1. Dominique Arlettaz: A vanishing conjecture for the homology of congruence subgroups

For any prime number p, let $\Gamma_{n,p}$ denote the congruence subgroup of $SL_n(\mathbb{Z})$ of level p, i.e., the kernel of the surjective homomorphism $f_p : SL_n(\mathbb{Z}) \to SL_n(\mathbb{F}_p)$ induced by the reduction mod p, and let us write $\Gamma_p = \varinjlim_n \Gamma_{n,p}$, where the limit is defined by upper left inclusions. Notice that the groups $\Gamma_{n,p}$ are not homology stable with $\mathbb{Z}$-coefficients (see [Ch]).

If p is odd, then the group Γ_p is torsion-free. Therefore, it is of particular interest to detect torsion classes in the integral (co)homology of Γ_p. It turns out that $H^*(\Gamma_p;\mathbb{Z})$ contains 2-torsion elements in arbitrarily large dimensions (see Corollary 1.10 of [Ar]). Groups like this are called groups with "very strange torsion" by S. Weintraub in 1986.

However, vanishing results for the (co)homology of Γ_p are also extremely useful. Let us propose the following

Conjecture 1.1. *For an odd integer n and an odd prime p, the homology group $H_n(\Gamma_p;\mathbb{Z})$ contains no q-torsion if q is a sufficiently large prime (in comparison with n), $q \neq p$.*

As far as I know, this problem is not solved, but one should notice its relationship with the study of the Dwyer-Friedlander map $\varphi_{\mathbb{Z}} : (K_n(\mathbb{Z}))_q \to K_n^{\text{ét}}(\mathbb{Z}[\frac{1}{q}])$ relating the q-torsion of algebraic K-theory to étale K-theory. This map is known to be surjective and it is conjecturally an isomorphism (this is a version of the Quillen-Lichtenbaum conjecture, see [DF], Theorem 8.7 and Remark 8.8).

For $q \neq p$, the Dwyer-Friedlander map and the reduction mod p induce the commutative diagram

$$\begin{array}{ccc} (K_n(\mathbb{Z}))_q & \xrightarrow{\varphi_{\mathbb{Z}}} & K_n^{\text{ét}}(\mathbb{Z}[\frac{1}{q}]) \\ \downarrow{\scriptstyle (f_p)_*} & & \downarrow \\ (K_n(\mathbb{F}_p))_q & \xrightarrow{\varphi_{\mathbb{F}_p}} & K_n^{\text{ét}}(\mathbb{F}_p) \end{array}$$

The map $\varphi_{\mathbb{F}_p}$ is an isomorphism, since the K-theory of finite fields is completely known. If we define $A_n = \ker \varphi_{\mathbb{Z}}$ and $B_n = \ker(f_p)_*$, this implies that A_n is contained in B_n.

On the other hand, one can show by using Postnikov decompositions that B_n is a direct summand of $(H_n(\Gamma_p;\mathbb{Z}))_q$ for large enough primes $q \neq p$ (see [Ar], Introduction and Theorem 2.1).

Consequently, the proof of the above conjecture for n odd and q a large enough prime would imply the vanishing of B_n and therefore the vanishing of A_n which provides the assertion that the Dwyer-Friedlander map $\varphi_{\mathbb{Z}}$ is an isomorphism.

References

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