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Robust Estimations for the Tail Index of Weibull-Type Distribution

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Abstract: Based on suitable left-truncated or censored data, two flexible classes of M-estimations of Weibull tail coefficient are proposed with two additional parameters bounding the impact of extreme contamination. Asymptotic normality with $\sqrt{n}$-rate of convergence is obtained. Its robustness is discussed via its asymptotic relative efficiency and influence function. It is further demonstrated by a small scale of simulations and an empirical study on CRIX.

Keywords: robust; Weibull tail coefficient; influence function; asymptotic relative efficiency; CRIX

MSC: 60G70; 62G32; 91B28

1. Introduction

The estimation of tail quantities plays an important role in extreme value statistics. One challenging problem is to select extreme sample fraction to balance the asymptotic variance and bias. Meanwhile, this requires a large and ideal sample from the underlying distribution. Indeed, in practical data analysis, it is not unusual to encounter outliers or mis-specifications of the underlying model which may have a considerable impact on the estimation results. A typical treatment is then required for instance by down-weighting its influence on the estimation in various standards, see e.g., Basu et al. (1998); Beran and Schell (2012); Vandewalle et al. (2004, 2007); Goegebeur et al. (2015); Liu and Tang (2010).

Given the wide applications of Weibull-type distributions and little studies on its robust estimations, this paper shall address this issue concerning its tail quantities. Let $X_1, \ldots, X_n$ be an independent and identically distributed sequence from parent $X \sim F(x)$ satisfying

$$1 - F(x) = \exp\{-x^{\alpha}\ell(x)\} \quad \text{for large } x, \tag{1}$$

where $\alpha > 0$ is the so-called Weibull tail coefficient (WTC) and $\ell(x)$ is a slowly varying function at infinity, i.e., (cf. Bingham et al. (1987))

$$\lim_{t\to\infty} \ell(tx)/\ell(x) = 1, \quad \forall x > 0.$$

Prominent instances of Weibull-type distributions of F are Gaussian ($\alpha = 2$), gamma, Logistic and exponential ($\alpha = 1$) and extended Weibull (any $\alpha > 0$) distributions (cf. Gardes and Girard (2008)). As an important subgroup of light-tailed distributions, Weibull-type distributions are of great use in hydrology, meteorology, environmental and actuarial science, to name but a few (cf. Arendarczyk and Dębicki (2011); Beirlant and Teugels (1992); Dębicki et al. (2018); Hashorva and Weng (2014)).

Meanwhile, the WTC governs the tail behavior of F, and the larger the WTC is, the faster the tail of F decays. Dedicated estimations of WTC have thus been proposed and most of them are based on an asymptotically vanishing sample fraction of high quantiles, which asymptotic normality is achieved under certain second-order condition specifying the rate of convergence of $\ell(tx)/\ell(t)$ to 1, see e.g., Girard (2004); Gardes and Girard (2008); Goegebeur et al. (2010); Asimit et al. (2010). Indeed, most data-sets from applied-oriented fields are relative large and with certain deviations from the pre-supposed model. For instance, it occurs with the slowly varying function $\ell(\cdot)$ (where $1-F(x) = \exp\{-x^\alpha \ell(x)\}$) in the left part of the distributions. To the best of our knowledge, it is new to investigate the robust Weibull tail estimations when only a small sample is available.

Inspired by the theory of robust inference in Huber (1964), we propose two classes of robust estimations of WTC. Denote for given $c_0 > 0$

$$h(t) = (c_0 t - 1)\ln t - 1, \quad t > 0. \tag{2}$$

Clearly, we have $g(x;\alpha) = -\alpha^{-1}h(x^\alpha)$ is the score function of

$$X \sim F_W(x;\alpha) = 1 - \exp\{-c_0 x^\alpha\}, \quad x > 0,\, \alpha > 0. \tag{3}$$

Please note that $h(t)$, $t > 0$ is not monotone and thus one cannot directly weaken the effect of outliers by bounding score function $g(x;\alpha)$. On the other hand, most interest of risk management lies principally in the extreme large risks. This motivates us to consider some tailored $h(t)$ according to certain left-truncated/censored Weibull distributions with the same Weibull tail coefficient α under considerations. Namely, we set below $t_0 = \arg\min_{t\geq 1} h(t)$ with h specified by (2) and

$$\tilde{h}(t) = h(t),\, t \geq 1, \quad h^*(t) = h(t),\, t \geq t_0, \tag{4}$$

which properties are stated as below.

Lemma 1. *Let $X \sim F_W(x;\alpha)$ and $\tilde{X} = X|\{X \geq 1\} \sim \tilde{F}_W(x;\alpha)$. Then $\tilde{h}^{\leftarrow}(y) = \inf\{t \geq 1 : \tilde{h}(t) \geq y\}, y \geq -1$ is strictly increasing, and $-\alpha^{-1}\tilde{h}(x^\alpha)$ is the score function of $\tilde{F}_W(x;\alpha)$. Moreover, $h^*(x^\alpha)$, $x \geq x_0 = t_0^{1/d_0}$ is strictly increasing provided that $\alpha \geq d_0 > 0$.*

Basically, both $\tilde{h}^{\leftarrow}$ and h^* are certain modifications of h via its valued interval and domain region. Now, we are ready to state our M-estimations of Weibull tail coefficient using the M-estimation process based on the alternative samples $\tilde{X}_i$'s and X_i^*'s respectively from $\tilde{X} := X|\{X \geq 1\} \sim \tilde{F}$ and $X^* := \max(X, x_0) \sim F^*$ where X_i's is a random sample from $X \sim F$. Set below $[y]_v^u = \min(\max(y,v),u),\}, v < u$ and $\Im$ is a set of distributions with support in $(0,\infty)$.

Definition 1. *Let $F_W(x;\alpha)$ and $\tilde{h}, h^*$ be given by (3) and (4), respectively. Define the psi-function $\tilde{\psi}$ as*

$$\begin{aligned}
\tilde{\psi}_{v,u}(y;\alpha) &= [\tilde{h}(y^\alpha)]_v^u - \int_1^\infty [\tilde{h}(z^\alpha)]_v^u d\tilde{F}_W(z;\alpha) \\
&= [\tilde{h}(y^\alpha)]_v^u - \left[v + \int_v^u \exp\{-c_0[\tilde{h}^{\leftarrow}(z) - 1]\}\, dz\right], \ -1 \leq v < u < \infty.
\end{aligned} \tag{5}$$

Then the functional $\tilde{T}(F)$ as the solution of the equation

$$\tilde{\lambda}_F(t) = \int_1^\infty \tilde{\psi}_{v,u}(y;t)d\tilde{F}(y) = 0, \quad F \in \Im,$$

is called huberized Weibull tail M-functional corresponding to $\tilde{\psi}$. The corresponding M-estimator $\tilde{T}_n = \tilde{T}_n^{(v,u)}(F_n)$, the solution of the equation

$$\tilde{\lambda}_{F_n}(t) = \sum_{j=1}^{m} \tilde{\psi}_{v,u}(\tilde{X}_j; t) = 0, \quad m = \#\{1 \le i \le n : X_i \ge 1\},$$

is the huberized Weibull tail M-estimator of α. If further $0 < d_0 \le \alpha \le d_1$, then define the psi-function ψ^* with $x_0 = \big(\arg\min_{t \ge 1} h(t)\big)^{1/d_0}$ and $v_0 = h(x_0^{d_1})$

$$\begin{aligned} \psi^*_{v,u}(y;\alpha) &= [h^*(y^\alpha)]_v^u - \int_{x_0}^{\infty} [h^*(z^\alpha)]_v^u dF_W^*(z;\alpha) \\ &= [h^*(y^\alpha)]_v^u - \left[v + \int_v^u \exp\{-c_0 (h^*)^{\leftarrow}(z)\}\, dz \right], \quad v_0 \le v < u < \infty. \end{aligned} \tag{6}$$

Then the functional $T^*(F), F \in \Im$ as the solution of the equation

$$\lambda_F^*(t) = \int_{x_0}^{\infty} \psi^*_{v,u}(y; t) dF^*(y) = 0,$$

is called huberized Weibull tail M-functional corresponding to ψ^*. The corresponding M-estimator $T_n^* = T_n^{*(v,u)}(F_n)$, the solution of the equation

$$\lambda^*_{F_n}(t) = \sum_{i=1}^{n} \psi_{v,u}(X_i^*; t) = 0,$$

is the huberized M-estimator of the Weibull tail coefficient α.

We remark that (5) and (6) hold since

$$\tilde{F}_W(x;\alpha) = 1 - \exp\{-c_0[x^\alpha - 1]\}, \quad x \ge 1; \quad F_W^*(x;\alpha) = \begin{cases} 1 - \exp\{-c_0 x^\alpha\}, & x > x_0, \\ 0, & x \le x_0. \end{cases}$$

Figure 1 illustrates the lower huberization by comparing the score function $\tilde{\psi}_{-1,\infty}(y;\alpha)$ of $\tilde{F}_W$ (recall Lemma 1) with $\tilde{\psi}_{v,\infty}(y;\alpha)$. We see that the contaminated Weibull density by Gamma (see (10) below for its definition) has almost the same shape as the pre-supposed Weibull one in the right tail, and therefore lower-huberized psi-function $\tilde{\psi}_{v,\infty}(y;\alpha)$ can restrict the influence of all observations below $y_0 = (\tilde{h}^{\leftarrow}(v))^{1/\alpha}$ instead of removing them completely. On the other hand, for all $y > y_0$, the $\tilde{\psi}_{v,\infty}(y;\alpha)$ is shifted downwards for the consistency purpose. One may similarly analyze the ψ^* function.

The paper principally investigates the asymptotic behavior of the proposed new classes of M-estimations of Weibull tail coefficient. Details are as follows.

In Section 2, we consider Weibull distributions in Theorems 1 and 2 and establish its asymptotic normality of the M-estimations $\tilde{T}_n$ and T_n^* with $\sqrt{n}$-rate of convergence, which is rather faster than that of most classical Weibull tail estimations such as the Hill-type estimation, see Theorem 2 in Girard (2004). Generally, we study related asymptotic properties in Theorems 3 and 4 when the underlying risk follows Weibull-type distributions specified in (1). Some bounded asymptotic bias may appear due to its deviations from the Weibull distributions.

In Section 3, using asymptotically relative efficiency (AEFF) and influence function (IF), we investigate the robustness (Theorem 5) and the bias, which are further related to the choices of flexible parameters v and u. These results are useful, especially when the practical regulators in risk management consider the trade-off between the robustness and consistency.

In Section 4, a small scale of Monte Carlo simulations and an empirical study concerning the CRIX proposed by Trimborn and Härdle (2016) are carried out. We see that both M-estimations are robust and perform very well even for small samples, in comparisons with the classical maximum likelihood estimations and Hill-type estimations of the Weibull tail coefficient. We expect the results would be beneficial to both financial practitioners and theoretical experts in risk management and extreme value statistics.

The rest of the paper is organized as follows. Main results are given in Section 2 followed with a section dedicated to the robust analysis. Sections 4 and 5 are devoted to a small scale of Monte Carlo simulations and an empirical studies on CRIX. All proofs of the results are postulated to Section 6.

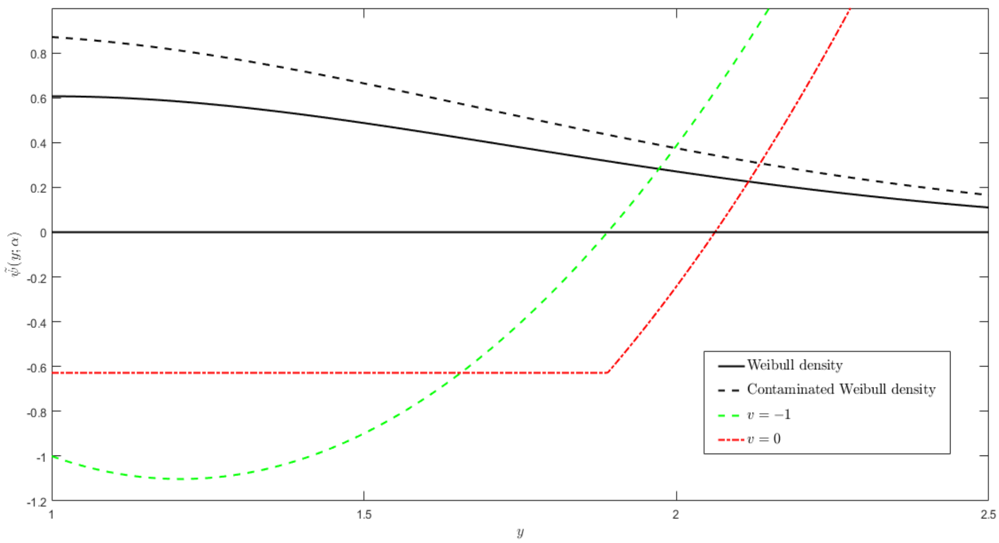

Figure 1. Psi-functions $\tilde{\psi}$ for the huberized M-estimators $\tilde{T}_n^{(v,\infty)}$. Here the truncated densify functions are generated from the Weibull $F_W(x;\alpha)$ and contaminated Weibull $F_\epsilon(x) = (1-\epsilon)F_W(x;\alpha) + \epsilon\Gamma(x;\lambda,\beta)$ with $\alpha = 2, c_0 = 0.5, \epsilon = 0.3, \lambda = 1, \beta = 1$.

2. Asymptotic Results

Throughout this section, we keep the same notation as in Introduction and write further $\xrightarrow{p}$ and $\xrightarrow{d}$ for the convergence in probability and in distribution, respectively. All the limits are taken as $n \to \infty$ unless otherwise stated.

Theorem 1. *Let $X_1, \ldots, X_n$ be a random sample from $X \sim F_W(x;\alpha_0) = 1 - \exp\{-c_0 x^{\alpha_0}\}$, $x > 0, c_0, \alpha_0 > 0$. Denote by $\tilde{X}_j \sim \tilde{X} = X|\{X \geq 1\}$, $1 \leq j \leq m = \#\{1 \leq i \leq n : X_i \geq 1\}$, and by $\tilde{T}_n = \tilde{T}_n^{(v,u)}, -1 \leq v < u < \infty$ the solution of*

$$\tilde{\lambda}_{(F_W)_n}(t) = \sum_{i=1}^{m} \tilde{\psi}_{v,u}(\tilde{X}_i;t) = 0.$$

Then $\tilde{T}_n \xrightarrow{p} \alpha_0$ and

$$\sqrt{n}(\tilde{T}_n - \alpha_0) \xrightarrow{d} N(0, \tilde{\sigma}^2_{v,u,\alpha_0;F_W}), \tag{7}$$

where, with $\tilde{\mu} = v + \int_v^u \exp\{-c_0[\tilde{h}^{\leftarrow}(z) - 1]\}dz$

$$\tilde{\sigma}^2_{v,u,\alpha_0;F_W} = e^{c_0}\frac{\alpha_0^2}{c_0^2}\frac{(v-\tilde{\mu})^2 + 2\int_{\tilde{h}^{\leftarrow}(v)}^{\tilde{h}^{\leftarrow}(u)}[\tilde{h}(s) - \tilde{\mu}]\exp\{-c_0(s-1)\}d\tilde{h}(s)}{\left[\int_{\tilde{h}^{\leftarrow}(v)}^{\tilde{h}^{\leftarrow}(u)} s\ln s \exp\{-c_0(s-1)\}d\tilde{h}(s)\right]^2}.$$

Remark 1. *As stated in Lemma 1, $-\alpha^{-1}\tilde{h}(x^\alpha), x \geq 1$ is the score function of $\tilde{F}_W(x;\alpha)$. Therefore, $\tilde{T}_n = \tilde{T}_n^{(v,u)}$ with $v = -1, u = \infty$ reduces to the maximum likelihood estimation of α. This fact will be used in Theorem 5 for the asymptotic relative efficiency analysis. Additionally, we have by laws of large numbers that $m = m(n)$ satisfies $m/n \xrightarrow{p} P\{X \geq 1\} = e^{-c_0}$.*

Theorem 2. *Let $X_1, \ldots, X_n$ be a random sample from $X \sim F_W(x;\alpha_0) = 1 - \exp\{-c_0 x^{\alpha_0}\}$, $x > 0$ and $0 < d_0 \leq \alpha_0 \leq d_1$. Denote by $X_i^* = \max(X_i, x_0), 1 \leq i \leq n$ with $x_0 = (\arg\min_{t \geq 1} h(t))^{1/d_0}$, and by $T_n^* = T_n^{*(v,u)}$, $n \in \mathbb{N}$, $u > v \geq v_0 = h(x_0^{d_1})$ the solution of*

$$\lambda^*_{(F_W)_n}(t) = \sum_{i=1}^{n} \psi^*_{v,u}(X_i^*;t) = 0.$$

Then $T_n^* \xrightarrow{p} \alpha_0$ and

$$\sqrt{n}(T_n^* - \alpha_0) \xrightarrow{d} N(0, \sigma^{*2}_{v,u,\alpha_0;F_W}),$$

where, with $\mu^* = v + \int_v^u \exp\{-c_0 (h^*)^{\leftarrow}(z)\}\, dz$

$$\sigma^{*2}_{v,u,\alpha_0;F_W} = \frac{\alpha_0^2}{c_0^2} \frac{(v-\mu^*)^2 + 2\int_{(h^*)^{\leftarrow}(v)}^{(h^*)^{\leftarrow}(u)} [h^*(s) - \mu^*] \exp\{-c_0 s\} dh^*(s)}{\left[\int_{(h^*)^{\leftarrow}(v)}^{(h^*)^{\leftarrow}(u)} s \ln s \exp\{-c_0 s\} dh^*(s)\right]^2}.$$

Remark 2. *(i) The difference between $\tilde{\psi}$ and ψ^* is that h^* is not the score function of F_W^*, the distribution of the censored risk at point x_0, where $0 < d_0 \leq \alpha_0 \leq d_1$ is needed to ensure the monotonicity of h^* and ψ^*, see details in (19) with $\alpha = \alpha_0$.*

(ii) The proposed M-estimations are principally based on suitable left-truncated and censored data, which are commonly used in survival analysis, see e.g., Kudu et al. (2017). Moreover, both consistency and robustness are obtained since we bound the psi-functions to weaken the influence of the extreme outliers for the exact Weibull models.

In what follows, we consider generally the Weibull-type risks and investigate asymptotic properties of the proposed M-estimations.

Theorem 3. *Let $X_1, \ldots, X_n$ be a random sample from $F(x) = 1 - \exp\{-c_0 x^{\alpha_0} \ell(x)\}$, $x > 0$. Suppose that there is a unique solution t_0 of $\tilde{\lambda}_F(t) = 0$. Then $\tilde{T}_n = \tilde{T}_n^{(v,u)}, -1 \leq v < u < \infty$, the solution of*

$$\tilde{\lambda}_{F_n}(t) = \sum_{i=1}^{m} \tilde{\psi}_{v,u}(\tilde{X}_i;t) = 0, \quad m = \#\{1 \leq i \leq n : X_i \geq 1\},$$

converges in probability to t_0. If further $\int_1^\infty \tilde{\psi}^2_{v,u}(x;t_0)\, d\tilde{F}(x) < \infty$ and $\tilde{\lambda}'_F(t) \neq 0$ hold in a neighbourhood of t_0, then

$$\sqrt{n}(\tilde{T}_n - t_0) \xrightarrow{d} N(0, \tilde{\sigma}^2_{v,u,t_0;F}), \tag{8}$$

where

$$\tilde{\sigma}^2_{v,u,t_0;F} = e^{c_0 \ell(1)} \frac{\int_1^\infty \tilde{\psi}^2_{v,u}(x;t_0) d\tilde{F}(x)}{[\tilde{\lambda}'_F(t_0)]^2}.$$

Theorem 4. *Let $X_1, \ldots, X_n$ be a random sample from $F(x) = 1 - \exp\{-c_0 x^{\alpha_0} \ell(x)\}$, $x > 0$ and $0 < d_0 \leq \alpha_0 \leq d_1, x_0 = (argmin_{t \geq 1} h(t))^{1/d_0}$. Suppose that there is a unique solution t_0 of $\lambda_F^*(t) = 0$. Then $T_n^* = T_n^{*(v,u)}, u > v \geq v_0 = h(x_0^{d_1})$, the solution of*

$$\lambda_{F_n}^*(t) = \sum_{i=1}^{n} \psi_{v,u}^*(X_i^*; t) = 0, \quad n \in \mathbb{N},$$

converges in probability to t_0. If further $\int_{x_0}^{\infty} \psi_{v,u}^{*2}(x; t_0)\, dF^*(x) < \infty$ and $(\lambda_F^*)'(t) \neq 0$ hold in a neighbourhood of t_0, then

$$\sqrt{n}(T_n^* - t_0) \xrightarrow{d} N(0, \sigma_{v,u,t_0;F}^{*2}), \tag{9}$$

where

$$\sigma_{v,u,t_0;F}^{*2} = \frac{\int_{x_0}^{\infty} \psi_{v,u}^{*2}(x; t_0) dF^*(x)}{[(\lambda_F^*)'(t_0)]^2}.$$

Please note that here the t_0, the unique solution of $\tilde{\lambda}_F$ and λ_F^* specified in Theorems 3 and 4, might not be equal to α_0. In other words, to maintain the robustness of the M-estimations is at cost of consistency. In the next section, we shall discuss the balance via the flexible parameters v and u.

3. Robustness

A simple criterion for choosing v and u in the M-estimations is the trade-off between the efficiency loss (that one is willing to put up with when data are generated by a Weibull distribution), and its asymptotic bias (when the underlying distribution deviates from the ideal Weibull distribution). We study below the relative asymptotic efficiency (AEFF) in Theorem 5, and then analyze its influence function. Both quantities are some functions of the flexible parameters v and u, which enable the risk regulators to balance the robustness and consistency.

As stated in Remark 1, the M-estimation $\tilde{T}_n^{(v,u)}$ with $v = -1, u = \infty$ reduces to the maximum likelihood estimation of α. Therefore, a straightforward application of Theorems 1 and 2 leads to the following theorem.

Theorem 5. *Under the same assumptions of Theorems 1 and 2, we have the relative asymptotic efficiency functions of $\tilde{T}_n^{(v,u)}$ and $T_n^{*(v,u)}$ (compared to $\tilde{T}_n^{(-1,\infty)}$, the maximum likelihood estimation) are given by*

$$\begin{aligned} AEFF(\tilde{T}_n^{(v,u)}) &= \frac{\tilde{\sigma}^2_{-1,\infty,\alpha_0;F_W}}{\tilde{\sigma}^2_{v,u,\alpha_0;F_W}} \\ &= \frac{\left[\int_{-1}^{\infty} \exp\{-c_0[\tilde{h}^{\leftarrow}(z) - 1]\}\, dz\right]^2 + 2\int_1^{\infty} [\tilde{h}(s) - \tilde{\mu}] \exp\{-c_0(s-1)\} d\tilde{h}(s)}{\left[\int_1^{\infty} s \ln s \exp\{-c_0(s-1)\} d\tilde{h}(s)\right]^2} \\ &\times \frac{\left[\int_{\tilde{h}^{\leftarrow}(v)}^{\tilde{h}^{\leftarrow}(u)} s \ln s \exp\{-c_0(s-1)\} d\tilde{h}(s)\right]^2}{(v - \tilde{\mu})^2 + 2\int_{\tilde{h}^{\leftarrow}(v)}^{\tilde{h}^{\leftarrow}(u)} [\tilde{h}(s) - \tilde{\mu}] \exp\{-c_0(s-1)\} d\tilde{h}(s)} \end{aligned}$$

and

$$
\begin{aligned}
AEFF(T_n^{*(v,u)}) &= \frac{\tilde{\sigma}^2_{-1,\infty,\alpha_0;F_W}}{\sigma^{*2}_{v,u,\alpha_0;F_W}} \\
&= e^{c_0}\,\frac{\left[\int_{-1}^{\infty}\exp\{-c_0[\tilde{h}^{\leftarrow}(z)-1]\}\,dz\right]^2 + 2\int_1^{\infty}[\tilde{h}(s)-\tilde{\mu}]\exp\{-c_0(s-1)\}d\tilde{h}(s)}{\left[\int_1^{\infty} s\ln s\exp\{-c_0(s-1)\}d\tilde{h}(s)\right]^2} \\
&\times \frac{\left[\int_{(h^*)^{\leftarrow}(v)}^{(h^*)^{\leftarrow}(u)} s\ln s\exp\{-c_0 s\}dh^*(s)\right]^2}{(v-\mu^*)^2 + 2\int_{(h^*)^{\leftarrow}(v)}^{(h^*)^{\leftarrow}(u)}[h^*(s)-\mu^*]\exp\{-c_0 s\}dh^*(s)}.
\end{aligned}
$$

Here $\tilde{\mu}$ and μ^* are given by Theorems 1 and 2, respectively.

Figure 2 illustrates the effect of v on the relative asymptotic efficiency of $\tilde{T}_n^{(v,\infty)}$ and $T_n^{*(v,\infty)}$ (compared to the MLE $\tilde{T}_n^{(-1,\infty)}$). For smaller v, the relative asymptotic effective loss of $\tilde{T}_n$ is rather smaller than that of T_n^*. While for larger v, both are asymptotically the same.

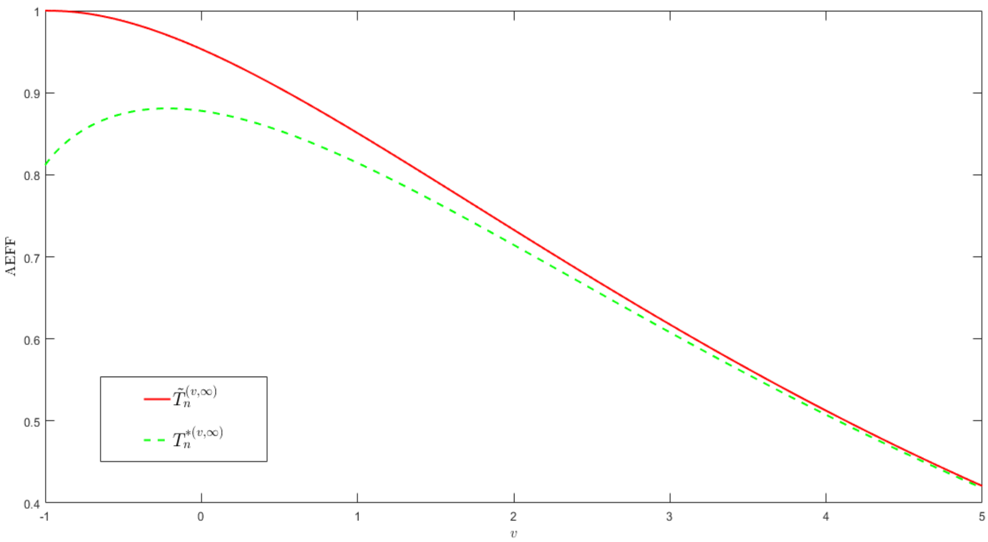

Figure 2. Relative asymptotic efficiency (AEFF) of $\tilde{T}_n^{(v,\infty)}$ and $T_n^{*(v,\infty)}$ compared to the MLE $\tilde{T}_n^{(-1,\infty)}$. Here $F_W(x;\alpha_0)$ is given by (1) with $c_0 = 1, \alpha_0 = 1$.

The influence function approach, known also as the "infinitesimal approach", is generally employed to quantify robustness. Recall that the influence function describes the effect of some functional $T(F)$ for F in an infinitesimal ϵ-contamination neighbourhood $\{F_\epsilon | F_\epsilon(x) = (1-\epsilon)F(x) + \epsilon G(x)\}$, is defined by

$$
IF(T;F,G) = \lim_{\epsilon\to 0}\frac{T((1-\epsilon)F+\epsilon G) - T(F)}{\epsilon} = \frac{\partial}{\partial\epsilon}T(F_\epsilon)\big|_{\epsilon=0}.
$$

We have

$$
IF(\tilde{T};F,G) = -\frac{\int_1^{\infty}\tilde{\psi}_{v,u}(y;\tilde{T}(F))d\tilde{G}(y)}{\tilde{\lambda}'_F(\tilde{T}(F))};\quad IF(T^*;F,G) = -\frac{\int_{x_0}^{\infty}\psi^*_{v,u}(y;T^*(F))dG^*(y)}{(\lambda^*_F)'(T^*(F))}.
$$

In Figure 3, we take $G(x) = \Gamma(x;\lambda,\beta)$ with scale parameter $\lambda = 0.5$ and shape parameter $\beta \in (0,5)$, which is a Weibull-type distribution with $\alpha = 1$. Its density function $g(x;\lambda,\beta)$ is given by

$$g(x;\lambda,\beta) = \frac{\lambda^\beta}{\Gamma(\beta)} x^{\beta-1}\exp\{-\lambda x\}, \quad x > 0. \tag{10}$$

We see that, the absolute values of the influence functions of both M-estimations $\tilde{T}_n$ and T_n^* are increasing in β, and decreasing with v. In other words, with increasing huberization and light-tail contamination, one gets the reduction of sensitivity to deviations from the Weibull model.

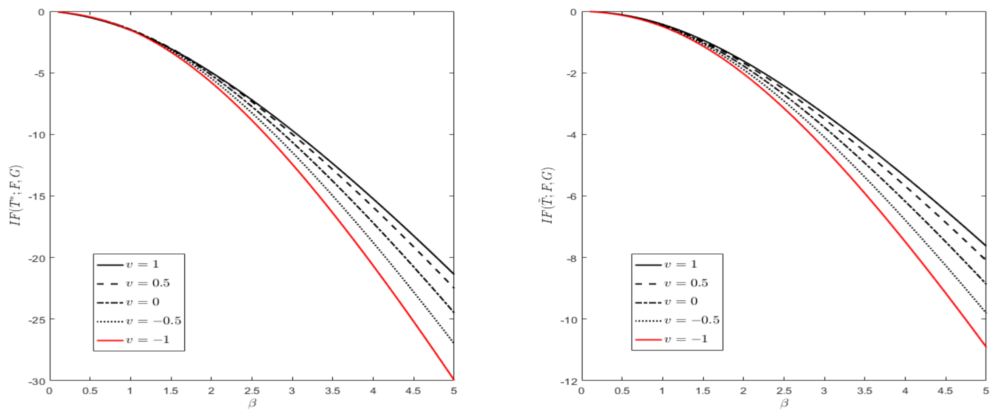

Figure 3. Influence functions $IF(T;F,G)$ for $T = \tilde{T}_n^{(v,\infty)}$ (**left**) and $T_n^{*(v,\infty)}$ (**right**). Here $G(x) = \Gamma(x;\lambda,\beta)$, $\lambda = 0.5, \beta \in (0,5), v = \pm 1, \pm 0.5, 0$ and $F(x) = F_W(x;\alpha_0)$ is given by (1) with $c_0 = 1, \alpha_0 = 1$.

4. Simulations

In this section, we carry out a simulation study to illustrate the small sample behavior of M-estimations $\tilde{T}_n^{(v,\infty)}$ and $T_n^{*(v,\infty)}$ compared to the maximum likelihood estimation $\widehat{\alpha}_{mle} = \tilde{T}_n^{(-1,\infty)}$ and the classical Hill-type estimation $\widehat{\alpha}_{Hill}^{(k_n)}$ of the Weibull tail coefficient given by (cf. Girard (2004))

$$\widehat{\alpha}_{Hill}^{(k_n)} = \frac{\frac{1}{k}\Sigma_{j=1}^k \log(\log\frac{n+1}{j}) - \log(\log\frac{n+1}{k+1})}{\frac{1}{k}\Sigma_{j=1}^k \log(X_{n-j+1,n}) - \log(X_{n-k,n})}, \quad k = 1,\ldots,n-1. \tag{11}$$

To analyze the robustness of the M-estimations, we generate $m = 1000$ samples of size $n = 30, 50, 80$ and 100 from Weibull distribution $F_W(x;\alpha) = 1 - \exp\{-c_0 x^\alpha\}, x > 0$ contaminated by Gamma distribution $\Gamma(x;\lambda,\beta)$ with contamination level $\epsilon \in (0,1)$, i.e., the underlying risk follows

$$F_\epsilon(x) = (1-\epsilon)F_W(x;\alpha) + \epsilon\Gamma(x;\lambda,\beta).$$

In the simulations, we take $c_0 = 0.5, 1, 2, d_0 = 1, d_1 = 2, \alpha = 1, 2$ and $\lambda = \beta = 0.5, \epsilon = 0.1, 0.3$. Table 1 lists the average estimations $\overline{\alpha}$, the sample variance s^2 and the ratio of mean squared error (MSE) of MLE, Hill-type estimation to that of $\tilde{T}_n$ and T_n^* with $v = 0$, i.e., $(\widehat{r}, \tilde{r}, r^*) = (\widehat{r}_0, \tilde{r}_0, r_0^*)$ is given by

$$\widehat{r}_v = \frac{MSE(\widehat{\alpha}_{Hill}^{(k_{opt})})}{MSE(\tilde{T}_n^{(v,\infty)})}, \quad \tilde{r}_v = \frac{MSE(\widehat{\alpha}_{mle})}{MSE(\tilde{T}_n^{(v,\infty)})}, \quad r_v^* = \frac{MSE(\widehat{\alpha}_{mle})}{MSE(T_n^{*(v,\infty)})}. \tag{12}$$

Here, we use alternatively $k_n = k_{opt}$ given by (since the traditional optimal choice of k_n in Girard (2004) is not available for small samples)

$$k_{opt} = \arg\min_{k_n \geq 1} MSE(\widehat{\alpha}_{Hill}^{(k_n)}).$$

The last column of Table 1 is the relative proportion of k_n for which $MSE(\widehat{\alpha}_{Hill}^{(k_n)}) \leq MSE(\tilde{T}_n^{(v,\infty)})$, denoted by p_{Hill}, is given by

$$p_{Hill} = \frac{\#\{1 \leq k_n \leq n-1 : MSE(\widehat{\alpha}_{Hill}^{(k_n)}) \leq MSE(\tilde{T}_n^{(v,\infty)})\}}{n-1} \times 100\%.$$

The p_{Hill} describes the percent that the Hill-type estimation outperforms the estimation $\tilde{T}_n^{(v,\infty)}$.

Table 1. Comparisons of $\tilde{T}_n$, T_n^* with $\widehat{\alpha}_{mle}$, $\widehat{\alpha}_{Hill}^{(k_n)}$. Here we take $m = 1000$ samples of size $n = 30, 50, 80, 100$ from $F_\epsilon(x) = (1-\epsilon)F_W(x;\alpha) + \epsilon\Gamma(x; 0.5, 0.5)$.

(ϵ, c_0, α)	n	$\bar{\alpha}_{mle}$	$\bar{\alpha}_{Hill}$	$\tilde{T}_n$	T_n^*	s^2_{mle}	s^2_{Hill}	$s^2_{\tilde{T}}$	$s^2_{T^*}$	$\widehat{r}$	$\tilde{r}$	r^*	p_{Hill}
(0.3, 1, 1)	30	0.9217	0.8255	1.0006	1.0005	0.0072	0.0451	0.0018	0.0015	68.6811	8.7774	8.3678	0.00
	50	0.8609	0.8435	1.0022	1.0061	0.0068	0.0289	0.0014	0.0015	23.1288	14.3407	13.4344	0.00
	80	0.8274	0.8326	1.0083	1.0075	0.0041	0.0176	0.0013	0.0016	9.7485	19.5972	19.2393	0.00
	100	0.8161	0.8368	1.0147	1.0119	0.0030	0.0148	0.0018	0.0015	6.2607	20.8143	20.0842	0.00
(0.1, 1, 1)	30	0.9885	0.9343	0.9942	0.9940	0.0007	0.0469	0.0007	0.0006	5.7287	1.2283	1.0561	0.00
	50	0.9834	0.9252	0.9949	0.9952	0.0009	0.0269	0.0007	0.0006	3.6407	1.8194	1.8000	0.00
	80	0.9776	0.9407	0.9962	0.9953	0.0009	0.0189	0.0005	0.0006	1.0006	2.5849	2.3829	0.00
	100	0.9735	0.9302	0.9964	0.9961	0.0010	0.0130	0.0005	0.0005	0.7138	3.2549	2.9623	0.15
(0.3, 1, 2)	30	1.2382	1.6039	1.9960	1.9919	0.0687	0.2408	0.0056	0.0050	74.7362	147.2932	126.8653	0.00
	50	1.1347	1.6443	2.0015	1.9963	0.0268	0.1576	0.0045	0.0042	22.7834	158.1670	165.1127	0.00
	80	1.0851	1.6853	2.0050	2.0039	0.0127	0.1219	0.0039	0.0038	7.8035	197.7217	180.3746	0.00
	100	1.0731	1.6709	2.0081	2.0073	0.0085	0.0883	0.0042	0.0038	5.1102	223.2889	186.3996	0.00
(0.1, 1, 2)	30	1.9245	1.8399	1.9903	1.9859	0.0169	0.2025	0.0026	0.0024	4.8833	9.3566	8.4148	0.00
	50	1.8459	1.8506	1.9900	1.9888	0.0239	0.1223	0.0022	0.0021	3.1233	21.4576	18.4649	0.00
	80	1.7654	1.8392	1.9873	1.9895	0.0228	0.0754	0.0017	0.0017	3.0758	39.6547	35.4584	0.00
	100	1.7249	1.8681	1.9898	1.9906	0.0190	0.0660	0.0018	0.0018	1.9142	43.1656	45.1557	0.00
(0.3, 2, 1)	30	0.9466	0.8640	0.9974	0.9987	0.0047	0.0429	0.0016	0.0021	82.5558	4.6014	3.5335	0.00
	50	0.9061	0.8881	0.9955	0.9965	0.0051	0.0298	0.0015	0.0013	22.7834	12.7013	10.1286	0.00
	80	0.8729	0.8848	0.9944	0.9955	0.0036	0.0172	0.0012	0.0010	3.2990	16.1109	16.0286	0.00
	100	0.8562	0.8938	0.9970	0.9978	0.0029	0.0152	0.0011	0.0011	1.7146	19.8407	18.1323	0.00
(0.1, 2, 1)	30	0.9880	0.9223	0.9953	0.9956	0.0005	0.2773	0.0483	0.0011	5.1904	0.8848	0.7232	0.00
	50	0.9852	0.9557	0.9941	0.9952	0.0007	0.1438	0.0261	0.0006	3.4681	1.2380	1.1071	0.00
	80	0.9808	0.9452	0.9940	0.9942	0.0008	0.1775	0.0165	0.0005	0.9524	2.0353	1.6522	0.10
	100	0.9762	0.9560	0.9927	0.9939	0.0009	0.0444	0.0148	0.0006	0.8589	2.3773	2.2494	0.12
(0.3, 2, 2)	30	1.3019	1.6018	1.9855	1.9853	0.0583	0.2434	0.0030	0.0026	85.6893	153.3920	159.4866	0.00
	50	1.2012	1.6589	1.9850	1.9850	0.0228	0.1209	0.0024	0.0021	14.7957	260.7202	270.0256	0.00
	80	1.1570	1.6701	1.9844	1.9838	0.0099	1.1006	0.0018	0.0017	2.5650	348.1278	345.6587	0.00
	100	1.1484	1.6771	1.9832	1.9830	0.0069	0.0712	0.0017	0.0017	1.4158	386.5744	370.8852	0.00
(0.1, 2, 2)	30	1.9238	1.8054	1.9885	1.9870	0.0161	0.1763	0.0027	0.0032	4.3161	5.8439	6.3442	0.00
	50	1.8519	1.8637	1.9886	1.9850	0.0212	0.1201	0.0031	0.0023	3.9388	17.6548	16.9291	0.00
	80	1.7646	1.8565	1.9869	1.9849	0.0200	0.0696	0.0017	0.0019	1.2588	37.7912	36.6744	0.00
	100	1.7402	1.8839	1.9849	1.9843	0.0181	0.0756	0.0017	0.0017	0.9791	47.8601	44.6477	0.05
(0.3, 0.5, 1)	30	0.9320	0.7989	1.0056	1.0065	0.0048	0.0565	0.0012	0.0014	7.2497	7.7231	6.7977	0.00
	50	0.8912	0.8250	1.0116	1.0096	0.0051	0.0468	0.0012	0.0013	8.8465	12.2309	10.2765	0.00
	80	0.8565	0.8130	1.0185	1.0188	0.0034	0.0261	0.0013	0.0012	2.8355	15.4294	14.8902	0.00
	100	0.8463	0.8368	1.0218	1.0232	0.0024	0.0252	0.0011	0.0012	1.3175	17.0045	16.5285	0.00
(0.1, 0.5, 1)	30	0.9874	0.8848	0.9968	0.9943	0.0005	0.0428	0.0006	0.0006	5.3788	1.2157	1.0236	0.00
	50	0.9853	0.9136	0.9972	0.9952	0.0005	0.0295	0.0005	0.0005	3.8457	1.5111	1.4974	0.00
	80	0.9799	0.9193	0.9977	0.9975	0.0006	0.0181	0.0004	0.0005	1.9436	2.3528	1.9708	0.00
	100	0.9783	0.9165	0.9991	0.9989	0.0006	0.0143	0.0005	0.0004	0.9241	2.2865	2.1840	0.10
(0.3, 0.5, 2)	30	1.3277	1.5964	2.0065	1.8144	0.0713	0.2504	0.0052	0.0004	61.5168	111.5918	15.1011	0.00
	50	1.2141	1.6243	2.0185	1.8083	0.0373	0.1607	0.0049	0.0003	31.3754	129.6489	17.7850	0.00
	80	1.1618	1.6596	2.0357	1.8042	0.0147	0.1047	0.0046	0.0002	13.2211	128.0530	18.8915	0.00
	100	1.1486	1.6707	2.0386	1.8035	0.0125	0.0974	0.0047	0.0002	4.5564	118.1708	19.1617	0.00
(0.1, 0.5, 2)	30	1.9443	1.8589	1.9900	1.8040	0.0093	0.2091	0.0020	0.0005	8.5329	6.7537	0.3454	0.00
	50	1.8936	1.8745	1.9935	1.7963	0.1316	0.6520	0.0020	0.0003	4.9060	12.6372	0.5832	0.00
	80	1.8329	1.8271	1.9958	1.7937	0.0720	0.6408	0.0018	0.0002	1.3524	22.1326	0.9232	0.00
	100	1.8125	1.8538	2.0024	1.7930	0.0670	2.8163	0.0017	0.0002	0.9561	26.7188	1.0363	0.08

We conclude from Table 1 that

(i) The bias of the proposed M-estimations is smaller than that of Hill-type estimation and MLE estimation (see columns 2–5 for details).
(ii) The sample variance s^2 of our estimations is very close to zero. Note by passing that even with the optimal choice of $k_n = k_{opt}$, the s^2 of Hill-type estimations is still relatively larger than the other (see columns 6–9 for details).
(iii) Since the ratios of MSE satisfy $\hat{r} \leq r^* \leq \tilde{r}$, we see that the best rank estimation is $\tilde{T}_n$, which coincides with the analysis of the relative efficiency (see columns 10–12 and Figure 2).
(iv) For $n = 30, 50$, the p_{Hill} is almost zero indicating that for very small samples $\tilde{T}_n^{(0,\infty)}$ outperforms Hill-type estimators $\widehat{\alpha}_{Hill}^{(k_n)}$ for almost all k_n's. For $n = 80$, p_{Hill} does not exceed 10% in most cases which means that there is a set K with at most s = 8 of $k_n \in K$ such that the Hill-type estimators would outperform $\tilde{T}_n^{(0,\infty)}$. Similar argument holds for $n = 100$. Hence, the M-estimations perform better even for small samples.

5. Empirical Study

The CRIX, a market index (benchmark), is designed by Trimborn and Härdle (2016). It enables each interested party to study the performance of the crypto market as a whole or single crypto market, and therefore attracts increasing attention of risk managers and regulators. We select the daily CRIX index during 31 July 2014–1 January 2018 (available on crix.berlin) and take all $n = 713$ positive log returns of CRIX multiplied by 15 to obtain a moderate amount of sample of size m around 35–50 greater than 1 for the M-estimation $\tilde{T}_n$ (recall scaled risks keep the same tail decay feature) as the original data sequence $X = (X_i, i = 1, \ldots, n)$.

In Figure 4 we employ the empirical mean excess function from extreme value theory to analyze its tail feature (set below $\mathbb{I}\{\cdot\}$ as the indicator function)

$$\widehat{m}_X(t) = \frac{\sum_{i=1}^{n} (X_i - t)\,\mathbb{I}\{X_i > t\}}{\sum_{i=1}^{n} \mathbb{I}\{X_i > t\}} \quad \text{for } t \text{ large},$$

where X_i's are the scaled daily log returns of CRIX. We see that the log mean excess function behaves linearly for large threshold, indicating the Weibull tail feature of the data-set (cf. Dierckx et al. (2009)).

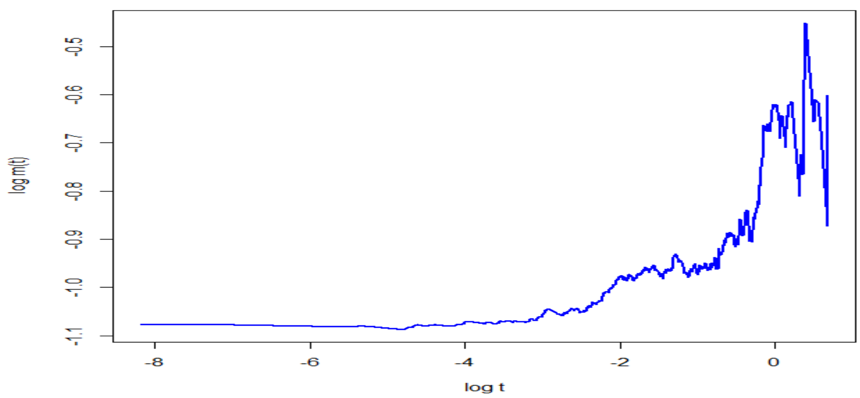

Figure 4. Graph of log mean excess function of scaled log returns of daily CRIX during 31 July 2014–1 January 2018.

Therefore, we illustrate the robustness of the proposed M-estimations $\tilde{T}_n^{(v,\infty)}$ and $T_n^{*(v,\infty)}$ with $(d_0, d_1) = (0.8848, 0.9898)$ as the 95% confidence interval via MLE, and $v = 0$ using the real data-set X and compare it with the Hill-type estimations $\widehat{\alpha}_{Hill}^{(k_n)}$ given by (11). Specifically, we consider the same contamination distribution $G(x) = \Gamma(x; 0.5, 0.5)$ and contamination level $\epsilon = 0.05i$, $i = 0, 1, \ldots, 10$.

Besides, the sample fraction k_n involved in the Hill-type estimations, is chosen via the bootstrap and maximum likelihood method as follows.

$$k_{opt}^{(1)} = \arg\min_{k_n \ge 1} |\widehat{\alpha}_{Hill}^{(k_n)} - \overline{\alpha}_{b-Hill}^{(k_n)}|, \quad k_{opt}^{(2)} = \arg\min_{k_n \ge 1} |\widehat{\alpha}_{Hill}^{(k_n)} - \widehat{\alpha}_{mle}|, \tag{13}$$

where $\overline{\alpha}_{b-Hill}^{(k_n)}$ is the average value of Hill-type estimations based on $m = 100$ bootstrap samples, and $\widehat{\alpha}_{mle}$ is the maximum likelihood estimation of the shape parameter α of Weibull distribution (see (1) for its definition). Due to the unknown Weibull tail coefficient α, we use alternatively the relative deviation of $\widehat{\alpha}$ at contamination level ϵ to $\epsilon + \delta_\epsilon$, denoted by $D(\widehat{\alpha})$ to study the relative robustness. Specifically,

$$D(\widehat{\alpha}) = Deviation(\widehat{\alpha}) = |\widehat{\alpha}(\epsilon + \delta_\epsilon) - \widehat{\alpha}(\epsilon)|, \tag{14}$$

where $\widehat{\alpha} = \tilde{T}_n, T_n^*, \widehat{\alpha}_{Hill}^{(1)}$ and $\widehat{\alpha}_{Hill}^{(2)}$ stand for the M-estimations and Hill-type estimations with optimal choice of k_n as in (13), accordingly.

From Table 2, we draw the following conclusions: (i) As expected, the proposed M-estimations are not sensitive to the contaminations, since the relative deviations of M-estimations are almost zero. Conversely, both Hill-type estimations with optimal choices of sample fraction have obvious deviations from no contamination to small contamination ($D(\widehat{\alpha}_{Hill}^{(1)}) = 0.1277, D(\widehat{\alpha}_{Hill}^{(2)}) = 0.2601$ for $\epsilon = 0$). (ii) The Hill-type estimation $\widehat{\alpha}_{Hill}^{(2)}$, with average value around 0.67, underestimates the α to some extent since the averages of the other three estimations are closer to 0.80.

Table 2. Estimations of Weibull tail coefficient and its relative deviations via contamination level $\epsilon = 0.05i, i = 0, \ldots, 10$. Data is the positive and scaled log returns of daily CRIX during 31 July 2014–1 January 2018.

ϵ	$\tilde{T}_n$	T_n^*	$\widehat{\alpha}_{Hill}^{(1)}$	$\widehat{\alpha}_{Hill}^{(2)}$	$D(\tilde{T}_n)$	$D(T_n^*)$	$D(\widehat{\alpha}_{Hill}^{(1)})$	$D(\widehat{\alpha}_{Hill}^{(2)})$
0.00	0.7711	0.7932	0.9202	0.9359	0.0072	0.0055	**0.1277**	**0.2601**
0.05	0.7783	0.7987	0.7925	0.6758	0.0056	0.0060	**0.0084**	**0.0246**
0.10	0.7839	0.8047	0.8009	0.6512	0.0002	0.0005	**0.0038**	**0.0028**
0.15	0.7841	0.8052	0.8047	0.6484	0.0026	0.0144	**0.0258**	**0.0172**
0.20	0.7867	0.8196	0.8305	0.6312	0.0093	0.0117	**0.0560**	0.0094
0.25	0.7960	0.8313	0.7745	0.6406	0.0046	0.0186	0.0168	0.0075
0.30	0.8006	0.8499	0.7577	0.6331	0.0046	0.0092	0.0038	0.0089
0.35	0.7960	0.8407	0.7539	0.6420	0.0120	0.0084	**0.0168**	0.0049
0.40	0.8080	0.8491	0.7707	0.6371	0.0008	0.0096	**0.0370**	0.0029
0.45	0.8072	0.8587	0.7337	0.6400	0.0069	0.0052	**0.0208**	**0.0096**
0.50	0.8003	0.8639	0.7545	0.6304	-	-	-	-

6. Proofs

Proof of Lemma 1. Firstly, we show that $h^*(t), t \ge t_0$ is strictly increasing. Indeed, $h(t) = (c_0 t - 1)\ln t - 1, t > 0$ is twice differentiable and

$$h'(t) = c_0(\ln t + 1) - \frac{1}{t}, \quad h''(t) = \frac{c_0}{t} + \frac{1}{t^2} > 0, \tag{15}$$

which imply that $h(t), t > 0$ is a convex function with a unique minimum $h(t_0^*)$ where $h'(t_0^*) = 0$. Therefore, we have $t_0 = \arg\min_{t \ge 1} h(t)$ exists and the unique solution $t_0 = \max(t_0^*, 1)$ and thus $h^*(t), t \in [t_0, \infty)$ is strictly increasing. Noting further for given $t_0 \ge 1$ that $t_0^{1/\alpha}$ is strictly decreasing in α, we have $h^*(x^\alpha)$ is strictly increasing in $[x_0, \infty)$ with $x_0 = t_0^{1/d_0} \ge t_0^{1/\alpha}$ since $\alpha \ge d_0$.

Secondly, note that $1 - \tilde{F}_W(x;\alpha) = \exp\{-c_0(x^\alpha - 1)\}, x \ge 1$. It follows by some elementary calculations that $-\alpha^{-1}\tilde{h}(x^\alpha)$ is the score function of $\tilde{F}_W(x;\alpha)$. Moreover, in view of (15), the minimizer t_0^* of h is decreasing in c_0. This together with the fact that $h(1) = -1$ implies that $\tilde{h}^{\leftarrow}(y) = \inf\{t \ge 1 : \tilde{h}(t) \ge y\}, y \in [-1, \infty)$ is strictly increasing. □

Proof of Theorem 1. It follows by (5) that $\tilde{\psi}_{v,u}(y;\alpha)$ is strictly increasing and continuous in α. Hence it suffices to show that

$$\tilde{\lambda}(\alpha) := \tilde{\lambda}_{F_W}(\alpha) = \int_1^\infty \tilde{\psi}_{v,u}(y;\alpha) d\tilde{F}_W(y;\alpha_0)$$

has an isolated root $\alpha = \alpha_0$. We have

$$\begin{aligned} \tilde{\lambda}(\alpha) &= \int_1^\infty [\tilde{h}(y^\alpha)]_v^u d\tilde{F}_W(y;\alpha_0) - \int_1^\infty [\tilde{h}(y^\alpha)]_v^u d\tilde{F}_W(y;\alpha) \\ &= \int_1^\infty [\tilde{h}(y^\alpha)]_v^u d\tilde{F}_W(y;\alpha_0) - \tilde{\mu}, \quad \tilde{\mu} := v + \int_v^u \exp\{-c_0[\tilde{h}^{\leftarrow}(z) - 1]\}\, dz. \end{aligned} \tag{16}$$

Next, it follows by a change of variable $t = \tilde{h}(y^\alpha)$ and integration by parts that

$$\begin{aligned} &\int_1^\infty [\tilde{h}(y^\alpha)]_v^u d\tilde{F}_W(y;\alpha_0) \\ &= \int_{-1}^v v d\tilde{F}_W([\tilde{h}^{\leftarrow}(t)]^{1/\alpha};\alpha_0) + \int_v^u t d\tilde{F}_W([\tilde{h}^{\leftarrow}(t)]^{1/\alpha};\alpha_0) + \int_u^\infty u d\tilde{F}_W([\tilde{h}^{\leftarrow}(t)]^{1/\alpha};\alpha_0) \\ &= v + \int_v^u \exp\{-c_0[(\tilde{h}^{\leftarrow}(t))^{\alpha_0/\alpha} - 1]\} dt. \end{aligned}$$

Hence, $\tilde{\lambda}(\alpha_0) = 0$ and

$$\tilde{\lambda}'(\alpha) = \frac{c_0\alpha_0}{\alpha^2} \int_{\tilde{h}^{\leftarrow}(v)}^{\tilde{h}^{\leftarrow}(u)} s^{\alpha_0/\alpha} \ln s \exp\{-c_0[s^{\alpha_0/\alpha} - 1]\} d\tilde{h}(s) > 0 \tag{17}$$

since $\tilde{h}^{\leftarrow}(s)$ is strictly increasing over $[1,\infty)$ and

$$s > \tilde{h}^{\leftarrow}(v) \geq \tilde{h}^{\leftarrow}(-1) = 1.$$

Consequently, the consistency of $\tilde{T}_n$ is obtained.

Next, we show the asymptotic normality of $\tilde{T}_n$. Set below (recall $\tilde{\mu}$ given in (16))

$$\tilde{\sigma}^2_{v,u}(\alpha) := \int_1^\infty \tilde{\psi}^2_{v,u}(y;\alpha) d\tilde{F}_W(y;\alpha_0) = \int_1^\infty ([\tilde{h}(y^\alpha)]_v^u - \tilde{\mu})^2 d\tilde{F}_W(y;\alpha_0).$$

Since

$$\begin{aligned} \tilde{\sigma}^2_{v,u}(\alpha) &= \int_{-1}^v (v-\tilde{\mu})^2 d\tilde{F}_W([\tilde{h}^{\leftarrow}(t)]^{1/\alpha};\alpha_0) + \int_v^u (t-\tilde{\mu})^2 d\tilde{F}_W([\tilde{h}^{\leftarrow}(t)]^{1/\alpha};\alpha_0) \\ &\quad + \int_u^\infty (u-\tilde{\mu})^2 d\tilde{F}_W([\tilde{h}^{\leftarrow}(t)]^{1/\alpha};\alpha_0) \\ &= (v-\tilde{\mu})^2 + 2\int_v^u (t-\tilde{\mu}) \exp\{-c_0[(\tilde{h}^{\leftarrow}(t))^{\alpha_0/\alpha} - 1]\} dt \\ &= (v-\tilde{\mu})^2 + 2\int_{\tilde{h}^{\leftarrow}(v)}^{\tilde{h}^{\leftarrow}(u)} (\tilde{h}(s)-\tilde{\mu}) \exp\{-c_0[s^{\alpha_0/\alpha} - 1]\} d\tilde{h}(s) \end{aligned}$$

is finite in a neighbourhood of α_0 and continuous at $\alpha = \alpha_0$. It follows thus by Theorem A, p. 251 in Serfling (1980) that $\tilde{T}_n$ is asymptotically normal distributed.

Furthermore, we have by (20)

$$\tilde{\lambda}'(\alpha_0) = \frac{c_0}{\alpha_0} \int_{\tilde{h}^{\leftarrow}(v)}^{\tilde{h}^{\leftarrow}(u)} s \ln s \exp\{-c_0(s-1)\} d\tilde{h}(s) > 0.$$

Hence, the asymptotic variance of $\sqrt{m}(\tilde{T}_n - \alpha_0)$ is given by

$$\tilde{\sigma}^2_{v,u,\alpha_0;F_W} = \frac{\tilde{\sigma}^2_{v,u}(\alpha_0)}{[\tilde{\lambda}'(\alpha_0)]^2}.$$

Please note that $m/n \xrightarrow{p} P\{X \geq 1\} = \exp\{-c_0\}$. We complete the proof of Theorem 1. □

Proof of Theorem 2. Similar arguments of Theorem 1 apply with $\tilde{\psi}, \tilde{F}_W$ and $\tilde{h}$ replaced by ψ^*, F^* and h^*, respectively. First we show the consistency of T^*_n. It follows by (6) that $\psi^*_{v,u}(y;\alpha)$ is strictly increasing and continuous in α. Hence it suffices to show that

$$\lambda^*(\alpha) := \lambda^*_{F_W}(\alpha) = \int_{x_0}^{\infty} \psi^*_{v,u}(y;\alpha) dF^*_W(y;\alpha_0)$$

has an isolated root $\alpha = \alpha_0$. We have

$$\begin{aligned}
\lambda^*(\alpha) &= \int_{x_0}^{\infty} \psi^*_{v,u}(y;\alpha) dF^*_W(y;\alpha_0) \\
&= \int_{x_0}^{\infty} [h^*(y^\alpha)]_v^u dF^*_W(y;\alpha_0) - \int_{x_0}^{\infty} [h^*(y^\alpha)]_v^u dF^*_W(y;\alpha) \\
&= \int_{x_0}^{\infty} [h^*(y^\alpha)]_v^u dF^*_W(y;\alpha_0) - \mu^*,\ \mu^* := v + \int_v^u \exp\{-c_0(h^*)^{\leftarrow}(z)\}\, dz. \qquad (18)
\end{aligned}$$

Next, it follows by a change of variable $t = h^*(y^\alpha)$ and integration by parts that

$$\begin{aligned}
\int_{x_0}^{\infty} [h^*(y^\alpha)]_v^u dF^*_W(y;\alpha_0) &= \int_{h^*(x_0^\alpha)}^{v} v dF_W([(h^*)^{\leftarrow}(t)]^{1/\alpha};\alpha_0) + \int_v^u t dF_W([(h^*)^{\leftarrow}(t)]^{1/\alpha};\alpha_0) \\
&\quad + \int_u^{\infty} u dF_W([(h^*)^{\leftarrow}(t)]^{1/\alpha};\alpha_0) + F_W(x_0;\alpha_0)[h^*(x_0^\alpha)]|_v^u \\
&= v[1 - F_W(x_0;\alpha_0)] + \int_v^u [1 - F_W([(h^*)^{\leftarrow}(t)]^{1/\alpha};\alpha_0)]dt + vF_W(x_0;\alpha_0) \\
&= v + \int_v^u \exp\{-c_0((h^*)^{\leftarrow}(t))^{\alpha_0/\alpha}\}dt, \qquad (19)
\end{aligned}$$

where in the second equality we use $h^*(x_0^\alpha) \leq h^*(x_0^{d_1}) = v_0 \leq v$. Hence, $\lambda^*(\alpha_0) = 0$ and

$$(\lambda^*)'(\alpha) = \frac{c_0\alpha_0}{\alpha^2} \int_{(h^*)^{\leftarrow}(v)}^{(h^*)^{\leftarrow}(u)} s^{\alpha_0/\alpha} \ln s \exp\{-c_0 s^{\alpha_0/\alpha}\} dh^*(s) > 0 \qquad (20)$$

since $h^*(s)$ is strictly increasing over $(x_0^{d_0}, \infty)$ and

$$s > (h^*)^{\leftarrow}(v) \geq (h^*)^{\leftarrow}(v_0) = x_0^{d_1} = t_0^{d_1/d_0} \geq 1.$$

Consequently, the consistency of T^*_n is obtained.

Next, we show the asymptotic normality of T^*_n. Set below (recall μ^* given by (18))

$$(\sigma^*_{v,u})^2(\alpha) := \int_{x_0}^{\infty} (\psi^*_{v,u})^2(y;\alpha) dF^*_W(y;\alpha_0) = \int_{x_0}^{\infty} ([h^*(y^\alpha)]_v^u - \mu^*)^2 dF^*_W(y;\alpha_0).$$

Since

$$
\begin{aligned}
(\sigma^*_{v,u})^2(\alpha) &= \int_{h^*(x_0^\alpha)}^{v} (v-\mu^*)^2 dF_W([(h^*)^{\leftarrow}(t)]^{1/\alpha};\alpha_0) + \int_v^u (t-\mu^*)^2 dF_W([(h^*)^{\leftarrow}(t)]^{1/\alpha};\alpha_0) \\
&\quad + \int_u^\infty (u-\mu^*)^2 dF_W([(h^*)^{\leftarrow}(t)]^{1/\alpha};\alpha_0) + F_W(x_0;\alpha_0)(v-\mu^*)^2 \\
&= (v-\mu^*)^2 + 2\int_v^u (t-\mu^*)\exp\{-c_0((h^*)^{\leftarrow}(t))^{\alpha_0/\alpha}\}dt \\
&= (v-\mu^*)^2 + 2\int_{(h^*)^{\leftarrow}(v)}^{(h^*)^{\leftarrow}(u)} (h^*(s)-\mu^*)\exp\{-c_0 s^{\alpha_0/\alpha}\}dh^*(s)
\end{aligned}
$$

is finite in a neighbourhood of α_0 and continuous at $\alpha = \alpha_0$, it follows by Theorem A, p. 251 in Serfling (1980) that T_n^* is asymptotically normal distributed.

Furthermore, we have by (20)

$$
(\lambda^*)'(\alpha_0) = \frac{c_0}{\alpha_0}\int_{(h^*)^{\leftarrow}(v)}^{(h^*)^{\leftarrow}(u)} s\ln s \exp\{-c_0 s\}dh^*(s) > 0.
$$

Hence, the asymptotic variance is given by

$$
\sigma^2_{v,u,\alpha_0;F} = \frac{(\sigma^*_{v,u}(\alpha_0))^2}{[(\lambda^*)'(\alpha_0)]^2}.
$$

We complete the proof of Theorem 2. □

Proof of Theorem 3. The result follows by analogous arguments as in the proof of Theorem 1. Since $\tilde{\psi}_{v,u}(x;\alpha)$ is strictly increasing and contionuous in α, the assumptions of Theorem 3 are sufficient for the consistency and asymptotic normality of $\tilde{T}_n$. Using further Lemma 7.2.1A and Theorem A (see p. 249 and 251 therein) by Serfling (1980), we complete the proof of Theorem 3. □

Proof of Theorem 4. The result follows by analogous arguments as in the proof of Theorem 2. Since $\psi^*_{v,u}(x;\alpha)$ is strictly increasing and continuous in α, the assumptions of Theorem 4 are sufficient for the consistency and asymptotic normality of T_n^*. Using further Lemma 7.2.1A and Theorem A (see p. 249 and 251 therein) by Serfling (1980), we complete the proof of Theorem 4. □

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