Semi-static hedging strategies for exotic options: An overview

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Abstract

In this paper we will give a survey on results for semi-static hedging strategies for exotic options under different model assumptions and also in a model-independent framework. Semi-static hedging strategies consist of rebalancing the underlying portfolio only at certain pre-specified timepoints during the lifetime of the hedged derivative, as opposed to classical dynamic hedging, where adjustments have to be made continuously in time. In many market situations (and in particular in times of limited liquidity) this alternative approach to the hedging problem is quite useful and has become an increasingly popular research topic over the last years. We summarize results on barrier options as well as strongly path-dependent options such as Asian or lookback options. Finally it is shown how perfect semi-static hedging strategies for discretely observed options can be developed in quite general Markov-type models.

1 Introduction

In their famous work of 1973, Black & Scholes [10] showed how a standard European option can be replicated by dynamically trading the underlying asset and investing in the riskless bond. The so-called delta hedging strategy is (with some modifications) still among the most widely used methods to manage and reduce the risk inherent in writing an option. However, already Merton [60] applied static "hedging" methods to relate different option prices, and also the Put-Call-Parity, which at least goes back to Stoll [76], can be interpreted as a consequence of a particular static hedging strategy. Recently, Derman & Taleb [39] outlined some pitfalls of the dynamic replication strategies and argued in favour of the more robust static replication methods, which in the meantime attained quite some attention in the practical and academic literature, starting in the mid 1990's with the work of Bowie & Carr [14] and Derman et al. [37].

The semi-static hedging approach structurally differs from the dynamic counterpart in terms of hedging instruments and trading times. While dynamic hedges base on the assumption of being able to trade continuously in time, semi-static portfolios do not need to be adjusted dynamically, but only at pre-specified (stopping) times. Furthermore, in contrast to classic dynamic hedging, they typically use plain vanilla (or other

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liquidly traded) options as hedging instruments. Studies comparing the performance of semi-static and dynamic hedging strategies (as for example undertaken by Tompkins [79], Nalholm & Poulsen [64], or Engelmann et al. [43]), indicate that the semi-static ones are in many situations more robust and outperform their dynamic counterparts.

Here, we aim to give an overview of the most prominent semi-static hedging strategies for options written on some asset (e.g. a stock or an exchange rate), for which some standard European options are liquid. In this text we focus on fully discrete strategies, i.e. we exclude the possibility of dynamic hedging (for combined dynamic-static hedging strategies see e.g. Ilhan & Sircar [53] and Ilhan et al. [52]). In principle, the outlined methods directly translate to the case where the options are written on baskets of assets, but one has to be aware that plain vanilla options with the basket as the underlying are needed to apply the strategies in practice.

Strategies, for which the composition of the portfolio is not changed after the initial composition except investing gains in the riskless asset will be called static, and those with a finite number of transactions will be termed semi-static (semi-static strategies can be further subclassified as done in Joshi [55]). We will also distinguish between model-dependent and model-independent strategies. The former depend on assumptions on the asset price process (like continuity) or need a fully specified model, while the latter are correct in any market, which is frictionless in the following sense:

Definition 1.1 (Frictionless market) *A market is called frictionless, if*

- *the no-arbitrage assumption holds*
- *all investors are price-takers*
- *all parties have the same access to relevant information*
- *there are no transaction costs, taxes or commissions*
- *all assets are perfectly divisible and can be traded at any time*
- *there are no restrictions on short-selling*
- *interest rates for borrowing and lending are identical*

Throughout the text we will assume frictionless markets. Although the methods work for time-dependent interest rates as well, we will (mainly for notational convenience) further assume that the riskless interest rate is a constant r.

Definition 1.2 (Weak and strong path-dependence) *An option is said to be weakly path-dependent, if the dynamics of the option price depend only on asset price and time, i.e. the stochastic differential equation (SDE) for the option price differs from the SDE of a vanilla option only in the boundary conditions. Otherwise the option is said to be strongly path-dependent.*

While for weakly path-dependent options quite general methods are available, the theory is somewhat more involved for strongly path-dependent options. Nevertheless, we will present hedging strategies for lookback options, as well as for Asian options and discretely sampled options in general.

The rest of the paper is structured as follows. Section 2 introduces a method to synthesize arbitrary (sufficiently regular) payoffs that depend solely on the asset price at some later time T. In Section 3 techniques for weakly path-dependent options will be outlined. The focus lies in particular on barrier options, which are the most liquid and best understood exotic options of this kind. Section 4 deals with two examples of strongly path-dependent options (the lookback and the Asian option) and shows some robust hedging strategies. In Section 5 discretely sampled options, which may be weakly path-dependent (e.g. discretely monitored barrier options) or strongly path-dependent (e.g. Asian options) will be considered. Finally, Section 6 concludes and points out some potential future research topics.

2 Hedging path-independent options

Consider now options with a path-independent payoff. To fix ideas let us denote the price process of some asset S up to time T by $\mathbb{S}_T = \{S_t\}_{0 \leq t \leq T}$ and the payoff function by p, so that $p(\mathbb{S}_T) = p(S_T)$. Moreover we assume, that there is a riskless bond available, which pays off 1 at time T.

Plain vanilla options with arbitrary strikes are liquid

First we suppose that standard European options with maturity T are liquid for all strike levels $K \geq 0$, and the payoff function p is assumed to be twice differentiable. Then the fundamental theorem of calculus implies

$$p(S) = p(K^*) + p'(K^*)(S - K^*) + \int_{K^*}^{S} p''(x)(S - x)\,dx.$$

Since $(S - x) = (S - x)^+ - (x - S)^+$, we can rewrite this further to

$$\begin{aligned} p(S) = {} & p(K^*) + p'(K^*)(S - K^*) \\ & + \int_{K^*}^{\infty} p''(x)(S - x)^+\,dx + \int_{0}^{K^*} p''(x)(x - S)^+\,dx. \end{aligned} \tag{1}$$

Note that (1) holds for any K^* and even extends to the case where p is not twice differentiable (in which case – e.g. for a Dirac-Delta payoff p – the above equality will hold in a distribution sense).
For hedging purposes, we can rewrite formula (1) as

$$\begin{aligned} p(S) = {} & p(K^*) + p'(K^*)(F - K^*) + p'(K^*)(S - F) \\ & + \int_{K^*}^{\infty} p''(x)(S - x)^+\,dx + \int_{0}^{K^*} p''(x)(x - S)^+\,dx, \end{aligned}$$

where F denotes the forward price of the asset. In this way the payoff of the contingent claim p is decomposed into four parts, two of which are hedgeable by static positions in the riskless bond and a forward on the asset. The other two are synthesized by "infinitesimal" positions in standard European options. In total we get the following static hedging strategy (cf. Carr & Madan [22]):

Payoff	Hedged by positions in
$p(K^*) + p'(K^*)(F - K^*)$	$(p(K^*) - (K^* - F)p'(K^*))$ bonds
$p'(K^*)(S_T - F)$	$p'(K^*)$ forwards
$\int_{K^*}^{\infty} p''(x)(S_T - x)^+ \, dx$	$p''(x)dx$ calls struck at x $\quad (\forall x \in [K^*, \infty))$
$\int_0^{K^*} p''(x)(x - S_T)^+ \, dx$	$p''(x)dx$ puts struck at x $\quad (\forall x \in [0, K^*))$

This means that if European calls and puts with maturity T are liquid for all strikes, any (sufficiently regular) contingent claim on the time-T-price of S can be replicated perfectly – regardless of the model assumptions and in particular also in incomplete market models (this fact was already noticed e.g. by Ross [71], Breeden & Litzenberger [11], Green & Jarrow [47], or Nachmann [62]).
A nice application of formula (1) was given by Carr & Madan [22], who show that a variance swap written on some forward F can be replicated by a contingent claim with payoff $\ln(F_T)$ and a dynamic trading strategy involving the forward (see also e.g. Dupire [42], Derman et al. [38], Carr & Lee [18, 19] for further details).
It is worth noting that equation (1) also yields a pricing formula for the contingent claim with payoff p. Denote by $\mathrm{C}(S_t, t; K, T)$ and $\mathrm{P}(S_t, t; K, T)$ the call and put price at current time t. Since due to the first fundamental theorem of asset pricing (see e.g. Delbaen & Schachermayer [35]) the price of the claim in an arbitrage-free setup is given as the risk-neutral expectation of the discounted payoff, we have (the forward contract has value 0 by definition)

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-r(T-t)}\, p(S_T)] &= \mathrm{e}^{-r(T-t)}\, p(K^*) + \mathrm{e}^{-r(T-t)}(F_t - K^*)p'(K^*) \\ &\quad + \int_{K^*}^{\infty} p''(x)\, \mathrm{C}(S_t, t; x, T)\, dx + \int_0^{K^*} p''(x)\, \mathrm{P}(S_t, t; x, T)\, dx, \qquad (2)\end{aligned}$$

where interchanging the order of integration is justified, whenever the integrands in this last formula are absolutely integrable.
A straightforward corollary of the above formula is the familiar Put-Call-Parity

$$\mathrm{C}(S_t, t; K, T) = \mathrm{e}^{-r(T-t)}(F_t - K) + \mathrm{P}(S_t, t; K, T), \qquad (3)$$

which is obtained by setting $p(S) = (S - K)^+$ and choosing $K^* > K$ (F_t denotes the time t-forward price of the asset). Equation (1) implies the following static hedging portfolio for the call payoff:

- a long position in a put with the same strike K and maturity T as the put
- a long position in a forward contract
- investing $\exp(-rT)(F_t - K)$ in the bond

It is easy to see that the payoff of this portfolio at time T is the same as provided by the vanilla put, and the hedging strategy is therefore correct.

Finitely many liquid strikes

The derivation of the perfect hedging strategies in the last section relied on the assumption that standard European options are liquid for arbitrary strikes. In reality, of course, there are only a finite number of maturity/strike combinations available on the market.

The first question in this more realistic context is, whether the option prices themselves are consistent with the no-arbitrage assumption. This question was for instance addressed by Carr & Madan [21] and in a general setting by Davis and Hobson [32] (see also Cousot [28]).
Now, given a set of traded option prices, one can derive no-arbitrage bounds for the price of contingent claims by solving a semi-definite optimization problem (see e.g. Hodges & Neuberger [51], or Bertsimas & Popescu [9], who both also consider more general options). The dual of this problem is then to find the most expensive sub- and the cheapest super-replicating strategy using static positions in liquid options. The latter question is obviously highly relevant in practical applications, as it shows an (in some sense) optimal way to cover the payoff in any case.
Alternatively the right hand side of equation (1) may be approximated by a sum, where the sampling points correspond to the liquid strikes. However, then one cannot ensure that the payoff of the hedging portfolio will be sufficient to cover the payoff of the contingent claim, unless the weights of the liquid options are chosen with great care.

3 Hedging barrier and other weakly path dependent options

The prototype of a weakly path-dependent option is the barrier option, which, starting from Bowie & Carr [14], has attracted by far the most attention in the literature of semi-static hedging. The payoff of a barrier option generally depends on whether the asset price has reached some pre-specified region up to maturity or not.
For example, the Down-and-Out call (DOC) with parameters strike K, barrier $B < S_0$ and maturity T has the payoff

$$p(\mathbb{S}_T) = (S_T - K)^+ \mathbb{I}_{\{\inf_{0\le t\le T} S_t > B\}}, \tag{4}$$

where $\mathbb{I}_A$ denotes the indicator function of the set A.
Analogously the payoff of a Down-and-In call (DIC) is defined by

$$p(\mathbb{S}_T) = (S_T - K)^+ \mathbb{I}_{\{\inf_{0\le t\le T} S_t \le B\}}.$$

In a similar manner the payoff of Up-and-Out and Up-and-In calls (UOC and UIC) are defined by

$$p(\mathbb{S}_T) = (S_T - K)^+ \mathbb{I}_{\{\sup_{0\le t\le T} S_t < B\}} \quad \text{and} \quad p(\mathbb{S}_T) = (S_T - K)^+ \mathbb{I}_{\{\sup_{0\le t\le T} S_t \ge B\}},$$

respectively, and the put counterparts are defined analogously.
Variants of the standard barrier options include double barrier options (where the barrier region is two-sided), or barrier options with rebates. Here we will mainly focus on the standard ones (note that Sbuelz [72] shows in the Black-Scholes-model how double barrier options can be replicated by single barrier options).

Since the plain vanilla option payoff obviously dominates the barrier option payoff, a first simple super-replicating portfolio for the considered a barrier option is a long position in a standard European option with the same strike and the same maturity.
Let us denote the time-t-price of a Down-and-Out and a Down-and-In call with strike K, barrier B and maturity T by $\mathrm{DOC}(S_t, t; K, T, B)$ and $\mathrm{DIC}(S_t, t; K, T, B)$, respectively. Since the sum of the payoff of a DOC and a DIC with the same parameters

equals the payoff of a plain vanilla call, we have by the law of one price

$$\mathrm{C}(S_t, t; K, T) = \mathrm{DOC}(S_t, t; K, T, B) + \mathrm{DIC}(S_t, t; K, T, B). \quad (5)$$

The same equation also holds for UOC and UIC prices and naturally for the corresponding put options as well.
In the remainder of this section the DOC will serve as the illustrating example. Extensions to other option types are often analogous.

In practice, early-exercise or cancellation features will additionally play a role. The interested reader is referred to Neuberger [65] for an optimization based treatment of super- and sub-replicating strategies for American options and to Davis et al. [33, 34] or Ben-Hameur et al. [8] for hedging strategies of installment options.

3.1 Model-dependent strategies: Perfect replication

Let us assume that the barrier option with maturity T is written on the time T forward price F_T of the asset and let us use the forward price as the underlying. For convenience, the prices of all options will w.l.o.g. be expressed with respect to this forward price.
Any barrier option can be adapted to this notation, simply by replacing the original constant barrier by a time-dependent one. The first hitting time τ is defined as the first time for which the forward asset price process is at or below the (now possibly time-dependent) barrier $B(t)$,

$$\tau = \inf \{t \leq T : F_t \leq B(t)\},$$

with the convention $\inf \emptyset = \infty$.

First we assume that the barrier $B(t) \equiv B$ for the forward is constant (i.e. the corresponding barrier for the spot price would be time-dependent) and we suppose the forward price process F_t to be continuous. The following is a nice and conceptionally simple semi-static hedging strategy for a DOC written on the forward price of an asset (with strike K, maturity T and barrier $B = K$ at the strike):

- Buy a standard European call struck at K and maturing at T and sell a European put with the same parameters.
- The first time the forward price touches the barrier close the positions.
- If the barrier is never hit, hold the call until expiry.

This perfectly replicates the payoff of the DOC, as follows by considering the two different sacenarios:

- The barrier is not reached until maturity. Then the payoff of the portfolio is the same as the one of the DOC, i.e. $(S_T - K)$, as necessarily $S_T = F_T \geq B = K$ (and the put has payoff 0).
- The barrier is hit at $\tau < T$. Then

  $$\mathrm{C}(F_\tau, \tau; K, T) = \mathrm{P}(F_\tau, \tau; K, T),$$

  because of the Put-Call-Parity (3) (recall that $F_\tau = K$). Thus the cost of closing the positions is 0 and the payoff of the DOC is replicated also in this case.

The price of the DOC is hence given by:

$$\mathrm{DOC}(F_0, 0; K, T, K) = \mathrm{C}(F_0, 0; K, T) - \mathrm{P}(F_0, 0; K, T). \tag{6}$$

This hedging strategy is rather simple (and – due to the involved continuity assumption – model-dependent). But it is not straight-forward to extend this to other types of barrier options (an exception is the barrier exchange option, see Lindset & Perrson [57]). Thus there is the need for a more general approach.

One of the most famous semi-static hedging strategies for barrier options was originally developed by Carr & Bowie in [14] (see also Carr & Chou [15, 16] and Carr et al. [17]) and is based on a relation between standard European calls and puts with *different* strikes:

Assumption 1 *For any $0 \leq t \leq T$ the implied volatility smile (for maturity T), plotted as a function of the log-moneyness, is symmetric. More precisely: Let K and H be such that the geometric mean of K and H is the forward price, $\sqrt{KH} = F_t$, then*

$$\sigma_{\text{imp}}(K) = \sigma_{\text{imp}}(H).$$

Theorem 3.1 (Put-Call-Symmetry, Carr et al. [17]) *Given a frictionless market, under Assumption 1 the following holds for all K, H with $\sqrt{KH} = F_t$:*

$$\mathrm{C}(F_t, t; K, T) = \sqrt{\frac{K}{H}} \cdot \mathrm{P}(F_t, t; H, T), \qquad \forall t \in [0, T].$$

We will shortly sketch the proof of the above theorem: Recall that the price of a call in the Black-Scholes model (with volatility σ) is given by

$$\mathrm{C}(F_t, t; K, T) = \mathrm{e}^{-r(T-t)}\big(F_t\Phi(d_+) - K\Phi(d_-)\big),$$

where F_t is the forward price of the asset,

$$d_\pm = \frac{\log(F_t/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$

and $\Phi(x)$ is the cumulative distribution function of the normal distribution. Due to $\sqrt{KH} = F_t$, the above is easily seen to be equivalent to

$$\begin{aligned}\mathrm{C}(S_t, t; K, T) &= \sqrt{\frac{K}{H}}\, \mathrm{e}^{-r(T-t)}\big(H\Phi(-d_-) - F_t\Phi(-d_+)\big) \\ &= \sqrt{\frac{K}{H}}\, \mathrm{P}(S_t, t; H, T).\end{aligned}$$

Thus as long as the implied volatility for vanilla options with strike K and H is identical (which is Assumption 1), the theorem remains valid.
Necessary and sufficient conditions on an asset price model to fulfill Assumption 1 are given in Carr & Lee [20]. A first example is of course the Black-Scholes model. Another is the following local volatility model for the forward price:

$$dF_t = F_t\sigma(F_t, t)dW_t^{\mathbb{Q}},$$

where W_t is a Brownian motion and

$$\sigma(F_t, t) = \sigma(F_0^2/F_t, t). \tag{7}$$

Condition (7) implies that the local volatility at a fixed time t plotted as a function of the log-returns is symmetric around 0, which explains also the symmetry of the implied volatility surface.

Starting from Theorem 3.1, Bowie & Carr [14] investigated how the Put-Call-Symmetry can be used to hedge barrier options (a similar relation between different call and put prices, termed Put-Call-Reversal, was used in Andreasen & Carr [6] to hedge long-term call options). Consider a DOC with strike K, maturity T and barrier $B < K$. By definition, the payoff of this option is $(S_T - K)^+$, if the barrier was never hit and 0 otherwise. In order to hedge the final payoff, a vanilla call with the same strike and maturity as the DOC should be bought. Clearly this is a super-hedge and another vanilla option can be sold such that one obtains a portfolio with value 0 at the barrier. Assuming the conditions for the Put-Call-Symmetry, we know that a put with the same maturity and strike B^2/K fulfills the geometric mean condition of the theorem. Therefore

$$\mathrm{C}(B, t; K, T) = \frac{K}{B}\,\mathrm{P}(B, t; B^2/K, T) \qquad \forall t < T.$$

Hence a portfolio consisting of a long position in the corresponding vanilla call and a short position in K/B puts with strike B^2/K has value 0, if the forward price is exactly at the barrier. Consequently a replicating portfolio for a DOC is to take a long position in a standard European call with strike K and maturity T and a short position in K/B puts with strike B^2/K and the same maturity. The corresponding trading strategy is to hold the portfolio until expiry, if the barrier is never hit, or to close the positions (at 0 cost) at the first hitting time $\tau \leq T$. Note that for the correctness of the replication strategy it is necessary that the forward price is exactly B at the first hitting time (which explains the continuity assumption).
Summing up, under the no-arbitrage principle this leads to the equation:

$$\mathrm{DOC}(S_0 0; K, T, B) = \mathrm{C}(S_0, 0; K, T) - \frac{K}{B}\,\mathrm{P}(B, 0; B^2/K, T), \qquad B < K. \tag{8}$$

This hedging procedure can be extended to other types of barrier options, but typically then becomes more complicated. For example, for an UOC one would also need to take a position in a binary call (i.e. an option, which pays 1 if the asset price at maturity exceeds the strike, and 0 otherwise). The latter can then be hedged by the strategies outlined in the previous section. If the assumption of a constant barrier on the forward price is relaxed, super- and sub-replication strategies can be derived. For the details we refer to Carr et al. [17] (see also the nice summary of Poulsen [68]). Kraft [56] considers the case, when only a finite number of options with prespecified strikes are available and identifies conditions under which a strategy based on the Put-Call-Symmetry is a semi-static superhedge.

Another hedging strategy for barrier options has its roots in papers of Derman et al. [36, 37]. In order to point out the main idea of the Derman-Ergener-Kani (D-E-K) algorithm, a binomial model is considered first, i.e. the asset price process is assumed to follow a binomial tree. In this setting the D-E-K-algorithm for a DOC with maturity T, strike K and barrier B is as follows:

- Identify the boundary nodes, i.e. the maturity and the barrier nodes and denote the resulting time grid by $t_1 < \ldots < t_n$.
- Buy a standard European option replicating the payoff of the barrier option, if the barrier boundary is not hit during the lifetime of the barrier option. In the case of a DOC with $B < K$ this would be a standard European call with the same strike and maturity as the barrier option.
- Choose n options with different maturities T_i and strikes K_i having payoff 0, if the barrier was not hit. In the example of a DOC, one could for instance choose puts with strikes $K_i \leq B$ and maturities $T_i \leq T$. Then solve the linear equation system
$$\mathbf{V}_i \cdot \mathbf{w} + \mathrm{C}(B, t_i; K, T) = 0, \quad i = 1, \ldots, n \tag{9}$$
for $\mathbf{w}$, where
$$\mathbf{V}_i := (\mathrm{P}(B, t_i; K_1, T_1), \mathrm{P}(B, t_i; K_2, T_2), \ldots, \mathrm{P}(B, t_i; K_n, T_n))$$
denotes the vector of the time-t_i-prices of the n additional options and $\cdot$ is the scalar product.

Note that due to (9) the portfolio consisting of the option replicating the final payoff and w_i options with strike K_i and maturity T_i can be sold at zero cost at every barrier node and thus the semi-static trading strategy of holding the portfolio until expiry in case the barrier has not been hit, or to close the positions at the first hitting time, is a perfect replication of the barrier option.

In the binomial framework the D-E-K-algorithm thus gives a perfect hedge for all kinds of barrier options – in particular also for double-barrier options and options with a time-dependent barrier – as long as enough different standard European options are liquid (i.e. at least as many as there are barrier nodes). However, since the prices in (9) are model-dependent (as they have to be calculated using a model – in the above case the binomial model), the entire portfolio is model-dependent.
For continuous-time models, the approach outlined in [37] is to discretize (in time) the asset price process and hedge the values at the discrete monitoring times. Toft & Xuan [78] examine the effectiveness of such a discretization for stochastic volatility and find that it works well for a small volatility of volatility, but not satisfactory in the opposite case combined with discontinuous final payoff of the barrier option (as e.g. for an UOC). We will come back to better alternatives in the next section.

A possible limit strategy for continuous-time price models can be achieved by assuming standard options with all strikes and maturities to be liquid. This strategy was developed by Andersen et al. [5] (see also Mijatović [61]), who observed that the key assumption on the price model is that the price of European options, besides their parameters, only depends on the current time t and asset price S_t (this assumption for instance rules out stochastic volatility models). The approach is, as the D-E-K-algorithm, very powerful in terms of the barrier structure and allows for arbitrary (sufficiently regular) functions of time.
For the ease of exposition we assume that the forward price again follows a local volatility model in the risk-neutral world (the method can be extended to more general Markov-type models), i.e.
$$\frac{dF_t}{F_t} = \sigma(F_t, t) dW_t, \tag{10}$$

where σ is now just assumed to be positive and sufficiently regular to admit a unique solution of equation (10).

Consider again the standard example of a DOC with a continuous time-dependent barrier $B(t)$ on the forward price of an asset with dynamics as specified in (10). Then the following PDE specifies the price function $\mathrm{G}(F_t, t)$ of a DOC with strike K, barrier $B(t)$ and maturity T started at time t (and has in particular not reached the barrier up to the time t):

$$\begin{aligned} \mathrm{G}_t(F,t) + \frac{1}{2}\sigma^2(F,t)F^2\,\mathrm{G}_{FF}(F,t) &= 0, \qquad t < T,\ F > B(t) \\ \mathrm{G}(F,t) &= 0, \qquad t < T,\ F \le B(t) \\ \mathrm{G}(F,T) &= (F-K)^+ \qquad \forall F, \end{aligned} \tag{11}$$

where we assumed again for simplicity that $B(T) < K$. Then

$$\mathrm{G}(F,t) = \mathrm{DOC}(F, t; K, T, \{B(s)\}_{t \le s \le T}),$$

in case the asset price has not crossed the barrier before t. If the function σ in (11) is sufficiently regular, $\mathrm{G}(F,t)$ is twice differentiable with respect to F, except at $F = B(t)$, where the (formal) second derivative is given by $\delta(F - B(t))\,\mathrm{G}_F^+(B(t),t)$ (δ denoting the Dirac Delta measure and G_F^+ the right derivative). Using (11) and the Meyer-Itô-formula (see e.g. Protter [69, Theorem IV.70]), one finds:

$$\begin{aligned} (F_T - K)^+ - \mathrm{G}(F_0, 0) = &\int_0^T 1_{\{F_t > B(t)\}}\,\mathrm{G}_F(F_t,t)F_t\sigma(F_t,t)\,dW_t \\ &+ \int_0^T \frac{1}{2}\delta(F_t - B(t))\,\mathrm{G}_F^+(B(t),t)B(t)^2\sigma^2(B(t),t)\,dt. \end{aligned}$$

Rearranging the terms gives:

$$\begin{aligned} &\mathrm{G}(F_0,0) + \int_0^T 1_{\{F_t > B(t)\}}\,\mathrm{G}_F(F_t,t)dF_t = \\ &\qquad (F_T - K)^+ - \int_0^T \frac{1}{2}\delta(F_t - B(t))\,\mathrm{G}_F^+(B(t),t)B(t)^2\sigma^2(B(t),t)\,dt. \end{aligned} \tag{12}$$

Using equation (12), a semi-static hedge for the DOC can be formulated: Take a long position in a call with strike K and maturity T and short positions in an amount of $\mathrm{G}_F^+(B(t),t)B(t)^2\sigma^2(B(t),t)/2$ options with maturity t ($\forall\ 0 \le t \le T$) and payoff $\delta(F_t - B(t))$. Note that the latter payoff can be approximated arbitrarily closely by butterfly spreads (we assumed options for all maturities and strikes to be liquid). The corresponding trading strategy is to hold the portfolio until expiry, if the barrier is not reached and to sell the portfolio at the first hitting time τ. This is in fact a perfect replication strategy, which can be seen as follows: If the barrier is not reached, then the payoff of the replicating portfolio equals the payoff of the call option and thus the payoff of the DOC. On the other hand, with (12) we have

$$\begin{aligned} \mathbb{E}\left[\mathrm{G}(F_0,0) + \int_0^T 1_{\{F_t > B(t)\}}\,\mathrm{G}_F(F_t,t)dF_t \,\middle|\, \mathcal{F}_\tau\right] &= \mathbb{E}[(F_T - K)^+ | \mathcal{F}_\tau] \\ &- \mathbb{E}\left[\int_0^T \frac{1}{2}\delta(F_t - B(t))\,\mathrm{G}_F^+(B(t),t)B(t)^2\sigma^2(B(t),t)\,dt \,\middle|\, \mathcal{F}_\tau\right], \end{aligned}$$

which can be reformulated as

$$\begin{aligned}
\mathrm{G}(F_0,0)+\int_0^{\tau} 1_{\{F_t>B(t)\}}\,\mathrm{G}_F(F_t,t)dF_t = &\\
&\mathrm{C}(B(\tau),\tau;K,T)-\frac{1}{2}\,\mathrm{G}_F^{+}(B(\tau),\tau)B(\tau)^2\sigma^2(B(\tau),\tau)\\
&-\int_{\tau+}^{T}\frac{1}{2}\,\mathrm{DC}(B(\tau),\tau;B(t),t)\,\mathrm{G}_F^{+}(B(t),t)B(t)^2\sigma^2(B(t),t)\,dt,
\end{aligned}$$

where we denoted the time-t-price of the payoff $\delta(F_T-K)$ by $\mathrm{DC}(F_t,t;K,T)$. Note that at this point it is crucial that the prices of European options only depend on the asset price at time τ in order to interpret the expectations as prices. Since

$$\begin{aligned}
\mathrm{G}(F_0,0)+\int_0^{\tau} 1_{\{F_t>B(t)\}}\,\mathrm{G}_F(F_t,t)dF_t&\\
+\frac{1}{2}\,\mathrm{G}_F^{+}(B(\tau),\tau)B(\tau)^2\sigma^2(B(\tau),\tau)&=\mathrm{G}(B(\tau),\tau)
\end{aligned}$$

and $\mathrm{G}(B(\tau),\tau)=0$ by definition, the value of the portfolio at time τ is 0. Thus we can close the positions at zero cost and replicate the payoff of the DOC.
This replication strategy can be generalized to all kinds of terminal payoffs, barrier regions, rebates and also to jump-diffusion processes. We refer to Andersen et al. [5] for details.
An interesting difference to hedging strategies using the Put-Call-Symmetry is that here not only the correctness, but also the strategy itself depends on the model. More precisely: the strategy $\mathrm{G}_F^{+}(B(t),t)$ has to be calculated using a model, while the strategy outlined before was independent of the exact specification of the local volatility, as long as Assumption 1 was fulfilled.

3.2 Model-dependent strategies: Approximations

The hedging strategies in the previous sections were perfect, if the model was correct; i.e. the risk of the option could be completely eliminated. In particular, this gave further insight into the underlying models and into relations between barrier options and standard European options. However, the perfect replication could only be achieved at the cost of more or less severe assumptions like liquidity of arbitrary standard European options, or restrictions on the model. In this section a different and in some sense more pragmatic approach to semi-static hedging is discussed.
In the previous section we discussed an algorithm for a binomial model (although already Derman et al. generalized the approach to continuous time models). A key point of this generalized D-E-K algorithm is that the barrier is hit exactly and that the prices of the European options only depend on the asset price. This assumption is too restrictive, as for example stochastic volatility or price jumps cannot be accommodated in this framework. Therefore, a lot of research in recent years has been devoted to generalize the assumptions, but keeping the main idea of matching the barrier option value at the first hitting time. Examples of such studies are Allen & Padovani [4], Fink [44], Nalholm & Poulsen [63, 64], Siven & Poulsen [75], Giese & Maruhn [46], or Maruhn & Sachs [58, 59].
To outline the basic ideas of these generalizations we focus again on a DOC. As in the D-E-K algorithm, we take a long position in a call with the same maturity and strike

as the DOC to cover the payoff at maturity in the case the barrier was not hit. Thus the aim is to take positions in other options, such that the value of the portfolio is 0, if the barrier is hit. Furthermore the additional options should not have any payoff, if the asset price stays above the barrier during the lifetime of the barrier option. In a binomial world this could be done by solving equation system (9). For general asset price dynamics, this equation system has to be replaced by something more general and the method of choice of the above cited studies is to consider different scenarios for the portfolio value at the hitting time.
To fix ideas, let Θ denote the set of possible scenarios for the evolution of the barrier option. Then an element $\theta \in \Theta$ consists of

- the time when the barrier is hit (the hitting time)
- the undershoot of the asset price under the barrier
- the form of the implied volatility surface at the hitting time (i.e. the prices of the used European options at the hitting time).

Thus we have to solve

$$\mathbf{V}_\theta \cdot \mathbf{w} + \mathrm{C}_\theta(S_{\tau_\theta}, \tau_\theta; K, T) = 0, \quad \theta \in \Theta, \tag{13}$$

where V_θ is the price vector of the options and $\mathrm{C}_\theta(S_{\tau_\theta}, \tau_\theta; K, T)$ is the price of the call option at the hitting time τ_θ under scenario θ. Note that the asset price $S_{\tau_\theta} \leq B(\tau_\theta)$ is allowed to undershoot the barrier.
As Θ is an infinite set, system (13) consists of a continuum of linear equations, which is obviously not feasible.
Hence the set Θ has to be discretized to make (13) a finite-dimensional equation system. This can be done by assuming a model and using Monte-Carlo simulation (see e.g. [63, 64, 75, 46]) or by an a-priori choice of "reasonable" scenarios (e.g. [4, 59]). The implied volatility surface at the hitting time, i.e. $\mathbf{V}_\theta$ and C_θ, can be calculated using a pre-specified model, or can be allowed to vary in some reasonable manner (e.g. by allowing the parameters of the model to take a set of values [59]) in order to robustify the hedge with respect to model risk.
After a discretization of (13), the number of equations, albeit finite, may be huge, if a large number of different scenarios is considered. However, since the number of scenarios is directly linked to the error due to discretization, it should in fact be chosen large in order to obtain robust results. As in general only a limited number of different options is liquid, we cannot hope to be able to solve even a discretized version (13) exactly. To overcome this problem, several approaches are feasible. Siven & Poulsen [75] (see also Nalholm & Poulsen [63, 64]) propose to minimize a risk measure of the hedging error under the budget condition that the price of the hedging portfolio is smaller or equal to the one of the barrier option. Alternatively, Giese & Maruhn [46] and Maruhn & Sachs [58, 59] suggest to modify the equation to an inequality, such that the trading strategy becomes a super-replication and then minimize the cost of this hedge.

3.3 Model-independent strategies: Robust strategies

In the preceding section the problem of robustifying the hedging strategies with respect to model risk was already mentioned. Now let us sharpen the question: what can be deduced on the price of the barrier option, if solely the no-arbitrage assumption is

assumed? In particular, robust super- and sub-replicating strategies for barrier options will be considered. Those strategies are called robust, because they over- (respectively under-)hedge the barrier option in any frictionless (and in particular arbitrage-free) market model. For barrier options, these were first developed by Brown et al. in [12] (for some recent extensions see e.g. Cox & Oblój [29, 30]).
In order to outline the approach, assume again a forward market model and a constant barrier on the forward price (the ideas can be extended to time-dependent barriers, but the price intervals implied by the hedges might increase). Consider again the DOC with strike K, maturity T and barrier B. If $K \geq B$, then a sub-replicating portfolio for the DOC is given by:

- a long position in a call with strike K and maturity T
- a short position in $\dfrac{\tilde{\gamma} \vee K - K}{\tilde{\gamma} \vee K - B}$ puts with strike B and maturity T
- a short position in $\dfrac{K - B}{\tilde{\gamma} \vee K - B}$ calls with strike $\tilde{\gamma} \vee K$,

where $a \vee b := \max\{a, b\}$ and

$$\tilde{\gamma} = \operatorname{argmax}_{\beta > B} \frac{\mathrm{P}(F_0, B, 0, T) - \mathrm{C}(F_0, \beta, 0, T)}{\beta - B}.$$

The corresponding trading strategy is to hold the portfolio until expiry, if the barrier is not reached, and otherwise to unwind the positions at the first hitting time.
To show the sub-replication property of this semi-static strategy, let us again distinguish the two cases:

- If the barrier is not hit, the payoff of the strategy is

$$(S_T - K)^+ - \frac{K - B}{\tilde{\gamma} \vee K - B}(S_T - \tilde{\gamma} \vee K)^+ \leq (S_T - K)^+,$$

  where the right-hand side corresponds to the payoff of the DOC.

- If the barrier is hit at time τ, then $F_t = B - x$, where $x \geq 0$ denotes the (possible) undershoot. Hence for the value V_τ of the portfolio we find

$$\begin{aligned} V_\tau &= \mathrm{C}(B - x, 0; K, T) - \frac{\tilde{\gamma} \vee K - K}{\tilde{\gamma} \vee K - B} \mathrm{P}(B - x, 0; B, T) \\ &\quad - \frac{K - B}{\tilde{\gamma} \vee K - B} \mathrm{C}(B - x, 0; \tilde{\gamma} \vee K, T) \\ &\leq \mathrm{C}(B - x, 0; K, T) - \frac{\tilde{\gamma} \vee K - K}{\tilde{\gamma} \vee K - B} \mathrm{C}(B - x, 0; B, T) \\ &\quad - \frac{K - B}{\tilde{\gamma} \vee K - B} \mathrm{C}(B - x, 0; \tilde{\gamma} \vee K, T) \\ &\leq 0, \end{aligned}$$

  where the first inequality follows from the Put-Call Parity (3) and the second one from the convexity (in K) of the call price (or alternatively by a direct look on the final payoff of the last portfolio).

If $K < B$, then a sub-replicating portfolio for the DOC is given by:

- a long position in a forward contract (with forward price F_0)
- a long position in $F_0 - K$ bonds
- a short position in $\frac{B-K}{\gamma - B}$ puts with maturity T and strike γ, where

$$\gamma = \mathrm{argmin}_{\beta > B} \frac{\mathrm{P}(S_0, \beta, 0, T)}{\beta - B}.$$

The trading strategy is the same as before and the subreplication property is again shown by distinguishing two cases:

- If the barrier is not hit, we have for the payoff of the strategy

$$(S_T - F_0) + (F_0 - K) - \frac{B-K}{\gamma - B}(\gamma - S_T)^+ \leq (S_T - K),$$

where the right-hand side corresponds to the payoff of the DOC, since $B > K$.

- If the barrier is hit at time τ, then $F_t = B - x$. Thus the time-τ-value of the long position in the forward is $\mathrm{e}^{-r(T-\tau)}(B - x - F_0)$ and hence for V_τ we find

$$\begin{aligned} V_\tau &= \mathrm{e}^{-r(T-\tau)}(B - x - K) - \frac{B-K}{\gamma - B}\,\mathrm{P}(\mathrm{e}^{-(r-\delta)(T-\tau)}(B-x), \tau; \gamma, T) \\ &\leq \mathrm{e}^{-r(T-\tau)}(B - K) - \frac{B-K}{\gamma - B}\,\mathrm{P}(\mathrm{e}^{-(r-\delta)(T-\tau)}\,B, \tau; \gamma, T) \\ &\leq \mathrm{e}^{-r(T-\tau)}(B - K) - \frac{B-K}{\gamma - B}\,\mathrm{e}^{-r(T-\tau)}(\gamma - B) \\ &= 0, \end{aligned}$$

where the first inequality follows from the monotonicity of the put price and the second one from the Put-Call Parity (3).

Super-replicating portfolios are in both cases found by omitting the barrier feature in the final payoff, i.e. they are given as the trivial super-hedges consisting of

- a long position in a call with maturity T and strike K, if $K \geq B$
- a long position in a call with maturity T and strike B and in $B - K$ binary calls (a path-independent option with payoff $\mathbb{I}_{\{S_T \geq B\}}$) with strike B, if $K < B$.

Of course the sub- and super-hedging strategies also imply price bounds on the DOC: If $K \geq B$ we have

$$\begin{aligned} \mathrm{C}(S_0, 0; K, T) - \frac{\tilde\gamma \vee K - K}{\tilde\gamma \vee K - B}\,\mathrm{P}(S_0, 0; B, T) - \frac{K - B}{\tilde\gamma \vee K - B}\,\mathrm{C}(S_0, 0; \tilde\gamma \vee K, T) \\ \leq \mathrm{DOC}(S_0, 0; K, T, B) \leq \mathrm{C}(S_0, 0; K, T), \end{aligned}$$

while in the case $K < B$ we have

$$\begin{aligned} \mathrm{e}^{-rT}(F_0 - K) - \frac{B-K}{\gamma - B}\,\mathrm{P}(S_0, 0; \gamma, T) \leq \mathrm{DOC}(S_0, 0; K, T, B) \\ \leq \mathrm{C}(S_0, 0; B, T) + (B - K)\,\mathrm{BC}(S_0, 0; B, T), \end{aligned}$$

where $\mathrm{BC}(S_t, t; B, T)$ denotes the time-t-price of a binary call with strike B, and $\tilde{\gamma}, \gamma$ are as before.

The particular role of these sub- and super-replication strategies is identified in the following result:

Theorem 3.2 (Brown et al. [12]) *Assume that European options for all strikes with maturity T are liquid and that there are no options available with maturity less than T. Then the bounds on the price of the DOC are sharp. This means that there are forward price processes, that are martingales and consistent with the prices of the European options, for which the price of the DOC is given by the lower (resp. upper) bound. In those market models the hedging strategies are perfect.*

The proof of this theorem is closely related to the Skorokhod problem (see also e.g. Obłój [67]).

4 Hedging strongly path-dependent options

In this section we consider strongly path-dependent options. We will in particular focus on lookback and Asian options. For lookback options we show how barrier options (and the corresponding hedging strategies) can be used to hedge them. For Asian options robust hedging strategies are shown, which are related to hedging strategies for basket options. A perfect (model-dependent) strategy for discretely sampled options (DSO), which include Asian or cliquet options, will be outlined in Section 5.

4.1 Lookback options

Let us consider a floating strike lookback option (LO) with terminal payoff

$$p(\mathbb{S}_T) = S_T - m_T,$$

where $m_T = \inf_{0 \leq s \leq T} S_s$.
This kind of lookback option can be hedged robustly in terms of barrier options, as was first observed by Carr et al. in [17] (see also Coleman et al. [27, 26] for static hedging strategies of similar insurance products). The hedging strategies are not exact, but rather sub- and super-replications and hence impose bounds on the price of the lookback option. Along the way to obtain the hedging strategies for the lookback option, we will also deal with some other exotic options with gradually increasing complexity.

A roll-down call (RDC) with maturity T and strike K_0 has two barriers $H_1 > H_2$. In contrast to a double-barrier option, these two barriers are both below the initial asset price and the strike. The payoff of the RDC is specified as follows: if neither of the two barriers is hit, then the payoff is $(S_T - K_0)^+$. If the nearer barrier (H_1) is hit, the strike of the option is rolled down to this barrier and the second barrier becomes an out-barrier. So hitting H_1 the RDC becomes a DOC with strike H_1 and out-barrier H_2. For some more details concerning roll-down-calls and also roll-up-puts, see e.g. Gastineau [45].
For the purpose of hedging the lookback option, the definition should be extended as follows. Let $H_1 > H_2 > \ldots > H_n$ be a decreasing sequence of barriers all below the current spot price and below the initial strike K_0. If no barrier is hit, then the payoff

is again $(S_T - K_0)^+$. If the first barrier is hit, the strike rolls down to some level $K_1 \in [H_1, K_0]$. If the second barrier is hit, the strike rolls down to a certain strike $K_2 \in [H_2, K_1]$. This rolling down process is repeated until the asset price reaches (or undershoots) H_n – the out-barrier. Thus hitting H_n knocks out the option.
The option defined above is called extended roll-down call (ERDC) and admits the following decomposition in terms of DOC's:

$$\begin{aligned}\mathrm{ERDC}(S_0, 0; K_0, T, \{K_i\}_{1\le i\le n-1}, \{H_i\}_{1\le i\le n}) = \\ \mathrm{DOC}(S_0, 0; K_0, T, H_1) \\ + \sum_{i=1}^{n-1}\Big(\mathrm{DOC}(S_0, 0; K_i, T, H_{i+1}) - \mathrm{DOC}(S_0, 0; K_i, T, H_i)\Big). \quad (14)\end{aligned}$$

This portfolio indeed matches the payoff of the ERDC exactly, since if $H_{i+1} < m_t \le H_i$ (recall $m_t = \inf_{0\le s\le t} S_s$), the sum in (14) starts with $i+1$ and the leading term $\mathrm{DOC}(S_0, 0; K_0, T, H_1)$ is replaced by $\mathrm{DOC}(S_t, t; K_i, T, H_{i+1})$. Thus the ERDC can be perfectly replicated by a finite number of barrier options.
Note that the representation above is model-independent and the semi-static hedge is perfect in all frictionless models. Therefore, to build a semi-static hedging strategy for ERDC's using European options, the corresponding hedging strategies for DOC's can be used.
The next option we want to consider is the ratchet call. The ratchet call (RC) is an ERDC with initial strike K_0 and the strikes K_i equal the barrier levels H_i for $1 \le i \le n$. Furthermore it cannot be knocked out. Thus the RC can be written as follows:

$$\begin{aligned}\mathrm{RC}(S_0, 0; K_0, T, \{K_i\}_{1\le i\le n}) = \\ \mathrm{ERDC}(S_0, 0; K_0, T, \{K_i\}_{1\le i\le n-1}, \{K_i\}_{1\le i\le n}) + \mathrm{DIC}(S_0, 0; K_n, T, K_n),\end{aligned}$$

where DIC stands for a down-and-in call.
Using the model-independent representation (14) of the ERCD, we find:

$$\begin{aligned}\mathrm{RC}(S_0, K_0, 0, T, \{K_i\}_{1\le i\le n}) = \\ \mathrm{DOC}(S_0, 0; K_0, T, K_1) + \sum_{i=1}^{n-1}\Big(\mathrm{DOC}(S_0, 0; K_i, T, K_{i+1}) - \mathrm{DOC}(S_0, 0; K_i, T, K_i)\Big) \\ + \mathrm{DIC}(S_0, 0; K_n, T, K_n). \quad (15)\end{aligned}$$

Consider now the floating strike lookback call, defined at the beginning of the section. Recall that the payoff at maturity T is given by $S_T - m_T$, where m_T is the minimum price of the asset up to T. This type of lookback calls can be seen as RC with a continuum of roll-down barriers and strikes. Due to this continuum of roll-down barriers we are not able to find an exact hedge, but we will give super- and sub-hedges for lookback calls.
It is clear that an RC with $K_0 = S_0$ and $K_n = 0$ undervalues the LO, because the strike of the lookback option at time t can only be below or equal the strike of the RC. Therefore a model-independent sub-hedge for a LO is given by:

$$\begin{aligned}\mathrm{LO}(S_0, 0; T) \ge \mathrm{RC}(S_0, 0; S_0, T, \{K_i\}_{1\le i\le n}) = \\ \mathrm{DOC}(S_0, 0; S_0, T, K_1) + \sum_{i=1}^{n-1}\Big(\mathrm{DOC}(S_0, , 0; K_i, T, K_{i+1}) - \mathrm{DOC}(S_0, 0; K_i, T, K_i)\Big) \\ + \mathrm{DIC}(S_0, 0; K_n, T, K_n). \quad (16)\end{aligned}$$

Again the barrier options might be hedged using the strategies of Section 3. Adding more roll-down barriers obviously increases the quality of the sub-hedge, i.e. a tighter bound is obtained.

In order to find a super-hedge for the LO, an ERDC with $K_i = H_{i+1}$ for $1 \leq i \leq n$ and $H_{n+1} = 0$ can be used. Such an ERDC is a super-hedge, because its strike is below or equal to the strike of the LO throughout the lifetime of the option. Hence we have:

$$\begin{aligned} \mathrm{LO}(S_0, 0; T) \leq \mathrm{ERDC}(S_0, 0; H_1, T, \{H_{i+1}\}_{1\leq i\leq n}, \{H_i\}_{1\leq i\leq n+1}) = \\ \mathrm{DOC}(S_0, 0; H_1, T, H_1) \\ + \sum_{i=1}^{n} \Big(\mathrm{DOC}(S_0, 0; H_{i+1}, T, H_{i+1}) - \mathrm{DOC}(S_0, 0; H_{i+1}, T, H_i) \Big). \end{aligned} \quad (17)$$

Another portfolio having the same payoff as the one stated above is given by:

$$\begin{aligned} &\mathrm{LO}(S_0, 0; T) \leq \\ &\quad \mathrm{C}(S_0, 0; H_1, T) - \mathrm{P}(S_0, 0; H_1, T) \\ &\quad + \sum_{i=1}^{n-1} \Big((H_i - H_{i+1})\, \mathrm{DIB}(S_0, 0; H_i, T) \Big) + H_n\, \mathrm{DIB}(S_0, 0; H_n, T), \end{aligned} \quad (18)$$

where DIB is a down-and-in digital option, that pays 1 at expiry if the barrier was hit before maturity. As for the sub-hedge the bounds become tighter the more H_i are added.

Note that (16) as well as (17) and (18) are model-independent super- and sub-hedging portfolios, respectively.

For fixed-strike lookback options one can use similar arguments and digital options to replicate the payoff (see e.g. Buchen & Konstandatos [13]). Then again the hedging strategies for weakly path-dependent options can be used.

It is worth noting that Hobson [48] found the best model-independent bounds for lookback options. However, the ideas are similar to the ones used for the derivation of the robust strategies for hedging barrier options and we omit the details here.

4.2 Asian options

Let K be the strike, $\{t_i\}_{1\leq n}$ the monitoring times and T the maturity of an Asian call (AC). Then the payoff p of the AC at T is given by:

$$p(\mathbb{S}_T) = \left(\frac{1}{n} \sum_{i=1}^{n} S_{t_i} - K \right)^+ = \frac{1}{n} \left(\sum_{i=1}^{n} S_{t_i} - nK \right)^+ . \quad (19)$$

First the best (cheapest) static super-hedging strategy for arithmetic Asian options involving one European call per monitoring time will be outlined. This strategy was originally introduced by Simon et al. [74] for complete markets and later generalized by Albrecher et al. [1] and Albrecher & Schoutens [3] (see also Vanmaele et al. [80] and Chen et al [25]).

Taking the operator $(\cdot)^+$ inside the sum in (19) clearly gives an upper bound and hence a static super-replication strategy for the AC, i.e.

$$(\sum_{i=1}^{n} S_{t_i} - nK)^+ = \Big((S_{t_1} - \mathcal{K}_1) + \ldots + (S_{t_n} + \mathcal{K}_n) \Big)^+ \leq \sum_{i=1}^{n} (S_{t_i} - \mathcal{K}_i)^+, \quad (20)$$

whenever $\sum_{i=1}^{n} \mathcal{K}_i = nK$. A simple first choice is $\mathcal{K}_i = K \; \forall \; 1 \leq i \leq n$. It is now a natural question to ask for the optimal choice of the $\mathcal{K}_i$'s that minimizes the price of the vanilla call portfolio. It turns out that the concept of comonotonicity is a helpful tool to this end:

Definition 4.1 (Stop-loss transform) *Let $F_X(x)$ be the distribution function of a non-negative random variable X. Then the stop-loss transform $\Phi_{F_X}(m)$ is defined by:*

$$\Phi_{F_X}(m) = \int_m^\infty (x-m)\, dF_X(x) = \mathbb{E}\left[(X-m)^+\right], \quad m > 0.$$

Definition 4.2 (Comonotone random vector) *Let $(X_1, X_2, \ldots, X_n)$ be a non-negative random vector with marginal distribution functions $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$. The vector is called comonotone, if the joint distribution function $F_{X_1,X_2,\ldots,X_n}$ is given by*

$$F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = \min\left\{F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)\right\}.$$

Note that the right-hand side of the above equation is a copula (often called lower Frechet-copula). For a general introduction to this field and proofs of the properties used below see e.g. Dhaene et al. [40, 41].
To simplify notation, let us assume that the marginal distribution functions are strictly increasing. Let $(X_1, \ldots, X_n)$ be a non-negative random vector with marginal distribution functions F_{X_i} and suppose $(Y_1, \ldots, Y_n)$ to be the comonotone vector with the same marginal distributions. Setting $S^C = \sum_{i=1}^{n} Y_i$ the following holds:

$$F_{S^C}^{-1}(x) = \sum_{i=1}^{n} F_{X_i}^{-1}(x), \quad 0 \leq x \leq 1. \tag{21}$$

A crucial result is now the following:

$$\Phi_{F_{S^C}}(m) = \sum_{i=1}^{n} \Phi_{F_{X_i}}\Big(F_{X_i}^{-1}\big(F_{S^C}(m)\big)\Big), \quad m \geq 0. \tag{22}$$

Let now $(Y_1, \ldots, Y_n)$ be the comonotone vector with the marginal distribution functions $F_{S_{t_1}}, \ldots, F_{S_{t_n}}$ and S^C the sum of the Y_i's. The following inequality holds for all $\sum_{i=1}^{n} \mathcal{K}_i = nK$:

$$\Phi_{F_{S^C}}(nK) = \mathbb{E}\left[\left(\sum_{i=1}^{n} Y_i - nK\right)^+\right] \leq \sum_{i=1}^{n} \mathbb{E}\left[(Y_i - \mathcal{K}_i)^+\right] = \sum_{i=1}^{n} \Phi_{S_{t_i}}(\mathcal{K}_i).$$

Using equation (22) and the relation $\Phi_{S_{t_i}}(\mathcal{K}_i) = \exp(rt_i)\,\mathrm{C}(S_0, 0; \mathcal{K}_i, T)$ we finally find:

$$\sum_{i=1}^{n} \exp(rt_i)\,\mathrm{C}(S_0, 0; F_{S_{t_i}}^{-1}\big(F_{S^C}(nK)\big), T) \leq \sum_{i=1}^{n} \exp(rt_i)\,\mathrm{C}(S_0, 0; \mathcal{K}_i, T).$$

As a consequence, the optimal choice for the call strikes in (20) is:

$$\mathcal{K}_i = F_{S_{t_i}}^{-1}\big(F_{S^C}(nK)\big).$$

Note that

$$\sum_{i=1}^{n} F_{S_{t_i}}^{-1}\left(F_{S^C}(nK)\right) = F_{S^C}^{-1}\left(F_{S^C}(nK)\right) = nK,$$

for all strictly increasing marginal cumulative distribution functions. Since the inverse function of F_{S^C} is given by (21) and the right-hand side is strictly increasing in x, it is computationally straight-forward to calculate $F_{S^C}(nK)$. Thus this hedging strategy is also simple to evaluate.

Numerical studies carried out in Albrecher et al. [1] for popular Lévy models and Albrecher & Schoutens [3] for stochastic volatility models suggest that the obtained bounds become tighter the deeper the options are in the money.
Of course the concept of comonotonicity can also be applied to other types of options, for which the payoff depends on a sum of possibly dependent random variables, e.g. basket options. For more details we refer to e.g. Hobson et al. [49, 50], or Chen et al. [25]. In those papers in particular also the (realistic) case is considered, when there are only finitely many strikes liquid in the market and one has to find the optimal combination of those.
This last question and especially the problem of finding a sub-replicating portfolio (which is associated to a lower price bound for the Asian option) was also addressed in Albrecher et al. [2]. Lower price bounds on the AC price were first considered by Curran [31] and Rogers & Shi [70], who pioneered a quite accurate method to determine lower price bounds in the Black-Scholes model based on the following idea: Jensen's inequality gives

$$\left(\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K\right)^{+} \geq \left(\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[S_{t_i}|Z\right] - K\right)^{+}, \qquad (23)$$

where Z is an arbitrary random variable. In the Black-Scholes model the choice

$$Z = \left(\prod_{i=1}^{n} S_i\right)^{\frac{1}{n}}$$

or variants thereof are very popular (because the distribution of the geometric mean is explicitly available in the Black-Scholes-model) and leads to tight lower price bounds, since the arithmetic and geometric average are strongly correlated (see e.g. Curran [31], Rogers & Shi [70], Nielsen & Sandmann [66], Thompson [77], or Vanmaele et al. [80]).
However, in contrast to upper price bounds based on the concept of comonotonicity, this method in general does not imply a subreplication strategy. Nevertheless, in Albrecher et al. [2] it is shown that using $Z = S_{t_1}$ together with (21) yields a robust sub-replicating portfolio consisting of $\sum_{i=1}^{n} e^{-r(T-t_i)}/n$ calls with maturity t_1 and strike $nK/(\sum_{j=1}^{n} e^{r(t_j-t_1)})$. The corresponding trading strategy is to do nothing, when $S_1 \leq n\,K/\sum_{i=1}^{n} e^{r(t_i-t_1)}$ or to buy $\frac{1}{n}\sum_{i=1}^{n} e^{-r(T-t_i)}$ assets in the case that $S_1 > n\,K/\sum_{i=1}^{n} e^{r(t_i-t_1)}$. The cost for this trade is exactly the payoff of the options in the portfolio plus $Ke^{-r(T-t_1)}$, which one should borrow. Then at each monitoring time t_i one should sell $e^{-r(T-t_i)}/n$ assets and invest the gain in the riskless bank account. At maturity T of the Asian call, the payoff of the trading strategy will be:

$$\left(\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K\right)\mathbb{I}_{\left\{\sum_{i=1}^{n} e^{r(t_i-t_1)}S_{t_1} > nK\right\}},$$

which is clearly dominated by the payoff of the AC.

5 Case study: Model-dependent hedging of discretely monitored options

In Section 2 a hedging portfolio for path-independent options was presented. As the payoff of discretely monitored options depends only on a finite number of monitoring times, such options can be understood as path-independent between the monitoring times and it is possible to construct a semi-static hedging strategy with adaptations of the portfolio only at the monitoring times. This kind of strategies was developed by Carr & Wu [24] and Joshi [55] (see also Joshi [54]). Similar to Section 2 we assume that standard European options are liquid for all strikes and monitoring times of the DSO. However, to apply the techniques for the path-independent options we need the extra assumption that the asset price process is Markovian. In particular, the prices of standard European options may only depend on the current option price.
We consider a recursion algorithm, that can be used to price such options and to obtain some Greeks (namely the Delta and the Gamma) of the exotic option through the corresponding ones of plain vanilla options. This algorithm is based on the assumption that the price of the exotic option depends in addition to the current price of the asset only on some summary statistic of the historic asset prices at the monitoring times, which is measurable with respect to the filtration generated by the asset price and updated only at the monitoring times of the option. If more than one statistic is needed to describe the price of the exotic option, the method is still feasible, but for notational convenience we focus on the case with a single one.
As an illustrating example, an AC with strike K, monitoring times $t_1, \ldots, t_n$ and maturity $T = t_n$ will be considered, but any other similar DSO's (like e.g. cliquets, or discretely monitored barrier and lookback options) could serve as well.
Let us denote the summary statistic of a generic DSO at time t_i by X_i. Due to the assumptions on X, X_i is $\mathcal{F}_{t_i}$-measureable and $X_t = X_i$ for $t_i \leq t \leq t_{i+1}$, $i = 0, 1, \ldots, n-1$. Furthermore we assume that

$$X_{i+1} = f(X_i, S_{t_{i+1}}), \tag{24}$$

which is fulfilled for all DSO's we are aware of.
For the AC, X_i is the running average A_i of the asset prices at time t_i, i.e.

$$X_i := A_i = \frac{1}{i}\sum_{j=1}^{i} S_{t_j} \quad \forall\, 1 \leq i \leq n.$$

Obviously here assumption (24) is fulfilled, since

$$A_{i+1} = \frac{i}{i+1}A_j + \frac{1}{i+1}S_{t_{i+1}} \quad \forall\, 1 \leq i \leq n-1.$$

Due to (24) we have for the payoff of the generic DSO

$$p(\mathbb{S}_T) = g(X_{n-1}, S_{t_n})$$

for some terminal payoff function g. Now, since X_{n-1} is known at time t_{n-1} we can use (1) to find

$$p(\mathbb{S}_T) = g(X_{n-1}, K^*) + g_S(X_{n-1}, K^*)(S - K^*) \\ + \int_{K^*}^{\infty} g_{SS}(X_{n-1}, x)(S - x)^+ \, dx + \int_0^{K^*} g_{SS}(X_{n-1}, x)(x - S)^+ \, dx, \tag{25}$$

where g_S and g_{SS} denote the first and second derivative of g with respect to its second argument. Note that, as in Section 2, the above describes a static hedge for the DSO at the time t_{n-1} and therefore also its time-t_{n-1}-value $V_{t_{n-1}}$ is settled.

$$V_{t_{n-1}}(X_{t_{n-1}}, S_{t_{n-1}}) = g(X_{n-1}, K^*) + g_S(X_{n-1}, K^*)(F_{t_{n-1}} - K^*) \\ + \int_{K^*}^{\infty} g_{SS}(X_{n-1}, x)\, \mathrm{C}(S_{t_{n-1}}, t_{n-1}; x, T) \, dx \\ + \int_0^{K^*} g_{SS}(X_{n-1}, x)\, \mathrm{P}(S_{t_{n-1}}, t_{n-1}; x, T) \, dx, \tag{26}$$

where $F_{t_{n-1}}$ is the time-t_{n-1}-forward price. For the AC we have:

$$p(\mathbb{S}_T) = \Big(\frac{n-1}{n} A_{n-1} + \frac{1}{n} S_T - K\Big)^+ = \frac{1}{n}\Big(S_T - (nK - (n-1)A_{n-1})\Big)^+.$$

and

$$V_{t_{n-1}}(X_{t_{n-1}}, S_{t_{n-1}}) = \frac{1}{n}\,\mathrm{C}\big(S_{t_{n-1}}, t_{n-1}; nK - (n-1)A_{n-1}, T\big),$$

where $\mathrm{C}(S_t, t; K, T) := \mathrm{e}^{-r(T-t)}(F_t - K)$ for $K < 0$ with F_t denoting again the forward price.

The formula (26) together with the Markov assumption implies that the value $V_{t_{n-1}}$ of the DSO at time t_{n-1} only depends on $X_{t_{n-1}}$ and $S_{t_{n-1}}$ and using again (24) we actually have (with a slight abuse of notation)

$$V_{t_{n-1}}(X_{t_{n-1}}, S_{t_{n-1}}) = V_{t_{n-1}}(X_{t_{n-2}}, S_{t_{n-1}}).$$

Thus we can use the right-hand-side of (26) as the new payoff function of a (modified) DSO and iterate the procedure until we reach the current time t_0 and obtain a replicating portfolio. The associated hedging strategy is to hold this portfolio until expiry t_1 of the standard European options involved and invest the payoff of these options to form the new replicating portfolio for the European-type option $V_{t_2}(X_1, S_{t_2})$. This strategy is of course self-financing.

Note that the replicating portfolio does not change until t_1 and thus one can calculate the price of the DSO, as well as the derivatives with respect to the asset price, with this portfolio. Unfortunately the derivatives with respect to other parameters of the DSO price in general cannot be calculated in the same manner, since changing those parameters would also have an effect on the portfolio.

It is worth noting, that the weights of the standard options in the hedging portfolio at the current time t_0 are determined by the Gamma of the DSO at time t_1.

The derivation of this hedging strategy of course relied on the Markov assumption on the model and the liquidity of arbitrary standard European options. However, even if those assumptions may fail in practice, the approach gives an idea of how to obtain approximative hedging strategies. More precisely: if neither the Markov assumption, nor the liquidity assumption is fulfilled, one might not be able to hedge the DSO perfectly. However, if alternatively a hedging strategy is chosen in order to minimize a certain risk measure for the hedging error in (25), the semi-static strategy might still outperform classical dynamic hedging strategies. Recently, this kind of approach was applied e.g. by Becherer & Ward [7].

6 Conclusion and future research

Semi-static hedging strategies can be a valuable alternative to classic dynamic replication approaches in various situations, often leading to a better performance and reduced model risk. Although over the last years various results have been obtained, there are many open research questions. From a theoretical perspective, it might be interesting to extend the discussed sharp price bounds to other classes of exotic options, like e.g. cliquet options. Also, improved price bounds in terms of other liquid options beyond standard European options are needed. It could be rewarding to investigate, which price bounds can be obtained if American put options, variance/volatility swaps or credit default swaps are liquid. A recent paper in this direction is e.g. Carr & Schoutens [23]. Another line of extension is to model the vanilla option prices themselves too. Schweizer & Wissel [73] recently found dynamics for option prices, that are consistent with the no-arbitrage assumption and one might be able to use this kind of modelling to design more advanced trading strategies with options.
Among further future research topics is the quantitative and systematic comparison of hedging performance of static and dynamic strategies, including historical back-testing of the strategies.

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