)^{-1}\left(\mu - r\mathbf{1}\right)\frac{1}{\gamma} + \left(1 - \frac{1}{\gamma}\right)\left(\sigma^{'}\right)^{-1}\sigma_L.$$

The existence of a unique solution can be checked by the way of the verification theorem, see Schmidli, 2007.

We are aware that many pension fund particularities are not taken into account in the ALM SC model treated in our example. In a first hand, the cash flows dynamics $-$ pension benefits and yearly contributions $-$ are not brought out. On the other hand, there is no pension fund realistic restrictions such as portfolio, legal, budget, risk, regulatory and operating constraints. That said, the problem has been simplified in order to ease understandings. *More realistic* ALM problems for pension funds using SC approaches and explanations can be found in Cairns, 2000 and Devolder et al., 2013 (Chapters 7 and 8).

1.7.2 Appendix 2: The Wilkie investment model

Let ∇ denote the backwards difference operator where

$$\nabla X\left(t\right) = X\left(t\right) - X\left(t-1\right).$$

Considering the consumer price index $Q\left(t\right)$ at time t, $\nabla \ln Q\left(t\right) = \ln Q\left(t\right) - \ln Q\left(t-1\right)$ measures the force of inflation over $(t-1, t)$. The variable $I\left(t\right) := \nabla \ln Q\left(t\right)$ follows an autoregressive process of order 1 ($AR1$) such that

$$\begin{aligned} I\left(t\right) &= \nabla \ln Q\left(t\right) \sim AR1\left(QMU,\ QA,\ QSD\right) \\ \Leftrightarrow I\left(t\right) &= QMU + QA\left(I\left(t-1\right) - QMU\right) + QSD \times QZ\left(t\right) \end{aligned} \tag{1.15}$$

and we can derive

$$Q\left(t\right) = Q\left(t-1\right) \times \exp\left\{I\left(t\right)\right\}.$$

Here, the expression $QSD \times QZ(t)$ is a zero-mean white noise with a standard deviation QSD whereas QMU and QA are respectively the fixed mean and the parameter of the $AR1$ model. The equation (1.15) assumes that the force of inflation only depends on its previous value. As specified in Geoghegan et al., 1992, the parameter estimation will show that retail price inflation has a normal distribution, with constant mean and standard deviation increasing at a decreasing rate over time toward an upper limit.

As shown in Figure 1.7, the price index directly influences the other variables of the model. The share yield process $Y(t)$ is directly dependent of the current level of the force of inflation $I(t)$ and its previous values contained in $YN(t)$. That is given by

$$\ln Y(t) = YW \times I(t) + YN(t)$$

where YW is an inflation factor and $YN(t)$ is a first order autoregressive process such that

$$\begin{aligned} & YN(t) \sim AR1(\ln YMU, YA, YSD) \\ \Leftrightarrow \ & YN(t) = \ln YMU + YA(YN(t-1) - \ln YMU) + YSD \times YZ(t) \end{aligned}$$

In the process $YN(t)$, $YSD \times YZ(t)$ is the zero-mean white noise with a standard deviation YSD, $\ln YMU$ is the mean and YA is the parameter of the $AR1$ model. The model estimation is made using historical data and similar from the Financial Times Actuaries All share index from 1919 to 1982.

The current level of the share dividend $D(t)$ is dependent of itself at time $t-1$ contained in $DM(t)$, the white noise of the share yield $YSD \times YZ(t)$ and the price inflation $I(t)$, and is given by

$$\begin{aligned} \nabla \ln D(t) = {} & DW \times DM(t) + DX \times I(t) + DMU \\ & + DY \times YSD \times YZ(t-1) + DB \times DSD \times DZ(t-1) + DSD \times DZ(t) \end{aligned}$$

where

$$DM(t) = DD \times I(t) + (1 - DD)\, DM(t-1), \tag{1.16}$$

$DSD \times DZ(t)$ is a zero-mean white noise with a standard deviation DSD and, DW, DX, DD, DMU, DB and DY are the model parameters. The effect of inflation can be identified through the expression $DW \times DM(t) + DX \times I(t)$. Once we have the dividend yield and share yield models, the share price at time t can be obtained by

$$P(t) = \frac{D(t)}{Y(t)} \text{ or } \ln P(t) = \ln D(t) - \ln Y(t).$$

We can next calculate the value of the share index $PR(t)$

$$PR(t) = PR(t-1)\left(\frac{P(t) + D(t)(1 - tax\ A)}{P(t-1)}\right)$$

where $tax\ A$ is the rate of tax on share dividends and the dividends, net of tax, are reinvested in shares.

The last variable of the model is the consols yield and it can be considered as the long term rate of return over bond. According to Wilkie, this variable is formed by the inflation rate and a real part term $CN(t)$. That is

$$C(t) = CW \times CM(t) + CMU \times \exp\{CN(t)\}$$

where the influence of the inflation rate is expressed through

$$CM(t) = CD \times I(t) + (1 - CD)\, CM(t-1)$$

and the process $CN(t)$ is defined by a three order autoregressive model given by

$$\begin{aligned} CN(t) = {} & CA_1 \times CN(t-1) + CA_2 \times CN(t-2) + CA_3 \times CN(t-3) \\ & + CY \times YSD \times YZ(t) + CSD \times CZ(t). \end{aligned}$$

The expression $CSD \times CZ(t)$ is a zero-mean white noise with a standard deviation CSD and the parameters of the model are CA_1, CA_2, CA_3, CY, CW and CD. The consols index is derived from the process $C(t)$ by

$$CR(t) = CR(t-1)\left(\frac{1}{C(t)} + 1\right) C(t-1).$$

The model has come to be widely used in actuarial work (mainly in the UK) and is hence a benchmark for future development. The fact that some parameter values are proposed is interesting in application: e.g. $QMU = 0.05$, $QA = 0.6$ and $QSD = 0.05$ in the UK; see Wilkie, 1984 for all parameter values. The model can be used for simulations of possible future extending for many years ahead.

Chapter 2

On Integrated Chance Constraints in ALM for Pension Funds

This chapter is based on the preprint paper Toukourou and Dufresne, 2015. The latter has been awarded as best paper by the International Actuarial Association at its annual colloquium in Oslo, April 2015.

2.1 Introduction

A pension fund is any plan, fund or scheme, established by a company, governmental institution or labour union, which provides retirement incomes. The actuarial present value of current and future payments constitutes the total liability of the fund. The pension fund receives contributions from its active members and/or the employer. This money (considered as the total wealth or total asset) is invested in a wide range of assets. The asset allocation is made in such a way that it guarantees, to a certain extent, the payments of future obligations. That is not so trivial: assets yield random returns and future benefits are not known with certainty. An asset liability management study (ALM) provides a rich theoretical background to address that issue. Its goal is to determine the adequate asset allocations and contribution rates in order to guarantee the payment of current and future pensions.

The use of ALM methods has a long tradition in pension funds. At the beginning, it has started with *deterministic methods*, whose goal is to determine the adequate allocation that protects the fund from any unexpected movements in cash flows and/or interest rate. The future cash flows are estimated and assumed to be certain; the wealth is mainly allocated to bonds considered as risk free and the assets are deterministic. Bonds are chosen in such a way that their related incomes correspond to yearly pension payments. The models are essentially based on immunization and cash-flow matching; see, for example, Koopmans, 1942 and Redington, 1952. The drawbacks of different deterministic methods are examined in

Hiller and Schaack, 1990. Two key points should be emphasized here. Firstly, it is difficult (if not impossible) to find bonds that enable a perfect match for a realistic pension fund problem. Secondly, these models could be costly since the yields on bonds are low.

A possible way to reduce the cost of the pension fund is to invest in riskier assets. The *mean-variance* portfolio of Harry Markowitz, 1952 provides a good compromise between the high yield and the pension funds level of risk. In this respect, the liability is evaluated deterministically and the decision maker has to determine the lowest risk portfolios in order to meet the estimated liability. This approach is known as the *asset only* method. The integration of the liability in the traditional mean-variance problem has led to the *surplus optimization* theory. The surplus is generally defined as the total asset minus the total liability. As interest rate and future payments are random, the total liability is a random variable influenced by many factors. The factors can be actuarial or economic. The economic risks are often related to inflation, salary growth and discount rates. As these risks also have an effect on the total amount of asset, it is important to include the random liability as part of the ALM model and not only as the target wealth for the pension fund portfolio optimization. More specifically, the correlations between assets and liabilities are thus considered in the determination of the optimal portfolio. For example, the assets, with higher covariance to liabilities, tend to reduce the risk exposition of the pension fund, see Sharpe and Tint, 1990 and Keel and Müller, 1995. Leibowitz and Henriksson, 1988 showed that an asset such as cash, which should typically reduce the riskiness of an all-asset portfolio, may actually increase the riskiness of a portfolio that includes liabilities. Including liabilities in the optimal asset allocation decision has a long story in finance theory. We refer the readers to Sharpe and Tint, 1990, Elton and Gruber, 1992, Leibowitz et al., 1992 among others. The multi-period ALM case has been discussed in Fama, 1970, Hakansson, 1970 and Hakansson, 1974 and Merton, 1969).

The pension fund problem is a long term problem with a horizon span of many decades. Hence, its model should be dynamic. Furthermore, regulations often impose many types of constraints. Those matters are hardly taken into account by surplus optimization methods. In practice, *Monte Carlo methods* are commonly used due to their ability to incorporate the above issues. Initially, they consist on defining a set of feasible allocations and contribution rates, and chosing the best one in some sense. The choice is based on the simulation of the future paths. Due to the technical innovations, these methods have significantly evolved with the work of Wilkie, 1995 and Ahlgrim et al., 2005 concerning economic scenario generation. Møller and Steffensen, 2007 provide different tools for valuing the pension fund liabilities. Recent years have also seen the emergence of methods known as *stochastic programming* (SP).

Based on a scenario approach, SP gives a flexible and powerful tool for ALMs. Its importance lies in its ability to easily incorporate various types of constraints, Zenios and Bertola, 2006. Moreover, assets and liabilities are all influenced by many sources of risk and the risk aversion is accommodated; the framework has a long time horizon split into subperiods (*multistage*); the portfolio can be rebalanced dynamically at the beginning of each subperiod; all these are incorporated in a single and consistent structure while satisfying operational or regulatory restrictions and policy requirements. SP for ALM is rooted with the work of Kusy and Ziemba, 1986 who showed, based on a five-year period application to the Vancouver City Saving Credit Union, that SLP is theoretically and operationally superior to a corresponding deterministic linear programming (LP) model. The authors have proved that the effort required for the implementation of ALM and its computational requirements are comparable to those of the deterministic model. For a Japanese insurance company, Carino et al., 1994 developed a model that enables the decision makers to include risk management tools as well as including the complex regulations imposed by the Japanese insurance laws and practices. Over the two years of experiment, the resulting investment strategy has been fruitful as it has yielded extra income of 42 basis points (US$79 million). More recently, Geyer and Ziemba, 2008 − for the Austrian pension fund of the electronics firm Siemens − has implemented a model that allows specific features such as state-dependent correlation matrices and fat-tailed asset return distributions. Considering a Finnish pension company, Hilli et al., 2007 focuses on the modeling of the stochastic factors and analyses the obtained numerical solution. Dert, 1995 pioneered the inclusion of *chance constraints* (CC) in multistage recourse models for pension funds. Chance constrained programs often lead to integer programming for which, it may be difficult to determine a tractable solution. As an alternative to chance constraints, Haneveld et al., 2010 proposed the *integrated chance constraints* (ICC), whose feasibility set is more handleable as it does not require integer programming. The literature of SP in ALM also includes Consigli and Dempster, 1998, Bogentoft et al., 2001, Drijver, 2005, Faleh, 2011, Aro and Pennanen, 2013 among others.

The ALM model in this paper is a MSP, for which, we minimize the total funding cost under risk, legal, budget, regulatory and operating constraints. The total funding cost is composed of regular and remedial contributions. Regular contribution constitutes a certain proportion (contribution rate) of the total salary whereas remedial contribution is an additional financial support provided by the employer (or a sponsor) whenever the solvency target is in question. More specifically, we focus on the risk constraints, which are of ICC type in this work. The ICC is computationaly of great interest; in particular when a quantitative risk measure is preferable. We define the *funding ratio* as the ratio of total assets over total liabilities. Our goal is to meet a certain funding ratio, called here *target funding ratio*, at the end of each subperiod. For a predefined target funding ratio, the ICC put an upper bound on the expected shortfall, i.e. the expected amount by which the goal is not attained. Haneveld et al., 2010 and Drijver, 2005 pionnered the application of ICC in ALM for pension fund. However, the risk parameter considered in their models is neither scale free, nor time dependent. Our model is close to Haneveld et al., 2010 with the particularity that the risk parameter is a linear function of total liabilities. Then, it becomes unvariant with respect to the size of the fund as well as time dependent. We define two types of ICC: the *one period* integrated chance constraint (OICC) and the *multiperiod* integrated chance constraint (MICC). As their names suggest, the OICC covers only one period whereas several periods are taken into account with the MICC. A multistage stochastic *linear* program is therefore developed for this purpose and a special mention is paid to the modeling of the MICC. We also measure how conservative is MICC comparing to the one period approach and the results strenghthen the evidence that MICC is a good alternative to OICC.

The rest of the paper is organized as follows. In Section 2.2, the theoretical background, the dynamics and the ALM optimization problem are extensively detailed. Section 2.3 defines the risk contraints and shows how CC leads to ICC. Moreover, OICC and MICC are introduced and their stochastic linear program reformulations are derived as well. In Section 2.4, a numerical example is examined from the perspective of a defined benefit fund that invests in stocks, real estate, bonds, deposits and cash. All numerical results are implemented using the solver CPLEX in the mathematical programming language AMPL. We first analyse the effect of the risk parameter on the optimal decisions. This section finishes by a brief comparison of the two ICCs.

2.2 Settings

2.2.1 Multistage recourse models

In this section, we describe the classical architecture of the multiperiod decision framework. The model's setup presented here ressembles mostly to Haneveld et al., 2010.

Since we aim for strategic decisions, we model the ALM process over a number of years and one set of decisions is taken each year. We discretize time accordingly so that the model has a (finite) number of one-year time periods. Consequently, we assume that the ALM model has a horizon of T years from the beginning, split in T subperiods of one year each. The resulting years are denoted by an index t, where time $t = 0$ is the current time. By year t $(t = 1, \cdots, T)$, we mean the span of time $[t-1, t)$. We define

$$\mathcal{T}_t := \{t, t+1, \cdots, T\}.$$

We assume that uncertain parameters (e.g. asset returns) can be modeled as random variables with known distributions. At each time $t \in \mathcal{T}_0$, the pension fund is allowed to make decisions (corresponding to yearly corrections), based on the actual knowledge of parameters. During each one-year period, a realization of the corresponding random parameters becomes known (e.g. assets return during that year). That is, the concept underlying our model is the following sequence of decisions and observations:

$$\overset{\text{decide}}{X_0} \rightsquigarrow \overset{\text{observe}}{\omega_1} \rightsquigarrow \overset{\text{decide}}{X_1} \rightsquigarrow \quad \cdots \quad \rightsquigarrow \overset{\text{observe}}{\omega_{T-1}} \rightsquigarrow \overset{\text{decide}}{X_{T-1}} \rightsquigarrow \overset{\text{observe}}{\omega_T}$$

where X_t is the vector of decision variables at time $t \in \mathcal{T}_0$, and vector ω_t, $t \in \mathcal{T}_1$ models all economic events which are the source of uncertainty and risk for the pension fund management, which, in our case, are asset returns as well as random contributions and liability streams. Time t is assumed to be the end of the financial year t. We assume that a financial year coincides with a calendar year. At time $t \in \mathcal{T}_0$, decisions X_t are taken with full knowledge of the past $[0, t]$ but with only probabilistic informations about the future $(t, T]$.

Uncertainty in the model is expressed through a finite number S of sample paths spanning from $t = 0$ until $t = T$ called scenarios. That is, we assume the random variable follows a discrete distribution with S possible outcomes. Each scenario represents a sequence of possible realizations of all uncertain parameters in the model. As explained above, ω_t is the stochastic vector process whose values are revealed in year t. Then, the set of all scenarios is the set of all realizations

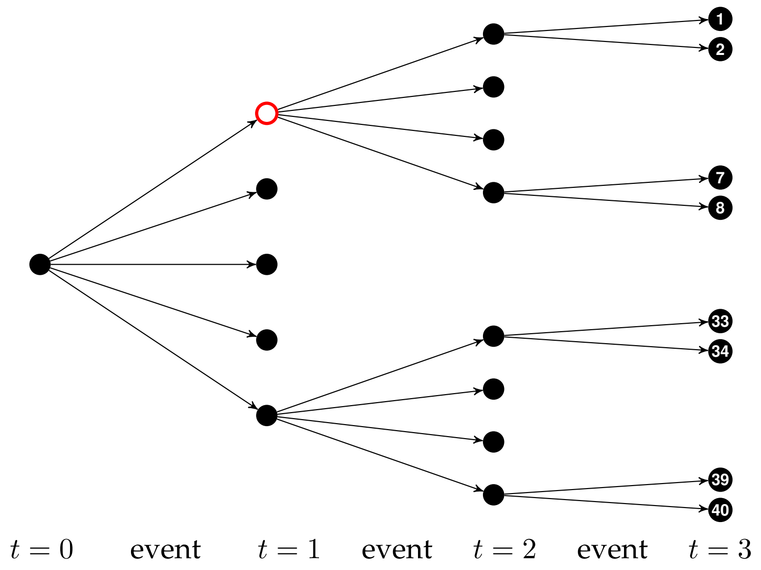


FIGURE 2.1: A scenario tree with 40 scenarios and 66 nodes.

$\omega^s := (\omega_1^s, \cdots, \omega_T^s)\,, s \in \mathcal{S} := \{1, \cdots, S\}$ of $\omega := (\omega_1, \cdots, \omega_T)$. Scenario s has a probability p^s, where $p^s > 0$ and $\sum_{s=1}^{S} p^s = 1$. It represents a description of possible future, starting just after $t = 0$. If we assume that we can observe the "state of the world" at time $t, (0 < t < T)$, then there is a unique history of realizations of $(\omega_1, \cdots, \omega_{t-1})$ leading to that state, but the future as seen from time t may unfold in several ways. That is, there are several distinct scenarios which share a common history up to time t. A suitable representation of the set of scenarios is given by a scenario tree (see Figure 2.1). In respect to Figure 2.1, we define the node as the possible outcome of the stochastic vector ω_t at a given time $t \in \mathcal{T}_0$. Each path of ω_t from $t = 0$ to $t = 3$ represents one scenario; each node of the scenario tree has multiple sucessors, in order to model the process of information being revealed progressively through time. By convention, the scenarios are numbered top-down by their end node. The arcs in the tree denote realizations in one time period. We assume here that, for a specific decision time $t \in \mathcal{T}_0$, the numbers of realizations in one time period descending from the current nodes are identical.

For example in Figure 2.1, we have a 3-year horizon scenario tree with 40 scenarios. Over the first period starting from time 0 to time 1, there are five possible realizations. From each of these realisations, we have four possible outcomes over the second year; each of them is a conditionnal realisation as it depends on the preceding node. Over the third period, each of the second period observations can lead to two possible outcomes. All this gives a branching structure of $1-5-4-2$ and leading to a total of $S = 5 \times 4 \times 2 = 40$ possible scenarios.

A multistage recourse model is an optimization problem defined on such a scenario tree. Considering the remaining future represented by the subtree rooted at (t, s), optimal decisions are taken for each node (t, s) of the event tree, given the informations available at that point. Optimality is defined in terms of current costs plus expected future costs, which are computed with respect to the appropriate conditional distributions, Vlerk et al., 2003.

Ideally, one would like to make different decisions for every path at every $t \in \mathcal{T}_0$, but this would lead to undesirable anticipativity in the model. The simplest way to avoid this is to make one single decision at each time t for all paths by adding explicit constraints. That is, for any two different scenarios s_1 and s_2 ($s_1, s_2 \in \mathcal{S}$ and $s_1 \neq s_2$) having the same history up to time $t \in \mathcal{T}_0$, we enforce $X_t^{s_1} = X_t^{s_2}$, where X_t^s is the decision X_t under scenario s. For example, at the empty circle of Figure 2.1, $X_1^1 = X_1^2 = \cdots = X_1^8$.

2.2.2 Dynamics

Assets

In this chapter, we are considering a dynamic asset allocation model applied to a DB plan in which one seeks to minimize the expected cost of funding. In this respect, dynamics for both assets and liabilities should clearly be specified.

At initial time $t = 0$, the exact levels of wealth and liability are available to the decision maker who has to decide, each period, how to rearrange his portfolio in order to cover liabilities and, at the same time, to achieve high returns on the financial market. The higher the returns are, the lower the contribution rate could be. Let denote A_t the total amount of wealth at time $t \in \mathcal{T}_0$. The total wealth is allocated into d classes of assets and in cash. Let $k \in \mathcal{K} := \{1, ..., d\}$ denote the asset class index. At each decision time $t \in \mathcal{T}_0$, a specified amount of $H_{k,t}$ is allocated to asset k and C_t is the cash amount. We can write

$$A_t = \sum_{k=1}^{d} H_{k,t} + C_t.$$

Through buying and selling, the investor restructures his portfolio at each time t. Once the t^{th} stage decision is made, the holdings $H_{k,t}$ can be calculated. The shares in the portfolio are then kept constant till the next decision time. The value of $H_{k,t}$ is affected by the returns on the market. Let define $\xi_{k,t} := 1 + r_{k,t}$ where $r_{k,t}$ is the random rate of return on asset class k over year t.

Over year t, the pension fund pays benefits to its non-active members and receives contributions from its active members or/and the employer (also called the sponsor). Benefits regroup pensions which are paid to retirees, disability and death annuities or lump sum, whereas contributions are composed of yearly payments from all the active members and/or sponsor to the plan. When it appears that the plan is unfunded according to its solvency target, the sponsor may finance the deficit. As in Vlerk et al., 2003, we name this funding here as remedial contribution. We then assume that whenever the solvency target is not fulfilled, a remedial contribution in cash can be obtained from the sponsor. In practice, it does not really work that way. For example in Vlerk et al., 2003, the remedial is only provided after two consecutive periods of underfunding. We will see in the model description that the parameters are set such that the remedial contribution variable is non-zero only under some conditions. In general for DB plans, future benefits and liabilities depend on company policy regulation and can be estimated whereas yearly contribution is defined as a certain proportion of the yearly salary. Asset allocation and contribution rate are defined with respect to the level of future benefits and liabilities (e.g. Switzerland). Kim, 2008 provides a rich source of informations concerning different types of pension plans and features. During year t, let Ben_t and W_t denote, respectively, the total amount of benefits paid and the level of salary. The variable cr_t is the decided contribution rate for year $t+1$. For returns and cash-flow variables, index t means that payments occur over year t but cash-flows are accounted at the end of year. Accordingly, the total asset dynamic is modeled as

$$A_t = \sum_{k=1}^{d} H_{k,t-1}\xi_{k,t} + C_{t-1}\left(1+r_f\right) + cr_{t-1}W_t - \mathrm{Ben}_t + Z_t = \sum_{k=1}^{d} H_{k,t} + C_t, \quad (2.1)$$

for $t \in \mathcal{T}_1$, where r_f is the risk free interest rate and Z_t is the remedial contribution at time t. Before receiving the remedial contribution at time t, the total wealth is defined as A_t^* and one can write

$$A_t^* = \sum_{k=1}^{d} H_{k,t-1}\xi_{k,t} + C_{t-1}\left(1+r_f\right) + cr_{t-1}W_t - \mathrm{Ben}_t = A_t - Z_t. \quad (2.2)$$

Asset returns $(\xi_t)_{t=1}^{T} := \begin{pmatrix} \xi_{1,t} & \cdots & \xi_{k,t} & \cdots & \xi_{d,t} \end{pmatrix}_{t=1}^{T}$, pension payments Ben_t and salary W_t are modeled as stochastic processes on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F})_{t=1}^{T}, \mathrm{P}\right)$. Obviously, at each decision time, A_t is a random variable whose distribution depends, on a first hand, on ξ_t, W_t and Ben_t, and on the second hand, on asset allocations before t. At a specific date t, the variable A_t is known as it can be observed.

The initial wealth is defined by $\bar{A}_0$ and known at initial time $t = 0$. According to (2.1), total wealth A_{t-1} at time $t-1$ is allocated into the d classes of assets and cash. Each asset k generates a return $\xi_{k,t}$ over period $[t-1, t]$. The initial wealth plus accumulated interest at the end of period will be augmented by the balance of external flows: contributions minus pension payments. This latter can be either positive or negative depending on the difference between contributions and benefits paid. A negative balance could be due to the fact that a company is no more hiring new employees. This may happen for various reasons: runoff, economic difficulties, etc. Obviously, the total contributions $cr_{t-1}W_t$ will decrease considerably as the total salaries decrease whereas Ben_t will tend to increase as people leave the fund. When at time t the total asset can not fulfill the pension fund solvency target, it may obtain a remedial contribution Z_t.

In order to be as close as possible to the reality on the financial market, one has to consider costs of trading activities. Therefore, we include proportional transaction costs $\bar{c}^B := \begin{pmatrix} \bar{c}_1^B & \dots & \bar{c}_k^B & \dots & \bar{c}_d^B \end{pmatrix}$ and $\bar{c}^S := \begin{pmatrix} \bar{c}_1^S & \dots & \bar{c}_k^S & \dots & \bar{c}_d^S \end{pmatrix}$ for purchases and sales, respectively. Inclusion of transaction costs will lead to some changes in asset dynamics. Thus, (2.1) is then replaced by

$$A_T = \sum_{k=1}^{d} H_{k,T-1}\xi_{k,T} + C_{T-1}(1+r_f) + cr_{T-1}W_T - \text{Ben}_T = A_T^* \tag{2.3}$$

over period $[T-1, T]$ and when $t \in \mathcal{T}_1 \setminus \{T\}$,

$$\begin{aligned} A_t &= \sum_{k=1}^{d} H_{k,t-1}\xi_{k,t} + C_{t-1}(1+r_f) + cr_{t-1}W_t - \text{Ben}_t + Z_t - \sum_{k=1}^{d} \left(\bar{c}_k^B B_{k,t} + \bar{c}_k^S S_{k,t}\right) \\ &= \xi_t H_{t-1} + C_{t-1}(1+r_f) + cr_{t-1}W_t - \text{Ben}_t + Z_t - \left(\bar{c}^B B_t + \bar{c}^S S_t\right) \\ &= A_t^* + Z_t - \left(\bar{c}^B B_t + \bar{c}^S S_t\right) \\ &= \mathbf{e} \cdot H_t + C_t \end{aligned} \tag{2.4}$$

where $\mathbf{e} := \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}$ is a $(1 \times d)$ vector. The vectors $H_t := \begin{pmatrix} H_{1,t} & \cdots & H_{k,t} & \cdots & H_{d,t} \end{pmatrix}^\top$, $B_t := \begin{pmatrix} B_{1,t} & \cdots & B_{k,t} & \cdots & B_{d,t} \end{pmatrix}^\top$ and $S_t := \begin{pmatrix} S_{1,t} & \cdots & S_{k,t} & \cdots & S_{d,t} \end{pmatrix}^\top$ of dimension $(d \times 1)$ each, are, respectively, amount of asset hold, bought and sold at each decision time $t \in \mathcal{T}_0$. In fact, (2.4) is obtained by substracting transaction costs in the first equality of (2.1). At time T, no more asset is bought or sold: $B_T = S_T = 0$; the value of the portfolio is determined by adding all values of assets including the last period returns and external flows. This justifies why there is no transaction cost in (2.3). The reader should notice that variables cr_t, Z_t, B_t, S_t and H_t are all decision variables. We denote by $\bar{H}_k$, the initial holding in asset k, $k \in \mathcal{K}$ and $\bar{H} := \begin{pmatrix} \bar{H}_1 & \cdots & \bar{H}_k & \cdots & \bar{H}_d \end{pmatrix}^\top$ is a $d \times 1$ vector. $\bar{C}_0$ is the initial cash amount. The first stage asset allocation is determined by

$$H_0 = \bar{H} + B_0 - S_0$$

with total asset

$$A_0 = \mathbf{e} \cdot \bar{H} + \bar{C}_0 + Z_0 - \left(\bar{c}^B B_0 + \bar{c}^S S_0\right) = \bar{A}_0 + Z_0 - \left(\bar{c}^B B_0 + \bar{c}^S S_0\right) = \mathbf{e} \cdot H_0 + C_0.$$

For $t \geq 1$,

$$H_t = \xi_t H_{t-1} + B_t - S_t$$

defines the dynamic of holding assets between two consecutive decision times. For any given (k, t), whenever $S_{k,t} > 0, B_{k,t} = 0$ and vice-versa. Transaction costs also influence the cash dynamics. Buying an amount x_k of asset k requires $x_k \left(1 + \bar{c}_k^B\right)$ of cash and selling the same amount of asset k results in $x_k \left(1 - \bar{c}_k^S\right)$ of cash. Initially,

$$C_0 = \bar{C}_0 + Z_0 - \left(\mathbf{e} + \bar{c}^B\right) B_0 + \left(\mathbf{e} - \bar{c}^S\right) S_0$$

and for $t \geq 1$,

$$C_t = C_{t-1}\left(1 + r_f\right) + cr_{t-1} W_t - \text{Ben}_t + Z_t - \left(\mathbf{e} + \bar{c}_k^B\right) B_t + \left(\mathbf{e} - \bar{c}_k^S\right) S_t$$

where we assume that $cr_{t-1} W_t$, Ben_t and Z_t come in cash.

Liability and external flows

As we consider a DB plan, total liabilities are the discounted expected value of future pre-defined payments. At a given time t, they represent the amount the fund has to own if it has to close at that time. This value has to be estimated with appropriate rules taking into account actuarial risks, pension fund provisions, and other relevant factors for the employer's line of business. Let L_t denote the total amount of liabilities at time t.

All quantitative models considered in this chapter will be applied to the planning problem of a large and stable pension fund. We can then assume that the fund keeps the same structure and number of members over the study period. Liability, contributions and benefits are therefore invariant with respect to actuarial risk over the period under study. Actuarial risks regroup the random events that affect the number of members into the fund. However, those variables are yearly indexed with the general increase of wages w_t. For $t \in \mathcal{T}_1$, we have:

$$L_t = L_{t-1}(1+w_t)\,;\; W_t = W_{t-1}(1+w_t) \;\text{ and } \mathrm{Ben}_t = \mathrm{Ben}_{t-1}(1+\kappa w_t) \qquad (2.5)$$

and their initial values L_0, W_0 and Ben_0 known at $t=0$; $\kappa \geq 0$ is a model parameter. In practice, the pension payments Ben_t are often indexed with a certain rate which is a function of the inflation rate. In order to reduce the complexity of our model, we assume that this indexation rate is a certain proportion of the salary increase as this latter is highly positively correlated to the inflation. From the above definitions, uncertainty, represented by vector $\Big((1+w_t)\,,\;\; \xi_t\Big)_{t=1}^{T}$, affects boths assets and liabilities. As often in the literature (e.g. Kouwenberg, 2001), we use a vector auto-regressive model (VAR model) such that:

$$\begin{aligned} & h_t = c + \Omega h_{t-1} + \epsilon_t, \;\; \epsilon_t \sim N(0,\Sigma)\,, \\ & h_t := \Big(\ln(1+w_t) \;\;\; \ln(\xi_{1t}) \;\;\; \cdots \;\;\; \ln(\xi_{kt}) \;\;\; \cdots \;\;\; \ln(\xi_{dt})\Big)^{\top}\,, \\ & \quad t \in \mathcal{T}_1 \end{aligned} \qquad (2.6)$$

where h_t is a $\{(d+1)\times 1\}$ vector of continuously compounded rate, c the $\{(d+1)\times 1\}$ vector of coefficients, Ω the $\{(d+1)\times(d+1)\}$ matrix of coefficients, ϵ_t the $\{(d+1)\times 1\}$ vector of error term and Σ the $\{(d+1)\times(d+1)\}$ covariance matrix. The parameter estimation of this model requires time series analysis. For example in Kouwenberg, 2001, annual observations of the total asset returns and the general wage increase from 1956 to 1994 are used to estimate the coefficients of the VAR model. The resulting estimates will serve in constructing the scenario tree which constitutes the workhorse of multistage stochastic programs.

2.2.3 The ALM problem

The total cost of funding is the sum of regular ($\sum cr_{t-1}W_t$) and remedial ($\sum Z_t$) contributions over the studied period. In this study, we are looking for the investment stragtegy H_t, contribution rate cr_t and remedial contribution Z_t for which the total expected cost of funding is minimized. The optimization is made under risk, legal, budget, regulatory and operating constraints. The constraints and objective of the ALM study will be presented in this section.

We denote by symbol $\mathbb{E}_t(x)$ the conditional expectation of random variable x with respect to the natural filtration $\mathcal{F}_t$ whereas $\mathrm{P}\{E\}$ denotes the occurence probability of event E. At each decision time t, the optimization problem consists in minimizing the total expected costs under the constraints considered in the following subsections. To simplify the notation, we omit the scenario index s.

Risk constraints

The pension fund wants to guarantee the participants a certain amount of pension. But the members also depend on the pension fund to actually provide for their needs in the future. Therefore, the safety of the portfolio is of paramount concern. This safety is translated into risk constraints.

A pension fund has long term obligations, up to decades, and therefore, its planning horizon is large, too. The main goal of an ALM is to find acceptable allocations which guarantee the solvency of the fund during the planning horizon. In general, solvency is measured by the funding ratio F_t (also called cover ratio) that we define for a given time t by

$$F_t := \frac{A_t^*}{L_t}.$$

Underfunding occurs when the funding ratio is less than one. The assertion $F_t \leq 1$ is equivalent to saying that the surplus at time t, i.e. $A_t^* - L_t$, is negative. When this occurs, the shortfall could be provided by the fund's sponsor or any other external contribution. That is the remedial contribution as in Haneveld et al., 2010. Depending on how the random vector $\omega_t := \Big((1 + w_t)\,, \quad \xi_t\Big)\,,\ t \in \mathcal{T}_1$, behaves, F_t may change over time. Therefore, the pension fund rebalances its assets portfolio and redefines its contribution rate in order to control the funding ratio. The higher F_t, the healthier the fund. However, the decision maker would like to avoid as much as possible the changes in contribution rates. We will see in the model description that the parameters can be set in order to limit those variations.

The long term objective of the pension fund consists in fullfilling both long and short (one year) term constraints. We define two types of funding ratio risk constraints in this paper. Their goal is to constrain the funding ratio to be larger, on average, than a predefined minimum $\gamma, \gamma \geq 0$. Namely, the expected shortfall $\mathbb{E}_{h-1}\left(A_h^* - \gamma L_h\right)^-\,,\ h > t$, is required to be less than a certain amount β_t known at time t. Here, $(a)^- := \max\{-a, 0\}$ is the negative part of $a \in \mathbb{R}$. Also in order to simplify understanding, the expression expected shortfall is used to name $\mathbb{E}_{h-1}\left(A_h^* - \gamma L_h\right)^-$. That is slightly different from its definition in actuarial science where γ is equal to one. The one period risk constraint (OICC[1]) is expressed by

$$\mathbb{E}_t \left(A^*_{t+1} - \gamma L_{t+1} \right)^- \leq \beta_t, \quad t \in \mathcal{T}_0 \setminus \{T\} \tag{2.7}$$

and for the multiperiod (MICC) approach,

$$\mathbb{E}_{h-1} \left(A^*_h - \gamma L_h \right)^- \leq \beta_t, \quad h \in \mathcal{T}_{t+1} \text{ and } \ t \in \mathcal{T}_0 \setminus \{T\}, \tag{2.8}$$

where β_t and γ are parameters defined by the pension fund. According to the short term approach (2.7), at each decision time t, the expected shortfall over the following period should be smaller than a certain amount β_t. Notice that, at time t, the short term risk only controls the expected shortfall of $\left(A^*_{t+1} - \gamma L_{t+1} \right)^-$ over the following one-year period.

When we want to control the expected shortfall over the whole remaining period up to maturity, the risk constraint (2.8) is a good measure of long term risk (multiperiod). That is, at time t, equation (2.8) means that the one period expected shortfall $\mathbb{E}_{h-1} \left(A_h - \gamma L_h \right)^-$ should be smaller than β_t at any future node with $h \in \mathcal{T}_{t+1}$. Equation (2.8) can be rewritten as

$$\max_{h \in \mathcal{T}_{t+1}} \mathbb{E}_{h-1} \left(A^*_h - \gamma L_h \right)^- \leq \beta_t, \quad t \in \mathcal{T}_0 \setminus \{T\}, \tag{2.9}$$

meaning at time t, that, the highest one year expected shortfall, over the remaining periods to maturity, has to be smaller than the amount β_t. Parameter β_t is set by the pension fund at time t as a function of the total assets and liabilities at that time; e.g. $\beta_t := f\left(A_t, L_t \right) = \alpha A_t, \ 0 \leq \alpha \leq 1$. Readers should notice that when $T = 1$, (2.7) is equivalent to (2.8).

In stochastic programming, constraints such as (2.7) and (2.8), i.e. bounding an expected shortfall, are named *integrated chance constraints* (ICC). They were proposed by Haneveld, 1986 as a quantitative alternative for *chance constraints* (CC). In Section 2.3, both ICC and CC will be discussed more specifically. Successful applications of the ICC in ALM for pension fund can be found in Drijver, 2005 and Haneveld et al., 2010. The authors assume that $\beta_t := \beta$ is unchanged over the studying period and in their numerical illustrations, the ICC is only applied to the first stage. In such case, one can prove that the OICC and the MICC are equivalent.

[1]OICC (resp. MICC) stands for One period Integrated Chance Constraint (resp. Multiperiod Integrated Chance Constraint) which will be more clearly defined in Section 2.3.2.

Instead, we remove that assumption in our work. Therefore, we define:

$$\beta_t := \alpha L_t \tag{2.10}$$

where $\alpha,\ 0 \leq \alpha \leq 1$, denotes a scale free risk parameter. Obviously, the feasible set of the MICC formulation is contained in the feasible set of the OICC formulation. The two constraints cannot be implemented in the same model at the same time. There, comes the other particularity of this paper: we analyse the multiperiod risk constraint and then measure how conservative it is comparing to the one period approach.

Other constraints

Risk constraints are important, but institutional and legal rules regarding pension fund operations in general are also relevant. As stated in Pflug and Swietanowski, 1998, institutional and legal rules are designed to restrict the risk of losses which would adversely affect pensionners. Hence, the following restrictions are integrated to the model.

Firstly, the fund is not allowed to sell an asset that is not owned. This is the *not short selling assets* constraints and can be expressed by

$$\begin{aligned}
&H_{k,t} \geq 0, \\
&B_{k,t} \geq 0, \\
&S_{k,t} \geq 0 \text{ for } k \in \mathcal{K} \text{ and } t \in \mathcal{T}_0.
\end{aligned}$$

The no short selling constraint goes with the *not borrowing cash* constraint expressed by

$$C_t \geq 0, \ \ t \in \mathcal{T}_0.$$

Secondly, at any time, the fund should dispose a minimum amount in cash in order to pay possible claims such as death benefits or pensions. This can be called *liquidity constraint* and is formulated in our model as

$$C_t\,(1 + r_f) + \mathbb{E}_t\,(cr_t W_{t+1} - Ben_{t+1}) \geq 0$$

which means that, on average, the cash allocation C_t at time t should be sufficient to cover the possible negative value of the cash flow balance over period $[t, t+1]$. Notice that the term *liquidity constraint* used here may have a different meaning in another context, e.g Fonseca et al., 2007 in a macroeconomic framework.

Thirdly, the fund is subject to *portfolio constraints* imposed by the legislator in order to keep a minimum control on its risk exposure. It consists on bounding the holding in asset k by setting upper and lower bounds, u_k and l_k respectively, on $H_{k,t}$. That is

$$l_k A_t \leq H_{k,t} \leq u_k A_t, \quad k \in \mathcal{K}, \; t \in \mathcal{T}_0 \setminus \{T\} . \tag{2.11}$$

For example in Switzerland[2], the amount allocated to stocks should not exceed fifty percent of total wealth. In such case, $l_{stocks} = 0$ and $u_{stocks} = 0.5A_t$ and constraint (2.11) is equivalent to

$$0 \leq H_{stocks,t} \leq 0.5A_t, \qquad t \in \mathcal{T}_0 \setminus \{T\} .$$

These bounds are also applicable to cash C_t and we obtain

$$l_c A_t \leq C_t \leq u_c A_t, \qquad t \in \mathcal{T}_0 \setminus \{T\} . \tag{2.12}$$

Notice that, in equations (2.11) and (2.12), upper and lower bounds can also be time dependent.

Finally in an asset allocation problem, dynamics and budget constraints, already defined in section 2.2.2 are inavoidable. If they were left out, the optimization program would be unbounded. The constraints presented in this subsection are common in any ALM stochastic programming implemention; see for e.g. Kusy and Ziemba, 1986, Carino et al., 1994, Consigli and Dempster, 1998, Bogentoft et al., 2001 and Dert, 1995 among others.

The optimization problem

As we are considering a DB fund, it is natural to assume that the aim of the fund is to minimize its costs while controlling the risk. Thus, the ALM model is a dynamic decision making optimization tool to minimize the total expected cost under risk and operating constraints. Decisions are taken at the beginning of each one-year period. Accordingly, the ALM model is developped as a multiperiod decision problem, for which, we are asked to come up with an optimal asset allocation, contribution rate and remedial contribution at the beginning of each year. Moreover, penalty costs are assigned to the undesirable events: remedial contributions, and

[2] OPP2 of April 18th, 1984, Art 55-b, (As of January 1st, 2012)

yearly absolute variation of contribution rates. All these components together constitute the objective function:

$$\min_{H,cr,Z} \mathbb{E}_0 \left[\sum_{t=0}^{T-1} v_{t+1} \left(cr_t W_{t+1} + \lambda_z Z_{t+1} \right) + \sum_{t=0}^{T-2} v_{t+1} \lambda_{\Delta_{cr}} \Delta_{cr_t} W_{t+1} \right] \tag{2.13}$$

where $\Delta_{cr_t} := \mid cr_{t+1} - cr_t \mid$ is the absolute variation of contribution rate from year t to $t+1$, v_t is the discount factor for a cash flow in year t, λ_z and $\lambda_{\Delta_{cr}}$ are, respectively, penalty parameters for remedial contribution and absolute variation of contribution rate. The variables cr_t and Δ_{cr_t} are bounded:

$$cr^l \leq cr_t \leq cr^u \quad \text{and} \quad \underline{\Delta_{cr}} \leq \Delta_{cr_t} \leq \bar{\Delta}_{cr}, \qquad t \in \mathcal{T}_0 \setminus \{T\}$$

where cr^l, $\underline{\Delta_{cr}}$ are the lower bounds and cr^u, $\bar{\Delta}_{cr}$ the upper bounds of cr_t, Δ_{cr_t}, respectively. The optimal decisions have to lead to a funding ratio greater than a certain minimum $\bar{F}$ (sometimes called target funding ratio) at the end of period of study T:

$$F_T = \frac{A_T}{L_T} \geq \bar{F}.$$

The entire ALM model, with objective and constraints, can be found in Appendix 1. An optimization program such as (2.13) is often referred to as a *here and now* problem. Uncertainty, characterized by $\omega_t = \Big((1 + w_t) \,, \quad \xi_t \Big) \,, \; t \in \mathcal{T}_1$, is approached by scenarios. Therefore, we define $\tilde{\omega}$ with a finite number S of possible realizations $\tilde{\omega}^s := (\tilde{\omega}_1^s, \cdots, \tilde{\omega}_T^s)\,, s \in \mathcal{S} := \{1, ..., S\}$, from $t = 0$ to $t = T$ with relative probability p^s.

The objective (2.13) is obviously linear as it can be rewritten as a linear combination of decision variables. We can also notice that dynamics and constraints (except risk constraints which have to be rewritten in a linear form for the stochastic program) presented in sections 2.2.2 and 2.2.3 are all linear in decision variables. If the risk constraints OICC (2.7) and MICC (2.8) were written in a linear form, the ALM problem would be a stochastic linear program (SLP), theoretically solvable by any SLP software depending on its size. In the next section, we will show how they can be turned into linear programs. The books Kall and Mayer, 2011, Shapiro et al., 2009 and Birge and Louveaux, 2011 provide good ressources to deal with such problems. When the size is big, resolution may require heuristic methods. Size is big means that number of asset classes is large or/and time horizon is long or/and number of scenarios is large. Decisions variables are H_t, B_t, S_t, C_t, cr_t and Z_t for $t \in \mathcal{T}_0$; but only first stage values H_0, B_0, S_0, C_0, cr_0 and Z_0 are crucial to the decision maker, since, almost surely, a true realization of the random data will be different from the set of generated scenarios.

By definition, the pension fund risk problem is often a shortfall problem. In such models, the relevant measure of risk for the firm is the expected amount (if any) by which goals are not met, Carino et al., 1994. The model considered in this paper has a general DB ALM structure such as explored in Haneveld et al., 2010 and Ziemba and Mulvey, 1998. Its main particularity consists in the integration of ICC by the way of OICC (2.7) and MICC (2.8). A successful implementation of constraints (2.7) in ALM for a DB fund can be found in Vlerk et al., 2003 and Haneveld et al., 2010. In their works, the optimization problem is solved assuming that the parameter β_t is constant: $\beta_t = \beta$. Furthermore, remedial contribution are provided only when funding ratio falls short in two consecutive years. Implementing this latter condition has lead to the use of binary variables. The authors propose a heuristic solution to the problem.

As we will explain in section 2.3.2, the parameter β is not scale free. A certain value of β does not have an equivalent meaning for two different pension funds. It can be too low for a certain fund whereas too high for an other one. In addition, the pension fund actual situation should be taken into account. Our paper is an extension of Haneveld et al., 2010. As a novelty, we assume that the risk parameter β_t varies with respect to time t and is defined as a proportion α of the actual level of liability at time t, see Equation (2.10). Roughly speaking, on average, the total asset should cover a proportion of magnitude $(1-\alpha)$ of liability at any time. In our model, remedial contributions can however be provided at anytime where solvency is in question, avoiding the use of binary variables, and indirectly, the need of heuristics. The penalty parameter λ_z punishes the abuse of remedial contributions.

The main features of this study turn around the following points:

- As in Haneveld et al., 2010 where optimal decision is analysed for different values of their risk parameter β, we first measure the effect of our risk parameter α on the decisions H_0, cr_0 and Z_0; this with respect to the OICC. In addition, for a fixed value α, the influence of the initial funding ratio is also explored.
- Secondly, as a safer alternative to the OICC, we propose the MICC (constraint (2.8)) and we then measure how constraining it is, compared to the OICC. In constraint (2.8), index h is a decision time index and we are not aware of any implementation of such constraint in ALM. The OICC considered in the first

item is actually extended to a multiperiod risk constraint, reinforcing the long term aspect of the pension fund's ALM.

In the rest of this paper, OICC (resp. MICC) will stand for the one period (resp. multiperiod) ICC itself as well as the ALM model with the OICC (resp. MICC).

2.3 Framework of the risk constraints

The most important constraints, of course, deal with the goal of the pension fund: in all circumstances keep a certain control on the funding ratio. This latter is expressed in terms of shortfall constraints which are of ICC type in this paper. Proposed by Haneveld, 1986, the ICC's formulation directly results from CC's. That is why, in this chapter, we firstly introduce CC and how it leads to ICC. Secondly, ICC is discussed and we show how constraints (2.7) and (2.8) are related to it. We finish this section by proposing simple linear reformulations of (2.7) and (2.8).

For the sake of clarity, we define the generic linear function $G : \mathbb{R}^d \times \Xi \to \mathbb{R}^m$ such that

$$G(X, \omega) := BX - D$$

where $X \in \mathcal{X}$ is an d-vector of decision variables, $\mathcal{X} \subset \mathbb{R}^d$ is a polyhedral and closed set and $\omega := (B, D) : \Omega \to \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^m$ is a random parameter on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$. The support of ω is defined as the smallest closed set $\Xi \subset \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^m$ having the property $\mathrm{P}(\omega \in \Xi) = 1$. For $i \in \mathcal{I} := \{1, \cdots, m\}$, the vector B is of dimension $\mathbb{R}^m \times \mathbb{R}^d$ such as $B := \begin{pmatrix} B_1 & \cdots B_i & \cdots & B_m \end{pmatrix}^\top$ with $B_i \in \mathbb{R}^d$ whereas $D := \begin{pmatrix} D_1 & \cdots D_i & \cdots & D_m \end{pmatrix}^\top$ with $D_i \in \mathbb{R}$. As supposed in our SP model, we assume that $\omega = (B, D)$ has a finite number S of possible realizations $\omega^s = (B^s, D^s), s \in \mathcal{S} = \{1, ..., S\}$ with respective probability p^s.

2.3.1 Chance constraints

Chance constraints (CC) models serve as tool for modeling risk and risk aversion in SPs. Let $\mathbf{0}$ be a m-dimensional vector of zeroes. Satisfying the constraint $G(X, \omega) \geq \mathbf{0}$ could lead to high costs or unfeasibility. This equation refers to a finite system of m inequalities. Instead, if the distribution of ω is known, one can formulate the condition that the probability of $G(X, \omega) \geq \mathbf{0}$ is sufficiently high, i.e. close enough to 1. That is

$$\mathrm{P}\{G(X, \omega) \geq \mathbf{0}\} \geq 1 - \epsilon \tag{2.14}$$

where the fixed parameter $(1-\epsilon) \in [0,1]$ is called probability level and is chosen by the decision maker in order to model the safety requirements. Equation (2.14) is the general form of chance (probabilistic) constraints and can be viewed as a compromise with the requirement of enforcing the constraint $G(X,\omega) \geq \mathbf{0}$ for all values $\omega \in \Xi$ of the uncertain data matrix.

When $m = 1$, $G(X,\omega) := g(X,\omega)$ is a scalar and equation (2.14) leads to

$$\mathrm{P}\{g(X,\omega) \geq 0\} \geq 1-\epsilon \tag{2.15}$$

with $g : \mathbb{R}^d \times \Xi \to \mathbb{R}$. Equation (2.15) is known as *individual* CC. For $m > 1$, we obtain

$$\mathrm{P}\{g_i(X,\omega) \geq 0, \;\; i \in \mathcal{I}\} \geq 1-\epsilon, \tag{2.16}$$

called *joint* CC. Chance-constrained programs have been pionnered by Charnes et al., 1958 in production planning. Since then, they have been extensively studied and have also been applied in many other areas such as telecommunication, finance, chemical processing and water ressources management. Despite important theoretical progress and practical importance, there could be major problems with numerical processing of CCs, see Ahmed and Shapiro, 2008 and Nemirovski and Shapiro, 2006.

Especially when ω has a discrete distribution, Raike, 1970 introduces a mixed-integer reformulation of CC. Assuming $m = 1$, equation (2.15) is equivalent to

$$\sum_{s=1}^{S} p^s \cdot \mathbf{1}_{(g(X,\omega^s)\geq 0)}(s) \geq 1-\epsilon$$

where $\mathbf{1}_{(g(X,\omega^s)\geq 0)}(s) = 1$ if $g(X,\omega^s) \geq 0$ and 0 otherwise. Now, we are able to write inequalities (2.15) in a mixed-integer program (MIP) formulation. We introduce binary variables $\delta^s, \;\; s \in \mathcal{S}$. They play the role of indicator function: $\delta^s = 1$ in scenario s if it holds that $g(X,\omega^s) < 0$ and equals 0 otherwise. In terms of these additional decision variables, the CC can be written as linear inequalities

$$g^s(X,\omega^s) + \delta^s M \geq 0, \;\; s \in \mathcal{S}, \tag{2.17}$$

$$\sum_{s=1}^{S} p^s \delta^s \leq \epsilon, \;\; s \in \mathcal{S}, \tag{2.18}$$

$$x \in X, \;\; \delta^s \in \{0,1\}, \;\; s \in \mathcal{S}, \tag{2.19}$$

where M is a sufficiently large number. If $\delta^s = 0$, then the constraint $g(X,\omega^s) \geq 0$ corresponding to the realization s in the sample is enforced. On the other hand, if $\delta^s = 1$, the constraint is satisfied for any candidate solution. The probability weighted average of these binary variables equals the risk of not meeting the condition $g(X, \omega^s) \leq 0$ with the decision X, which should be at most ϵ.

This formulation is well known in SP and has first been applied to ALM for pension funds by Dert, 1995. It also holds for the joint CC case where $m > 1$. In fact, $\{g_i(X, \omega) \geq 0, \ \ i \in I\}$ is equivalent to

$$\min_{i \in I} \{g_i(X, \omega)\} \geq 0$$

and can also be writen as linear inequalities

$$g_i^s(X, \omega^s) + \delta_i^s M \geq 0, \ \ i \in I, \ \ s \in \mathcal{S}, \tag{2.20}$$

$$\Delta^s \geq \delta_i^s, \qquad i \in I, \ \ s \in \mathcal{S}, \tag{2.21}$$

$$\sum_{s=1}^{S} p^s \Delta^s \leq \epsilon, \qquad s \in \mathcal{S}, \tag{2.22}$$

$$x \in X, \ \ \Delta^s, \delta_i^s \in \{0, 1\}, \ \ i \in I, \ \ s \in \mathcal{S}. \tag{2.23}$$

Even with these linear settings (2.17) – (2.19) and (2.20) – (2.23), implementing this constraint with a reasonnable number of scenarios can be computationally challenging as the feasible set is obviously not linear, neither convex. That is due to the increase in complexity from MIP that arises from the introduction of at least one binary variable per each of the S scenarios. Efficient solution algorithms are proposed in chapter 4 of Kall and Mayer, 2011, Luedtke, 2014, Luedtke et al., 2010, Tanner and Ntaimo, 2010, Prékopa et al., 1998 and Ruszczyński, 2002.

Note that the CC, as described above, only considers the qualitative aspect of the risk, i.e. attention is only paid to whether the integrand is satisfied or not. A better approach can be to control the quantitative aspect of the fail, i.e the size of negative values of G^s. That is often the case for pension funds where sponsors want to know approximatively how much they are willing to contribute in the following periods. Due to an idea of Haneveld, 1986, binaries δ^s are dropped and the integrated chance constraint has been proposed.

2.3.2 Integrated Chance Constraint

The MIP constraints (2.20) to (2.23) are hardly implementable due to the integrality conditions in (2.23). For problems involving binary (or general integer) decision variables, a natural approach is to relax the integrality and solve the resulting relaxation, see Vlerk et al., 2003. If we relax the integrality constraints and substitute

$y^s := \delta^s M$ and $\beta := \alpha M$, we obtain

$$B^s X + y^s \geq D^s, \quad s \in \mathcal{S} \tag{2.24}$$

$$\sum_{s=1}^{S} p^s y^s \leq \beta, \tag{2.25}$$

$$y^s \geq 0, \quad s \in \mathcal{S} \tag{2.26}$$

$$X \in \mathcal{X}, \tag{2.27}$$

where the parameter β is non-negative. By (2.24), for each s, the non-negative variable y^s is not less than the shortfall $(B^s x - D^s)^-$, where $(a)^- := \max\{-a, 0\}$ is the negative part of $a \in \mathbb{R}$. The inequality (2.25) therefore puts an upper bound β on the expected shortfall. That is, the system (2.24) – (2.27) is equivalent to

$$\mathbb{E}\,(BX - D)^- = \sum_{s=1}^{S} p^s \left(B^s X - D^s\right)^- \leq \beta. \tag{2.28}$$

Such constraint is called integrated chance constraint (ICC) and has been introduced by Haneveld, 1986 as an alternative to CC. However, Haneveld et al., 2010, Vlerk et al., 2003 and Drijver, 2005 have pionnered its application to ALM for pension funds and since then, it has been implemented in practice.

By definition, the feasible set, defined by linear inequalities (2.24) – (2.27) is a polyedron (convex) as it contains only continuous decision variables, see Haneveld and Vlerk, 2002. Thus, it can usually be solved efficiently using an appropriate software. Constraints (2.24) – (2.27) are very attractive from an algorithm point of view. Haneveld and Vlerk, 2002 propose a faster algorithm for big size problems. ICC is a good alternative to CC from different perspectives:

- Firstly, CC only measures the probability of shortage whereas ICC uses the probability distribution to measure the expected magnitude of the shortage. We can say that ICC takes into account both quantitative and qualitative aspects of the shortage whereas CC only considers its qualitative side. CC says only if there is underfunding or not and especially in practice, it could be important to limit the amount of remedial contributions the sponsor is willing to provide in years after.

- Secondly, ICC and CC somehow ressemble, respectively, to the so-called *conditional value-at-risk* (CVaR) and *value-at-risk* (VaR). Conversely to CVaR which is known as *coherent* (Rockafellar and Uryasev, 2002), it is well known that

VaR is not a coherent risk measure as it does not fulfill the subadditivity condition. Therefore, ICC possesses more attractive risk properties than CC. To learn more about coherent risk measures, see Artzner et al., 1999.

- We should also add that, if the risk aversion parameter is changed, the feasible region in case of ICC changes smoothly, while this region changes in a rough way in case of CC, Drijver, 2005.
- Finally, we should admit that the parameter ϵ of CCs is scale free, and corresponds to risk notion which is more familiar to pension fund managers. It is not the case for ICC. Our solution to this problem is to set β as a proportion α of liability.

From now, and without loss of generality, we assume $m > 1$. Therefore, equation (2.28) can be rewritten as

$$\mathbb{E}\left\{(B_i X - D_i)^- \, , \; i \in I\right\} \leq \beta$$

which is the *joint* form of ICC, see Haneveld and Vlerk, 2002. When index i is a decision stage index with conditional expectation at stage i, we obtain a multistage program and variable X becomes stage dependent (X_i). That is, at stage $j \in I \backslash \{m\}$:

$$\mathbb{E}_i \left(B_{i+1} X_i - D_{i+1}\right)^- \leq \beta_j, \;\; i \in \{j, j+1, \cdots, m-1\} \tag{2.29}$$

which is equivalent to the MICC (2.8) for $I = \mathcal{T}_0$ and $B_{h+1} X_h - D_{h+1} = A_{h+1} - \gamma L_{h+1}$. At time t, that is:

$$\mathbb{E}_h \left(A^*_{h+1} - \gamma L_{h+1}\right)^- \leq \beta_t, \; h \in \mathcal{T}_t \setminus \{T\}\,. \tag{2.30}$$

The parameter β_t is then set at time t and will remain applicable until T. As decision is taken at each stage, the MICC inequality (2.8) shows a collection of inequality (2.30) going from $t = 0$ to $t = T - 1$. Similarly, when $m = 1$, one can proove that equation (2.28) leads to the OICC (2.7).

2.3.3 OICC and MICC: Scenario tree interpretation

Section 2.2.1 briefly explains our scenario tree model. We recall that the node (t, s) corresponds to a certain scenario s at decision time t. To avoid anticipativity, we have to consider that many pairs (t, s) might correspond to the same node on the scenario tree picture. For example in Figure 2.1, the nodes $(1,1), (1,2), \cdots, (1,8)$ correspond graphically to the empty red cercle. At each node (t, s), the fund's manager has to rebalance the asset portfolio and fix the contribution rate. These decisions are taken considering the actual scenario and possible future paths as well as the risk constraints.

OICC

In principle, considering a certain node (t, s), the OICC constraint (2.7) would be implemented as follows:

$$\mathbb{E}_{t,s}\left(A^{*s}_{t+1} - \gamma L^{s}_{t+1}\right)^{-} := \sum_{s' \in \mathcal{S}} p^{s'}_{t,s}\left(A^{*s'}_{t+1} - \gamma L^{s'}_{t+1}\right)^{-} \leq \alpha L^{s}_{t} \tag{2.31}$$

where $p^{s'}_{t,s}$ stands for the conditional probability to reach node $\left(t+1, s'\right)$ going from (t, s) and $p^{s'}_{t,s} = 0$ for any scenario s' of $t+1$ not descending from (t, s). As in Vlerk et al., 2003, we include the linear inequality (2.31) in every subproblem (t, s), $t < T$ of our multistage recourse model. At (t, s), they reflect the short-term risk constraint, stating that the expected funding shortfall over the following period $(t+1)$ is at most αL^{s}_{t}. In other words, on average, the pension fund should be able to cover the proportion $(1-\alpha)$ of its total liability. The increase of α will relax the feasibility set of the optimization problem.

MICC

Considering the node (t, s), the MICC constraint can be formulated in the following way:

$$\mathbb{E}_{h-1,s}\left(A^{*s}_{h} - \gamma L^{s}_{h}\right)^{-} \leq \alpha L^{s}_{t}, \quad h \in \mathcal{T}_{t+1} \text{ and } t \in \mathcal{T}_{0} \setminus \{T\} \tag{2.32}$$

with

$$\mathbb{E}_{h-1,s}\left(A^{*s}_{h} - \gamma L^{s}_{h}\right)^{-} = \sum_{s' \in \mathcal{S}} p^{s'}_{h-1,s}\left(A^{*s'}_{h} - \gamma L^{s'}_{h}\right)^{-}.$$

Under (2.32), at each node (t, s), decisions are taken such that the descending nodes's one-period expected shorfall are smaller than αL^{s}_{t} (defined at current node). Such constraint permits to have a certain control of the cover ratio over the whole remaining periods: $[t+1, T]$; whereas (2.31) only covers one period: $[t, t+1]$. For example, at initial time $t = 0$, the minimum cost is determined under the condition that the expected shortfall at any node in the tree (as descendant of the initial node) is smaller than $\beta_0 = \alpha L^{s}_{0}$ as in Haneveld et al., 2010:

$$\sum_{s' \in \mathcal{S}} p^{s'}_{t,s}\left(A^{*s'}_{t+1} - \gamma L^{s'}_{t+1}\right)^{-} \leq \beta_0, \quad t \in \mathcal{T}_0 \setminus \{T\}, \ s \in \mathcal{S}.$$

Futhermore, at each node $(t,s)\,, t \in \mathcal{T}_1 \setminus \{T\}\,, s \in \mathcal{S}$, we add the restriction:

$$\sum_{s^{'} \in \mathcal{S}} p_{t,s}^{s^{'}} \left(A_{t+1}^{*s^{'}} - \gamma L_{t+1}^{s^{'}} \right)^{-} \leq \alpha L_t^s.$$

That is how we implement (2.32) at initial node. If we repeat the same procedure at each node of the tree, we can then propose a simpler SP reformulation:

Proposition 2.3.1. *Constraint* (2.32) *is equivalent to the following statement:*
At each node $(t,s)\,,\ t < T,\ s \in \mathcal{S}$

$$\sum_{s^{'} \in \mathcal{S}} p_{t,s}^{s^{'}} \left(A_{t+1}^{*s^{'}} - \gamma L_{t+1}^{s^{'}} \right)^{-} \leq \min_{0 \leq t' \leq t} \alpha L_{t'}^s. \tag{2.33}$$

That is, at a given node $(t,s)\,, t < T, s \in \mathcal{S}$, the expected shortfall over the next period should be less or equal to the smallest value of $\alpha L_{t'}^s$ calculated over the preceding nodes $(t',s)\,, t' \leq t$. This is based on the fact that, in the multiperiod framework, the decision taken at node (t,s) is influenced by the history of ω_t^s up to time t, in particular β_t^s at preceding nodes. Inequality (2.33) is linear and describes a polyhedral set. The proof of proposition 2.3.1 is straightforward when we go backward in time starting from nodes $(T-1,s)$, see Appendix 2 for an example based sketch of proof. At each node (t,s), as we know the history of β_t^s up to time t, one can determine the smallest $\beta_{t'}^s,\ t' \leq t$. Therefore, implementation of MICC consists in including the linear constraint (2.33) at each node (t,s).

2.4 Numerical illustrations

This section contains computational results for the SP model. Let's recall that we are dealing with a DB pension fund whose objective is to minimize the total expected costs under constraints. The study will focus on risk constraints which are of ICC type. Firstly, based on the OICC, the effects of risk parameter and cover ratio on the optimal decisions are analysed. Prima facie, the MICC appears to be a safer and more restrictive than OICC. Based on the same analysis as before, the cost of conservativeness is subsequently measured.

For this study, consider a hypothetical large pension fund which may invest into $d = 4$ classes of asset ordered by level of risk:

1. Deposits,
2. Bonds,
3. Real estate,

TABLE 2.1: Data on the asset classes

Asset classes	k	l_k	u_k	$\bar{c}$	Initial Investments	%
Cash	$-$	0	1	0	$4'950$	4.5
Deposits	1	0	0.7	0.00150	$16'500$	15
Bonds	2	0.1	1	0.00150	$38'500$	35
Real estate	3	0	0.30	0.00425	$17'600$	16
Stocks	4	0	0.50	0.00425	$32'450$	29.5

TABLE 2.2: Values of the other deterministic parameters

$\lambda_Z = 350$	$\lambda_{\Delta_{cr}} = 1$	$r_f = 0.008$	$v_t = (1 + r_f)^{-t}$
$\underline{\Delta_{cr}} = -0.08$	$cr^l = -0.08$	$\bar{\Delta}_{cr} = 0.05$	$cr^u = 0.3$
$\bar{A}_0 = 110'000$	$\gamma = 1.05$	$\bar{F} = 1.05$	

4. Stocks.

We are aware of the fact that the number of assets is often much larger in practice. That said, only four classes of assets are considered here in order to reduce the complexity of the model. After investing in these asset classes, the rest is held in cash. The deterministic properties of asset classes are described in Table 2.1. Investment limits are defined with respect to practical rules of liquidity and diversification; transaction costs are taken from Haneveld et al., 2010 with $\bar{c}^S = \bar{c}^B = \bar{c}$, whereas the initial investments are defined considering general statistics of pension fund's assets allocation in Switzerland, see Towers Watson, 2013 (where we assume that "real estate" corresponds to "other assets"). The portfolio constraints are defined in term of proportion and all amounts are assumed to be in thousands of Swiss francs. The values of the other deterministic parameters are shown in Table 2.2

The time horizon $T = 5$ years is split into five periods of one year each. Consequently, the considered ALM model has five stages, allowing for decisions at $t = 0$ (now) up to time $t = 4$. The random vector ω_t follows a VAR process, approximated in our case by a multistage scenario tree. In the following considerations, we first present the descriptive statistics of our model. Then, we discuss the numerical results obtained from our study.

2.4.1 Scenarios

The implementation of the scenario tree requires a careful specification of the VAR process. For this purpose, we use the estimation results obtained in Kouwenberg, 2001. More specifically, the author estimates this process based on annual observations of the total asset returns and the general wage increase from 1956 to 1994. Table 3.1 displays descriptive statistics of the time series whereas Table 2.4 shows

TABLE 2.3: Statistics, time series 1956-1994, Kouwenberg, 2001

	Statistics			
Assets	Mean	S.D.	Skewness	Kurtosis
Wages	0.061	0.044	0.434	2.169
Deposits	0.055	0.025	0.286	2.430
Bonds	0.061	0.063	0.247	3.131
Real estate	0.081	0.112	−0.492	7.027
Stocks	0.102	0.170	0.096	2.492

TABLE 2.4: Residual correlations of VAR-model, Kouwenberg, 2001

Assets	Wages	Deposits	Bonds	Real estate	Stocks
Wages	1				
Deposits	0.227	1			
Bonds	−0.152	−0.268	1		
Real estate	−0.008	−0.179	0.343	1	
Stocks	−0.389	−0.516	0.383	0.331	1

the estimated correlation matrix of the residuals. Future returns for financial planning models can be constructed by sampling from the error distribution of the VAR model and applying the estimated equations of Table 2.5. We refer to Kouwenberg, 2001 for further details on this model estimation and for building the tree as well. For this purpose, we specify a branching structure of $1-10-6-6-4-4$. This scenario tree has one initial node at time 0 and 10 succeeding nodes at time $1, \cdots,$ resulting in $10 \times 6 \times 6 \times 4 \times 4 = S = 5760$ path from 0 to 5, each with probability $p^s = \frac{1}{5760}$.

2.4.2 Numerical results

This section presents the outputs of our study. All numerical results were implemented using the solver CPLEX in the mathematical programming language AMPL. The ALM models are formulated as large LP-problems with $616'321$ variables. In the model with the OICC, there are $995'347$ constraints and $3'041'032$ nonzeros in the constraint matrix whereas they are respectively $1'002'317$ and $3'105'602$ in the MICC. On average, the solution times are 381 seconds and 448 seconds, respectively, for OICC and MICC.

As a result of the ALM analysis, we are supposed to provide the first stage optimal decisions: a contribution rate, a remedial contribution and asset allocation that minimize the total cost. In the first part of this section, we analyse the effects of the risk parameter α and the initial funding ratio F_0 on the optimal decision. The optimization is made with respect to OICC. The values of α ranges from 0 to 0.085 whereas the initial funding ratio $F_0 = \frac{\bar{A}_0}{L_0}$ vary from 0.5 to 1.5. In order to vary F_0,

TABLE 2.5: Coefficient of the VAR model, Kouwenberg, 2001

$\ln\left(1+\text{wages}_t\right)$	$=$	0.018 (2.058)	$+$	$0.693 \ln\left(1+\text{wages}_{t-1}\right) + e_{1t}$ (5.789)	$\sigma_{1,t} = 0.030$
$\ln\left(1+\text{deposits}_t\right)$	$=$	0.020 (2.865)	$+$	$0.644 \ln\left(1+\text{deposits}_{t-1}\right) + e_{2t}$ (5.448)	$\sigma_{2,t} = 0.017$
$\ln\left(1+\text{bonds}_t\right)$	$=$	0.058 (6.241)	$+$	e_{3t}	$\sigma_{3,t} = 0.060$
$\ln\left(1+\text{real estate}_t\right)$	$=$	0.072 (4.146)	$+$	e_{5t}	$\sigma_{5,t} = 0.112$
$\ln\left(1+\text{stocks}_t\right)$	$=$	0.086 (3.454)	$+$	e_{6t}	$\sigma_{6,t} = 0.159$

we change the initial liability L_0 accordingly, as $\bar{A}_0$ is specified from Table 2.2. In the second part, we compare the OICC to the MICC.

One-period Integrated Chance Constraint

In order to analyse the impact of the risk parameter α, we fix the value of the initial cover ratio. That is:

$$L_0 := 120'000 \Rightarrow F_0 = \frac{\bar{A}_0}{L_0} = 0.9166, \Rightarrow \underline{\text{under covered}}.$$

In what follows, the letter O at the top of a symbol stands for OICC whereas M is related to MICC. Figure 2.2 shows the evolution of the contribution rate cr_0. The value of cr_0 is particularly high as the institution is underfunded. We observe that when $\alpha \leq \alpha_*^O := 0.025$ (O at the top stands for OICC.), the contribution rate is at its maximum: $cr^u = 0.3$ as specified in Table 2.2. From 0.025, cr_0 decreases linearly until it reaches the value 0.27 at $\alpha = \bar{\alpha}^O := 0.04$, and remains unchanged thereafter. According to the objective (2.13), the total cost is composed of the total contribution and of the total remedial contribution, these, over the period under study. Remedial contribution should be seen as an external financial support which may come from the sponsor of the pension fund. Figure 2.3 displays the allocation of the total costs into the two types of contribution. The proportion of remedial contribution linealy decreases from 9% at $\alpha = 0$ to reach 0% at $\alpha_*^O = 0.025$ and stays constant for $\alpha \geq \alpha_*^O$. Indeed, Figures 2.2 and 2.3 help understand how the ALM model parameters have been defined. It is conventional to assume that, from a certain level of risk and for a fix cover ratio, the sponsor will no more provide any financial support to the fund. In this model, the penalty parameter λ_z has been set such that the total remedial contribution is zero for $\alpha \geq \alpha_*^O$. Consequently, cr_0 decreases from $\alpha = \alpha_*^O$. It remains equal to 0.27 for $\alpha \geq \bar{\alpha}^O$ due to the target cover ratio at maturity: $F_5 \geq \bar{F} = 1.05$.

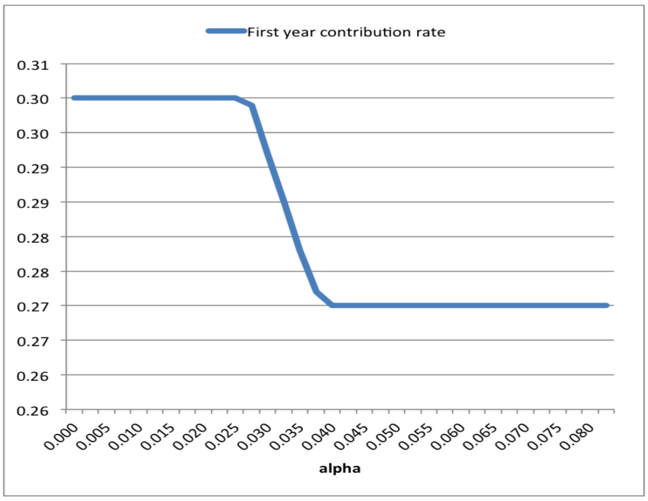


FIGURE 2.2: OICC: Contribution rate as function of α

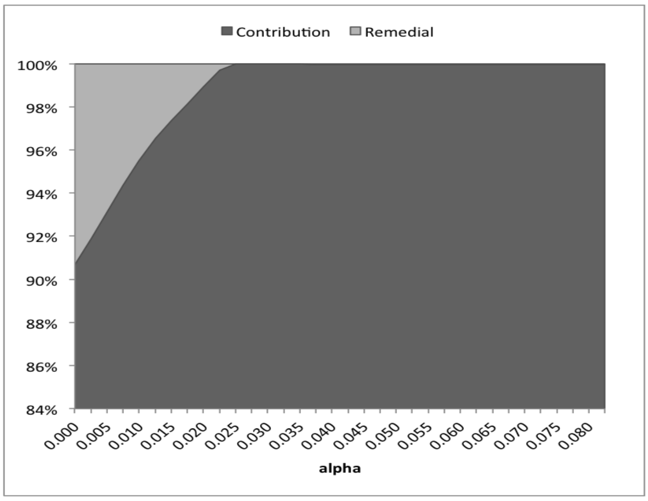


FIGURE 2.3: OICC: Initial cost allocation in function of α

Figure 2.4 describes the optimal asset allocation for different values of α. Assets are ordered with respect to their level of risk. For small values of the risk parameter, the proportion of riskier assets (stocks and real estate) tends to increase with an increase of α. When it approaches α_*^O, as the remedial contribution is already low, the decision maker starts reducing the risk exposition of its portfolio. However, the proportion of bonds is increased in order to improve the performance of the asset portfolio. The risk exposition is then progressively reduced until $\alpha = \bar{\alpha}^O$, from

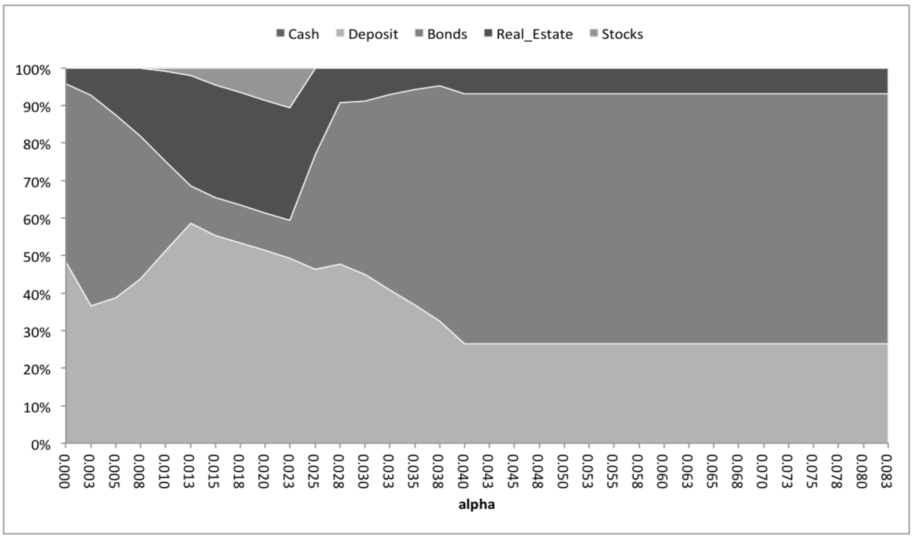


FIGURE 2.4: OICC: Asset allocation in function of α

which, it remains unchanged thereafter. The value $\bar{\alpha}^O$ can be seen as the smallest value of α, above what, the OICC influence is no more significant, i.e. contribution rate, remedial contribution and asset allocation stay constant.

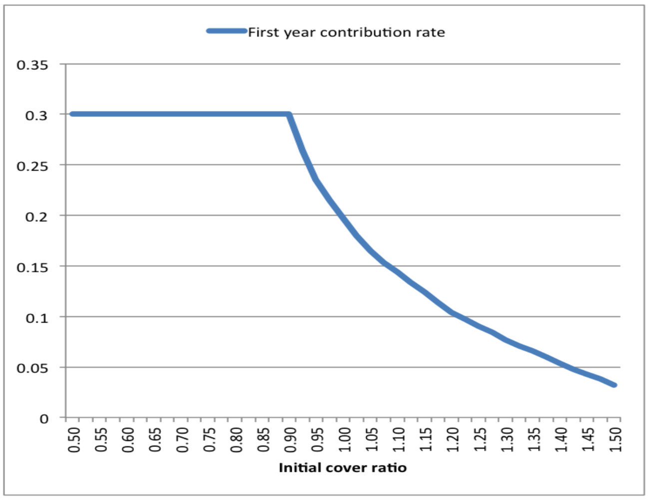


FIGURE 2.5: OICC: Contribution rate in function of F_0

Next we consider the effect of the initial funding ratio on the first stage optimal decision. We vary the value of L_0 so that the initial funding ratio F_0 lies between 0.5 and 1.5, and we assume $\alpha = 0.035$. Figure 2.5 displays the evolution of the contribution rate cr_0 whereas Figure 2.6 shows how the total cost is distributed into regular and remedial contributions. As explained earlier, it is conventional to assume that, above a certain funding ratio $F_0^{O^*}$, the remedial contribution is zero and

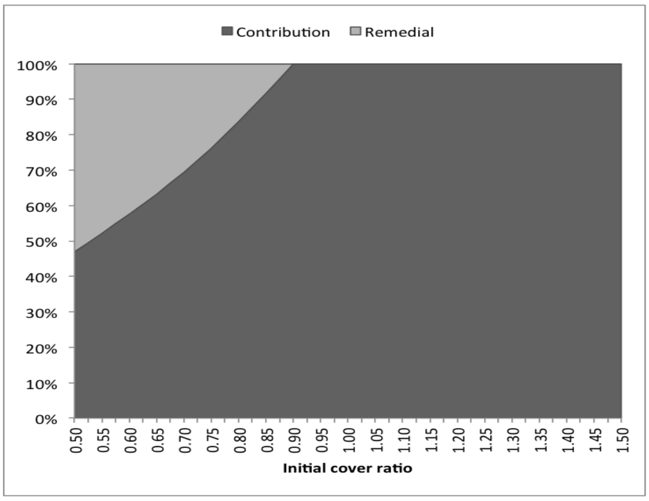


FIGURE 2.6: OICC: Initial cost allocation in function of F_0

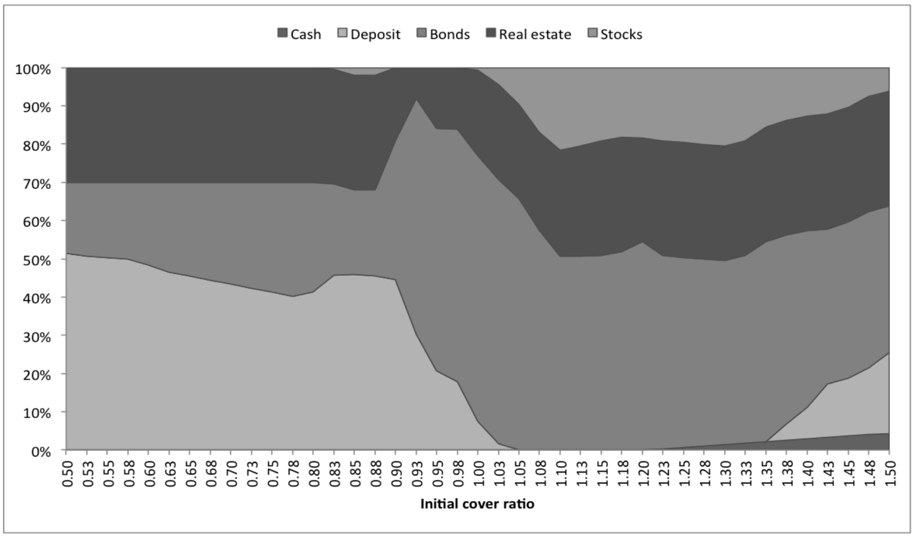


FIGURE 2.7: OICC: Asset allocation in function of F_0

contribution rate decreases as well. From Figures 2.5 and 2.6, it can be seen that the ALM model is set such that $F_0^{O^*} := 0.9$.

Figure 2.7 describes the behaviour of the first stage optimal asset allocation with respect to F_0. When $F_0 < F_0^{O^*}$, the optimal asset allocation is stable: approximatively 30% in riskier assets. From $F_0 = F_0^{O^*}$, the cover ratio is high enough to no more obtain remedial contribution and to reduce the contribution rate. However, the decision maker has to act in a riskier way in order to meet pension fund liabilities. As a result, the risk exposition increases up to 50% at $F_0 = 1.275$. An important target of our model is to guarantee a funding ratio $F_5 \geq \bar{F}$ by minimizing the total cost and risk level. For larger values of F_0, with a higher chance to fulfill the condition $F_5 \geq \bar{F}$, the contribution rate and the risk exposition (i.e. proportion of higher risk assets) decrease. We recall that the objective is not to maximize the wealth, but to minimize the total cost. Thus, the wealthier the fund is, the more prudent the allocation will be.

Multiperiod Integrated Chance Constraint

In this section, we firstly present the results of the ALM model with the MICC, and secondly, we compare with the OICC. For the MICC analysis, assumptions are similar to the ones made for the model with OICC. Figures 2.8 to 2.13 display the results of the analysis. The effect of the risk parameter α is measured in Figures 2.8, 2.9 and 2.10 whereas Figures 2.11, 2.12 and 2.13 analyse the initial funding ratio impact. Although the first stage optimal decisions are different, they behave similarly. The parameters α_*^M, $\bar{\alpha}^M$ and $F_0^{M^*}$ (respectively α_*^O, $\bar{\alpha}^O$ and $F_0^{O^*}$ for OICC) slightly differ:

$$\alpha_*^M := 0.027; \qquad \bar{\alpha}^M := 0.07; \qquad F_0^{M^*} := 0.9.$$

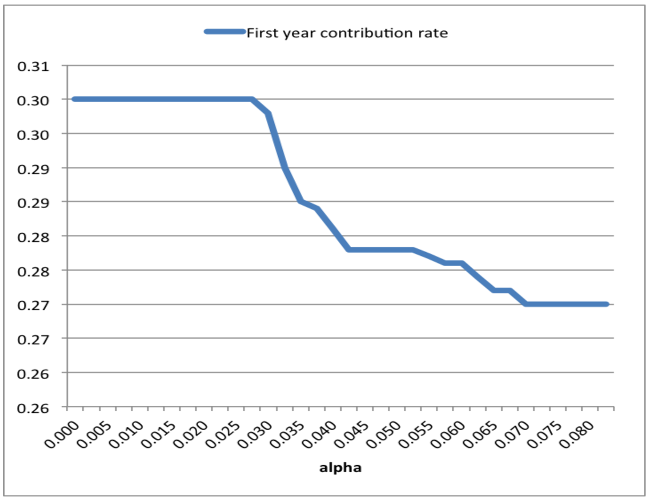


FIGURE 2.8: MICC: Contribution rate in function of α

The feasible set of the MICC formulation is contained in the feasible set of the OICC formulation. That is why the above parameters are greater or equal to their analogues in OICC. Notice that, according to Figures 2.4 and 2.10 and for $\alpha \geq \bar{\alpha}^O$ in OICC and $\alpha \geq \bar{\alpha}^M$ in MICC, the asset allocations are the same; showing that when α is above $\bar{\alpha}^O$ (resp. $\bar{\alpha}^M$), the OICC (resp. MICC) has no influence on the initial ALM model. In general, the optimal decisions obtained from OICC and MICC slightly differ. For example, when $\alpha = 0.05$ and $F_0 = 0.9166$, the first stage optimal

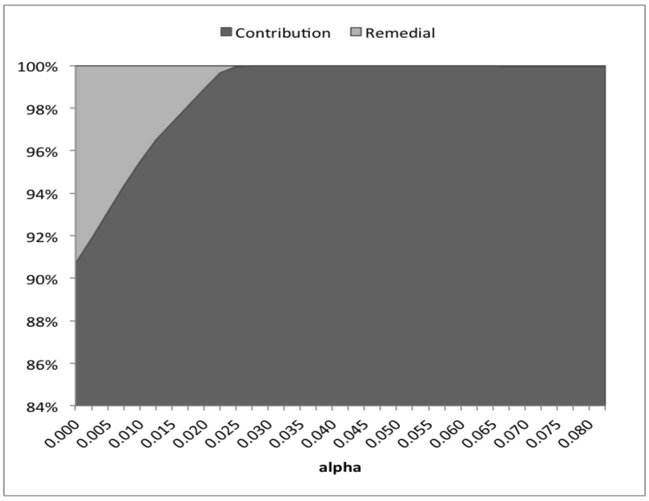


FIGURE 2.9: MICC: Initial cost allocation in function of α

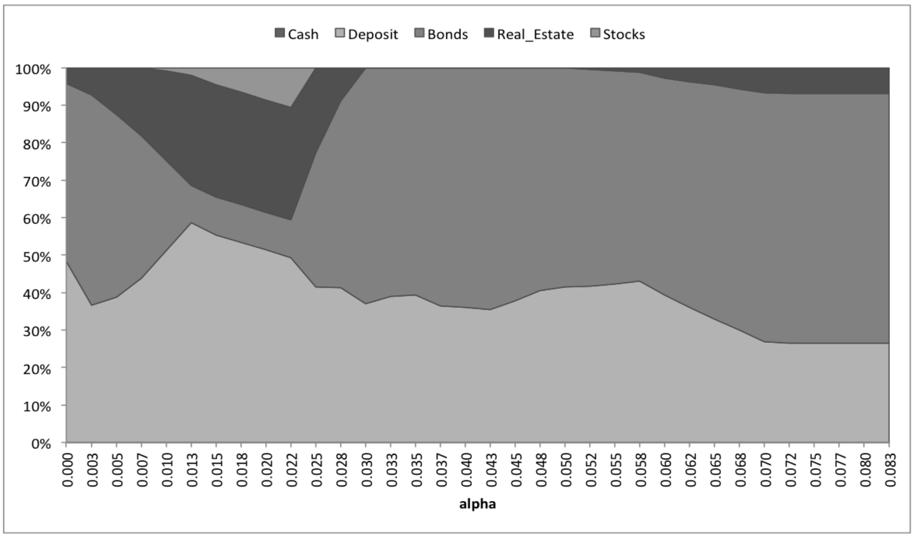


FIGURE 2.10: MICC: Asset allocation in function of α

decision of the MICC is:

$$H_0 := \begin{pmatrix} 0.41 & 0.59 & 0 & 0 \end{pmatrix}^\top; \qquad cr_0 = 0.279 \qquad \text{and} \qquad Z_0 = 0$$

whereas for the OICC:

$$H_0 := \begin{pmatrix} 0.26 & 0.66 & 0.08 & 0 \end{pmatrix}^\top; \qquad cr_0 = 0.270; \qquad \text{and} \qquad Z_0 = 0.$$

Asset allocations are calculated as percentage of the total asset. According to the above example, the decisions related to the OICC approach are riskier than the ones of MICC, especially regarding the asset allocation. In what follows, we will try to quantify the cost of this risk reduction. For a pension fund, this can be done

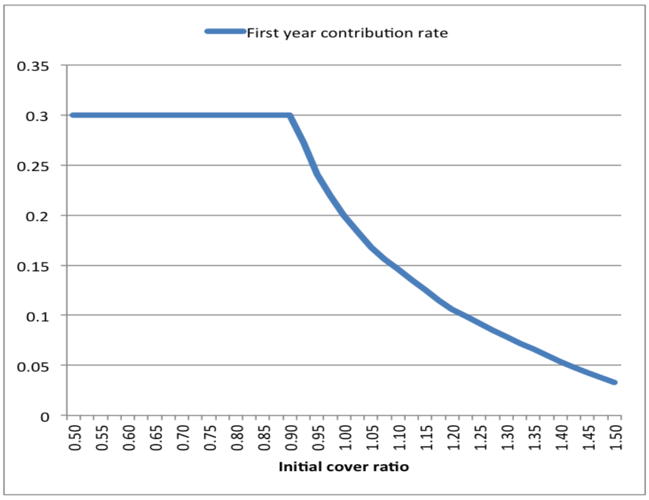


FIGURE 2.11: MICC: Contribution rate in function of F_0

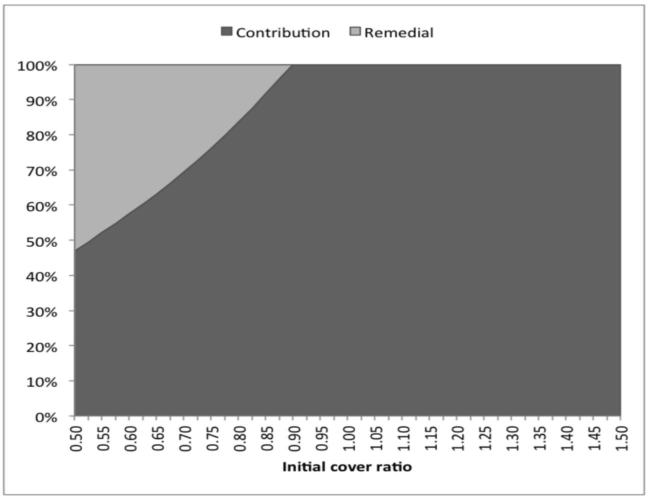


FIGURE 2.12: MICC: Initial cost allocation in function of F_0

by measuring the difference in term of the total cost (regular contribution + remedial contribution). Hence, Figure 2.14 compares the total costs of OICC and MICC whereas Figure 2.15 displays the contribution rate difference, all this with respect to α. When $\alpha \leq \alpha_*^O = 0.025$ or $\alpha \geq \bar{\alpha}^M = 0.07$, the contribution rate and total cost are equal for both models. For $\alpha_*^O \leq \alpha \leq \bar{\alpha}^M$, OICC and MICC slightly differ, i.e. MICC costs more for a maximum variation of $2'000$ (less than 2% of total asset) and 1.5%, respectively, for the total cost and the contribution rate. Consequently, although being more conservative, the MICC has to be considered as a serious contender for optimal ALM for the two following reasons:

- it is safer, and
- the cost of this safety is less than 2% of total asset.

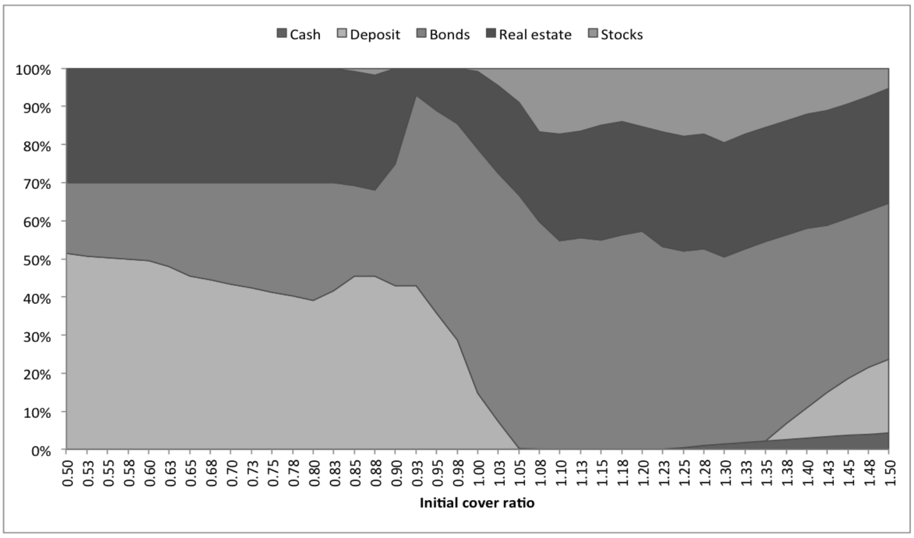


FIGURE 2.13: MICC: Asset allocation in function of F_0

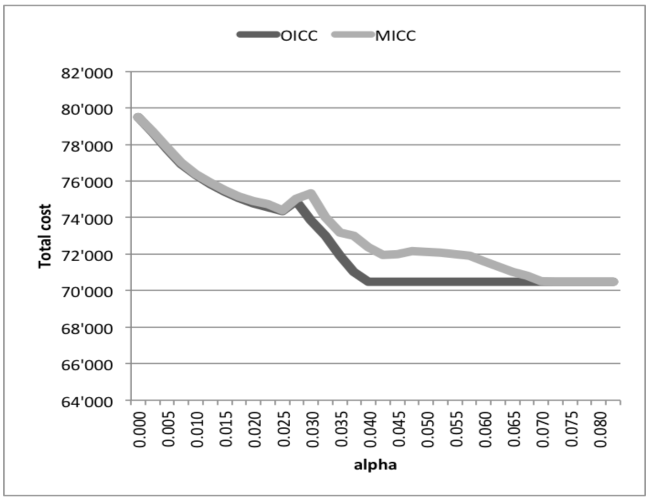


FIGURE 2.14: Comparison of OICC and MICC in function of α: Total cost

Figures 2.16 and 2.17, which compare the effect of F_0 on the OICC and on the MICC, confirm that result. However, it would be interesting to analyse the impact of the first stage decisions on the other stages in order to conclude which approach is better.

2.5 Appendices

2.5.1 Appendix 1: The ALM program description

We start this section by defining indices, variables and parameters of the model. Secondly, the ALM model with the objective and the constraints are also displayed.

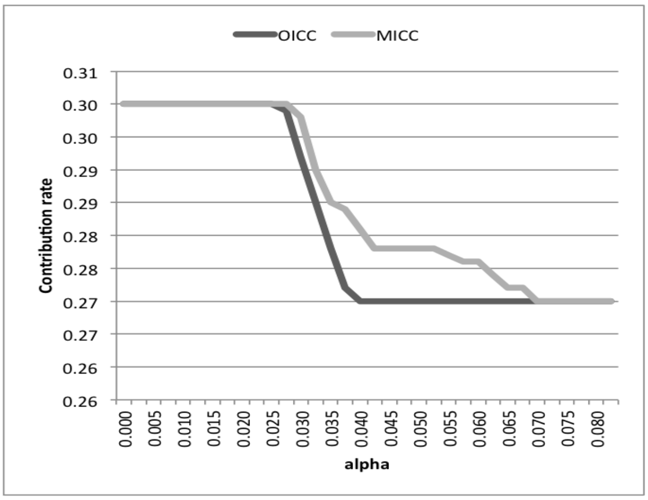


FIGURE 2.15: Comparison of OICC and MICC in function of α: Contribution rate cr_0

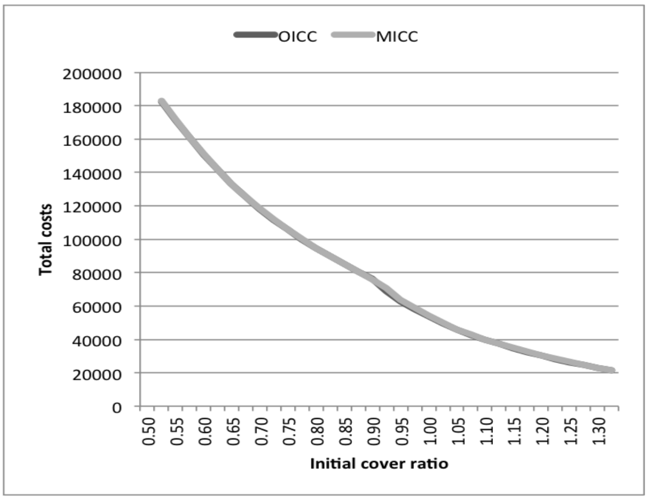


FIGURE 2.16: Comparison of OICC and MICC in function of F_0: Total cost

Indices

t	time index, $t = 0, 1, \cdots, T$
s	scenarios index, $s = 1, \cdots, S$
k	index of asset classes, $k = 1, \cdots, d$

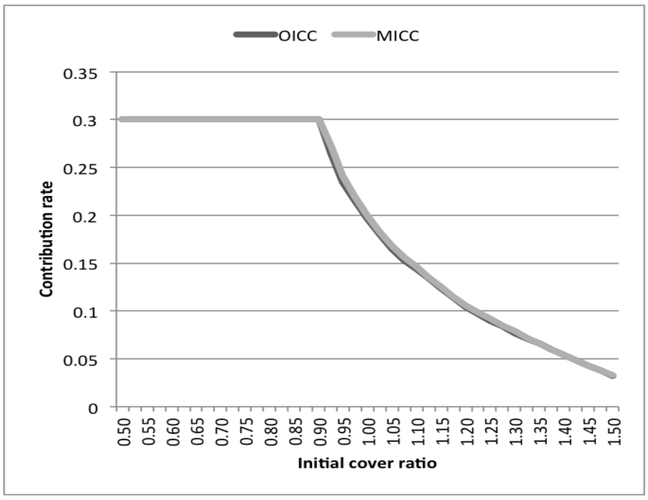


FIGURE 2.17: Comparison of OICC and MICC in function of F_0: Contribution rate cr_0

Decision variables

Z_t^s	remedial contribution by the sponsor at time t in scenario s
C_t^s	total cash amount at the beginning of year t in scenario s
$H_{k,t}^s$	value of the investments hold in asset class $k,$ at the beginning of year t and in scenario s
$B_{k,t}^s$	value of the asset class $k,$ bought at the beginning of year t and in scenario s
$S_{k,t}^s$	value of the asset class $k,$ sold at the beginning of year t and in scenario s
cr_t^s	contribution rate for year $t+1$ in scenario s
A_t^s	total asset value at time t in scenario s
A_t^{*s}	total asset value just before the asset allocation and the remedial contribution at time t in scenario s
$\Delta_{cr_t}^s$	variation (increase or decrease) of contribution rate from year t to $t+1$ in scenario s

Random parameters

$r_{k,t}^s$	random rate of return on asset class k over year t in scenario s and $\xi_{k,t} := 1 + r_{k,t}$
W_t^s	random total wages of active participants in year t in scenario s
L_t^s	random value of liabilities at time t in scenario s
Ben_t^s	random total benefit payments to active participants in year t in scenario s

Deterministic parameters

T	time horizon
S	number of scenarios
d	number of asset classes
α	risk parameter defined by either the sponsor or the regulator
$\bar{c}_k^B$	proportional transaction cost for purchasing an asset class k
$\bar{c}_k^S$	proportional transaction cost for purchasing an asset class k
l_k	lower bound on the proportion of asset class $k$in the total asset portfolio
u_k	upper bound on the proportion of asset class $k$in the total asset portfolio
l_c	lower bound on the proportion of cash $k$in the total asset portfolio
u_c	upper bound on the proportion of cash $k$in the total asset portfolio
cr^l	lower bound on the contribution rate
cr^u	upper bound on the contribution rate
$\underline{\Delta_{cr}}$	lower bound on the yearly absolute variation of the contribution rate
$\bar{\Delta}_{cr}$	upper bound on the yearly absolute variation of the contribution rate
$\bar{F}$	target funding ratio
γ	lower bound on the funding ratio
r_f	risk free interest rate
λ_z	penalty parameter for remedial contribution
$\lambda_{\Delta_{cr}}$	penalty parameter for absolute variation of contribution rate
$\bar{H}_{k,0}$	value of the initial allocation in asset class k
$\bar{C}_0$	initial cash amount

Objective

The objective of the model is to determine the asset allocation, contribution rate and remedial contributions that minimize the total cost defined as follows

$$\min_{H,cr,Z} \mathbb{E}_0 \left[\sum_{t=0}^{T-1} v_{t+1} \left(cr_t W_{t+1} + \lambda_z Z_{t+1} \right) + \sum_{t=0}^{T-2} v_{t+1} \lambda_{\Delta_{cr}} \Delta_{cr_t} W_{t+1} \right]$$

under the following constraints. In the model description, anytime we use indices s and/or k is equivalent to saying for any $s = 1, \cdots, S$ and/or for any $k = 1, \cdots, d$.

Budget constraints and total value of the assets

$$A_0 = \sum_{k=1}^{d} \bar{H}_{k,0} + \bar{C}_0 + Z_0 - \sum_{k=1}^{d} \left(\bar{c}_k^B B_{k,0} + \bar{c}_k^S S_{k,0} \right) = \sum_{k=1}^{d} H_{k,0} + C_0$$

$$A_T^s = \sum_{k=1}^{d} H_{T-1,k}^s \xi_{T,k}^s + C_{T-1}^s \left(1 + r_f\right) + cr_{T-1}^s W_T^s - \mathrm{Ben}_T^s = A_T^{*s}$$

$$A_t^s = \sum_{k=1}^{d} H_{k,t-1}^s \xi_{k,t}^s + C_{t-1}^s \left(1 + r_f\right) + cr_{t-1}^s W_t^s - \mathrm{Ben}_t^s + Z_t^s - \sum_{k=1}^{d} \left(\bar{c}_k^B B_{k,t}^s + \bar{c}_k^S S_{k,t}^s \right); \quad t = 1, \cdots, T-1$$

$$A_t^s = A_t^{*s} + Z_t^s - \sum_{k=1}^{d} \left(\bar{c}_k^B B_{k,t}^s + \bar{c}_k^S S_{k,t}^s \right) = \sum_{k=1}^{d} H_{k,t}^s + C_t^s \; ; \qquad t = 1, \cdots, T-1$$

Asset classes dynamics

$$H_{k,0} = \bar{H}_{k,0} + B_{k,0} - S_{k,0}$$

$$H_{k,t}^s = \xi_{k,t}^s H_{k,t-1}^s + B_{k,t}^s - S_{k,t}^s \; ; \qquad t = 1, \cdots, T$$

Cash dynamics

$$C_0 = \bar{C}_0 + Z_0 - \sum_{k=1}^{d} \left(1 + \bar{c}_k^B\right) B_{k,0} + \sum_{k=1}^{d} \left(1 - \bar{c}_k^S\right) S_{k,0}$$

$$C_t^s = C_{t-1}^s \left(1 + r_f\right) + cr_{t-1}^s W_t^s - \mathrm{Ben}_t^s + Z_t^s - \sum_{k=1}^{d} \left(1 + \bar{c}_k^B\right) B_{k,t}^s + \sum_{k=1}^{d} \left(1 - \bar{c}_k^S\right) S_{k,t}^s \; ; \quad t = 1, \cdots, T$$

Not short selling assets and not borrowing cash constraints

$$H^s_{k,t} \geq 0 \ ; \ \ B^s_{k,t} \geq 0 \ ; \ \ S^s_{k,t} \geq 0 \ ; \ \ C^s_t \geq 0 \ ; \qquad t = 0, \cdots, T$$

Liquidity constraints

$$C^s_t \left(1 + r_f\right) + \mathbb{E}_{t,s} \left(cr^s_t W_{t+1} - Ben_{t+1}\right) \geq 0 \ ; \qquad t = 0, \cdots, T-1$$

Portfolio constraints

$$\begin{aligned} l_k A^s_t \leq H^s_{k,t} \leq u_k A^s_t \ ; & \qquad t = 0, \cdots, T-1 \\ l_c A^s_t \leq C^s_t \leq u_c A^s_t \ ; & \qquad t = 0, \cdots, T-1 \end{aligned}$$

Constraints on contribution rates

$$cr^l \leq cr^s_t \leq cr^u \quad \text{and} \quad \underline{\Delta_{cr}} \leq \Delta^s_{cr_t} \leq \bar{\Delta}_{cr} \ ; \qquad t = 0, \cdots, T-1$$

The decision variables are subject to the non-anticipativity constraints. The integrated chance constraints defined in Section 2.2.3 control the risk-level of the model. They also have to be included in the model as explained in section 2.3.3.

2.5.2 Appendix 2: Proposition 2.3.1, an example based sketch of proof

We consider the event tree of Figure 2.18 with a time horizon $T = 3$ and a branching structure of $1 - 5 - 4 - 2$, leading to $S = 5 \times 4 \times 2 = 40$ scenarios. A node is a possible outcome of the stochastic event at a given time. The starting and ending nodes of the tree are round whereas the others are rectangular. At each $t \in \{2, \ 3\}$, the rectangular nodes are named according to time and following a top-down alphabetic order. For example, the rectangle $(1, a)$ describe the outcome at the first node at time 1. At each node, the economical values such as total asset, total liability and expected shortfall can be determined. For the sake of clarity, we recall that, due to non-anticipativity, the node $(1, a)$ is equivalent to the node $(1, s')\,,\ s' \in \{1, \cdots, 8\}$ as described before, the node $(1, b)$ is equivalent to the node $(1, s')\,,\ s' \in \{9, \cdots, 16\}$ and so on for the other nodes. The other notations used here are similar to the ones in the paper. We define

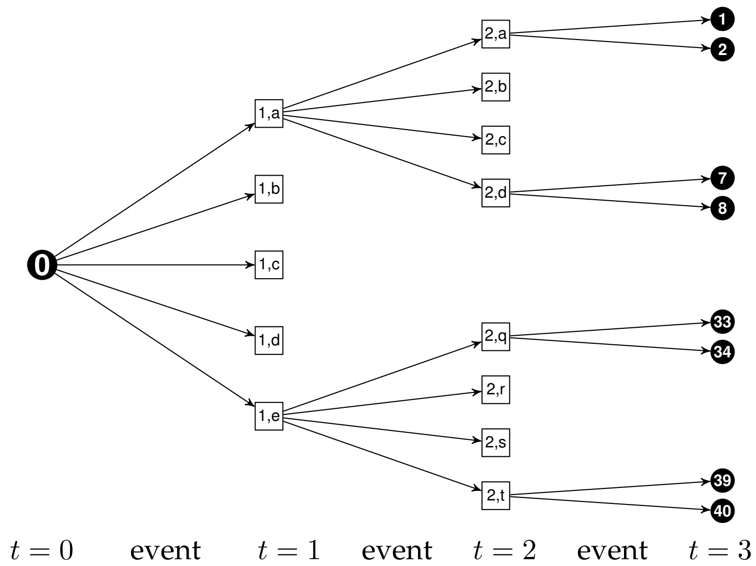


FIGURE 2.18: A scenario tree with 40 scenarios and 66 nodes.

$$\Lambda_{(t,s)} := \sum_{s^{'} \in \mathcal{S}} p_{t,s}^{s^{'}} \left(A_{t+1}^{*s^{'}} - \gamma L_{t+1}^{s^{'}} \right)^{-}$$

where $p_{t,s}^{s^{'}}$ stands for the conditional probability to reach node $\left(t+1, s^{'}\right)$ going from (t, s) and $p_{t,s}^{s^{'}} = 0$ for any scenario $s^{'}$ of $t+1$ not descending from (t, s). The MICC defined in equation (2.32) is then

$$\{\Lambda_{h-1,s}, \ \ h \in \mathcal{T}_{t+1}\} \leq \beta_{t,s}, \ \ t \in \{0, \ 1, \ 2\}\,.$$

The value $\beta_{t,s} := \alpha L_t^s$ is the ICC upper limit computed at time t in scenario s. According to this set of constraints,

From initial node at $t := 0$

$$h \in \{1, \ 2, \ 3\} \Rightarrow \Lambda_{0,s} \leq \beta_0, \ \Lambda_{1,s} \leq \beta_0, \text{ and } \Lambda_{2,s} \leq \beta_0, \quad s \in \mathcal{S}. \tag{2.34}$$

At $t := 1$,

from node $(1, a)$, $h \in \{2, \ 3\} \Rightarrow \Lambda_{1,s} \leq \beta_{1,s}$, and $\Lambda_{2,s} \leq \beta_{1,s}$, $\quad s \in \{1, \ 2, \cdots, \ 8\}$

from node $(1, b)$, $h \in \{2, \ 3\} \Rightarrow \Lambda_{1,s} \leq \beta_{1,s}$, and $\Lambda_{2,s} \leq \beta_{1,s}$, $\quad s \in \{9, \ 10, \cdots, \ 16\}$

$\vdots \qquad \vdots$

$\vdots \qquad \vdots$

from node $(1, e)$, $h \in \{2, \ 3\} \Rightarrow \Lambda_{1,s} \leq \beta_{1,s}$, and $\Lambda_{2,s} \leq \beta_{1,s}$, $\quad s \in \{33, \ 34, \cdots, \ 40\}\,.$

At $t := 2$

$$
\begin{aligned}
\text{from node } (2,a),\ h=3 \Rightarrow &\ \text{ and } \Lambda_{2,s} \le \beta_{2,s}, \quad s \in \{1,\ 2\} \\
\text{from node } (2,b),\ h=3 \Rightarrow &\ \text{ and } \Lambda_{2,s} \le \beta_{2,s}, \quad s \in \{3,\ 4\} \\
\vdots \qquad\qquad & \qquad \vdots \\
\vdots \qquad\qquad & \qquad \vdots \\
\text{from node } (2,t),\ h=3 \Rightarrow &\ \text{ and } \Lambda_{2,s} \le \beta_{2,s}, \quad s \in \{39,\ 40\}\,.
\end{aligned}
$$

Therefore, we obtain for any $s \in \mathcal{S}$ that

$$
\begin{aligned}
&\text{at time } 0,\ \Lambda_{0,s} \le \beta_0 && \Leftrightarrow \Lambda_{0,s} \le \beta_0 \\
&\text{at time } 1,\ \Lambda_{1,s} \le \beta_0,\ \Lambda_{1,s} \le \beta_{1,s} && \Leftrightarrow \Lambda_{1,s} \le \min\{\beta_0,\ \beta_{1,s}\} \\
&\text{at time } 2,\ \Lambda_{2,s} \le \beta_0,\ \Lambda_{2,s} \le \beta_{1,s},\ \Lambda_{2,s} \le \beta_{1,s} && \Leftrightarrow \Lambda_{2,s} \le \min\{\beta_0,\ \beta_{1,s},\ \beta_{2,s}\}\,.
\end{aligned}
$$

The obtained result leads obviously to the proposition 2.3.1 in the paper. Considering such an example is therefore without loss of generality. For another event tree with different time horizon and branching structure, the proposition can be prooved using the same procedure.

Chapter 3

On Bivariate Lifetime Modeling in Life Insurance Applications

This chapter is based on the preprint paper Toukourou et al., 2016.

3.1 Introduction

Insurance and annuity products covering several lives require the modelling of the joint distribution of future lifetimes. Commonly in actuarial practice, the future lifetimes among a group of people are assumed to be independent. This simplifying assumption is not supported by real insurance data as demonstrated by numerous investigations. Joint life annuities issued to married couples offer a very good illustration of this fact. It is well known that husband and wife tend to be exposed to similar risks as they are likely to have the same living habits. For example, Parkes et al., 1969 and Ward, 1976 have brought to light the increased mortality of widowers, often called the *broken heart syndrome*. Many contributions have shown that there could be a significant difference between risk-related quantities, such as risk premiums, evaluated according to dependence or independence assumptions. Denuit and Cornet, 1999 have measured the effect of lifetime dependencies on the present value of a widow pension benefit. Based on the data collected in cemeteries, not only do their estimation results confirm that the mortality risk depends on the marital status, but also show that the amounts of premium are reduced approximately by 10 per cent compared to model which assumes independence. According to data from a large Canadian insurance company, Frees et al., 1995 have demonstrated that there is a strong positive dependence between joint lives. Their estimation results indicate that annuity values are reduced by approximately 5 per cent compared to a model with independence.

Introduced by Sklar, 1959, copulas have been widely used to model the dependence structure of random vectors. In the particular case of bivariate lifetimes, frailty models can be used to describe the common risk factors between husband and wife. Oakes, 1989 has shown that the bivariate distributions generated by frailty models are a subclass of Archimedean copulas. This makes this particular copula family very attractive for modelling bivariate lifetimes. We refer to Nelsen, 2007 for a general introduction to copulas and Albrecher et al., 2011; Constantinescu et al., 2011, for applications of Archimedean copula in risk theory. The Archimedean copula family has been proved valuable in numerous life insurance applications, see e.g., Frees et al., 1995; Brown and Poterba, 1999; Carriere, 2000. In Luciano et al., 2008, the marginal distributions and the copula are fitted separately and, the results show that the dependence increases with age.

It is known that the level of association between variables is characterized by the value of the dependence parameter. In this paper, a special attention is paid to this dependence parameter. Youn and Shemyakin, 1999 have introduced the age difference between spouses as an argument of the dependence parameter of the copula. In addition, the sign of the age difference is of great interest in our model. More precisely, we presume that the gender of the older member of the couple has an influence on the level of dependence between lifetimes. In order to confirm our hypothesis, four families of Archimedean copulas are discussed namely, Gumbel, Frank, Clayton and Joe copulas, all these under a Gompertz distribution assumption for marginals. The parameter estimations are based on the maximum likelihood approach using data from a large Canadian insurance company, the same set of data used by Frees et al., 1995. Following Joe and Xu, 1996 and Oakes, 1989, a two-step technique, where marginals and copula are estimated separately, is applied. The results make clear that the dependence is higher when husband is older than wife.

Once the marginal and copula parameters are estimated, one needs to assess the goodness of fit of the model. For example, the likelihood ratio test is used in Carriere, 2000 whereas the model of Youn and Shemyakin, 1999 is based on the Akaike Information Criterion (AIC). In this paper, following Gribkova and Lopez, 2015 and Lawless, 2011, we implement a whole goodness of fit procedure to validate the model. Based on the Cramèr-von Mises statistics, the Gumbel copula, whose dependence parameter is a function of the age difference and its sign gives the best fit.

The rest of the paper is organized as follows. Section 3.2 discusses the main characteristics of the dataset and provides some key facts that motivate our study. Section 3.3 describes the maximum likelihood procedure used to estimate the marginal distributions. The dependence models are examined in Section 3.4. In a first hand, we describe the copula models whose parameter are estimated. Secondly, a bootstrap algorithm is proposed for assessing the goodness of fit of the model. Considering several products available on the life insurance market, numerical applications with real data, including best estimate of liabilities, risk capital and stop loss premiums are presented in Section 3.5. Section 3.6 concludes the paper.

3.2 Motivation

As already shown in Maeder, 1996, being in a married couple can significantly influence the mortality. Moreover, the remaining lifetimes of male and female in the couple are dependent, see e.g., Carriere, 2000; Frees et al., 1995. In this contribution, we aim at modelling the dependence between the lifetimes of a man and a woman within a married couple. Common dependence measures, which will be used in our study, are: the Pearson's correlation coefficient r, the Kendall's Tau τ, and the Spearman's Rho ρ. In order to develop these aspects, data[1] from a large Canadian life insurance company are used. The dataset contains information from policies that were in force during the observation period, i.e. from December 29, 1988 to December 31, 1993. Thus, we have $14'947$ contracts among which $14'889$ couples (one male and one female) and the remaining 58 are contracts where annuitants are both male (22 pairs) or both female (36 pairs). The same dataset has been analysed in Frees et al., 1995; Carriere, 2000; Youn and Shemyakin, 1999; Gribkova and Lopez, 2015 among others, also in the framework of modelling bivariate lifetime. Since we are interested in the dependence within the couple, we focus our attention on the male-female contracts.

We refer the readers to Frees et al., 1995 for the data processing procedure. The dataset is left truncated as the annuitant informations are recorded only from the date they enter the study; this means that insured who have died before the beginning of the observation period were not taken into account in the study. The dataset is also right censored in the sense that most of the insured were alive at the end of the study. Considering our sample as described above, some couples having several contracts could appear many times. By considering each couple only once, our dataset consists of $12'856$ different couples for which, we can draw the following informations:

- the entry ages x_m and x_f for male and female, respectively,

[1] We wish to thank the Society of Actuaries, through the courtesy of Edward (Jed) Frees and Emiliano Valdez, for allowing the use of the data in this paper.

- the lifetimes under the observation period t_m and t_f for male and female, respectively, and
- the binary right censoring indicator δ_m and δ_f for male and female, respectively,
- the couple's benefit in Canadian Dollar (CAD) amount within a last survivor contract.

The entry age is the age at which, the annuitant enters the study. The lifetime at entry age corresponds to the lapse of time during which the individual was alive over the period of study. Therefore, for a male (resp. female) aged x_m (resp. x_f) at entry and whose data is not censored i.e. $\delta_m = 0$ (resp. $\delta_f = 0$), $x_m + t_m$ (resp. $x_f + t_f$) is the age at death. When the data is right censored i.e. $\delta_m = 1$ (resp. $\delta_f = 1$), the number $x_m + t_m$ (resp. $x_f + t_f$) is the age at the end of the period of study (December 31, 1993). The lifetime is usually equal to 5.055 years corresponding to the duration of the study period; but it is sometimes less as some people may entry later or die before the end of study. Benefit is paid each year until the death of the last survivor. Its value will be used as an input for the applications of the model to insurance products in Section 3.5.2. Some summary statistics of the age distribution of our dataset are displayed in Table 3.1. It can be seen that the average

	Males age		Females age	
Statistics	Entry	Death	Entry	Death
Number	$12'856$	$1'349$	$12'856$	484
Mean	67.9	74.41	64.95	73.76
Std. dev.	6.38	7.18	7.26	7.87
$Median$	67.68	74.18	65.27	73.09
$10^{th} percentile$	60.34	66.00	55.92	64.24
$90^{th} percentile$	75.41	83.21	73.42	83.92

TABLE 3.1: Summary of the univariate distribution statistics.

entry age is 66.4 for the entire population, 67.9 for males and 64.9 for female; 90% of annuitants are older than 57.9 at entry and males are older than females by 3 years on average. Among the $12'856$ couples considered, there are 1349 males and 484 females who die during the study period. In addition, there are 11'228 couples where both annuitants are alive at the end of the observation while both spouses are dead for 205 couples. Based on these 205 couples, the empirical dependence measures are displayed in the last row of Table 3.2. The values show that the ages at death of spouses are positively correlated.

		Dependence measures		
	Number	r	ρ	τ
$x_m > x_f$	154	0.90	0.88	0.72
$x_m < x_f$	51	0.88	0.86	0.69
Total	205	0.82	0.80	0.62

TABLE 3.2: Empirical dependence measures with respect to the gender of the elder partner.

From the existing literature, see e.g., Denuit and Cornet, 1999; Youn and Shemyakin, 1999; Denuit et al., 2001, the dependence within a couple is often influenced by three factors:

- the **common lifestyle** that husband and wife follow, for example their eating habits,
- the **common disaster** that affects simultaneously the husband and his wife, as they are likely to be in the same area when a catastrophic event occurs,
- the **broken-heart factor** where the death of one would precipitate the death of the partner, often due to the vacuum caused by the passing away of the companion.

Based on the common disaster and the broken-heart, Youn and Shemyakin, 1999 have introduced the *age difference between spouses*. Their results show that the model captures some additional association between lifetime of the spouses that would not be reflected in a model without age difference. It is also observed that, the higher the age difference is, the lower is the dependence. Referring to the same dataset, Table 3.3 confirms their results, with $|d|$ the absolute value of d and $d = x_m - x_f$.

		Dependence measures		
	Number	r	ρ	τ
$0 \leq \|d\| < 2$	83	0.97	0.96	0.84
$2 \leq \|d\| < 4$	50	0.94	0.94	0.82
$\|d\| \geq 4$	72	0.72	0.63	0.50

TABLE 3.3: Empirical dependence measures with respect to the age difference.

Our study follows the same lines of idea as these authors. In addition to the age difference, we believe that the gender of the elder partner may have an impact on their lifetimes dependencies. Indeed, the fact that the husband is older than the wife may influence their relationship, and indirectly, the dependence factors cited above. The results displayed in Table 3.2 clearly show that the spouse lifetime dependencies are higher when d is positive, i.e. when husband is older than wife. The variable *gender of the elder member* is measured through the sign of the age difference d. Table 3.4 displays the empirical Kendall's τ with respect to the age difference and to the gender of the elder partner. One can notice that the coefficients can vary for more than 30% depending on who is the older member of the couple.

τ	Total	$0 \leq \|d\| < 2$	$2 \leq \|d\| < 4$	$\|d\| \geq 4$
$x_m \geq x_f$	0.72	0.89	0.89	0.55
No. of $(x_m \geq x_f)$	154	53	41	60
$x_m < x_f$	0.69	0.86	0.86	0.74
No. of $(x_m < x_f)$	51	30	9	12

TABLE 3.4: Kendal'Tau correlation coefficients by age and gender of the elder partner.

In what follows, a bivariate lifetime model will verify our hypothesis. To do this, marginal distributions for each of the male and female lifetimes are firstly defined and secondly the copula models are introduced. The estimation methods will be detailed in the Section 3.3 and Section 3.4.

3.3 Marginal distributions

3.3.1 Background

The lifetime of a newborn shall be modelled by a positive continuous random variable, say X with distribution function (df) F and survival function S. The symbol (x) will be used to denote a live aged x and $T(x) = (X - x)|X > x$ is the remaining lifetime of (x). The actuarial symbols ${}_tp_x$ and ${}_tq_x$ are, respectively, the survival function and the df of $T(x)$. Indeed, the probability, for a live (x), to remain alive t more years is given by

$$ {}_tp_x = \mathbb{P}(X > x + t \mid X > x) = \frac{\mathbb{P}(X > x + t)}{\mathbb{P}(X > x)} = \frac{S(x+t)}{S(x)}. $$

When X has a probability density function f, then $T(x)$ has a probability density function given by

$$ f_x(t) = {}_tp_x \, \mu(x+t). $$

where $\mu(.)$ is the hasard rate function, also called *force of mortality*.

Several parametric mortality laws such as De Moivre, constant force of mortality, Gompertz, Inverse-Gompertz, Makeham, Gamma, Lognormal and Weibull are used in the literature; see Bowers et al., 1986. The choice of a specific mortality model is determined mainly by the caracteristics of the available data and the objective of the study. It is well known that the De Moivre law and the constant force of mortality assumptions are interesting for theoretical purposes whereas Gompertz and Weibull are more appropriate for fitting real data, especially for population of age over 30. The data set exploited in this paper regroups essentially policyholders who are at least middle-aged. That is why, in our study, the interest is on the Gompertz law whose caracteristics are defined as follows

$$\mu(x) = Bc^x \qquad \text{and} \qquad S(x) = \exp\left(-\frac{B}{\ln c}\left(c^x - 1\right)\right)$$

with $B > 0, \quad c > 1, \quad x \geq 0$. In addition, Frees et al., 1995 and Carriere, 2000 have shown that the Gompertz mortality law fits our dataset very well, see Figure 3.1. For estimation purposes, the Gompertz law has been reparametrized as follows (see Carriere, 1994)

$$e^{-m/\sigma} = \frac{B}{\ln c} \qquad \text{and} \qquad e^{1/\sigma} = c$$

from which we obtain

$$\begin{aligned}
\mu(x+t) &= \frac{1}{\sigma}\exp\left(\frac{x+t-m}{\sigma}\right), \\
{}_tp_x &= \exp\left(e^{\frac{x-m}{\sigma}}\left(1 - e^{\frac{t}{\sigma}}\right)\right), \\
f_x(t) &= \exp\left(e^{\frac{x-m}{\sigma}}\left(1 - e^{\frac{t}{\sigma}}\right)\right)\frac{1}{\sigma}\exp\left(\frac{x+t-m}{\sigma}\right), \\
F_x(t) &= 1 - \exp\left(e^{\frac{x-m}{\sigma}}\left(1 - e^{\frac{t}{\sigma}}\right)\right), \qquad (3.1)
\end{aligned}$$

where the mode $m > 0$ and the dispersion parameter $\sigma > 0$ are the new parameters of the distribution.

3.3.2 Maximum likelihood procedure

In what follows, we will use the following notation:

- The index j indicates the gender of the individual, i.e. $j = m$ for male and $j = f$ for female.

- $\theta_j = (m_j, \sigma_j)$ denotes the vector of unknown Gompertz parameters for a given gender j.
- n is the total number of couples in our data set. Hereafter, a couple means a group of two persons of opposite gender that have signed an insurance contract and i is the couple index with $1 \leq i \leq n$.
- For a couple i, t_j^i is the remaining lifetime observed in the collected data. Indeed, for an individual of gender j aged x_j, the remaining lifetime $T_j^i(x)$ is a random variable such that

$$T_j^i(x_j) = \min\left(t_j^i, B_j^i\right) \qquad \text{and} \qquad \delta_j^i = \mathbf{1}_{\{t_j^i \geq B_j^i\}},$$

where B_j^i is a random censoring point of the individual of gender j in the couple i.

Consider a couple i where the male and female were, respectively, aged x_m and x_f at contract initiation date. For each gender $j = m, f$, the contribution to the likelihood is given by

$$L_j^i(\theta_j) = \left[{}_{B_j^i}p_{x_j}(\theta_j)\right]^{\delta_j^i} \left[f_{x_j}^i\left(t_j^i, \theta_j\right)\right]^{1-\delta_j^i}. \tag{3.2}$$

We recall that the dataset is left truncated that is why likelihood function in (3.2) has therefore to be conditional on survival to the entry age x_j, see e.g. Carriere, 2000. Therefore, the overall likelihood function can be written as follows

$$L_j(\theta_j) = \prod_{i=1}^{n} L_j^i(\theta_j), \qquad j = m, f. \tag{3.3}$$

By maximizing the likelihood function in (3.3) using our dataset, the MLE estimates of the Gompertz df are displayed in Table 3.5. Standard errors are relatively low and estimation shows that the modal age at death is larger for females than for males. This latter can be explained by the fact that women have a longer life expectency than men. A good way to analyse how well the model performs is to compare with the *Kaplan-Meier (KM) product-limit estimator* of the dataset. We recall that the KM technique is an approach which consists in estimating non-parametrically the survival function from the empirical data.

$\hat{\theta}$	Estimate	Std. error
$\hat{m}_m$	86.378	0.289
$\hat{m}_f$	92.175	0.527
$\hat{\sigma}_m$	9.833	0.415
$\hat{\sigma}_f$	8.114	0.392

TABLE 3.5: Gompertz parameter estimates.

Figure 3.1 compares, for the female group, the KM estimator of the survival function to the one obtained from the Gompertz distribution estimated above. Since almost all the annuitants are older than 40 at entry, all the distributions are conditional on survival to age 40. The survival functions are plotted as a function of age x (from $x = 40$ to $x = 110$). The Gompertz curve is smooth whereas the KM is jagged. The figures clearly show that the estimated Gompertz model is a valid choice for approximating the KM curve.

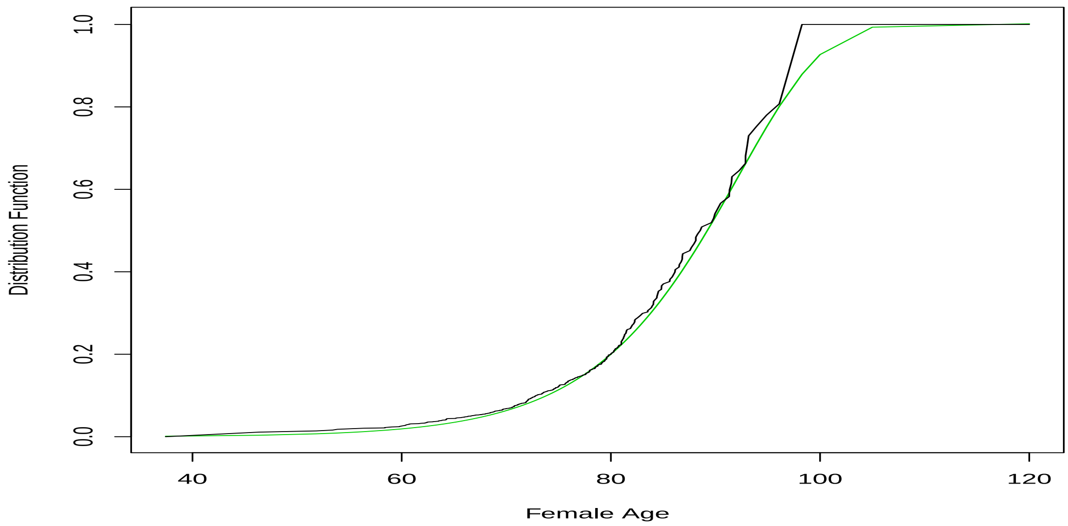


FIGURE 3.1: Gompertz and Kaplan-Meier fitted female distribution functions

3.4 Dependence Models

3.4.1 Background

Copula models were introduced by Sklar, 1959 in order to specify the joint df of a random vector by separating the behavior of the marginals and the dependence structure. Without loss of generality, we focus on the bivariate case. We denote by $T(x_m)$ and $T(x_f)$ the future lifetime respectively for man and woman. If $T(x_m)$ and $T(x_f)$ are positive and continuous, there exists a unique copula $C : [0,1]^2 \to [0,1]$ which specifies the joint df of the bivariate random vector $(T(x_m), T(x_f))$ as follows

$$\mathbb{P}(T(x_m) \leq t_1, T(x_f) \leq t_2) = C\left(\mathbb{P}\left(T(x_m) \leq t_1\right), \mathbb{P}\left(T(x_f) \leq t_2\right)\right) = C({}_{t_1}q_{x_m}, {}_{t_2}q_{x_f}).$$

Similarly, the survival function of $(T(x_m), T(x_f))$ is written in terms of copulas and marginal survival functions. This is given by

$$\begin{aligned}\mathbb{P}(T(x_m) > t_1, T(x_f) > t_2) &= \tilde{C}({}_{t_1}p_{x_m}, {}_{t_2}p_{x_f}) \\ &= {}_{t_1}p_{x_m} + {}_{t_2}p_{x_f} - 1 + C({}_{t_1}q_{x_m}, {}_{t_2}q_{x_f}). \end{aligned} \quad (3.4)$$

A broad range of parametric copulas has been developed in the literature. We refer to Nelsen, 2007 for a review of the existing copula families. The Archimedean copula family is very popular in life insurance applications, especially due to its flexibility in modelling dependent random lifetimes, see e.g. Frees et al., 1995; Youn and Shemyakin, 1999 . If ϕ is a convex and twice-differentiable strictly decreasing function, the df of an Archimedean copula is given by

$$C_\phi(u,v) = \phi^{-1}(\phi(u) + \phi(v)),$$

where $\phi : [0,1] \to [0,\infty]$ is the generator of the copula satisfying $\phi(1) = 0$ with $u, v \in [0,1]$. In this paper, four well known copulas are discussed. Firstly, the Gumbel copula generated by

$$\phi(t) = (-\ln(t))^{-\alpha}, \quad \alpha > 1,$$

which yields the copula

$$C_\alpha(u,v) = \exp\{-[(-\ln(u))^\alpha + (-\ln(v))^\alpha]^{1/\alpha}\}, \quad \alpha > 1. \quad (3.5)$$

Secondly, we have the Frank copula

$$C_\alpha(u,v) = -\frac{1}{\alpha}\ln\left(1 + \frac{(e^{-\alpha u} - 1)(e^{-\alpha v} - 1)}{(e^{-\alpha} - 1)}\right), \quad \alpha \neq 0, \quad (3.6)$$

with generator

$$\phi(t) = -\ln\left(\frac{e^{-\alpha t} - 1}{e^{-\alpha} - 1}\right), \quad \alpha \neq 0.$$

Thirdly, the Clayton copula is associated to the generator

$$\phi(t) = t^{-\alpha} - 1, \quad \alpha > 0$$

and is given by

$$C_\alpha(u, v) = (u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}, \quad \alpha > 0. \tag{3.7}$$

Finally, for the Joe copula, $\alpha > 1$ and

$$C_\alpha(u, v) = 1 - \Big((1-u)^\alpha + (1-v)^\alpha - (1-u)^\alpha(1-v)^\alpha\Big)^{1/\alpha}, \tag{3.8}$$

with the generator $\phi(t) = -\ln(1 - (1-t)^{-\alpha}), \alpha > 1$.

Clearly, the parameter α in (3.5)-(3.8) determines the dependence level between the two marginal distributions. In our case, that would be the lifetimes of wife and husband. Youn and Shemyakin, 1999 have utilized a Gumbel copula where the association parameter α depends on d as follows

$$\alpha(d) = 1 + \frac{\beta_0}{1 + \beta_2 d^2}, \quad \beta_0, \beta_2 \in \mathbb{R} \tag{3.9}$$

where $d = x_m - x_f$ with x_m and x_f the ages for male and female, respectively.

In our model for α, in addition to this specification, the gender of the elder partner, represented by the sign of d, is also taken into account. This latter is captured through the second term of the denominator $\beta_1 d$ in equations (3.10) and (3.11). Thus, for our model the copula association parameter for the Frank and the Clayton is expressed by

$$\alpha(d) = \frac{\beta_0}{1 + \beta_1 d + \beta_2 |d|}, \quad \beta_0, \beta_1, \beta_2 \in \mathbb{R}. \tag{3.10}$$

Since the copula parameter α in the Gumbel and Joe copulas is restricted to be greater than 1, the corresponding dependence parameter in (3.11) is allowed to have an intercept of 1 and we write

$$\alpha(d) = 1 + \frac{\beta_0}{1 + \beta_1 d + \beta_2 |d|}, \quad \beta_0, \beta_1, \beta_2 \in \mathbb{R}. \tag{3.11}$$

It can be seen that if $\beta_1 < 0$, the dependence parameter is lower when husband is younger than wife, i.e. $d < 0$. Also when d tends to infinity, the dependence parameter goes to 0 for Frank and Clayton and 1 for the Gumbel copula, thus tending towards the independence assumption. Note in passing that instead of taking d^2 as in equation (3.9), we use $|d|$ in both (3.10) and (3.11) for the representation of the absolute age difference.

3.4.2 Estimation of Parameters

The maximum likelihood procedure has been widely used to fit lifetime data to copula models, see e.g., Lawless, 2011; Shih and Louis, 1995; Carriere, 2000. A priori, this method consists in estimating jointly the marginal and copula parameters at once. However, given the huge number of parameters to be estimated at the same time, this approach is computationally intensive. Therefore, we adopt a procedure that allows the determination of marginal and copula parameters, separately. In this respect, Joe and Xu, 1996 have proposed a two step technique which, firstly estimates the marginal parameters $\theta_j, j = m, f$, and the copula parameter $\alpha(d)$ in the second step. This is referred to as the *inference functions for margins* (IFM) method. Specifically, the survival function of each lifetime is evaluated by maximazing the likelihood function in (3.3). For each couple i with x_m^i and x_f^i, let $u_i := {}_{t_m^i}p_{x_m^i}(\hat{\theta}_m)$ and $v_i := {}_{t_f^i}p_{x_f^i}(\hat{\theta}_f)$ be the resulting marginal survival functions for male and female, respectively. Considering the right-censoring feature of the two lifetimes as indicated by δ_m^i and δ_f^i, the estimates $\widehat{\alpha(d)}$ of the copula parameters are obtained by maximizing the likelihood function

$$L(\alpha(d)) := L(\alpha) = \prod_{i=1}^{n} \left[\frac{\partial^2 \tilde{C}_\alpha(u_i, v_i)}{\partial u_i \partial v_i}\right]^{(1-\delta_m^i)(1-\delta_f^i)} \left[\frac{\partial \tilde{C}_\alpha(u_i, v_i)}{-\partial u_i}\right]^{(1-\delta_m^i)\delta_f^i} \times \left[\frac{\partial \tilde{C}_\alpha(u_i, v_i)}{-\partial v_i}\right]^{\delta_m^i(1-\delta_f^i)} \left[\tilde{C}_\alpha(u_i, v_i)\right]^{\delta_m^i \delta_f^i}. \tag{3.12}$$

A similar two-step technique, known as the *Omnibus semi-parametric procedure* or the *pseudo-maximum likelihood*, was also introduced by Oakes, 1989. In this procedure, the marginal distributions are considered as nuisance parameters of the copula model. The first step consists in estimating the two marginals survival functions non-parametrically using the KM method. After rescaling the resulting estimates by $\frac{n}{n+1}$, we obtain the pseudo-observations $(U_{i,n}, V_{i,n})$ where

$$U_{i,n} = \frac{\hat{S}_m(x_m^i + t_m^i)}{\hat{S}_m(x_m^i)} \quad \text{and} \quad V_{i,n} = \frac{\hat{S}_f(x_f^i + t_f^i)}{\hat{S}_f(x_f^i)}.$$

In the second step, the copula estimation is achieved by maximizing the following function

$$
\begin{aligned}
L(\alpha(d)) := L(\alpha) \quad = \quad & \prod_{i=1}^{n} \left[\frac{\partial^2 \tilde{C}_\alpha(U_{i,n}, V_{i,n})}{\partial U_{i,n} \partial V_{i,n}} \right]^{(1-\delta_m^i)(1-\delta_f^i)} \quad \left[\frac{\partial \tilde{C}_\alpha(U_{i,n}, V_{i,n})}{-\partial U_{i,n}} \right]^{(1-\delta_m^i)\delta_f^i} \\
& \times \left[\frac{\partial \tilde{C}_\alpha(U_{i,n}, V_{i,n})}{-\partial V_{i,n}} \right]^{\delta_m^i (1-\delta_f^i)} \quad \left[\tilde{C}_\alpha(U_{i,n}, V_{i,n}) \right]^{\delta_m^i \delta_f^i} . \qquad (3.13)
\end{aligned}
$$

Genest et al., 1995 and Shih and Louis, 1995 have shown that the stemmed estimators of the copula parameters are consistent and asymptotically normally distributed. Due to their computational advantages, the IFM and the Omnibus approaches are used in our estimations. By comparing the results stemming from the two techniques, we can analyse to which extent a certain copula is a reliable model for bivariate lifetimes within a couple. Table 3.6 and Table 3.7 display the copula estimations based on our dataset.

Copula parameters	$\alpha(d)$						α
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\alpha}(-2)$	$\hat{\alpha}(0)$	$\hat{\alpha}(2)$	$\hat{\alpha}$
Gumbel	1.027	-0.024	0.036	1.917	2.027	2.003	1.993
Frank	7.359	-0.017	0.023	6.813	7.359	7.272	7.065
Clayton	2.461	-0.302	0.464	0.972	2.461	1.857	1.960
Joe	1.488	-0.063	0.063	2.189	2.488	2.488	2.389

TABLE 3.6: IFM method: copula parameters estimate $\alpha(d)$ and α.

Copula parameters	$\alpha(d)$						α
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\alpha}(-2)$	$\hat{\alpha}(0)$	$\hat{\alpha}(2)$	$\hat{\alpha}$
Gumbel	0.976	-0.022	0.030	1.884	1.976	1.960	1.924
Frank	7.294	-0.016	0.021	6.791	7.294	7.223	6.828
Clayton	1.924	-0.169	0.296	0.997	1.924	1.534	1.117
Joe	1.409	-0.0505	0.0581	2.158	2.409	2.388	2.352

TABLE 3.7: Omnibus approach: copula parameters estimate $\alpha(d)$ and α.

The estimated values from the IFM and the omnibus estimations are quite close for the Gumbel, the Frank and the Joe copulas. The important difference observed in the Clayton case indicates that this copula is probably not appropriate for modelling the bivariate lifetimes in our dataset. The negative sign of $\hat{\beta}_1$ in all cases demonstrates that if husband is older than wife (i.e. $d > 0$), their lifetimes are more likely to be correlated. The positive sign of $\hat{\beta}_2$ suggests that the higher the age difference is, the lesser is the level of dependence between lifetimes. The parameters

$\hat{\beta}_1$ and $\hat{\beta}_2$ have opposing effects on $\hat{\alpha}(d)$. That is why the maximum level of dependence is attained when $d = 0$, i.e. when wife and husband have exactly the same age. Our estimate of $\alpha(d)$ under the Gumbel copula is quite similar to the results in the model of Youn and Shemyakin, 1999 where $\hat{\beta}_0 = 1.018, \hat{\beta}_1 = 0$ and $\hat{\beta}_2 = 0.021$. Column 8 contains the estimation output when the dependence parameter α does not depend on d. When $d = 0$, $\alpha(0) = \beta_0$ (or $1 + \beta_0$ for Gumbel and Joe) and that is equivalent to the case where the dependence parameter is not in function of the age difference. By comparing the sixth and the eighth columns, it can be seen that the model without age difference underestimates the lifetime dependence level between spouses.

3.4.3 Goodness of fit

A goodness of fit procedure is performed in order to assess the robustness of our model. For this purpose, the model, including age difference and gender of the elder member within the couple with $\alpha(d)$, is compared to two other types, namely the one where the copula parameter does not depend on d and the model of Youn and Shemyakin, 1999. Many approaches for testing the goodness of fit of copula models are proposed in the litterature, see e.g., Genest et al., 2009; Berg, 2009. We refer to Genest et al., 2009 for an overview of the existing methods. There are several contributions highlighting the properties of the empirical copula, especially when the data are right censored, the contributions Dabrowska, 1988; Prentice et al., 2004; Gribkova and Lopez, 2015 are some examples. In our framework, the goodness of fit approach is based on the non parametric copula introduced by Gribkova and Lopez, 2015 as follows

$$C_n(u_1, u_2) = \frac{1}{n} \sum_{i=1}^{n} (1 - \delta_m^i)(1 - \delta_f^i) W_{in} \mathbb{1}_{\{T(x_m^i) \leq \hat{F}_{m,n}^{-1}(u_1), T(x_f^i) \leq \hat{F}_{f,n}^{-1}(u_2)\}}, \tag{3.14}$$

where $W_{in} = \frac{1}{S_{B_m}(\max(T_m^i, T_f^i - \epsilon_i)-)}$ and S_{B_m} is the survival function of the right censored random variable B_m that is estimated using KM approach; $\epsilon_i = B_f^i - B_m^i$. The term $\hat{F}_{j,n}^{-1}$ is the KM estimator of the quantile function of $T(x_j^i), j = m, f$. The particularity of equation (3.14) is that, the uncensored observations are twice weighted (with $1/n$ and W_{in}) unlikely to the original empirical copula where the same weight $1/n$ is assigned to each observation. The weight W_{in} is devoted to compensate right censoring. Based on the p-value, the goodness of fit test indicates to which extent a certain parametric copula is close to the empirical copula C_n. We adopt the Cramèr-von Mises statistic to assess the adequacy of the hypothetical copula to the empirical one, namely

$$\mathcal{V}_n = \int_{[0,1]^2} K_n(v) dK_n(v), \tag{3.15}$$

where $K_n(v) = \sqrt{n}(C_n(v) - C_{\hat{\alpha}(d)}(v))$ is the empirical copula process. Genest et al., 2009 have proposed an empirical version of equation (3.15) which is given by

$$\widehat{\mathcal{V}}_n = \sum_{i=1}^{n} (C_n(u_{1i}, u_{2i}) - C_{\hat{\alpha}(d)}(u_{1i}, u_{2i}))^2. \tag{3.16}$$

The assertion, the bivariate lifetime within the couple is described by the studied copula, is then tested under the null hypothesis H_0. Since the Cramèr-von Mises statistic $\widehat{\widehat{\mathcal{V}}}_n$ does not possess an explicit df, we implement a bootstrap procedure to evaluate the p-value as presented in the following pseudo-algorithm. For some large integer K, the following steps are repeated for every $k = 1, \ldots, K$:

- **Step 1:** Generate lifetimes from the hypothetical copula, i.e. $(U_i^b, V_i^b), i = 1, \ldots, n$ is generated from $C_{\hat{\alpha}(d)}$. If the IFM method is used to determine $\hat{\alpha}(d)$, then the two lifetimes are produced from the Gompertz distribution

$$(t_m^{b,i} = F_{x_m}^{-1}(U_i^b, \hat{\theta}_m), t_f^{b,i} = F_{x_f}^{-1}(V_i^b, \hat{\theta}_f)),$$

  where $\hat{\theta}_j,\ j = m, f$ are taken from Table 3.5, while, for the omnibus, the corresponding lifetimes are generated with the KM estimators of the quantile functions of $T\left(x_j\right),\ j = m, f$

$$(t_m^{b,i} = \hat{F}_{m,n}^{-1}(U_i^b),\ t_f^{b,i} = \hat{F}_{f,n}^{-1}(V_i^b)).$$

- **Step 2:** Generate the censored variables $B_m^{b,i}$ and $B_f^{b,i}, i = 1, \ldots, n$ from the empirical distribution of B_m and B_f respectively.

- **Step 3:** Considering the same data as used for the estimation, replicate the insurance portfolio by calculating $T^b(x_m^i) = \min(t_m^{b,i}, B_m^{b,i}), \quad \delta_m^{b,i} = \mathbb{1}_{\{t_m^{b,i} \geq B_m^{b,i}\}}$, $T^b(x_f^i) = \min(t_f^{b,i}, B_f^{b,i}), \quad \delta_f^{b,i} = \mathbb{1}_{\{t_f^{b,i} \geq B_f^{b,i}\}}$ for each couple i of ages x_m^i and x_f^i.

- **Step 4:** If the IFM approach is chosen in **Step 1**, the parameters of the marginals and the hypothetical copula parameters are estimated from the bootstrapped data $(T^b(x_m^i), T^b(x_f^i), \delta_m^{b,i}, \delta_f^{b,i})$ by maximizing (3.2) and (3.12) whereas under the omnibus approach, the hypothetical copula parameters are estimated from the bootstrapped data as well by maximizing equation (3.13).

- **Step 5:** Compute the Cramèr-von Mises statistics $\widehat{\mathcal{V}}^b_{n,k}$ using (3.16).
- **Step 6:** Evaluate the estimate of the p-value as follows

$$\hat{p} = \frac{1}{K+1} \sum_{k=1}^{K} \mathbb{1}_{\{\widehat{\mathcal{V}}^b_{n,k} \geq \widehat{\mathcal{V}}_n\}}.$$

Based on 1000 bootstrap samples, the results of the goodness of fit is summarized in Table 3.8. It can be seen that for both IFM and Omnibus, our model have a greater p-value than the model without age difference, showing that age difference between spouses is an important dependence factor of their joint lifetime. Under the Gumbel model in Youn and Shemyakin, 1999 where $\beta_1 = 0$, the p-value is evaluated at 0.678. For the Gumbel copula in Table 3.8, the p-value in the model with $\alpha(d)$ is slightly higher, strenghthening the evidence that the sign of d captures some additional association between spouses.

	IFM		Omnibus	
Copula parameters	α	$\alpha(d)$	α	$\alpha(d)$
Gumbel	0.647	0.679	0.639	0.670
Frank	0.518	0.525	0.521	0.530
Clayton	0.111	0.163	0.120	0.158
Joe	0.321	0.338	0.318	0.329

TABLE 3.8: Goodness of fit test: p-value of each copula model.

At a critical level of 5%, the three copula families are accepted, even though the Clayton copula performs inadequately. Actually, as pointed out in the work of Gribkova and Lopez, 2015, the important percentage of censored data in the sample results in a huge loss of any GoF test. Therefore, these results can not efficiently assess the lifetime dependence within a couple. Nevertheless, the calculated p-values may give an idea about which direction to go. In this regards, since the Gumbel and Frank copulas have the highest p-value, they are good candidates for addressing the dependence of the future lifetimes of husband and wife in this Canadian life insurer portfolio.

3.5 Insurance applications

3.5.1 Joint life insurance contracts

Multiple life actuarial calculations is common in the insurance practice. Hereafter, (x) stands for the husband aged x whereas (y) is the wife. Considering a couple (xy), $T(xy)$ describes the remaining time until the first death between (x) and (y)

and, it is known as the *joint-life status*. Conversely, $T\left(\overline{xy}\right)$ is the time until death of the *last survivor*. The variables $T\left(\overline{xy}\right)$ and $T\left(xy\right)$ are random and we can write

$$T\left(xy\right) = \min\left(T\left(x\right), T\left(y\right)\right) \text{ whereas } T\left(\overline{xy}\right) = \max\left(T\left(x\right), T\left(y\right)\right).$$

As in the single life model, the survival probabilities are given by

$$ {}_tp_{xy} = \mathbb{P}\left(T\left(xy\right) > t\right) \quad \text{and} \quad {}_tp_{\overline{xy}} = \mathbb{P}\left(T\left(\overline{xy}\right) > t\right). \tag{3.17}$$

Clearly, if $T\left(x\right)$ and $T\left(y\right)$ are independent, then

$$ {}_tp_{xy} = {}_tp_x \, {}_tp_y \quad \text{and} \quad {}_tp_{\overline{xy}} = 1 - {}_tq_x \, {}_tq_y.$$

The curtate life expectancies, for $T\left(xy\right)$ and $T\left(\overline{xy}\right)$ respectively, are given by

$$e_{xy} = \mathbb{E}\left(T\left(xy\right)\right) = \sum_{t=1}^{\infty} {}_tp_{xy} \quad \text{and} \quad e_{\overline{xy}} = \mathbb{E}\left(T\left(\overline{xy}\right)\right) = \sum_{t=1}^{\infty} {}_tp_{\overline{xy}},$$

with the following relationship

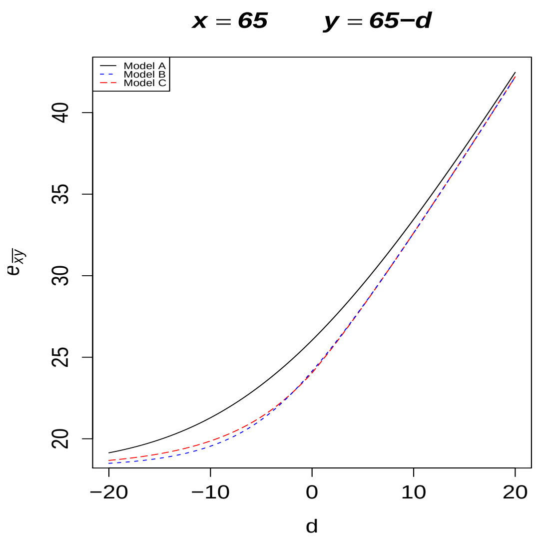
$$e_{\overline{xy}} = e_x + e_y - e_{xy}.$$

Figures 3.2 and 3.3 compare the evolution of $e_{\overline{xy}}$ as a function of the age difference $d = x - y$, under the following models:

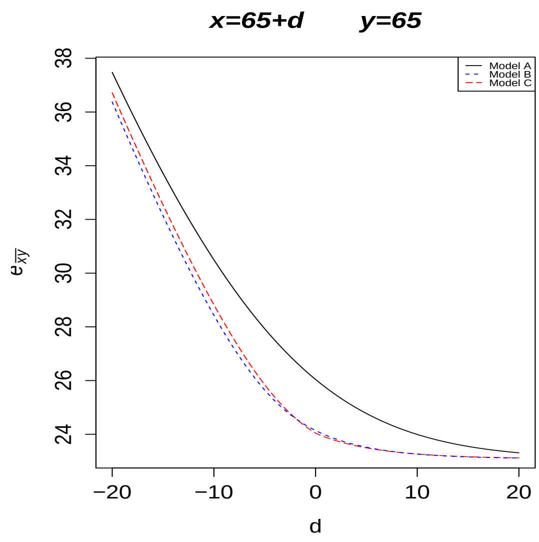
- Model A: $T\left(x\right)$ and $T\left(y\right)$ are independent;
- Model B: $T\left(x\right)$ and $T\left(y\right)$ are dependent with a constant copula parameter $\alpha = \alpha_0$;
- Model C: $T\left(x\right)$ and $T\left(y\right)$ are dependent with a copula parameter $\alpha\left(d\right)$ as described in (3.10) and (3.11).

On the left (resp. right), the graphs were constructed under the assumption of $x = 65$ (resp. $y = 65$) for the husband (resp. wife) and the age difference d ranges from -20 to 20 as more than 99% of our portfolio belongs to this interval. The fixed age is set to 65 because this is the retirement age in many countries. The analysis was made under the four families of copula described in Section 3.4. In general, it can be seen that the life expectancy of the last survivor $e_{\overline{xy}}$ increases when $e_{\overline{xy}} = e_{\overline{65:65-d}}$ whereas it decreases when $e_{\overline{xy}} = e_{\overline{65+d:65}}$. This result strengthens the evidence that the sign of d has an effect on annuity values. For example, when $|d| = 10$ under the Gumbel copula,

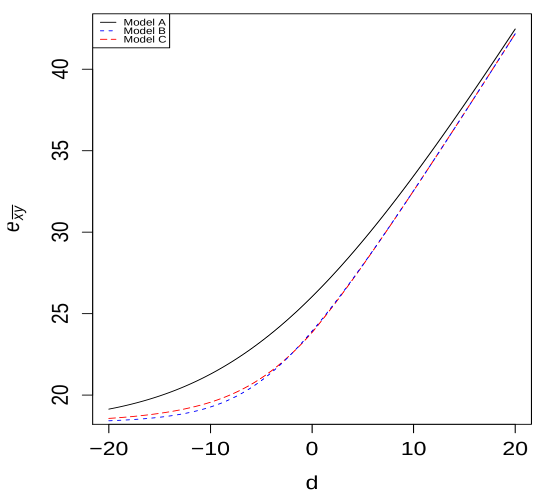
$$e_{\overline{65:55}} = 32.62 \geq e_{\overline{55:65}} = 28.82.$$



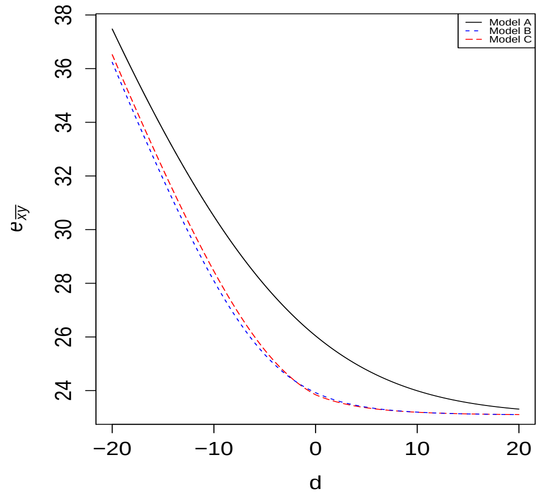
(A) Gumbel copula: $x = 65$



(B) Gumbel copula: $y = 65$

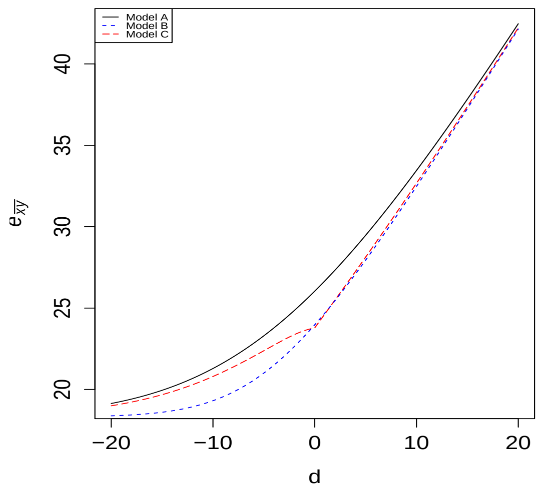


(C) Frank copula: $x = 65$

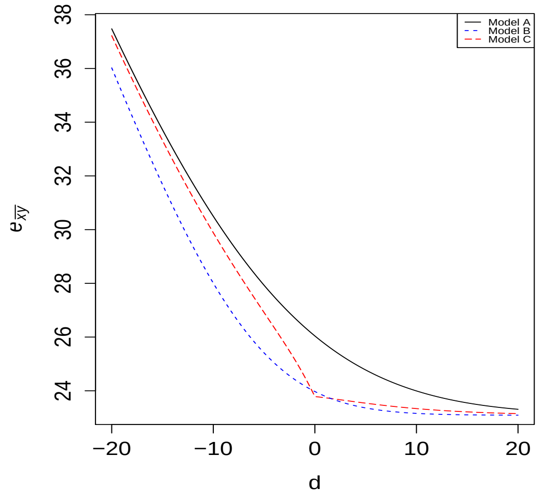


(D) Frank copula: $y = 65$

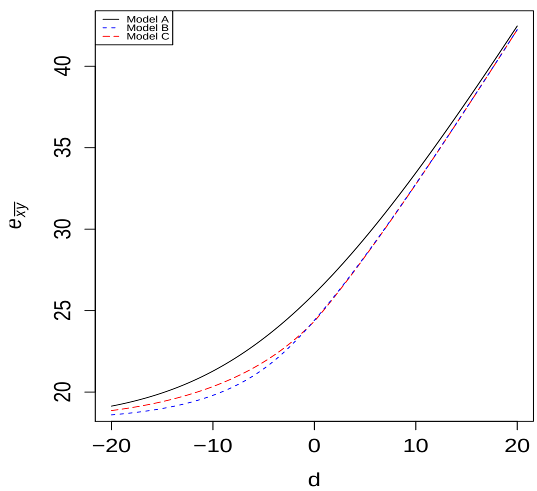
FIGURE 3.2: Comparison of $e_{\overline{xy}}$ under model A, B and C: Gumbel and Frank copulas



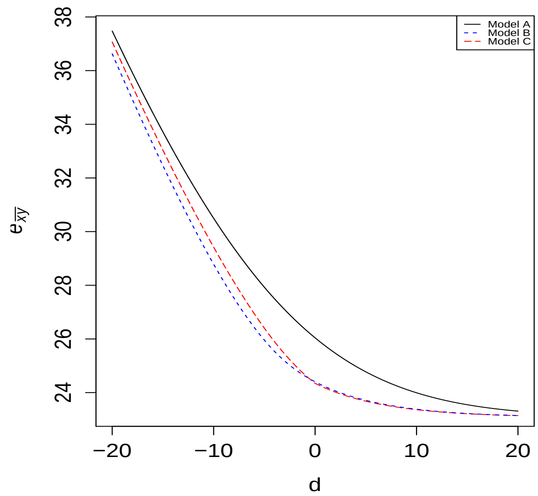
(A) Clayton copula: $x = 65$



(B) Clayton copula: $y = 65$



(C) Joe copula: $x = 65$



(D) Joe copula: $y = 65$

FIGURE 3.3: Comparison of $e_{\overline{xy}}$ under model A, B and C: Clayton and Joe copulas

When comparing the models A, B and C, it can be seen that the life expectancy $e_{\overline{xy}}$ is clearly overvalued under the model A of independence assumption, thus confirming the results obtained in Frees et al., 1995; Youn and Shemyakin, 1999; Denuit and Cornet, 1999. Now, let us focus our attention on models B and C considering only Gumbel, Frank and Joe copulas as it has been shown in the previous section that the Clayton copula might not be appropriate for the Canadian insurer's data. In all graphs, the life expectancy is always lower or equal under model B and the rate of decreases may exceed 2%. The largest decrease is observed when $d < 0$, i.e. when husband is younger than wife.

In order to illustrate the importance of these differences, we consider four types of multiple life insurance products. Firstly, Product 1 is the *joint life annuity* which pays benefits until the death of the first of the two annuitants. For a husband (x) and his wife (y) who receive continuously a rate of 1, the present value of future obligations and its expectation are given by

$$\bar{a}_{\overline{T(xy)}|} = \frac{1 - \exp\left(-\delta T\left(xy\right)\right)}{\delta} \qquad \text{and} \qquad \bar{a}_{xy} = \mathbb{E}\left(\bar{a}_{\overline{T(xy)}|}\right)$$

where δ is the constant instantaneous interest rate (also called force of interest). The variable $\bar{a}_{\overline{T(xy)}|}$ can be seen as the insurer liability regarding (xy). Product 2 is the last survivor annuity which pays a certain amount until the time of the second death $T\left(\overline{xy}\right)$. In that case, the present value of future annuities and its expectation are given by

$$\bar{a}_{\overline{T(\overline{xy})}|} = \frac{1 - \exp\left(-\delta T\left(\overline{xy}\right)\right)}{\delta} \qquad \text{and} \qquad \bar{a}_{\overline{xy}} = \mathbb{E}\left(\bar{a}_{\overline{T(\overline{xy})}|}\right).$$

In practice, payments often start at a higher level when both beneficiaries are alive. It drops at a lower level on the death of either and continues until the death of the survivor. This case is emphasized by product 3 where the rate is 1 when both annuitant are alive and reduces to $\frac{2}{3}$ after the first death. Product 3 is actually a combination of the two first annuities. Thus, the insurer liabilities and its expectation are given by

$$V\left(\overline{xy}\right) = \frac{1}{3}\bar{a}_{\overline{T(xy)}|} + \frac{2}{3}\bar{a}_{\overline{T(\overline{xy})}|} \qquad \text{and} \qquad \mathbb{E}\left(V\left(\overline{xy}\right)\right) = V_{\overline{xy}} = \frac{1}{3}\bar{a}_{xy} + \frac{2}{3}\bar{a}_{\overline{xy}}$$

where $\mathbb{E}\left(\bar{a}_{\overline{T(\overline{xy})}|}\right) = \bar{a}_{\overline{xy}}$.

Fourthly, imagine a family or couple whose income is mainly funded by the husband. The family may want to guarantee its source of income for the eventual death of the husband. For this purpose, the couple may buy the so called *reversionary annuity* for which the payments start right after the death of (x) until the death of (y). No payment is made if (y) dies before (x). As for Product 3, the reversionary annuity (Product 4) is also a combination of some specific annuity policies and the total obligations of the insurer and its expectation are computed as follows

$$\bar{a}_{\overline{T(x)|T(y)}|} = \bar{a}_{\overline{T(y)}|} - \bar{a}_{\overline{T(xy)}|} \qquad \text{and} \qquad \bar{a}_{x|y} = \mathbb{E}\left(\bar{a}_{\overline{T(x)|T(y)}|}\right) = \bar{a}_y - \bar{a}_{xy}. \tag{3.18}$$

In what follows, considering each of the insurance products 1, 2, 3 and 4, comparison of models A, B and C will be discussed. The analysis will include the valuation of the best estimate (BE) of the aggregate liability of the insurer as well as the quantification of risk capital and stop loss premiums.

3.5.2 Risk Capital & Stop-Loss Premium

In the enterprise risk management framework, insurers are required to hold a certain capital. This amount, known as the *risk capital*, is used as a buffer against unexpected large losses. The value of this capital is quantified in a way that the insurer is able to cover its liabilities with a high probability. For instance, under Solvency II, it is the *Value-at-Risk* (VaR) at a tolerance level of 99.5% of the insurer total liability, while for the Swiss Solvency Test (SST), it is the *Expected Shortfall* (ES) at 99%. Let L be the aggregate liability of the insurer. At a confidence level α, the VaR is given by

$$VaR_L(\alpha) = \inf \left\{ l \in \mathbb{R} : \mathbb{P}\left(L \leq l\right) \geq \alpha \right\},$$

whilst the ES is

$$ES_L(\alpha) = \mathbb{E}\left(L | L > VaR_L(\alpha)\right).$$

These risk measures will serve to compare models A, B and C for each type of product. As the insurance portfolio is made of n policyholders, we define

$$L = \sum_{i=1}^{n} L_i,$$

where L_i represents the total amount due to a couple i of (x_i) and (y_i). The dataset used in the calculations is the same as those used for the model estimations and described in Section 3.2. In principle, the couple i receives the amount b_i at the beginning of each year until the death of the last survivor. However, in our applications, b_i will be the continuous benefit rate in CAD for each type of product. For example, in the particular case of Product 3,

$$L_i = b_i V\left(\overline{x_i y_i}\right) = b_i \left(\frac{1}{3} \bar{a}_{\overline{T(x_i, y_i)}|} + \frac{2}{3} \bar{a}_{\overline{T(\overline{x_i, y_i})}|} \right).$$

Since there is no explicit form for the distribution of L, a simulation approach will serve to evaluate the insurer aggregate liability. The pseudo-algorithm used for simulations is presented in the following steps:

- **Step 1:** For each couple i, generate (U_i, V_i) from the the copula model (model A or model B or model C).

- **Step 2:** For each couple i with x_i and y_i, generate the future lifetime $T(x_i), T(y_i)$ from the Gompertz distribution as follows

$$T(x_i) = F_{x_i}^{-1}(U_i, \hat{\theta}_m) \qquad \text{and} \qquad T(y_i) = F_{y_i}^{-1}(V_i, \hat{\theta}_f), \tag{3.19}$$

where $\hat{\theta}_j,\ j = m, f$ are taken from Table 3.5.

- **Step 3:** Evaluate the liability L_i for each couple $i = 1, \dots, n$.

- **Step 4:** Evaluate the aggregate liability of the insurer $L = \sum_{i=1}^{n} L_i$.

Due to its goodness of fit performance, the Gumbel copula will be used in the calculations for Models B and C. Mortality risk is assumed to be the only source of uncertainty and we consider a constant force of interest of $\delta = 5\%$.

For each product described in Subsection 3.5.1, Step 1-4 are repeated 1000 times in order to generate the distribution of L. In addition to the risk capital measured as under the Solvency II and the SST framework, the BE of the aggregate liability of the insurer (i.e. $BE = \mathbb{E}(L)$), the Coefficient of Variation (CoV) and the Stop-Loss premium $SL = \mathbb{E}((L - \zeta)_+)$ are also evaluated, where ζ is the deductible. For the portfolio of Product 1, Product 2, Product 3 and Product 4, the amount of ζ in millions CAD are respectively 4, 4.5, 4.2, 1.7. Results are presented in Table $3.9 - 3.12$ according to each product. For the ease of understanding all values have been converted to a per Model A basis (the corresponding amounts are presented in Appendix 3.7.1).

As we could expect, the Model A with independent lifetime assumption misjudges the total liability of the insurer. The highest differences are observable with Product 4 where it reaches 20% for the BE, 30% for the risk capitals and 71% for the stop loss premiums. By comparing Model B and Model C, the findings tell minor differences. The variation noticed in Figure 3.2 (when $d < 0$) are practically non-existent in the aggregate values for most of the products under investigation. In other words, while the effects of the age difference and its sign are noticeable on the individual liability (see Subsection 3.5.1), the effects on the aggregate liability are merely small. This is due to the law of large number and to the high proportion of couple with $d > 0$ in our portfolio (70%). Actually, the compensation of the positive and negative effects of the age difference on the lifetimes dependency in the whole portfolio mitigates its effects on the aggregate liability. However, it should be noted that the relative difference exceeds 1.4% for the $VaR_L(0.95)$ in Table 3.12.

Product 1	BE	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	1.0000	0.6497	1.0000	1.0000	1.0000
Model B	1.0708	0.6279	1.4072	1.0235	1.0223
Model C	1.0721	0.6276	1.4157	1.0240	1.0228

TABLE 3.9: Relative BE and risk capital for the joint life annuity (Product 1) portfolio.

Product 2	BE	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	1.0000	0.5039	1.0000	1.0000	1.0000
Model B	0.9518	0.5251	0.9220	0.9988	0.9991
Model C	0.9510	0.5257	0.9204	0.9989	0.9991

TABLE 3.10: Relative BE and risk capital for the last survivor annuity (Product 2) portfolio.

Product 3	BE	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	1.0000	0.5039	1.0000	1.0000	1.0000
Model B	0.9820	0.5425	1.2148	1.0154	1.0146
Model C	0.9818	0.5431	1.2191	1.0159	1.0150

TABLE 3.11: Relative BE and risk capital for the last survivor annuity (Product 3) portfolio.

Product 4	BE	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	1.0000	0.5039	1.0000	1.0000	1.0000
Model B	0.8072	1.0692	0.2877	0.7077	0.7222
Model C	0.8039	1.0586	0.2731	0.6978	0.7135

TABLE 3.12: Relative BE and risk capital for the contingent annuity (Product 4) portfolio.

3.6 Conclusion

In this paper, we propose both parametric and semi-parametric techniques to model bivariate lifetimes commonly seen in the joint life insurance practice. The dependence factors between lifetimes are examined namely the age difference between spouses and the gender of the elder partner in the couple. Using real insurance data, we develop an appropriate estimator of the joint distribution of the lifetimes of spouses with copula models in which the association parameters have been allowed to incorporate the aforementioned dependence factors. A goodness of fit procedure clearly shows that the introduced models outperform the models without age factors. The results of our illustrations, focusing on valuation of joint life insurance products, suggest that lifetimes dependence factors should be taken into account when evaluating the best estimate of the annuity products involving spouses.

3.7 Appendix

3.7.1 Risk measures for the aggregate liability of the insurer

Product 1	Mean	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	1′815′490	0.649	31′393	5′031′430	5′083′090
Model B	1′944′105	0.628	44′177	5′149′873	5′196′529
Model C	1′946′400	0.627	44′443	5′152′233	5′199′015

TABLE 3.13: Risk capital for the joint life annuity (Product 1) portfolio in CAD.

Product 2	Mean	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	2′663′056	0.487	61′826	5′557′880	5′590′822
Model B	2′534′628	0.525	57′007	5′551′368	5′585′636
Model C	2′532′504	0.526	56′906	5′551′814	5′585′818

TABLE 3.14: Risk capital for the last survivor annuity (Product 2) portfolio in CAD.

Product 3	Mean	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	2′380′534	0.504	50′205	5′275′035	5′316′415
Model B	2′337′787	0.543	60′990	5′356′069	5′394′256
Model C	2′337′136	0.543	61′206	5′358′722	5′396′062

TABLE 3.15: Risk capital for the last survivor annuity (Product 3) portfolio in CAD.

Product 4	Mean	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	667′479	1.248	93′413	4′123′250	4′200′646
Model B	538′811	1.069	26′871	2′918′125	3′033′624
Model C	536′592	1.059	25′514	2′877′347	2′997′130

TABLE 3.16: Risk capital for the life contingent annuity (Product 4) portfolio in CAD.

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