EXACT ASYMPTOTICS FOR TYPE I BIVARIATE ELLIPTICAL DISTRIBUTIONS

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ABSTRACT. Let (S_1, S_2) be a bivariate spherical random vector with associated random radius which has distribution function in the Gumbel max-domain of attraction. In this paper we obtain an exact asymptotic expansion of the tail probability $\boldsymbol{P}\{S_1 > u_n, \rho_n S_1 + \sqrt{1-\rho_n^2} S_2 > v_n\}, \rho_n \in (-1, 1)$ with $u_n, v_n, n \geq 1$ constants letting $u_n \to \infty$ and $\rho_n \to \rho \in (-1, 1)$. As an application of our result the limit distribution of the joint and the partial excess distribution is obtained.

1. INTRODUCTION

Let (S_1, S_2) be a spherical random vector with associated random radius $R := \sqrt{S_1^2 + S_2^2} > 0$ almost surely. Basic properties of spherical random vectors are obtained in Cambanis et al. (1981). So if $R > 0$ almost surely, then we have the stochastic representation

$$(S_1, S_2) \stackrel{d}{=} (RO_1, RO_2),$$

with (O_1, O_2) uniformly distributed on the unit circle of $\mathbb{R}^2$ being further independent of R ($\stackrel{d}{=}$ stands for equality of distribution functions).
A natural generalisation of this class is the class of elliptical random vectors, defined as linear combination of spherical random vectors. Elliptical random vectors are both from the theoretical and the practical point of view very interesting. This class is very large, including the prominent Gaussian and Kotz distribution.
Throughout this paper we consider elliptical random vectors $(X_0, Y_0), (X_1, Y_1), \ldots$ in $\mathbb{R}^2$ with stochastic representation

$$(1) \qquad (X_n, Y_n) \stackrel{d}{=} (S_1, \rho_n S_1 + \sqrt{1-\rho_n^2} S_2), \quad \rho_n \in (-1, 1), \quad n \geq 0.$$

The basic distribution properties of elliptical random vectors are well-known, see e.g., Kotz (1975), Cambanis et al. (1981), Anderson and Fang (1990), Fang et. al (1990), Fang and Zhang (1990), Szabłowski (1990), Berman (1992), Gupta and Varga (1993), Kano (1994), Kotz and Ostrovskii (1994) among several others.
The main asymptotic properties of bivariate elliptical random vectors are derived by Berman (1982,1983) culminating in his excellent monograph Berman (1992). Berman's focus was the asymptotic properties of the Berman processes. The work of Berman has been therefore not referred for a long time in the literature of multivariate distributions. Similar results for bivariate spherical random vectors are

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obtained in Carnal (1970), Gale (1980), Eddy and Gale (1981) in the context of convex hull asymptotics.
In this paper we are interested in the exact asymptotics of the tail probability

$$\boldsymbol{P}\{X_n > u_n, Y_n > v_n\} = \quad u_n, v_n \in \mathbb{R}, n \geq 1 \tag{2}$$

letting u_n tend to ∞.
Intuitively, since the associated random radius R is the only unknown component of the elliptical random vectors, we expect that its tail asymptotic behaviour determines the asymptotic behaviour of (2). This is the case for the Gaussian random vectors (see e.g., Hashorva and Hüsler (2003) or Hashorva (2005a)).
Indeed the Gaussian case has been treated in very many papers. The result for the case $u_n = v_n, n \geq 1$ is given in Berman (1962). See Dai and Mukherjea (2001) or Hashorva (2005a) for further references.
The square of the associated random radius of a d-dimensional Gaussian vector is chi-squared distributed with d degrees of freedom. From the extreme value theory we know that R in the Gaussian case has distribution function F in the max-domain of attraction of the Gumbel distribution function $\Lambda(x) := \exp(-\exp(-x)), x \in \mathbb{R}$.
Motivated by this fact, in the recent paper Hashorva (2006b) an asymptotic expansion of the tail probability for a general multivariate setup is obtained. Those results can be applied to our case when ρ_n does not depend on n.
Making use of a tractable formula for the bivariate elliptical distributions we obtain in this paper the asymptotic expansion of the tail probability of interest allowing ρ_n to depend on n, and provide a simpler proof than that in the aforementioned paper.
Further, we apply our result to study the asymptotics of bivariate excess distributions.

2. Preliminaries

In this section we present some standard notation and give few preliminary results. The main results are given in Section 3, followed by the proofs in Section 4 (last one).
Given a random variable Y with distribution function H, we shall denote this alternatively as $Y \sim H$. If F is the Gamma distribution with positive parameters a, b we write $Y \sim Gamma(a, b)$.
Next, let $(X_n, Y_n), n \geq 0$ be a bivariate elliptical random vector as in (1), and write throughout this paper $(X, Y), \rho$ instead of $(X_0, Y_0), \rho_0$.
We assume in the following that the associated random radius R has distribution function F such that $F(0) = 0$. Further, we impose a certain asymptotic restriction on the distribution function F, namely we suppose that there exists a positive scaling function w such that

$$\lim_{u \uparrow x_F} \frac{1 - F(u + x/w(u))}{1 - F(u)} = \exp(-x), \quad \forall x \in \mathbb{R} \tag{3}$$

is valid with $x_F \in (0, \infty]$ the upper endpoint of F. The above condition is equivalent (see the standard monographs de Haan (1970), Leadbetter et al. (1983), Galambos (1987), Resnick (1987), Reiss (1989), Falk et al. (2004) or Kotz and Nadarajah (2005), or de Haan and Ferreira (2006)) with the fact that F is in the Gumbel max-domain of attraction, meaning the sample maxima of a random sample with

underlying distribution function F converges in distribution (after an affine normalisation) to a Gumbel random variable.
We refer to (X, Y) in the case F satisfies (3) as Type I elliptical random vector. The scaling function w can be defined by

$$w(u) \quad := \quad \frac{1 - F(u)}{\int_u^{x_F} [1 - F(s)]\, ds}, \quad u \in (0, x_F]. \tag{4}$$

Further, uniformly on the compact sets of $z \in \mathbb{R}$

$$\lim_{u \uparrow x_F} \frac{w(u + z/w(u))}{w(u)} \quad = \quad 1, \tag{5}$$

and

$$\lim_{u \uparrow x_F} uw(u) \quad = \quad \infty. \tag{6}$$

In view of Lemma 6.2 of Berman (1982) (given also in Lemma 12.1.2 in Berman (1992))

$$aS_1 + bS_2 \stackrel{d}{=} \sqrt{a^2 + b^2} S_1, \quad \forall a, b \in \mathbb{R},$$

hence for $(X_n, Y_n) \stackrel{d}{=} (S_1, \rho_n S_1 + \sqrt{1 - \rho_n^2} S_2)$ with $\rho_n \in (-1, 1), n \geq 0$ (as in (1)) we have

$$X_n \stackrel{d}{=} Y_n \stackrel{d}{=} S_1. \tag{7}$$

Applying Theorem 12.3.1 of Berman (1992) we obtain $(n \to \infty)$

$$\boldsymbol{P}\{X > u_n\} = \boldsymbol{P}\{S_1 > u_n\} \quad = \quad (1 + o(1)) \left(\frac{1}{u_n w(u_n)} \right)^{1/2} \frac{1}{\sqrt{2\pi}} [1 - F(u_n)], \tag{8}$$

provided that $\lim_{n \to \infty} u_n = x_F \in (0, \infty]$ and F satisfies (3) with the scaling function w. Consequently, when (X_n, Y_n) is a Type I elliptical random vector, then the asymptotic tail behaviour of its components is known. In the special case that (X_n, Y_n) has independent components we have

$$\boldsymbol{P}\{X_n > u_n, Y_n > v_n\} = \boldsymbol{P}\{X_n > u_n\} \boldsymbol{P}\{Y_n > v_n\}, \quad n \geq 1,$$

hence for this instance there is nothing to investigate.
Provided that (X_n, Y_n) has a density function, we know that X_n and Y_n are independent (see e.g., Fang et al. (1990), Hashorva et al. (2007)) only when X_n and Y_n are standard Gaussian random variables. Therefore the above simplification of our problem of interest is only possible for a trivial case. In the case (X_n, Y_n) is Gaussian and X_n, Y_n are correlated ($\rho_n \neq 0$), the exact asymptotics of the probability of interest is know (see e.g., Hashorva (2005a)). The general elliptical case is derived in Hashorva (2006b).
Next, we consider briefly the bivariate Gaussian case and then give a conditional limiting result which will be utilised in Section 3. Assume for simplicity that

$$v_n = a u_n, \quad a \in (-\infty, 1], \quad n \geq 1,$$

and $u_n, n \geq 1$ is a positive sequence converging to infinity.

It turns out that the correlation ρ plays via the Savage condition (see e.g., Hashorva and Hüsler (2003)) a crucial role in determining the joint tail asymptotic behaviour of (X, Y). In the bivariate case this condition is very simple to formulate, namely if $a > \rho$ we have

$$\boldsymbol{P}\{X > u_n, Y > au_n\} = (1+o(1))C_{a,\rho}\frac{\exp(-(u_n\boldsymbol{\alpha}_{a,\rho})^2/2)}{2\pi u_n^2}, \quad n \to \infty,$$

with

$$\boldsymbol{\alpha}_{a,\rho} := \sqrt{(1-2a\rho+a^2)/(1-\rho^2)} > 1, \; C_{a,\rho} := \frac{(1-\rho^2)^{3/2}}{(1-a\rho)(a-\rho)} > 0. \tag{9}$$

If $a \le \rho$ then

$$\boldsymbol{P}\{X > u_n, Y > au_n\} = (1+o(1))\mathbf{1}_{\rho,a}\frac{\exp(-u_n^2/2)}{\sqrt{2\pi}u_n}, \quad n \to \infty \tag{10}$$

is valid with $\mathbf{1}_{\rho,a} := 1/2$ if $\rho = a$ and $\mathbf{1}_{\rho,a} := 1$, otherwise.
If the Savage condition holds, i.e., $\rho > a$ then $\boldsymbol{\alpha}_{a,\rho} > 1$, implying that the joint tail asymptotics is faster than the convergence rate to 0 of $\boldsymbol{P}\{X > u_n\}$. Moreover, the speed of the convergence is governed by $\boldsymbol{\alpha}_{a,\rho}$, which is actually the attained minimum of a related quadratic programming problem (see e.g., Hashorva (2005a)). If the Savage condition does not hold, then (10) shows that the asymptotics is of the same rate as of $\boldsymbol{P}\{X > u_n\}, n \to \infty$. The later asymptotics is well-known and related to Mills Ratio (see e.g., Berman (1962)).
In the Gaussian case (3) holds with $w(t) = t, t > 0$, hence we may write (10) using further (8)

$$\begin{aligned}&\boldsymbol{P}\{X > u_n, Y > au_n\}\\ &= (1+o(1))\mathbf{1}_{\rho,a}\boldsymbol{P}\{X > u_n\}\\ &= (1+o(1))\mathbf{1}_{\rho,a}\left(\frac{1}{u_n w(u_n)}\right)^{1/2}\frac{1}{\sqrt{2\pi}}[1-F(u_n)], \quad n \to \infty\end{aligned}$$

showing that the asymptotics is defined by $1 - F(u_n)$ and $u_n w(u_n)$. This is the case for Type I elliptical random vectors in general as shown in Hashorva (2006b) (corresponding to our case ρ not depending on n). We shall present in this paper another proof of that result (see Theorem 2 below), and consider further the case ρ_n depends on n.
Finally for ease of reference we present next a conditional limiting theorem proved in Theorem 4.1 of Berman (1983) (see also Theorem 12.4.1 of Berman (1992)). That result first appears in Lemma 8.2 of Berman (1982) (with some additional restrictions). More special case are dealt with in Gale (1980), Eddy and Gale (1981). See for details Abdous et al. (2005), Abdous et al. (2006), Hashorva (2006a), or Hashorva et al. (2007).
Recent deep articles on the subjects are Heffernan and Tawn (2004), Butler and Tawn (2005) and Heffernan and Resnick (2005).

We denote throughout the paper a Gaussian random variable with mean 0 and variance $\sqrt{1-\rho^2} \in (0,1]$ by Z_ρ, i.e.,

$$Z_\rho \stackrel{d}{=} \sqrt{1-\rho^2}W, \tag{11}$$

where W is a standard Gaussian random variable.

Theorem 1. *[Berman (1992)] Let (S_1, S_2) be a spherical bivariate random vector with associated random radius $R := \sqrt{S_1^2 + S_2^2} > 0$ almost surely. If the distribution* *function F of R satisfies* (3) *with the scaling function w, then we have for any $\rho \in (-1,1)$ and $u_n < x_F, n \ge 1$ such that* $\lim_{n\to\infty} u_n = x_F$

$$q_n\Big(\rho S_1 + \sqrt{1-\rho^2} S_2 - \rho u_n\Big)\big|S_1 > u_n \quad \overset{d}{\to} \quad Z_\rho, \quad n \to \infty, \tag{12}$$

with $q_n := \sqrt{w(u_n)/u_n}$ *and Z_ρ as in* (11).

For the case $\rho = 0$ the proof is given in Theorem 12.4.1 of Berman (1992). The case $\rho \in (-1,1)$ is proved in Berman (1992) in Theorem 12.5.1 (see (12.5.5)). In fact from (12.5.7) therein we have the convergence in probability

$$q_n|S_1 - u_n|\big|S_1 > u_n \quad \overset{p}{\to} \quad 0, \quad n \to \infty, \tag{13}$$

hence the proof of the case $\rho \neq 0$ is a simple consequence of Theorem 12.4.1 of Berman (1992) and (13).

3. Main Results

In this section we consider bivariate elliptical random vectors $(X,Y),(X_1,X_2),\ldots$ with stochastic representation (1) and associated random radius $R \sim F$. We consider for simplicity only the case F has an infinite upper endpoint.
Given two sequences $u_n, v_n, n \ge 1$ we derive in the main result below an asymptotic expansion for $\boldsymbol{P}\{X_n > u_n, Y_n > v_n\}$ letting u_n tend to ∞. For $v_n, n \ge 1$ we require that $\lim_{n\to\infty} v_n/u_n = a \in (-\infty,1]$. As illustrated by the Gaussian example above the pseudo-correlation coefficient ρ (recall (1)) plays a central role for the asymptotics via the Savage condition.
The main assumption in this section is that (X_n,Y_n) is a Type I elliptical random vector, i.e., the distribution function F of the associated random radius R is in the Gumbel max-domain of attraction.

Theorem 2. *Let $(X,Y),(X_1,Y_1),\ldots$ be Type I bivariate elliptical random vector with stochastic representation* (1)*, where $\rho, \rho_n \in (-1,1), n \ge 1$, and let $u_n, v_n \in \mathbb{R}, n \ge 1$ be given constants such that $\lim_{n\to\infty} u_n = \infty$. Assume that the associated random radius $R \sim F$ is almost surely positive with F in the Gumbel max-domain of attraction satisfying* (3) *with the positive scaling function w, and upper endpoint $x_F = \infty$. Suppose further that $\lim_{n\to\infty} \rho_n = \rho \in (-1,1)$ and let Z_ρ be as in* (11).
i) If for some $z \in [-\infty,\infty)$

$$\lim_{n\to\infty} q_n[v_n - \rho_n u_n] = z \tag{14}$$

holds with $q_n := \sqrt{w(u_n)/u_n}, n \ge 1$, then for any sequence $y_n \in \mathbb{R}, n \ge 1$ such that $\lim_{n\to\infty} y_n = y \in [-\infty,\infty)$

$$\begin{aligned}
&\boldsymbol{P}\{X_n > u_n, Y_n > v_n + y_n/q_n\} \\
&= (1+o(1))\boldsymbol{P}\{Z_\rho > y+z\}\frac{1}{\sqrt{2\pi}}\left(\frac{1}{u_n w(u_n)}\right)^{1/2}[1-F(u_n)] \qquad (15)\\
&= (1+o(1))\boldsymbol{P}\{Z_\rho > y+z\}\boldsymbol{P}\{X > u_n\} \qquad (16)
\end{aligned}$$

holds as $n \to \infty$.
ii) Set $a_n := v_n/u_n, n \ge 1$ and suppose further that $a_n \in (\rho_n, 1]$ for all large n and

$$\lim_{n\to\infty} a_n = a \in (\rho, 1]. \tag{17}$$

Then we have

$$\begin{aligned}
&\boldsymbol{P}\{X_n > u_n, Y_n > v_n\} \\
(18)\qquad &= (1+o(1))\frac{\boldsymbol{\alpha}_{a,\rho} C_{a,\rho}}{2\pi}\frac{1}{u_n w(u_n^*)}[1-F(u_n^*)] \\
(19)\qquad &= (1+o(1))\frac{\boldsymbol{\alpha}_{a,\rho}^2 C_{a,\rho}}{\sqrt{2\pi}}\left(\frac{1}{u_n^* w(u_n^*)}\right)^{1/2}\boldsymbol{P}\{X > u_n^*\}, \quad n\to\infty,
\end{aligned}$$

with $\boldsymbol{\alpha}_{a,\rho}, C_{a,\rho}$ *as in* (9) *and* $u_n^* := \boldsymbol{\alpha}_{n,a,\rho} u_n, n \ge 1$ *where*

$$(20)\qquad \boldsymbol{\alpha}_{n,a,\rho} := \sqrt{(1-2a_n\rho_n + a_n^2)/(1-\rho^2)} \quad\to\quad \boldsymbol{\alpha}_{a,\rho} > 1, \quad n\to\infty.$$

Remarks 1. a) Since F is in the Gumbel max-domain of attraction we have

$$\lim_{t\to\infty}\frac{1-F(ct)}{1-F(t)} = 0, \quad \forall c>1,$$

hence (8) yields also

$$\lim_{t\to\infty}\frac{\boldsymbol{P}\{X_n > ct\}}{\boldsymbol{P}\{X_n > t\}} = \lim_{t\to\infty}\frac{\boldsymbol{P}\{S_1 > ct\}}{\boldsymbol{P}\{S_1 > t\}} = 0, \quad \forall c>1.$$

Consequently by (20)

$$\lim_{n\to\infty}\frac{\boldsymbol{P}\{X_n > u_n^*\}}{\boldsymbol{P}\{X_n > u_n\}} = 0.$$

Further (6) implies

$$(21)\qquad \lim_{n\to\infty} u_n w(u_n) = \lim_{n\to\infty} u_n w(u_n^*) \quad=\quad \infty,$$

hence the asymptotics in (18) is faster than the one in (15).
b) If the distribution function F has a finite upper endpoint $x_F \in (0,\infty)$, then the first statement above still holds for $u_n \to x_F$ as $n\to\infty$ and $v_n, n\ge 1$ satisfying further

$$(22)\qquad u_n^2 - 2\rho_n u_n v_n + v_n^2 \quad<\quad 1-\rho_n^2, \quad n\ge 1.$$

c) Our asymptotic results in the above theorem confirm (for the case $\rho_n = \rho, n\ge 1$) the ones previously obtained in Hashorva (2006b).
Note that if ρ_n depends on n, then the rate of convergence in (18) depends explicitly on ρ_n. Furthermore, the conditions leading to both statements above need to be formulated with ρ_n instead of ρ (see (14)).

In the special case (which is common in applications), namely $v_n = u_n a, \rho_n = \rho, n\ge 1$ with $a\in(-\infty,1]$ and $\rho\in(-1,1)$ we have:

Corollary 3. *Under the assumptions and the notation of Theorem 2, if further* $a_n = a \in (-\infty, 1]$ *for all large* n *then we have:*
i) In the case $a < \rho$

$$(23)\qquad \lim_{n\to\infty}\frac{\boldsymbol{P}\{X > u_n, Y > a u_n + y_n/q_n\}}{\boldsymbol{P}\{X > u_n\}} \quad=\quad 1.$$

ii) In the case $a = \rho$

$$(24)\qquad \lim_{n\to\infty}\frac{\boldsymbol{P}\{X > u_n, Y > a u_n + y_n/q_n\}}{\boldsymbol{P}\{X > u_n\}} \quad=\quad \boldsymbol{P}\{Z_\rho > y\}.$$

iii) If $a \in (\rho, 1]$ then we have as $n \to \infty$

$$\begin{aligned} &\boldsymbol{P}\{X > u_n, Y > au_n\} \\ (25) \quad &= (1+o(1)) \frac{\boldsymbol{\alpha}_{a,\rho}^{3/2} C_{a,\rho}}{\sqrt{2\pi}} \left(\frac{1}{u_n w(\boldsymbol{\alpha}_{a,\rho} u_n)} \right)^{1/2} \boldsymbol{P}\{X > \boldsymbol{\alpha}_{a,\rho} u_n\}, \end{aligned}$$

with $C_{a,\rho}, \boldsymbol{\alpha}_{a,\rho}$ as in Theorem 2.

Corollary 4. *Under the assumptions of Theorem 2 we have:*
i) In the case $\rho \in (0,1)$

$$(26) \quad \boldsymbol{P}\{X > u_n, Y > y\} = (1+o(1))\boldsymbol{P}\{X > u_n\}, \quad n \to \infty$$

is valid for any $y \in \mathbb{R}$.
ii) In the case $\rho = 0$ and

$$(27) \quad \lim_{n\to\infty} \left(\frac{w(u_n)}{u_n} \right)^{1/2} = b \in [0, \infty),$$

we have for $y \in \mathbb{R}$

$$(28) \quad \boldsymbol{P}\{X > u_n, Y > y\} = (1+o(1))\boldsymbol{P}\{Z_\rho > by\}\boldsymbol{P}\{X > u_n\}, \quad n \to \infty.$$

iii) If $\rho \in (-1, 0)$ then for any $y > 0$

$$\begin{aligned} &\boldsymbol{P}\{X > u_n, Y > y\} \\ (29) \quad &= (1+o(1)) \left(\frac{(1-\rho^2)^3}{2\pi\rho^2} \right)^{1/2} \left(\frac{1}{u_n w(u_n^*))} \right)^{1/2} \boldsymbol{P}\{X > u_n^*\}, \quad n \to \infty, \end{aligned}$$

is valid with

$$u_n^* := \sqrt{(y^2 - 2\rho u_n y + u_n^2)/(1-\rho^2)}, \quad n \in \mathbb{N}.$$

Remark 1. The above corollary is important also in a distributional context. In view of (26) and (29), X and Y cannot be independent if (X, Y) is a Type I elliptical random vector with pseudo-correlation $\rho \in (-1, 1), \rho \neq 0$.
When X, Y are independent with (X, Y) Type I, then we have thus $\rho = 0$, hence if further (27) holds, then (28) implies that $Y \stackrel{d}{=} Z_\rho / b$.

We consider next an illustrating example:
Example 1. [**Kotz Type III**] Let $(X, Y) = R(O_1, \rho O_1 + \sqrt{1-\rho^2} O_2)$, with R a positive random radius independent of the bivariate random vector (O_1, O_2) which is uniformly distributed on the unit circle of $\mathbb{R}^2$. We call $\boldsymbol{X}$ a Kotz Type III elliptical random vector if further

$$\boldsymbol{P}\{R > u\} = (1+o(1))Ku^N \exp(-ru^\delta), \quad K > 0, \delta \in \mathbb{R}, N \in \mathbb{R}, \quad u \to \infty,$$

with $\delta \le 0$ if $N < 0$. We consider next only the case $\delta > 0$. Define the function w by

$$w(u) = r\delta u^{\delta-1}, \quad u > 0.$$

For any $x \in \mathbb{R}$ we have

$$\frac{\boldsymbol{P}\{R > u + x/w(u)\}}{\boldsymbol{P}\{R > u\}} = (1+o(1)) \exp\Big(-ru^\delta \Big[\Big(1 + \frac{x}{r\delta u^\delta}\Big)^\delta - 1 \Big] \Big) \to \exp(-x)$$

as $u \to \infty$, implying that F is in the Gumbel max-domain of attraction with the scaling function w. In view of (8) we have

$$\boldsymbol{P}\{X > u\} \quad = \quad (1+o(1))\frac{K}{\sqrt{2r\delta\pi}} u^{N-\delta/2}\exp(-ru^\delta), \quad u \to \infty.$$

Let $u_n \to \infty$ and $y_n \to y \in \mathbb{R}$ be two given sequence and let $a \in (-\infty, 1]$ be a given constant. Then by the above results we have if $\rho < a$

$$\begin{aligned}&\boldsymbol{P}\Big\{X > u_n, Y > au_n + y_n\sqrt{r\delta u_n^{\delta-1}/u_n}\Big\}\\ &= \quad (1+o(1))\frac{K}{\sqrt{2r\delta\pi}} u_n^{N-\delta/2}\exp(-ru_n^\delta), \quad n \to \infty.\end{aligned}$$

If $a = \rho$ similar asymptotics follows where the constant is additionally multiplied by $\boldsymbol{P}\{Z_\rho > y\}$.
Assuming that $a \in (\rho, 1]$ we obtain as $n \to \infty$

$$\boldsymbol{P}\{X > u_n, Y > au_n\} \quad = \quad (1+o(1))\frac{K\boldsymbol{\alpha}_{a,\rho}^{2-\delta+N}C_{a,\rho}}{2r\delta\pi} u_n^{N-\delta}\exp(-r(\boldsymbol{\alpha}_{a,\rho}u_n)^\delta).$$

Note in passing that the Gaussian case corresponds to the choice of parameters

$$K = 1, N = 0, \delta = 2, r = 1/2.$$

4. Approximation of Excess Distribution

Consider $(X, Y), (X_1, Y_1), \ldots$ Type I elliptical bivariate random vector with stochastic representation (1) and associated random radius $R \sim F$. Let $u_n, n \geq 1$ be a positive sequence such that $\lim_{n\to\infty} u_n = x_F, |u_n| < x_F, n \geq 1$. The random variable $X_n - u_n | X_n > u_n$ is the excess of X_n above the threshold u_n given X_n jumps the threshold.
An immediate consequence of the assumption F is in the Gumbel max-domain of attraction with the positive scaling function w is the convergence in distribution of the corresponding excess random variables above the threshold u_n

$$w(u_n)(X_n - u_n)|X_n > u_n \overset{d}{\to} U, \quad w(u_n)(Y_n - u_n)|Y_n > u_n \overset{d}{\to} U, \tag{30}$$

with $U \sim Gamma(1,1)$ a unit Exponential random variable.

Another interesting situation arises when we additionally condition on the other component being large, i.e., considering the joint excess bivariate random sequence (with respect to $u_n, v_n, n \geq 1$)

$$(X_{u_n,Y,v_n}, Y_{v_n,X,u_n}) \quad := \quad (X_n - u_n, Y_n - v_n)\big|X_n > u_n, Y_n > v_n, \quad n \geq 1.$$

With the above notation we can re-write (12) and (13) as

$$\Big(q_n(Y_n - \rho_n u_n), q_n(X_n - u_n)\Big)\Big|X_n > u_n \quad \overset{d}{\to} \quad \Big(Z_\rho, 0\Big), \quad n \to \infty, \tag{31}$$

where $q_n := \sqrt{w(u_n)/u_n}, n \geq 1$ and Z_ρ as in (11).
Convergence in distribution is stated in the next theorem, which is a slight modification of Berman's result presented in the previous section.

Theorem 5. *Let (S_1, S_2) be a Type I bivariate spherical random vector with associated random radius $R \sim F$, where F satisfies (3) with upper endpoint $x_F \in (0, \infty]$*

and the scaling function w. *Let* $u_n < x_F, n \ge 1, \rho_n \in (-1,1)$ *be constants such that* $\lim_{n\to\infty} u_n = x_F$ *and set*

$$X_n := S_1, \quad Y_n := \rho_n S_1 + \sqrt{1-\rho_n^2} S_2, n \ge 1.$$

If $\lim_{n\to\infty} \rho_n = \rho \in (-1,1)$, *then we have the convergence in distribution* $(n \to \infty)$

$$\Big(q_n\big(Y_n - \rho_n u_n\big), w(u_n)(X_n - u_n)\Big)\Big| X_n > u_n \quad \stackrel{d}{\to} \quad (Z_\rho, U), \quad n \to \infty, \tag{32}$$

where $q_n := \sqrt{w(u_n)/u_n}, n \ge 1$, *and* Z_ρ *as in* (11) *independent of* $U \sim Gamma(1,1)$.

Hashorva (2006b) obtains in Theorem 5.1 the convergence in distribution of the joint excess sequence $(X_{u_n,Y,u_n}, Y_{u_n,X,u_n}), n \ge 1$. A similar (independent) result appears in Asimit and Jones (2007) under the further restriction that the scaling function w is regularly varying and F has an infinite upper endpoint.
We apply our previous results to derive several approximations in the next theorem.

Theorem 6. *Let* $F, (X,Y), \rho, Z_\rho, (X_n, Y_n), \rho_n, a_n, u_n, v_n, n \ge 1$ *be as in Theorem 2*, F *satisfies* (3) *with the scaling function* w *and upper endpoint* $x_F = \infty$, *and let* $h_{ni}, n \ge 1, i = 1,2$ *be positive constants such that*

$$\lim_{n\to\infty} q_n h_{ni} \quad = \quad c_i \in [0,\infty), \quad i = 1,2, \tag{33}$$

with $q_n := \sqrt{w(u_n)/u_n}, n \ge 1$.
i) If (14) *holds with* $z \in [-\infty, \infty)$, *then we have for any* $x, y \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{\boldsymbol{P}\{X_{u_n,Y,v_n} > h_{n1}x, Y_{v_n,X,u_n} > h_{n2}y\}}{\boldsymbol{P}\{X > u_n + h_{n1}x | X > u_n\}} \quad = \quad \overline{\Phi}_{\rho,z}(c_2 y - \rho c_1 x), \tag{34}$$

where $\overline{\Phi}_{\rho,z}(s) := \boldsymbol{P}\{Z_\rho > s + z\}/\boldsymbol{P}\{Z_\rho > z\}, s \in \mathbb{R}$.
Furthermore, in the case that $z \in \mathbb{R}$ *we have the convergence in distribution*

$$\Big(w(u_n) X_{u_n,Y,v_n}, q_n Y_{v_n,X,u_n}\Big) \quad \stackrel{d}{\to} \quad (U, V_z), \quad n \to \infty, \tag{35}$$

with $U \sim Gamma(1,1), V_z \sim 1 - \overline{\Phi}_{\rho,z}$.
ii) Set $u_n^* := \sqrt{(u_n^2 - 2\rho_n u_n v_n + v_n^2)/(1-\rho_n^2)}, n \in \mathbb{N}$. *If further* (17) *is satisfied, we then have the convergence in distribution*

$$\Big(w(u_n^*) X_{u_n,Y,v_n}, w(u_n^*) Y_{v_n,X,u_n}\Big) \quad \stackrel{d}{\to} \quad (U_1, U_2), \quad n \to \infty, \tag{36}$$

where

$$U_1 \sim Gamma(1, \frac{1 - a\rho}{\boldsymbol{\alpha}_{a,\rho}(1-\rho^2)}), \quad U_2 \sim Gamma(1, \frac{a - \rho}{\boldsymbol{\alpha}_{a,\rho}(1-\rho^2)}),$$

with U_1, U_2 *being further independent and* $\boldsymbol{\alpha}_{a,\rho} := \sqrt{(1 - 2a\rho + a^2)/(1-\rho^2)} > 1$.

We give next an illustrating example.
Example 2. Let $X, Y, \rho, F, w, u_n, u_n^*, v_n, n \ge 1$ be as in Theorem 6. Assume that for all $c > 1$

$$\lim_{u\to\infty} \frac{w(cu)}{w(u)} \quad = \quad c^\lambda, \quad \lambda \in (-1, \infty). \tag{37}$$

If (17) holds, then we obtain (the convergence above is locally uniformly)

$$\lim_{n\to\infty} \frac{w(u_n^*)}{w(u_n)} = \frac{w(u_n \boldsymbol{\alpha}_{a,\rho})}{w(u_n)} = \boldsymbol{\alpha}_{a,\rho}^\lambda > 1,$$

hence for any x, y positive

$$\begin{aligned}
&\lim_{n\to\infty} \boldsymbol{P}\Big\{X > u_n + \frac{x}{w(u_n)}, Y > v_n + \frac{y}{w(u_n)} \big| X > u_n, Y > v_n\Big\} \\
&= \lim_{n\to\infty} \boldsymbol{P}\Big\{X > u_n + \frac{\boldsymbol{\alpha}_{a,\rho}^{-\lambda} x}{w(u_n)}, Y > v_n + \frac{\boldsymbol{\alpha}_{a,\rho}^{-\lambda} y}{w(u_n)} \big| X > u_n, Y > v_n\Big\} \\
&= \exp\Big(-\boldsymbol{\alpha}_{a,\rho}^{-\lambda}([\overline{K}_{a,\rho} x + K_{a,\rho} y]\Big) \\
&= \exp\Big(-\frac{1 - a\rho}{\boldsymbol{\alpha}_{a,\rho}^{\lambda+1}(1-\rho^2)} x + \frac{a - \rho}{\boldsymbol{\alpha}_{a,\rho}^{\lambda+1}(1-\rho^2)} y\Big).
\end{aligned}$$

Note in passing that if (X, Y) is a Kotz Type III elliptical random vector with $\delta > 0$ then (37) holds with $\lambda = \delta - 1$.

5. Proofs

For the proof of Theorem 2 we need the next lemma, which could be of some interest on its own.

Lemma 7. *Let F be a univariate distribution function with upper endpoint $x_F \in (0, \infty]$ such that F satisfies (3) with the positive scaling function w. Let further $a_n < b_n \le x_F, u_n, r_n, n \ge 1$ be four sequences of positive constants such that $u_n^* := a_n u_n < x_F, \forall n \ge 1$*

$$\lim_{n\to\infty} u_n^* = x_F, \quad \text{and} \quad \lim_{n\to\infty} u_n w(u_n^*)[b_n - a_n] = \eta \in [0, \infty]. \tag{38}$$

If further $\psi_n, h_n, n \ge 1$ are positive measurable functions such that for all large n

$$\psi_n(a_n + x/(u_n w(u_n^*))) = r_n h_n(x), \quad \forall x > 0, \tag{39}$$

where

$$\lim_{n\to\infty} h_n(x) = h(x),$$

and for all n large and any $x > 0$

$$h_n(x) \le K \max(x^{\lambda_1}, x^{\lambda_2}), \quad K \in (0, \infty), \lambda_i \in (-1, \infty), i = 1, 2,$$

is satisfied, then we have for any $\xi_n \to \xi \in [0, \infty)$ with $\xi \le \eta \le \infty$

$$\int_{a_n + \xi_n/(u_n w(u_n^*))}^{b_n} [1 - F(u_n x)]\psi_n(x)\, dx = (1 + o(1)) \frac{r_n[1 - F(u_n^*)]}{u_n w(u_n^*)} I(h, \eta, \xi)$$

as $n \to \infty$ with $I(h, \eta, \xi) := \int_\xi^\eta h(x) \exp(-x)\, ds \in [0, \infty)$.

Proof. Set for any $n \in \mathbb{N}$

$$u_n^* := a_n u_n, \quad t_n := u_n w(u_n^*), \quad \eta_n := t_n[b_n - a_n].$$

Since $\lim_{n\to\infty} u_n^* = x_F$ the assumption on F implies

$$\lim_{n\to\infty} \frac{1 - F(u_n^* + x/w(u_n^*))}{1 - F(u_n^*)} = \exp(-x), \quad \forall x \in \mathbb{R}.$$

Next, applying Fatou Lemma we obtain

$$
\begin{aligned}
&\liminf_{n\to\infty} \int_{a_n+\xi_n/w(u_n^*)}^{b_n} [1-F(u_n x)]\psi_n(x)\,dx \\
&\geq \liminf_{n\to\infty} t_n^{-1}\int_{\xi_n}^{\eta_n} [1-F(u_n[a_n+x/t_n])]\psi_n(a_n+x/t_n)\,dx \\
&\geq [1-F(u_n^*)]r_n t_n^{-1}\int_{\xi}^{\eta} \liminf_{n\to\infty} \frac{1-F(u_n^*+x/w(u_n^*))}{1-F(u_n^*)} h_n(x)\,dx \\
&= (1+o(1))r_n[1-F(u_n^*)]t_n^{-1}\int_{\xi}^{\eta} \exp(-x)h(x)\,dx, \quad n\to\infty.
\end{aligned}
$$

Since for all x positive it follows that $h(x)\leq K\max(x^{\lambda_1},x^{\lambda_2})$ we have

$$
0\leq \int_{\xi}^{\eta} \exp(-x)h(x)\,dx<\infty.
$$

The proof for the lim sup follows (non-trivially) along the lines of the proof of Lemma 4.2, 4.3, 4.5 in Hashorva (2006a) utilising ideas and results in Berman (1992). The case $\lambda\geq 0$ follows easily with the arguments from Berman (1992) (see (12.3.7) therein). The case $\alpha\in(-1,0)$ is established using further the fact that for any $\varepsilon>0, x\in[0,1]$

$$
\Big|\frac{1-F(\tau_n+x/w(\tau_n))}{1-F(\tau_n)}-\exp(-x)\Big|<\varepsilon\exp(-x)
$$

holds uniformly for any sequence $\tau_n<x_F, n\geq 1$, such that $\lim_{n\to\infty}\tau_n=x_F$, hence the proof. ☐

Proof of Theorem 2 Set for $n\in\mathbb{N}$

$$
a_n:=v_n/u_n,\quad w_n:=w(u_n),\quad q_n:=\sqrt{w_n/u_n}.
$$

(6) implies $\lim_{n\to\infty}u_n w_n=\infty$. In view of (12) we have

$$
q_n(Y_n-\rho_n u_n)|X_n>u_n \quad \overset{d}{\to} \quad Z_\rho, \quad n\to\infty,
$$

with $Z_\rho/\sqrt{1-\rho^2}$ a standard Gaussian random variable. Consequently for all n large we have

$$
\begin{aligned}
&\boldsymbol{P}\{X_n>u_n, Y_n>v_n+y_n/q_n\} \\
&= \boldsymbol{P}\{X>u_n\}\boldsymbol{P}\{q_n(Y_n-\rho_n u_n)>q_n[a_n u_n+y_n/q_n-\rho_n u_n]|X_n>u_n\} \\
&= \boldsymbol{P}\{X>u_n\}\boldsymbol{P}\{q_n(Y-\rho_n u_n)>y_n+\sqrt{u_n w_n}[a_n-\rho_n]|X>u_n\} \\
&= (1+o(1))\boldsymbol{P}\{Z_\rho>y+z\}\boldsymbol{P}\{X>u_n\}, \quad n\to\infty.
\end{aligned}
$$

Using now (8) establishes the first claim.
ii) For simplicity assume that $\rho,\rho_n\in[0,1], n\geq 1$ and $v_n, n\geq 1$ is a positive sequence. The other case follows with similar arguments.
In view of Lemma 3.3 of Hashorva (2005b) we obtain

$$
\begin{aligned}
&\boldsymbol{P}\{X_n>u_n, Y_n>a_n u_n\} \\
&= \frac{1}{2\pi}\int_{\beta_n}^{\pi/2}[1-F(x/\cos(\alpha))]\,d\alpha+\frac{1}{2\pi}\int_{\psi_n-\pi/2}^{\beta_n}[1-F(y/\cos(\alpha-\psi_n))]\,d\alpha \\
&=: I_{n1}+I_{n2},
\end{aligned}
$$

with $\beta_n := \arctan((a_n - \rho_n)/\sqrt{1-\rho_n^2}), \psi_n := \arccos(\rho_n), n \geq 1$.
Define next for any $n \in \mathbb{N}$

$$\boldsymbol{\alpha}_{n,a,\rho} := 1/\cos(\beta_n) = \sqrt{(1 - 2\rho_n a_n + a_n^2)/(1-\rho_n^2)} \geq 1,$$

and

$$u_n^* := \boldsymbol{\alpha}_{n,a,\rho} u_n \quad w_n^* := w(u_n^*).$$

By the assumptions

$$\lim_{n\to\infty} \beta_n = \beta := \arctan((a-\rho)/\sqrt{1-\rho^2}),$$

and

$$\lim_{n\to\infty} \boldsymbol{\alpha}_{n,a,\rho} = 1/\cos(\beta) = \sqrt{(1-2\rho a + a^2)/(1-\rho^2)} > 1, \quad \lim_{n\to\infty} \boldsymbol{\alpha}_{n,a,\rho} u_n = \infty.$$

A simpler formula as the above one for the bivariate tail probability is given in Abdous et al. (2006), Klüppelberg et al. (2007). Transforming the variables (borrowing the idea and the formula from Abdous et al. (2006)) we obtain applying Lemma 7

$$\begin{aligned} I_{n1} &= \frac{1}{2\pi}\int_{1/\cos(\beta_n)}^{\infty} [1 - F(u_n x)] \frac{1}{x} \frac{1}{\sqrt{x^2-1}}\, dx \\ &= (1+o(1)) \frac{1-F(u_n^*)}{2\pi u_n w_n^*} \frac{1}{1/\cos(\beta)} \frac{1}{\sqrt{(1/\cos(\beta))^2 - 1}} \int_0^{\infty} \exp(-x)\, dx \\ &= (1+o(1)) \frac{1-F(u_n^*)}{2\pi u_n w_n^*} \frac{1}{\boldsymbol{\alpha}_{a,\rho}} \frac{1}{\sqrt{(\boldsymbol{\alpha}_{a,\rho})^2 - 1}}, \quad n\to\infty, \end{aligned}$$

and similarly for any $a > 0$

$$\begin{aligned} I_{n2} &= \frac{1}{2\pi}\int_{a_n/\cos(\beta_n)}^{\infty} [1 - F(a_n u_n y)] \frac{1}{y} \frac{1}{\sqrt{y^2-1}}\, dy \\ &= (1+o(1)) \frac{1-F(u_n^*)}{2\pi u_n w_n^*} \frac{1}{\boldsymbol{\alpha}_{a,\rho}} \frac{1}{\sqrt{(\boldsymbol{\alpha}_{a,\rho})^2/a^2 - 1}}, \quad n\to\infty. \end{aligned}$$

Consequently we may write as $n \to \infty$ using further (8)

$$\begin{aligned} &\boldsymbol{P}\{X_n > u_n, Y_n > a_n u_n\} \\ &= (1+o(1)) \frac{1}{\boldsymbol{\alpha}_{a,\rho}} \Big[\frac{1}{\sqrt{(\boldsymbol{\alpha}_{a,\rho})^2 - 1}} + \frac{1}{\sqrt{(\boldsymbol{\alpha}_{a,\rho})^2/a^2 - 1}} \Big] \frac{1-F(u_n^*)}{2\pi u_n w_n^*} \\ &= (1+o(1)) \frac{\boldsymbol{\alpha}_{a,\rho} C_{a,\rho}}{2\pi} \frac{1-F(u_n^*)}{u_n w_n^*} \\ &= (1+o(1)) \frac{\boldsymbol{\alpha}_{a,\rho}^2 C_{a,\rho}}{\sqrt{2\pi}} \Big(\frac{1}{u_n^* w_n^*}\Big)^{1/2} \frac{1-F(u_n^*)}{\sqrt{2\pi \boldsymbol{\alpha}_{n,a,\rho} u_n w_n^*}} \\ &= (1+o(1)) \frac{\boldsymbol{\alpha}_{a,\rho}^2 C_{a,\rho}}{\sqrt{2\pi}} \Big(\frac{1}{u_n^* w_n^*}\Big)^{1/2} \boldsymbol{P}\{X > u_n^*\}, \quad n\to\infty. \end{aligned}$$

In the case $a = 0$ we obtain the same asymptotics since $I_{n2} = o(I_{n1}), n \to \infty$. Thus the claim follows. □

PROOF OF COROLLARY 3 In view of (21)

$$\lim_{n\to\infty} \sqrt{u_n w(u_n)}[a-\rho] = 0, \quad \text{if } a = \rho$$

or

$$\lim_{n\to\infty} \sqrt{u_n w(u_n)}[a-\rho] = -\infty, \quad \text{if } a < \rho$$

implying that (14) holds with $z=0$ or $z=-\infty$, respectively. Hence the proof follows immediately from Theorem 2. □

Proof of Corollary 4 i) Since $u_n \to \infty$ we get using (6)

$$\lim_{n\to\infty} \sqrt{u_n w(u_n)}[y/u_n-\rho] = -\infty,$$

hence (14) holds with $z=-\infty$. Applying Theorem 2 establishes the first claim.
ii) In this case (14) holds with $z=by$, thus the claim follows again by a direct application of the mentioned theorem.
iii) For all large n we have $a_n := y/u_n > \rho$ and further $\lim_{n\to\infty} a_n = 0$. Utilising again Theorem 2 establishes the proof. □

Proof of Theorem 5 (5) and (8) imply that both S_1 and S_2 are in the Gumbel max-domain of attraction with the scaling function w. Consequently

$$\lim_{n\to\infty} \frac{\boldsymbol{P}\{S_1 > u_n + x r_n/w(u_n)\}}{\boldsymbol{P}\{S_1 > u_n\}} = 0$$

for any $r_n, n \ge 1$ tending to ∞. Hence the proof follows easily from (12). □

Proof of Theorem 6 Set for $n \ge 1$

$$q_n := \sqrt{w(u_n)/u_n}, \quad u_n' := u_n + h_{n1}x, \quad y_n = y q_n h_{n2}.$$

By the assumptions and using (6) we obtain for any $x, y \in \mathbb{R}$

$$\lim_{n\to\infty} u_n' = \infty, \quad \lim_{n\to\infty} y_n = c_2 y$$

and

$$\lim_{n\to\infty} \frac{u_n'}{u_n} = \lim_{n\to\infty}[1 + x h_{n1}/u_n] = \lim_{n\to\infty}[1 + x\frac{(1+o(1))c_1}{\sqrt{u_n w(u_n)}}] = 1,$$

hence

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{w(u_n')}{u_n'}\right)^{1/2}[v_n - \rho_n u_n'] &= \lim_{n\to\infty} q_n[v_n - \rho_n u_n] - \rho_n x \lim_{n\to\infty} h_{n1} q_n \\
&= z - \rho c_1 x,
\end{aligned}$$

consequently applying Theorem 2 we obtain for any $x, y \in \mathbb{R}$

$$\begin{aligned}
&\boldsymbol{P}\{X_n > u_n + h_{n1}x, Y_n > v_n + h_{n2}y\} \\
&= \boldsymbol{P}\{X_n > u_n', Y_n > v_n + y_n\sqrt{u_n/w(u_n)}\} \\
&= (1+o(1))\boldsymbol{P}\{Z_\rho > c_2 y - \rho c_1 x + z\}\boldsymbol{P}\{X > u_n + h_{n1}x\}, \quad n \to \infty.
\end{aligned}$$

Thus we have if x, y are positive (recall (7))

$$\begin{aligned}
&\boldsymbol{P}\{X_n > u_n + h_{n1}x, Y_n > v_n + h_{n2}y \big| X_n > u_n, Y_n > v_n\} \\
&= \frac{\boldsymbol{P}\{X_n > u_n + h_{n1}x, Y_n > v_n + h_{n2}y\}}{\boldsymbol{P}\{X_n > u_n, Y_n > v_n\}} \\
&= (1+o(1))\overline{\Phi}_{\rho,z}(c_2 y - \rho c_1 x)\frac{\boldsymbol{P}\{X > u_n + h_{n1}x\}}{\boldsymbol{P}\{X > u_n\}}, \quad n \to \infty,
\end{aligned}$$

with $\overline{\Phi}_{\rho,z} := \boldsymbol{P}\{Z_\rho > s + z\}/\boldsymbol{P}\{Z_\rho > z\}, s \in \mathbb{R}$ and $Z_\rho/\sqrt{1-\rho^2}$ a standard Gaussian random variable, hence (34) follows.

Next, if the sequence $h_{n1}, n \ge 1$ is asymptoticaly equivalent with $w(u_n), n \ge 1$, i.e., $\lim_{n\to\infty} h_{n1} w(u_n) = 1$, then

$$\lim_{n\to\infty} q_n h_{n1} = 0,$$

and further $\forall x \in \mathbb{R}$ (recall (8))

$$\lim_{n\to\infty} \frac{1 - F(u_n + h_{n1}x)}{1 - F(u_n)} = \lim_{n\to\infty} \frac{\boldsymbol{P}\{X > u_n + x/w(u_n)\}}{\boldsymbol{P}\{X > u_n\}} = \exp(-x).$$

Consequently if additionally $\lim_{n\to\infty} h_{n2} w(u_n) = 1$ we obtain for any $x, y \in [0, \infty)$

$$\lim_{n\to\infty} \boldsymbol{P}\{X_n > u_n + h_{n1}x, Y_n > v_n + h_{n2}y \big| X_n > u_n, Y_n > v_n\} = \exp(-x)\overline{\Phi}_{\rho,z}(y).$$

We thus have the convergence in distribution

$$w(u_n)(X_n - u_n)\big| X_n > u_n, Y_n > v_n \quad \stackrel{d}{\to} \quad U, \quad n \to \infty,$$

with U a unit exponential random variable, and for any $z \in \mathbb{R}$

$$\sqrt{w(u_n)/u_n}(Y_n - v_n)\big| X_n > u_n, \quad Y_n > v_n \quad \stackrel{d}{\to} \quad V_z, \quad n \to \infty,$$

where V_z is a positive random variable with survival function $\overline{\Phi}_{\rho,z}(y), y \ge 0$. Furthermore, the joint convergence in distribution holds, hence (34) follows.
ii) Since $\lim_{n\to\infty} u_n w(u_n) = \infty$ we have that (14) holds with $z = \infty$, hence Theorem 2 implies for any $x, y \in \mathbb{R}$ as $n \to \infty$

$$\begin{aligned} &\boldsymbol{P}\{X_n > u_n + h_{n1}y, Y_n > v_n + h_{n2}y\} \\ &\quad = \quad (1 + o(1)) \frac{\boldsymbol{\alpha}_{a,\rho} C_{a,\rho}}{2\pi} \Big(\frac{1}{u_n w(t_n^*)} \Big) [1 - F(t_n^*)], \end{aligned}$$

with $C_{a,\rho}$ defined in (9) and

$$t_n^* = u_n^* + (1 + o(1))[(\boldsymbol{\alpha}_{a,\rho} - aK_{a,\rho})x + K_{a,\rho}y]/w(u_n^*), \quad n \to \infty,$$

where $u_n^* := \boldsymbol{\alpha}_{n,a,\rho} u_n, \boldsymbol{\alpha}_{n,a,\rho} := \sqrt{(1 - 2\rho_n a_n + a_n^2)/(1 - \rho_n^2)}$, and

$$\boldsymbol{\alpha}_{a,\rho} := \sqrt{(1 - 2a\rho + a^2)/(1 - \rho^2)} > 1, \quad K_{a,\rho} := \frac{a - \rho}{\boldsymbol{\alpha}_{a,\rho}(1 - \rho^2)} > 0.$$

Since

$$\boldsymbol{\alpha}_{a,\rho} - aK_{a,\rho} = \frac{1 - a\rho}{\boldsymbol{\alpha}_{a,\rho}(1 - \rho^2)} =: \overline{K}_{a,\rho} > 0$$

we may further write ($n \to \infty$)

$$\begin{aligned} &\boldsymbol{P}\Big\{X_n > u_n + \frac{x}{w(u_n^*)}, Y_n > v_n + \frac{y}{w(u_n^*)}\Big\} \\ &\quad = \quad (1 + o(1)) \frac{\boldsymbol{\alpha}_{a,\rho} C_{a,\rho}}{2\pi} \exp(-\overline{K}_{a,\rho}x - K_{a,\rho}y) \Big(\frac{1}{u_n w(u_n^*)} \Big)^{1/2} [1 - F(u_n^*)], \end{aligned}$$

thus the proof follows. □

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