UNIFORM TAIL APPROXIMATION OF HOMOGENOUS FUNCTIONALS OF GAUSSIAN FIELDS

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Abstract: Let $X(t), t \in \mathbb{R}^d$ be a centered Gaussian random field with continuous trajectories and set $\xi_u(t) = X(f(u)t), t \in \mathbb{R}^d$ with f some positive function. Classical results establish the tail asymptotics of $\mathbb{P}\{\Gamma(\xi_u) > u\}$ as $u \to \infty$ with $\Gamma(\xi_u) = \sup_{t\in[0,T]^d} \xi_u(t), T > 0$ by requiring that $f(u)$ tends to 0 as $u \to \infty$ with speed controlled by the local behaviour of the correlation function of X. Recent research shows that for applications more general functionals than supremum should be considered and the Gaussian field can depend also on some additional parameter $\tau_u \in K$, say $\xi_{u,\tau_u}(t), t \in \mathbb{R}^d$. In this contribution we derive uniform approximations of $\mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > u\}$ with respect to τ_u in some index set K_u, as $u \to \infty$. Our main result have important theoretical implications; two applications are already included in [12, 13]. In this paper we present three additional ones, namely i) we derive uniform upper bounds for the probability of double-maxima, ii) we extend Piterbarg-Prisyazhnyuk theorem to some large classes of homogeneous functionals of centered Gaussian fields ξ_u, and iii) we show the finiteness of generalized Piterbarg constants.

Key Words: fractional Brownian motion; supremum of Gaussian random fields; stationary processes; double maxima; uniform double-sum method; generalized Piterbarg constants.

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1. Introduction

Let $X(t), t \geq 0$ be a centered stationary Gaussian process with continuous trajectories, unit variance and correlation function r satisfying for some $\alpha \in (0, 2]$

$$1 - r(t) \sim |t|^\alpha, \quad t \to 0, \quad \text{and } r(t) < 1, \quad \forall t > 0.$$

We write $\sim$ for asymptotic equivalence when the argument tends to 0 or infinity.

The seminal paper [24] established for any T positive and $q(u) = u^{-2/\alpha}$

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X(t) > u\right\} \sim T\mathcal{H}_\alpha \frac{\mathbb{P}\{X(0) > u\}}{q(u)} \tag{1}$$

as $u \to \infty$, where $\mathcal{H}_\alpha$ is the *Pickands constant* defined by

$$\mathcal{H}_\alpha = \lim_{T\to\infty} \frac{1}{T}\mathcal{H}_\alpha[0,T] \in (0,\infty), \quad \text{with } \mathcal{H}_\alpha[0,T] = \mathbb{E}\left\{\sup_{t\in[0,T]} e^{\sqrt{2}B_\alpha(t) - t^\alpha}\right\},$$

with B_α a standard fractional Brownian motion with Hurst index $\alpha/2$; see the recent contributions [6, 7, 10, 19, 20] for the main properties of Pickands and related constants.

While the original proof of Pickands utilizes a discretisation approach, in [25, 26] the asymptotics (1) was derived by establishing first the exact asymptotics on the short interval $[0, q(u)T]$, namely (see e.g., Lemma 6.1 in [26])

$$\mathbb{P}\left\{\sup_{t\in[0,q(u)T]} X(t) > u\right\} \sim \mathcal{H}_\alpha[0,T]\mathbb{P}\{X(0) > u\}, \quad u \to \infty \tag{2}$$

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and then using the *double-sum method*. A completely independent proof for the stationary case, based on the notion of *sojourn time*, was derived by Berman (see [3, 4]).

In this contribution we develop the *uniform double-sum method*. Originally, introduced by Piterbarg for non-stationary case, see e.g., [26], the *double-sum method* is a powerful tool in derivation of the exact asymptotics of the tail distribution of supremum for non-stationary Gaussian processes (and fields). With no loss of generality, for a given centered Gaussian process $Y(t), t \in [0, S]$ with continuous trajectories, the crucial steps of this method are:

a) application of Slepian inequality that allows for uniform approximation as $u \to \infty$ (uniformity is with respect to $k \le N(u)$) of summands of $\mathbb{P}\left\{\sup_{t\in[kTq(u),(k+1)Tq(u)]} Y(t) > u\right\}$ by $\mathbb{P}\left\{\sup_{t\in[0,Tq(u)]} X^{\epsilon}(t) > u_k\right\} =: p(u_k)$, for appropriately chosen stationary process $X^{\varepsilon}, \varepsilon > 0$;

b) uniform approximation for $k \le N(u)$ of $p(u_k)$ as $u \to \infty$;

c) uniformly tight upper bounds for the probability of double supremum

$$\mathbb{P}\left\{\sup_{t\in[kTq(u),(k+1)Tq(u)]} Y(t) > u, \sup_{t\in[lTq(u),(l+1)Tq(u)]} Y(t) > u\right\} \tag{3}$$

for $k, l \in \mathcal{A}_u$, where the set $\mathcal{A}_u$ is suitably chosen.

The deep contribution [18] showed that while dealing with supremum of Gaussian processes on the half-line it is convenient to replace Slepian inequality by a uniform version of the tail asymptotics of threshold-dependent Gaussian processes. Omitting technical details, [18] derives the exact asymptotics and a uniform upper bound of

$$\mathbb{P}\left\{\sup_{t\in[0,T]} \xi_{u,\tau_u}(t) > g_{u,\tau_u}\right\}$$

as $u \to \infty$, with respect to $\tau_u \in K_u$, for ξ_{u,τ_u} being centered Gaussian processes indexed by u and τ_u, see also Lemma 5.1 in [16]. This uniform counterpart of (2) is crucial when the processes X_{u,τ_u} are parameterised by u and τ_u.

Recent contributions show strong need for analysis of distributional properties of more general continuous functionals than supremum, as e.g., $\sup_{t\in[0,T]} \inf_{s\in[0,S]} X(s+f(u)t), S > 0$, see [9, 11] or $\inf_{s\in\mathcal{A}_u} \sup_{t\in\mathcal{B}_u} Y(s,t)$, see [14, 16].

The lack of Slepian-type results for general continuous functionals Γ can be overcome by the derivation of uniform approximations with respect to τ_u of the tail distribution of $\Gamma(\xi_{u,\tau_u})$ as $u \to \infty$. Therefore, the principal goal of this contribution is to derive uniform approximations for the tail of homogeneous continuous functionals Γ of general Gaussian random fields. Specifically, we shall consider Γ defined on $C(E)$, the space of continuous functions on E with $E \subset \mathbb{R}^d, d \ge 1$ a compact set containing the origin. In Theorem 2.1 we derive the following uniform asymptotics

$$\lim_{u\to\infty} \sup_{\tau_u\in K_u} \left| \frac{\mathbb{P}\left\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\right\}}{\Psi(g_{u,\tau_u})} - C \right| = 0, \tag{4}$$

where $\xi_{u,\tau_u}(t), t \in E, \tau_u \in K_u$ is a centered Gaussian random field, C is a positive finite constant, and Ψ denotes the survival function of an $N(0,1)$ random variable. This result allows us to derive counterparts of (1) for a class of homogeneous functionals of centered Gaussian fields satisfying some weak asymptotic conditions. Additionally, in Section 3.1 we derive a uniform upper bound for the double maxima for general Gaussian fields parameterised by u and τ_u. That extends and unifies the known upper bounds for (3).

Brief organisation of the rest of the paper: main results of this contribution and related discussions are presented in Section 2. We dedicate Section 3 to applications. Finally, we display the proofs of all the results in Section 4, postponing some technical calculations to Appendix.

2. Main Result

We begin this section with some motivations for the investigation of distributional properties of functionals of threshold-dependent Gaussian random fields. For this purpose we focus on supremum of non-centered Gaussian process. Then we introduce the class of functionals that are of our interest and provide the main result of this contribution; see Theorem 2.1.

Numerous articles, e.g., [8, 18, 21, 22], developed techniques for the approximation, as $u \to \infty$, of the so-called ruin probability

$$p(u) = \mathbb{P}\left\{ \sup_{t \in \mathcal{T}} (X(t) - ct) > u \right\}, \tag{5}$$

where X is a centered continuous Gaussian process, $c > 0$ is some constant and $\mathcal{T} = [0, \infty)$ or $\mathcal{T} = [0, T], T > 0$. Originally the *double-sum method* was designed to handle supremum of centered Gaussian processes. For our case, this method still works under the following modifications. First, we rewrite the original problem in the language of a centered, threshold-dependent family of Gaussian processes $Z_u(t) = \frac{X(t)}{u+ct}$, $u > 0$ as follows

$$p(u) \quad = \quad \mathbb{P}\left\{ \sup_{t \in \mathcal{T}} Z_u(t) > 1 \right\}. \tag{6}$$

Then, one checks that, for suitably chosen $w(u)$ and $N(u)$,

$$\begin{aligned} p(u) \quad &\sim \quad \mathbb{P}\left\{ \text{There exists } |k| \le N(u) : \sup_{t \in [0, w(u)S]} Z_u(t + kSw(u)) > 1 \right\} \\ &\sim \quad \sum_{|k| \le N(u)} \mathbb{P}\left\{ \sup_{t \in [0,S]} Y_{u,k}(t) > v_k(u) \right\} =: \sum_{|k| \le N(u)} p_k(u) \end{aligned} \tag{7}$$

as $u \to \infty$ and $S \to \infty$ respectively, where

$$Y_{u,k}(t) = Z_u(w(u)t + w(u)kS)v_k(u), \quad v_k(u) = \inf_{t \in [0,S]} \frac{1}{\sqrt{Var(Z_u(w(u)t + w(u)kS))}}.$$

Finally, since usually $\lim_{u \to \infty} N(u) = \infty$, then in order to determine the asymptotics of $p(u)$ it is necessary to derive the asymptotics of $p_k(u)$, as $u \to \infty$, uniformly for $|k| \le N(u)$.

In this section, we consider a more general situation focusing on the validity of (4) for centered Gaussian random fields.

Next, let $E \subset \mathbb{R}^d$ be a compact set including the origin and write $C(E)$ for the set of real-valued continuous functions defined on E. Let $\Gamma : C(E) \to \mathbb{R}$ be a real-valued continuous functional satisfying

F1: there exists $c > 0$ such that $\Gamma(f) \le c \sup_{t \in E} f(t)$ for any $f \in C(E)$;

F2: $\Gamma(af + b) = a\Gamma(f) + b$ for any $f \in C(E)$ and $a > 0, b \in \mathbb{R}$.

Note that **F1-F2** cover the following important examples:

$$\Gamma = \sup, \quad \inf, \quad a \sup + (1-a) \inf, \quad a \in \mathbb{R}.$$

We shall consider a family of centered Gaussian random fields ξ_{u,τ_u} given by

$$\xi_{u,\tau_u}(t) = \frac{Z_{u,\tau_u}(t)}{1 + h_{u,\tau_u}(t)}, \quad t \in E, \tau_u \in K_u,$$

with Z_{u,τ_u} a centered Gaussian random field with unit variance and continuous trajectories, and $h_{u,\tau_u} \in C_0(E)$, where $C_0(E)$ is the Banach space of all continuous functions f on E such that $f(0) = 0$ equipped with the sup-norm. In order to avoid trivialities, the thresholds g_{u,τ_u} will be chosen such that

$$\lim_{u \to \infty} \mathbb{P}\left\{ \Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u} \right\} = 0.$$

In order to derive the asymptotics of $\mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\}$ as $u \to \infty$ we shall first condition on $\xi_{u,\tau_u}(0) = g_{u,\tau_u} - \frac{w}{g_{u,\tau_u}}$, yielding that

$$\mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\} = \frac{e^{-g^2_{u,\tau_u}/2}}{\sqrt{2\pi} g_{u,\tau_u}} \int_{\mathbb{R}} e^{w - \frac{w^2}{2g^2_{u,\tau_u}}} \mathbb{P}\{\Gamma(\chi_{u,\tau_u}) > w\} \, dw,$$

where

$$\chi_{u,\tau_u}(t) = g_{u,\tau_u}(\xi_{u,\tau_u}(t) - g_{u,\tau_u}) + w \Big| \Big(\xi_{u,\tau_u}(0) = g_{u,\tau_u} - \frac{w}{g_{u,\tau_u}}\Big).$$

Note that

$$\chi_{u,\tau_u}(t) \stackrel{d}{=} \frac{g_{u,\tau_u}}{1 + h_{u,\tau_u}(t)} \Big(Z_{u,\tau_u}(t) - r_{u,\tau_u}(t,0) Z_{u,\tau_u}(0)\Big) + \mathbb{E}\{\chi_{u,\tau_u}(t)\}, \quad t \in E,$$

where $\stackrel{d}{=}$ means equality of distributions.

Next, we shall impose the following assumptions (see also [16][Lemma 5.1] and [18][Lemma 2]) to ensure the weak convergence of $\{\chi_{u,\tau_u}(t), t \in E\}$, as $u \to \infty$.

C0: The positive constants g_{u,τ_u} are such that $\lim_{u\to\infty} \inf_{\tau_u \in K_u} g_{u,\tau_u} = \infty$.

C1: There exists $h \in C_0(E)$ such that

$$\lim_{u\to\infty} \sup_{\tau_u \in K_u, t \in E} |g^2_{u,\tau_u} h_{u,\tau_u}(t) - h(t)| = 0. \tag{8}$$

C2: There exists $\theta_{u,\tau_u}(s,t)$ such that

$$\lim_{u\to\infty} \sup_{\tau_u \in K_u} \sup_{s \neq t \in E} \left| g^2_{u,\tau_u} \frac{Var\,(Z_{u,\tau_u}(t) - Z_{u,\tau_u}(s))}{2\theta_{u,\tau_u}(s,t)} - 1 \right| = 0 \tag{9}$$

and for some centered Gaussian random field $\eta(t), t \in \mathbb{R}^d$ with continuous trajectories and $\eta(0) = 0$

$$\lim_{u\to\infty} \sup_{\tau_u \in K_u} |\theta_{u,\tau_u}(s,t) - Var(\eta(t) - \eta(s))| = 0, \quad \forall s, t \in E. \tag{10}$$

C3: There exists $a > 0$ such that

$$\limsup_{u\to\infty} \sup_{\tau_u \in K_u} \sup_{s \neq t, s,t \in E} \frac{\theta_{u,\tau_u}(s,t)}{\sum_{i=1}^d |s_i - t_i|^a} < \infty \tag{11}$$

and

$$\lim_{\epsilon \downarrow 0} \limsup_{u\to\infty} \sup_{\tau_u \in K_u} \sup_{\|t-s\| < \epsilon, s,t \in E} g^2_{u,\tau_u} \mathbb{E}\{[Z_{u,\tau_u}(t) - Z_{u,\tau_u}(s)] Z_{u,\tau_u}(0)\} = 0. \tag{12}$$

If X is a centered Gaussian process with stationary increments sastifying **AI-AII** in [16], then $Y_{u,k}(t), t \in [0,S], |k| \le N(u)$ in (7) satisfies **C0-C3**; see also [18].

The intuitive explanation behind these assumptions is as follows: **C1** and (12) in **C3** are used to guarantee the uniform convergence of the function $\mathbb{E}\{\chi_{u,\tau_u}(t)\}$ for $t \in E$ as $u \to \infty$. Utilising further **C2**, the convergence of finite-dimensional distributions (fidi's) of $\chi_{u,\tau_u}(t), t \in E$ to those of $\eta(t), t \in E$ can be shown. Moreover, the tightness follows by (11) in **C3**.

Given $h \in C_0(E)$ and the functional Γ satisfying **F1-F2**, for η introduced in **C2**, we define a new constant

$$\mathcal{H}^{\Gamma}_{\eta,h}(E) := \mathbb{E}\left\{e^{\Gamma(\eta^h)}\right\}, \quad \eta^h(t) := \sqrt{2}\eta(t) - Var(\eta(t)) - h(t), \tag{13}$$

which by **F1** is finite. For notational simplicity we set below

$$\mathcal{H}_\eta(E) = \mathcal{H}^{\sup}_{\eta,0}(E).$$

We present next the main result of this section. Recall that Ψ stands for the survival function of an $N(0,1)$ random variable.

Theorem 2.1. *Under assumptions* **C0-C3** *and* **F1-F2**, *if further* $\mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\} > 0$ *for all* $\tau_u \in K_u$ *and all* u *large, then*

$$\lim_{u\to\infty} \sup_{\tau_u \in K_u} \left| \frac{\mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\}}{\Psi(g_{u,\tau_u})} - \mathcal{H}^{\Gamma}_{\eta,h}(E) \right| = 0. \tag{14}$$

Remark 2.2. *i) Under the assumptions of Theorem 2.1 we have*

$$\limsup_{u\to\infty} \sup_{\tau_u \in K_u} \frac{\mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\}}{\Psi(g_{u,\tau_u})} < \infty, \tag{15}$$

which coincides with the results of Lemma 5.1 in [16] and extends Lemma 2 in [18].
ii) Condition **C2** *and (12) in* **C3** *are equivalent to* **C2** *and*

$$\lim_{u\to\infty} \sup_{t\in E, \tau_u \in K_u} \left| g^2_{u,\tau_u} Var(Z_{u,\tau_u}(t) - Z_{u,\tau_u}(0)) - 2Var(\eta(t)) \right| = 0. \tag{16}$$

iii) Condition **C2** *can be formulated also for the degenerated case* $\eta(t) = 0, t \in \mathbb{R}^d$ *almost surely. The claim of Theorem 2.1 holds also for such* η.

Next we give a simplified version of Theorem 2.1. Instead of **C2-C3**, we assume that

$$\lim_{u\to\infty} \sup_{\tau_u\in K_u} \sup_{s\neq t, s,t\in E} \left| g^2_{u,\tau_u} \frac{Var\left(Z_{u,\tau_u}(t) - Z_{u,\tau_u}(s)\right)}{2\sum_{i=1}^d \frac{c_i\sigma_i^2(q_i(u)|s_i-t_i|)}{\sigma_i^2(q_i(u))}} - 1 \right| = 0, \tag{17}$$

where $q_i(u), i = 1,\dots,d$ are some functions of u with $q_i(u) > 0$ for u large enough and $\lim_{u\to\infty} q_i(u) = \varphi_i \in [0,\infty]$ with

$$\varphi_i = \begin{cases} 0, & 1 \le i \le d_1 \\ (0,\infty), & d_1+1 \le i \le d_2, \\ \infty, & d_2+1 \le i \le d \end{cases}$$

and $c_i \ge 0, 1 \le i \le d$. Moreover, $\sigma_i, 1 \le i \le d$ are regularly varying at 0 with indices $\alpha_{i,0}/2 \in (0,1]$ respectively and $\sigma_i(0) = 0$, $\sigma_i(t) > 0, t > 0$, $1 \le i \le d$; $\sigma_i, d_2+1 \le i \le d$ are bounded on any compact interval and regularly varying at ∞ with indices $\alpha_{i,\infty}/2 \in (0,1]$, respectively; $\sigma_i^2(t), d_1+1 \le i \le d_2$ are continuous and non-negative definite, implying that there exist centered Gaussian processes $\eta_i, d_1+1 \le i \le d_2$ with continuous sample path and stationary increments such that $Var(\eta_i(t)) := \sigma_i^2(t), d_1+1 \le i \le d_2$. We refer to, e.g., [8, 18, 21, 22], where particular examples of Gaussian processes that satisfy the above regularity assumptions are investigated; see also [23] for characterisation of such processes in terms of max-stable stationary processes.

Proposition 2.3. *Suppose that* **C0-C1** *and* **F1-F2** *hold. If (17) holds with* $\sum_{i=1}^d c_i > 0$ *and* $\mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\} > 0$ *for all* $\tau_u \in K_u$ *and all* u *large, then (14) holds with*

$$\eta(t) = \sum_{i=1}^{d_1} \sqrt{c_i} B_{\alpha_{i,0}}(t_i) + \sum_{i=d_1+1}^{d_2} \sqrt{c_i} \frac{\eta_i(\varphi_i t_i)}{\sigma_i(\varphi_i)} + \sum_{i=d_2+1}^{d} \sqrt{c_i} B_{\alpha_{i,\infty}}(t_i), \tag{18}$$

where $B_{\alpha_{i,0}}, 1 \le i \le d_1$, $\eta_i, d_1+1 \le d_2$ *and* $B_{\alpha_{i,\infty}}, d_2+1 \le i \le d$ *are mutually independent.*

Remark 2.4. *i) Condition (17) is satisfied by a large class of important processes that are investigated in the literature, see e.g. [8, 12, 16, 18, 21].*
ii) Under the assumptions of Theorem 2.1

$$\lim_{u\to\infty} \sup_{\tau_u\in K_u} \left| \frac{\mathbb{P}\{\Gamma_i(\xi_{u,\tau_u}) > u, i = 1,\dots,d\}}{\Psi(g_{u,\tau_u})} - \mathcal{H}^{\Gamma_1,\dots,\Gamma_d}_{\eta,h} \right| = 0, \tag{19}$$

with $\Gamma_i, i \le d$ continuous functionals satisfying **F1-F2** *and*

$$\mathcal{H}_{\eta,h}^{\Gamma_1,\dots,\Gamma_d} = \int_{\mathbb{R}} e^w \mathbb{P}\left\{\Gamma_i(\eta^h) > w, i=1,\dots,d\right\} dw \in (0,\infty).$$

Moreover, (19) *holds also in the case that η is degenerated, i.e., $\eta(t)=0, t\in\mathbb{R}^d$ almost surely.*

Finally, we present below a version of Theorem 2.1 under slightly different and more explicit assumptions. We keep the same notation as in Theorem 2.1 and moreover let $\sigma^2_{u,\tau_u}(t) := Var(\xi_{u,\tau_u}(t))$.

D1: Condition **C0** holds for g_{u,τ_u} and $\sigma_{u,\tau_u}(0) = 1$ for all $\tau_u \in K_u$ and all $u>0$, and there exists some $h \in C_0(E)$ such that

$$\lim_{u\to\infty} \sup_{t\in E,\tau_u\in K_u} |g^2_{u,\tau_u}(1-\sigma_{u,\tau_u}(t)) - h(t)| = 0.$$

D2: There exists a centered Gaussian random field $\eta(t), t\in\mathbb{R}^d$ with continuous sample paths, $\eta(0)=0$ such that for any $s,t\in E$ and $\tau_u\in K_u$

$$\lim_{u\to\infty} \sup_{\tau_u\in K_u} \left| g^2_{u,\tau_u} Var(\xi_{u,\tau_u}(t) - \xi_{u,\tau_u}(s)) - 2Var(\eta(t)-\eta(s)) \right| = 0, \tag{20}$$

and

$$\lim_{u\to\infty} \sup_{t\in E,\tau_u\in K_u} \left| g^2_{u,\tau_u} Var(\xi_{u,\tau_u}(t) - \xi_{u,\tau_u}(0)) - 2Var(\eta(t)) \right| = 0. \tag{21}$$

D3: There exist positive constants G,ν,u_0 such that for any $u>u_0$

$$\sup_{\tau_u\in K_u} g^2_{u,\tau_u} Var(\xi_{u,\tau_u}(t) - \xi_{u,\tau_u}(s)) \le G\|t-s\|^\nu$$

holds for all $s,t\in E$.

Theorem 2.5. *If* **D1-D3** *and* **F1-F2** *are satisfied, then (14) holds.*

3. Applications

3.1. **Upper Bounds for Double Supremum.** Uniform bounds for the tail distribution of bivariate maxima of Gaussian processes play a key role in the double-sum technique of V.I. Piterbarg; see, e.g., [26, 27]. More precisely, of interest is to find an optimal upper bound for

$$D(\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2,u) := \mathbb{P}\left\{ \sup_{t\in\lambda_1+\mathcal{E}_1} X_u(t) > m_{\lambda_1}(u), \sup_{t\in\lambda_2+\mathcal{E}_2} X_u(t) > m_{\lambda_2}(u) \right\},$$

which is valid for all large u with λ_i's and $\mathcal{E}_i$'s controlled by E_u by requiring that $\lambda_i+\mathcal{E}_i \subset E_u$, with E_u a compact subset of $\mathbb{R}^d$. Further, the thresholds $m_{\lambda_1}(u), m_{\lambda_2}(u)$ are assumed to satisfy

$$\lim_{u\to\infty} m(u) = \infty, \quad \lim_{u\to\infty} \sup_{\lambda_i+\mathcal{E}_i\subset E_u} \left| \frac{m_{\lambda_i}(u)}{m(u)} - 1 \right| = 0, \quad i=1,2 \tag{22}$$

for some positive function m.

Set below $F(A,B) = \inf_{s\in A,t\in B}\|s-t\|$ with A,B two non-empty subsets of $\mathbb{R}^d$ and $\|\cdot\|$ the Euclidean norm. Let $\mathbb{K} = \{(\lambda_1,\lambda_2) : \lambda_i+\mathcal{E}_i\subset E_u, i=1,2\}$.

Theorem 3.1. *Let $X_u(t), t\in E_u\subset\mathbb{R}^d$ be a family of centered Gaussian random fields with continuous trajectories, variance 1 and correlation function r_u. Suppose that there exist positive constants $S_1,\mathcal{C}_1,\mathcal{C}_2,\beta$ and $\alpha\in(0,2]$ such that for u sufficiently large*

$$m^2(u)(1-r_u(s,t)) \ge \mathcal{C}_1\|s-t\|^\beta, \|s-t\|\ge S_1, \quad s,t\in E_u, \tag{23}$$

and

$$m^2(u)(1-r_u(s,t)) \le \mathcal{C}_2\|s-t\|^{\alpha}, \quad s,t\in E_u, s-t\in[-1,1]^d. \tag{24}$$

Moreover, there exists $\delta>0$ *such that for* u *large enough*

$$r_u(s,t) > \delta - 1, \quad s,t\in E_u. \tag{25}$$

If further (22) holds, then there exists $\mathcal{C}>0$ *such that for all* u *large enough*

$$\sup_{(\lambda_1,\lambda_2)\in\mathbb{K},\mathcal{E}_i\subset[0,S_2]^d,\mathcal{E}_i\neq\emptyset,i=1,2} \frac{e^{\frac{\mathcal{C}_1F^{\beta}(\lambda_1+\mathcal{E}_1,\lambda_2+\mathcal{E}_2)}{8}}D(\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2,u)}{{S_2}^{2d}\Psi(m_{\lambda_1,\lambda_2}(u))} \quad\le\quad \mathcal{C}, \tag{26}$$

with $S_2>1$, $m_{\lambda_1,\lambda_2}(u)=\min(m_{\lambda_1}(u),m_{\lambda_2}(u))$ *and* $\mathcal{C}$ *a positive constant independent of* S_2,u.

Next assume that $\kappa_i(t)>0, t>0, 1\le i\le 2d$ are some non-negative locally bounded functions and define

$$g_u(s,t)=\sum_{i=1}^{d}\frac{\kappa_i(q_i(u)|s_i-t_i|)}{\kappa_i(q_i(u))} \quad\text{and}\quad \widetilde{g}_u(s,t)=\sum_{i=1}^{d}\frac{\kappa_{i+d}(q_{i+d}(u)|s_i-t_i|)}{\kappa_{i+d}(q_{i+d}(u))}.$$

Further, let $q_i(u)>0, u>0$ be such that

$$\lim_{u\to\infty} q_i(u)=\varphi_i\in[0,\infty], \quad 1\le i\le 2d.$$

Corollary 3.2. *Let* $X_u(t), t\in E_u$ *be centered Gaussian random fields with continuous trajectories, variance* 1 *and correlation function* r_u *satisfying (25). Assume further that (22) holds. If further for* u *sufficiently large*

$$\mathcal{C}_3 g_u(s,t)\le m^2(u)(1-r_u(s,t))\le \mathcal{C}_4\widetilde{g}_u(s,t), \quad s,t\in E_u, \tag{27}$$

with $\mathcal{C}_3,\mathcal{C}_4>0$ *and* $\kappa_i, 1\le i\le 2d$, *being regularly varying both at* 0 *and at* ∞ *with indices* $\alpha_{i,0}>0$ *and* $\alpha_{i,\infty}>0$, *respectively, then there exists* $\mathcal{C}>0$ *such that for* u *large enough (26) holds with* $\beta=\frac{1}{2}\min_{i=1,\dots,2d}\min(\alpha_{i,0},\alpha_{i,\infty},2)$ *and* $\mathcal{C}_1$ *a fixed positive constant.*

Corollary 3.3. *Let* $X_u(t), t\in E_u\subset\mathbb{R}^d$ *be centered Gaussian random fields with continuous trajectories, variance* 1 *and correlation function* r_u *satisfying (25) and (27) with* $\varphi_i=0, 1\le i\le 2d$ *and* $\kappa_i, 1\le i\le 2d$ *being regularly varying at* 0 *with indices* $\alpha_{i,0}>0$. *If further (22) and*

$$\limsup_{u\to\infty}\sup_{s,t\in E_u}\max\nolimits_{i=1,\dots,2d} q_i(u)|s_i-t_i|<\infty \tag{28}$$

hold, then there exist positive constants $\mathcal{C},\mathcal{C}_1$ *such that for* u *large enough (26) holds with* $\beta=\frac{1}{2}\min(2,\min_{i=1,\dots,2d}\alpha_{i,0})$.

Remark 3.4. *i) Under the assumptions of Theorem 3.1, using the idea of [15, 28], since for* $\gamma\in(0,1)$

$$D(\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2,u) \quad\le\quad \mathbb{P}\left\{\sup_{s\in\lambda_1+\mathcal{E}_1,t\in\lambda_2+\mathcal{E}_2}(\gamma X_u(s)+(1-\gamma)X_u(t))>m_{\lambda_1,\lambda_2,\gamma}(u)\right\},$$

with $m_{\lambda_1,\lambda_2,\gamma}(u)=\gamma m_{\lambda_1}(u)+(1-\gamma)m_{\lambda_2}(u)$, *then in some cases* (26) *can be improved by putting* $4\gamma(1-\gamma)\mathcal{C}_1$ *instead of* $\mathcal{C}_1$ *and* $m_{\lambda_1,\lambda_2,\gamma}(u)$ *instead of* $m_{\lambda_1,\lambda_2}(u)$, *respectively.*
ii) A particular example is $\kappa_i(x)=x^{\alpha_i}, \alpha_i\in(0,2]$. *For such a case, the result of Corollary 3.3 yields the claim of Lemma 9.14 in [27], see also Lemma 6.3 in [26].*

3.2. **Tail Approximation of $\Gamma_{E_u}(X_u)$.** In many applications the tail asymptotics of general functionals of Gaussian random fields X_u indexed by thresholds $u>0$ is of interest. In this section we present an application of Theorem 2.1 concerned with the tail asymptotics of $\Gamma_{E_u}(X_u)$, where

$$E_u:=\left(\prod_{i=1}^{d}[a_i(u),b_i(u)]\right)\times E$$

is also parametrised by u, with E a compact subset of $\mathbb{R}^n, n\in\mathbb{N}$. Without loss of generality, we assume $0\in E$. The functional Γ_{E_u} is defined as follows:
Let $\Gamma^*: C(E)\to\mathbb{R}$ be a real-valued continuous functional satisfying **F1-F2** with $c=1$ in **F1**. For any compact set $A\subset\mathbb{R}^d$ define

$$\Gamma_{A\times E}(f)=\sup_{s\in A}\Gamma^*(f(s,t)),\quad f\in C(A\times E).$$

It follows that $\Gamma_{A\times E}$ is a continuous functional and satisfies **F1–F2** with $c=1$ in **F1**. Examples of Γ^* are

$$\Gamma^*=\sup,\quad \inf,\quad a\sup+(1-a)\inf,\quad a\le 1.$$

We shall consider $X_u(s,t),(s,t)\in E_u$, a family of centered continuous Gaussian random fields with variance function $\sigma_u(s,t)$ and correlation function $r_u(s,t,s',t')$ satisfying as $u\to\infty$

$$\sigma_u(0,0)=1,\quad 1-\sigma_u(s,0)\sim\sum_{i=1}^{d}\frac{|s_i|^{\beta_i}}{g_i(u)},\quad s\in\prod_{i=1}^{d}[a_i(u),b_i(u)] \tag{29}$$

and

$$\lim_{u\to\infty}\sup_{s\in\prod_{i=1}^{d}[a_i(u),b_i(u)],t\neq 0,t\in E}\left|\frac{1-\frac{\sigma_u(s,t)}{\sigma_u(s,0)}}{\sum_{i=d+1}^{d+n}\frac{|t_i|^{\beta_i}}{g_i(u)}}-1\right|=0, \tag{30}$$

where $\beta_i>0$ and $g_i(u)$ is a function of u satisfying $\lim_{u\to\infty}g_i(u)=\infty$ for $1\le i\le d+n$. Moreover, there exists $m(u)$ such that $\lim_{u\to\infty}m(u)=\infty$ and

$$\lim_{u\to\infty}\sup_{(s,t),(s',t')\in E_u,(s,t)\neq(s',t')}\left|\frac{m^2(u)(1-r_u(s,t,s',t'))}{\sum_{i=1}^{d}\frac{c_i\sigma_i^2(q_i(u)|s_i-s_i'|)}{\sigma_i^2(q_i(u))}+\sum_{i=d+1}^{d+n}\frac{c_i\sigma_i^2(q_i(u)|t_i-t_i'|)}{\sigma_i^2(q_i(u))}}-1\right|=0, \tag{31}$$

where $c_i>0$, $q_i(u)>0$, $\lim_{u\to\infty}q_i(u)=\varphi_i\in[0,\infty], 1\le i\le d+n$, and σ_i are the variance functions of η_i's, centered continuous Gaussian processes with stationary increments, $\eta_i(0)=0$, satisfying further the following assumptions:
A1: $\sigma_i^2(t)$ is regularly varying at ∞ with index $2\alpha_{i,\infty}\in(0,2)$ and is continuously differentiable over $(0,\infty)$ with $\dot{\sigma}_i^2(t)$ being ultimately monotone at ∞.
A2: $\sigma_i^2(t)$ is regularly varying at 0 with index $2\alpha_{i,0}\in(0,2]$.
Moreover, we shall assume that

$$\lim_{u\to\infty}\frac{|a_i(u)|^{\beta_i}}{g_i(u)}=\lim_{u\to\infty}\frac{|b_i(u)|^{\beta_i}}{g_i(u)}=0,\quad 1\le i\le d+n.$$

Let

$$V_{\varphi_i}(t_i)=\begin{cases}\sqrt{c_i}B_{\alpha_{i,0}}(t_i), & \varphi_i=0\\ \frac{\sqrt{c_i}}{\sigma_i(\varphi_i)}\eta_i(\varphi_i t_i), & \varphi_i\in(0,\infty),\\ \sqrt{c_i}B_{\alpha_{i,\infty}}(t_i), & \varphi_i=\infty\end{cases}\quad 1\le i\le d+n. \tag{32}$$

In the sequel, we shall denote

$$\mathcal{P}_\eta^h(E)=\mathcal{H}_{\eta,h}^{\sup}(E),\quad \mathcal{H}_\eta(E)=\mathcal{H}_{\eta,0}^{\sup}(E)$$

and set

$$\mathcal{P}_{\eta}^{h}=\lim_{S\to\infty}\mathcal{P}_{\eta}^{h}([0,S]),\quad \widehat{\mathcal{P}}_{\eta}^{h}=\lim_{S\to\infty}\mathcal{P}_{\eta}^{h}([-S,S]),\quad \mathcal{H}_{\eta}=\lim_{S\to\infty}S^{-1}\mathcal{H}_{\eta}([0,S])$$

if the limits exist. We refer to [12, 17, 26] for the properties of *Piterbarg constants* $\mathcal{P}_{\eta}^{h}$ and *Pickands constants* $\mathcal{H}_{\eta}$. Next, suppose that

$$\lim_{u\to\infty}\frac{m^2(u)}{g_i(u)}=\gamma_i\in[0,\infty]$$

and for all u large $\mathbb{P}\left\{\Gamma_{E_u}(X_u)>m(u)\right\}>0$.

Theorem 3.5. *Let $X_u(s,t),(s,t)\in E_u\subset\mathbb{R}^{d+n}$ be a family of centered Gaussian random fields with continuous trajectories satisfying (29)-(31) and*

$$\gamma_i=\begin{cases}0, & \text{if } 1\le i\le d_1,\\ \infty, & \text{if } d_2+1\le i\le d,\end{cases}\qquad \gamma_i\in(0,\infty),\ d_1+1\le i\le d_2,\qquad \gamma_i\in[0,\infty),\ d+1\le i\le d+n.$$

If further for $1\le i\le d_1$

$$\lim_{u\to\infty}\frac{(m(u))^{2/\beta_i}a_i(u)}{(g_i(u))^{1/\beta_i}}=y_{i,1},\quad \lim_{u\to\infty}\frac{(m(u))^{2/\beta_i}b_i(u)}{(g_i(u))^{1/\beta_i}}=y_{i,2},\quad \lim_{u\to\infty}\frac{(m(u))^{2/\beta_i}(a_i^2(u)+b_i^2(u))}{(g_i(u))^{2/\beta_i}}=0,$$

with $-\infty\le y_{i,1}<y_{i,2}\le\infty$, for $d_1+1\le i\le d_2$, $a_i(u)\le 0\le b_i(u),\lim_{u\to\infty}a_i(u)=a_i\in[-\infty,0],\lim_{u\to\infty}b_i(u)=b_i\in[0,\infty]$ and $a_i(u)\le 0\le b_i(u)$ for $d_2+1\le i\le d$, then

$$\begin{aligned}&\mathbb{P}\left\{\Gamma_{E_u}(X_u)>m(u)\right\}\\ (33)\qquad &\sim\prod_{i=1}^{d_1}\mathcal{H}_{V_{\varphi_i}}\prod_{i=d_1+1}^{d_2}\mathcal{P}_{V_{\varphi_i}}^{h_i}[a_i,b_i]\mathcal{H}_{\widetilde{V}_{\varphi},\widetilde{h}}^{\Gamma^*}(E)\prod_{i=1}^{d_1}\int_{y_{i,1}}^{y_{i,2}}e^{-|s|^{\beta_i}}ds\prod_{i=1}^{d_1}\left(\frac{g_i(u)}{m^2(u)}\right)^{1/\beta_i}\Psi(m(u)),\end{aligned}$$

where

$$(34)\qquad \widetilde{V}_{\varphi}(t)=\sum_{i=1}^{n}V_{\varphi_{d+i}}(t_i),\quad \widetilde{h}(t)=\sum_{i=1}^{n}\gamma_{d+i}|t_i|^{\beta_{d+i}},\quad h_i(s_i)=\gamma_i|s_i|^{\beta_i},\quad d_1+1\le i\le d_2.$$

Remark 3.6. *Theorem 3.5 extends and unifies both the previous findings of [8, 18, 21, 22] and in particular Theorem 8.2 in [26].*

3.3. **Generalized Piterbarg Constants.** Let $X(t),t\ge 0$ be a centered Gaussian process with stationary increments and continuous trajectories. Suppose that the variance function $\sigma^2(t)=Var(X(t))$ is strictly positive for all $t>0$ and $\sigma(0)=0$. Define next

$$\mathcal{P}_X^b([0,S],[0,T])=\mathbb{E}\left\{\sup_{t\in[0,T]}\inf_{s\in[0,S]}e^{\sqrt{2}X(t-s)-(1+b)\sigma^2(|t-s|)}\right\},$$

where b,S,T are positive constants. In the special case, that $X=B_\alpha$ is a fractional Brownian motion (fBm) with Hurst index $\alpha/2\in(0,1]$, the generalized Piterbarg constant

$$\mathcal{P}_{B_\alpha}^b(S)=\lim_{T\to\infty}\mathcal{P}_{B_\alpha}^b([0,S],[0,T])\in(0,\infty)$$

determines the asymptotics of Parisian ruin of the corresponding risk model, see [11]. Note that the classical Piterbarg constant corresponds to the case $S=0$. Our next result shows that $\mathcal{P}_X^b(S)\in(0,\infty)$ for a general Gaussian process with stationary increments.

Proposition 3.7. *If $X(t), t \geq 0$ is a centred Gaussian process with stationary increments and variance function satisfying* **A1** *with regularly varying index $2\alpha_\infty \in (0, 2]$ and* **A2** *with regularly varying index $2\alpha_0 \in (0, 2)$, then for any b, S positive we have*

$$\lim_{T \to \infty} \mathcal{P}_X^b([0, S], [0, T]) \in (0, \infty).$$

4. Proofs

Hereafter, by $\mathbb{Q}$, $\mathbb{Q}_i, i = 1, 2, \dots$ we denote positive constants which may differ from line to line.

Proof of Theorem 2.1 Since we assume that $\mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\} > 0$ for all u large and any $\tau_u \in K_u$, then by conditioning

$$\begin{aligned} \mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\} &= \int_{\mathbb{R}} \mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u} | \xi_{u,\tau_u}(0) = x\} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx \\ &= \frac{e^{-g_{u,\tau_u}^2/2}}{\sqrt{2\pi} g_{u,\tau_u}} \int_{\mathbb{R}} e^{w - \frac{w^2}{2g_{u,\tau_u}^2}} \mathbb{P}\{\Gamma(\chi_{u,\tau_u}) > w\}\, dw \\ &=: \frac{e^{-g_{u,\tau_u}^2/2}}{\sqrt{2\pi} g_{u,\tau_u}} \mathcal{I}_{u,\tau_u}, \end{aligned}$$

with $\mathcal{I}_{u,\tau_u} > 0$ for all u large and

$$\chi_{u,\tau_u}(t) = \zeta_{u,\tau_u}(t) | (\zeta_{u,\tau_u}(0) = 0), \quad \zeta_{u,\tau_u}(t) = g_{u,\tau_u}(\xi_{u,\tau_u}(t) - g_{u,\tau_u}) + w.$$

Hence the proof follows by showing that $\mathcal{H}_{\eta,h}^{\Gamma}(E)$ is finite and

$$\lim_{u \to \infty} \sup_{\tau_u \in K_u} |\mathcal{I}_{u,\tau_u} - \mathcal{H}_{\eta,h}^{\Gamma}(E)| = 0. \tag{35}$$

Weak convergence of $\Gamma(\chi_{u,\tau_u})$. We have that $\chi_{u,\tau_u}(0) = 0$ almost surely. Setting $r_{u,\tau_u}(s,t) = Cor(Z_{u,\tau_u}(s), Z_{u,\tau_u}(t))$ we may write

$$\chi_{u,\tau_u}(t) \stackrel{d}{=} \frac{g_{u,\tau_u}}{1 + h_{u,\tau_u}(t)} \Big(Z_{u,\tau_u}(t) - r_{u,\tau_u}(t,0) Z_{u,\tau_u}(0) \Big) + \mathbb{E}\{\chi_{u,\tau_u}(t)\}, \quad t \in E,$$

where $\stackrel{d}{=}$ means equality of the fidi's. Since

$$(1 + h_{u,\tau_u}(t)) \mathbb{E}\{\chi_{u,\tau_u}(t)\} = -g_{u,\tau_u}^2 (1 - r_{u,\tau_u}(t,0)) - g_{u,\tau_u}^2 h_{u,\tau_u}(t) + w(1 - r_{u,\tau_u}(t,0) + h_{u,\tau_u}(t))$$

by **C1, C3** for some arbitrary M positive, uniformly with respect to $t \in E, \tau_u \in K_u, w \in [-M, M]$

$$(1 + h_{u,\tau_u}(t)) \mathbb{E}\{\chi_{u,\tau_u}(t)\} \quad \to \quad -(\sigma_\eta^2(t) + h(t)), \ u \to \infty \tag{36}$$

and also for any $s, t \in E$ uniformly with respect to $\tau_u \in K_u, w \in [-M, M]$

$$\begin{aligned} & Var\Big((1 + h_{u,\tau_u}(t)) \chi_{u,\tau_u}(t) - (1 + h_{u,\tau_u}(s)) \chi_{u,\tau_u}(s) \Big) \\ &= g_{u,\tau_u}^2 \left[\mathbb{E}\left\{ \Big(Z_{u,\tau_u}(t) - Z_{u,\tau_u}(s) \Big)^2 \right\} - \left(\mathbb{E}\{ Z_{u,\tau_u}(0) [Z_{u,\tau_u}(t) - Z_{u,\tau_u}(s)] \} \right)^2 \right] \\ &\to 2Var(\eta(t) - \eta(s)), \quad u \to \infty. \end{aligned} \tag{37}$$

Consequently, by Lemma 4.1 in [29] the fidi's of $(1 + h_{u,\tau_u}(t)) \chi_{u,\tau_u}(t), t \in E$ converge to those of $\eta^h(t), t \in E$ as $u \to \infty$ uniformly for $\tau_u \in K_u, w \in [-M, M]$ where $M > 0$ is fixed (recall $\eta^h(t) = \sqrt{2}\eta(t) - Var(\eta(t)) - h(t)$). Condition **C3** together with the uniform convergence in (36) guarantee that Proposition 9.7 in [27] can be applied to yield the uniform tightness of $(1+h_{u,\tau_u}(t))\chi_{u,\tau_u}(t), t\in E$ and thus $\{(1+h_{u,\tau_u}(t))\chi_{u,\tau_u}(t), t\in E\}$ weakly converges to $\{\eta^h(t), t\in E\}$, as $u\to\infty$, uniformly with respect to $\tau_u\in K_u$. Further, since

$$\lim_{u\to\infty}\sup_{t\in E,\tau_u\in K_u} h_{u,\tau_u}(t)=0,$$

then $\{\chi_{u,\tau_u}(t), t\in E\}$ converges weakly to $\{\eta^h(t), t\in E\}$ as $u\to\infty$, uniformly with respect to $\tau_u\in K_u$. Consequently, since we assume that Γ is a continuous functional, by the continuous mapping theorem $\Gamma(\chi_{u,\tau_u})$ converges in distribution to $\Gamma(\eta^h)$ as $u\to\infty$ uniformly with respect to $\tau_u\in K_u$.

Convergence of (35). Denote by $\mathbb{A}=\{w:\mathbb{P}\left\{\Gamma(\eta^h)>w\right\}$ is discontinuous at $w\}$, then $\mathbb{A}$ is an countable set with measure 0. Hence for any $w\in\mathbb{R}\setminus\mathbb{A}$

$$\lim_{u\to\infty}\sup_{\tau_u\in K_u}\left|\mathbb{P}\left\{\Gamma(\chi_{u,\tau_u})>w\right\}-\mathbb{P}\left\{\Gamma(\eta^h)>w\right\}\right|=0$$

and by **C0**

$$\lim_{u\to\infty}\sup_{\tau_u\in K_u,w\in[-M,M]} e^w\left[1-e^{-\frac{w^2}{2g^2_{u,\tau_u}}}\right]\le\frac{e^M M^2}{2\liminf_{u\to\infty}\inf_{\tau_u\in K_u} g^2_{u,\tau_u}}\to 0,\quad u\to\infty$$

implying

$$\begin{aligned}
&\lim_{u\to\infty}\sup_{\tau_u\in K_u}\left|\int_{-M}^{M}\left[e^{w-\frac{w^2}{2g^2_{u,\tau_u}}}\mathbb{P}\left\{\Gamma(\chi_{u,\tau_u})>w\right\}-e^w\mathbb{P}\left\{\Gamma(\eta^h)>w\right\}\right]dw\right|\\
&\le\lim_{u\to\infty}\sup_{\tau_u\in K_u}\int_{-M}^{M}e^w(1-e^{-\frac{w^2}{2g^2_{u,\tau_u}}})\mathbb{P}\left\{\Gamma(\eta^h)>w\right\}dw\\
&\quad+\lim_{u\to\infty}\sup_{\tau_u\in K_u}\left|\int_{-M}^{M}\left[e^{w-\frac{w^2}{2g^2_{u,\tau_u}}}\left(\mathbb{P}\left\{\Gamma(\chi_{u,\tau_u})>w\right\}-\mathbb{P}\left\{\Gamma(\eta^h)>w\right\}\right)\right]dw\right|\\
&\le e^M\lim_{u\to\infty}\int_{-M}^{M}\sup_{\tau_u\in K_u}\left|\mathbb{P}\left\{\Gamma(\chi_{u,\tau_u})>w\right\}-\mathbb{P}\left\{\Gamma(\eta^h)>w\right\}\right|dw=0.
\end{aligned}$$

Using (36) for $\delta\in(0,1/c)$, $|w|>M$ with M sufficiently large and all u large we have

$$\sup_{\tau_u\in K_u,t\in E}(1+h_{u,\tau_u}(t))\mathbb{E}\left\{\chi_{u,\tau_u}(t)\right\}\le\delta|w|.$$

Moreover, in view of (37) and (11) in **C3** we have that for u sufficiently large

$$\begin{aligned}
Var\Big((1+h_{u,\tau_u}(t))\chi_{u,\tau_u}(t)-(1+h_{u,\tau_u}(s))\chi_{u,\tau_u}(s)\Big)\ &\le\ g^2_{u,\tau_u}\mathbb{E}\left\{\Big(Z_{u,\tau_u}(t)-Z_{u,\tau_u}(s)\Big)^2\right\}\\
&\le\ \mathbb{Q}\sum_{i=1}^{d}|s_i-t_i|^a.
\end{aligned}$$

Consequently, by Piterbarg inequality (see e.g., Theorem 8.1 in [26]) we obtain for some $\varepsilon\in(0,1)$, $\delta\in(0,1/c)$ with c given in **F1**, and all u large

$$\begin{aligned}
&\int_{|w|>M}e^{w-\frac{w^2}{2g^2_{u,\tau_u}}}\mathbb{P}\left\{\Gamma(\chi_{u,\tau_u})>w\right\}dw\\
&\le\ \int_{|w|>M}e^w\mathbb{P}\left\{c\sup_{t\in E}(1+h_{u,\tau_u}(t))(\chi_{u,\tau_u}(t)-\mathbb{E}\left\{\chi_{u,\tau_u}(t)\right\})>w-c\sup_{t\in E,\tau_u\in K_u}(1+h_{u,\tau_u}(t))\mathbb{E}\left\{\chi_{u,\tau_u}(t)\right\}\right\}dw\\
&\le\ e^{-M}+\int_M^\infty e^w\Psi((1-\varepsilon)(1/c-\delta)w)\,dw\\
&=:\ A(M)\to 0,\quad M\to\infty.
\end{aligned}$$

Moreover, by Borell-TIS inequality (see e.g., [1])

$$\begin{aligned}\int_{|w|>M} e^w \mathbb{P}\left\{\Gamma(\eta^h) > w\right\} dw &\le \int_{|w|>M} e^w \mathbb{P}\left\{c\sup_{t\in E}\eta^h(t) > w\right\} dw\\ &\le e^{-M} + \int_M^\infty e^w \mathbb{P}\left\{\sqrt{2}c\sup_{t\in E}\eta(t) > w - c\sup_{t\in E}\left(Var(\eta(t)) + h(t)\right)\right\} dw\\ &\le e^{-M} + \int_M^\infty e^{w-\frac{(w-a)^2}{2\sup_{t\in E} Var(\sqrt{2}c\eta(t))}} dw\\ &=: B(M) \to 0, \quad M\to\infty,\end{aligned}$$

with $a = \sqrt{2}c\mathbb{E}\left\{\sup_{t\in E}\eta(t)\right\} - c\sup_{t\in E}\left(Var(\eta(t)) + h(t)\right) < \infty$. Hence (35) follows from

$$\begin{aligned}\sup_{\tau_u\in K_u}\left|\mathcal{I}_{u,\tau_u} - \mathcal{H}^{\Gamma}_{\eta,h}(E)\right| &\le \sup_{\tau_u\in K_u}\left|\int_{-M}^{M}\left[e^{w-\frac{w^2}{2g^2_{u,\tau_u}}}\mathbb{P}\left\{\Gamma(\chi_{u,\tau_u}) > w\right\} - e^w\mathbb{P}\left\{\Gamma(\eta^h) > w\right\}\right] dw\right|\\ &\quad + A(M) + B(M)\\ &\to A(M) + B(M), \quad u\to\infty,\\ &\to 0, \quad M\to\infty,\end{aligned}$$

establishing the proof. □

Proof of Proposition 2.3 It follows from Remark 2.2 ii) that it suffices to prove (10), (11) and (16). Without loss of generality, in the following derivation we assume that $c_i > 0, 1 \le i \le d$. By (17), we have

$$\theta_{u,\tau_u}(s,t) = \sum_{i=1}^{d}\frac{c_i\sigma_i^2(q_i(u)|s_i-t_i|)}{\sigma_i^2(q_i(u))}, \quad (s,t)\in E.$$

By uniform convergence theorem (UCT) for regularly varying functions, see [5], (10) holds with η defined in (18). Next we verify (11). For $0 < \beta < \min(\min_{1\le i\le d}\alpha_{i,0}, \min_{d_2+1\le i\le d}\alpha_{i,\infty})$ we have

$$\sum_{i=1}^{d}\frac{c_i\sigma_i^2(q_i(u)|s_i-t_i|)}{\sigma_i^2(q_i(u))} = \sum_{i=1}^{d} c_i\frac{f_i(q_i(u)|s_i-t_i|)}{f_i(q_i(u))}|s_i-t_i|^{\beta/2},$$

with $f_i(t) = \frac{\sigma_i^2(t)}{t^{\beta/2}}, t > 0$. Note that f_i is regularly varying at 0 with index $\alpha_{i,0} - \beta/2 > 0$ for $1 \le i \le d$ and for $d_2 + 1 \le i \le d$, f_i is regularly varying at ∞ with index $\alpha_{i,\infty} - \beta/2 > 0$. By UCT for any $M > 0$ we have

$$\lim_{u\to\infty}\max\nolimits_{i=1,\dots,d_1}\sup_{0<|s_i-t_i|\le M}\left|\frac{f_i(q_i(u)|s_i-t_i|)}{f_i(q_i(u))} - |s_i-t_i|^{\alpha_{i,0}-\beta/2}\right| = 0.$$

Using the fact that f_i is bounded on compact intervals for $d_2 + 1 \le i \le d$, again by UCT, for any $M > 0$

$$\lim_{u\to\infty}\max\nolimits_{i=d_2+1,\dots,d}\sup_{0<|s_i-t_i|\le M}\left|\frac{f_i(q_i(u)|s_i-t_i|)}{f_i(q_i(u))} - |s_i-t_i|^{\alpha_{i,\infty}-\beta/2}\right| = 0.$$

Moreover, since f_i is regularly varying at 0 with index $\alpha_{i,0} - \beta > 0$ and $\varphi_i \in (0,\infty), d_1 + 1 \le i \le d_2$, then for any $M > 0$ and u large enough

$$\max\nolimits_{d_1+1\le i\le d_2}\sup_{0<|s_i-t_i|\le M}\frac{f_i(q_i(u)|s_i-t_i|)}{f_i(q_i(u))} < \infty.$$

Thus we conclude that for u large enough

$$\sum_{i=1}^{d}\frac{c_i\sigma_i^2(q_i(u)|s_i-t_i|)}{\sigma_i^2(q_i(u))} \le \mathbb{Q}\sum_{i=1}^{d}|s_i-t_i|^{\beta/2}, \quad s,t\in E,$$

which confirms (11). We are now left to prove (16). In light of (17) and UCT, we have

$$\lim_{u\to\infty}\sup_{t\in E\setminus\{0\},\tau_u\in K_u}\left|g^2_{u,\tau_u}Var(Z_{u,\tau_u}(t) - Z_{u,\tau_u}(0)) - 2Var(\eta(t))\right|$$

$$\leq \lim_{u\to\infty} \sup_{t\in E\setminus\{0\},\tau_u\in K_u} \left|\frac{g^2_{u,\tau_u} Var(Z_{u,\tau_u}(t)-Z_{u,\tau_u}(0))}{2\theta_{u,\tau_u}(0,t)}-1\right|\left|2\theta_{u,\tau_u}(0,t)\right|$$

$$+\lim_{u\to\infty} \sup_{t\in E,\tau_u\in K_u}\left|2\theta_{u,\tau_u}(0,t)-2Var(\eta(t))\right|=0,$$

which implies that (16) holds. This completes the proof. □

Proof of Theorem 2.5 We check that **C0-C3** hold. Clearly, **C0** is satisfied by the assumptions. We observe that

$$\xi_{u,\tau_u}(t)=\frac{\overline{\xi}_{u,\tau_u}(t)}{1+h_{u,\tau_u}(t)},\quad t\in E,\tau_u\in K_u,$$

with

$$\overline{\xi}_{u,\tau_u}(t)=\frac{\xi_{u,\tau_u}(t)}{\sigma_{u,\tau_u}(t)},\quad h_{u,\tau_u}(t)=\frac{1-\sigma_{u,\tau_u}(t)}{\sigma_{u,\tau_u}(t)},$$

which together with **D1** immediately implies that **C1** is valid. Let next for $u>0$

$$\theta_{u,\tau_u}(s,t)=\frac{g^2_{u,\tau_u}}{2}Var(\overline{\xi}_{u,\tau_u}(t)-\overline{\xi}_{u,\tau_u}(s)).$$

Direct calculations yield

$$\theta_{u,\tau_u}(s,t)=I_{1,u,\tau_u}(s,t)+I_{2,u,\tau_u}(s,t)+I_{3,u,\tau_u}(s,t),\quad s,t\in E,$$

where

$$I_{1,u,\tau_u}(s,t)=\frac{g^2_{u,\tau_u}}{2}\frac{Var(\xi_{u,\tau_u}(t)-\xi_{u,\tau_u}(s))}{\sigma^2_{u,\tau_u}(t)},\quad I_{2,u,\tau_u}(s,t)=\frac{g^2_{u,\tau_u}}{2}\frac{(\sigma_{u,\tau_u}(t)-\sigma_{u,\tau_u}(s))^2}{\sigma^2_{u,\tau_u}(t)},$$

$$I_{3,u,\tau_u}(s,t)=g^2_{u,\tau_u}\frac{\sigma_{u,\tau_u}(t)-\sigma_{u,\tau_u}(s)}{\sigma^2_{u,\tau_u}(t)\sigma_{u,\tau_u}(s)}\mathbb{E}\left\{(\xi_{u,\tau_u}(s)-\xi_{u,\tau_u}(t))\xi_{u,\tau_u}(s)\right\}.$$

It follows from **D1** that

$$\lim_{u\to\infty}\sup_{s,t\in E,\tau_u\in K_u} I_{2,u,\tau_u}(s,t)\leq \lim_{u\to\infty}\sup_{s,t\in E,\tau_u\in K_u} g^2_{u,\tau_u}\frac{(\sigma_{u,\tau_u}(t)-1)^2+(1-\sigma_{u,\tau_u}(s))^2}{\sigma^2_{u,\tau_u}(t)}=0.$$

Further, by **D1,D2**

$$\lim_{u\to\infty}\sup_{\tau_u\in K_u}\left|I_{1,u,\tau_u}(s,t)-Var(\eta(t)-\eta(s))\right|=0,\quad s,t\in E$$

and

$$\lim_{u\to\infty}\sup_{\tau_u\in K_u}\left|I_{3,u,\tau_u}(s,t)\right|\leq \lim_{u\to\infty}\sup_{\tau_u\in K_u} g^2_{u,\tau_u}\frac{|\sigma_{u,\tau_u}(t)-\sigma_{u,\tau_u}(s)|}{\sigma^2_{u,\tau_u}(t)}\sqrt{Var(\xi_{u,\tau_u}(s)-\xi_{u,\tau_u}(t))}=0,\quad s,t\in E.$$

Thus we confirm that **C2** holds. Moreover, by **D3** and the fact that

$$(\sigma_{u,\tau_u}(t)-\sigma_{u,\tau_u}(s))^2\leq Var(\xi_{u,\tau_u}(t)-\xi_{u,\tau_u}(s))$$

we obtain

$$\lim_{u\to\infty}\sup_{\tau_u\in K_u}\sup_{s\neq t,s,t\in E}\frac{\theta_{u,\tau_u}(s,t)}{||t-s||^\nu}\leq \mathbb{Q}\lim_{u\to\infty}\sup_{\tau_u\in K_u}\sup_{s\neq t,s,t\in E}\frac{g^2_{u,\tau_u}Var(\xi_{u,\tau_u}(t)-\xi_{u,\tau_u}(s))}{||t-s||^\nu}<\infty.$$

Using again **D1,D2** we obtain

$$\lim_{u\to\infty}\sup_{t\in E,\tau_u\in K_u}\left|I_{1,u,\tau_u}(0,t)-Var(\eta(t))\right|=0,$$

$$\lim_{u\to\infty}\sup_{t\in E,\tau_u\in K_u} I_{2,u,\tau_u}(0,t)=0,\quad \lim_{u\to\infty}\sup_{t\in E,\tau_u\in K_u}\left|I_{3,u,\tau_u}(0,t)\right|=0,$$

which imply

$$\lim_{u\to\infty}\sup_{t\in E,\tau_u\in K_u}\left|\theta_{u,\tau_u}(0,t)-Var(\eta(t))\right|=0.$$

Hence **C3** is satisfied with (16) instead of (12). In view of Remark 2.2 the proof is completed. □

Proof of Theorem 3.1 Recall that $F(A,B)=\inf_{s\in A, t\in B}\|s-t\|$ with A,B two non-empty subsets of $\mathbb{R}^d$ and $\|\cdot\|$ the Euclidean norm. Clearly, for any u positive

$$\mathbb{P}\left\{\sup_{t\in\lambda_1+\mathcal{E}_1} X_u(t)>m_{\lambda_1}(u), \sup_{t\in\lambda_2+\mathcal{E}_2} X_u(t)>m_{\lambda_2}(u)\right\}\le \mathbb{P}\left\{\sup_{s\in\lambda_1+\mathcal{E}_1, t\in\lambda_2+\mathcal{E}_2}(X_u(s)+X_u(t))>2m_{\lambda_1,\lambda_2}(u)\right\},$$

where $m_{\lambda_1,\lambda_2}(u)=\min(m_{\lambda_1}(u), m_{\lambda_2}(u))$. By (23) and (25), we have that for u sufficiently large and $F(\lambda_1+\mathcal{E}_1,\lambda_2+\mathcal{E}_2)>S_1$, with S_1 large enough,

$$2\delta\le Var(X_u(s)+X_u(t))=4-2(1-r_u(s,t))\le 4-\frac{2\mathcal{C}_1F^{\beta}(\lambda_1+\mathcal{E}_1,\lambda_2+\mathcal{E}_2)}{m^2(u)}.$$

Moreover, by (24) and the above inequality,

$$\begin{aligned}1-Cor(X_u(s)+X_u(t), X_u(s')+X_u(t')) &\le \frac{Var(X_u(s)+X_u(t)-X_u(s')-X_u(t'))}{2\sqrt{Var(X_u(s)+X_u(t))}\sqrt{Var(X_u(s')+X_u(t'))}}\\ &\le \delta^{-1}(1-r_u(s,s')+1-r_u(t,t'))\\ &\le \mathcal{C}_2\frac{\delta^{-1}d^{\alpha/2}}{m^2(u)}\sum_{i=1}^d(|s_i-s_i'|^{\alpha}+|t_i-t_i'|^{\alpha})\end{aligned}$$

holds for $s,t,s',t'\in[0,1]^d$. Let $X_u^*(s,t), s,t\in\mathbb{R}^d, u>0$ be a family of centered Gaussian random fields with unit variance and correlation satisfying

$$r_u(s,t)=e^{-\frac{2\delta^{-1}d^{\alpha/2}\mathcal{C}_2}{m^2(u)}\sum_{i=1}^d(|s_i|^{\alpha}+|t_i|^{\alpha})},\quad s,t\in\mathbb{R}^d$$

and let further

$$m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2}:=\frac{2m_{\lambda_1,\lambda_2}(u)}{\sqrt{4-\frac{2\mathcal{C}_1F^{\beta}(\lambda_1+\mathcal{E}_1,\lambda_2+\mathcal{E}_2)}{m^2(u)}}},\quad I_{i_1,\dots,i_d}=\prod_{j=1}^d[i_j,i_j+1].$$

For all u large we have

$$\begin{aligned}&\mathbb{P}\left\{\sup_{s\in\lambda_1+\mathcal{E}_1,t\in\lambda_2+\mathcal{E}_2}(X_u(s)+X_u(t))>2m_{\lambda_1,\lambda_2}(u)\right\}\\ &\le \mathbb{P}\left\{\sup_{s\in\lambda_1+\mathcal{E}_1,t\in\lambda_2+\mathcal{E}_2}\overline{X_u(s)+X_u(t)}>m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2}\right\}\\ &\le \mathbb{P}\left\{\sup_{s\in\lambda_1+[0,S_2]^d,t\in\lambda_2+[0,S_2]^d}\overline{X_u(s)+X_u(t)}>m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2}\right\}\\ &\le \sum_{i_1,i_2,\dots,i_d,i_1',i_2',\dots,i_d'=0}^{[S_2]}\mathbb{P}\left\{\sup_{s\in\lambda_1+I_{i_1,\dots,i_d},t\in\lambda_2+I_{i_1',\dots,i_d'}}\overline{X_u(s)+X_u(t)}>m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2}\right\}\\ &\le \sum_{i_1,i_2,\dots,i_d,i_1',i_2',\dots,i_d'=0}^{[S_2]}\mathbb{P}\left\{\sup_{s\in\lambda_1+I_{i_1,\dots,i_d},t\in\lambda_2+I_{i_1',\dots,i_d'}}X_u^*(s,t)>m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2}\right\}\\ &= (S_2+1)^{2d}\mathbb{P}\left\{\sup_{s,t\in[0,1]^d}X_u^*(s,t)>m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2}\right\},\end{aligned}\tag{38}$$

where we used Slepian inequality (see, e.g., [1, 2]) to derive (38). Hence in order to complete the proof, we need to apply Proposition 2.3 to the family of Gaussian random fields $\{X_u^*(s,t),(s,t)\in[0,1]^{2d}\}$. Let

$$K_u=\{(\lambda_1,\lambda_2),\lambda_i+\mathcal{E}_i\subset E_u, i=1,2\}.$$

Note that

$$\lim_{u\to\infty}\sup_{(\lambda_1,\lambda_2)\in K_u}\sup_{(s,t)\neq(s',t'),(s,t),(s',t')\in[0,1]^{2d}}\left|\frac{(m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2})^2 Var\left(X_u^*(s,t)-X_u^*(s',t')\right)}{2\sum_{i=1}^d 2\delta^{-1}d^{\alpha/2}\mathcal{C}_2(\sum_{i=1}^d|s_i-s_i'|^\alpha+\sum_{i=1}^d|t_i-t_i'|^\alpha)}-1\right|=0.$$

Since conditions **C0-C1** are clearly satisfied, then Proposition 2.3 implies

$$\lim_{u\to\infty}\sup_{(\lambda_1,\lambda_2)\in K_u}\left|\frac{1}{\Psi\left(m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2}\right)}\mathbb{P}\left\{\sup_{s,t\in[0,1]^{2d}}X_u^*(s,t)>m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2}\right\}-\mathcal{H}_\eta([0,1]^{2d})\right|=0,$$

where

$$\eta(s,t)=\sum_{i=1}^d\sqrt{2\delta^{-1}d^{\alpha/2}\mathcal{C}_2}B_\alpha^{(i)}(s_i)+\sum_{i=d+1}^{2d}\sqrt{2\delta^{-1}d^{\alpha/2}\mathcal{C}_2}B_\alpha^{(i)}(t_{i-d}),$$

with $B_\alpha^{(i)}, 1\le i\le 2d$ independent fBm's with index α. Thus we establish the claim for $F(\lambda_1+\mathcal{E}_1,\lambda_2+\mathcal{E}_2)>S_1$. For $F(\lambda_1+\mathcal{E}_1,\lambda_2+\mathcal{E}_2)\le S_1$, we have

$$\mathbb{P}\left\{\sup_{s\in\lambda_1+\mathcal{E}_1}X_u(s)>m_{\lambda_1}(u),\ \sup_{t\in\lambda_2+\mathcal{E}_2}X_u(t)>m_{\lambda_2}(u)\right\}\le\mathbb{P}\left\{\sup_{t\in\lambda_1+[-S_1,S_2+S_1]^d}X_u(t)>m_{\lambda_1,\lambda_2}(u)\right\}.$$

By (24) and Slepian inequality

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{s\in\lambda_1+[-S_1,S_2+S_1]^d}X_u(s)>m_{\lambda_1,\lambda_2}(u)\right\}\\
&\le(S_2+2S_1+1)^d\mathbb{P}\left\{\sup_{s\in[0,1]^d}X_u^*(\delta^{1/\alpha}s,0,\dots,0)>m_{\lambda_1,\lambda_2}(u)\right\}\\
&\sim(S_2+2S_1+1)^d\mathcal{H}_\lambda([0,1]^d)\Psi(m_{\lambda_1,\lambda_2}(u)),\quad u\to\infty,
\end{aligned}$$

with $\lambda(s)=\sqrt{\delta}\eta(s,0,\dots,0)$. This completes the proof. □

Proof of Corollary 3.2 Let $\beta=\frac{1}{2}\min_{i=1,\dots,2d}\min(\alpha_{i,0},\alpha_{i,\infty},2)$ and $f_i(t)=\kappa_i(t)/t^\beta$. Clearly, f_i's are regularly varying at 0 with index $\alpha_{i,0}-\beta>0$ and regularly varying at ∞ with index $\alpha_{i,\infty}-\beta>0$. With this notation we have

$$\frac{\kappa_i(q_i(u)|s_i-t_i|)}{\kappa_i(q_i(u))}=\frac{f_i(q_i(u)|s_i-t_i|)}{f_i(q_i(u))}|s_i-t_i|^\beta,\quad s_i\neq t_i, i=1,\dots,2d. \tag{39}$$

Next we focus on $\frac{f_i(q_i(u)|s_i-t_i|)}{f_i(q_i(u))}$. We consider the upper bound and lower bound respectively.

Lower bound. For $\varphi_i=0$ we define $g_i(t)=1/f_i(1/t)$. Then g_i is both regularly varying at 0 with index $\alpha_{i,\infty}-\beta>0$ and regularly varying at ∞ with index $\alpha_{i,0}-\beta>0$. By the assumption on κ_i's, further g_i is bounded over any compact interval and by UCT

$$\lim_{u\to\infty}\sup_{|s_i-t_i|\ge1}\left|\frac{g_i(\frac{1}{q_i(u)|s_i-t_i|})}{g_i(\frac{1}{q_i(u)})}-\left(\frac{1}{|s_i-t_i|}\right)^{\alpha_{i,0}-\beta}\right|=0$$

implying that for u large enough

$$\frac{g_i(\frac{1}{q_i(u)|s_i-t_i|})}{g_i(\frac{1}{q_i(u)})}\le2,\qquad\frac{1}{|s_i-t_i|}\le1.$$

Consequently, for u sufficiently large

$$\frac{f_i(q_i(u)|s_i-t_i|)}{f_i(q_i(u))}=\frac{g_i(\frac{1}{q_i(u)})}{g_i(\frac{1}{q_i(u)|s_i-t_i|})}\ge\frac{1}{2},\quad|s_i-t_i|\ge1.$$

Next, if $\varphi_i\in(0,\infty)$, then by the fact that $\lim_{t\to\infty}f_i(t)=\infty$, there exists $S_1>0$ and M_i' such that for u sufficiently large

$$\frac{f_i(q_i(u)|s_i-t_i|)}{f_i(q_i(u))}>M_i',\quad|s_i-t_i|>S_1.$$

For $\varphi = \infty$, Potter's theorem (see e.g., [5][Theorem 1.5.6]) implies that for any $0 < \epsilon < \alpha_{i,\infty} - \beta$ there exists $M_i'' > 0$ and $S_1' > 1$ such that for u sufficiently large

$$\frac{f_i(q_i(u)|s_i - t_i|)}{f_i(q_i(u))} > M_i''|s_i - t_i|^{\alpha_{i,\infty}-\beta-\epsilon} \geq M_1'', \quad |s_i - t_i| > S_1'.$$

Consequently, there exists $S > 1$ and $M > 0$ such that for u sufficiently large

$$\frac{\kappa_i(q_i(u)|s_i - t_i|)}{\kappa_i(q_i(u))} \geq M|s_i - t_i|^{\beta}, \quad |s_i - t_i| > S, i = 1, \ldots, d.$$

Further, for u large enough

$$g_u(s,t) \geq d^{-\frac{\beta}{2}} M\|s - t\|^{\beta}, \quad \|s - t\| > \sqrt{d}S. \tag{40}$$

Upper bound. If $\varphi_i \in \{0, \infty\}$, then using again UCT we have that

$$\sup_{|s_i - t_i| \leq 1} \frac{f_i(q_i(u)|s_i - t_i|)}{f_i(q_i(u))} \leq C$$

is valid for all u large enough and some constant C. Further, since f_i is locally bounded, then the above holds also if $\varphi_i \in (0, \infty)$. This implies that for some $M' > 0$

$$\widetilde{g}_u(s,t) \leq M' \sum_{i=1}^{d} |s_i - t_i|^{\beta} \leq dM'\|s - t\|^{\beta}, \quad s - t \in [-1, 1]^d, \tag{41}$$

which combined with (40) and Theorem 3.1 establishes the claim. □

Proof of Corollary 3.3 The claim follows straightforwardly using the arguments of Corollary 3.2 for the case $\varphi_i = 0$. □

Proof of Theorem 3.5 Without loss of generality, we assume that $a_i = -\infty$, $b_i = \infty$ for $d_1 + 1 \leq i \leq d_2$. Set in the following

$$I_k = \prod_{i=1}^{d_1} [k_i S, (k_i + 1)S], \quad k = (k_1, \ldots, k_{d_1}),$$

$$J_l = \prod_{i=d_1+1}^{d_2} [l_i S, (l_i + 1)S] \times \prod_{i=d_2+1}^{d} [l_i T, (l_i + 1)T], \quad l = (l_{d_1+1}, \ldots, l_d),$$

$$J^* = \prod_{i=d_1+1}^{d_2} [-S, S] \times \prod_{i=d_2+1}^{d} [-T, T], \widetilde{J} = \prod_{i=d_1+1}^{d_2} [-S, S] \times \{0\}, \quad 0 \in \mathbb{R}^{d-d_2}.$$

Further, define

$$I_k^* = I_k \times J^* \times E, \quad \widetilde{I}_k = I_k \times \widetilde{J} \times E, \quad I_{k,l} = I_k \times J_l \times E,$$

$$K_u^{\pm} = \left\{ k, \frac{a_i(u)}{S} \mp 1 \leq k_i \leq \frac{b_i(u)}{S} \pm 1, 1 \leq i \leq d_1 \right\},$$

$$L_u = \left\{ l, \frac{a_i(u)}{S} - 1 \leq l_i \leq \frac{b_i(u)}{S} + 1, d_1 + 1 \leq i \leq d_2, \quad \frac{a_i(u)}{T} - 1 \leq l_i \leq \frac{b_i(u)}{T} + 1, d_2 + 1 \leq i \leq d, J_l \nsubseteq J^* \right\}.$$

For some $\varepsilon \in (-1, 1)$ and $u > 0$ set

$$\Theta_\epsilon(u) := \prod_{i=1}^{d_1} \int_{y_{i,1}}^{y_{i,2}} e^{-(1-\epsilon)|s|^{\beta_i}} ds \prod_{i=1}^{d_1} \left(\frac{g_i(u)}{m^2(u)} \right)^{1/\beta_i} \Psi(m(u)).$$

Observe that

$$X_u(s,t) = \frac{\sigma_u(s,t)\overline{X}_u(s,t)}{\sigma_u(0,0)}, \quad \frac{\sigma_u(0,0)}{\sigma_u(s,t)} = \frac{\sigma_u(0,0)}{\sigma_u(s,0)} \frac{\sigma_u(s,0)}{\sigma_u(s,t)}.$$

Using (29) and (30), there exists $e_{u,1}(s)$ and $e_{u,2}(s,t)$ such that as $u\to\infty$

$$\sup_{s\in\prod_{i=1}^d[a_i(u),b_i(u)]}|e_{u,1}(s)|=o(1),\quad \sup_{(s,t)\in E_u}|e_{u,2}(s,t)|=o(1),$$

and

$$\begin{aligned}\frac{\sigma_u(0,0)}{\sigma_u(s,0)}&=1+(1+e_{u,1}(s))\sum_{i=1}^d\frac{|s_i|^{\beta_i}}{g_i(u)},\quad s\in\prod_{i=1}^d[a_i(u),b_i(u)],\\ \frac{\sigma_u(s,0)}{\sigma_u(s,t)}&=1+(1+e_{u,2}(s,t))\sum_{i=d+1}^{d+n}\frac{|t_i|^{\beta_i}}{g_i(u)},\quad (s,t)\in E_u.\end{aligned}$$

Note that by **F2** for Γ^*

$$\Gamma_{E_u}(X_u(s,t))=\sup_{s\in\prod_{i=1}^d[a_i(u),b_i(u)]}\Gamma^*(X_u(s,t))=\sup_{s\in\prod_{i=1}^d[a_i(u),b_i(u)]}\sigma_u(s,0)\Gamma^*\left(\overline{X}_u(s,t)\frac{\sigma_u(s,t)}{\sigma_u(s,0)}\right).$$

Thus, by **F2** for Γ^*, and the property of sup functional we have that for $0<\epsilon<1/2$ and u sufficiently large

$$\mathbb{P}\left\{\Gamma_{E_u}(X_u^{+\epsilon})>m(u)\right\}\le\mathbb{P}\left\{\Gamma_{E_u}(X_u)>m(u)\right\}\le\mathbb{P}\left\{\Gamma_{E_u}(X_u^{-\epsilon,y})>m(u)\right\},\tag{42}$$

where for $(s,t)\in E_u$

$$\begin{aligned}X_u^{-\epsilon,y}(s,t)&=\frac{\overline{X}_u(s,t)}{(1+\sum_{i=1}^{d_1}(1-\epsilon)\frac{|s_i|^{\beta_i}}{g_i(u)})(1+\sum_{i=d_1+1}^{d_2}(1-\epsilon)\frac{|s_i|^{\beta_i}}{g_i(u)}+\sum_{i=d_2+1}^{d}y\frac{|s_i|^{\beta_i}}{m^2(u)})}\\ &\quad\times\frac{1}{(1+(1+e_{u,2}(s,t))\sum_{i=d+1}^{d+n}\frac{|t_i|^{\beta_i}}{g_i(u)})},\end{aligned}$$

and

$$X_u^{+\epsilon}(s,t)=\frac{\overline{X}_u(s,t)}{(1+\sum_{i=1}^{d_1}(1+\epsilon)\frac{|s_i|^{\beta_i}}{g_i(u)})(1+\sum_{i=d_1+1}^{d}(1+\epsilon)\frac{|s_i|^{\beta_i}}{g_i(u)})(1+(1+e_{u,2}(s,t))\sum_{i=d+1}^{d+n}\frac{|t_i|^{\beta_i}}{g_i(u)})}.$$

Upper bound. By the property of sup functional, we have that

$$\begin{aligned}\mathbb{P}\left\{\Gamma_{E_u}(X_u^{-\epsilon,y})>m(u)\right\}&\le\sum_{k\in K_u^+}\mathbb{P}\left\{\Gamma_{I_k^*}(X_u^{-\epsilon,y})>m(u)\right\}+\sum_{(k,l)\in K_u^+\times L_u}\mathbb{P}\left\{\Gamma_{I_{k,l}}(X_u^{-\epsilon,y})>m(u)\right\}\\ &\le\sum_{k\in K_u^+}\mathbb{P}\left\{\Gamma_{I_0^*}(\xi_{u,k})>m_{u,k}\right\}+\sum_{(k,l)\in K_u^+\times L_u}\mathbb{P}\left\{\Gamma_{I_{0,0}}(\xi_{u,k,l})>m_{u,k,l}\right\},\end{aligned}\tag{43}$$

where

$$\begin{aligned}\xi_{u,k}(s,t)&=\frac{\overline{X}_u(s+kS,t)}{(1+\sum_{i=d_1+1}^{d_2}(1-\epsilon)\frac{|s_i|^{\beta_i}}{g_i(u)}+\sum_{i=d_2+1}^{d}y\frac{|s_i|^{\beta_i}}{m^2(u)})(1+(1+e_{u,2}(s,t))\sum_{i=d+1}^{d+n}\frac{|t_i|^{\beta_i}}{g_i(u)})},\quad (s,t)\in I_0^*,\\ \xi_{u,k,l}(s,t)&=\frac{\overline{X}_u(s+(k,l)(S,T),t)}{1+(1+e_{u,2}(s,t))\sum_{i=d+1}^{d+n}\frac{|t_i|^{\beta_i}}{g_i(u)}},\quad (s,t)\in I_{0,0},\\ m_{u,k}&=m(u)\left(1+\sum_{i=1}^{d_1}(1-\epsilon)\frac{|k_i^*S|^{\beta_i}}{g_i(u)}\right),\\ m_{u,k,l}&=m(u)\left(1+\sum_{i=1}^{d_1}(1-\epsilon)\frac{|k_i^*S|^{\beta_i}}{g_i(u)}+\sum_{i=d_1+1}^{d_2}(1-2\epsilon)\frac{|l_i^*S|^{\beta_i}}{g_i(u)}+\sum_{i=d_2+1}^{d}y/2\frac{|l_i^*S|^{\beta_i}}{m^2(u)}\right),\end{aligned}$$

with $kS=(k_1S,\dots,k_{d_1}S,0,\dots,0)\in\mathbb{R}^d$ and

$$(k,l)(S,T)=(k_1S,\dots,k_{d_1}S,l_{d_1+1}S,\dots,l_{d_2}S,l_{d_2+1}T,l_dT)\in\mathbb{R}^d,$$

$$k_i^*=\min(|k_i|,|k_i+1|),\quad 1\le i\le d_1,\ l_i^*=\min(|l_i|,|l_i+1|),\quad d_1+1\le i\le d.$$

In order to apply Proposition 2.3, by (31), set

$$\theta_{u,k}(s,t,s',t')=\sum_{i=1}^{d}\frac{c_i\sigma_i^2(q_i(u)|s_i-s_i'|)}{\sigma_i^2(q_i(u))}+\sum_{i=d+1}^{d+n}\frac{c_i\sigma_i^2(q_i(u)|t_i-t_i'|)}{\sigma_i^2(q_i(u))},\quad (s,t),(s',t')\in I_0^*,$$

$$h_{u,k}(s,t)=\left(\sum_{i=d_1+1}^{d_2}(1-\epsilon)\frac{|s_i|^{\beta_i}}{g_i(u)}+\sum_{i=d_2+1}^{d}y\frac{|s_i|^{\beta_i}}{m^2(u)}+\sum_{i=d+1}^{d+n}\frac{|t_i|^{\beta_i}}{g_i(u)}\right)(1+o(1)),\quad (s,t)\in I_0^*,$$

$$g_{u,k}=m_{u,k},\quad K_u=K_u^+,\quad E=I_0^*.$$

First we note that condition **C0** holds straightforwardly. One can easily check that **C1** holds with

$$h_\epsilon(s,t)=\sum_{i=d_1+1}^{d_2}(1-\epsilon)\gamma_i|s_i|^{\beta_i}+\sum_{i=d_2+1}^{d}y|s_i|^{\beta_i}+\sum_{i=d+1}^{d+n}\gamma_i|t_i|^{\beta_i},\quad (s,t)\in I_0^*. \tag{44}$$

Thus in view of **A1-A2** and by Proposition 2.3, we have

$$\lim_{u\to\infty}\sup_{k\in K_u^+}\left|\frac{\mathbb{P}\left\{\Gamma_{I_0^*}(\xi_{u,k})>m_{u,k}\right\}}{\Psi(m_{u,k})}-\mathcal{H}^{\Gamma}_{V_\varphi,h_\epsilon}(I_0^*)\right|=0, \tag{45}$$

with h_ϵ defined in (44) and $V_\varphi(s,t)=\sum_{i=1}^{d}V_{\varphi_i}(s_i)+\sum_{i=1}^{n}V_{\varphi_{d+i}}(t_i)$ with V_{φ_i} defined in (32). Similarly, we have

$$\lim_{u\to\infty}\sup_{(k,l)\in K_u^+\times L_u}\left|\frac{\mathbb{P}\left\{\Gamma_{I_{0,0}}(\xi_{u,k,l})>m_{u,k,l}\right\}}{\Psi(m_{u,k,l})}-\mathcal{H}^{\Gamma}_{V_\varphi,\widetilde{h}}(I_{0,0})\right|=0, \tag{46}$$

with $\widetilde{h}(s,t)=\sum_{i=1}^{n}\gamma_{i+d}|t_i|^{\beta_{i+d}}$. Further, as $u\to\infty$

$$\begin{aligned}\sum_{k\in K_u^+}\mathbb{P}\left\{\Gamma_{I_0^*}(\xi_{u,k})>m_k(u)\right\} &\sim \mathcal{H}^{\Gamma}_{V_\varphi,h_\epsilon}(I_0^*)\sum_{k\in K_u^+}\Psi(m_{u,k})\\ &\sim \mathcal{H}^{\Gamma}_{V_\varphi,h_\epsilon}(I_0^*)\Psi(m(u))\sum_{k\in K_u^+}e^{-\sum_{i=1}^{d_1}(1-\epsilon)m^2(u)\frac{|k_i^*S|^{\beta_i}}{g_i(u)}}\\ &\sim S^{-d_1}\mathcal{H}^{\Gamma}_{V_\varphi,h_\epsilon}(I_0^*)\Theta_\epsilon(u)\end{aligned} \tag{47}$$

and

$$\begin{aligned}&\sum_{(k,l)\in K_u^+\times L_u}\mathbb{P}\left\{\Gamma_{I_{0,0}}(\xi_{u,k,l})>m_{u,k,l}\right\}\\ &\sim \mathcal{H}^{\Gamma}_{V_\varphi,\widetilde{h}}(I_{0,0})\sum_{(k,l)\in K_u^+\times L_u}\Psi(m_{u,k,l})\\ &\le \mathcal{H}^{\Gamma}_{V_\varphi,\widetilde{h}}(I_{0,0})\sum_{k\in K_u^+}\Psi(m_{u,k})\sum_{l\in L_u}e^{-m^2(u)(\sum_{i=d_1+1}^{d_2}(1-2\epsilon)\frac{|l_i^*S|^{\beta_i}}{g_i(u)}+\sum_{i=d_2+1}^{d}y/2\frac{|l_i^*T|^{\beta_i}}{m^2(u)})}(1+o(1))\\ &\le \mathcal{H}^{\Gamma}_{V_\varphi,\widetilde{h}}(I_{0,0})\sum_{k\in K_u^+}\Psi(m_{u,k})\sum_{l\in L_u}e^{-\sum_{i=d_1+1}^{d_2}(1-2\epsilon)\gamma_i|l_i^*S|^{\beta_i}-\sum_{i=d_2+1}^{d}y/2|l_i^*T|^{\beta_i}}(1+o(1))\\ &\le S^{-d_1}\mathcal{H}^{\Gamma}_{V_\varphi,\widetilde{h}}(I_{0,0})\left(\sum_{i=d_1+1}^{d_2}e^{-\mathbb{Q}S^{\beta_i}}+\sum_{i=d_2+1}^{d}e^{-y\mathbb{Q}T^{\beta_i}}\right)\Theta_\epsilon(u)(1+o(1)).\end{aligned} \tag{48}$$

<u>*Lower bound*</u>. By the property of sup functional and Bonferroni inequality, we obtain

$$\begin{aligned}\mathbb{P}\left\{\Gamma_{E_u}(X_u^{+\epsilon})>m(u)\right\} &\ge \sum_{k\in K_u^-}\mathbb{P}\left\{\Gamma_{\widetilde{I}_k}(X_u^{+\epsilon})>m(u)\right\}\\ &\quad -\sum_{k,q\in K_u^-,k\neq q}\mathbb{P}\left\{\Gamma_{\widetilde{I}_k}(X_u^{+\epsilon})>m(u),\Gamma_{\widetilde{I}_q}(X_u^{+\epsilon})>m(u)\right\}.\end{aligned} \tag{49}$$

Similarly as (47), we have

$$\sum_{k\in K_u^-}\mathbb{P}\left\{\Gamma_{\widetilde{I}_k}(X_u^{+\epsilon})>m(u)\right\}\sim S^{-d_1}\mathcal{H}^{\Gamma}_{V_\varphi,h_\epsilon^*}(\widetilde{I}_0)\Theta_{-\epsilon}(u), \tag{50}$$

with $h_\epsilon^*(s,t)=\sum_{i=d_1+1}^{d_2}(1+\epsilon)\gamma_i|s_i|^{\beta_i}+\sum_{i=1}^{n}\gamma_{i+d}|t_i|^{\beta_{i+d}}, (s,t)\in\widetilde{I}_0$. Finally, we focus on the double-sum term. It follows from **F1**, that

$$\begin{aligned}&\sum_{k,q\in K_u^-,k\neq q}\mathbb{P}\left\{\Gamma_{\widetilde{I}_k}(X_u^{+\epsilon})>m(u),\Gamma_{\widetilde{I}_k}(X_u^{+\epsilon})>m(u)\right\}\\ &\le \sum_{k,q\in K_u^-,k\neq q}\mathbb{P}\left\{\sup_{(s,t)\in\widetilde{I}_k}X_u^{+\epsilon}(s,t)>m(u),\ \sup_{(s,t)\in\widetilde{I}_q}X_u^{+\epsilon}(s,t)>m(u)\right\}\\ &\le \sum_{k,q\in K_u^-,k\neq q}\mathbb{P}\left\{\sup_{(s,t)\in\widetilde{I}_k}\overline{X}_u(s,t)>m_{u,k},\ \sup_{(s,t)\in\widetilde{I}_q}\overline{X}_u(s,t)>m_{u,q}\right\}.\end{aligned}$$

Let for $u>0$

$$\mathcal{T}_1=\{(k,q),k,q\in K_u^-,k\neq q,\widetilde{I}_k\cap\widetilde{I}_q\neq\emptyset\},\quad \mathcal{T}_2=\{(k,q),k,q\in K_u^-,\widetilde{I}_k\cap\widetilde{I}_q=\emptyset\}.$$

Without loss of generality, we assume that $q_1=k_1+1, S>1$. Then $\widetilde{I}_k=\widetilde{I}_k'\bigcup\widetilde{I}_k''$ with

$$\widetilde{I}_k'=[k_1S,(k_1+1)S-\sqrt{S}]\times\prod_{i=2}^{d_1}[k_iS,(k_i+1)S]\times\widetilde{J}\times E,$$

$$\widetilde{I}_k''=[(k_1+1)S-\sqrt{S},(k_1+1)S]\times\prod_{i=2}^{d_1}[k_iS,(k_i+1)S]\times\widetilde{J}\times E.$$

Consequently,

$$\begin{aligned}&\mathbb{P}\left\{\sup_{(s,t)\in\widetilde{I}_k}\overline{X}_u(s,t)>m_{u,k},\ \sup_{(s,t)\in\widetilde{I}_q}\overline{X}_u(s,t)>m_{u,q}\right\}\\ &\le\mathbb{P}\left\{\sup_{(s,t)\in\widetilde{I}_k'}\overline{X}_u(s,t)>m_{u,k},\ \sup_{(s,t)\in\widetilde{I}_q}\overline{X}_u(s,t)>m_{u,q}\right\}+\mathbb{P}\left\{\sup_{(s,t)\in\widetilde{I}_k''}\overline{X}_u(s,t)>m_{u,k}\right\}.\end{aligned}$$

Similarly as in (45), we have

$$\lim_{u\to\infty}\sup_{k\in K_u^-}\left|\frac{\mathbb{P}\left\{\sup_{(s,t)\in\widetilde{I}_k''}\overline{X}_u(s,t)>m_{u,k}\right\}}{\Psi(m_{u,k})}-\mathcal{H}^{\sup}_{V_\varphi,h_\epsilon^*}(\widehat{I}_0)\right|=0,$$

with $\widehat{I}_0=[0,\sqrt{S}]\times[0,S]^{d_1-1}\times\widetilde{J}\times E$.

Let $\beta=\min(\min_{i=1}^{d+n}\alpha_{i,0},\min_{i=1}^{d+n}\alpha_{i,\infty})$. By (31) and Corollary 3.2, there exists $\mathcal{C}>0$ and $\mathcal{C}_1>0$ such that

$$\begin{aligned}&\mathbb{P}\left\{\sup_{(s,t)\in\widetilde{I}_k'}\overline{X}_u(s,t)>m_{u,k},\ \sup_{(s,t)\in\widetilde{I}_q}\overline{X}_u(s,t)>m_{u,q}\right\}\\ &\le\mathcal{C}(S+|E|+1)^{2(d_2+n)}e^{-\mathcal{C}_1S^{\beta/2}}\Psi(m_{u,k,q}^*)\end{aligned}$$

and for $(k,q)\in\mathcal{T}_2$

$$\begin{aligned}&\mathbb{P}\left\{\sup_{(s,t)\in\widetilde{I}_k}\overline{X}_u(s,t)>m_{u,k},\ \sup_{(s,t)\in\widetilde{I}_q}\overline{X}_u(s,t)>m_{u,q}\right\}\\ &\le\mathcal{C}(S+|E|+1)^{2(d_2+n)}e^{-\mathcal{C}_1F^{\beta}(\widetilde{I}_k,\widetilde{I}_q)}\Psi(m_{u,k,q}^*),\end{aligned}$$

with $m^*_{u,k,q}=\min(m_{u,k},m_{u,q})$. Since each $\widetilde{I}_k$ has at most 3^{d_1} neighbours, then for S and u sufficiently large

$$\begin{aligned}
&\sum_{(k,q)\in\mathcal{T}_1}\mathbb{P}\left\{\sup_{(s,t)\in\widetilde{I}_k}\overline{X}_u(s,t)>m_{u,k},\ \sup_{(s,t)\in\widetilde{I}_q}\overline{X}_u(s,t)>m_{u,q}\right\}\\
&\le 3^d\sum_{k\in K_u^-}\mathcal{H}^{\sup}_{V_\varphi,h^*_\epsilon}(\widehat{I}_0)\Psi(m_{u,k})+\sum_{(k,q)\in\mathcal{T}_1}\mathcal{C}(S+|E|+1)^{2(d_2+n)}e^{-\mathcal{C}_1S^{\beta/2}}\Psi(m^*_{u,k,q})\\
&\le \mathbb{Q}\sum_{k\in K_u^-}\left(\mathcal{H}^{\sup}_{V_\varphi,h^*_\epsilon}(\widehat{I}_0)+e^{-\frac{\mathcal{C}_1S^{\beta/2}}{2}}\right)\Psi(m_{u,k})\\
&\le \mathbb{Q}S^{-d_1}\left(\mathcal{H}^{\sup}_{V_\varphi,h^*_\epsilon}(\widehat{I}_0)+e^{-\frac{\mathcal{C}_1S^{\beta/2}}{2}}\right)\Theta_\epsilon(u).
\end{aligned}\tag{51}$$

Moreover, for all u large

$$\begin{aligned}
&\sum_{(k,q)\in\mathcal{T}_2}\mathbb{P}\left\{\sup_{(s,t)\in\widetilde{I}_k}\overline{X}_u(s,t)>m_{u,k},\ \sup_{(s,t)\in\widetilde{I}_q}\overline{X}_u(s,t)>m_{u,q}\right\}\\
&\le \sum_{(k,q)\in\mathcal{T}_2}\mathcal{C}(S+|E|+1)^{2(d_2+n)}e^{-\mathcal{C}_1F^\beta(\widetilde{I}_k,\widetilde{I}_q)}\Psi(m_{u,k,q})\\
&\le \sum_{k\in K_u^-}\Psi(m_{u,k})\mathbb{Q}S^{\mathbb{Q}_1}\sum_{q\neq 0}e^{-\mathcal{C}_1(S^2\sum_{i=1}^{d_1}q_i^2)^{\beta/2}}\\
&\le \mathbb{Q}S^{\mathbb{Q}_1}e^{-\mathbb{Q}_2S^\beta}\Theta_\epsilon(u).
\end{aligned}\tag{52}$$

Inserting (43–52) into (42) and dividing each term by $\Theta_0(u)$, we have, with $\epsilon\to 0$

$$\begin{aligned}
&S^{-d_1}\mathcal{H}^{\Gamma}_{V_\varphi,h^*_0}(\widetilde{I}_0)-\mathbb{Q}S^{-d_1}\left(\mathcal{H}^{\sup}_{V_\varphi,h^*_0}(\widehat{I}_0)+e^{-\frac{\mathcal{C}_1S^{\beta/2}}{2}}\right)-\mathbb{Q}S^{\mathbb{Q}_1}e^{-\mathbb{Q}_2S^\beta}\\
&\le \liminf_{u\to\infty}\frac{\mathbb{P}\left\{\Gamma_{E_u}(X_u)>m(u)\right\}}{\Theta_0(u)}\\
&\le \lim_{T\to 0}\lim_{y\to\infty}\limsup_{u\to\infty}\frac{\mathbb{P}\left\{\Gamma_{E_u}(X_u)>m(u)\right\}}{\Theta_0(u)}\\
&\le \lim_{T\to 0}S^{-d_1}\mathcal{H}^{\Gamma}_{V_\varphi,h_0}(I_0^*)+\lim_{T\to 0}\lim_{y\to\infty}S^{-d_1}\mathcal{H}^{\Gamma}_{V_\varphi,\widetilde{h}}(I_0^*)\left(\sum_{i=d_1+1}^{d_2}e^{-\mathbb{Q}S^{\beta_i}}+\sum_{i=d_2+1}^{d}e^{-y\mathbb{Q}T^{\beta_i}}\right)\\
&= S^{-d_1}\mathcal{H}^{\Gamma}_{V_\varphi,h^*_0}(\widetilde{I}_0)\left(1+\sum_{i=d_1+1}^{d_2}e^{-\mathbb{Q}S^{\beta_i}}\right).
\end{aligned}\tag{53}$$

Note further that

$$\mathcal{H}^{\sup}_{V_\varphi,h^*_0}(\widehat{I}_0)=\mathcal{H}_{V_{\varphi_1}}([0,\sqrt{S}])\prod_{i=2}^{d_1}\mathcal{H}_{V_{\varphi_i}}[0,S]\prod_{i=d_1+1}^{d_2}\mathcal{P}^{h_i}_{V_{\varphi_i}}[0,S]\mathcal{H}^{\Gamma^*}_{\widetilde{V}_\varphi,\widetilde{h}}(E)\tag{54}$$

and

$$\mathcal{H}^{\Gamma}_{V_\varphi,h^*_0}(\widetilde{I}_0)=\prod_{i=1}^{d_1}\mathcal{H}_{V_{\varphi_i}}[0,S]\prod_{i=d_1+1}^{d_2}\mathcal{P}^{h_i}_{V_{\varphi_i}}[0,S]\mathcal{H}^{\Gamma^*}_{\widetilde{V}_\varphi,\widetilde{h}}(E),\tag{55}$$

with $V_{\varphi_i},\widetilde{V}_\varphi$ and $\widetilde{h}$ defined in (32) and (34). Using further the fact that (see e.g., Theorem 3.1 in [8])

$$\lim_{S\to\infty}\frac{\mathcal{H}_{V_{\varphi_i}}[0,S]}{S}=\mathcal{H}_{V_{\varphi_i}}\in(0,\infty),\quad 1\le i\le d_1$$

and letting $S\to\infty$ on the left side of (53), we have

$$\prod_{i=1}^{d_1}\mathcal{H}_{V_{\varphi_i}}\prod_{i=d_1+1}^{d_2}\lim_{S\to\infty}\mathcal{P}^{h_i}_{V_{\varphi_i}}[-S,S]\mathcal{H}^{\Gamma^*}_{\widetilde{V}_\varphi,\widetilde{h}}(E)\le S^{-d_1}\mathcal{H}^{\Gamma}_{V_\varphi,h^*_0}(\widetilde{I}_0)\left(1+\sum_{i=d_1+1}^{d_2}e^{-\mathbb{Q}S^{\beta_i}}\right)<\infty.$$

Thus we conclude that

$$\lim_{S\to\infty} \mathcal{P}_{V_{\varphi_i}}^{h_i}[-S,S] \in (0,\infty), \quad d_1+1 \le i \le d_2,$$

which establishes the claim by letting $S\to\infty$ on both sides of (53). For other cases of $a_i, b_i, d_1+1\le i\le d_2$, the proof is similar as above. □

Proof of Proposition 3.7 We have that for any S,T positive

$$0 < \mathcal{P}_X^b([0,S],[0,T]) \le \mathcal{P}_X^{b\sigma^2(t)}[0,T].$$

In order to complete the proof it suffices to prove that $\lim_{T\to\infty}\mathcal{P}_X^{b\sigma^2(t)}[0,T]<\infty$. For this purpose, define for any $S>0, u>1$

$$Y_u(t) = \frac{\overline{X}(u(t+1))}{1+\frac{b\sigma^2(ut)}{2\sigma^2(u)}}, \quad t\in[0,u^{-1}\ln u].$$

Note that

$$1-Cor\,(X(ut),X(us)) = \frac{\sigma^2(u|t-s|)-(\sigma(ut)-\sigma(us))^2}{2\sigma(ut)\sigma(us)} = \frac{\sigma^2(u|t-s|)-(u\dot{\sigma}(u\theta)(t-s))^2}{2\sigma(ut)\sigma(us)},$$

with $\theta\in[s,t]$. By **A1** and Theorem 1.7.2 in [5], it follows that

$$\lim_{u\to\infty}\frac{u\dot{\sigma}(u)}{\sigma(u)} = \alpha_\infty.$$

If we set $f(t)=t^2/\sigma^2(t)$, then by Lemma 5.2 in [16] it follows that f is bounded over any compact set and regularly varying at ∞ with index $2-2\alpha_\infty>0$. Consequently, UCT implies for any $S>0$

$$\lim_{u\to\infty}\sup_{t\in(0,S]}\left|\frac{f(ut)}{f(u)}-|t|^{2-2\alpha_\infty}\right| = 0$$

and therefore as $u\to\infty$

$$\begin{aligned} 1-Cor\,(X(ut),X(us)) &\sim \frac{\sigma^2(u|t-s|)}{2\sigma(ut)\sigma(us)}\left(1-\frac{\alpha_\infty^2}{\theta^2}\frac{\sigma^2(u\theta)(t-s)^2}{\sigma^2(u|t-s|)}\right)\\ &= \frac{\sigma^2(u|t-s|)}{2\sigma(ut)\sigma(us)}\left(1-\alpha_\infty^2\frac{f(u|t-s|)}{f(u\theta)}\right)\sim\frac{\sigma^2(u|t-s|)}{2\sigma^2(u)} \end{aligned} \tag{56}$$

for $s,t\in[1,1+u^{-1}\ln u]$. Let further

$$I_k(u) = [ku^{-1}S, u^{-1}(k+1)S], \quad 0\le k\le N(u), \text{ with } N(u):=[S^{-1}\ln u]+1.$$

It follows that for S sufficiently large

$$p_0(u) \le \mathbb{P}\left\{\sup_{t\in[0,u^{-1}\ln u]} Y_u(t) > \sqrt{2}\sigma(u)\right\} \le p_0(u)+\sum_{k=1}^{N(u)} p_k(u), \tag{57}$$

where

$$p_0(u) = \mathbb{P}\left\{\sup_{t\in I_0(u)} Y_u(t) > \sqrt{2}\sigma(u)\right\},$$

$$p_k(u) = \mathbb{P}\left\{\sup_{t\in I_k(u)} \overline{X}(u(t+1)) > \sqrt{2}\sigma(u)\left(1+\frac{b\sigma^2(kS)}{4\sigma^2(u)}\right)\right\}, \quad k\ge 1.$$

In order to apply Theorem 2.1, in view of (56) we set (using the notation in Theorem 2.1)

$$K_u = \{k: 0\le k\le N(u)\}, \quad E=[0,S], \quad g_{u,k} = \sqrt{2}\sigma(u)\left(1+\frac{b\sigma^2(kS)}{4\sigma^2(u)}\right), k\in K_u, \tag{58}$$

$$Z_{u,k}(t) = \overline{X}(u(u^{-1}kS+u^{-1}t+1)), \quad k\in K_u,$$

$$\theta_{u,k}(s,t) = g_{u,k}^2\frac{\sigma^2(|t-s|)}{2\sigma^2(u)}, \quad s,t\in E, k\in K_u,$$

$$h_{u,0}(t)=\frac{b\sigma^2(t)}{2\sigma^2(u)},\quad t\in E,\quad h_{u,k}=0,\quad k\in K_u\setminus\{0\},\quad \eta=X.$$

C0 and **C2** are obviously fulfilled. **C1** is also satisfied with

$$g_{u,0}^2h_{u,0}(t)\to b\sigma^2(t),\quad u\to\infty$$

uniformly with respect to $t\in E$ and

$$g_{u,k}^2h_{u,k}(t)=0,\quad t\in E,k\in K_u\backslash\{0\},\quad u>0$$

Next we shall verify **C3**. Clearly by **A2** for u sufficiently large

$$\theta_{u,k}(s,t)=g_{u,k}^2\frac{\sigma^2(|t-s|)}{2\sigma^2(u)}\le 2\sigma^2(|t-s|)\le Q|t-s|^{\alpha_0},\quad s,t\in E,k\in K_u.$$

Moreover, by (56)

$$\begin{aligned}
&\sup_{k\in K_u}\sup_{\|t-s\|<\epsilon,s,t\in E}g_{u,k}^2\mathbb{E}\left\{\left[Z_{u,k}(t)-Z_{u,\tau}(s)\right]Z_{u,k}(0)\right\}\\
&\le \sup_{k\in K_u}\sup_{\|t-s\|<\epsilon,s,t\in E}g_{u,k}^2\left(\frac{\sigma^2(t)}{2\sigma^2(u)}(1+o(1))-\frac{\sigma^2(s)}{2\sigma^2(u)}(1+o(1))\right)\\
&\le \sup_{k\in K_u}\sup_{\|t-s\|<\epsilon,s,t\in E}\frac{g_{u,k}^2}{2\sigma^2(u)}\left(|\sigma^2(t)-\sigma^2(s)|+o(1)\right)\to 0,\quad u\to\infty,\epsilon\downarrow 0.
\end{aligned}$$

Thus **C3** is satisfied. Therefore, in light of Theorem 2.1, we have that

$$\lim_{u\to\infty}\frac{p_0(u)}{\Psi(\sqrt{2}\sigma(u))}=\mathcal{P}_X^{b\sigma^2(t)}[0,S]$$

and

$$\lim_{u\to\infty}\sup_{k\in K_u/\{0\}}\left|\frac{p_k(u)}{\Psi\left(\sqrt{2}\sigma(u)\left(1+\frac{b\sigma^2(kS)}{4\sigma^2(u)}\right)\right)}-\mathcal{H}_X[0,S]\right|=0. \tag{59}$$

Dividing (57) by $\Psi(\sqrt{2}\sigma(u))$, letting $u\to\infty$ and by **A1**, we have that for sufficiently large S_1

$$\begin{aligned}
\mathcal{P}_X^{b\sigma^2(t)}[0,S] &\le \mathcal{P}_X^{b\sigma^2(t)}[0,S_1]+\mathcal{H}_X[0,S_1]\sum_{k=1}^{\infty}e^{-\frac{b\sigma^2(kS_1)}{2}}\\
&\le \mathcal{P}_X^{b\sigma^2(t)}[0,S_1]+\mathcal{H}_X[0,S_1]\sum_{k=1}^{\infty}e^{-Q_1(kS_1)^{\alpha_\infty}}\\
&\le \mathcal{P}_X^{b\sigma^2(t)}[0,S_1]+\mathcal{H}_X[0,S_1]e^{-Q_2S_1^{\alpha_\infty}}.
\end{aligned}$$

Next, letting $S\to\infty$ leads to

$$\lim_{S\to\infty}\mathcal{P}_X^{b\sigma^2(t)}[0,S]\le\mathcal{P}_X^{b\sigma^2(t)}[0,S_1]+\mathcal{H}_X[0,S_1]e^{-Q_2S_1^{\alpha_\infty}}<\infty$$

establishing the claim.

□

5. Appendix

Proof of Remark 2.2 ii). First we suppose that **C2** and (12) hold. Our aim is to prove (16). By (12), the continuity of $\sigma_\eta^2(t), t \in E$ and the compactness of E, for any $c > 0$, there exists a constant $\epsilon := \epsilon_c > 0$ such that

$$\limsup_{u\to\infty} \sup_{\tau_u \in K_u} \sup_{\|t-s\|<\epsilon, s,t\in E} \left| g_{u,\tau_u}^2 Var\left(b_u(s)\right) - g_{u,\tau_u}^2 Var\left(b_u(t)\right)\right| < c/3,$$

with $b_u(t) = Z_{u,\tau_u}(t) - Z_{u,\tau_u}(0)$ and further

$$\sup_{\|t-s\|<\epsilon, s,t\in E} \left|\sigma_\eta^2(t) - \sigma_\eta^2(s)\right| < c/3.$$

By the compactness of E, we can find $E_c \subset E$ which has a finite number of elements such that for any $t \in E$

$$O_\epsilon(t) \cap E_c \neq \emptyset, \quad O_\epsilon(t) := \{s \in \mathbb{R}^d : \|t - s\| < \epsilon\}.$$

For any $t \in E$, with $t' \in O_\epsilon(t) \cap E_c$

$$\begin{aligned}\left|g_{u,\tau_u}^2 Var\left(b_u(t)\right) - 2\sigma_\eta^2(t)\right| \quad &\leq \quad \left|g_{u,\tau_u}^2 Var\left(b_u(t)\right) - g_{u,\tau_u}^2 Var\left(b_u(t')\right)\right| \\ &\quad + 2\left|\sigma_\eta^2(t) - \sigma_\eta^2(t')\right| + \left|g_{u,\tau_u}^2 Var\left(b_u(t')\right) - 2\sigma_\eta^2(t')\right|.\end{aligned}$$

It follows from **C2** that

$$\lim_{u\to\infty} \sup_{\tau_u\in K_u} \left|g_{u,\tau_u}^2 Var(b_u(t)) - 2\sigma_\eta^2(t)\right| = 0, \quad t \in E.$$

Consequently, we have

$$\begin{aligned}&\limsup_{u\to\infty} \sup_{\tau_u\in K_u} \sup_{t\in E}\left|g_{u,\tau_u}^2 Var\left(b_u(t)\right) - 2\sigma_\eta^2(t)\right| \\ &\leq \limsup_{u\to\infty} \sup_{\tau_u\in K_u} \sup_{\|t-s\|<\epsilon, s,t\in E}\left|g_{u,\tau_u}^2 Var\left(b_u(s)\right) - g_{u,\tau_u}^2 Var\left(b_u(t)\right)\right| \\ &\quad +2 \sup_{\|t-s\|<\epsilon, s,t\in E}\left|\sigma_\eta^2(t) - \sigma_\eta^2(s)\right| + \limsup_{u\to\infty} \sup_{\tau_u\in K_u} \sup_{t\in E_c}\left|g_{u,\tau_u}^2 Var\left(b_u(t)\right) - 2\sigma_\eta^2(t)\right| \\ &\leq c.\end{aligned}$$

Hence letting c to 0 yields (16).

Next, supposing that **C2** and (16) hold, we prove (12). By the continuity of $\sigma_\eta^2(t), t \in E$ and the compactness of E, for any $c > 0$, there exists a constant $\epsilon > 0$ such that

$$\sup_{\|t-s\|<\epsilon, s,t\in E} \left|\sigma_\eta^2(t) - \sigma_\eta^2(s)\right| < c/3.$$

For any $s, t \in E$

$$\begin{aligned}\left|g_{u,\tau_u}^2 Var\left(b_u(s)\right) - g_{u,\tau_u}^2 Var\left(b_u(t)\right)\right| \quad &\leq \quad \left|g_{u,\tau_u}^2 Var\left(b_u(s)\right) - 2\sigma_\eta^2(s)\right| + 2|\sigma_\eta^2(s) - \sigma_\eta^2(t)| \\ &\quad + \left|2\sigma_\eta^2(t) - g_{u,\tau_u}^2 Var\left(b_u(t)\right)\right|.\end{aligned}$$

Consequently, by (16)

$$\begin{aligned}&\limsup_{u\to\infty} \sup_{\tau_u\in K_u} \sup_{\|t-s\|<\epsilon, s,t\in E}\left|g_{u,\tau_u}^2 Var\left(b_u(s)\right) - g_{u,\tau_u}^2 Var\left(b_u(t)\right)\right| \\ &\leq \quad 2\limsup_{u\to\infty} \sup_{\tau_u\in K_u} \sup_{t\in E}\left|g_{u,\tau_u}^2 Var\left(b_u(t)\right) - 2\sigma_\eta^2(t)\right| + 2\sup_{\|t-s\|<\epsilon, s,t\in E}\left|\sigma_\eta^2(t) - \sigma_\eta^2(s)\right| \\ &\leq \quad c.\end{aligned}$$

Letting $c \to 0$, the above establishes (12), which completes the proof. □

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