STRUCTURED REINSURANCE DEALS WITH REFERENCE TO RELATIVE MARKET PERFORMANCE*

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ABSTRACT. In this paper we study a specific type of structured reinsurance deals, for which the indemnification scheme is contingent upon the performance of the cedent, for instance measured in terms of his loss ratio compared to the average loss ratio of the market. We show that this type of deals may be efficiently used to manage risk in the presence of financial distress cost when the cover is provided to a cohort of insurers with positively correlated loss experience. In addition to theoretical results we quantitatively illustrate the potential performance improvement in a numerical example.

1. INTRODUCTION

Reinsurance is considered to be one of the key strategic levers used to manage risk and optimise capital and its cost in order to preserve and enhance shareholder value. Along with other risk levers, such as underwriting portfolio mix, diversification, asset mix, funding composition and dividend policy, it is often used to address the following key issues:

(1) Risk-taking: what risks to write where and how much.
(2) Risk retention: what part of risk to retain and what part to transfer.
(3) Funding and cost of funding: how to fund the retained risk (i.e., debt, equity, reinsurance or hybrids) and at what cost.

Conventional forms of reinsurance – whilst being efficient in finding optimal risk management strategies to address mainly (2) and (3) and to some extent (1) – are uni-dimensional focusing on insurance risk only and thus result in a local optimisation. In contrast, structured reinsurance deals are more flexible allowing to manage different parts of enterprise risk of the insurance company in a holistic way and thus enable global risk optimisation. Reinsurance companies offer such deals in various forms, concrete examples being catastrophe bonds, finite risk solutions or multi-line products. For a broad overview, we refer to Culp [11] and Albrecher et al. [1].

A frequently used type of reinsurance deal consists of an indemnification scheme that – in addition to the insurance risk – depends on the realization of one or more observable random variables (triggers) which are related to other sources of risk the insurer is exposed to. Structured deals of this type are sometimes referred to as *contingent covers*, i.e. the nature of the risk transfer applying is contingent upon the realization of such additional random variable(s). Contingent covers have already been studied by several authors and under different denominations. For example, Gründl and Schmeiser [18] discussed several approaches to pricing double-trigger contracts, which refer to reinsurance deals for which the indemnity is triggered if both the reinsured loss exceeds a fixed deductible and some capital market index falls below a pre-defined threshold value. Asimit et al. [4] considered a framework in which an insurer shares his risk with a reinsurer according to an indemnification scheme that may vary with the risk environment, and showed that for a wide

1991 *Mathematics Subject Classification.* Primary: 91B28; Secondary: 91G05.
Key words and phrases. Structured reinsurance, contingent reinsurance, optimal reinsurance, relative market performance, financial distress cost.
* Partly supported by the Swiss National Science Foundation Project 200021_191984.

class of risk measures layer-type indemnities with parameters that vary with the risk environment can minimize the resulting sum of risk measures of insurer and reinsurer. In other contributions, contingent covers are considered but the random variables they depend on are not related to any source of risk the insurer is already exposed to, and thus the contingent nature of the indemnification scheme is used as a purely mathematical tool. For instance, Gajek and Zagrodny [17] found that for an insurer being endowed with a fixed budget to purchase reinsurance, if the loss to be reinsured has discrete components, then covers contingent upon the realization of an independent random event can provide the insurer with a lower ruin probability than those who are not. Under the name of random treaties, Guerra and Centeno [19] used contingent reinsurance as a mathematical tool (and intermediate step) to solve an optimal reinsurance problem for deterministic treaties. In a framework with regulatory solvency constraints and cost of capital, Albrecher and Cani [2] studied a form of stop-loss cover, where the deductible is randomized according to an independent external mechanism and they showed that this can yield higher expected profits than traditional stop-loss covers.

While these three latter contributions reveal interesting properties of contingent covers, the main purpose of such covers in practice is to increase the efficiency of the risk transfer, by focusing it on the scenarios where the cedent expects to need it the most regarding his overall financial risk, and hence to appropriately mitigate that risk for a smaller reinsurance premium. It is therefore intuitive that for a contingent cover to be relevant, the random variable(s) it depends on should somehow be related to the overall financial result of the insurer. Among many different sources of potential losses contributing to the overall financial result, we would like to distinguish:

- Insurance (core) risk, which consists of the risks the insurer expects to make money on and generate return on capital (underwriting, reserving, catastrophe).
- Peripheral (non-core) risks, such as investment risk and operational risk, which are contained and actively managed to reduce leakages and additional drag on capital.
- Frictional cost, which emerges as the result of insufficient and/or inefficient control of risk.

In the actuarial literature, the peripheral risks of an insurer are sometimes referred to as background risks. While in practice peripheral risks and frictional cost should be considered as two distinct sources of potential losses, from a mathematical point of view the latter can also be treated as a materialisation of background risk. Multiple contributions have shown that the presence of background risk can influence the choice of reinsurance made by an insurer. For instance, Dana and Scarsini [12], Lu et al. [22] and Chi and Wei [9] showed that the optimal traditional (non-contingent) reinsurance cover can be influenced by the presence of a background risk. Fan [14] considered an insurer maximizing his expected utility and facing a background risk, and proved the stop-loss reinsurance with a deductible being contingent upon the realization of the background risk to be optimal in this setup.

In this paper we consider a structured reinsurance deal whose indemnification scheme is contingent upon the performance of the insurer buying it, for instance measured in terms of his loss ratio relative to the average loss ratio of the market, or relative performance for short. There are several arguments for doing so, which we explain hereafter.
On the one hand, as the insurer incurs larger insurance losses, his solvency is at stake, which may put him into financial distress. During such periods, the insurer will face additional expenses (the path to ruin is costly), such as the ones related to the intervention of the regulator or the increased difficulty of issuing new debt or acquiring new business. These expenses, once combined together, are referred to as the *financial distress cost* (see e.g. Froot et al. [15]), which is a particular type of non-negligible frictional cost. Moreover, since the market stakeholders partly assess the performance of insurers by benchmarking them to one another, the ones who perform below the average will typically go through worse financial distress periods and thus have a larger financial distress

cost. For this reason, but also simply because a bad relative performance is in general the consequence of large insurance losses, the overall financial risk of an insurer can be expected to be well related with his relative performance, which makes the latter an interesting candidate to be used in a contingent cover.
On the other hand, since the worse the relative performance of a particular insurer is, the better the one of the other insurers will be, a reinsurance company selling covers of this kind to several insurers in a given market will benefit from some degree of hedging.

It is hence assumed that a key driver of the need (demand) for this special type of reinsurance is the financial distress cost (this is also in line with findings in Krvavych and Sherris [21]). We will consider the above simple structured reinsurance deal as a toy model and assess the benefits it might bring when used for optimising capital resources to enhance firm value. In particular, we want to quantify the potential improvements, for both the insurer(s) and the reinsurer, when the latter offers such covers simultaneously to various insurers. To do so, in a simple yet realistic model, we consider a representative insurer who manages his risk according to a scenario-based approach and seeks to minimize his reinsurance premium. We solve the resulting optimal reinsurance problem and explicitly derive the optimal reinsurance cover, both when contingent covers are available and when they are not. The optimal contingent and traditional covers are then compared, first theoretically and then quantitatively, by means of a numerical application.

The rest of the paper is organised as follows. In Section 2, we formulate a model in which the insurer faces both an insurance risk and financial distress cost, and we then describe the two forms of reinsurance he has at his disposal, namely contingent and traditional covers. In Section 3, we formulate the optimization problem that allows us to determine both the optimal contingent cover and the optimal traditional one. In Section 4, the optimal contingent and traditional reinsurance covers are explicitly derived, and their properties are discussed and compared. In Section 5, we give a concrete application in which we are able to determine the cases where the optimal contingent cover outperforms the traditional one, and we quantify the eventual improvements. Finally, we provide a conclusion in Section 6.

2. PRELIMINARIES

2.1. **The Model.** In this paper, all random variables are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $P_X \geq 0$ be the total premium received by an insurer for covering an (annual aggregate) insurance loss. This insurance loss is represented by the non-negative random variable X with distribution function $F_X(x) = \mathbb{P}(X \leq x)$.

As a tool to manage his risk, the insurer may purchase a reinsurance cover, under which he will cede the portion R (the ceded loss) of his insurance loss to a reinsurance company (the reinsurer), and in turn pay the corresponding reinsurance premium P_R. That reinsurance premium is defined as $P_R = \pi[R]$, where the functional $\pi : \Omega \to \mathbb{R}_+$ satisfying $\pi[0] = 0$ is the premium principle, that is, the rule determining the amount to be paid by the insurer for ceding R.

Under a reinsurance cover, the part of X and P_X being retained by the insurer are thus $D = X - R$ (the retained loss) and $P_D = P_X - P_R$ (the retained premium). Here, for simplicity the insurer is assumed to face only one other source of loss, namely the financial distress cost. The latter is assumed to depend on the retained loss and is therefore denoted Y_D. The cash-flows are considered to occur according to the following sequence: the insurer first receives P_X and pays P_R at the same time ($t = 0$), and then one year later ($t = 1$) he pays D and Y_D. We refer to the sum

$$H_D := D + Y_D$$

as the insurer's retained risk.

The financial distress cost is modelled as a function of the excess of the retained loss over the retained premium, and its concrete shape depends on the insurer's relative performance described by the discrete random variable Z, leading to

$$Y_D = g_Z\big((D - P_D)_+\big) = \sum_{z \in \mathcal{Z}} g_z\big((D - P_D)_+\big) \cdot \mathbb{1}\{Z = z\},$$

where $g_z : \mathbb{R}_+ \to \mathbb{R}_+$ is the financial distress cost function for the risk scenario z and $\mathbb{1}\{Z = z\}$ the indicator function of the event $Z = z$. We assume that

(A1) $$g_z(x) \text{ is continuous and increasing in } x, \text{ with } g_z(0) = 0, \text{ for all } z.$$

(throughout this paper, we write "increasing" for "non-decreasing" and "decreasing" for "non-increasing"). As a consequence, the financial distress cost is non-negative, and can only be greater than 0 if the retained loss of the insurer exceeds the retained premium. In practice, the shape of each function g_z would typically be linked to the market capitalization of the insurer, as studied by Froot [16].

The insurer is part of an insurance market with a total of $n \geq 2$ insurers. Let V and $\overline{V}_n$ be variables measuring at $t = 1$ the performance realized during the elapsed year by the insurer and the market, respectively. Typically V is a function of X, and $\overline{V}_n$ is based on public data. For instance, in the concrete example in Section 5, we will consider the insurer's loss ratio $V = X/P_X$, and $\overline{V}_n$ will be the average loss ratio of the market.

The relative performance of the insurer is now modelled as $Z := s(V, \overline{V}_n)$, where $s : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a bivariate step function. It is assumed that both the insurer and the reinsurer know the function s. The relative performance Z is thus a random variable at $t = 0$, whose realization can be known with no ambiguity by both the insurer and the reinsurer at $t = 1$. Correspondingly, Z is a discrete random variable that represents mutually exclusive scenarios for the realization of the relative performance of the insurer at $t = 1$, and we denote its domain by $\mathcal{Z}$.

Consider a risk scenario $z \in \mathcal{Z}$. The random variable $(X \mid Z = z)$ has the conditional distribution function $F_{X|Z=z}(x) = \mathbb{P}(X \leq x \mid Z = z)$, and we denote its Value-at-Risk at one particular $\beta \in [0, 1]$ by

$$\rho_z[X] := \mathrm{VaR}_\beta[X|Z = z] = \inf\{x : F_{X|Z=z}(x) \geq \beta\}.$$

Then, we make the two additional assumptions

(A2) $$\rho_z[X] \text{ increases in } z,$$

and

(A3) $$g_z(x) \text{ increases in } z, \text{ for all } x,$$

so Z can be seen as measuring how bad the relative performance of the insurer has been. From this, we finally define $\sup \mathcal{Z}$ as the worst-case scenario.

Remark 2.1. Studies on the optimal choice of reinsurance in the presence of background risk (or a related risk factor) often consider the property of stochastic increasingness (see e.g. Dana and Scarsini [12], Lu et al. [22] and Chi and Wei [9]). Note that in our framework the stochastic increasingness of X in Z – i.e. having $\mathbb{P}(X > x \mid Z = z)$ that increases in z, for all x – is a sufficient, but not a necessary condition for (A2).

2.2. **Traditional and Contingent Reinsurance.** The ceded loss of a traditional reinsurance cover is computed as

$$R = f(X)$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a pre-defined deterministic function, referred to as the ceded loss function. For instance, the ceded loss function of a quota-share (qs) cover at $x \in \mathbb{R}_+$ is $f^{qs}(x) = a \cdot x$, where $a \in [0, 1]$ is the proportionality factor, and the one of a (traditional) bounded stop-loss (*tbsl*) (also referred to as limited stop-loss or one-layer reinsurance cover) is

$$f^{tbsl}(x; d, \ell) = \min\{(x - d)_+, \ell\},$$

where $d \geq 0$ is the deductible (or retention) and $\ell \geq 0$ the limit (or layer). The reader interested in a broader overview of traditional reinsurance covers can refer to Albrecher et al. [1].

In contrast to traditional reinsurance, under a contingent cover the type of the ceded loss function (or the values of the parameters involved in it) can depend on the realization of one or more random variables. For a reinsurance cover being contingent upon the realization of Z, the ceded loss is computed as

$$R = f_Z(X) = \sum_{z \in \mathcal{Z}} f_z(X) \cdot \mathbb{1}\{Z = z\},$$

where f_Z denotes the contingent ceded loss function that depends on Z (so f_z is the ceded loss function when $Z = z$).

Remark 2.2. Any traditional ceded loss function f is a special case of a contingent one with $f_z = f$ for all z. Consequently, any result that holds in general for contingent ceded loss functions, also holds for the traditional ones.

The contingent version of a bounded stop-loss ($cbsl$) has ceded loss function

$$f_Z^{cbsl}(x; d_Z, \ell_Z) = \min\{(x - d_Z)_+, \ell_Z\} = \sum_{z \in \mathcal{Z}} \min\{(x - d_z)_+, \ell_z\} \cdot \mathbb{1}\{Z = z\},$$

where $d_z \geq 0$ and $\ell_z \geq 0$ are the pre-defined deductible and limit that apply if $Z = z$, and $d_Z = \sum_{z \in \mathcal{Z}} d_z \cdot \mathbb{1}\{Z = z\}$ and $\ell_Z = \sum_{z \in \mathcal{Z}} \ell_z \cdot \mathbb{1}\{Z = z\}$ are the resulting contingent deductible and contingent limit.

3. Choice of Reinsurance

3.1. **The Optimization Problem.** Sometimes insurers are willing (or constrained by law) to measure and manage their risk in several distinct risk scenarios, rather than on the average over all the risk scenarios. When an insurer does so, but contingent reinsurance is not available (for instance because the realized risk scenario is not observable), he usually has no other choice than purchasing the traditional cover that fits his needs under the worst-case risk scenario and therefore tends to pay a high reinsurance premium. A typical example of a non-observable realized risk scenario is the one of model risk (or model uncertainty, ambiguity), that is, when an insurer considers multiple probability models to describe his risk but cannot determine which is the correct one (for a suggestion of mixing of quantile levels over different such scenarios, see e.g. Cohignac and Kazi-Tani [10]). In that case, if the insurer wants to manage his risk over all the considered models by purchasing a reinsurance cover, then he will have to make his choice under the worst one. In the literature, this situation is referred to as worst-case or minimax (maxmin) optimization. For recent studies on that topic we refer to Asimit et al. [3], Birghila and Pflug [6] and Birghila et al. [5]. If, by contrast, the realized risk scenario is observable (for instance, whether a hailstorm occurs or not in a particular geographical region and time period), then using it in a contingent reinsurance cover allows the insurer to choose a potentially different indemnification scheme for each risk scenario, which lowers the reinsurance premium but still achieves the desired mitigation of risk.

The nature of the financial distress cost motivates a scenario-based measurement of risk for each z separately. For the purposes of this paper, we thus consider, for all z, the concrete conditional risk measure

$$\rho_z[H_D] = \text{VaR}_\beta[H_D|Z = z],$$

where β is typically large. If the insurer enters into a reinsurance deal at $t = 0$, then $\rho_z[H_D]$ corresponds to the funds needed at $t = 1$ in order to limit his ruin probability to at most $1 - \beta$ in scenario z. The insurer then sets his maximal acceptable level of riskiness $k \geq 0$, and chooses the reinsurance cover so that none of the resulting conditional risk measures exceeds it. Concretely, if the insurer does so and has the funds k (and hence the capital $k - P_D$) available at $t = 1$, then for any realization of Z his ruin probability is at most $1 - \beta$, which represents an additional level of safety over simply having a ruin probability of at most $1 - \beta$.

Remark 3.1. For an insurer managing his risk according to the approach described above, the choice of the maximal acceptable level of risk k will depend on several factors, such as the resulting retained premium, the capital available and the objective function. While the determination of this choice is a question of interest, it is outside the scope of this paper. Therefore, in order to keep our results general, we will consider the maximal acceptable level of risk to be chosen exogenously and assume k to be given. However, the optimality results from Section 4 can be used in a second step to determine k endogenously.

Since in general there will be several covers reducing all the conditional risk measures to at most k, we are led to the mathematical problem of choosing the contingent cover with the smallest premium. Also, in order to avoid ending up with a cover that has undesirable properties, as in Asimit et al. [4] we restrict this choice to ceded loss functions in the set

$$\mathcal{C}^{(1)} = \Big\{ f_Z : 0 \leq f_z(x) \leq x \text{ and both } f_z(x) \text{ and } x - f_z(x) \text{ are increasing functions, for all } z \Big\},$$

where the first condition ensures that the insurer cedes a loss that is neither negative nor greater than the original loss, and the second one reduces moral hazard from both parties within each risk scenario. The constraints in $\mathcal{C}^{(1)}$ are in fact an extension of those suggested by Huberman et al. [20], that were for traditional reinsurance covers only.

Remark 3.2. While restricting the choice of ceded loss functions to $\mathcal{C}^{(1)}$ reduces moral hazard within each risk scenario, the insurer may still have an incentive to misreport X and hence V in order to increase his ceded loss $f_Z(X)$. In practice, there may, however, be ways to deal with this issue applying a proper level of governance and auditing.

In mathematical terms, the optimal contingent reinsurance cover can be formulated as

$$f_Z^{(1)}(\,\cdot\,; k) = \underset{f_Z(\,\cdot\,;k) \in \mathcal{C}^{(1)}(k)}{\arg\min} \quad \pi[f_Z(X)], \tag{1}$$

where

$$\mathcal{C}^{(1)}(k) = \Big\{ f_Z \in \mathcal{C}^{(1)} : \rho_z[H_{X - f_Z(X)}] \leq k, \text{ for all } z \Big\}$$

is the set of admissible contingent ceded loss functions for a given maximal acceptable level of riskiness k.

Remark 3.3. If instead of reducing the conditional risk measure $\rho_z[H_D]$ to at most k for each risk scenario z, we considered it to be done for $\text{VaR}_\beta[H_D]$ globally, then in many cases ceding less risk in more dangerous risk scenarios is the optimal strategy for minimizing the reinsurance premium (see e.g. some solutions in Asimit et al. [4]). A contingent cover with such a property might, however, be viewed as a theoretical optimization tool rather than a risk management instrument that could be implemented in practice. So among other advantages, the scenario-based approach above can be seen as a way to prevent the optimal contingent cover to have this possibly undesirable property.

Regarding the traditional reinsurance cover that will be used as a benchmark for the optimal contingent one, we consider the same problem as (1), but with $\mathcal{C}^{(1)}$ replaced by

$$\mathcal{C}^{(2)} = \Big\{ f : 0 \le f(x) \le x \text{ and both } f(x) \text{ and } x - f(x) \text{ are increasing functions} \Big\}.$$

The best choice of the benchmark cover can then be formulated as

$$f^{(2)}(\,\cdot\,;k) = \operatorname*{arg\,min}_{f(\,\cdot\,;k)\in\mathcal{C}^{(2)}(k)} \pi[f(X)], \tag{2}$$

where

$$\mathcal{C}^{(2)}(k) = \Big\{ f(x) \in \mathcal{C}^{(2)} : \rho_z[H_{X-f(X)}] \le k, \text{ for all } z \Big\}.$$

Remark 3.4. When the relative performance and the financial distress cost are ignored, the constraints on the conditional risk measures in $\mathcal{C}^{(2)}(k)$ can be replaced by the single condition $\mathrm{VaR}_\beta[X - f(X)] \le k$. If reinsurance is priced according to the expected value principle, the optimal reinsurance problem (2) then becomes the dual problem of (3.10) in Chi and Tan [8], which consists in the minimization of $\mathrm{VaR}[X - f(X)] + \pi[f(X)]$ under the set of admissible ceded loss functions $\mathcal{C}^{(2)}$. Problem (2) is thus closely related to (3.10) in Chi and Tan [8], and as a consequence so is problem (1), as it is simply a relaxed version of (2) (the difference being the set of admissible ceded loss functions, with $\mathcal{C}^{(2)}(k) \subseteq \mathcal{C}^{(1)}(k)$). Note that problems (1) and (2) slightly differ in spirit from the formulation of many other optimal reinsurance problems in the literature, in that the measure of risk is to be made acceptable rather than minimal (see also [1, Sec.8], the review paper Cai and Chi [7] and references therein). In our formulation, the insurer can make an explicit trade-off between the reinsurance premium and his maximal acceptable measure of riskiness k, see also Remark 3.1.

3.2. **The Premium Principle.** In this paper, problems (1) and (2) will be solved for monotone premium principles.

Definition 3.1. *The premium principle π is said to be monotone if for any two ceded losses R and $\tilde{R}$, having $R \le \tilde{R}$ almost surely always results in $\pi[R] \le \pi[\tilde{R}]$.*

An important property of monotone premium principles is that they preserve the order relationships of contingent ceded loss functions.

Definition 3.2. *If two contingent ceded loss functions f_Z and $\tilde{f}_Z$ satisfy $f_z(x) \le \tilde{f}_z(x)$ for all x and z, then f_Z is said to be pointwise smaller than $\tilde{f}_Z$, and we write $f_Z \preceq \tilde{f}_Z$.*

We state this property formally:

Property 3.1. *If π is a monotone premium principle, then for any two ceded loss functions f_Z and $\tilde{f}_Z$ satisfying $f_Z \preceq \tilde{f}_Z$, we have $\pi[f_Z(X)] \le \pi[\tilde{f}_Z(X)]$.*

A well-known instance of a monotone premium principle is the expected value premium principle

$$\pi[R] = (1+\theta)\,\mathbb{E}[R],$$

where $\mathbb{E}[R]$ is the expectation of R and $\theta \ge 0$ the safety loading factor. Another example is the risk-adjusted premium principle introduced by Wang [25] (also referred to as Wang or distortion premium principle)

$$\pi[R] = \int_0^\infty w(\mathbb{P}(R > x))\,\mathrm{d}x,$$

with $w(u)$ being a non-negative increasing and concave function such that $w(0) = 0$ and $w(1) = 1$.

4. Optimality Results

4.1. **Preliminaries.** We begin by recalling Theorem 1 from Dhaene et al. [13], according to which for any increasing and continuous function $\eta(x)$ and $\beta \in [0,1]$, the Value-at-Risk of a random variable X satisfies

$$\text{VaR}_\beta[\eta(X)] = \eta(\text{VaR}_\beta[X]). \tag{3}$$

Given the definition of the financial distress costs Y_D, the retained risk of the insurer can alternatively be expressed as

$$H_D = h_Z(D; P_D) = \sum_{z \in \mathcal{Z}} h_z(D; P_D) \cdot \mathbb{1}\{Z = z\},$$

where

$$h_z(x; p) = x + g_z\big((x - p)_+\big)$$

is a function of x that depends on both p and z. More specifically, we have that

$$h_z(x; p) \text{ is a continuous and strictly increasing function of } x, \tag{4}$$

from (A1),

$$h_z(x; p) \text{ decreases in } p, \tag{5}$$

and finally

$$h_z(x; p) \text{ increases in } z, \tag{6}$$

from (A3).

Note that in absence of reinsurance, the insurer's retained risk is $H_X = h_Z(X; P_X)$.

We then define the random variables

$$h_Z^{-1}(u; p) = \sum_{z \in \mathcal{Z}} h_z^{-1}(u; p) \cdot \mathbb{1}\{Z = z\}$$

where $h_z^{-1}(u; p) = \inf\{x : h_z(x; p) \geq u\}$ is the inverse function of $h_z(x; p)$, and

$$\rho_Z[X] = \sum_{z \in \mathcal{Z}} \rho_z[X] \cdot \mathbb{1}\{Z = z\}.$$

For $i = 1, 2$, the minimal reinsurance premium and the related retained premium resulting from problem (i) are denoted

$$P_R^{(i)}(k) = \pi[f_Z^{(i)}(X; k)] \quad \text{and} \quad P_D^{(i)}(k) = P_X - P_R^{(i)}(k),$$

and the retained risk of the insurer is

$$H_D^{(i)}(k) = h_Z\big(X - f_Z^{(i)}(X; k); P_D^{(i)}(k)\big),$$

where $f_Z^{(2)}(\,\cdot\,; k) = f^{(2)}(\,\cdot\,; k)$.

Let us derive the following results.

Proposition 4.1. *For $i = 1, 2$, let π be a monotone premium principle and k the maximal acceptable level of riskiness. If the set $\mathcal{C}^{(i)}(k)$ of candidates is non-empty, then problem (i) admits a solution.*

Proof. For $i = 1, 2$, let $j \geq 1$ be an integer and $z \in \mathcal{Z}$. Let further $f_Z(\,\cdot\,; k, x_{j-1,z}, x_{j,z})$ be the pointwise smallest element of $\mathcal{C}^{(i)}(k)$ for $x \in [x_{j-1,z}, x_{j,z})$ and $Z = z$ (or an arbitrarily chosen one of them, if there are several), where $x_{0,z} = 0$ and $x_{j-1,z} \leq x_{j,z}$ for all j and z. Since by definition

$$f_Z(\,\cdot\,; k, x_{j-1,z}, x_{j,z}) \in \mathcal{C}^{(i)}(k), \text{ for all } j \text{ and } z, \tag{7}$$

the functions $f_z(x; k, x_{j-1,z}, x_{j,z})$ and $x - f_z(x; k, x_{j-1,z}, x_{j,z})$ are both increasing in x, and hence they are both continuous in x too, which yields

$$f_z(x_{j,z}; k, x_{j-1,z}, x_{j,z}) = f_z(x_{j,z}; k, x_{j,z}, x_{j+1,z}), \text{ for all } j \text{ and } z. \tag{8}$$

Let then

$$f_Z(x; k) = \sum_{z \in \mathcal{Z}} f_z(x; k) \cdot \mathbb{1}\{Z = z\},$$

where

$$f_z(x; k) = \sum_{j \geq 1} f_z(x; k, x_{j-1,z}, x_{j,z}) \cdot \mathbb{1}\{x \in [x_{j-1,z}, x_{j,z})\}.$$

Given (7), we know that $f_z(x; k)$ and $x - f_z(x; k)$ are both increasing in each interval $(x_{j-1,z}, x_{j,z})$. Therefore, since (8) yields that $f_z(x; k)$ is continuous in x at $x = x_{j,z}$, for all j and z, we have that $f_z(x; k)$ and $x - f_z(x; k)$ are both increasing in x, which yields

$$f_Z(\,\cdot\,; k) \in \mathcal{C}^{(i)}(k). \tag{9}$$

By definition, we have that $f_Z(\,\cdot\,; k) \preceq \tilde{f}_Z(\,\cdot\,; k)$, for any $\tilde{f}_Z(\,\cdot\,; k) \in \mathcal{C}^{(i)}(k)$, and hence from (9) we deduce that $f_Z(\,\cdot\,; k)$ is the pointwise smallest element in $\mathcal{C}^{(i)}(k)$. Since π is a monotone premium principle, Property 3.1 yields $f_Z^{(i)}(\,\cdot\,; k) = f_Z(\,\cdot\,; k)$, which proves the result. □

Proposition 4.2. *For $i = 1, 2$, let π be a monotone premium principle and $k_{\inf}^{(i)}$ the smallest maximal acceptable level of riskiness for which problem (i) admits a solution. Then problem (i) admits a solution for any $k \geq k_{\inf}^{(i)}$, and the resulting minimal reinsurance premium $P_R^{(i)}(k)$ decreases in k, down to 0 for $k \geq \rho_{\sup \mathcal{Z}}[H_X]$.*

Proof. For $i = 1, 2$, consider the maximal acceptable levels of riskiness k and $\tilde{k}$ satisfying $k_{\inf}^{(i)} \leq k \leq \tilde{k}$. By definition, for any $f_Z(\,\cdot\,; k) \in \mathcal{C}^{(i)}(k)$ we have $f_Z(\,\cdot\,; k) \in \mathcal{C}^{(i)}(\tilde{k})$, leading to

$$\mathcal{C}^{(i)}(k) \subseteq \mathcal{C}^{(i)}(\tilde{k}). \tag{10}$$

Since by definition problem (i) admits a solution for the maximal acceptable level of riskiness $k_{\inf}^{(i)}$, the set $\mathcal{C}^{(i)}(k_{\inf}^{(i)})$ is non-empty and hence from (10), neither is $\mathcal{C}^{(i)}(k)$, which, given Proposition 4.1, proves that problem (i) admits a solution for any k satisfying $k \geq k_{\inf}^{(i)}$.

Therefore, since k and $\tilde{k}$ satisfy $k_{\inf}^{(i)} \leq k \leq \tilde{k}$, the solutions $f_Z^{(i)}(\,\cdot\,; k)$ and $f_Z^{(i)}(\,\cdot\,; \tilde{k})$ are both defined, and from (10), they satisfy $f_Z^{(i)}(\,\cdot\,; k) \preceq f_Z^{(i)}(\,\cdot\,; \tilde{k})$. Given that π is monotone, from Property 3.1 this last relationship yields $P_R^{(i)}(k) \leq P_R^{(i)}(\tilde{k})$, which proves that $P_R^{(i)}$ decreases in k for any $k \geq k_{\inf}^{(i)}$.

Finally, consider the case where the insurer cedes no loss to the reinsurer, and hence $R = 0$ for any realization of X and Z. The corresponding contingent ceded loss function is

$$f_Z(x) = 0, \text{ for all } x \text{ and any realization of } Z, \tag{11}$$

which belongs to both $\mathcal{C}^{(i)}$. Since by definition $\pi[0] = 0$, the reinsurance premium resulting from (11) is

$$P_R = 0, \tag{12}$$

and for each z, the related conditional risk measure is thus $\rho_z[H_X]$. By virtue of (3) and (4), the latter can be rewritten as

$$\rho_z[H_X] = h_z(\rho_z[X]; P_X). \tag{13}$$

By considering properties (4) and (6) in (13), and since $\rho_z[X]$ is assumed to increase in z, the conditional risk measure $\rho_z[H_X]$ increases in z, which leads to

$$\rho_z[H_X] \leq \rho_{\sup \mathcal{Z}}[H_X], \text{ for all } z. \tag{14}$$

As a consequence, if

$$k \geq \rho_{\sup \mathcal{Z}}[H_X], \tag{15}$$

then from (14) we know that (11) belongs to $\mathcal{C}^{(i)}(k)$. Moreover, since by the definition of π the reinsurance premium must be non-negative, from (12) we deduce that under (15), the contingent ceded loss function (11) is a solution to problem (i), which proves that $P_R^{(i)}(k) = 0$ whenever $k \geq \rho_{\sup \mathcal{Z}}[H_X]$. □

Lemma 4.1. *For any $k \geq 0$ and $v \geq 0$, the pointwise smallest $f \in \mathcal{C}^{(2)}$ satisfying $f(v) \geq v - k$ is $f(x) = \min\{(x-k)_+, \ell\}$, where $\ell = (v-k)_+$.*

Proof. Let us first consider the case $0 \leq k < v$: the smallest value $f(v)$ satisfying $f(v) \geq v - k$ then is $f(v) = v - k$. For any $f \in \mathcal{C}^{(2)}$, we have

$$0 \leq f(x) - f(y) \leq x - y, \text{ for all } y \leq x,$$

which can be partitioned into

$$0 \leq f(v) - f(x) \leq v - x, \text{ for } x \in [0, v), \tag{16}$$

and

$$0 \leq f(x) - f(v) \leq x - v, \text{for } x \in [v, \infty). \tag{17}$$

With $f(v) = v - k$, inequalities (16) and (17) become

$$x - k \leq f(x) \leq v - k, \text{for } x \in [0, v),$$

and

$$v - k \leq f(x) \leq x - k, \text{for } x \in [v, \infty),$$

from what we deduce that the pointwise smallest $f \in \mathcal{C}^{(2)}$ satisfying $f(v) \geq v - k$ is

$$\begin{aligned} f(x) &= \begin{cases} 0 & \text{, for } x < k, \\ x - k & \text{, for } k \leq x < v, \\ v - k & \text{, for } x \geq v, \end{cases} \\ &= \min\{(x-k)_+, v - k\}. \end{aligned} \tag{18}$$

For the case $0 \leq v \leq k$, we have $v - k \leq 0$, which means that the pointwise smallest $f \in \mathcal{C}^{(2)}$ satisfying $f(v) \geq v - k$ is

$$f(x) = 0, \text{ for all } x. \tag{19}$$

Finally, if we let $\ell = (v-k)_+$, reassembling (18) and (19) establishes the result. □

4.2. **Optimal Contingent Reinsurance Cover.** In the following theorem, we derive the optimal reinsurance cover when contingent covers are allowed.

Theorem 4.1. *Let π be a monotone premium principle and k the maximal acceptable level of riskiness satisfying $k \geq k_{\inf}^{(1)} = \inf\{k \,:\, h_{\sup \mathcal{Z}}^{-1}\big(k; P_D^{(1)}(k)\big) \geq 0\}$. Then $f_Z^{(1)}(\,\cdot\,;k) = f_Z^{cbsl}\big(\,\cdot\,; d_Z(k), \ell_Z(k)\big)$, i.e. a contingent bounded stop-loss cover is optimal, with contingent deductible $d_Z(k) = h_Z^{-1}\big(k; P_D^{(1)}(k)\big)$ and contingent limit $\ell_Z(k) = \big(\rho_Z[X] - d_Z(k)\big)_+$.*

Proof. By Proposition 4.2, we know that for $k \geq k_{\inf}^{(1)}$, the solution $f_Z^{(1)}(\,\cdot\,;k)$ is defined and thus belongs to $\in \mathcal{C}^{(1)}(k)$, meaning that

$$\rho_z[H_D^{(1)}(k)] \leq k, \text{ for all } z. \tag{20}$$

but also

$$0 \leq f_z^{(1)}(x;k) \leq x, \text{ for all } z, \tag{21}$$

and

$$f_z^{(1)}(x;k) \text{ and } x - f_z^{(1)}(x;k) \text{ are increasing functions, for all } z. \tag{22}$$

From (3) and (4), each inequality $\rho_z[H_D^{(1)}(k)] \leq k$ can be rewritten as

$$\rho_z[X - f_z^{(1)}(X;k)] \leq h_z^{-1}\big(k; P_D^{(1)}(k)\big),$$

which, since (3) and (22) yield $\rho_z[X - f_z^{(1)}(X;k)] = \rho_z[X] - f_z^{(1)}(\rho_z[X];k)$, means that (20) is equivalent to

$$f_z^{(1)}(\rho_z[X];k) \geq \rho_z[X] - h_z^{-1}\big(k; P_D^{(1)}(k)\big), \text{ for all } z. \tag{23}$$

By reassembling (21) for $x = \rho_z[X]$ with (23), we obtain

$$\rho_z[X] \geq f_z^{(1)}(\rho_z[X];k) \geq \rho_z[X] - h_z^{-1}\big(k; P_D^{(1)}(k)\big), \text{ for all } z,$$

from what we deduce that since $f_Z^{(1)}(\,\cdot\,;k)$ is defined, we have

$$h_z^{-1}\big(k; P_D^{(1)}(k)\big) \geq 0, \text{ for all } z. \tag{24}$$

Given (6), for the inequalities in (24) to be satisfied, it is necessary and sufficient that

$$h_{\sup \mathcal{Z}}^{-1}\big(k; P_D^{(1)}(k)\big) \geq 0. \tag{25}$$

Moreover, from (4), (5) and Proposition 4.2, we have that

$$h_{\sup \mathcal{Z}}^{-1}\big(k; P_D^{(i)}(k)\big) \text{ is strictly increasing in } k, \text{ for } i = 1, 2, \tag{26}$$

and therefore (25) is equivalent to $k \geq \inf\{k : h_{\sup \mathcal{Z}}^{-1}\big(k; P_D^{(1)}(k)\big) \geq 0\}$, which proves that

$$k_{\inf}^{(1)} = \inf\{k : h_{\sup \mathcal{Z}}^{-1}\big(k; P_D^{(1)}(k)\big) \geq 0\}.$$

Subsequently, since X is a positive random variable, we have

$$\rho_z[X] \geq 0, \text{ for all } z. \tag{27}$$

If we now define $f_z^{cbsl}\big(x; d_z(k), \ell_z(k)\big) = \min\{\big(x - d_z(k)\big)_+, \ell_z(k)\}$, where $d_z(k) = h_z^{-1}\big(k; P_D^{(1)}(k)\big)$ and $\ell_z(k) = \big(\rho_z[X] - d_z(k)\big)_+$, then from Lemma 4.1 and given (24) and (27), we have that $f_z^{cbsl}\big(\,\cdot\,; d_z(k), \ell_z(k)\big)$ is, for all z, the pointwise smallest f_z satisfying (21), (22) and (23). As a result,

$$f_Z^{cbsl}\big(\,\cdot\,; d_Z(k), \ell_Z(k)\big) = \sum_{z \in \mathcal{Z}} f_z^{cbsl}\big(\,\cdot\,; d_z(k), \ell_z(k)\big) \cdot \mathbb{1}\{Z = z\}$$

is the pointwise smallest contingent ceded loss function in $\mathcal{C}^{(1)}(k)$, which, from Property 3.1 and since π is a monotone premium principle, proves that $f_Z^{(1)}(\,\cdot\,;k) = f_Z^{cbsl}\Big(\,\cdot\,;d_Z(k),\ell_Z(k)\Big)$ for $k \geq k_{\inf}^{(1)}$. $\square$

Several observations concerning the optimal contingent ceded loss function are in order. Firstly, for each scenario z, the deductible $d_z(k)$ corresponds to the threshold amount at which the insurance loss X yields a retained risk of k for the insurer. Since $d_z(k) = h_z^{-1}(k; P_D^{(1)}(k))$, from (4) we have $d_z(k) \leq k$ for all z. Moreover, if $k \leq P_D^{(1)}(k)$, or if for a particular scenario z there is no financial distress cost ($g_z(x) = 0$ for all x), then we have $d_z(k) = k$. Subsequently, if $d_z(k) < \rho_z[X]$, then the related limit $\ell_z(k)$ is strictly positive ($\ell_z(k) > 0$), whereas whenever $d_z(k) \geq \rho_z[X]$ one has no risk transfer for the scenario z ($\ell_z(k) = 0$).

Secondly, given property (6), the deductible $d_z(k)$ decreases in z, and thus, as $\rho_z[X]$ is assumed to increase in z, the limit $\ell_z(k)$ also increases in z. From this we get

$$f_z^{(1)}(\,\cdot\,;k) \preceq f_{\tilde{z}}^{(1)}(\,\cdot\,;k), \text{ whenever } z \leq \tilde{z}, \tag{28}$$

meaning that the worse the relative performance of the insurer is, the more extended his optimal reinsurance cover will be.

Thirdly, under $f_Z^{(1)}(\,\cdot\,;k)$, the retained risk of the insurer has, for each scenario z, the conditional distribution

$$\begin{aligned} F_{H_D^{(1)}(k)|Z=z}(x) &= \mathbb{P}\big(H_D^{(1)}(k) \leq x \;\big|\; Z = z\big) \\ &= \mathbb{P}\Big(X - f_z^{(1)}(X;k) \leq h_z^{-1}\big(x; P_D^{(1)}(k)\big) \;\Big|\; Z = z\Big) \\ &= \begin{cases} F_{X|Z=z}\Big(h_z^{-1}\big(x; P_D^{(1)}(k)\big)\Big) & \text{, if } x < k, \\ F_{X|Z=z}\Big(h_z^{-1}\big(x; P_D^{(1)}(k)\big) + \ell_z(k)\Big) & \text{, otherwise.} \end{cases} \end{aligned}$$

At $x = k$, this conditional distribution amounts to

$$\begin{aligned} F_{H_D^{(1)}(k)|Z=z}(k) &= F_{X|Z=z}\Big(h_z^{-1}\big(k; P_D^{(1)}(k)\big) + \ell_z(k)\Big) \\ &= F_{X|Z=z}\big(d_z(k) + \ell_z(k)\big) \\ &= F_{X|Z=z}\Big(d_z(k) + \big(\rho_z[X] - d_z(k)\big)_+\Big) \\ &\begin{cases} = 1 - \alpha & \text{, if } \ell_z(k) > 0, \\ \geq 1 - \alpha & \text{, otherwise,} \end{cases} \end{aligned} \tag{29}$$

and has the probability mass

$$F_{X|Z=z}\big(d_z(k) + \ell_z(k)\big) - F_{X|Z=z}\big(d_z(k)\big).$$

The unconditional distribution function of the retained risk of the insurer is then

$$\begin{aligned} F_{H_D^{(1)}(k)}(x) &= \mathbb{P}\big(H_D^{(1)}(k) \leq x\big) \\ &= \sum_{z \in \mathcal{Z}} F_{H_D^{(1)}(k)|Z=z}(x) \cdot \mathbb{P}(Z = z), \end{aligned}$$

which, at $x = k$, amounts to

$$F_{H_D^{(1)}(k)}(k) \begin{cases} = 1-\alpha & \text{, if } \ell_z(k) > 0 \text{ for all } z, \\ \geq 1-\alpha & \text{, otherwise} \end{cases}$$

and has the probability mass

$$\sum_{z\in\mathcal{Z}} \Big(F_{X|Z=z}\big(d_z(k) + \ell_z(k)\big) - F_{X|Z=z}\big(d_z(k)\big)\Big) \cdot \mathbb{P}(Z = z).$$

Finally regarding the conditional risk measures, from (29) we have

$$\rho_z[H_D^{(1)}(k)] \begin{cases} = k & \text{, if } \ell_z(k) > 0, \\ \leq k & \text{, otherwise.} \end{cases}$$

Hence, for each scenario z for which reinsurance is required ($\ell_z(k) > 0$), the optimal contingent bounded stop-loss only just satisfies the related constraint on the conditional risk ($\rho_z[H_D^{(1)}(k)] = k$).

4.3. **Optimal Benchmark Traditional Reinsurance Cover.** In the following theorem, we derive the optimal reinsurance cover when contingent covers are not allowed.

Theorem 4.2. *Let π be a monotone premium principle and k be the maximal acceptable level of riskiness satisfying $k \geq k_{\inf}^{(2)} = \inf\big\{k : h^{-1}_{\sup\mathcal{Z}}\big(k; P_D^{(2)}(k)\big) \geq 0\big\}$. Then $f^{(2)}(\,\cdot\,; k) = f^{tbsl}\big(\,\cdot\,; d(k), \ell(k)\big)$, i.e. a traditional bounded stop-loss cover is optimal among all traditional covers, with deductible $d(k) = h^{-1}_{\sup\mathcal{Z}}\big(k; P_D^{(2)}(k)\big)$ and limit $\ell(k) = \big(\rho_{\sup\mathcal{Z}}[X] - d(k)\big)_+$.*

Proof. By Proposition 4.2, we know that for $k \geq k_{\inf}^{(2)}$, the solution $f^{(2)}(\,\cdot\,; k)$ is defined and thus belongs to $\in \mathcal{C}^{(2)}(k)$, so that

$$\rho_z[H_D^{(2)}(k)] \leq k, \text{ for all } z, \tag{30}$$

but also

$$0 \leq f^{(2)}(x; k) \leq x, \tag{31}$$

and

$$f^{(2)}(x; k) \text{ and } x - f^{(2)}(x; k) \text{ are increasing functions.} \tag{32}$$

Following the same steps as in the proof of Theorem 4.1, we obtain that $\rho_z[H_D^{(2)}(k)] \leq k$ can be rewritten as

$$\rho_z[X] - f^{(2)}(\rho_z[X]) \leq h_z^{-1}\big(k; P_D^{(2)}(k)\big).$$

On the one hand, given (A2) and (32), the left-hand side of that last inequality increases in z. On the other hand, from (6) its right-hand side decreases in z. As a result, in order to fulfil (30), it is necessary and sufficient that $f^{(2)}(\,\cdot\,; k)$ satisfies

$$f^{(2)}(\rho_{\sup\mathcal{Z}}[X]) \geq \rho_{\sup\mathcal{Z}}[X] - h^{-1}_{\sup\mathcal{Z}}\big(k; P_D^{(2)}(k)\big). \tag{33}$$

By reassembling (31) for $x = \rho_{\sup\mathcal{Z}}[X]$ and $z = \sup\mathcal{Z}$ with (33), we obtain

$$\rho_{\sup\mathcal{Z}}[X] \geq f^{(2)}(\rho_{\sup\mathcal{Z}}[X]; k) \geq \rho_{\sup\mathcal{Z}}[X] - h^{-1}_{\sup\mathcal{Z}}\big(k; P_D^{(2)}(k)\big),$$

from which we deduce

$$h^{-1}_{\sup\mathcal{Z}}\big(k; P_D^{(2)}(k)\big) \geq 0. \tag{34}$$

With this, by applying the same reasoning as in the proof of Theorem 4.1, we can prove that

$$k_{\inf}^{(2)} = \inf\big\{k : h^{-1}_{\sup\mathcal{Z}}\big(k; P_D^{(2)}(k)\big)\big\}. \tag{35}$$

If we now define $f^{tbsl}\big(x; d(k), \ell(k)\big) = \min\big\{\big(x - d(k)\big)_+, \ell(k)\big\}$, where $d(k) = h^{-1}_{\sup \mathcal{Z}}\big(k; P_D^{(2)}(k)\big)$ and $\ell(k) = \big(\rho_{\sup \mathcal{Z}}[X] - d(k)\big)_+$, then from Lemma 4.1 and given (27) and (34), we have that $f^{tbsl}\big(\,\cdot\,; d(k), \ell(k)\big)$ is the pointwise smallest f satisfying (31), (32) and (33), and hence it is the pointwise smallest ceded loss function in $\mathcal{C}^{(2)}(k)$, which by Property 3.1 for a monotone premium principle proves the result. □

Remark 4.1. In their Theorem 3.2, Chi and Tan [8] proved the traditional bounded stop-loss to solve the problem of minimizing $\mathrm{VaR}_\beta[X - f(X)] + \pi[f(X)]$, when the set of admissible ceded loss functions is $\mathcal{C}^{(2)}$ and reinsurance is priced according to the expected value premium principle. As outlined in Remark 3.4, that problem is similar in spirit to both Problems (1) and (2) of the present paper, and the appearance of bounded stop-loss structures in our optimal solutions is therefore intuitive.

We conclude this section with the following observations concerning the optimal benchmark cover:
Firstly, the deductible $d(k)$ corresponds to the amount at which the insurance loss X yields a retained risk of k for the insurer, given that he has the relative performance $Z = \sup \mathcal{Z}$. Since $d(k) = h^{-1}_{\sup \mathcal{Z}}(k; P_D^{(2)}(k))$, from (4) we have $d(k) \le k$, and if $k \le P_D^{(2)}(k)$, or if there is no financial distress cost ($g_z(x) = 0$ for all x and z), then we have $d(k) = k$. Subsequently, if the deductible $d(k)$ satisfies $d(k) < \rho_{\sup \mathcal{Z}}[X]$, then the limit $\ell(k)$ is strictly positive ($\ell(k) > 0$). On the other hand, if $d(k) \ge \rho_{\sup \mathcal{Z}}[X]$, then the benchmark traditional bounded stop-loss implies no risk transfer ($\ell(k) = 0$).

Secondly, under $f^{(2)}(\,\cdot\,; k)$, the retained risk of the insurer has, for each scenario z, the conditional distribution

$$
\begin{aligned}
F_{H_D^{(2)}(k)|Z=z}(x) &= \mathbb{P}\big(H_D^{(2)}(k) \le x \mid Z = z\big) \\
&= \mathbb{P}\Big(X - f^{(2)}(X; k) \le h_z^{-1}\big(x; P_D^{(2)}(k)\big) \;\Big|\; Z = z\Big) \\
&= \begin{cases} F_{X|Z=z}\Big(h_z^{-1}\big(x; P_D^{(2)}(k)\big)\Big) & , \text{ if } x < h_z\big(d(k); P_D^{(2)}(k)\big), \\ F_{X|Z=z}\Big(h_z^{-1}\big(x; P_D^{(2)}(k)\big) + \ell(k)\Big) & , \text{ if } x \ge h_z\big(d(k); P_D^{(2)}(k)\big), \end{cases}
\end{aligned}
$$

where $h_{\sup \mathcal{Z}}\big(d(k); P_D^{(2)}(k)\big) = k$ and hence

$$
h_z\big(d(k); P_D^{(2)}(k)\big) \le h_{\tilde{z}}\big(d(k); P_D^{(2)}(k)\big) \le k, \text{ whenever } z \le \tilde{z} \le \sup \mathcal{Z},
$$

from (6).

At $x = h_z\big(d(k); P_D^{(2)}(k)\big)$, this conditional distribution amounts to

$$
\begin{aligned}
F_{H_D^{(2)}(k)|Z=z}\Big(h_z\big(d(k); P_D^{(2)}(k)\big)\Big) &= F_{X|Z=z}\big(d(k) + \ell(k)\big) \\
&= F_{X|Z=z}\Big(d(k) + \big(\rho_z[X] - d(k)\big)_+\Big) \\
&\begin{cases} = 1 - \alpha & , \text{ if } \ell(k) > 0 \text{ and } z = \sup \mathcal{Z}, \\ \ge 1 - \alpha & , \text{ otherwise}, \end{cases}
\end{aligned}
\tag{36}
$$

and has the probability mass

$$
F_{X|Z=z}\big(d(k) + \ell(k)\big) - F_{X|Z=z}\big(d(k)\big).
$$

The unconditional distribution function of the retained risk of the insurer is then

$$F_{H_D^{(2)}(k)}(x) = \mathbb{P}\big(H_D^{(2)}(k) \leq x\big) \\ = \sum_{z\in\mathcal{Z}} F_{H_D^{(2)}(k)|Z}(x|z) \cdot \mathbb{P}(Z = z),$$

and has, at each

$$x \in \big\{h_z\big(d(k); P_D^{(2)}(k)\big),\ z \in \mathcal{Z}\big\},$$

the probability masses of at least

$$\Big\{\Big(F_{X|Z=z}\big(d(k) + \ell(k)\big) - F_{X|Z=z}\big(d(k)\big)\Big) \cdot \mathbb{P}(Z = z),\ z \in \mathcal{Z}\Big\}.$$

Finally regarding the conditional risk measures, from (36) we have

$$\rho_z[H_D^{(2)}(k)] \begin{cases} = k & \text{, if } \ell(k) > 0 \text{ and } z = \sup\mathcal{Z}, \\ \leq k & \text{, otherwise,} \end{cases} \tag{37}$$

meaning that when reinsurance is required ($\ell(k) > 0$), the benchmark cover only just satisfies the constraint on the conditional risk measure for the worst-case scenario ($\rho_{\sup\mathcal{Z}}[H_D^{(2)}(k)] = k$).

4.4. **Comparison.** We now compare the optimal contingent bounded stop-loss and its benchmark, when the same monotone premium principle π and maximal acceptable level of riskiness k apply. We assume the latter to be such that the solutions $f_Z^{(1)}(\,\cdot\,; k)$ and $f^{(2)}(\,\cdot\,; k)$ are both defined.

We start by showing that an optimal contingent cover always leads to a smaller reinsurance premium than an optimal traditional cover:

Proposition 4.3. *Let π be a monotone premium principle and k the maximal acceptable level of riskiness. Then $k_{\inf}^{(1)} \leq k_{\inf}^{(2)}$ and for all $k \geq k_{\inf}^{(2)}$*

$$P_R^{(1)}(k) \leq P_R^{(2)}(k). \tag{38}$$

Proof. By Proposition 4.2, problems (1) and (2) both admit a solution for $k \geq \max\{k_{\min}^{(1)}, k_{\min}^{(2)}\}$. As $\mathcal{C}^{(2)}(k) \subseteq \mathcal{C}^{(1)}(k)$, problem (2) is hence just a constrained version of problem (1), so that

$$P_R^{(1)}(k) \leq P_R^{(2)}(k), \text{ for } k \geq \max\{k_{\inf}^{(1)}, k_{\inf}^{(2)}\}. \tag{39}$$

To complete the proof, it only remains to show that $k_{\inf}^{(1)} \leq k_{\inf}^{(2)}$. By the definitions of $k_{\inf}^{(1)}$ and $k_{\inf}^{(2)}$ and from (26), we have

$$h_{\sup\mathcal{Z}}^{-1}\big(k; P_D^{(1)}(k)\big) \geq 0, \text{ for } k \geq k_{\inf}^{(1)},$$

and

$$h_{\sup\mathcal{Z}}^{-1}\big(k; P_D^{(2)}(k)\big) \geq 0, \text{ for } k \geq k_{\inf}^{(2)},$$

and hence (5) and (39) yield

$$h_{\sup\mathcal{Z}}^{-1}\big(k; P_D^{(1)}(k)\big) \geq h_{\sup\mathcal{Z}}^{-1}\big(k; P_D^{(2)}(k)\big) \geq 0, \text{ for } k \geq \max\{k_{\inf}^{(1)}, k_{\inf}^{(2)}\}. \tag{40}$$

Let us now assume that $k_{\inf}^{(1)} > k_{\inf}^{(2)}$. In that case, since

$$h_{\sup\mathcal{Z}}\big(k_{\inf}^{(1)}; P_D^{(1)}(k_{\inf}^{(1)})\big) = 0 \text{ and } h_{\sup\mathcal{Z}}\big(k_{\inf}^{(2)}; P_D^{(2)}(k_{\inf}^{(2)})\big) = 0,$$

from (26) and (40) we would have

$$0 \geq h_{\sup\mathcal{Z}}\big(k_{\inf}^{(1)}; P_D^{(2)}(k_{\inf}^{(1)})\big) > 0,$$

which is not possible, so that necessarily $k_{\inf}^{(1)} \leq k_{\inf}^{(2)}$. □

Remark 4.2. Proposition 4.3 proves that for any given maximal acceptable level of riskiness $k \geq k_{\inf}^{(2)}$, the optimal contingent bounded stop-loss is worth a smaller reinsurance premium than its benchmark, which yields a larger retained premium and hence a greater risk bearing capacity for the insurer. Under the optimal contingent bounded stop-loss, the insurer can thus typically afford to choose a larger maximal acceptable level of riskiness, which, according to Proposition 4.2, will make that reinsurance cover even cheaper than its benchmark. However, since the choice of the maximal acceptable level of riskiness is not dealt with in this paper (cf. Remark 3.1), we will pursue our analysis by comparing the optimal contingent cover and its benchmark under the same k.

With this result, we are now able to compare in more detail $f_Z^{(1)}(\,\cdot\,;k)$ and $f^{(2)}(\,\cdot\,;k)$. Indeed, since (38) yields $P_D^{(1)}(k) \geq P_D^{(2)}(k)$, given (5) we have $d_{\sup \mathcal{Z}}(k) \geq d(k)$ and thus $\ell_{\sup \mathcal{Z}}(k) \leq \ell(k)$, which results in

$$f_{\sup \mathcal{Z}}^{(1)}(\,\cdot\,;k) \preceq f^{(2)}(\,\cdot\,;k) \tag{41}$$

and hence

$$f_Z^{(1)}(\,\cdot\,;k) \preceq f^{(2)}(\,\cdot\,;k), \tag{42}$$

from (28). That last relationship implies by definition

$$f_Z^{(1)}(X;k) \leq f^{(2)}(X;k), \text{ for any realization of } X \text{ and } Z, \tag{43}$$

which can be seen as the counterpart of (38). That is, while the optimal contingent bounded stop-loss is cheaper than its benchmark, it also yields a smaller ceded loss.

Remark 4.3. We can distinguish two factors responsible for the difference between $P_R^{(1)}(k)$ and $P_R^{(2)}(k)$. On the one hand, by definition the optimal contingent bounded stop-loss varies with Z, which makes it being adapted to the need of the insurer (in terms of reduction of the conditional risk measures) for each scenario. In contrast, the benchmark cover cannot vary with Z and, as shown by (37), it fits the need of the insurer only for the worst-case scenario, leaving him over-reinsured for the other scenarios, which contributes to a higher premium. On the other hand, the optimal contingent bounded stop-loss cover results in a larger retained premium than its benchmark, which for the worst-case scenario yields a smaller financial distress cost. As a result, while one could expect the ceded loss functions $f_{\sup \mathcal{Z}}^{(1)}(\,\cdot\,;k)$ and $f^{(2)}(\,\cdot\,;k)$ to be equal (they both serve the insurer's need under the worst-case scenario), their order relationship is given by (41), and the latter contributes to making the benchmark cover being more expensive than the optimal contingent bounded stop-loss.

At this point, it is interesting to note that whereas (43) results in

$$X - f_Z^{(1)}(X;k) \geq X - f^{(2)}(X;k), \text{ for any realization of } X \text{ and } Z,$$

the optimal contingent bounded stop-loss does not necessarily yield a larger retained risk for the insurer. Indeed, due to (38), it can happen that the financial distress cost

$$Y_D^{(2)}(k) = g_Z\Big(\big(X - f^{(2)}(X;k) - P_D^{(2)}(k)\big)_+\Big)$$

exceeds

$$Y_D^{(1)}(k) = g_Z\Big(\big(X - f_Z^{(1)}(X;k) - P_D^{(1)}(k)\big)_+\Big),$$

for some X and Z, in which case the difference $Y_D^{(2)}(k) - Y_D^{(1)}(k)$ may be large enough to result in $H_D^{(1)}(k) < H_D^{(2)}(k)$. However, when that happens, the difference $H_D^{(2)}(k) - H_D^{(1)}(k)$ will be non-negligible only if the retained risk of the insurer is dominated by the financial distress cost.

In any case, by the design of problems (1) and (2), the optimal contingent bounded stop-loss and its benchmark both bring all the conditional risk measures to at most the maximal acceptable level of riskiness k and hence, from this viewpoint, they mitigate the risk equivalently. However, the contingent cover does it for a smaller reinsurance premium and thus leaves more potential profits for the insurer. In Section 5, we will quantify the difference between $P_R^{(1)}(k)$ and $P_R^{(2)}(k)$ in a concrete example.

Remark 4.4. In the light of Remark 4.3, it is intuitive that the main factors determining how small $P_R^{(1)}(k)$ will be relative to $P_R^{(2)}(k)$ are, on the one hand, how over-reinsured the insurer will be for the scenarios different from $\sup \mathcal{Z}$, and on the other hand, how much weight is given to these scenarios. If reinsurance is priced according to the expected value premium principle (as it will be in the concrete example from Section 5), the difference between $P_R^{(2)}(k)$ and $P_R^{(1)}(k)$ can be expressed as

$$P_R^{(2)}(k) - P_R^{(1)}(k) = (1+\theta) \cdot \sum_{z \in \mathcal{Z}} \Delta_z(k),$$

where $\Delta_z(k) = \big(\mathbb{E}[f^{(2)}(X;k)|Z=z] - \mathbb{E}[f_z^{(1)}(X;k)|Z=z]\big) \cdot \mathbb{P}(Z=z)$. Correspondingly, this difference will be substantial if $\Delta_z(k)$ is large for some z (which occurs when $\rho_z[X]$ is significantly smaller than $\rho_{\sup \mathcal{Z}}[X]$, as that increases the difference between $\ell_z(k)$ and $\ell(k)$ and subsequently between $f_z^{(1)}(\,\cdot\,;k)$ and $f^{(2)}(\,\cdot\,;k)$) and at the same time $\mathbb{P}(Z=z)$ is large.

Finally, from the reinsurer's perspective, while (38) means that offering $f_Z^{(1)}(\,\cdot\,;k)$ instead of $f^{(2)}(\,\cdot\,;k)$ yields less potential profits, it also yields less risk, cf. (43). Whether selling contingent covers instead of traditional ones will improve the risk-to-profit of the reinsurer will hence depend on the concrete situation, particularly on the degree of negative dependence between the relative performance of each insurer. However, we will show in the next section that in several realistic cases, selling contingent covers can indeed improve the risk-to-profit of the reinsurer.

Remark 4.5. Note that in the absence of financial distress costs ($g_z(x) = 0$ and $h_z(x;p) = h_z^{-1}(x;p) = x$ for all p, x and z), there is in fact no need for Z to still model the relative performance of the insurer. Any other contingent cover based on an external (discrete) Z can then also be considered, with the above results still being applicable, as long as one assures (A2) to hold.

5. Numerical Illustration

In this section we will consider a concrete numerical illustration in detail. We assume an insurance market with $n = 3$ (and later $n = 5$) insurers. They are all assumed to be identical in distribution.

5.1. **Concrete Model Specifications.** For the marginal distribution of X_i ($i = 1, \dots, n$) representing the aggregate loss of insurer i, we consider the following composite (splicing) model (see e.g. Scollnik [24]) with density function

$$b(x) = \lambda \cdot \varphi(x;\mu,\sigma,s) + (1-\lambda) \cdot \nu(x;\alpha,s),$$

where

$$\varphi(x;\mu,\sigma,s) = \frac{\frac{1}{x} \cdot \exp\left(-\frac{1}{2} \cdot \left(\frac{\ln x - \mu}{\sigma}\right)^2\right)}{\int_0^s \frac{1}{y} \cdot \exp\left(-\frac{1}{2} \cdot \left(\frac{\ln y - \mu}{\sigma}\right)^2\right) \mathrm{d}y} \cdot \mathbb{1}\{0 < x \leq s\},$$

is the density function of a Log-Normal(μ, σ) random variable, upper-truncated at $s > 0$, and

$$\nu(x;\alpha,s) = \alpha \cdot \frac{s^\alpha}{x^{\alpha+1}} \cdot \mathbb{1}\{x > s\},$$

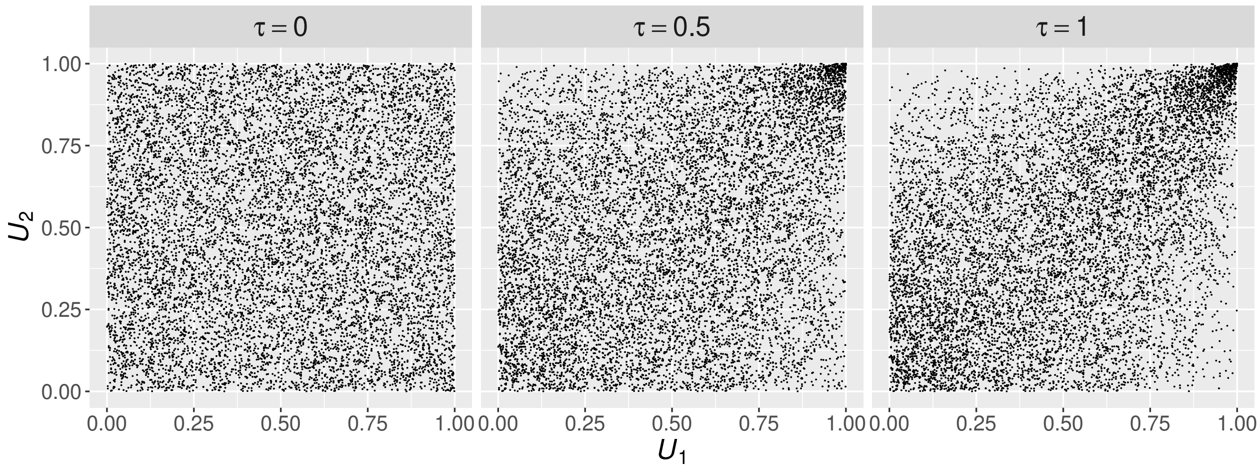

FIGURE 1. Random samples of the random vector (U_1, U_2), with Uniform$[0,1]$ marginals and F-Clayton(τ) copula, for $\tau = 0, 0.5, 1$. On these plots, each random sample contains 10^4 realizations.

the density function of a Pareto(α,s) random variable. This density function b allows to model the loss with a Log-Normal bulk of the distribution and a Pareto tail, which is often considered to be realistic. For the parameters, we set $\alpha = 2.2$ and $s = 1'800$, and then choose λ, μ and σ so that $\mathbb{E}[X_i] = 1'000$ and b is continuous and differentiable at $x = s$, which yields $\lambda \approx 0.9009$, $\mu \approx 6.5728$ and $\sigma \approx 0.6476$. The resulting standard deviation is $\sqrt{\text{Var}[X_i]} \approx 1'780$.

The dependence structure of the random vector $(X_1, X_2, ..., X_n)$ is modelled according to an Archimedean survival copula with generator $\phi(t) = t^{-1/\tau} - 1$, which is an n-dimensional Flipped-Clayton (or F-Clayton, for short) copula with parameter τ (see e.g. Nelsen [23]). In addition to interpolate between independence ($\tau = 0$) and comonotonicity ($\tau = \infty$), this copula allows for right-tail dependence and is therefore appropriate to our context, since it is not rare for reinsurers to face tail dependence when reinsuring losses. In the following applications, we will consider three dependence scenarios: mutual independence ($\tau = 0$), medium tail dependence ($\tau = 0.5$) and strong tail dependence ($\tau = 1$). Figure 1 illustrates random samples drawn from a bivariate F-Clayton(τ) copula for these three cases.

As the above specifications do not allow an explicit expression of the joint distribution function of $(X_1, ..., X_n)$, we will consider the respective empirical joint distribution based on 10^7 sample points instead for all calculations.

For simplicity of notation, we now re-identify X_1 with X. The total premium P_X and the reinsurance premium P_R are both assumed to be computed according to the expected value principle (with safety loading $\theta_X = 0.2$ and $\theta_R = 0.5$, respectively). The total premium is thus $P_X = 1'200$ and the retained premium after reinsurance amounts to

$$P_D = 1'200 - 1.5 \cdot \mathbb{E}[R].$$

As mentioned in Section 3.2, the expected value principle is monotone and hence $f_Z^{cbsl}\big(\,\cdot\,; d_Z(k), \ell_Z(k)\big)$ and $f^{tbsl}\big(\,\cdot\,; d(k), \ell(k)\big)$ are indeed the solutions to problems (1) and (2), respectively.

In practice, a conditional distribution $F_{X|Z=z}$ and a financial distress cost function g_z must be estimated for all $z \in \mathcal{Z}$. If too many risk scenarios are considered in Z, this can be a difficult task and may significantly increase model risk. For this reason, we consider here the simple case of the binary measure of relative performance

$$Z = \mathbb{1}\{V > 1.5 \cdot \frac{n}{0.5+n} \cdot \overline{V}_n\} = \mathbb{1}\{V > 1.5 \cdot \overline{V}_{n-1}\},$$

where $V_i = X_i/P_{X_i}$ is the loss ratio of insurer i (and hence $V = X/P_X$), $\overline{V}_n = \frac{1}{n} \cdot \sum_{i=1}^n V_i$ the average loss ratio of the market and $\overline{V}_{n-1} = \frac{1}{n-1} \cdot \sum_{i=2}^n V_i$ the average loss ratio of the insurer's $n-1$ competitors. The insurer is thus viewed as realizing a notably bad relative performance if his loss ratio is greater than 1.5 times the average loss ratio of his competitors. The resulting domain of Z is correspondingly $\mathcal{Z} = \{0, 1\}$. Notice that, since the insurers are assumed to be identical, they all receive the same total premium and hence the relative performance simplifies to

$$Z = \mathbb{1}\{X > 1.5 \cdot \overline{X}_{n-1}\},$$

where $\overline{X}_{n-1} = \frac{1}{n-1} \cdot \sum_{i=2}^n X_i$.

For the financial distress cost, we assume

$$Y_D = 0.5 \cdot Z \cdot (D - P_D)_+.$$

Therefore, if both a bad relative performance occur ($Z = 1$) and the retained loss exceeds the retained premium, then each two additional monetary units of retained loss result in one monetary unit of financial distress cost. The resulting financial distress cost function $g_z(x) = 0.5 \cdot z \cdot x$ is continuous in x and increasing in both x and z with $g_z(0) = 0$, in accordance with assumptions (A1) and (A3). Also, we have $h_z(x; p) = x + 0.5 \cdot z \cdot (x - p)_+$, which yields that the optimal contingent bounded stop-loss has deductibles

$$d_z(k) = \begin{cases} k & , \text{ for } z = 0, \\ \frac{2}{3} \cdot k + \frac{1}{3} \cdot \min\{k, P_D^{(1)}(k)\} & , \text{ for } z = 1, \end{cases}$$

and limits

$$\ell_z(k) = \begin{cases} \big(\rho_0[X] - d_0(k)\big)_+ & , \text{ for } z = 0, \\ \big(\rho_1[X] - d_1(k)\big)_+ & , \text{ for } z = 1, \end{cases}$$

while the benchmark has deductible

$$d(k) = \frac{2}{3} \cdot k + \frac{1}{3} \cdot \min\{k, P_D^{(2)}(k)\},$$

and limit

$$\ell(k) = \big(\rho_1[X] - d(k)\big)_+,$$

cf. Section 4. For the operators ρ_0 and ρ_1, we set $\beta = 0.995$.

In order to illustrate the model described above, we show in Table 1 the values of the conditional risk measures $\rho_0[H_X]$ and $\rho_1[H_X]$, together with the probability $\mathbb{P}(Z = 1)$, for $n = 3, 5$ and $\tau = 0, 0.5, 1$. Note that in all cases $\rho_z[H_X]$ increases in z, in accordance with assumption (A2).

Figure 2 depicts the plots of the conditional distribution functions $F_{H_X|Z=0}$ and $F_{H_X|Z=1}$, as well as the unconditional distribution function F_{H_X} for each choice of n and τ. We observe from Table 1 and Figure 2 that there is a significant difference between the conditional distributions $F_{H_X|Z=0}$ and $F_{H_X|Z=1}$ (and hence between their 99.5%-quantile $\rho_0[H_X]$ and $\rho_1[H_X]$), with the unconditional F_{H_X} being in between. The significant difference between $F_{H_X|Z=0}$ and $F_{H_X|Z=1}$ indicates that the measure Z of the relative performance distinguishes two risk scenarios in which the risk faced by the insurer (and hence his need for reinsurance) is clearly distinct, which makes

	$n=3$			$n=5$		
τ	0	0.5	1	0	0.5	1
$\mathbb{P}(Z=1)$	0.2713	0.2440	0.2103	0.2253	0.2066	0.1749
$\rho_0[H_X]$	$2'238$	$4'339$	$5'814$	$2'092$	$4'227$	$5'782$
$\rho_1[H_X]$	$18'351$	$16'932$	$14'475$	$19'962$	$18'529$	$15'780$

TABLE 1. Conditional risk measures $\rho_0[H_X]$ and $\rho_1[H_X]$ resulting from the model inputs, together with the the probability that the insurer incurs a bad relative performance $\mathbb{P}(Z=1)$, for $n=3,5$ and $\tau=0,0.5,1$.

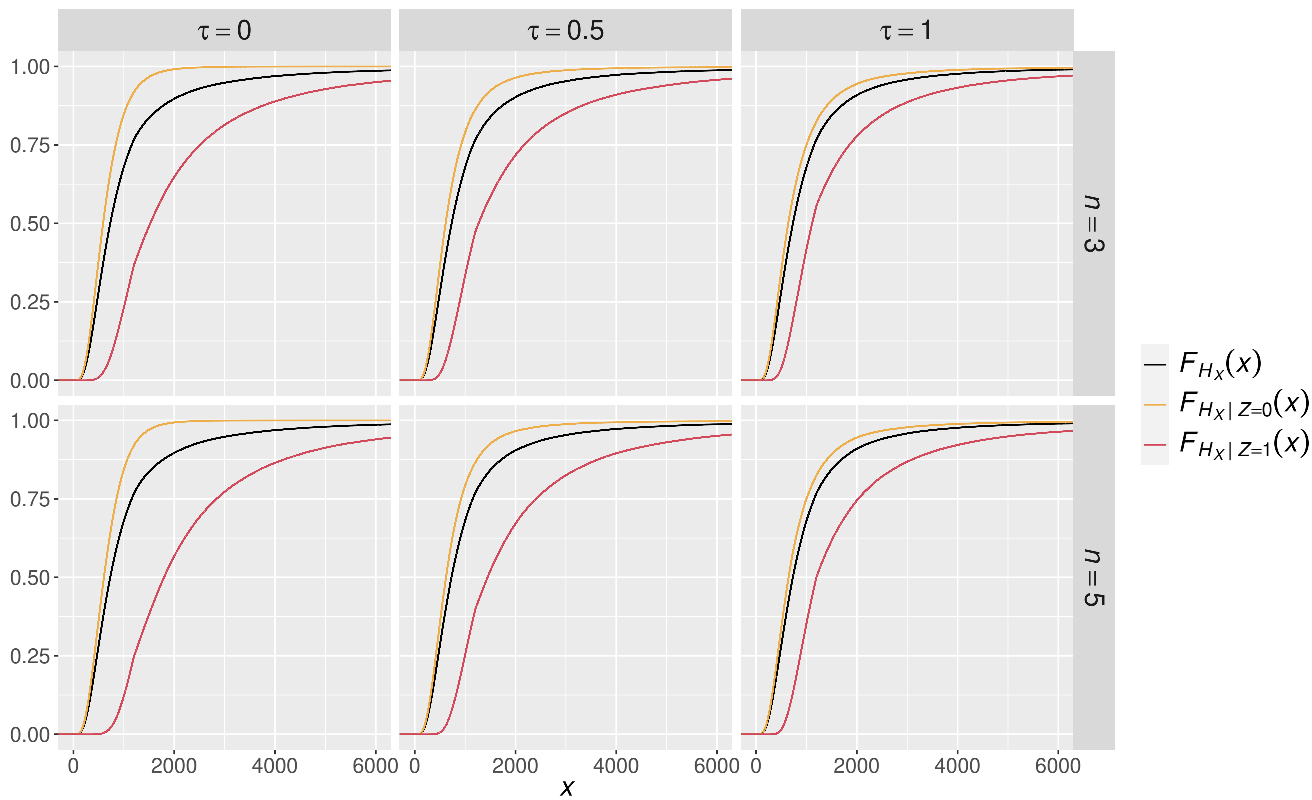

FIGURE 2. Conditional distribution functions $F_{H_X|Z=0}$ and $F_{H_X|Z=1}$, together with the unconditional distribution function F_{H_X}, for $n=3,5$ and $\tau=0,0.5,1$.

Z being an appropriate candidate to be used in a contingent reinsurance cover. The distance between the curves naturally decreases with τ and increases with n: The parameter τ models the strength of the right-tail dependence between the X_i's. For larger τ, the realizations of the X_i's will be closer to each other and hence less information on X will be carried by the events $Z=0$ and $Z=1$. At the same time, increasing the market size n lowers the variance of $\overline{X}_{n-1}$ and hence allows for potentially larger deviations of X from $1.5 \cdot \overline{X}_{n-1}$, which explains that the distance between $F_{H_X|Z=1}$ and F_{H_X} increases with n.

The same effects also drive the probability to experience a bad relative performance $\mathbb{P}(Z=1)$ as given in Table 1. That probability decreases in both τ and n. For larger τ the realizations of the

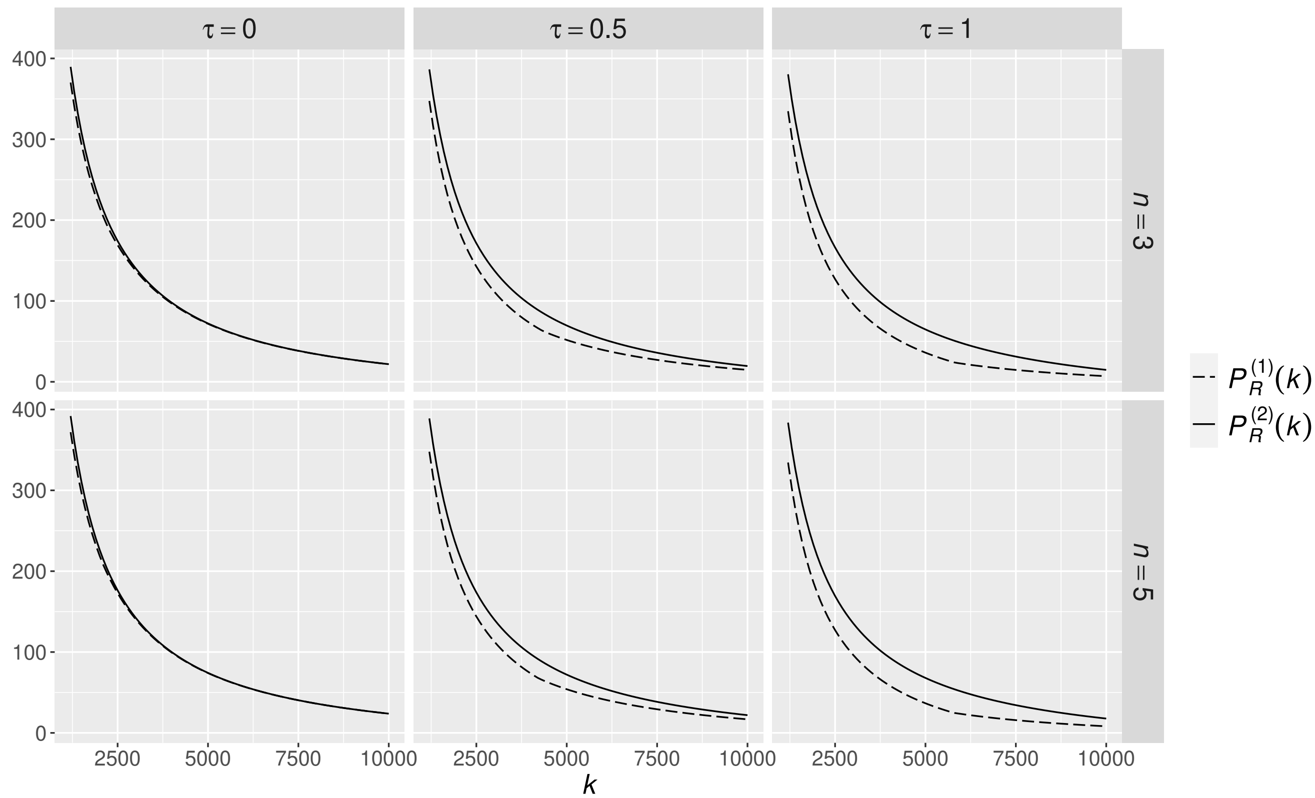

FIGURE 3. Reinsurance premiums, for $n = 3, 5$ and $\tau = 0, 0.5, 1$.

X_i's will be closer to each other, making the exceedance of X over $1.5 \cdot \overline{X}_{n-1}$ less likely. Since increasing n lowers the variance of $\overline{X}_{n-1}$, that also makes it less likely that $Z = 1$ occurs caused by a small realization of $\overline{X}_{n-1}$.

5.2. **The insurer's viewpoint.** In this section, we analyse the position of the insurer, when he purchases either the optimal contingent bounded stop-loss or its benchmark. The quantities to follow turn out to vary considerably with the given maximal acceptable level of riskiness. For the clarity of the plots, we thus consider the intermediate range of maximal acceptable levels of riskiness $k \in [1'200, 10'000]$. For $n = 3, 5$ and $\tau = 0, 0.5, 1$, the lower bound is greater than both $k_{\inf}^{(1)}$ and $k_{\inf}^{(2)}$, which ensures that both $f_Z^{(1)}(\,\cdot\,; k)$ and $f^{(2)}(\,\cdot\,; k)$ are defined.

In Figure 3, we plot the reinsurance premiums $P_R^{(1)}(k)$ and $P_R^{(2)}(k)$ for $n = 3, 5$ and $\tau = 0, 0.5, 1$. In accordance with Proposition 4.3, it shows that for all the considered maximal acceptable levels of riskiness, the reinsurance premium for the optimal contingent bounded stop-loss is smaller than the one for its benchmark. If we then compare the difference between $P_R^{(1)}(k)$ and $P_R^{(2)}(k)$ for the various values of τ and n, we notice that it notably increases with τ. The reason for this is the following: By (28) and (43), for any given monotone premium principle π and maximal acceptable level of riskiness k we have

$$f_0^{(1)}(\,\cdot\,; k) \preceq f_1^{(1)}(\,\cdot\,; k) \preceq f^{(2)}(\,\cdot\,; k), \tag{44}$$

and the only difference between $f_1^{(1)}(\,\cdot\,; k)$ and $f^{(2)}(\,\cdot\,; k)$ is the retained premium involved in the related parameters $d_1(k)$, $\ell_1(k)$, $d(k)$ and $\ell(k)$. In the present example, for all the considered maximal acceptable levels of riskiness the distance between $f_1^{(1)}(\,\cdot\,; k)$ and $f^{(2)}(\,\cdot\,; k)$ is very

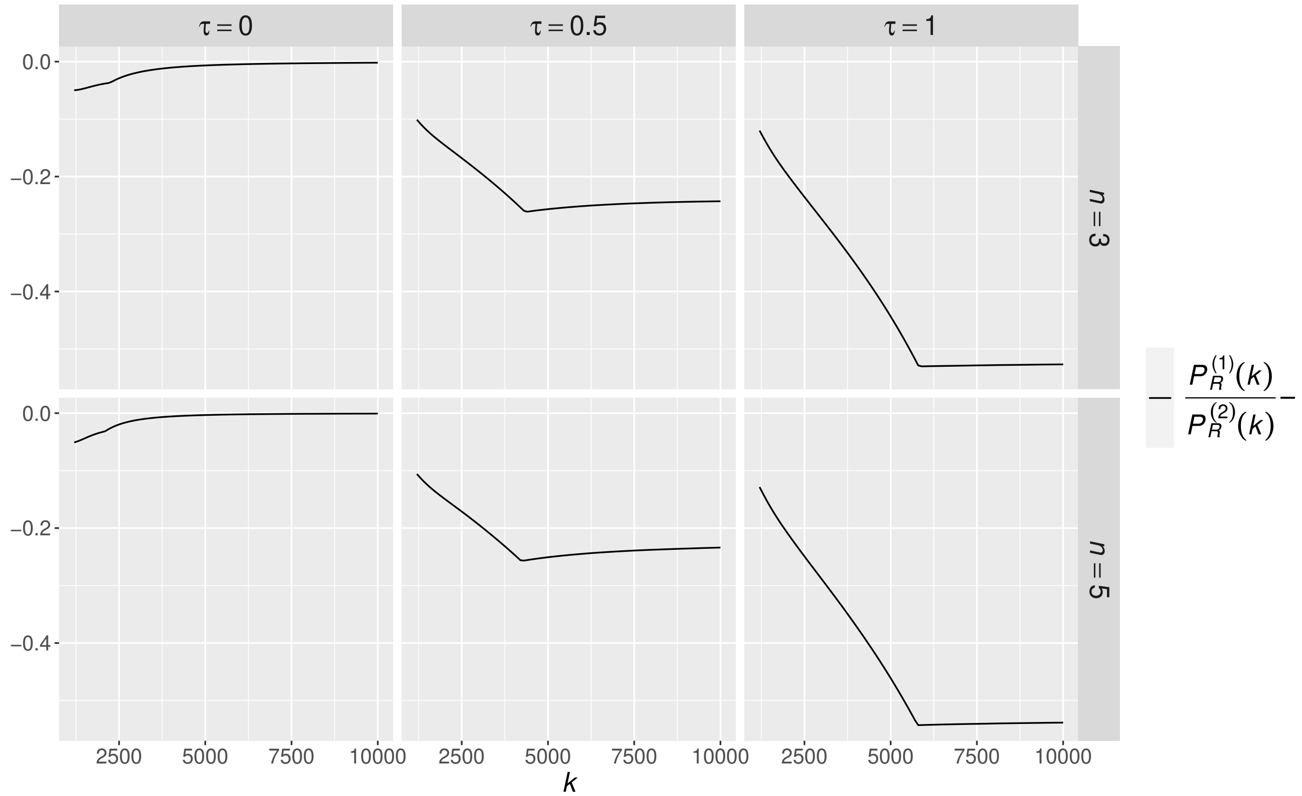

FIGURE 4. Relative difference of the reinsurance premiums, for $n = 3, 5$ and $\tau = 0, 0.5, 1$.

small, and hence so is $\Delta_1(k)$. As a result, given that here $P_R^{(2)}(k) - P_R^{(1)}(k) = 1.5 \cdot \big(\Delta_0(k) + \Delta_1(k)\big)$, what prevails in the latter difference is

$$\Delta_0(k) = \underbrace{\big(\mathbb{E}[f^{(2)}(X;k)|Z=0] - \mathbb{E}[f_0^{(1)}(X;k)|Z=0]\big)}_{(a)} \cdot \underbrace{\mathbb{P}(Z=0)}_{(b)}.$$

where (a) quantifies how much the benchmark makes the insurer being over-reinsured with respect to the contingent cover and (b) is the related weight. As shown in Table 1, the difference between $\rho_0[X]$ and $\rho_1[X]$ decreases in τ, which leads to (a) being decreasing in τ (cf. Remark 4.4). On the other hand, the probability $\mathbb{P}(Z = 0) = 1 - \mathbb{P}(Z = 1)$ increases in τ (see Table 1 and the respective discussion above). While these two effects are conflicting, it turns out that the increase in τ of (b) dominates, which leads $P_R^{(2)}(k) - P_R^{(1)}(k)$ to increase with τ.

Since in absolute terms, the above curves are quite close to each other, it may be more instructive to consider the relative difference $P_R^{(1)}(k)/P_R^{(2)}(k) - 1$ instead, which is plotted in Figure 4 for $n = 3, 5$ and $\tau = 0, 0.5, 1$. The bend appearing in each plot occurs at $k = \rho_0[X]$, and its presence can be understood as follows. While the deductibles $d_0(k)$, $d_1(k)$ and $d(k)$ are all increasing in k, the limits $\ell_0(k)$, $\ell_1(k)$ and $\ell(k)$ are all decreasing in k, which makes the reinsurance premiums $P_R^{(1)}(k)$ and $P_R^{(2)}(k)$ both to be decreasing in k, as shown by Figure 3 and in accordance with Proposition 4.2. However, whereas for $k < \rho_0[X]$, the limits $\ell_0(k)$, $\ell_1(k)$ and $\ell(k)$ are all strictly decreasing in k, for $k \in [\rho_0[X], \rho_1[H_X])$, the limit $\ell_0(k)$ is constant at 0 and hence only $\ell_1(k)$ and $\ell(k)$ remain strictly decreasing. As a result, for $k \in [\rho_0[X], \rho_1[H_X])$ the reinsurance premium

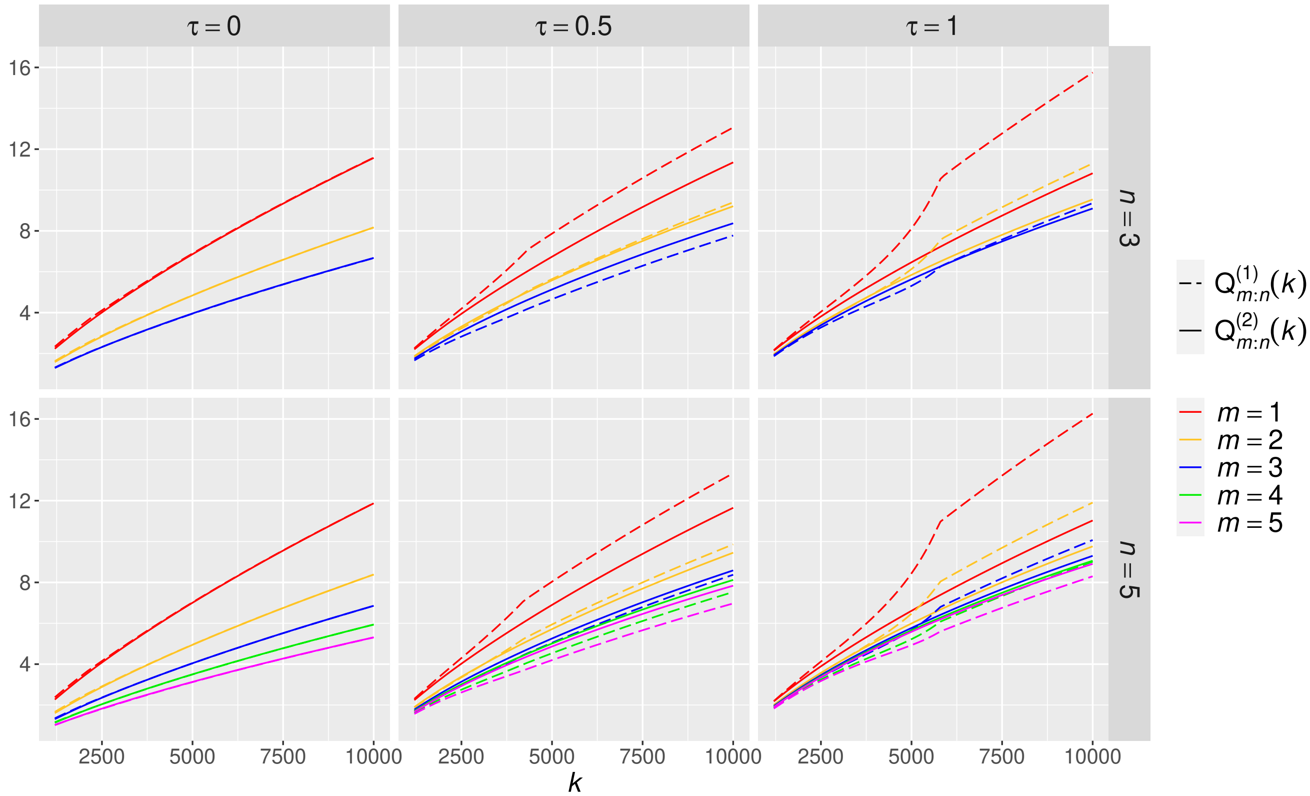

FIGURE 5. Standard-deviation of the reinsurer's loss ratio, for $n = 3, 5$ and $\tau = 0, 0.5, 1$.

$P_R^{(1)}(k)$ decreases less in k than $P_R^{(2)}(k)$ does, resulting in that particular bend in the relative difference curve $P_R^{(1)}(k)/P_R^{(2)}(k) - 1$ at $k = \rho_0[X]$.

Finally, Figure 3 already showed that the more likely it is for the worst-case scenario not to occur, the cheaper (and hence the more advantageous for the insurer) the optimal contingent bounded stop-loss will be in comparison to its benchmark. Figure 4 confirms it, showing that the relative difference can be about -25% for $\tau = 0.5$ and -55% for $\tau = 1$. That is, an increased degree of dependence among insurers is in fact advantageous for the performance of this cover. Also, we would like to emphasize that similar results could be obtained using other measures of relative performance, as long as they imply one or more large $\Delta_z(k)$ (cf. Remark 4.4).

5.3. **The reinsurer's viewpoint.** In order to assess whether selling contingent covers instead of traditional ones improves the risk-to-profit measure of the reinsurer or not, we will consider his loss ratio in two distinct (extreme) situations: either the reinsurance covers he sells are all of optimal contingent bounded stop-loss type, or they are all benchmark covers. The resulting reinsurer's loss ratios are then

$$W_{m:n}^{(1)}(k) = \frac{\sum_{i=1}^{m} f_{Z_i}^{(1)}(X_i;k)}{\sum_{i=1}^{m} P_{R_i}^{(1)}(k)} \quad \text{and} \quad W_{m:n}^{(2)}(k) = \frac{\sum_{i=1}^{m} f_{Z_i}^{(2)}(X_i;k)}{\sum_{i=1}^{m} P_{R_i}^{(2)}(k)},$$

where $m \in \{1, ..., n\}$ is the number of insurers that the reinsurer sold covers to and Z_i is the relative performance of the i^{th} insurer.

At $t = 0$, these two loss ratios are random variables which have, for any m and $k \geq k_{\inf}^{(2)}$ the common expectation

$$\mathbb{E}[W^{(1)}_{m:n}(k)] = \mathbb{E}[W^{(2)}_{m:n}(k)] = 2/3. \tag{45}$$

In order to characterise the difference between $W^{(1)}_{m:n}(k)$ and $W^{(2)}_{m:n}(k)$, we will thus focus on their standard-deviation, denoted by

$$Q^{(1)}_{m:n}(k) = \sqrt{\mathrm{Var}[W^{(1)}_{m:n}(k)]} \quad \text{and} \quad Q^{(2)}_{m:n}(k) = \sqrt{\mathrm{Var}[W^{(2)}_{m:n}(k)]}.$$

Remark 5.1. Since reinsurance is priced according to the expected value principle with a unique risk margin of 0.5, the standard deviations of the reinsurer's loss ratio $Q^{(1)}_{m:n}(k)$ and $Q^{(2)}_{m:n}(k)$ can alternatively be expressed as

$$Q^{(1)}_{m:n}(k) = \frac{\sqrt{\mathrm{Var}\big[\sum_{i=1}^m f^{(1)}_{Z_i}(X_i;k)\big]}}{1.5\cdot\mathbb{E}\big[\sum_{i=1}^m f^{(1)}_{Z_i}(X_i;k)\big]} \quad \text{and} \quad Q^{(2)}_{m:n}(k) = \frac{\sqrt{\mathrm{Var}\big[\sum_{i=1}^m f^{(2)}(X_i;k)\big]}}{1.5\cdot\mathbb{E}\big[\sum_{i=1}^m f^{(2)}(X_i;k)\big]}, \tag{46}$$

meaning that $Q^{(1)}_{m:n}(k)$ and $Q^{(2)}_{m:n}(k)$ are just the scaled (by a factor $2/3$) coefficients of variation of the total reinsurance claims $\sum_{i=1}^m f^{(1)}_{Z_i}(X_i;k)$ and $\sum_{i=1}^m f^{(2)}(X_i;k)$, respectively.

In Figure 5, we plot $Q^{(1)}_{m:n}(k)$ and $Q^{(2)}_{m:n}(k)$ for $n = 3, 5$ and $\tau = 0, 0.5, 1$. We firstly observe that for all n and τ, $Q^{(1)}_{m:n}(k)$ and $Q^{(2)}_{m:n}(k)$ are both decreasing in m, which reveals and quantifies the diversification effect of pooling risks for the reinsurer. We notice that for all n and τ, $Q^{(1)}_{m:n}(k)$ and $Q^{(2)}_{m:n}(k)$ are both increasing in k. The reason for this is the following: As k increases, the part of the X_i's that is transferred to the reinsurer decreases ($\ell_0(k)$, $\ell_1(k)$ and $\ell(k)$ are all decreasing in k) and is shifted to the right tail ($d_0(k)$, $d_1(k)$ and $d(k)$ are all increasing in k), which reduces proportionally more the expectations of $\sum_{i=1}^m f^{(1)}_{Z_i}(X_i;k)$ and $\sum_{i=1}^m f^{(2)}(X_i;k)$ than their standard deviations and hence makes $Q^{(1)}_{m:n}(k)$ and $Q^{(2)}_{m:n}(k)$ both increase in k, from (46).

In Figure 6 we consider the relative difference $Q^{(1)}_{m:n}(k)/Q^{(2)}_{m:n}(k) - 1$ for $n = 3, 5$ and $\tau = 0, 0.5, 1$, which like for the premium differences before may be more instructive to study. We firstly notice that, as for $P^{(1)}_R(k)/P^{(2)}_R(k) - 1$, the curves $Q^{(1)}_{m:n}(k)/Q^{(2)}_{m:n}(k) - 1$ all contain a bend, which occurs at $k = \rho_0[X]$ in every case. The reason for this is the following: For the considered maximal acceptable levels of riskiness, the limit $\ell_0(k)$ decreases faster than $\ell_1(k)$ in k when $k < \rho_0[X]$, and slower when $k \geq \rho_0[X]$. The difference between $\ell_0(k)$ and $\ell_1(k)$ thus increases in k for $k < \rho_0[X]$, and decreases for $k \geq \rho_0[X]$. Then, given that for all i the potential difference between $f^{(1)}_0(X_i;k)$ and $f^{(1)}_1(X_i;k)$ depends directly and positively on the one between $\ell_0(k)$ and $\ell_1(k)$, increasing k when $k < \rho_0[X]$ will add some variability to each $f^{(1)}_{Z_i}(X_i;k)$ and hence also to $\sum_{i=1}^m f^{(1)}_{Z_i}(X_i;k)$, while for $k \geq \rho_0[X]$ it will remove some. As a result, $Q^{(1)}_{m:n}(k)$ tends to increase in k faster for $k < \rho_0[X]$ than it does for $k \geq \rho_0[X]$, which results in the notable bend that occurs in the relative difference $Q^{(1)}_{m:n}(k)/Q^{(2)}_{m:n}(k) - 1$ at $k = \rho_0[X]$.

We observe that for a single reinsurance deal ($m = 1$), this relative difference is always positive, meaning that the loss ratio of each optimal contingent bounded stop-loss cover has a greater standard deviation than the one of the benchmark. This comes from the fact the optimal contingent bounded stop-loss inherits from its property of varying with the relative performance some variability that the benchmark does not have. As a result, while on the one hand the contingent bounded stop-loss reduces the expectation of the ceded loss over the one of the benchmark, on the other hand it reduces proportionally less its standard deviation. The ceded loss $f^{(1)}_Z(X;k)$ therefore has a greater coefficient of variation than $f^{(2)}(X;k)$, which from (46) yields

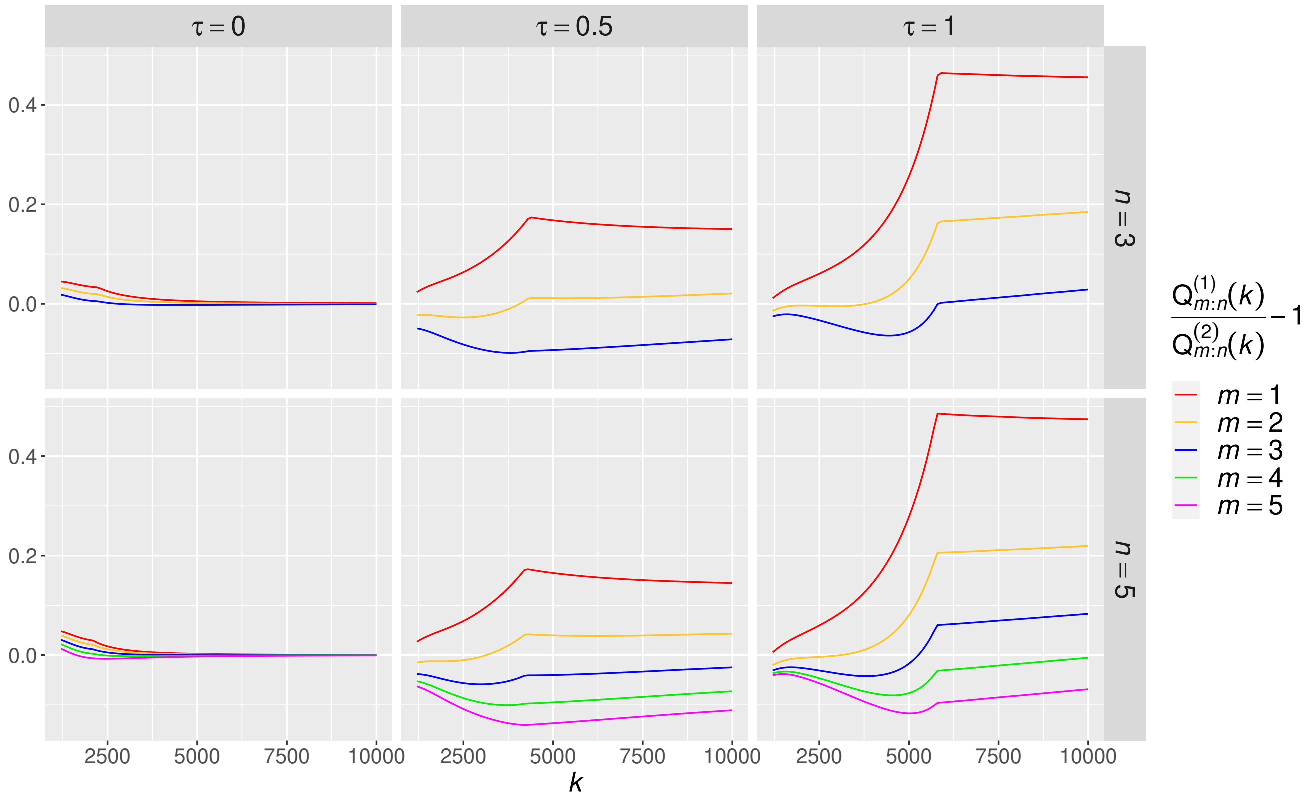

FIGURE 6. Relative difference of the standard-deviations of the reinsurer's loss ratio, for $n = 3, 5$ and $\tau = 0, 0.5, 1$.

$Q^{(1)}_{1:n}(k)/Q^{(2)}_{1:n}(k) - 1 > 0.$

We note that $Q^{(1)}_{m:n}(k)/Q^{(2)}_{m:n}(k) - 1$ decreases in m. The explanation for that is as follows: On the one hand, by construction the Z_i's tend to be negatively correlated, which introduces some degree of negative dependence between the ceded losses $f^{(1)}_{Z_i}(X_i; k)$. On the other hand, since the ceded loss function $f^{(2)}(\,\cdot\,; k)$ is increasing, the ceded losses $f^{(2)}(X; k)$, ..., $f^{(2)}(X_m; k)$ have the same dependence structure as X, ..., X_m. Because of this, when the reinsurer sells reinsurance covers to more insurers in the market (when m increases), if these covers are the optimal contingent bounded stop-loss, then he benefits from a larger diversification effect than if they are the benchmark ones. The standard deviation $Q^{(1)}_{m:n}(k)$ thus decreases faster than $Q^{(2)}_{m:n}(k)$ in m, which makes $Q^{(1)}_{m:n}(k)/Q^{(2)}_{m:n}(k) - 1$ to be decreasing in m.

Finally, when m approaches n, the relative difference $Q^{(1)}_{m:n}(k)/Q^{(2)}_{m:n}(k)-1$ turns negative. Therefore, if the reinsurer has a large market share, then while keeping the same expectation of the loss ratio (cf. (45)), in several cases selling contingent covers rather than the traditional benchmark ones makes him benefit from a smaller standard deviation of his loss ratio. Figure 6 illustrates that the improvement is substantial when the insurers' losses are positively dependent. This is particularly noteworthy, as in this case positive dependence has a favourable impact for both the insurers and the reinsurer, which is rather uncommon in risk sharing constructions. Also, it suggests that such a favourable effect for the reinsurer may still be obtained using another measure of relative performance Z, as long as one ensures that it sufficiently introduce negative dependence among the contingent covers.

6. Conclusion

In this paper we studied the efficiency of contingent reinsurance covers as a particular example of structured reinsurance deals. Since for insurers the performance relative to other market participants is quite important in terms of potential financial distress costs, we investigated a reinsurance form that pays more in scenarios where the financial distress cost is increased. On the marginal side of the insurer, this can lead to a performance improvement, and for a reinsurer offering similar covers to several market participants there also can be a beneficial diversification effect. Under certain assumptions on the performance and risk measures involved, we proved optimality results of such a cover from the viewpoint of the insurer. We further illustrated the results in a detailed numerical example, where we also showed the hedging effect for the reinsurer writing several simultaneous such contracts to market participants.
It was the purpose of this paper to propose a new perspective for the analysis and the intuitive understanding of potential advantages of this structured reinsurance deal, which is why we deliberately chose a rather simple model that allowed to keep the calculations tractable and led to explicit results. Naturally, there are various directions in which the present results can be extended. Next to possibly different performance and risk measures than the ones considered in the paper, it could also be interesting to generalize the analysis to other reinsurance premium principles, and to reinsurance pricing techniques that are more specific to the individual reinsurer's situation rather than applying a general principle. Furthermore, it will be interesting to see to what extent the results of this paper still hold in more heterogeneous (re)insurance markets.

Acknowledgements. We would like to thank two anonymous referees for their careful reading and interesting suggestions to improve the presentation of the manuscript.

References

[1] Hansjörg Albrecher, Jan Beirlant, and Jozef L Teugels. *Reinsurance: Actuarial and Statistical Aspects*. John Wiley & Sons, 2017.
[2] Hansjörg Albrecher and Arian Cani. On randomized reinsurance contracts. *Insurance: Mathematics and Economics*, 84:67–78, 2019.
[3] Alexandru V Asimit, Valeria Bignozzi, Ka Chun Cheung, Junlei Hu, and Eun-Seok Kim. Robust and Pareto optimality of insurance contracts. *European Journal of Operational Research*, 262(2):720–732, 2017.
[4] Alexandru Vali Asimit, Tim J Boonen, Yichun Chi, and Wing Fung Chong. Risk sharing with multiple indemnity environments. *Available at SSRN 3616746*, 2020.
[5] Corina Birghila, Tim J Boonen, and Mario Ghossoub. Optimal insurance under maxmin expected utility. *Available at SSRN 3711743*, 2020.
[6] Corina Birghila and Georg Ch Pflug. Optimal XL-insurance under Wasserstein-type ambiguity. *Insurance: Mathematics and Economics*, 88:30–43, 2019.
[7] Jun Cai and Yichun Chi. Optimal reinsurance designs based on risk measures: A review. *Statistical Theory and Related Fields*, 4(1):1–13, 2020.
[8] Yichun Chi and Ken Seng Tan. Optimal reinsurance under var and cvar risk measures: a simplified approach. *ASTIN Bulletin: The Journal of the IAA*, 41(2):487–509, 2011.
[9] Yichun Chi and Wei Wei. Optimal insurance with background risk: An analysis of general dependence structures. *Finance and Stochastics*, 24(4):903–937, 2020.
[10] Thierry Cohignac and Nabil Kazi-Tani. Quantile mixing and model uncertainty measures. *Bulletin Francais d'Actuariat*, 36(1):103–121, 2019.
[11] Christopher L Culp. *Structured finance and insurance: the ART of managing capital and risk*, volume 339. John Wiley & Sons, 2011.
[12] Rose-Anne Dana and Marco Scarsini. Optimal risk sharing with background risk. *Journal of Economic Theory*, 133(1):152–176, 2007.
[13] Jan Dhaene, Michel Denuit, Marc J Goovaerts, Rob Kaas, and David Vyncke. The concept of comonotonicity in actuarial science and finance: theory. *Insurance: Mathematics and Economics*, 31(1):3–33, 2002.
[14] Yulian Fan. The solution of the optimal insurance problem with background risk. *Journal of Function Spaces*, 2019.

[15] Kenneth Froot, Gary Venter, and John Major. Capital and value of risk transfer. In *14th Annual International AFIR Colloquium*, pages 181–195, 2004.
[16] Kenneth A Froot. The intermediation of financial risks: Evolution in the catastrophe reinsurance market. *Risk Management and Insurance Review*, 11(2):281–294, 2008.
[17] Lesław Gajek and Dariusz Zagrodny. Reinsurance arrangements maximizing insurer's survival probability. *Journal of Risk and Insurance*, 71(3):421–435, 2004.
[18] Helmut Gründl and Hato Schmeiser. Pricing double-trigger reinsurance contracts: financial versus actuarial approach. *Journal of Risk and Insurance*, 69(4):449–468, 2002.
[19] Manuel Guerra and Maria L Centeno. Are quantile risk measures suitable for risk-transfer decisions? *Insurance: Mathematics and Economics*, 50(3):446–461, 2012.
[20] Gur Huberman, David Mayers, and Clifford W Smith Jr. Optimal insurance policy indemnity schedules. *The Bell Journal of Economics*, pages 415–426, 1983.
[21] Yuriy Krvavych and Michael Sherris. Enhancing insurer value through reinsurance optimization. *Insurance: Mathematics & Economics*, 38(3):495–517, 2006.
[22] Zhiyi Lu, Shengwang Meng, Leping Liu, and Ziqi Han. Optimal insurance design under background risk with dependence. *Insurance: Mathematics and Economics*, 80:15–28, 2018.
[23] Roger B. Nelsen. *An introduction to copulas*. Springer Series in Statistics. Springer, New York, second edition, 2006.
[24] David PM Scollnik. On composite lognormal-Pareto models. *Scandinavian Actuarial Journal*, 2007(1):20–33, 2007.
[25] Shaun Wang. Premium calculation by transforming the layer premium density. *ASTIN Bulletin: The Journal of the IAA*, 26(1):71–92, 1996.

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