Optimal dividend strategies for a compound Poisson process under transaction costs and power utility *

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Abstract:
We characterize the value function of maximizing the total discounted utility of dividend payments for a compound Poisson insurance risk model when strictly positive transaction costs are included, leading to an impulse control problem. We illustrate that well known simple strategies can be optimal in the case of exponential claim amounts. Finally we develop a numerical procedure to deal with general claim amount distributions.

1 Control problem with transaction costs

Let $(\Omega, \mathcal{F}, P)$ be a probability space carrying all the stochastic quantities defined below. The uncontrolled risk reserve process $R = (R_t)_{t\geq 0}$ of an insurance portfolio in the Cramér-Lundberg model is given by

$$R_t = x + ct - \sum_{n=1}^{N_t} Y_i,$$

where $N = (N_t)_{t\geq 0}$ is a homogeneous Poisson process with intensity $\lambda > 0$. The sequence of claim amounts is $\{Y_i\} \overset{iid}{\sim} F_Y$, where F_Y is a probability distribution on $(0, \infty)$ with a continuous distribution function. We assume $\{Y_i\}$ to be independent of the claim counting process $N = (N_t)_{t\geq 0}$. The deterministic components are the initial capital x and the premium intensity c.
As an extension of the classical model, assume that the insurance company is allowed to pay out dividends to its shareholders, but with the constraint that for every payment z a transaction cost $(1-k)z + K$ has to be paid, which consists of a proportional cost $(1-k)z$ (with $k \in (0, 1)$) and a fixed amount $K > 0$ (see e.g. [4], where this type of transaction costs is used in a different model context). Consequently, the value of a payment of size z is reduced to $kz - K$. This can also be interpreted as a tax payment on the dividend with rate $1 - k$, which has to be paid directly at the payment. One immediately observes that only payments of size greater than $a := K/k$ are reasonable and only a finite number of actions in bounded time intervals will be feasible. The following definition taken from Korn [12] fixes the class of appropriate control strategies.

Definition 1.1. *An impulse control $S = \{(\tau_i, Z_i)\}_{i\in\mathbb{N}}$ is a sequence of increasing intervention times τ_i and associated control actions Z_i, which fulfills the following four conditions:*

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- *$0 \le \tau_i \le \tau_{i+1}$ a.s. for all $i \in \mathbb{N}$,*
- *τ_i is a stopping time with respect to the filtration $\mathcal{F}_t = \sigma\{R^S_{s-} \mid s \le t\}$ for $t \ge 0$,*
- *Z_i is measurable with respect to $\mathcal{F}_{\tau_i}$,*
- *$P\left(\lim_{i\to\infty} \tau_i \le T\right) = 0$ for all $T \ge 0$.*

Denote with $\mathcal{S}^*$ the set of impulse controls and with $\mathcal{S}^*_n$ the set for which at most n interventions are used ($\tau_{n+1} = \infty$). The controlled process $R^S = (R^S_t)_{t\ge 0}$ for $S \in \mathcal{S}^*$, is then defined by

$$R^S_t = x + ct - \sum_{n=1}^{N_t} Y_n - \sum_{i=1}^{\infty} I_{\{\tau_i < t\}} Z_i.$$

In particular we have $R^S_{\tau_i} = R^S_{\tau_i -} - Z_i$. The time of ruin of the controlled reserve is denoted by $\tau^S = \inf\{t > 0 \mid R^S_t < 0\}$ for a strategy $S \in \mathcal{S}^*$, whereas τ denotes the time of ruin of the uncontrolled reserve. After the event of ruin the free reserve process is stopped, $R^S_t = R^S_{\tau^S}$ for $t \ge \tau^S$.
Following [4], we model the utility of the shareholders by a power function. Then, in our context, the value of a dividend strategy S is of the form

$$V_S(x) = \mathbb{E}_x \left(\sum_{i=1}^{\infty} e^{-\delta \tau_i} u(Z_i) I_{\{\tau_i < \tau^S\}} \right),$$

where $\delta > 0$ is a constant discount factor and $u(z) = \frac{1}{\gamma}(kz - K)^\gamma$ with $\gamma \in (0, 1]$.
As mentioned above we need some further constraints on $\mathcal{S}^*$ to formulate a reasonable optimization problem, namely $Z_i > a$ and $R^S_{\tau_i} = R^S_{\tau_i -} - Z_i \ge 0$, thus $Z_i \in (a, R^S_{\tau_i -}]$. In the terminology of [6], $\mathcal{K}(x) = \{y \mid y \in (a, x]\}$ denotes the set of applicable actions at some point $x \ge 0$ with $\mathcal{K}(x) = \emptyset$ for $x \in [0, a]$.
We call the set of admissible controls $\mathcal{S}$ and $\mathcal{S}_n$, respectively. The maximization problem consists of identifying

$$V(x) = \sup_{S \in \mathcal{S}} V_S(x). \tag{1}$$

In [6], Davis develops a general theory of optimal impulse control for piecewise deterministic Markov processes and refers to articles by Lenhart [14] and Gạtarek [7] on that topic. Cadenillas et al. [4] look at the maximization problem (1) for a mean-reverting diffusion process and solve the problem by calculating an explicit solution to the associated quasi-variational inequalities. Other examples from finance dealing with portfolio optimization under various types of transaction costs are Korn [11, 12] and Irle and Saß [9]. Although similar in flavor, these results are not directly applicable in the present situation, since either the bounds on the value functions or the class of considered processes differ substantially from our setup. The papers by Bensoussan et al. [3] and Benkherouf & Bensoussan [2] deal with inventory models with diffusion and compound Poisson demand with ordering policies. While upward impulses are used in these papers, a dividend payment as described above is a downward impulse, so that these models can not be interpreted as a dividend problem in an insurance context. They rather correspond to the problem of minimizing capital injections, where the investor is allowed to give upward impulses to the reserve process in order to avoid ruin, see

Kulenko & Schmidli [13]. Furthermore such inventory models do not include the complication of a ruin event which affects the dividend problem.
For diffusion risk reserve processes, dividend maximization problems including transaction costs are for example investigated by Jeanblanc-Picqué & Shiryaev [10], Paulsen [16, 17] or He & Liang [8]. In [15] Loeffen studies the dividend maximization problem including transaction costs and linear utility for spectrally-negative Lévy processes (including the compound Poisson reserve process or jump-diffusions) by probabilistic means and gives a condition when a strategy of a simple form is the optimal one.[1]

In this paper we tackle the control problem (1) by analytical means and develop a concise theoretical framework for its solution. In Section 2, basic properties such as linear boundedness and continuity of the value function are derived. These results are used in Section 3 to characterize the value function as a fixed point of an associated optimal stopping operator. In Section 4 the problem is studied from the viewpoint of quasi-variational inequalities and we prove that the value function is the smallest solution to this set of inequalities. These inequalities are then used in Section 5 to derive an optimal strategy in the case of exponentially distributed claim amounts and a fixed parameter set. Finally the construction of a numerical procedure for getting approximations of the optimal value for general claim amount distributions is illustrated.

2 Properties of the Value Function

In this section some basic properties of the value function are proved. We can derive immediately that V is increasing, because for $0 \leq y \leq x$ we have that $\mathcal{K}(y) \subset \mathcal{K}(x)$, so that by the definition of the value function V and the utility function u we get that with identical intervention times always a higher pay-off can be realized. The definition of the set of admissible strategies $\mathcal{S}$ implies $V \geq 0$ for all $x \geq 0$. The next lemma identifies further properties of V.

Lemma 2.1. *V is locally Lipschitz-continuous and bounded by a linear function.*

Proof. Let $x > y \geq 0$ and $S_x^\epsilon = \{(\tau_i, Z_i)\}_{i\in\mathbb{N}}$ be an ϵ-optimal strategy for initial capital x ($V(x) \leq V_{S_x}(x) + \epsilon$). We define the strategy S_y for an initial capital y in the following way. As long as R^{S_y} stays below x, pay out nothing and if x is reached, apply strategy S_x^ϵ.
Let θ_x denote the first time when R^{S_y} hits x. On $\{\theta_x < \tau^{S_y}\}$, we get $S_y = \{(\tau_i + \theta_x, Z_i)\}_{i\in\mathbb{N}}$ and because the sum of stopping times is a stopping time, S_y is admissible. We have

$$\begin{aligned} 0 \leq V(x) - V(y) \leq V(x) - V_{S_y}(y) &\leq V(x) - (V(x) - \epsilon)\mathbb{E}_y\left(e^{-\delta\theta_x} I_{\{\theta_x < \tau^{S_y}\}}\right) \\ &= V(x)(W(x) - W(y)) + \epsilon W(y). \end{aligned} \tag{2}$$

The function W is defined as $W(y) := \mathbb{E}_y\left(e^{-\delta\theta_x} I_{\{\theta_x < \tau^{S_y}\}}\right)$ and it is a differentiable solution

[1] We point out that there is a paper by Zou et al. [22] published in an earlier issue of this journal that deals with the dividend maximization problem for linear utility and both fixed and proportional transaction costs for a risk reserve of jump-diffusion type. However, although stated otherwise, the authors actually construct a solution for the problem when payments continue after ruin. This leads them to the conclusion that a certain strategy of a simple form is optimal for every claim size distribution, which is in contradiction with [15] where an example with Gamma distributed claim amounts is presented for which such a simple strategy is in fact not optimal.

(see proof of Lemma 2.48, page 91 of [20]) to

$$0 = cW'(y) + \lambda \int_0^y W(y-z)dF_Y(y) - (\delta+\lambda)W(y).$$

Further, by definition $W(y) \le W(x) = 1$.

Therefore we get from (2) that $0 \le V(x) - V(y) \le V(x)C(x)(x-y) + \epsilon$, where $C(x)$ is a *local* constant depending on $W'(x)$. Because this inequality holds for every $\epsilon > 0$, we have $V(x) - V(y) \le C^*(x)(x-y)$, where $C^*(x)$ is another *local* constant.

From that we obtain that V is locally Lipschitz continuous on $[0,\infty)$, and consequently that V is differentiable almost everywhere. Because of the Lipschitz continuity on compact sets, the above calculations give bounds for $V'(x)$ which is therefore integrable on compact sets. This yields that V is absolutely continuous, see [21].

The next step is to determine a linear bound for V. Because $u(z)$ is linearly bounded, $u(z) \le mz + n$ for some m, $n \in \mathbb{R}^+$, we have

$$V(x) = \sup_{S\in\mathcal{S}} \mathbb{E}_x \left(\sum_{i=1}^{\infty} e^{-\delta\tau_i} u(Z_i) I_{\{\tau_i < \tau^S\}} \right) \le \sup_{S\in\mathcal{S}} \mathbb{E}_x \left(\sum_{i=1}^{\infty} e^{-\delta\tau_i} (mZ_i + n) I_{\{\tau_i < \tau^S\}} \right).$$

Using the process $L^S = (L_t^S)_{t\ge 0}$ defined by

$$L_t^S = \sum_{n=1}^{\infty} Z_i \, I_{\{\tau_i < t\wedge\tau^S\}},$$

we can bound the second sum for a fixed strategy S by

$$m \int_0^{\tau^S} e^{-\delta t} dL_t^S + n \int_0^{\infty} e^{-\delta t} dt. \tag{3}$$

Notice that the process L^S represents the accumulated dividend payments due to the admissible impulse strategy S and is an admissible dividend strategy for the classical dividend maximization problem without transaction costs as well, see Azcue & Muler [1]. Let $V^c(x)$ denote this classical value function. We obtain by the above mentioned relation that

$$V(x) \le mV^c(x) + \frac{n}{\delta}.$$

Since $V^c(x)$ is linearly bounded (see Proposition 2.1 of [1]), we get a linear bound for $V(x)$. □

Let us define an operator M acting on a function f by

$$Mf(x) = \sup_{y\in\mathcal{K}(x)} \{u(y) + f(x-y)\}. \tag{4}$$

It gives the optimal value of an intervention at some surplus height $x \ge 0$. Because of the definition of the set of admissible strategies $\mathcal{S}$, we have $\mathcal{K}(x) = \emptyset$ for $0 \le x \le a$, in which case we define $Mf = 0$. For negative arguments $x < 0$ there also can not be an admissible intervention. Correspondingly we set throughout $Mf = 0$ whenever $x < 0$.

The definitions of the set $\mathcal{S}$ and V by (1) give that

$$V(x) \ge MV(x)$$

holds for all $x \ge 0$ and indicate that at points x' where it would be optimal to intervene, $V(x') = MV(x')$ should hold. The next lemma presents basic properties of $Mf : [0,\infty) \to [0,\infty)$ for a suitable function f.

Lemma 2.2. *Let f be an increasing, continuous and linearly bounded function, then Mf, as defined in (4), is (as a function in $x \in [0, \infty)$) also bounded by a linear function, increasing and continuous.*

Proof. Let $x > a$ and $m,\ n \in \mathbb{R}$ such that $f(x) \le mx + n$. We have,

$$\begin{aligned} Mf(x) = \sup_{y\in(a,x]} \{u(y) + f(x-y)\} &\le \sup_{y\in(a,x]} \{u(y) + m(x-y) + n\} \le mx + u(x) + n \\ &\le m_1 x + n_1, \text{ for some } m_1,\ n_1 \in \mathbb{R}, \end{aligned}$$

where we use that also $u(x)$ is linearly bounded by $kx + u\left(\frac{K+1}{k}\right)$ for every x. The case $x \le a$ needs no special treatment because $Mf(x) = 0$.

Let $x > \bar{x} > a$ and let $y^* \in (a, \bar{x}]$ such that $\sup_{\bar{y}\in(a,\bar{x}]} \{u(\bar{y}) + f(\bar{x} - \bar{y})\} \le u(y^*) + f(\bar{x} - y^*) + \epsilon$ for some $\epsilon > 0$. Then

$$\begin{aligned} Mf(x) - Mf(\bar{x}) &= \sup_{y\in(a,x]} \{u(y) + f(x-y)\} - \sup_{\bar{y}\in(a,\bar{x}]} \{u(\bar{y}) + f(\bar{x}-\bar{y})\} \\ &\ge \sup_{y\in(a,\bar{x}]} \{u(y) + f(x-y)\} - \sup_{\bar{y}\in(a,\bar{x}]} \{u(\bar{y}) + f(\bar{x}-\bar{y})\} \\ &\ge u(y^*) + f(x-y^*) - u(y^*) - f(\bar{x}-y^*) \ge 0, \end{aligned}$$

where the last but one inequality holds because $\epsilon > 0$ can be chosen arbitrarily small.
For proving the continuity, fix some $\epsilon > 0$. Because f and u are continuous, we can choose $\delta_1 > 0$ such that for $0 < x - \tilde{x} < \delta_1/2$ ($x, \tilde{x} > a$) and $|y' - \tilde{y}| < \delta_1/2$ we have that $|u(y') - u(\tilde{y})| < \epsilon/3$ and $|f(x - y') - f(\tilde{x} - \tilde{y})| < \epsilon/3$ hold. We will state the definitions of y' and $\tilde{y}$ immediately (the last inequality is based on the continuity of f, the fact that $|x - y' + (-\tilde{x} + \tilde{y})| < \delta_1$ and an appropriate choice of δ_1).
We choose $y' \in (a, x]$ such that $Mf(x) \le u(y') + f(x - y') + \epsilon/3$. Because $x - \tilde{x} < \delta_1/2$ we can choose a $\tilde{y} \in (a, \tilde{x}]$ with $|y' - \tilde{y}| < \delta_1/2$. Then we have

$$0 \le Mf(x) - Mf(\tilde{x}) \le \frac{\epsilon}{3} + u(y') + f(x-y') - u(\tilde{y}) - f(\tilde{x}-\tilde{y}) < \epsilon.$$

The continuity of f and u hence ensures the continuity of Mf for $x \in [0, a]$. □

3 Characterization of V

In this section we will characterize V as a fixed point to a related optimal stopping operator. The proof uses a construction which allows to interpret V as a gain when applying iterated optimal stopping.
Such a characterization is inspired by Davis [6], where such a fixed point characterization is shown for a minimization problem with a bounded pay-off function u. Korn [11] uses this approach to characterize the solution of a portfolio optimization problem with an underlying diffusion process.
Let $\mathcal{T}$ denote the set of a.s. finite stopping times with respect to $\{\mathcal{F}_t\}_{t\ge0}$.

Remark 3.1. In the sequel of this section, an optimal stopping operator with stopping times from the class $\mathcal{T}$ is defined. But notice that when looking at the original problem, the stopping times (part of admissible strategies) do not have to be in $\mathcal{T}$ (as the use of τ_{n+1}

in the definition of $\mathcal{S}_n$ demonstrates). However, this has no consequences, since the linear bounds of V and MV indicate that, due to discounting, the use of an unbounded stopping time results in a profit equal to zero.

The following operator gives the value of the related optimal stopping problem,

$$\mathcal{M}f(x) = \sup_{\theta \in \mathcal{T}} \mathbb{E}_x \left(e^{-\delta\theta} \, Mf(R_\theta) \right). \tag{5}$$

Since the free reserve process is stopped at ruin, $Mf(R_\theta) = 0$ if $\theta \geq \tau$.
At first we have to develop some basic properties of the operator $\mathcal{M}$.

Lemma 3.1. *Let f be an increasing, linearly bounded and continuous function, then the same holds for $\mathcal{M}f$ as a function in $x \in [0, \infty)$.*

Proof. We start with deriving the linear bound, using $Mf(x) \leq mx + n$ and $R_\theta \leq x + c\theta$,

$$\begin{aligned} 0 \leq \mathcal{M}f(x) &= \sup_{\theta \in \mathcal{T}} \mathbb{E}_x \left(e^{-\delta\theta} \, Mf(R_\theta) \right) \\ &\leq \sup_{\theta \in \mathcal{T}} \mathbb{E}_x \left(e^{-\delta\theta} (m(x + c\theta) + n) \right) \leq mx + n + \sup_{\theta \in \mathcal{T}} \mathbb{E}_x \left(e^{-\delta\theta} c\theta \right) \\ &\leq mx + n + \frac{c}{e\delta}. \end{aligned}$$

The monotonicity of $\mathcal{M}f$ follows by the monotonicity of Mf.
Now let $0 < x - x' = \delta_1$ and let R^x and $R^{x'}$ be the uncontrolled processes starting at the indicated points with times of ruin $\tau(x)$, $\tau(x')$. If we look pathwise on these processes we get that $R^x_t(\omega) - R^{x'}_t(\omega) = \delta_1$ for each $\omega \in \Omega$ as long as $t < \tau(x') \leq \tau(x)$. Now let θ^* be an ϵ-optimal stopping time for initial capital x. Then we derive

$$0 \leq \mathcal{M}f(x) - \mathcal{M}f(x') \leq \mathbb{E} \left(e^{-\delta\theta^*} Mf(R^x_{\theta^*}) - e^{-\delta\theta^*} Mf(R^{x'}_{\theta^*}) \right) + \epsilon. \tag{6}$$

At this point we have to distinguish three cases:

1. $\theta^* \geq \tau(x) \geq \tau(x')$: here we have $Mf(R^x_{\theta^*}) = Mf(R^{x'}_{\theta^*}) = 0$,
2. $\theta^* < \tau(x') \leq \tau(x)$: here $R^x_{\theta^*} - R^{x'}_{\theta^*} = \delta_1$,
3. $\tau(x') \leq \theta^* < \tau(x)$: here we have that $Mf(R^{x'}_{\theta^*}) = 0$ and $R^x_{\tau(x')} \leq \delta_1$ because $R^{x'}_{\tau(x')} = R^{x'}_{\theta^*}$.

The above observations show that we only have to take care of the Cases 2 and 3. Case 2 is described by the set of all paths

$$\mathcal{P}_{\delta_1} = \left\{ \omega \in \Omega \mid R^x_t(\omega) \geq \delta_1 \text{ for all } t \in [0, \theta^*] \right\}.$$

Case 3 then is described by

$$\mathcal{P} = \left\{ \omega \in \Omega \mid R^x_t(\omega) \geq 0 \;\forall\, t \in [0, \theta^*] \text{ and } \exists\, s \in [0, \theta^*] \text{ such that } R^x_s < \delta_1 \right\}.$$

Let t^* be the smallest time such that $R^x_{t^*} < \delta_1$, then $t^* = \tau(x')$. We derive that the right-hand side of (6) is smaller or equal to

$$\mathbb{E}\left(I_{\mathcal{P}_{\delta_1}}\left(Mf(R^x_{\theta^*}) - Mf(R^{x'}_{\theta^*})\right)\right) + \mathbb{E}\left(I_{\mathcal{P}} Mf(R^x_{\theta^*})\right) + \epsilon \leq$$
$$\mathbb{E}\left(I_{\mathcal{P}_{\delta_1}}\left(Mf(R^x_{\theta^*}) - Mf(R^{x'}_{\theta^*})\right)\right) + \mathbb{E}\left(I_{\mathcal{P}} Mf\left(R^x_{\tau(x')+(\theta^*-\tau(x'))}\right)\right) + \epsilon$$
$$= \mathbb{E}\left(I_{\mathcal{P}_{\delta_1}}\left(Mf(R^x_{\theta^*}) - Mf(R^{x'}_{\theta^*})\right)\right) + \mathbb{E}\left(\mathbb{E}\left(I_{\mathcal{P}}\, Mf\left(R^x_{\tau(x')+(\theta^*-\tau(x'))}\right) \,\middle|\, \mathcal{F}_{\theta^*-\tau(x')}\right)\right) + \epsilon$$
$$\leq \mathbb{E}\left(I_{\mathcal{P}_{\delta_1}}\left(Mf(R^x_{\theta^*}) - Mf(R^{x'}_{\theta^*})\right)\right) + Mf(\delta_1) + \epsilon. \quad (7)$$

The last inequality holds due to the strong Markov property of the uncontrolled process R^x and because on $\mathcal{P}$ we have that $R^x_{\tau(x')} < \delta_1$. Now we can use that Mf is continuous and an appropriate choice of δ_1 (i.e. letting $\delta_1 \to 0$) gives the continuity of $\mathcal{M}f$ in x'. $\square$

Now we are able to prove the main result of this section. It verifies the proposed connection between V and the *implicit* optimal stopping problem.

Theorem 3.2. *Let $v_0(x) = 0$ for all $x \in [0,\infty)$ and define functions $v_n : [0,\infty) \to [0,\infty)$ by $v_n(x) = \mathcal{M}v_{n-1}(x)$ for $x \in [0,\infty)$ and $v_n(x) = 0$ for $x < 0$. Then the following holds:*

$$\lim_{n\to\infty} v_n(x) = V(x) \quad \forall\, x \in [0,\infty),$$
$$V(x) = \mathcal{M}V(x). \quad (8)$$

Proof. Note that $v_0 = 0$ is equal to the payoff of a strategy with no intervention. Now look at $v_1(x)$,

$$v_1(x) = \mathcal{M}v_0(x) = \sup_{\theta\in\mathcal{T}} \mathbb{E}_x\left(e^{-\delta\theta} \sup_{y\in(a,R_\theta]} \{u(y) + v_0(R_\theta - y)\}\right)$$
$$= \sup_{\theta\in\mathcal{T}} \mathbb{E}_x\left(e^{-\delta\theta} u(R_\theta) I_{\{R_\theta > a\}}\right) \geq v_0.$$

If only one intervention is allowed, followed by a gain equal to zero, it is optimal to pay out the whole reserve at the *best* stopping time, therefore $v_1(x) = \sup_{S\in\mathcal{S}_1} V_S(x)$.
Define the sequence of functions $\{v_n\}_{n\in\mathbb{N}}$ by $v_n(x) = \mathcal{M}v_{n-1}(x)$. The properties of the operator $\mathcal{M}$ (see Lemma 3.1) and v_0 guarantee monotonicity, continuity and linear boundedness of the functions v_n for all $n \in \mathbb{N}$ by induction.
Assume that $v_{n-1}(x) = \sup_{S\in\mathcal{S}_{n-1}} V_S(x)$, then

$$v_n(x) = \sup_{\theta\in\mathcal{T}} \mathbb{E}_x\left(e^{-\delta\theta} \sup_{y\in(a,R_\theta]} \{u(y) + v_{n-1}(R_\theta - y)\}\right)$$
$$= \sup_{\theta\in\mathcal{T}} \mathbb{E}_x\left(e^{-\delta\theta} \sup_{y\in(a,R_\theta]} \left\{u(y) + \sup_{S\in\mathcal{S}_{n-1}} V_S(R_\theta - y)\right\}\right). \quad (9)$$

In (9), at most n decisions are taken, so that clearly $v_n(x) \leq \sup_{S\in\mathcal{S}_n} V_S(x)$. On the other hand, let $S^* = \{(\tau_i^*, Z_i^*)\}_{1\leq i\leq n} \in \mathcal{S}_n$ be an ϵ-optimal strategy within $\mathcal{S}_n$, then we get

$$\sup_{S\in\mathcal{S}_n} V_S(x) < \epsilon + V_{S^*}(x) = \epsilon + \mathbb{E}_x\left(e^{-\delta\tau_1^*} u(Z_1^*) I_{\{\tau_1^* < \tau^{S^*}\}} + e^{-\delta\tau_1^*} \sum_{i=2}^n e^{-\delta(\tau_i^*-\tau_1^*)} u(Z_i^*) I_{\{\tau_i^* < \tau^{S^*}\}}\right)$$
$$\leq \epsilon + \mathbb{E}_x\left(e^{-\delta\tau_1^*} u(Z_1^*) I_{\{\tau_1^* < \tau^{S^*}\}} + e^{-\delta\tau_1^*} v_{n-1}(R_{\tau_1^*} - Z_1^*)\right) \leq \epsilon + v_n(x).$$

Consequently $v_n(x) = \sup_{S \in \mathcal{S}_n} V_S(x)$ holds for all $n \in \mathbb{N}$ by induction.
Because $v_n(x) \leq V(x)$ and the monotonicity of $\{v_n(x)\}_{n\in\mathbb{N}}$ for all $x \in [0,\infty)$, we get that the sequence converges pointwise to some function, $\lim_{n\to\infty} v_n(x) = v^*(x)$.
As a last step we have to show that $v^*(x) = V(x)$.
For that purpose, let $S = \{(\tau_i, Z_i)\}_{i\in\mathbb{N}} \in \mathcal{S}$ and define $S_n = \{(\tau_i, Z_i)\}_{1\leq i\leq n} \in \mathcal{S}_n$, so that S_n is equal to S up to the nth intervention and hence

$$\Big| V_S(x) - V_{S_n}(x) \Big| = \mathbb{E}_x \left(\sum_{i=n+1}^{\infty} e^{-\delta\tau_i} u(Z_i) I_{\{\tau_i < \tau^S\}} \right).$$

Since V is linearly bounded the sum has finite expectation and converges for $n = 0$ with probability 1. Therefore we get that for $n \to \infty$ the right-hand side converges to zero by dominated convergence. Finally we have

$$V(x) = \sup_{S\in\mathcal{S}} V_S(x) = \sup_{S \in \bigcup_{n\in\mathbb{N}} \mathcal{S}_n} V_S(x).$$

For the fixed point property, just observe

$$v_{n+1}(x) = \sup_{\theta\in\mathcal{T}} \mathbb{E}_x \left(e^{-\delta\theta} M v_n(R_\theta) \right), \tag{10}$$

$$\begin{aligned} MV(x) &= \sup_{y\in(a,x]} \{u(y) + \sup_{n\in\mathbb{N}} v_n(x-y)\} \\ &= \sup_{n\in\mathbb{N}} M v_n(x). \end{aligned} \tag{11}$$

Because $v_n \leq v_{n+1}$, we have that Mv_n is increasing in n, and dominated convergence in (10) yields $V(x) = \mathcal{M}V(x)$. □

Remark 3.2. Let w be a positive, absolutely continuous and linearly bounded function. Suppose w is another fixed point of $\mathcal{M}$. We have $w \geq v_0 = 0$ and assuming $w \geq v_n$ we get by induction and the monotonicity of $\mathcal{M}$,

$$w - v_{n+1} = \mathcal{M}w - \mathcal{M}v_n \geq 0.$$

Therefore $w \geq v_n$ holds for all $n \in \mathbb{N}$ and consequently $w \geq V$.

This theorem provides a characterization of the value function V defined by (1) as the smallest fixed point of an optimal stopping operator. The constructive nature of the proof shows that the optimal impulse control can be approximated by iterated solutions of optimal stopping problems.

4 A QVI point of view

We observed above that at points $x \geq 0$ where it would be optimal to intervene, we should have $MV(x) - V(x) = 0$. On the other hand, if it would be optimal not to intervene in an open interval around a point x, conditioning on the first claim occurrence in a small time interval $[0,h]$ and letting $h \to 0$ should result in

$$\mathcal{L}V(x) = cV'(x) + \lambda \left(\int_0^x V(x-y)\, dF_Y(y) - V(x) \right) - \delta V(x) = 0.$$

These observations motivate heuristically the so-called quasi-variational-inequalities (QVI):

$$\begin{aligned}\mathcal{L}V &\leq 0,\\ MV - V &\leq 0,\\ (\mathcal{L}V)(MV - V) &= 0,\end{aligned}$$

or equivalently

$$\max\{\mathcal{L}V, MV - V\} = 0. \tag{12}$$

This equation will allow for a better computability of V than the characterization through iterated optimal stopping of the last section. But first one has to determine if and in which sense V is a solution to (12).

Proposition 4.1. *The value function V fulfills the QVI (12) a.e.*

Proof. From the definition of V in (1) we immediately get $MV(x) - V(x) \leq 0$. Further, Theorem 3.2 shows that

$$V(x) = \sup_{\theta \in \mathcal{T}} \mathbb{E}_x \left(e^{-\delta\theta} \sup_{y \in (a, R_\theta]} \{u(y) + V(R_\theta - y)\} \right)$$

such that at a point where it is optimal to intervene we have $V(x) = MV(x)$.
Let $\theta \in \mathcal{T}$ and $\mathcal{S}^\theta \subset \mathcal{S}$ be the set of admissible impulse strategies without an intervention before time θ. By the Markov property of the uncontrolled reserve $R = (R_t)_{t \geq 0}$ and the definition of $V(x)$ we have that

$$V(x) \geq \sup_{S \in \mathcal{S}^\theta} V_S(x) = \mathbb{E}_x \left(e^{-\delta\theta} V(R_\theta) I_{\{\theta < \tau\}} \right).$$

Consequently we get for some $h > 0$ and T_1 being the time of the first claim occurrence,

$$V(x) \geq \mathbb{E}_x \left(e^{-\delta(T_1 \wedge h)} V(R_{T_1 \wedge h}) \right)$$

and after some manipulations

$$0 \geq \frac{V(x + ch) - V(x)}{ch} - \frac{1 - e^{-h(\delta+\lambda)}}{h} V(x + ch) + \frac{1}{h} \int_0^h \lambda e^{-t(\delta+\lambda)} \int_0^{x+ct} V(x + ct - y)\, dF_Y(y) dt.$$

Using the absolute continuity of V, which implies the existence of V' a.e., we can take the limit $h \to 0$ and obtain

$$0 \geq cV'(x+) - (\delta + \lambda)V(x) + \lambda \int_0^x V(x - y)\, dF_Y(y). \tag{13}$$

On the other hand we get, starting at $x - c\tilde{h}$, for some sufficiently small $\tilde{h} > 0$,

$$0 \geq cV'(x-) - (\delta + \lambda)V(x) + \lambda \int_0^x V(x - y)\, dF_Y(y). \tag{14}$$

The left-hand sides of (13) and (14) coincide a.e., but we can state $\mathcal{L}V \leq 0$ for $x \geq 0$ in general, by either fixing the right-hand or left-hand derivative as the density of V, denoted by V'.

Now we want to determine the behaviour of V around some point x where it is not optimal to intervene (of course $(0, a)$ is an interval of such points) and indeed we can follow the classical proof of the fact that the probability of ruin is differentiable as long as F_Y is continuous (see [19]).
Since $MV(x) - V(x)$ is continuous and negative for such points, we can choose some $h > 0$ small enough such that for initial capital $x' \in (x - ch, x + ch)$ an immediate intervention is not optimal. We get, again by conditioning on the first claim occurrence before time $h > 0$,

$$V(x) = e^{-(\delta+\lambda)h} V(x + ch) + \int_0^h \lambda e^{-(\delta+\lambda)t} \int_0^{x+ct} V(x + ct - y) \, dF_Y(y) \, dt,$$

which results in

$$c\frac{V(x+ch) - V(x)}{ch} = \frac{1 - e^{-(\delta+\lambda)h}}{h} V(x + ch) - \frac{1}{h} \int_0^h \lambda e^{-(\delta+\lambda)t} \int_0^{x+ct} V(x + ct - y) \, dF_Y(y) \, dt.$$

Because V is continuous and linearly bounded, the limit $h \to 0$ of the right-hand side exists, therefore V is differentiable from the right with

$$cV'(x+) = (\delta + \lambda)V(x) - \lambda \int_0^x V(x - y) dF_Y(y).$$

For initial capital $x - ch$, we can proceed as before to derive

$$cV'(x-) = (\delta + \lambda)V(x) - \lambda \int_0^{x-} V(x - y) dF_Y(y).$$

If Y has a continuous distribution function, we get that V' exists in x and that $\mathcal{L}V(x) = 0$ for x.
From now on choose either of the one-sided derivatives of V as density and denote it by V'. Since V, because it is absolutely continuous and linearly bounded, is in the domain of the generator of the uncontrolled reserve. This has the consequence that for a bounded stopping time θ the following equality holds:

$$\begin{aligned} V(x) = \mathbb{E}_x \Big(& e^{-\delta\theta} V(R_\theta) \\ & - \int_0^\theta e^{-\delta s} \left[cV'(R_{s-}) - (\delta + \lambda)V(R_{s-}) + \lambda \int_0^{R_{s-}} V(R_{s-} - y) \, dF_Y(y) \right] ds \Big). \end{aligned}$$

Combining this expression for $V(x)$ with its fixed point property (8), we get

$$\begin{aligned} 0 = \sup_{\theta \in \mathcal{T}} \mathbb{E}_x \Big(& e^{-\delta\theta} \left(MV(R_\theta) - V(R_\theta) \right) \\ & + \int_0^\theta e^{-\delta s} \left[cV'(R_{s-}) - (\delta + \lambda)V(R_{s-}) + \lambda \int_0^{R_{s-}} V(R_{s-} - y) \, dF_Y(y) \right] ds \Big). \end{aligned} \tag{15}$$

Of course the first summand in the above equation is smaller or equal to zero (at optimal intervention times it is optimal to jump). Further we know from above that $\mathcal{L}V \leq 0$. Therefore we obtain - in the case where immediate stopping is not optimal and as we have seen above - that $\mathcal{L}V = 0$ when the argument runs through non-intervention areas, such that the second summand is equal to zero.
Summarizing, at points where it is optimal to pay out optimal lump sums (at optimal

stopping times) the relation $V(x) = MV(x)$ holds. Between such actions (after the ith intervention, the fixed point argument can be applied again starting at $R^{S*}_{\tau_i+}$), the process follows the dynamics of the uncontrolled reserve and because of the above observations $\mathcal{L}V(x) = 0$ is fulfilled at points where $MV(x) - V(x) < 0$. Therefore V fulfills (12) a.e.. $\square$

For the moment, let g be an increasing, absolutely continuous and linearly bounded solution to the QVI (12), as V is. We can define in an abstract way the following admissible impulse control:

Definition 4.1 (QVI control)**.** *The strategy $S^g \in \mathcal{S}$ defined by*

$$\begin{aligned}
\tau_1^g &= \inf\left\{t > 0 \mid Mg(R_t^{S^g}) = g(R_t^{S^g})\right\},\\
Z_1^g &= \operatorname{argmax}\left\{u(z) + g(R^{Sg}_{\tau_1^g} - z) \mid z \in (a, R^{Sg}_{\tau_1^g}]\right\}
\end{aligned}$$

and for $n \geq 2$

$$\begin{aligned}
\tau_n^g &= \inf\left\{t > \tau_{n-1}^g \mid Mg(R_t^{S^g}) = g(R_t^{S^g})\right\},\\
Z_n^g &= \operatorname{argmax}\left\{u(z) + g(R^{Sg}_{\tau_n^g} - z) \mid z \in (a, R^{Sg}_{\tau_n^g}]\right\}.
\end{aligned}$$

is called QVI control.

Remark 4.1. We see from the definition of the QVI control that the ith intervention takes place at time τ_i^g which only takes the knowledge of $R^{Sg}_{\tau_i^g-}$ into account (an intervention has to be decided before observing a possible claim at the same time). Note that g and Mg are continuous and R^{S^g} has càdlàg paths between interventions. Because of the relation to the optimal stopping problem given by (5), we have that $\tau_{i-1} < \tau_i$, as otherwise the decision at τ_{i-1} cannot be optimal (this relation is pointed out in the proof of the following proposition). Finally, the prescribed choice of $Z_i^g > a$ (a maximizer exists because u and V are continuous) and the fact that at most $(x + cT)/a$ interventions in the time interval $[0, T]$ can be done, ensure the admissibility of such a strategy S^g.

We can state the following verification theorem.

Proposition 4.2. *The strategy S^V is optimal and $V \geq V_S$ for all $S \in \mathcal{S}$. Further, we have that V is the smallest increasing, absolutely continuous and linearly bounded solution to the QVI (12).*

Proof. Let $R^S = (R_t^S)_{t\geq 0}$ denote the free reserve controlled by a strategy $S = \{(\tau_i, Z_i)_{i\in\mathbb{N}}\} \in \mathcal{S}$ and let τ^S be the associated time of ruin. By following the dynamics of the uncontrolled reserve between interventions and taking the dividend payments into account, we get the following

$$\begin{aligned}
&\mathbb{E}_x\left(e^{-\delta(t\wedge\tau^S)}V(R^S_{t\wedge\tau^S})\right) = V(x)\\
&+\mathbb{E}_x\left(\sum_{i=1}^{\infty} I_{\tau_i<(t\wedge\tau^S)}\int_{\tau_{i-1}}^{\tau_i} e^{-\delta s}\left(cV'(R^S_{s-}) - (\delta+\lambda)V(R^S_{s-}) + \lambda\int_0^{R^S_{s-}} V(R^S_{s-} - y)dF_Y(y)\right)ds\right)\\
&\qquad + \mathbb{E}_x\left(\sum_{i=1}^{\infty} e^{-\delta\tau_i}\left(V(R^S_{\tau_i} - Z_i) - V(R^S_{\tau_i})\right) I_{\tau_i<(t\wedge\tau^S)}\right). \qquad (16)
\end{aligned}$$

Because V solves the QVI (12) ($\mathcal{L}V \leq 0$) and for a general strategy S we have $V(R^S_{\tau_i}) \geq u(Z_i) + V(R^S_{\tau_i} - Z_i)$, we arrive at

$$\mathbb{E}_x\left(e^{-\delta(t\wedge\tau^S)}V(R^S_{t\wedge\tau^S})\right) + \mathbb{E}_x\left(\sum_{i=1}^{\infty} e^{-\delta\tau_i}u(Z_i)I_{\tau_i<(t\wedge\tau^S)}\right) \leq V(x).$$

Letting $t \to \infty$ and using the linear boundedness of V, we get by bounded and monotone convergence

$$V_S(x) \leq V(x).$$

Looking at the strategy S^V, we get $V(R^{S^V}_{\tau^V_i}) = u(Z^V_i) + V(R^{S^V}_{\tau^V_i} - Z^V_i)$ and for $s \in (\tau^V_{i-1}, \tau^V_i)$ we have

$$\mathcal{L}V(R^{S^V}_{s-}) = 0.$$

From (16) we obtain as above that $V_{S^V}(x) = V(x)$ which proves the optimality of this strategy.
Similarly, as done for V in the derivation of (15), one can obtain for every fixed point g of the operator $\mathcal{M}$ (5), which is dominating V, that it is a solution to (12). On the other hand, we have to prove that a solution to (12) is a fixed point of (5) and therefore dominates V.
From now on let g be an increasing, absolutely continuous and linearly bounded solution to the QVI (12) and S^g the associated QVI control. We observe that for $s < \tau^g_1$ the reserve controlled by S^g follows the dynamics of the uncontrolled reserve and that g is in the domain of its generator,

$$g(x) = \mathbb{E}_x\left(e^{-\delta\tau^g_1}g(R_{\tau^g_1}) - \int_0^{\tau^g_1} e^{-\delta s}\mathcal{L}g(R_{s-})ds\right). \tag{17}$$

We have by construction of S^g that the integral part of the right hand side is equal to zero. Now, in view of (5),

$$\mathcal{M}g(x) = \sup_{\theta\in\mathcal{T}} \mathbb{E}_x\left(e^{-\delta\theta}\, Mg(R_\theta)\right).$$

Observe from (12) that in general $Mg(R_\theta) \leq g(R_\theta)$. But for $\theta = \tau^g_1$ we have $Mg(R_{\tau^g_1}) = g(R_{\tau^g_1})$. Therefore we conclude from (17) that

$$g(x) = \mathbb{E}_x\left(e^{-\delta\tau^g_1}Mg(R_{\tau^g_1})\right) = \sup_{\theta\in\mathcal{T}} \mathbb{E}_x\left(e^{-\delta\theta}\, Mg(R_\theta)\right)$$

holds, i.e. g is a fixed point of $\mathcal{M}$, and therefore dominates V. We have proven that V is the smallest solution to the QVI (12) in the set of increasing, absolutely continuous and linearly bounded functions. □

5 Computations

5.1 Strategies of the form $b = \{b_1, b_2\}$

From other results on impulse control problems for diffusion processes (e.g. [4], [10] and [17]) one knows that strategies of the following type could be optimal: below a certain level

$b_2 > a$ the process is not controlled, when hitting this level (from below) a fixed amount is paid out as dividend. This fixed amount can be represented as another level $0 \le b_1 < b_2 - a$ such that the payment is of size $b_2 - b_1$. Let the value of such a strategy $b = \{b_1, b_2\}$ be $V_b(x)$. We get for $x < b_2$ by the Markov property and by the continuously increasing paths of the uncontrolled reserve process that

$$V_b(x) = \mathbb{E}_x\left(e^{-\delta\theta(b_2)}\right) V_b(b_2),$$

where $\theta(w) = \inf\{t > 0 \mid R_t = w \text{ and } R_s \ge 0 \text{ for } 0 \le s \le t\}$. From e.g. [20] we know that $f(x) = \mathbb{E}_x\left(e^{-\delta\theta(b_2)}\right)$ is the unique solution to $\mathcal{L}f = 0$ and $f(b_2) = 1$. Using $V_b(b_2) = u(b_2 - b_1) + V_b(b_1)$ we get

$$V_b(b_2) = \frac{u(b_2 - b_1)}{1 - f(b_1)}.$$

It remains to deal with the case $x > b_2$. In the related literature usually the continuation $u(x - b_1) + V_b(b_1)$ is chosen. But in our *utility* framework, and from an optimization point of view (and a numerical example below), the choice $\sup_{\{x-b_2<z<x-a\}} u(z) + V_b(x - z)$ seems to be more appropriate. Fixing b we have

$$V_b(x) = \begin{cases} f(x)\frac{u(b_2-b_1)}{1-f(b_1)}, & x \le b_2, \\ \sup_{\{x-b_2<z<x-a\}} u(z) + f(x-z)\frac{u(b_2-b_1)}{1-f(b_1)}, & x > b_2. \end{cases}$$

In [15], Loeffen derives similar expressions for values of such strategies, for the case $\gamma = 1$, in terms of scale functions of spectrally negative Lévy processes. In order to find a maximizing strategy within this restricted class, he proposes to maximize $\frac{u(b_2-b_1)}{1-f(b_1)}$ over admissible pairs $b = \{b_1, b_2\}$. Furthermore he shows that if the Lévy measure has a log-convex density, then there exists a strategy $b^* = \{b_1^*, b_2^*\}$ which is optimal in the set of all admissible strategies. But notice that for Gamma distributed claims he gives an example where a $b = \{b_1, b_2\}$ strategy can not be optimal.
In the case of a mean-reverting diffusion risk reserve process and $\gamma \in (0, 1]$, additional strong assumptions for the optimality of a $b = \{b_1, b_2\}$ strategy are needed, see [4]. Paulsen [17] proves for a general diffusion risk reserve model and linear utility that if an optimal strategy exists, it is of the $b = \{b_1, b_2\}$ type.

For exponentially distributed claim amounts it is possible to compute $V_b(x)$ for some $b = \{b_1, b_2\}$, up to the *supremum* continuation, explicitly. Because of the results from [15] we know that for $\gamma = 1$ and exponentially distributed claim amounts a certain $b = \{b_1, b_2\}$ strategy is the optimal one. For $0 < \gamma < 1$ such a result is not proven up to now, but one expects that the general methods used in [15] should also be applicable for this case. With the QVI formulation of Section 4 one can at least show for certain parameter sets that such a strategy is optimal.
For instance, for $c = 2.5$, $\lambda = 1$, $\delta = 0.03$, $k = 0.99$, $K = 0.1$, $\gamma = 0.7$ and $Exp(2)$ distributed claims the strategy given by $b^* = (4.61, 4.95)$ is indeed optimal (by verifying that V_{b^*} fulfills the QVI (12)).
Figure 1 shows the value function V_{b^*} for these parameters and Figure 2 shows the difference of the possible continuations from b_2^* onwards. One observes that the *supremum* continuation improves the common one.

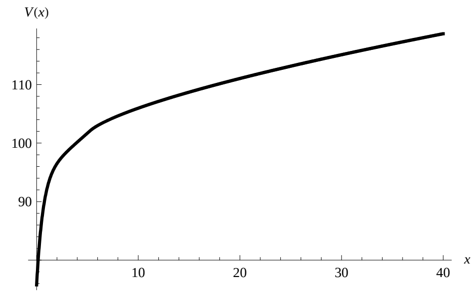

Figure 1: Value function

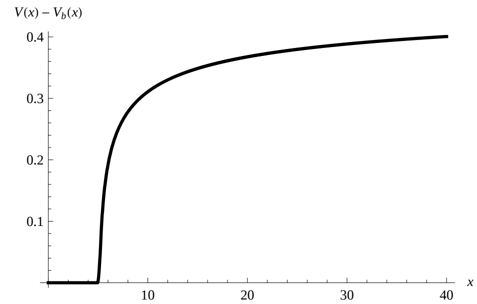

Figure 2: Difference of *sup* and *simple* continuation

5.2 A numerical scheme - policy iteration

For getting (approximate) solutions to the optimization problem for claim amount distributions beyond the exponential distribution, we implement a policy iteration algorithm (also known as Howard algorithm) for this specific problem. The idea is mainly based on Rogers [18] and Chancelier et al. [5].

At first we need to fix a discretization width $h > 0$ and some $N \in \mathbb{N}$ for approximating V on the interval $[0, hN]$. Note that a strategy divides the set $\{0, h, \dots, Nh\}$ into two parts A (points at which no dividends are paid) and B (points at which a certain dividend is paid). Let us denote B as *active* set and A as *non-active* set. The starting point of the algorithm is an admissible strategy $S_0 \in \mathcal{S}$ (with associated A_0 and B_0) and its associate value v_0, which we can in the best case compute explicitly (a simple type of strategy as introduced above seems to be appropriate in most cases). If

$$\max\{\mathcal{L}v_0, Mv_0 - v_0\} = 0$$

holds, one is finished. If not, we go through each point $i\,h \in A_0$, $0 \le i \le N$ and check if $v_0(ih) \le Mv_0(ih)$ holds. If this is the case, this point is moved to B_1 and in the end we set $A_1 = \{0, h, \dots, hN\} \setminus B_1$ (updating *active* and *non-active* sets). In the following step we compute v_1 as a solution to

$$cf'(x) - (\delta + \lambda)v_0(x) + \lambda \int_0^x f(x-y)\, dF_Y(y) = 0, \tag{18}$$

for $x \in A$ and set $v_1(x) = Mv_0(x)$ for $x \in B_1$ (note that as suggested by Rogers for solving a classical control problem, using $-(\delta + \lambda)v_0(x)$ instead of $-(\delta + \lambda)f(x)$ in (18) makes the procedure for solving the integro-differential equation more stable). This algorithm gives us an admissible strategy S_1, taking the local maximizers from Mv_0 and its value v_1 into account. Iterating this procedure (maybe up to a number n of iterations such that $\sup_{x\in\{0,h,\dots,hN\}}(v_n(x) - v_{n-1}(x))$ is smaller than a specified level) gives us an approximation of V and some sort of recommendable admissible strategy. For verifying this procedure one can plug the constructed solution into the QVI (12) for getting some sort of numerical verification, but notice that because of the numerical local maximizers and the non-local structure of the first part of the QVI, this can become a vague statement.

Of course it will be hard, or even impossible to determine the optimal strategy exactly, but one will at least get a good guess of the structure of the optimal policy and an approximation for V. The constructed sequence of values of admissible strategies is increasing and bounded

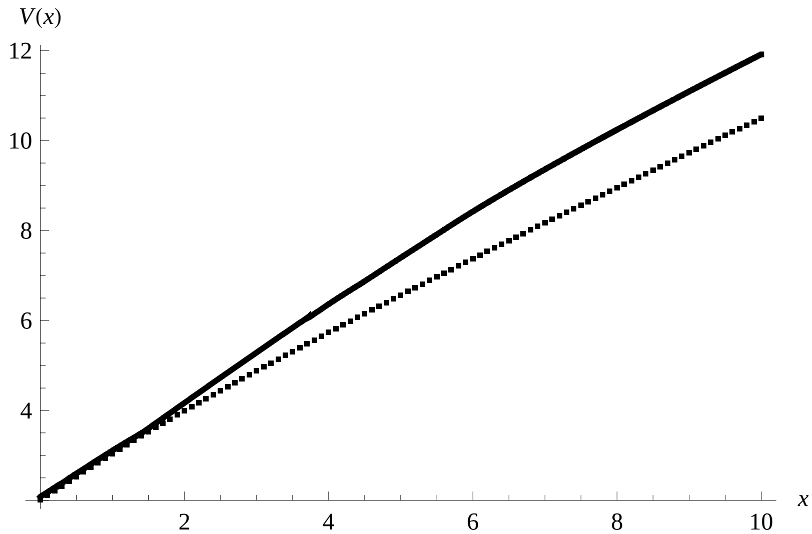

Figure 3: Value functions of iterated and best simple strategy

by the value function of the optimization problem, which itself is linearly bounded, and therefore it converges. Because at every step an admissible strategy is constructed, the procedure converges (formally) to the correct (the smallest) solution of (12).

Remark 5.1. When calculating the solution in the nth step of the algorithm, v_{n-1} and the *payoff* Mv_{n-1} are known and one determines v_n as a solution to

$$\max\{\mathcal{L}v_n, Mv_{n-1} - v_n\} = 0.$$

This is the way one proceeds when calculating V via the iterated optimal stopping characterization when writing the optimal stopping problem in a QVI form. This representation can be made rigorous by the same means as used in Section 4 for proving the QVI representation of V. When doing this computation formally in an exact way one needs to determine optimal *active* and *non-active* sets at each step while policy iteration adapts these sets from one iteration to another.

By these means we are able to demonstrate that as in the classical dividend maximization problem for the compound Poisson model (see [1] or [20]), it is possible that the optimal strategy is of a *band* structure. Figure 3 and 4 show the value function (full line: policy iterated one, dotted line: best $b = \{b_1, b_2\}$ strategy) and its associated *band* type strategy for the following set of parameters: $c = 21.4,\ \lambda = 10,\ \delta = 0.1,\ k = 0.95,\ K = 0.05,\ \gamma = 0.9$ and $Erlang(2,1)$ distributed claim amounts. For the linear utility case this parameter set turns out to admit no $b = \{b_1, b_2\}$ type optimal strategy, see [15]. In Figure 4 positive values mark the action set whereas zero values mark the non action set. The depicted heights of the action points are the heights of the (approximative) optimal dividend payments.

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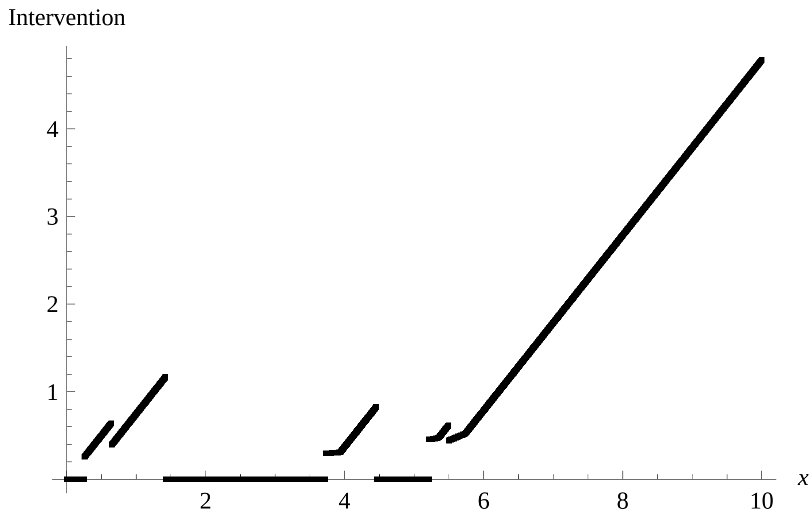

Figure 4: Strategy obtained by iteration

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