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# Mixed participating and unit-linked life insurance contracts: design, pricing and optimal strategy

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**ABSTRACT**

In many countries, the decline in interest rates has reduced the interest in traditional participating life insurance contracts with investment guarantees and has led to a shift to unit-linked policies without guarantees. We design a novel mixed insurance contract splitting premium payments between a participating and a unit-linked fund. An additional guarantee fee is applied on the unit-linked return in order to increase the investment guarantee of the participating fund. In a utility-based framework, using power utility and prospect theory as preference functions, we show that the mixed product is usually perceived more attractive than a full investment in either the unit-linked or the participating contract. The guarantee fee is beneficial for conservative investors interested in stronger protection against losses. This is also interesting from a marketing perspective: By the increase of the guarantee in the participating product, zero or negative guaranteed rates can be avoided.

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## 1. Introduction

In many countries, the industry of life insurance has traditionally focused on products with guarantees. The typical example is given by participating life insurance contracts (or life insurance with profit), where the insurer offers ex ante a return guarantee and afterwards provides the customer a participation in case the actual return exceeds this return guarantee. These participation schemes may have different forms such as reversionary bonus or terminal bonus (Bacinello 2001, Bohnert & Gatzert 2011, Hieber et al. 2019). The premiums are usually invested in the general fund of the insurer with no possible financial decision from the customer. In parallel to this activity, new forms of life insurance products have emerged for decades in many markets under the form of unit-linked contracts or equity-linked contracts. In these products, the financial risk is transferred to the investor allowing a more risky investment strategy. The investor can then realize his choices between several funds. In some countries, additional guarantees can be included in the product (Boyle & Schwartz 1977, Brennan & Schwartz 1979, Hansen & Miltersen 2002, Bauer et al. 2008, Bacinello et al. 2011, Mackay et al. 2015, Hieber 2017).

The recent evolution of the financial market, characterized by persistent low (or even negative) interest rates and volatile equity markets, threatens the existence of attractive guarantees from the life insurer and makes, therefore, the offer of participating contracts very challenging. Then, the unit-linked products could appear as the only future for life insurance. Removing completely

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the guarantees has been indeed an important trend for insurers in many markets these last decades. Nevertheless, it is confirmed by many empirical and experimental studies that the customers are risk-averse and specifically interested in a downside protection by guarantees (Arkes et al. 2008, Gatzert et al. 2011, Knoller 2016, Ru$\beta$ & Schelling 2021). The demand and interest for guarantees from the customers have not disappeared. The importance to provide some form of guarantees has been also emphasized by the European Commission in the context of the recent draft regulation on a new Pan European Pension Product (European Insurance and Occupational Pensions Authority 2021), where the providers should propose possibilities of capital protection for the customers. One solution to keep some form of guarantees in this challenging financial environment is hybrid or mixed products including partial or lighter guarantees. This is a promising area of development, in order to find intermediate solutions between pure participating products and pure equity-linked policies. As observed by Hambardzumyan & Korn (2019), finding a good compromise between the search for potential returns and the possibility to generate guarantees is a major issue for the life insurance industry.

The ability of the product to generate guarantees can also play an important role when legal guarantees are imposed by the regulator or incentivized by the state (for instance in Belgium, the minimum guaranteed rate to offer in occupational pensions, Devolder & De Valeriola 2017, or in Germany the reduced taxation for products with a so-called money-back guarantee, Alexandrova et al. 2017).

Mixed or hybrid products have already been considered in the literature (Bohnert 2013, Bohnert et al. 2014, Bohnert & Gatzert 2014) and developed in practice (for instance 'Select Products' in Germany, Alexandrova et al. 2017). 'Select Products' constitute a static hybrid insurance product where the investor can choose between the investment in a unit-linked index participation and a traditional participating life insurance contract. The detailed product design varies from insurer to insurer but the contracts typically also include a money-back guarantee (i.e. a 0% maturity guarantee) and possibly caps or floors on the unit-linked fund return, similar to equity-indexed annuities in the US (see, e.g. Bernard & Boyle 2011). Apart from static hybrid products, so-called 3-fund dynamic hybrid products have been introduced to the German insurance market (Kochanski & Karnarski 2011, Mahayni & Schneider 2016, Hambardzumyan & Korn 2019). The dynamic version includes a period rebalancing between the investment possibilities, typically a participating life insurance contract, a guarantee fund and a (unit-linked) equity fund. The dynamic rebalancing is used to finance and replicate additional guarantees, i.e. a money-back guarantee. The design allows to also adapt the product to the individual policyholder's needs (see also the related case of so-called flexibility riders in Chen et al. 2019).

In Belgium, new products have emerged these last years, called 'products of Branch 44', mixing a part in Branch 21 (participating life insurance contract with guaranteed rate) and a part in Branch 23 (pure unit-linked product) (Van Maldegem 2020).

In this paper, we introduce hybrid products based on a combination of a participating product and a pure unit-linked product. The latter is a product without any maturity guarantee, thereafter called 'unit-linked product'. Instead of considering just the superposition of these two parts, we create an automatic link between them, by a double mechanism. First, we allow contracts with or without periodical rebalancing between the two accounts. We also permit transfer from the unit-linked part to the guaranteed part by introducing a continuous fee on the unit-linked part. This technique is similar to the concept of guarantee fees in variable annuities (see, e.g. Bauer et al. 2008) and to well-known structured funds in finance offering only a part of the return of the market in exchange for a money-back guarantee. Our guarantee fee is chosen by the customer and works as a subsidy from the unit-linked part to increase the investment guarantee in the participating part. In the environment of low interest rates, this kind of internal subsidy inside the contract can boost the guaranteed part. In particular, we show that this technique permits the insurer to offer guarantees higher than the risk-free rate. This could be precious for two reasons. First, from a marketing point of view in a situation of negative returns, the insurer could propose some products with positive guaranteed rates (even if technically possible, life insurance contracts with a negative interest rate guarantee seem unrealistic). Second, we assume that the continuous fee in the unit-linked part is a fixed percentage that is due in both bad and good market scenarios. This fee is used to increase the downside protection of the investment boosting the guaranteed return in the participating contract. This leads to return distributions with a higher asymmetry, i.e. a lower return in good market scenarios but a higher downside protection. Such asymmetric return distributions are popular for many customers ( Ruβ & Schelling 2021). For the participating part, we look at various forms of guarantees (annual or maturity guarantees, and intermediate guarantees on a predefined number of years).

We analyze in the paper successively the point of view of the insurer and of the customer. Using a classical Black and Scholes market without mortality risk, we compute the fair value of various hybrid contracts. Generalizing the fairness relation between the parameters of a participating contract as developed by Bacinello (2001), we obtain different forms of equivalent relations depending on the parameters of the two parts and on the kind of guarantee chosen by the customer. The numerical results show the ability of the product to offer a range of guaranteed levels, including guarantees higher than the risk-free rate. In a second part, we look at the value of these contracts for the customer, using successively a utility approach and the cumulative prospect theory. These two preference functions were also considered by Døskeland & Nordahl (2008) and Chen et al. (2015) comparing several insurance products and a pure unit-linked product according to their expected utility. The analysis of the paper by Chen et al. (2015) shows that by incorporating realistic aspects like mortality, long time horizons and high risk-aversion, the demand for products with guarantee increases. Our results suggest that a hybrid product leads to a higher expected utility than the two special cases of a pure unit-linked and a participating product interpreted in Døskeland & Nordahl (2008) and Chen et al. (2015) for both expected utility and prospect theory. In particular, we obtain, for various risk aversion, optimal proportions between participating and unit-linked parts and optimal level of guarantee fees.

We organize the remainder of this paper as follows. In Section 2, we present the main financial assumptions used throughout the paper. Section 3 describes the mechanism of the hybrid product considered. In Section 4, we compute fair values and obtain various fairness relations between the parameters of the contract, depending on the kind of bonus. In Section 5, we look at optimal contracts for the customer in terms of mixing degree and level of fees. Section 6 concludes.

## 2. Basic financial and mortality assumptions

The insurance company has to take into account the mortality risk as well as the financial risk in order to fulfill their contractual obligations towards the customers. In this paper, we focus on the financial risk and ignore mortality risk assuming that these two risks are independent and that the mortality risk will not have an effect on the fairness relation of the contract (as illustrated, for example, in Bacinello 2001).

To follow the evolution of the wealth and the price dynamics, we assume a continuous-time model with Black–Scholes economy (Black & Scholes 1973) with the following assets:

(1) A money market account which is a risk-free asset denoted by $B$ earning a constant interest rate $r \in \mathbb{R}$ with dynamics:

$$\mathrm{d}B_t = r\,B\,\mathrm{d}t, \quad B_0 = 1.$$

(2) Two risky funds $G$ and $H$, where $G$ is a general fund of the company (in general with low volatility), not directly traded on the market and used as a reference for the participating contracts. However, $H$ is an investment fund (with high volatility) traded on the market and used to compute the value of the unit-linked contract. These two funds are described as follows:

(2.a) The fund $G$ is following a geometric Brownian motion with dynamics:

$$\mathrm{d}G_t = \mu_G\,G_t\,\mathrm{d}t + \sigma_G\,G_t\,\mathrm{d}W_t^G, \quad G_0 = 1, t \in [0, T], \tag{1}$$

where $W^G$ is a standard Brownian motion under the real probability measure $\mathbb{P}$ and $\sigma_G > 0$.In the following, we will discretize time in $n \in \mathbb{N}$ periods of equal length $\Delta s = \frac{T}{n}$.

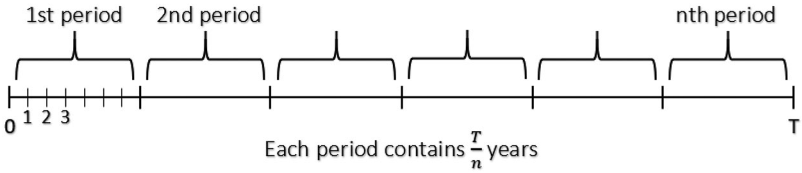


**Figure 1.** Representation of a $\frac{T}{n}$-years product.

Figure 1 illustrates $n$ periods with maturity $T$ where each period is made up of $\Delta s = \frac{T}{n}$ years. These periods are counted as $t = 1, 2, \ldots, n$.The period return of $G$ is defined as follows:

$$g_{t\cdot\Delta s} = \frac{G_{t\cdot\Delta s}}{G_{(t-1)\cdot\Delta s}} - 1, \quad t = 1, 2, \ldots, n. \tag{2}$$

It is well known that the solution of the stochastic differential equation (1), under the real probability measure $\mathbb{P}$, is $G_{t\cdot\Delta s} = G_0 \exp((\mu_G - \frac{\sigma_G^2}{2}) \cdot t \cdot \Delta s + \sigma_G \cdot W^G_{t\cdot\Delta s})$. Hence, $1 + g_{t\cdot\Delta s} = \exp((\mu_G - \frac{\sigma_G^2}{2}) \cdot \Delta s + \sigma_G(W^G_{(t+1)\cdot\Delta s} - W^G_{t\cdot\Delta s}))$ are independent and log-normally distributed for all $t$.

(2.b) The fund $H$ is following a geometric Brownian motion with dynamics:

$$\mathrm{d}H_t = \mu_H H_t \,\mathrm{d}t + \sigma_H \rho H_t \,\mathrm{d}W^G_t + \sigma_H \sqrt{1-\rho^2} H_t \,\mathrm{d}W_t, \quad H_0 = 1, t \in [0, T], \tag{3}$$

where $\rho \in ]-1, 1[$ is the parameter of correlation between the log-returns of the two funds $H$ and $G$, and $W$ is a standard Brownian motion under the real probability measure $\mathbb{P}$, independent of $W^G$ and $\sigma_H > 0$. We write $\mathrm{d}W^H_t = \rho \cdot \mathrm{d}W^G_t + \sqrt{1-\rho^2} \cdot \mathrm{d}W_t$.Similarly to the return of the fund $G$, the period return of the risky fund is defined as follows:

$$h_{t\cdot\Delta s} = \frac{H_{t\cdot\Delta s}}{H_{(t-1)\cdot\Delta s}} - 1, \quad t = 1, 2, \ldots, n. \tag{4}$$

Noting that the solution of the stochastic differential equation (3), under the real probability measure $\mathbb{P}$, is $H_{t\cdot\Delta s} = H_0 \exp((\mu_H - \frac{\sigma_H^2}{2}) \cdot t \cdot \Delta s + \sigma_H \rho W^G_{t\cdot\Delta s} + \sigma_H \sqrt{1-\rho^2} W_{t\cdot\Delta s})$ and then $1 + h_{t\cdot\Delta s} = \exp((\mu_H - \frac{\sigma_H^2}{2}) \cdot \Delta s + \sigma_H \cdot \rho \cdot (W^G_{(t+1)\cdot\Delta s} - W^G_{t\cdot\Delta s}) + \sigma_H \cdot \sqrt{1-\rho^2} \cdot (W_{(t+1)\cdot\Delta s} - W_{t\cdot\Delta s}))$ are also independent and log-normally distributed for all $t$.

Since we assume that the interest rate $r$ is deterministic and constant, the uncertainty is limited to the stochastic evolution of the portfolio and is reflected by the standard Brownian motions $W^G$ and $W$ under the real probability measure $\mathbb{P}$. These standard Brownian motions are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in the time interval $[0, T]$.

We further assume that the investment is riskier in the second fund than in the first fund, that is: $\mu_H > \mu_G > r$, $\mu_G$ and $\mu_H \in \mathbb{R}$ and $\sigma_H > \sigma_G$.

We consider the financial market to be complete and arbitrage-free. Let $\mathbb{Q}$ be the risk-neutral measure used for pricing in this market as per Harrison & Kreps (1979). Using Girsanov's theorem, we can write the change of measure between the real-world measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$

as

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\,|_{\mathbb{E}_t} = Z_t = e^{-\frac{1}{2}\|\theta\|^2 t - \theta \overline{W}_t}, \quad \text{where } \theta = \begin{pmatrix} \frac{\mu_G - r}{\sigma_G} \\ \frac{\mu_H - r}{\sigma_H\sqrt{1-\rho^2}} - \frac{\rho(\mu_G - r)}{\sigma_G\sqrt{1-\rho^2}} \end{pmatrix} \quad \text{and} \quad \overline{W}_t = \begin{pmatrix} W_t^G \\ W_t \end{pmatrix}$$

is a two-dimensional $\mathbb{P}$-Brownian motion.

Then, $\begin{pmatrix} \widetilde{W}_t^G \\ \widetilde{W}_t \end{pmatrix} = \widetilde{W}_t = \overline{W}_t + \theta \cdot t$ is a two-dimensional $\mathbb{Q}$-Brownian motion.

## 3. Structure of the policy

We consider a life insurance contract with a single premium $P$ paid at the inception of the contract. The contract pays a lump sum at maturity $T > 0$. A life insurance company decides on investing the customers' contributions such that a part is invested dynamically in a participating life insurance contract (for example, the 'Branch 21' insurance in Belgium) while the remaining is invested in a pure unit-linked contract (known as 'Branch 23' in Belgium). In the following, we point out the difference between these two contracts:

A participating life insurance contract is a rather safe investment since it guarantees a minimum investment return known in advance, in addition to a possible bonus participating in the investment profits of the insurer's asset portfolio. More precisely, if the real rate of return exceeds the guaranteed rate, the difference between a part of the real return and the guaranteed rate is granted to the insured. In each period, the insured receives a surplus return of

$$\delta_{t\cdot\triangle s} = \max(\beta \cdot \text{real rate of return}_{t\cdot\triangle s} - \text{guaranteed rate of return}, 0),$$

where $\beta \in [0, 1]$ is the participation level assumed to be constant over time, the guaranteed minimum rate of return, thereafter called $\tilde{i}$, is a technical interest rate that the insurance company determines at contract initiation while the real rate of return is a market rate acquired from the insurer's asset portfolio. We look at different guarantee periods ranging between one year (annual guarantee), several years (maturity guarantee) or simply by an arbitrary guarantee period $\triangle s$ (see Figure 1). We note that the rate of return $\tilde{i}$ is the total guarantee given on a period of $\frac{T}{n}$ and can be computed as $\tilde{i} = (1+i)^{\frac{T}{n}} - 1$, for an annual technical interest rate $i$.

The participating life insurance policy invests into the fund $G$ defined in Section 2 and its value is computed by investing the premium with the guaranteed interest rate and adding the bonus rate in good years. So, the total return on the participating fund is the guaranteed interest rate $\tilde{i}$ in addition to a surplus return $\delta_{t\cdot\triangle s} = (\beta \cdot g_{t\cdot\triangle s} - \tilde{i})^+$.

A pure unit-linked contract is more attractive for investors who are likely to take more risks to pursue higher returns since the return depends on the investment funds to which this policy is linked to. Referring to Bacinello (2002), this type of insurance offers advantages to the customers in terms of higher returns and the flexibility to choose or change the financial asset to which the policy is linked as well as it allows the insurance company to offer competitive products in the market. In contrast to classical life insurance products, the returns of a pure unit-linked contract are more volatile which makes the benefit at maturity uncertain and not guaranteed. There is no investment guarantee provided by the insurer.

We assume that the investment in the unit-linked part is subject to a continuous guarantee fee $\varepsilon$ used to increase the guarantee of the participating part of the mixed product. The dynamics of this investment $F_t := e^{-\varepsilon t} H_t$ is given by

$$\mathrm{d}F_t = (\mu_H - \varepsilon)\, F_t\, \mathrm{d}t + \sigma_H\, \rho\, F_t\, \mathrm{d}W_t^G + \sigma_H \sqrt{1-\rho^2}\, F_t\, \mathrm{d}W_t, \quad H_0 = 1,\ t \in [0, T], \tag{5}$$

where the period return of the unit-linked fund is

$$f_{t\cdot\triangle s} = \frac{F_{t\cdot\triangle s}}{F_{(t-1)\cdot\triangle s}} - 1, \quad t = 1, 2, \ldots, n. \tag{6}$$

Under the real probability measure $\mathbb{P}$, $1+f_{t\cdot\Delta s} = \exp((\mu_H - \varepsilon - \frac{\sigma_H^2}{2})\cdot\Delta s + \sigma_H\cdot\rho\cdot(W^G_{(t+1)\cdot\Delta s} - W^G_{t\cdot\Delta s}) + \sigma_H\cdot\sqrt{1-\rho^2}\cdot(W_{(t+1)\cdot\Delta s} - W_{t\cdot\Delta s}))$ are independent and log-normally distributed for all $t$.

The guarantee fee $\varepsilon$ is a continuous-time dividend extracted from process $H$ and credited to the guaranteed part.

Basically, the benefit granted to the customer is a function of the guaranteed interest rate, the surplus return and the rate of return from the investment in a risky fund.

More specifically, the contract is a combination of the participating life insurance and the unit-linked contract where

- a constant proportion $x \in ]0,1]$ of the financial portfolio is invested dynamically in a participating life insurance contract with period return $\tilde{i}+\delta_{t\cdot\Delta s}$. The real return is modeled by a fund $G$;
- a constant proportion $1-x$ is invested in a risky fund $F$ without guarantee.

With this investment, we analyze two different cases:

- *Rebalancing case* : After each time period, the investment share is rebalanced such that, again, a constant proportion $x \in ]0,1]$ is invested in the participating contract. Here is the benefit at maturity for a mixed product:

$$V_T = V_0\cdot\prod_{t=1}^{n}\left[x\cdot\left(\tilde{i}+\left(\beta\cdot g_{t\cdot\frac{T}{n}}-\tilde{i}\right)^+\right)+(1-x)\cdot\left(1+f_{t\cdot\frac{T}{n}}\right)\right]. \quad (7)$$

- *Non − rebalancing case* : The proportion $x$ is set once at the beginning of the contract. No further action is taken, so the benefit at maturity is written as follows:

$$V_T = V_0\cdot\left[x\cdot\prod_{t=1}^{n}\left(\tilde{i}+\left(\beta\cdot g_{t\cdot\frac{T}{n}}-\tilde{i}\right)^+\right)+(1-x)\cdot\prod_{t=1}^{n}\left(1+f_{t\cdot\frac{T}{n}}\right)\right]. \quad (8)$$

## 4. Valuation of the contract

Apart from the single premium, no payments occur until the maturity of the contract. Let us point out also that the guarantee can be given by the insurer on an annual basis ($n = T$), periodical basis or at maturity ($n = 1$). In order to price and evaluate the different forms of contract and define their parameters, we designate two main concepts:

(a) *The retrospective reserve*: it corresponds to the initial premium capitalized each period, following the specifications of the contract.

We will denote this reserve by $V_{t\cdot\frac{T}{n}}$ $(t = 0,1,\ldots,n)$ with initial condition: $V_0 = P$. We give the example of an annual guarantee contract ($n = T$):

(a) •For a pure participating life insurance contract ($x = 1$) with an annual guarantee, this reserve is given by

$$V_{t\cdot\frac{T}{n}} = V_t = V_0\cdot\prod_{k=0}^{t-1}\left(1+i+(\beta\cdot g_{k+1}-i)^+\right) = V_{t-1}\cdot\left(1+i+(\beta\cdot g_t-i)^+\right).$$

For a pure unit-linked contract ($x = 0$) with an annual guarantee, this reserve is given by

$$V_{t \cdot \frac{T}{n}} = V_t = V_0 \cdot \prod_{k=0}^{t-1} \left(1 + f_{k+1}\right).$$

The last two formulas can be easily generalized for a mixed contract ($x \in ]0, 1[$):
*For a rebalanced portfolio:*

$$V_{t \cdot \frac{T}{n}} = V_0 \cdot \prod_{k=0}^{t-1} \Bigg[ x \cdot \left( (1+i)^{\frac{T}{n}} + \left( \beta \cdot g_{(k+1) \cdot \frac{T}{n}} - ((1+i)^{\frac{T}{n}} - 1) \right)^{+} \right)$$
$$+ (1-x) \cdot \left( 1 + f_{(k+1) \cdot \frac{T}{n}} \right) \Bigg].$$

*For a non-rebalanced portfolio:*

$$V_{t \cdot \frac{T}{n}} = V_0 \cdot \Bigg[ x \cdot \prod_{k=0}^{t-1} \left( (1+i)^{\frac{T}{n}} + \left( \beta \cdot g_{(k+1) \cdot \frac{T}{n}} - ((1+i)^{\frac{T}{n}} - 1) \right)^{+} \right)$$
$$+ (1-x) \cdot \prod_{k=0}^{t-1} \left( 1 + f_{(k+1) \cdot \frac{T}{n}} \right) \Bigg].$$

We note that the final retrospective reserve $V_T$ corresponds to the benefit paid at maturity to the customer. Particularly, we obtain the reserve of a pure participating contract and a pure unit-linked contract by, respectively, extracting $x = 1$ and $x = 0$ from the above mixed contract.

(b) *The prospective reserve (or fair valuation)*: it corresponds to the discounted risk-neutral expectation of the final retrospective reserve.
We denote the fair value at time $t \cdot \frac{T}{n}$ by $FV_{t \cdot \frac{T}{n}}$ ($t = 0, 1, \ldots, n$), i.e.

$$FV_{t \cdot \frac{T}{n}} = e^{-r(T - t \cdot \frac{T}{n})} \cdot \mathbb{E}^{\mathbb{Q}}[V_T \,|\, \mathcal{F}_{t \cdot \frac{T}{n}}].$$

We say that the contract is fairly priced if and only if the initial fair value is equal to the initial retrospective reserve:

$$FV_0 = V_0 = P.$$

In the next sections, we develop this condition of fairness for different kinds of mixed contracts.

### 4.1. Maturity guarantee

As known, the timing of the allocation of the guarantee and the profit sharing can be annually, periodically or can be applied only at maturity. In the latter case where $n = 1$, the investment is developing throughout the years until maturity without assessing the performance of the funds which means that this is a product with a maturity guarantee (that is sometimes called terminal bonus). In common sense, we can deduce that the rebalancing of the portfolio during the investment years does not make any sense. For this reason, we will price hereinafter the case of a non-rebalanced portfolio where the guarantee and the profit sharing are computed at maturity, while the rebalancing case will be hidden in such a sort of product.

#### 4.1.1. Maturity guarantee without rebalancing

In this type of contract, the guarantee does not have to interfere during the years of investment and so there is somehow a compensation between good and bad years. We can conclude that this is a kind of static investment that leads to the following benefit at maturity:

$$V_T = V_0\left[x\cdot\left((1+i)^T + \left(\beta\cdot\left(\frac{G_T}{G_0}-1\right)-((1+i)^T-1)\right)^+\right)+(1-x)\cdot\left(\frac{F_T}{F_0}\right)\right]$$

(see Equation (7) with $n = 1$ and $\tilde{i} = (1+i)^{\frac{T}{n}} - 1$).

The value of this contract is assumed by computing the time 0 price of $V_T$ under the risk-neutral expectation: $FV_0(V_T) = \mathbb{E}^{\mathbb{Q}}[e^{-rT}\cdot V_T]$. This end gives, after replacing the retrospective reserve by its expression:

$$\mathbb{E}^{\mathbb{Q}}[e^{-rT}\cdot V_T] = x\cdot V_0\cdot\left[e^{-rT}\cdot(1+i)^T + \beta\cdot e^{-rT}\cdot\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{G_T}{G_0}-\left(1+\frac{(1+i)^T-1}{\beta}\right)\right)^+\right]\right]$$
$$+ (1-x)\cdot V_0\cdot e^{-rT}\cdot\mathbb{E}^{\mathbb{Q}}\left[\frac{F_T}{F_0}\right].$$

The term $c_0^{\{T\}} = e^{-rT}\cdot\mathbb{E}^{\mathbb{Q}}[(\frac{G_T}{G_0}-(1+\frac{(1+i)^T-1}{\beta}))^+]$ is the time 0 value of a European call option with a maturity $T$, an initial price of 1 of the underlying fund $G$ and a strike price $1+\frac{(1+i)^T-1}{\beta}$. The value of this call option is given by Black–Scholes under the following formula:

$$c_0^{\{T\}} = \Phi(d_1) - \left(1+\frac{(1+i)^T-1}{\beta}\right)\cdot e^{-rT}\cdot\Phi(d_2),$$

$$\text{where } d_1 = \frac{(r+\frac{\sigma_G^2}{2})T - \ln(1+\frac{(1+i)^T-1}{\beta})}{\sigma_G\sqrt{T}},\quad d_2 = d_1 - \sigma_G\sqrt{T}$$

and $\Phi$ is the cumulative distribution function of a standard normal variable.

In addition, $\frac{F_T}{F_0}$ expresses $1+$ return of the fund $F$ from the inception of the contract till maturity. Based on Equation (6), by taking $n = 1$, the $\mathbb{Q}$-expectation of $\frac{F_T}{F_0}$ gives $e^{(r-\varepsilon)T}$.

After these indications, the contract is fairly priced by equalizing the expected present value of benefits to the single premium paid by the insured ($FV_0(V_T) = P$):

$$x\cdot V_0\cdot\left[e^{-rT}\cdot(1+i)^T + \beta\cdot c_0^{\{T\}}\right] + (1-x)\cdot V_0\cdot e^{-rT}\cdot e^{(r-\varepsilon)T} = V_0.$$

Eventually, the premium equivalence is as follows:

$$x\cdot\left[((1+i)e^{-r})^T + \beta\cdot c_0^{\{T\}}\right] + (1-x)\cdot e^{-\varepsilon T} = 1. \tag{9}$$

This relation gives us a fair condition between all the parameters in order to generate a fair contract. It creates a link between the two components of the mixed product. In particular, we can interpret the relation as a feedback between the guaranteed rate $i$ and the guarantee fee $\varepsilon$, depending on the allocation level $x$ between these two parts.

### 4.2. $\frac{T}{n}$-years guarantee

We introduce a new product in which the bonus is incorporated not at maturity but each period, as represented in Figure 1. It is considered as a periodical guarantee contract since the guarantee and the profit sharing are investigated and computed each period $\Delta s$ (or each $\frac{T}{n}$ years). Similarly, in this type of contract, we differentiate between periodically rebalancing or not rebalancing our portfolio which is discussed, respectively, in Sections 4.2.1 and 4.2.2.

#### 4.2.1. $\frac{T}{n}$-years guarantee with rebalancing

Normally, the portfolio is rebalanced just after providing the guarantee and computing the profit sharing. Therefore, we rebalance the portfolio each period in order to maintain the original level of risk. To construct the benefit, we reinvest the capitalized amount each period with the same proportion, $x$ in a participating contract and $1-x$ in a pure unit-linked contract:

$$V_T = V_0 \cdot \prod_{t=1}^{n} \left[ x \cdot \left( (1+i)^{\frac{T}{n}} + \left( \beta \cdot \left( \frac{G_{t\frac{T}{n}}}{G_{(t-1)\frac{T}{n}}} - 1 \right) - ((1+i)^{\frac{T}{n}} - 1) \right)^{+} \right) + (1-x) \cdot \left( \frac{F_{t\frac{T}{n}}}{F_{(t-1)\frac{T}{n}}} \right) \right]. \tag{10}$$

The time 0 price of $V_T$ is computed by applying the martingale approach which is exploited by the following $\mathbb{Q}$-expectation: $\mathbb{E}^{\mathbb{Q}}[e^{-rT} \cdot V_T]$. By replacing $V_T$ by its expression in Equation (10), taking into consideration the stochastic independence of the periodical rates $g_{t \cdot \triangle s}$ for all $t$, as well as for $f_{t \cdot \triangle s}$. After algebraic manipulations, we get

$$V_0 \cdot \prod_{t=1}^{n} \left[ x \cdot \left( (e^{-r} \cdot (1+i))^{\frac{T}{n}} + \beta \cdot e^{-r\frac{T}{n}} \cdot \mathbb{E}^{\mathbb{Q}} \left[ \left( \frac{G_{t\frac{T}{n}}}{G_{(t-1)\frac{T}{n}}} - \left( 1 + \frac{(1+i)^{\frac{T}{n}} - 1}{\beta} \right) \right)^{+} \right] \right) + (1-x) \cdot e^{-r\frac{T}{n}} \cdot \mathbb{E}^{\mathbb{Q}} \left[ \frac{F_{t\frac{T}{n}}}{F_{(t-1)\frac{T}{n}}} \right] \right].$$

The term

$$c_0^{\{\frac{T}{n}\}} = e^{-r\frac{T}{n}} \cdot \mathbb{E}^{\mathbb{Q}} \left[ \left( \frac{G_{t\frac{T}{n}}}{G_{(t-1)\frac{T}{n}}} - \left( 1 + \frac{(1+i)^{\frac{T}{n}} - 1}{\beta} \right) \right)^{+} \right]$$

is obviously the time 0 value of a European call option with maturity $\frac{T}{n}$ with the following formula:

$$c_0^{\{\frac{T}{n}\}} = \Phi(d_1) - \left( 1 + \frac{(1+i)^{\frac{T}{n}} - 1}{\beta} \right) \cdot e^{-r\frac{T}{n}} \cdot \Phi(d_2),$$

where

$$d_1 = \frac{(r + \frac{\sigma_G^2}{2})\frac{T}{n} - \ln(1 + \frac{(1+i)^{\frac{T}{n}} - 1}{\beta})}{\sigma_G \sqrt{\frac{T}{n}}}, \quad d_2 = d_1 - \sigma_G \sqrt{\frac{T}{n}}$$

and $\Phi$ is the cumulative distribution function of a standard normal variable.

Recalling Equation (6), the $\mathbb{Q}$-expectation of

$$\frac{F_{t\frac{T}{n}}}{F_{(t-1)\frac{T}{n}}}$$

gives $e^{\frac{T}{n}(r-\varepsilon)}$.

To this end, we can deduce the explicit formula of the premium equivalence:

$$x \cdot \left[ ((1+i) \cdot e^{-r})^{\frac{T}{n}} + \beta \cdot c_0^{\{\frac{T}{n}\}} \right] + (1-x) \cdot e^{-\varepsilon\frac{T}{n}} = 1. \tag{11}$$

It is worth mentioning that we have computed the reserve at time 0 to obtain the fairness relation between the parameters. As well, we can compute the reserve at any moment where the moment in this case is expressed periodically. For instance, we can express the time $t \cdot \frac{T}{n}$ price of $V_T$ by solving

$$FV_{t \cdot \frac{T}{n}} = e^{-r(T - t \cdot \frac{T}{n})} \cdot \mathbb{E}^{\mathbb{Q}}[V_T \mid \mathcal{F}_{t \cdot \frac{T}{n}}], \quad t \leqslant n.$$

By replacing Equation (10) in the $\mathbb{Q}$-expectation and conditioning on the information available at time $t \cdot \frac{T}{n}$, we obtain the following:

$$FV_{t \cdot \frac{T}{n}} = V_{t \cdot \frac{T}{n}} \cdot \prod_{s=t}^{n-1} \left[ x \cdot \left( (e^{-r} \cdot (1+i))^{\frac{T}{n}} + \beta \cdot e^{-r\frac{T}{n}} \cdot \mathbb{E}^{\mathbb{Q}} \left[ \left( \frac{G_{(s+1)\frac{T}{n}}}{G_{s\frac{T}{n}}} - \left( 1 + \frac{(1+i)^{\frac{T}{n}} - 1}{\beta} \right) \right)^{+} \right] \right) \right.$$
$$\left. + (1-x) \cdot e^{-r\frac{T}{n}} \cdot \mathbb{E}^{\mathbb{Q}} \left[ \frac{F_{(s+1)\frac{T}{n}}}{F_{s\frac{T}{n}}} \right] \right].$$

Hence, the fair value will be $FV_{t \cdot \frac{T}{n}} = V_{t \cdot \frac{T}{n}} \cdot [x \cdot [((1+i) \cdot e^{-r})^{\frac{T}{n}} + \beta \cdot c_0^{\{\frac{T}{n}\}}] + (1-x) \cdot e^{-\varepsilon \frac{T}{n}}]^{n-t}$.

Consequently, if the contract is fairly priced at origin following condition (11), it remains fair in terms of the reserve at any time, by stating $FV_{t \cdot \frac{T}{n}} = V_{t \cdot \frac{T}{n}}$.

### 4.2.2. $\frac{T}{n}$-years guarantee without rebalancing

If the portfolio is not rebalanced each period, each contract is capitalized separately with the proportion invested in each asset at the beginning of the contract. At maturity, the structure of the retrospective reserve is in the following form:

$$V_T = V_0 \cdot \left[ x \cdot \prod_{t=1}^{n} \left( (1+i)^{\frac{T}{n}} + \left( \beta \cdot \left( \frac{G_{t\frac{T}{n}}}{G_{(t-1)\frac{T}{n}}} - 1 \right) - \left( (1+i)^{\frac{T}{n}} - 1 \right) \right)^{+} \right) \right.$$
$$\left. + (1-x) \cdot \prod_{t=1}^{n} \left( \frac{F_{t\frac{T}{n}}}{F_{(t-1)\frac{T}{n}}} \right) \right].$$

To value this contract, we calculate the time 0 price of $V_T$ under the risk-neutral measure and then deduce the premium equivalence in closed form. The fair value of the contract is written as follows:

$$FV_0(V_T) = e^{-rT} \cdot \mathbb{E}^{\mathbb{Q}}[V_T]$$
$$= V_0 \cdot \left[ x \cdot \prod_{t=1}^{n} \left( (e^{-r} \cdot (1+i))^{\frac{T}{n}} + \beta \cdot e^{-r\frac{T}{n}} \cdot \mathbb{E}^{\mathbb{Q}} \left[ \left( \frac{G_{t\frac{T}{n}}}{G_{(t-1)\frac{T}{n}}} - \left( 1 + \frac{(1+i)^{\frac{T}{n}} - 1}{\beta} \right) \right)^{+} \right] \right) \right.$$
$$\left. + (1-x) \cdot \prod_{t=1}^{n} e^{-r\frac{T}{n}} \cdot \mathbb{E}^{\mathbb{Q}} \left[ \frac{F_{t\frac{T}{n}}}{F_{(t-1)\frac{T}{n}}} \right] \right].$$

To write the premium equivalence, we simply connect the last expression to the premium paid at inception after considering the $\mathbb{Q}$-expectations as in Section 4.2.1 to obtain the following fairness relation:

$$x \cdot \left[ ((1+i) \cdot e^{-r})^{\frac{T}{n}} + \beta \cdot c_0^{\{\frac{T}{n}\}} \right]^{n} + (1-x) \cdot e^{-\varepsilon T} = 1. \quad (12)$$

### 4.2.3. Special case: annual guarantee ($n = T$)

We note that the annual guarantee (sometimes referred as reversionary bonus) is a special case from the $\frac{T}{n}$-years guarantee contract and is obtained by considering $n = T$ in Equations (11) and (12) for rebalancing and non-rebalancing contracts, respectively.

The premium equivalence of an annual guarantee with rebalancing is then

$$x \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1-x) \cdot e^{-\varepsilon} = 1,$$

and without rebalancing is

$$x \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1)^T + (1-x) \cdot e^{-\varepsilon T} = 1.$$

### 4.3. Numerical illustrations

In this section, we provide numerical solutions to the fairness relations obtained in each case. More precisely, we present some examples to Equations (9), (11) and (12) with respect to the technical interest rate $i$. The idea is to boost the minimum guaranteed rate by making a partial investment, which means, by investing a part in the unit-linked contract. This is because by increasing the guarantee fee $\varepsilon$, we are reinforcing the guaranteed rate $i$. For this reason, we will find thereafter solutions for the fairness relations with respect to $i$ according to different values of the investment share $x$ for each type of guarantee.

We consider a life insurance contract with a maturity $T = 20$ years. For the upcoming work, we suppose that the risk-free rate is $r = 1.5\%$ and the volatility of the guarantee fund is $\sigma_G = 3\%$. We note that the initial reserve $V_0$ does not need to be specified in the calculation of the fair guaranteed rate for the different forms of contracts.

#### 4.3.1. Numerical insights: $\frac{T}{n}$-years guarantee

By considering the general form of a partial guarantee contract which is the $\frac{T}{n}$-years guarantee with a number of periods $n = 4$, we vary the share $x$ invested in the participating contract and find the guaranteed rates $i$ that satisfy fairness relations (11) and (12) obtained in the theoretical part of the rebalancing and non-rebalancing cases, respectively, supposing that the insurance company has fixed all other parameters. Furthermore, we analyze the impact of the participation level $\beta$ and the guarantee fee $\varepsilon$ on these findings.

Briefly, we assess the change of $i$ with respect to different investment shares between the participating life insurance contract and the pure unit-linked contract. For this valuation, we consider a contract of 20 years divided in 4 periods with 5 successive years.

It is obvious from Table 1 the capacity of the insurance company to increase the guarantee by investing more in the unit-linked contract. In addition, we notice that a higher participation level leads to a lower guarantee; however, the more the company imposed fees the more it is able to give guarantees. This is confirmed in the first column of Table 1 where no fees are charged to the customer, which leads to a fixed guarantee regardless of the asset allocation $x$.

Besides, we can observe that the difference in the values of the guaranteed rates between a contract with rebalancing and without rebalancing can be almost negligible especially when a higher proportion is invested in the participating life insurance contract.

We note that, by taking $n = T$ and $n = 1$, we come across the annual guarantee and the maturity guarantee types, respectively. Hence, we will thereafter analyze the above points for these specific types of contracts.

#### 4.3.2. Numerical insights: annual guarantee

In the context of finding a solution to the fairness relation of an annual guarantee contract with respect to $i$, we assess the impact of the participation level $\beta$ for given values of $\varepsilon$, $r$ and $\sigma_G$ as well as the impact of the guarantee fee $\varepsilon$ for given values of $\beta$, $r$ and $\sigma_G$ on both rebalancing and non-rebalancing cases (Table 2 ).

We point out that, in both cases, the fair guaranteed rate $i$ increases with the decrease of the share in the participating contract $x$. This is trivial because the more the insurance company invests in the

**Table 1.** Fair guaranteed rates $i$ (expressed in %) with respect to the share in the participating contract $x$ for different participation levels $\beta$ and guarantee fees $\varepsilon$ - $n = 4$ *(T/n-years guarantee)*.

| | | With rebalancing | | | Without rebalancing | | |
|---|---|---|---|---|---|---|---|
| $x$ | $\beta = 70\%$ $\varepsilon = 0\%$ | $\beta = 70\%$ $\varepsilon = 0.25\%$ | $\beta = 50\%$ $\varepsilon = 0.25\%$ | $\beta = 70\%$ $\varepsilon = 0.5\%$ | $\beta = 70\%$ $\varepsilon = 0.25\%$ | $\beta = 50\%$ $\varepsilon = 0.25\%$ | $\beta = 70\%$ $\varepsilon = 0.5\%$ |
| 0.10 | 1.18 | 3.68 | 3.69 | 5.67 | 3.37 | 3.38 | 4.70 |
| 0.20 | 1.18 | 2.47 | 2.50 | 3.44 | 2.38 | 2.42 | 3.16 |
| 0.30 | 1.18 | 2.01 | 2.08 | 2.63 | 1.97 | 2.05 | 2.50 |
| 0.40 | 1.18 | 1.75 | 1.87 | 2.19 | 1.73 | 1.85 | 2.12 |
| 0.50 | 1.18 | 1.58 | 1.73 | 1.90 | 1.57 | 1.72 | 1.86 |
| 0.60 | 1.18 | 1.46 | 1.64 | 1.69 | 1.45 | 1.64 | 1.66 |
| 0.70 | 1.18 | 1.36 | 1.57 | 1.53 | 1.36 | 1.57 | 1.51 |
| 0.80 | 1.18 | 1.29 | 1.52 | 1.39 | 1.29 | 1.52 | 1.38 |
| 0.90 | 1.18 | 1.23 | 1.48 | 1.28 | 1.23 | 1.48 | 1.27 |
| 1.00 | 1.18 | 1.18 | 1.45 | 1.18 | 1.18 | 1.45 | 1.18 |

**Table 2.** Fair guaranteed rates $i$ (expressed in %) with respect to the share in the participating contract $x$ for different participation levels $\beta$ and guarantee fees $\varepsilon$ - $n = 20$ *(annual guarantee)*.

| | | With rebalancing | | | Without rebalancing | | |
|---|---|---|---|---|---|---|---|
| $x$ | $\beta = 70\%$ $\varepsilon = 0\%$ | $\beta = 70\%$ $\varepsilon = 0.25\%$ | $\beta = 50\%$ $\varepsilon = 0.25\%$ | $\beta = 70\%$ $\varepsilon = 0.5\%$ | $\beta = 70\%$ $\varepsilon = 0.25\%$ | $\beta = 50\%$ $\varepsilon = 0.25\%$ | $\beta = 70\%$ $\varepsilon = 0.5\%$ |
| 0.10 | 0.11 | 3.67 | 3.78 | 6.06 | 3.19 | 3.34 | 4.65 |
| 0.20 | 0.11 | 2.09 | 2.41 | 3.38 | 1.93 | 2.29 | 2.93 |
| 0.30 | 0.11 | 1.42 | 1.90 | 2.32 | 1.34 | 1.84 | 2.10 |
| 0.40 | 0.11 | 1.02 | 1.62 | 1.70 | 0.98 | 1.59 | 1.57 |
| 0.50 | 0.11 | 0.76 | 1.43 | 1.27 | 0.73 | 1.42 | 1.18 |
| 0.60 | 0.11 | 0.56 | 1.31 | 0.94 | 0.55 | 1.30 | 0.89 |
| 0.70 | 0.11 | 0.41 | 1.21 | 0.67 | 0.40 | 1.20 | 0.64 |
| 0.80 | 0.11 | 0.29 | 1.14 | 0.46 | 0.29 | 1.13 | 0.44 |
| 0.90 | 0.11 | 0.19 | 1.08 | 0.27 | 0.19 | 1.08 | 0.26 |
| 1.00 | 0.11 | 0.11 | 1.03 | 0.11 | 0.11 | 1.03 | 0.11 |

unit-linked contract, the less is its commitment to the customer, and thus, it is capable to give a higher guarantee. Moreover, we observe that, for a fixed guarantee fee $\varepsilon$, a higher participation level produces a low fair technical rate. For instance in the rebalancing case, taking $x = 50\%$ and $\varepsilon = 0.25\%$, we can guarantee to $i = 1.43\%$ for a participation level $\beta = 50\%$ and to $i = 0.76\%$ for a $\beta = 70\%$. However, for a fixed $\beta$, it is obvious that taking a higher fee results in a higher guarantee. Take an example in the rebalancing case, for $x = 50\%$ and $\beta = 70\%$, we can give a guarantee of $i = 1.27\%$ for $\varepsilon = 0.50\%$ and $i = 0.76\%$ for $\varepsilon = 0.25\%$. Whilst we have to note that in the case of $x = 1$, the technical interest rate is constant for different values of $\varepsilon$ and that is because we are going back to the classical life insurance product where the $\varepsilon$ does not have any effect. Similarly, the technical interest rate is constant when $\varepsilon$ is equal to 0 which confirms that $\varepsilon$ is used to boost $i$. Furthermore, we notice that the guarantee is slightly better in the rebalanced case, for all different values of $\beta$ and $\varepsilon$, since by rebalancing the portfolio we are avoiding somehow the exposure to undesirable risk. Overall, we can say that, in a partial guarantee product, we can get a much higher guaranteed rate.

We present, in Figure 2, the effect of the guarantee fee on the guaranteed rate taking all other parameters fixed.

It is noticeable that, in both cases, the fee does not have any effect when investing in a participating life insurance contract and then the more we invest in the unit-linked contract, the more the guaranteed rate is increasing. In addition, it is increasing with the increase of the guarantee fee $\varepsilon$, and besides, it goes up faster in a rebalanced portfolio than in a non-rebalanced portfolio.

Consequently, this figure confirms that the guarantee fee is financing the participating insurance contract to boost the guaranteed rate.

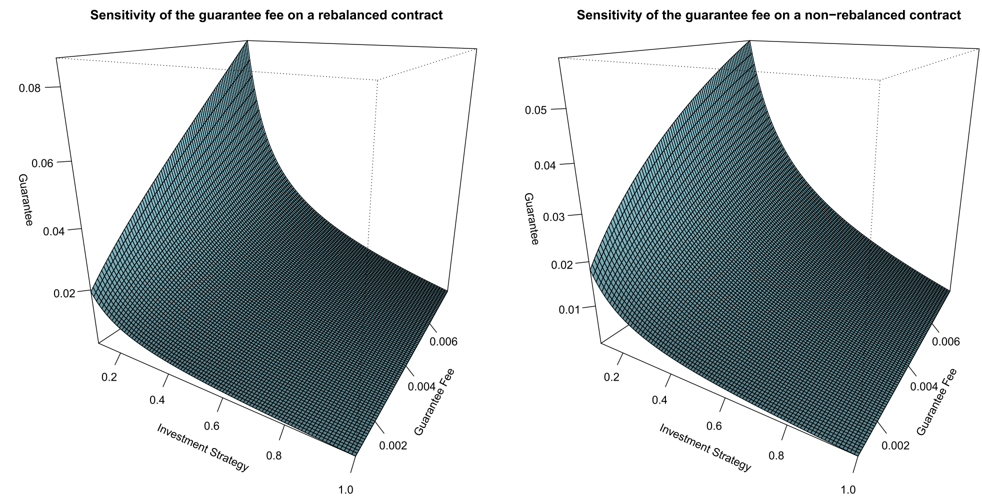


**Figure 2.** Sensitivity of the guarantee fee on the annual guarantee contract.

**Table 3.** Fair guaranteed rates *i* (expressed in %) with respect to *x* for different $\sigma_G$.

| | With rebalancing | | Without rebalancing | |
|---|---|---|---|---|
| $x$ | $\sigma_G = 3\%$ | $\sigma_G = 1\%$ | $\sigma_G = 3\%$ | $\sigma_G = 1\%$ |
| 0.10 | 3.67 | 3.79 | 3.19 | 3.38 |
| 0.20 | 2.09 | 2.52 | 1.93 | 2.41 |
| 0.30 | 1.42 | 2.08 | 1.34 | 2.03 |
| 0.40 | 1.02 | 1.84 | 0.98 | 1.82 |
| 0.50 | 0.76 | 1.69 | 0.73 | 1.68 |
| 0.60 | 0.56 | 1.58 | 0.55 | 1.58 |
| 0.70 | 0.41 | 1.50 | 0.40 | 1.50 |
| 0.80 | 0.29 | 1.44 | 0.29 | 1.44 |
| 0.90 | 0.19 | 1.39 | 0.19 | 1.39 |
| 1.00 | 0.11 | 1.35 | 0.11 | 1.35 |

We now examine the impact of the volatility coefficient $\sigma_G$ on the technical interest rate $i$ in order to satisfy the fairness relations of the rebalanced and non-rebalanced annual guarantee contract for given values of $\beta = 70\%$ and $\varepsilon = 0.25\%$ (Table 3).

It is obvious that for a very low $G$-fund volatility, the fairness relation is satisfied with a higher guaranteed rate comparing to the results for a $\sigma_G = 3\%$. For instance, with a high participation level of $\beta = 70\%$ and a guarantee fee of $\varepsilon = 0.25\%$ and with an investment of $x = 70\%$ of the premium in participating life insurance product, a volatility coefficient of $\sigma_G = 3\%$ leads to a fair technical rate around $i = 0.4\%$ while a volatility coefficient of $\sigma_G = 1\%$ leads to a fair technical rate of $i = 1.50\%$ in the with and without rebalancing cases.

In addition, we notice the effect of the volatility in the classical insurance product which means when making a full investment in the participating life contract ($x = 100\%$), the guaranteed rate is increased when we decrease $\sigma_G$.

### *4.3.3. Numerical insights: maturity guarantee*

Once again, we are going to give some solutions with respect to the guaranteed rate to hold the fair relation of the maturity guarantee without rebalancing determined by Equation (9) in Section 4.1.1. Table 4 analyzes the effect of the participating level and the guarantee fee on the positive guaranteed rates.

**Table 4.** Fair guaranteed rates $i$ (expressed in %) with respect to the share in the participating contract $x$ for different participation levels $\beta$ and guarantee fees $\varepsilon$ - $n = 1$ *(maturity guarantee)*.

| | | Without rebalancing | | |
|---|---|---|---|---|
| $x$ | $\beta = 70\%$ $\varepsilon = 0\%$ | $\beta = 70\%$ $\varepsilon = 0.25\%$ | $\beta = 50\%$ $\varepsilon = 0.25\%$ | $\beta = 70\%$ $\varepsilon = 0.5\%$ |
| 0.10 | 1.43 | 3.38 | 3.38 | 4.70 |
| 0.20 | 1.43 | 2.42 | 2.42 | 3.16 |
| 0.30 | 1.43 | 2.05 | 2.06 | 2.53 |
| 0.40 | 1.43 | 1.85 | 1.87 | 2.19 |
| 0.50 | 1.43 | 1.72 | 1.75 | 1.96 |
| 0.60 | 1.43 | 1.63 | 1.67 | 1.80 |
| 0.70 | 1.43 | 1.57 | 1.61 | 1.68 |
| 0.80 | 1.43 | 1.51 | 1.57 | 1.58 |
| 0.90 | 1.43 | 1.47 | 1.53 | 1.50 |
| 1.00 | 1.43 | 1.43 | 1.50 | 1.43 |

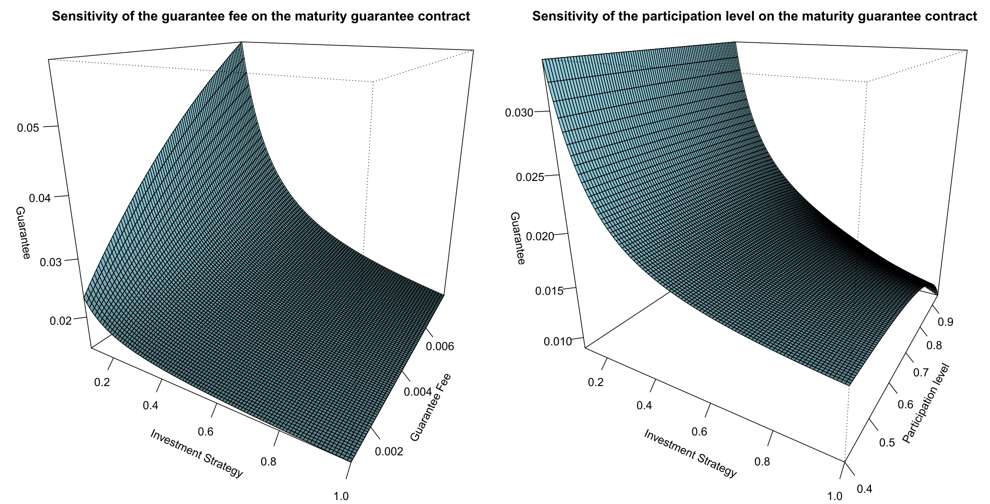


**Figure 3.** Sensitivity of the guarantee fee and the participation level on the maturity guarantee contract.

By fixing the guarantee fee $\varepsilon$ and looking at the different technical rates with respect to the participation level $\beta$, we notice that the more the investor is sharing the profits with the insurance company the lower will be the guarantee. For instance, when $\varepsilon$ is fixed at 0.25% and investing with a proportion of $x = 0.7$ of the premium, the fair guarantee is $i = 1.61\%$ for $\beta = 50\%$ against $i = 1.57\%$ for $\beta = 70\%$. Whilst by fixing the participation level $\beta$ and concentrating on the effect of the guarantee fee $\varepsilon$, the guarantee is absolutely increasing with the guarantee fee. We can check this result from Table 4 by taking $\beta = 70\%$ and $x = 0.7$, an $\varepsilon$ of 0.25% gives a fair guarantee of 1.57% while an $\varepsilon$ of 0.5% results in a fair guarantee of $i = 1.68\%$. The results can be checked graphically.

The first chart in Figure 3 confirms the increase of the guaranteed rate with the increase of the guarantee fee $\varepsilon$ as well as with a greater investment in the unit-linked part. On the other side, the impact of the participation level $\beta$ on the guaranteed rate is negative that is decreasing the more the insurance company is sharing its profits with the customer.

### *4.3.4. Comparison between annual and maturity guarantee contracts*

It is worth making a little comparison between the findings of the annual guarantee and the maturity guarantee. Two points can be noticeable:

**Table 5.** Solutions of Equations (9) with respect to the guaranteed rate $i$ for different maturities $T$ and volatilities $\sigma_G$.

| Maturity | Volatility (%) | Guarantee (%) |
|---|---|---|
| 20 | 10 | 0.60 |
| | 7 | 1.00 |
| | 3 | 1.43 |
| Volatility (%) | Maturity | Guarantee (%) |
| 3 | 20 | 1.43 |
| | 10 | 1.35 |
| | 1 | 0.11 |

(1) The participation level $\beta$ causes a small effect on the results of the technical interest rates in a fair maturity guarantee contract compared to the annual guarantee contract.
(2) When considering a pure participating insurance product, the guarantee is much higher in the maturity guarantee contract than it is in the annual guarantee. Table 5 displays different values of $T$ and $\sigma_G$ and their impacts on the fair guaranteed rates derived from Equation (9) for $\beta = 70\%$, $\varepsilon = 0.25\%$ and $x = 1$. Table 5 shows that the guarantee decreases with the decrease of the maturity $T$ for a given value of $\sigma_G$. Hence, the high value of the technical interest rate in the maturity guarantee is proved since the annual guarantee is a maturity guarantee with $T = 1$. In addition, the choice of a portfolio with a low volatility coefficient on a long maturity leads to a higher value of the guaranteed rate.

#### *4.3.5. Sensitivity analysis*

After analyzing the annual and maturity contracts, we can go back to the $\frac{T}{n}$-years guarantee contract to check the sensitivity of the volatility of the fund $G$, the maturity $T$ and the number of periods $n$ on the values of the technical interest rate with respect to $x$ and fixing all other parameters.

Starting by the impact of the volatility, we consider a $\frac{T}{n}$-years guarantee contract with the same assumptions as before in order to look at the variation of the technical rate.

Figure 4 shows the variation of the guarantee for different values of the parameter $\sigma_G$ and with respect to the investment strategy $x$. We notice that the volatility can have a significant effect when we invest more in a participating life insurance contract. Furthermore, the insurance company can give a better guarantee by investing in a portfolio less volatile in both contracts, with and without rebalancing.

Subsequently, we will examine the impact of the number of periods and maturity on the guaranteed interest rate. It is detected by fixing all the parameters and changing the interest rate according to the number of periods $n$ for two different maturities $T$ with respect to an investment in the participating life insurance contract and in the pure unit-linked contract.

In Figure 5, we consider a $\frac{T}{n}$-years guarantee contract with rebalancing. First, we fix the time to maturity and all other parameters and then we look at the variation of the guaranteed interest rate $i$ with respect to $x$ and for different values of $n$. We notice that the guarantee is increasing when we take a fewer number of periods $n$. It is worth mentioning that in the rebalancing portfolio, the number of periods is fixed from $n > 1$, since for $n = 1$ we come across the maturity guarantee contract. On the other side, to check the effect of the maturity $T$ on the guaranteed rate $i$ for a different number of periods, we observe that the guaranteed rate is increasing with the increase of $T$.

#### *4.3.6. Short-term versus long-term contracts*

Based on all the above different scenarios, we summarize with a comparison between a short- and a long-term contract where we consider, in each one, two investment cases in order to observe the variation in the guaranteed rate according to the different types of contracts that are supposed not to be rebalanced. In this comparison, we choose the same parameters $\beta$, $\sigma_G$, $\varepsilon$ and $n$ as 70%, 3%, 0.25%

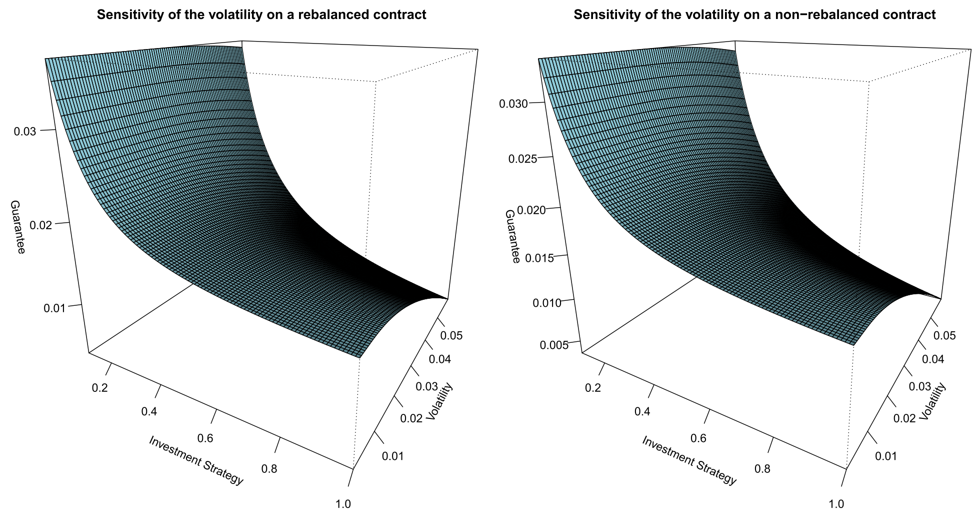


**Figure 4.** Sensitivity of the volatility $\sigma_G$ on a $\frac{T}{n}$-years guarantee contract.

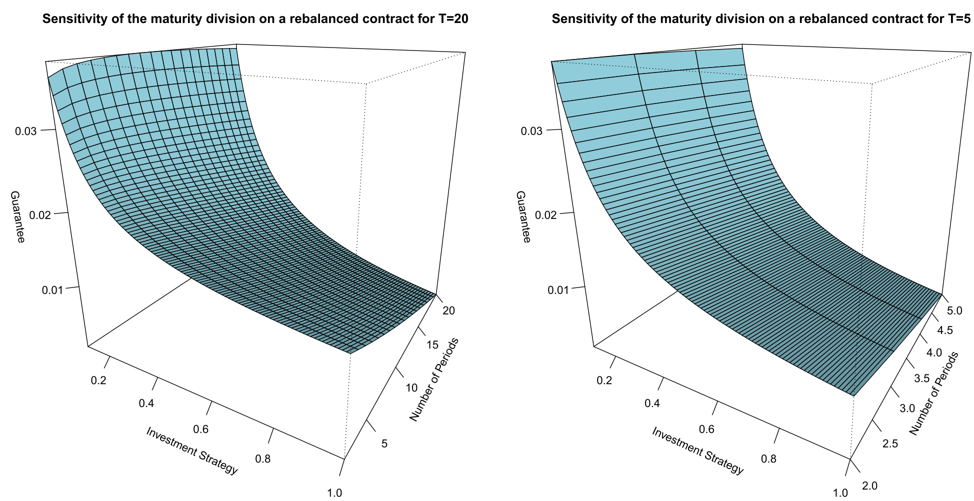


**Figure 5.** Sensitivity of the number of periods $n$ and the maturity $T$ on a $\frac{T}{n}$-years guarantee contract.

and 4, respectively. Whilst regarding the interest rate, we are going to construct the first table with the value assumed previously in the numerical illustrations $r = 1.5\%$ (Table 6) and in the second table we try to get closer to the current situation and take $r = 0.5\%$ (Table 7).

From a general look at the above two tables, it is obvious that the guarantee can be boosted in different ways:

- By considering a long-term contract because a higher time to maturity $T$ leads to a higher guarantee, except for the annual guarantee contract where the maturity $T$ does not have any effect.
- By combining a participating life insurance contract to a unit-linked contract.
- By turning into a maturity guarantee where the surplus does not occur until the end of the contract.

**Table 6.** Fair guaranteed rates $i$ (expressed in %) for different maturities $T$ when $r = 1.5\%$.

| | Pure participating product ($x = 1$) | Mixed product ($x = 0.5$) |
|---|---|---|
| *Short-term contract $T = 5$, $r = 1.5\%$* | | |
| Annual guarantee | 0.11 | 0.73 |
| $\frac{5}{4}$-years guarantee | 0.34 | 0.94 |
| Maturity guarantee | 1.18 | 1.58 |
| *Long-term contract $T = 20$, $r = 1.5\%$* | | |
| Annual guarantee | 0.11 | 0.73 |
| $\frac{20}{4}$-years guarantee | 1.18 | 1.57 |
| Maturity guarantee | 1.43 | 1.72 |

**Table 7.** Fair guaranteed rates $i$ (expressed in %) for different maturities $T$ when $r = 0.5\%$.

| | Pure participating product ($x = 1$) | Mixed product ($x = 0.5$) |
|---|---|---|
| *Short-term contract $T = 5$, $r = 0.5\%$* | | |
| Annual guarantee | -1.88 | -0.75 |
| $\frac{5}{4}$-years guarantee | -1.54 | -0.50 |
| Maturity guarantee | -0.24 | 0.39 |
| *Long-term contract $T = 20$, $r = 0.5\%$* | | |
| Annual guarantee | -1.88 | -0.78 |
| $\frac{20}{4}$-years guarantee | -0.24 | 0.38 |
| Maturity guarantee | 0.26 | 0.66 |

As a conclusion from the above suggested products and the numerical illustrations, we are able to mention some remarks and inferences. First of all, referring to the relations obtained, we can put the light on the exclusion of the parameters $\sigma_H$ and $\rho$ from the pricing part of each type of product. Hence, the changes in these parameters did not have an impact on the numerical solutions, particularly on the fair guaranteed interest rates. Second, we observe two dimensions that allow us to present a higher guarantee. The first dimension includes the concept of a partial guarantee which consists of investing a part of the premium in a pure unit-linked contract where a fee is charged to finance the participating life insurance contract in order to get a higher guarantee. While the second dimension consists of expanding the first dimension and turning it into a maturity guarantee product where a higher guarantee can be presented.

## 5. Customer utility and optimal investment

After pricing the different types of contracts, we aim to maximize the customers' utility based on individuals' preferences in order to put the above products in an optimal utility framework. Under this framework, the affiliate is the one responsible for his own investment strategy, and so decides to invest a part of his initial wealth in the participating fund and the remaining in the risky fund.

To measure the preferences in terms of risk and return, we maximize the expected utility of the terminal wealth over the decision variable which is the investment strategy chosen by the affiliate. More precisely, it is the proportion $x$ of the premium to invest in the participating life insurance contract. Hence, for each of the above fair relations we introduce the following optimization problem:

$$\max_{x\in[0,1]} \quad \mathbb{E}\left[U(V_T)\right], \tag{13}$$

where $U$ is the utility function of the affiliate assuming that $U' > 0$ and $U'' < 0$.

**Table 8.** CEQ and optimal investment in the different types of contracts with respect to the risk aversion $\gamma$.

| | $\gamma = 3$ | | | $\gamma = 4$ | | |
|---|---|---|---|---|---|---|
| | Optimal $x$ (%) | Guarantee $i$ (%) | CEQ | Optimal $x$ (%) | Guarantee $i$ (%) | CEQ |
| *With rebalancing* | | | | | | |
| Annual guarantee | 30 | 1.42 | 2160 | 48 | 0.80 | 1987 |
| $\frac{20}{4}$-years guarantee | 28 | 2.07 | 2134 | 47 | 1.62 | 1957 |
| *Without rebalancing* | | | | | | |
| Annual guarantee | 27 | 1.49 | 2147 | 46 | 0.82 | 1968 |
| $\frac{20}{4}$-years guarantee | 25 | 2.14 | 2126 | 46 | 1.62 | 1944 |
| Maturity guarantee | 26 | 2.17 | 2115 | 48 | 1.75 | 1938 |

We mention that the previous maximization problem is solved subject to the following constraint:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rT} \cdot V_T\right] = V_0 = P,$$

which means that the fair pricing relation is satisfied.

### 5.1. Power utility

We assume that the preferences of the customer are modeled by a constant relative risk aversion (CRRA) utility function with a risk aversion coefficient $\gamma$. The latter one is used to measure the level of risk of the affiliate and his behavior towards the uncertainty of the amount of money received at maturity. The utility function is described as a power utility function of the form

$$U(z) = \frac{z^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \gamma \neq 1.$$

We note that the higher the customer's unwillingness to take a risk, the greater the value of $\gamma$. Based on Davies (1981), the best suggestion for $\gamma$ may typically be between 3 and 5.

In order to solve numerically problem (13) for each type of contract, we refer to the same contract we introduced in Section 4.3 for which $T = 20$ years, $r = 1.5\%$, $\sigma_G = 3\%$, $\beta = 70\%$ and $\varepsilon = 0.25\%$. In addition, we suppose the following values for the asset parameters: $\mu_G = 3\%$, $\mu_H = 7\%$ and $\sigma_H = 15\%$, whereas the correlation is $\rho = 0.1$ and the initial value is $V_0 = 1000$.

The numerical results are based on the Monte Carlo simulations. We refer in our computation to the certainty equivalent concept defined as $U(CEQ) = \mathbb{E}[U(V_T)]$. The investor is indifferent between receiving a safe amount CEQ at time $T$ and the random payout $V_T$ from the insurance contract. The certainty equivalent allows us to easily compare risky payoffs. Under this framework, we compare the CEQ to the different types of contracts according to various parameters as indicated below.

#### 5.1.1. Optimality with respect to the risk aversion

Table 8 displays the optimal asset allocation given a fair guaranteed rate and based on the highest value of CEQ for the different types of contracts. The results show that the higher CEQ is obtained the lower the guarantee.

We note that the optimal asset allocation $x$ obtained in Table 8, for instance, $x = 0.3$, confirms that the mixed product is more attractive than the pure participating ($x = 1$) and unit-linked ($x = 0$) products. In addition, we notice that the risk aversion has a strong effect on the asset allocation. If the customer is risk-averse, the larger share of the premium is invested in the participating fund and, equivalently, he is less guaranteed. Comparing both cases, with and without rebalancing, better results are obtained by rebalancing the portfolio yearly or periodically because we take the gains from high average return investments and reinvest them in assets with low average returns.

However, if we look at the table vertically, we remark that switching from one product to the other does not affect significantly the investment strategy of the investor but we can deduce that by increasing the number of periods $n$ we get higher certainty equivalents. For instance, the table shows that the annual guarantee ($n = T$) performs better than the maturity guarantee ($n = 1$) given a fair guaranteed rate and an optimal asset allocation. The results obtained are confirmed by the fact that the guarantee and the surplus return are computed inside the contract on a shorter basis for a higher value of $n$.

We want to also connect our results to the literature on participating and unit-linked insurance contracts and optimal asset allocation, as, for example, Døskeland & Nordahl (2008) and Chen et al. (2015). In Døskeland & Nordahl (2008), the riskiness of the underlying risky funds can be controlled by the investment strategy of the insurance company. More specifically, it is possible to invest or borrow at a risk-free rate $r$; a constant share $\theta_G > 0$ of funds is invested in some risky asset. In our previous setting, this means that the fund $G$ is replaced by the fund $\tilde{G}$:

$$\begin{aligned}\frac{\mathrm{d}\tilde{G}_t}{\tilde{G}_t} &= \theta_G \cdot \frac{\mathrm{d}G_t}{G_t} + (1-\theta_G)\cdot r\mathrm{d}t \\ &= \big(r + \theta_G(\mu_G - r)\big)\mathrm{d}t + \theta_G \cdot \sigma_G \mathrm{d}W_t^G, \quad \tilde{G}_0 = 1,\ t \in [0,T].\end{aligned}$$

The parameter $\theta_G$ is a free parameter that may help to further increase the contract's expected utility; if $\theta_G = 1$ we are back to our setting. The same can also be done for the unit-linked fund $H$ that is replaced by $\tilde{H}$:

$$\mathrm{d}\tilde{H}_t = \big(r + \theta_H(\mu_H - r)\big)\,\tilde{H}_t\,\mathrm{d}t + \theta_H\sigma_H\,\rho\,\tilde{H}_t\,\mathrm{d}W_t^G + \theta_H\sigma_H\sqrt{1-\rho^2}\,\tilde{H}_t\,\mathrm{d}W_t, \quad \tilde{H}_0 = 1,\ t \in [0,T], \tag{14}$$

where $\theta_H > 0$ is the share invested in the risky fund. Following Døskeland & Nordahl (2008), we choose $\theta_G^*$ such that the utility of a participating contract (i.e. the case $x = 1$ in our setting) is maximized. Secondly, we choose $\theta_H^*$ such that the utility of a unit-linked contract with a guarantee fee of $\varepsilon = 0$ is maximized (i.e. the case $x = 0$ in our setting). Given the two parameters $\theta_G^*$ and $\theta_H^*$, we can now compute the utility of a mixed product for any share $x \in [0,1]$. As limiting cases $x = 0$ and $x = 1$, we obtain the same optimal contracts as considered in Døskeland & Nordahl (2008). As expected, the optimal participating contract leads to a higher certainty equivalent than the optimal unit-linked investment. However, we also see that a mixed product may lead to a higher certainty equivalent than these two extremes. Figure 6 gives the certainty equivalent as a function of the share $x$ using the same parameter set as in Table 8 with a risk aversion $\gamma = 3$. We note that the large values of certainty equivalents observed in Figure 6 are due to the optimal choice of asset allocations $\theta_G^*$ and $\theta_H^*$. The optimal mixing share is $x = 0.62$, confirming our previous results that mixed participating products can be more attractive than participating and unit-linked products, respectively.

#### *5.1.2. Optimality with respect to the volatility of the risky fund*

On the other side, we present the sensitivity of the parameters of the risky fund on the certainty equivalents. Table 9 shows that, for the same risk appetite, the optimal asset allocation is decreasing with the choice of a fund that is less volatile, which means that the part invested in the risky fund is higher when its volatility decreases. In addition, as shown above, the certainty equivalents increase for greater investment in the unit-linked contract.

Consequently, it is worth noting that the risk aversion and the risky asset parameters are the most important parameters in measuring the preferences of the affiliate in terms of risk and return since they identify the profile of the customers.

#### *5.1.3. Optimality with respect to the parameter of correlation*

In addition to the sensitivity of the risk aversion and the volatility of the risky asset we check the impact of the correlation parameter $\rho$ on the optimal investment by fixing the risk aversion $\gamma$ to 4

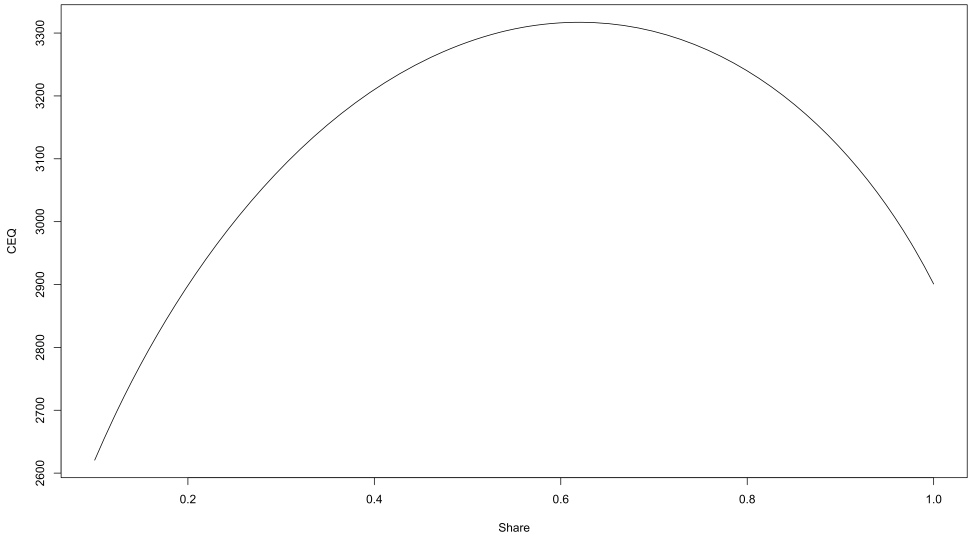


**Figure 6.** CEQ as a function of the share $x$ – Maturity guarantee.

**Table 9.** CEQ and optimal investment in the different types of contracts with respect to the volatility of the risky fund $\sigma_H$.

| | $\sigma_H = 15\%$ | | $\sigma_H = 20\%$ | | $\sigma_H = 25\%$ | |
|---|---|---|---|---|---|---|
| | Optimal $x$ (%) | CEQ | Optimal $x$ (%) | CEQ | Optimal $x$ (%) | CEQ |
| *With rebalancing,* | $\gamma = 4$ | | | | | |
| Annual guarantee | 48 | 1985 | 71 | 1784 | 82 | 1697 |
| $\frac{20}{4}$-years guarantee | 47 | 1958 | 71 | 1749 | 83 | 1662 |
| *Without rebalancing,* | $\boldsymbol{\gamma = 4}$ | | | | | |
| Annual guarantee | 46 | 1970 | 72 | 1755 | 84 | 1665 |
| $\frac{20}{4}$-years guarantee | 46 | 1943 | 72 | 1729 | 84 | 1640 |
| Maturity guarantee | 48 | 1938 | 73 | 1730 | 85 | 1644 |

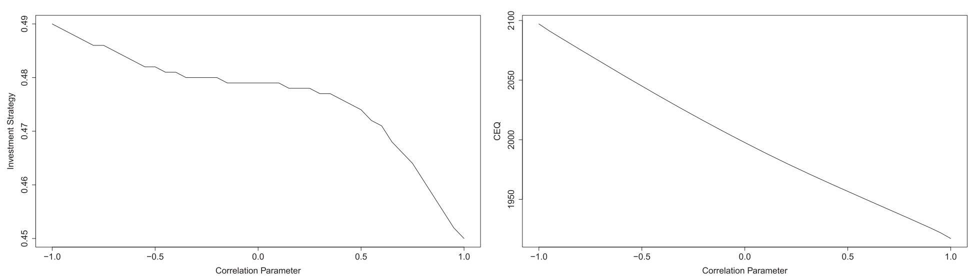


**Figure 7.** Sensitivity of the correlation parameter on the optimal investment in an annual guarantee contract.

and considering an annual guarantee contract. The first plot in Figure 7 shows that the correlation is not a significant measure of the preference of the customer; however, it affects the CEQ as shown in the second plot of Figure 7. The CEQ is decreasing with the increase of $\rho$ and we mention that in the case of perfectly negative correlation the portfolio is diversified which yields a maximum risk reduction and as a result a higher certainty equivalent.

**Table 10.** Optimal asset allocation parameter $x$ and optimal certainty equivalents (CEQ) for different contracts with respect to the guarantee fee.

| | $\varepsilon = 0\%$ | | $\varepsilon = 0.25\%$ | | $\varepsilon = 0.5\%$ | |
|---|---|---|---|---|---|---|
| | Optimal $x$ (%) | CEQ | Optimal $x$ (%) | CEQ | Optimal $x$ (%) | CEQ |
| *With rebalancing,* | $\gamma = 4$ | | | | | |
| Annual guarantee | 47 | 2006 | 48 | 1987 | 48 | 1969 |
| $\frac{20}{4}$-years guarantee | 46 | 1987 | 47 | 1957 | 47 | 1932 |
| *Without rebalancing,* | $\gamma = 4$ | | | | | |
| Annual guarantee | 47 | 1991 | 46 | 1968 | 45 | 1951 |
| $\frac{20}{4}$-years guarantee | 46 | 1975 | 46 | 1944 | 45 | 1918 |
| Maturity guarantee | 46 | 1981 | 48 | 1938 | 47 | 1904 |

**Table 11.** Different sets of parameter values (expressed in %) for the riskless and the risky assets.

| | Scenario 1 | Scenario 2 | Scenario 3 |
|---|---|---|---|
| $\mu_G$ | 2 | 2 | 2 |
| $\sigma_G$ | 6 | 6 | 6 |
| $\mu_H$ | 6 | 6 | 6 |
| $\sigma_H$ | 14 | 20 | 15.5 |

### *5.1.4. Optimality with respect to the guarantee fee*

In this section, we look at the effect of the guarantee fee $\varepsilon$ on the optimal asset allocation decision and the corresponding CEQ for the customer. Opposite to the effect of the risk aversion parameter, Table 10 shows that the guarantee fee does hardly affect the optimal investment strategy of the affiliate. In terms of the CEQ of the optimal contract, we analyze different parameter sets and find situations where the introduction of the guarantee fee can increase the CEQ but also situations where the case $\varepsilon = 0\%$ leads to the highest CEQ (in Table 10, for example, the CEQ is decreasing in $\varepsilon$). Overall, we find three possible effects of $\varepsilon$ on the CEQ of the optimal contract.

To demonstrate this, let us consider an annual guarantee contract with three different scenarios defined by a set of different parameters for the riskless asset $G$ and the risky asset $F$ illustrated in Table 11. All the other parameters are set as in the previous examples.

In Figure 8, we display graphically the impact of the guarantee fee $\varepsilon$ on the CEQ for the three parameter choices from Table 11. For $\varepsilon \in [0, 0.01]$, we observe a decreasing, an increasing (first two figures, scenarios 1, 2) or a hump-shaped effect (third figure, scenario 3) on the CEQ. The increase of CEQ with $\varepsilon$ can be explained by the fact that the constant fee $\varepsilon$ due in both good and bad years is used to reduce the risk in the part invested in the participating contract by increasing the investment guarantee.

In conclusion, we point out that the main parameters for the optimal investment framework are the risk aversion that represents the individual risk and the risky asset parameters. Therefore, the product answers different profiles of customers from small to large risk aversion and allows for different unit-linked funds based on each profile. The mixed product turns out to be more attractive than a single investment in a participating or a unit-linked contract. Further, we observe that the introduction of a guarantee fee can sometimes (but not always) help to further improve the optimal product's CEQ for the customer. In the next section, we confirm these results using prospect theory as preference functions. Prospect theory takes into account risk preferences of customers who are more sensitive to losses but are (at the same time) risk-seeking with respect to gains. Such a behavior is observed in many empirical or experimental studies (see, e.g. Tversky & Kahneman 1992).

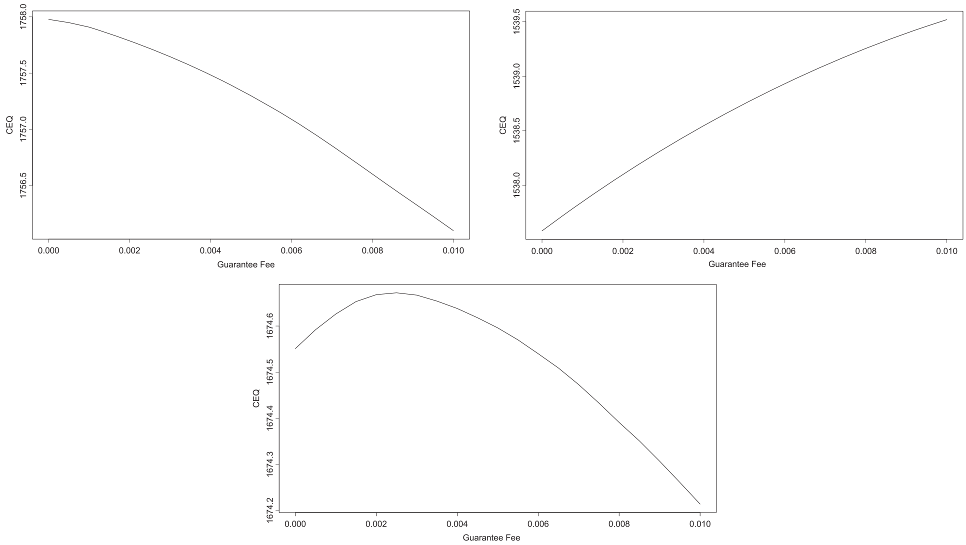


**Figure 8.** Different effects of the guarantee fee on the CEQ for the annual guarantee contract.

### 5.2. Cumulative prospect theory

In this part, we describe an alternative behavioral model proposed by Tversky & Kahneman (1992) called the cumulative prospect theory (CPT) that describes the human behaviors under risk and uncertainty. Generally, people tend to value gains and losses differently. Experimental studies confirm that they tend to be risk-averse with respect to losses (e.g. in an insurance product) but risk-seeking with respect to gains (e.g. in a lottery) (see also Tversky & Kahneman 1992). In prospect theory, this is accounted for by a reference point $\Gamma$ that separates the portfolio values in a gain part ($V_T - \Gamma$ is positive) and a loss part ($V_T - \Gamma$ is negative). The different view towards gains and losses is accounted for by an S-shaped preference function:

$$U(z) = \begin{cases} (z - \Gamma)^{\phi} & z \geq \Gamma, \\ -\lambda \cdot (\Gamma - z)^{\phi} & z < \Gamma, \end{cases}$$

where $\phi$ is the sensitivity parameter that measures gains or losses. $z$ refers to the final retrospective reserve where some values fall above the reference point and the others fall below. $\lambda$ represents the sensitivity to losses over gains and takes into account that losses 'hurt more'. Prospect theory helps to explain the high demand for guarantees in life and pension insurance – an observation that cannot be explained by a power utility preference function. The second aspect of human behavior according to prospect theory is a probability distortion: The likelihood of extreme events (e.g. a stock market crash) is overweighted while the likelihood of 'normal' events is underweighted. This effect is modeled by a weighting function $w$: $[0, 1] \longrightarrow [0, 1]$ that is strictly increasing with $w(0) = 0$ and $w(1) = 1$. It holds that $w(p) > p$ for small values of $p$ and $w(p) < p$ for high values of $p$. We rely on Prelec (1998) to define the weighting function as $w(p) = e^{-(-\log p)^{\varphi}}$ where $\varphi$ is a free parameter that controls the curvature of the function. Given the weighting function $w$ and the function $U$, the prospect utility function is defined as[1]

$$CPT(x) := \int_{-\infty}^{0} U(x)\, \mathrm{d}\big(w(F(x))\big) + \int_{0}^{\infty} U(x)\, \mathrm{d}\big(-w(1 - F(x))\big)$$

[1] To estimate the utility by Monte-Carlo simulation with $N$ simulation runs, denote by $V_T^{(1)} \leq V_T^{(2)} \leq \ldots \leq V_T^{(N)}$ the ordered realizations of terminal wealth. If there is no distortion, expected utility is simply obtained as $\mathbb{E}[U(V_T)] = \frac{1}{N}\sum_{j=1}^{N} U(V_T^{(j)})$. If extreme

**Table 12.** CEQ and optimal investment in the different types of contracts with respect to the sensitivity parameter $\phi$.

| | $\phi = 0.15$ | | | $\phi = 0.18$ | | |
|---|---|---|---|---|---|---|
| | Optimal $x$ (%) | Guarantee $i$ (%) | CEQ | Optimal $x$ (%) | Guarantee $i$ (%) | CEQ |
| *With Rebalancing* | | | | | | |
| Annual Guarantee | 85 | 0.24 | 1869 | 79 | 0.30 | 2259 |
| $\frac{20}{4}$-years Guarantee | 85 | 1.26 | 1814 | 82 | 1.28 | 2191 |
| *Without Rebalancing* | | | | | | |
| Annual Guarantee | 83 | 0.25 | 1896 | 80 | 0.29 | 2328 |
| $\frac{20}{4}$-years Guarantee | 86 | 1.25 | 1833 | 80 | 1.29 | 2227 |
| Maturity Guarantee | 82 | 1.50 | 1797 | 76 | 1.53 | 2200 |

with distribution function of terminal wealth $F(x) = P(V_T \leq x)$. This is a natural generalization (cf. Hens & Rieger 2010, Ru$\beta$ & Schelling 2018) of the discrete case introduced by Tversky & Kahneman (1992). Without probability weighting (i.e. for $\varphi = 1$ and $w(x) = x$), the prospect utility function reduces to

$$CPT(V_T) = \int_{-\infty}^{\infty} U(x)\,\mathrm{d}F(x) = E[U(V_T)].$$

To solve the problem numerically, we refer to the concept of certainty equivalent defined previously to a better comparison. We fix the same parameter values as in the power utility part, however, there are several studies that estimate the parameters of CPT and based on these studies and on the long time horizon $T = 20$ chosen in this paper, we assign the following values to our parameters: $\Gamma = (1+r)^T \cdot V_0$ as a reference point, $\lambda = 2.25$ as estimated by Tversky & Kahneman (1992) and $\varphi = 0.78$ inspired by several studies that use Prelec (1998) to estimate the probability weighting function. Under this framework, we compare the CEQ of the different types of contracts with respect to the sensitivity parameter $\phi$.

Table 12 shows that, consistently with the results of the power utility, the share in the participating fund $x$ is almost the same for different types of contracts for a given sensitivity parameter $\phi$ while the CEQ is increasing in the number of periods $n$. On the other hand, the increase of the sensitivity parameter $\phi$, which indicates an increased sensitivity to gains than to losses, generates a higher CEQ as well as a greater investment in the unit-linked contract.

### 5.2.1. Sensitivity analysis with respect to the sensitivity parameter and the guarantee fee

In the following, we analyze the effect of the guarantee fee $\varepsilon$ for different values of the sensitivity parameter $\phi$ by considering an annual guarantee contract. We notice that the CEQ is decreasing with the guarantee fee while the asset allocation $x$ is hardly affected (Table 13). This result depends on the careful choice of CPT parameters.

Under the power utility and CPT framework, we get higher CEQ by considering a mixed product under the annual guarantee contract. However, with the parameters we choose in this paper, the investor has a preference to invest more in the participating contract under a CPT framework, while he prefers to invest equally or more in the unit-linked contract under a power utility framework.

scenarios are perceived more likely, the expected utility is estimated as

$$\mathbb{E}[CPT(V_T)] = w(p) \cdot U(V_T^{(1)}) + \sum_{j=1}^{N-1} \Big[ w\big((j+1) \cdot p\big) - w\big(j \cdot p\big) \Big] \cdot U(V_T^{(j+1)}),$$

where $p = \frac{1}{N}$ is the true probability of each Monte-Carlo realization.

**Table 13.** CEQ and optimal investment with respect to the sensitivity parameter and the guarantee fee.

| | $\phi = 0.12$ | | | $\phi = 0.15$ | | | $\phi = 0.18$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon$ | Optimal $x$ (%) | Guarantee $i$ (%) | CEQ | Optimal $x$ (%) | Guarantee $i$ (%) | CEQ | Optimal $x$ (%) | Guarantee $i$ (%) | CEQ |
| 0.1% | 87 | 0.22 | 1556 | 84 | 0.25 | 1877 | 76 | 0.34 | 2267 |
| 0.25% | 86 | 0.23 | 1554 | 85 | 0.24 | 1869 | 79 | 0.30 | 2259 |
| 0.8% | 89 | 0.20 | 1543 | 87 | 0.22 | 1851 | 83 | 0.26 | 2238 |

## 6. Conclusion

In this paper, we have designed a new life insurance product that combines features of participating and unit-linked life insurance contracts and is adapted to the needs in a low interest rate environment. The product is not just a superposition of participating and unit-linked contracts but includes a continuous guarantee fee on the unit-linked part that permits a transfer from the unit-linked part to the participating part.

Under this framework, we have considered different forms of contracts, varying the guarantee period (annual guarantee, maturity guarantee or periodical guarantee) and the type of investment (rebalancing or non-rebalancing). In each contract, a minimum return is guaranteed to the customer along with a participation in the profits of the insurer. We have developed fair parameter combinations under a given valuation measure, demonstrating that our product allows for fair guaranteed rates in the participating part that may even exceed the risk-free rate. This design leads to attractive guaranteed rates also in a low interest rate environment. The guaranteed rate is increasing with a greater investment in the unit-linked part, a higher value of the guarantee fee as well as a lower participation in the profits of the insurer. Furthermore, the model shows that the guaranteed rate can be boosted by considering a long-term contract or by turning it into a maturity guarantee contract. Eventually, the interference of the guarantee fee in the model improved the guarantee on the participating part.

Moreover, we have analyzed the set of fair products from the customer's viewpoint working with preference functions. We have discussed the optimal share in the different parts as well as the effect of the guarantee fee on the attractiveness of the mixed product. We use two behavioral models, the utility approach and the cumulative prospect theory. Our numerical results show that the mixed product is in almost all cases more attractive than a simple participating or unit-linked contract. In some cases, the introduction of the guarantee fee can further improve the attractiveness of the product. The numerical results further show that the annual guarantee contract typically leads to the highest CEQ and that the main parameters that affect the optimal investment strategy of the affiliate are the risk aversion and the risky asset parameters. In addition, the product proves that the optimal level of fees strongly depends on the risk/return profiles of the unit-linked and participating funds involved.

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## References


Alexandrova M., Bohnert A., Gatzert N. & Russ J. (2017). Equity-linked life insurance based on traditional products: the case of select products. *European Actuarial Journal* **7**(2), 379–404.

Arkes H. R., Hirshleifer D., Jiang D. & Lim S. (2008). Reference point adaptation: tests in the domain of security trading. *Organizational Behavior and Human Decision Processes* **105**(1), 67–81.

Bacinello A. R. (2001). Fair pricing of life insurance participating policies with a minimum interest rate guaranteed. *ASTIN Bulletin* **31**(2), 275–298.

Bacinello A. R., Millossovich P., Olivieri A. & Pitacco E. (2011). Variable annuities: a unifying valuation approach. *Insurance: Mathematics and Economics* **49**(3), 285–297.

Bacinello A. R. & Persson S. A. (2002). Design and pricing of equity-linked life insurance under stochastic interest rates. *The Journal of Risk Finance* **3**(**2**), 6–21.

Bauer D., Kling A. & Russ J. (2008). A universal pricing framework for guaranteed minimum benefits in variable annuities. *ASTIN Bulletin: The Journal of the IAA* **38**(2), 621–651.

Bernard C. & Boyle P. (2011). A natural hedge for equity indexed annuities. *Annals of Actuarial Science* **5**(2), 211–230.

Black F. & Scholes M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy* **81**, 637–654.

Bohnert A. (2013). The market of dynamic hybrid products in Germany: concept, risk-return profiles, and market overview. *Zeitschrift für die gesamte Versicherungswissenschaft* **102**(5), 555–575.

Bohnert A., Born P. & Gatzert N. (2014). Dynamic hybrid products in life insurance: assessing the policyholder's viewpoint. *Insurance: Mathematics and Economics* **59**, 87–99.

Bohnert A. & Gatzert N. (2011). Risk assessment of surplus appropriation schemes in participating life insurance. *Zeitschrift für die gesamte Versicherungswissenschaft* **100**(5), 679–687.

Bohnert A. & Gatzert N. (2014). Fair valuation and risk assessment of dynamic hybrid products in life insurance: a portfolio consideration. *Geneva Papers on Risk and Insurance – Issues and Practice* **39**(1), 148–172.

Boyle P. P. & Schwartz E. S. (1977). Equilibrium prices of guarantees under equity-linked contracts. *Journal of Risk and Insurance* **44**, 639–660.

Brennan M. J. & Schwartz E. S. (1979). Alternative investment strategies for the issuers of equity linked life insurance policies with an asset value guarantee. *Journal of Business* **52**, 63–93.

Chen A., Hentschel F. & Klein J. (2015). A utility- and CPT-based comparison of life insurance contracts with guarantees. *Journal of Banking and Finance* **61**, 327–339.

Chen A., Hieber P. & Nguyen T. (2019). Constrained non-concave utility maximization: an application to life insurance contracts with guarantees. *European Journal of Operational Research* **273**(3), 1119–1135.

Davies J. (1981). Uncertain lifetime, consumption, and dissaving in retirement. *Journal of Political Economy* **89**, 561–577.

Devolder P. & De Valeriola S. (2017). Minimum protection in DC funding pension plans and Margrabe options. *Risks* ( Special Issue Actuarial and Financial Risks in Life Insurance, Pensions and Household Finance) **5**, 1–14.

Døskeland T. & Nordahl H. (2008). Optimal pension insurance design. *Journal of Banking and Finance***32**, 382–392.

European Insurance and Occupational Pensions Authority (2021). Pan-European Personal Pension Product (PEPP). Europa.eu, Available at: https://www.eiopa.europa.eu/browse/pensions/pan-european-personal-pension-product-pepp_en

Gatzert N., Huber C. & Schmeiser H. (2011). On the valuation of investment guarantees in unit-linked life insurance: a customer perspective. *The Geneva Papers on Risk and Insurance – Issues and Practice* **36**, 3–29.

Hambardzumyan H. & Korn R. (2019). Dynamic hybrid products with guarantees – an optimal portfolio framework. *Insurance: Mathematics and Economics* **84**, 54–66.

Hansen M. & Miltersen K. R. (2002). Minimum rate of return guarantees: the Danish case. *Scandinavian Actuarial Journal* **4**, 280–318.

Harrison M. J. & Kreps D. (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory* **20**, 381–408.

Hens T. & Rieger M. O. (2010). *Financial economics: A concise introduction to classical and behavioral finance, 1st edition*. Berlin: Sringer.

Hieber P. (2017). Cliquet-style return guarantees in a regime switching Lévy model. *Insurance: Mathematics and Economics* **72**, 138–147.

Hieber P., Natolski J. & Werner R. (2019). Fair valuation of cliquet-style return guarantees in (homogeneous and) heterogeneous life insurance portfolios. *Scandinavian Actuarial Journal* **6**, 478–507.

Knoller C. (2016). Multiple reference points and the demand for principal-protected life annuities: an experimental analysis. *Journal of Risk and Insurance* **83**(1), 163–179.

Kochanski M. & Karnarski B. (2011). Solvency capital requirement for hybrid products. *European Actuarial Journal* **1**(2), 173–198.

Mackay A., Augustyniak M., Bernard C. & Hardy M. (2015). Risk management of policyholder behavior in equity-linked life insurance. *Journal of Risk and Insurance* **84**(2), 661–690.

Mahayni A. & Schneider J. C. (2016). Minimum return guarantees, investment caps, and investment flexibility. *Review of Derivatives Research* **19**(2), 85–111.
Prelec A. (1998). The probability weighting function. *Econometrica* **60**, 497–528.
Ruß J. & Schelling S. (2018). Multi cumulative prospect theory and the demand for cliquet-style guarantees. *Journal of Risk and Insurance* **85**(4), 1103–1125.
Ruß J. & Schelling S. (2021). Return smoothing in life insurance from a client perspective. *Insurance: Mathematics and Economics* **36**, 245.
Tversky A. & Kahneman D. (1992). Advances in prospect theory: cumulative representation of uncertainty. *Journal of Risk and Uncertainty* **5**, 297–323.
Van Maldegem P. (2020). Branche 44, choisissez vous-même le niveau de risque. L'Echo 18 février 2020. Available at: https://www.lecho.be/les-marches/fonds/opinions/branche-44-choisissez-vous-meme-le-niveau-de-risque/1020-8932.html