RUIN PROBABILITIES AND AGGREGRATE CLAIMS DISTRIBUTIONS FOR SHOT NOISE COX PROCESSES

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Abstract

We consider a risk process R_t where the claim arrival process is a superposition of a homogeneous Poisson process and a Cox process with a Poisson shot noise intensity process, capturing the effect of sudden increases of the claim intensity due to external events. The distribution of the aggregate claim size is investigated under these assumptions. For both light-tailed and heavy-tailed claim size distributions, asymptotic estimates for infinite-time and finite-time ruin probabilities are derived. Moreover, we discuss an extension of the model to an adaptive premium rule that is dynamically adjusted according to past claims experience.

Keywords: adative premium rule; adjustment coefficient; convex ordering; Cramér-Lundberg approximation; exponential change of measure; Gärtner-Ellis theorem; large deviations; phase-type distribution; saddlepoint approximation; subexponential distribution;

1 Introduction

Let us consider the following risk model for the surplus process R_t of an insurance portfolio:

$$R_t \;=\; u + c\,t - \sum_{j=1}^{N_t} X_j, \tag{1}$$

where u is the initial capital, c is the premium density which is assumed to be constant, the claim amounts $\{X_j\}_{j\geq 1}$ are positive i.i.d. random variables (with distribution function F_X and kth moment $\mu_{X,k}$), which are also independent of N_t, the number of claims up to time $t \geq 0$. Let $S_t = \sum_{j=1}^{N_t} X_j$ denote the aggregate claim

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size at time t. In this paper, we will investigate the risk process under the assumption that N_t is a doubly stochastic Poisson process (Cox process) with a Poisson shot noise intensity process of the form

$$\lambda_t = \lambda + \sum_{n \in \mathbb{N}} h(t - T_n, Y_n) + \nu_t, \tag{2}$$

where $\{T_n\}_{n \in \mathbb{Z}}$ is the sequence of occurrence times of a homogeneous Poisson process of rate ρ, $\{Y_n\}_{n \in \mathbb{Z}}$ is an i.i.d. sequence of positive random variables (with distribution function F_Y and kth moment $\mu_{Y,k}$) independent of the Poisson process, and the function $h(t, x)$ is non-negative with $h(t, x) = 0$ for $t < 0$. The term $\nu_t \geq 0$ represents initial conditions and is a stochastic process, independent of $\sum_{n \in \mathbb{N}} h(t - T_n, Y_n)$ ($\lambda > 0$ is assumed to be constant).

A particular simple example of (2) are multiplicative shots of the form

$$\lambda_t = \lambda + \sum_{n \in \mathbb{N}} Y_n \, g(t - T_n) + \nu_t, \tag{3}$$

where $g(t)$ is again a non-negative function with $g(t) = 0$ for $t < 0$, $G(t) = \int_0^t g(s) \, ds$ and (w.l.o.g.) $G(\infty) = 1$. The main motivation for the introduction of ν_t in (2) is the particular choice

$$\nu_t = \sum_{n \in \mathbb{Z}^-} h(t - T_n, Y_n), \tag{4}$$

which corresponds to the stationary version of (2). In this case ν_t carries the response to Poisson shots from the past $t < 0$. However, ν_t can also represent some other perturbations that do not matter asymptotically. For modelling purposes, the choice

$$\nu_t = 0 \; a.s. \; \forall \, t \geq 0 \tag{5}$$

might be considered appropriate in many situations (meaning that there are no remaining claims from previous catastrophes when setting up a portfolio).

Following Dassios & Jang [7], one interpretation of this model is as follows: In addition to the occurrence of "normal" claims described by a homogeneous Poisson process with rate λ, there are also claims triggered by external events (such as natural catastrophes) occurring at times $\{T_n\}_{n \in \mathbb{N}}$ (which are assumed to follow a homogeneous Poisson process with rate ρ). The model captures the effect that these events lead to a dramatic increase of the number of claims, whereas the individual claim sizes are assumed to have the same distribution F_X as the "normal" claims (at the expense of more cumbersome notation, one can easily extend the model to a different distribution function for the latter). Due to reporting lags of the claims that originate from a given external event, the resulting increase in intensity will develop according to the function $h(t - T_n, Y_n)$. However, also less dramatic interpretations are possible, and in particular, shot-noise modeling in a variety of implementations occurs in the literature when dealing with the delay in settlement of individual claims.

A survey of shot-noise modeling in risk theory and finance is given by Kühn [19]. Except for Dassios & Jang [7], who as here consider a Cox process with a shot-noise

intensity, the risk process is most often modelled directly as a shot-noise process. In this setting, Klüppelberg & Mikosch [17] gave a diffusion approximation, whereas ruin estimates are in Bremaud [5] and Macci *et al.* [20, 21]. Dassios & Jang [7] studied the case $g(s) = e^{-\delta s}$ of the multiplicative model and used the theory of piecewise deterministic Markov processes (PDMP) to obtain the distribution of the aggregate claim amount under an equivalent Esscher measure. An estimation procedure for λ_t in this specific setting is considered in [8]. Ruin estimates for Cox processes that allow a PDMP approach can be found in Embrechts *et al.* [10].

The present paper deals both with ruin and the aggregate claims. In order to outline the content, we need some definitions. Let $H(t, y) = \int_0^t h(s, y) \, ds$ and $\overline{H}(t, Y) = H(\infty, Y) - H(t, Y)$. From Campbell's formula,

$$\mathbb{E}(\lambda_t) \;=\; \lambda + \rho \int_0^t \mathbb{E}h(t - s, Y_1) \, ds + \mathbb{E}(\nu_t) = \; \lambda + \rho \, \mathbb{E}H(t, Y_1) + \mathbb{E}(\nu_t),$$

and thus $\lim_{t \to \infty} \mathbb{E}(\lambda_t) = \lambda + \rho \, \mathbb{E}H(\infty, Y_1) := \beta$, which we will assume to be finite. The limiting average claim amount arriving per unit time is $\mu := \beta \mu_X$, where $\mu_X := \mu_{X,1}$. Throughout the paper we will assume the net profit condition $c > \mu$.

We start in Section 2 with some analytic identities for moment generating functions which are useful for both ruin and the aggregate claims. Section 3 then deals with the aggregate claims. For the stationary case, we show that the standard Cramér-Lundberg risk process with arrival intensity β is a lower bound in the sense of convex ordering; the formulation is in a more general stationary Cox process setting. We find alternative forms of the distribution of S_t and sketch some applications. Finally, a saddlepoint approximation is outlined.

The rest of the paper is then devoted to the ruin problem. Let $\tau(u) = \inf\{t \geq 0 : R_t < 0\}$ be the ruin time for the risk process and let $\psi(u) = \mathbb{P}(\tau(u) < \infty)$ be the infinite horizon ruin probability, $\psi(u, t) = \mathbb{P}(\tau(u) \leq t)$ the corresponding finite horizon one. The analysis is carried out both for light-tailed claims (Section 4) and heavy-tailed claims (Section 5).

In the light-tailed case, a fast route to the aymptotics of both $\psi(u)$ and $\psi(u, t)$ is provided by large deviations theory as implemented in Glynn & Whitt [13] and Nyrhinen [24]. In particular, given the analytic estimates from Section 2, it is almost immediate to establish the existence of a γ which plays the role of the adjustment coefficient in the sense that $\log \psi(u)/u \to -\gamma$, $u \to \infty$. We relate this γ to the adjustment coefficient of the Cramér-Lundberg process with intensity β mentioned above both in terms of inequalities and asymptotics. Then the finite horizon case is discussed.

It is basic to note that γ coincides with the adjustment coefficient of a Cramér-Lundberg risk process $\widetilde{R}_t$ with arrival intensity $\lambda + \rho$ and a claim size distribution $F_{\widetilde{X}}$ which is a mixture of F_X and a mixed compound Poisson sum of the X_i, with Poisson parameter $H(\infty, Y_1)$. This is intuitive, since $\widetilde{R}_t$ is obtained from R_t by collecting all claim arrivals triggered by a catastrophic event to the moment where this event occurs. In particular, a sample path comparison immediately yields $\psi(u) \leq \widetilde{\psi}(u)$, in obvious notation, and thus by Lundberg's inequality applied to $\widetilde{R}_t$ we have $\psi(u) \leq$

$e^{-\gamma u}$. Less trivial is a lower bound: we show that $\psi(u) \geq C_- e^{-\gamma u}$ for some C_-. This also strengthens the logarithmic asymptotics obtained via large deviations and in particular excludes prefactors to $e^{-\gamma u}$ like u^α with $-\infty < \alpha < \infty$, $e^{c_1 u^\alpha}$ with $0 < \alpha < 1$ etc.

The process $\widetilde{R}_t$, referred to as the *batch process* in the rest of the paper, also plays a major role in the heavy-tailed case, where we are able to show a result slightly stronger than for the light-tailed case, namely that both $\psi(u)$ and $\psi(u,t)$ have exactly the same asymptotics as for the batch process. This is achieved by bounding $\psi(u)$ from below using another batch process $\check{R}_t$ with all the claims originating from the same catastrophic event shifted to some suitable later point in time.

One should note that the stationary version of (2) is a Poisson cluster process and hence, in particular, for the stationary model the infinite-time ruin estimate for heavy tails given in Theorem 5.2 is a consequence of Theorem 3.1 in Asmussen et al. [4], see Theorem 12.6.3 of Rolski *et al.* [25]. However, the explicit structure of our model allows a direct approach that also covers the non-stationary case as well as a finite time horizon, see Section 5.

Finally, in Section 6 we outline one further application of large deviations theory for a modification of the model where the premimum rate is not fixed at c but dynamically adjusted according to past claims experience.

2   Moment-generating functions

The integrated process $\Lambda_t = \int_0^t \lambda_s \, ds$ is given by

$$\Lambda_t = \lambda\, t + \sum_{n \in \mathbb{N}} H(t - T_n, Y_n) + \int_0^t \nu_s \, ds \, .$$

For the stationary case (i.e. with (4)), this amounts to

$$\Lambda_t = \lambda\, t + \sum_{n \in \mathbb{N}} H(t - T_n, Y_n) + \sum_{n \in \mathbb{Z}^-} (H(t - T_n, Y_n) - H(-T_n, Y_n)).$$

From the definition of a Cox process with integrated intensity process Λ_t, the moment generating function (MGF) of the aggregate claims S_t is given by

$$\mathbb{E}(e^{\alpha S_t}) = \mathbb{E}_{\Lambda_t}(e^{(M_X(\alpha)-1)\Lambda_t}).$$

Let us first look at a slightly more general situation, where the distribution of a claim X_j is allowed to depend on the time T_j when it actually occurs, i.e. $X_j = X_j(T_j)$.

Proposition 2.1. *Suppose (5) and $X_j = X_j(T_j)$, $j = 1, 2, \dots$ Then the MGF of the aggregate claim size at time t with underlying Poisson shot noise intensity process (2) is given by*

$$\mathbb{E}(e^{\alpha\, S_t}) = \mathbb{E}(e^{\alpha \sum_{j=1}^{N_t} X_j(T_j)}) = \exp\big(\lambda \int_0^t (M_{X(w)}(\alpha) - 1)\, dw + \rho \int_0^t (\mu(s) - 1)\, ds\big)$$

with $\mu(s) = \mathbb{E}_Y\left(e^{\int_s^t h(w-s,Y)(M_{X(w)}(\alpha)-1)\, dw}\right).$

Proof. We decompose the intensity process (2) into a sum of inhomogeneous Poisson intensities, each corresponding to the response h of a particular shot. Let Z_i ($i = 1, 2, \ldots$) denote the MGF of the aggregate claims up to time t that are due to the intensity response to the shot at T_i. Then Z_i ($i = 1, 2, \ldots$) are independent. Fixing the time t, $\mathbb{E}(Z_i) = \mu(s)$, where $s = T_i < t$ (note that $\mu(s)$ depends on t). Denote $V_s := \prod_{i=1}^{N_s} Z_i$ and $f(s) := \mathbb{E}(V_s)$. Clearly, $\mathbb{E}(e^{\alpha S_t}) = e^{\lambda \int_0^t (M_{X(w)}(\alpha) - 1) \, dw} f(t)$, since the summand λ in (2) corresponds to an additional independent compound Poisson process with time-dependent claims. It remains to determine $f(t)$. The usual conditioning on whether an event occurs in $(s, s + h]$ or not gives

$$f(s + h) = (1 - \rho \, h) f(s) + \rho \, h f(s) \mu(s),$$

so the logarithmic derivative of f is $\rho(\mu(s) - 1)$, i.e. $f(t) = \exp(\rho \int_0^t (\mu(s) - 1) ds)$.
$\qquad\square$

Corollary 2.2. *For the aggregate claim size S_t in model (1) with (5), we have*

$$\log \mathbb{E}(e^{\alpha S_t}) = \lambda \, t \, (M_X(\alpha) - 1) + \rho \int_0^t \left(\mathbb{E}_Y(e^{(M_X(\alpha) - 1)H(s,Y)}) - 1 \right) ds \qquad (6)$$

(see also Daley & Vere-Jones [6]). For a general process ν_t, (6) extends to

$$\log \mathbb{E}(e^{\alpha S_t}) = \lambda \, t \, (M_X(\alpha) - 1) + \rho \int_0^t \left(\mathbb{E}_Y(e^{(M_X(\alpha) - 1)H(s,Y)}) - 1 \right) ds$$
$$+ \log \mathbb{E} \left(e^{(M_X(\alpha) - 1) \int_0^t \nu_s \, ds} \right). \qquad (7)$$

For later use, we emphasize the following consequence:

Corollary 2.3. *Assume that ν_t satisfies*

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left(e^{(M_X(\alpha) - 1) \int_0^t \nu_s \, ds} \right) = 0, \qquad (8)$$

then $\frac{1}{t} \log \mathbb{E} e^{\alpha(S_t - ct)} \to \kappa(\alpha)$ as $t \to \infty$ where

$$\kappa(\alpha) = \lambda(M_X(\alpha) - 1) - \alpha \, c + \rho \left(\mathbb{E}_Y(e^{(M_X(\alpha) - 1)H(\infty,Y)}) - 1 \right). \qquad (9)$$

Proof. From (7) we have

$$\kappa_t(\alpha) := \log \mathbb{E}(e^{\alpha(S_t - ct)}) = (\lambda(M_X(\alpha) - 1) - \alpha \, c) \, t$$
$$+ \rho \int_0^t \left(\mathbb{E}_Y(e^{(M_X(\alpha) - 1)H(s,Y)}) - 1 \right) ds + \log \mathbb{E} \left(e^{(M_X(\alpha) - 1) \int_0^t \nu_s \, ds} \right). \qquad (10)$$

Thus $\kappa(\alpha) := \lim_{t \to \infty} \kappa_t(\alpha)/t$ is given by (9).
$\qquad\square$

Proposition 2.4. *If $\mathbb{E}\left(\overline{H}(t, Y) \, e^{(M_X(\alpha) - 1)H(\infty,Y)} \right) < \infty$ for some $t > 0$ and $\int_0^\infty \left(\mathbb{E}_Y(e^{(M_X(\alpha) - 1)(H(t+w,Y) - H(w,Y))}) - 1 \right) dw < \infty$ for all $t > 0$, then the stationary version of the intensity process λ_t with (4) fulfills (8).*

Proof. Let

$$m(s,t) = \mathbb{E} \prod_{n \in \mathbb{Z}^- : T_n \in [-s,0]} e^{(M_X(\alpha)-1)(H(t-T_n,Y_n)-H(-T_n,Y_n))},$$

so that $\mathbb{E}\left(e^{(M_X(\alpha)-1)\int_0^t \nu_s \, ds}\right) = m(\infty,t)$. As in the proof of Proposition 2.1, conditioning on an event in $[-s-h,-s)$ of the Poisson process on the negative half-line yields

$$m(s+h,t) = (1-\rho h)\, m(s,t) + \rho h\, m(s,t)\, \mathbb{E}_Y(e^{(M_X(\alpha)-1)(H(t+s,Y)-H(s,Y))}),$$

which leads to

$$m(s,t) = \exp\left(\rho \int_0^s (\mathbb{E}_Y(e^{(M_X(\alpha)-1)(H(t+w,Y)-H(w,Y))}) - 1)\, dw\right).$$

De'l Hopital's rule gives

$$
\begin{aligned}
\lim_{t\to\infty} \frac{\log m(\infty,t)}{t} &= \lim_{t\to\infty} \rho\,(M_X(\alpha)-1) \int_0^\infty \mathbb{E}_Y\left(h(t+w,Y)\, e^{(M_X(\alpha)-1)(H(t+w,Y)-H(w,Y))}\right) dw \\
&\leq \rho\,(M_X(\alpha)-1) \lim_{t\to\infty} \int_0^\infty \mathbb{E}_Y\left(h(t+w,Y)\, e^{(M_X(\alpha)-1)H(\infty,Y)}\right) dw \\
&= \rho\,(M_X(\alpha)-1) \lim_{t\to\infty} \mathbb{E}_Y\left(\overline{H}(t,Y)\, e^{(M_X(\alpha)-1)H(\infty,Y)}\right) = 0,
\end{aligned}
$$

where the latter follows by monotone convergence under the assumption of the proposition. $\qquad\square$

Corollary 2.5. *For the multiplicative model* (3), *assume that* $M_Y(M_X(\alpha)-1) < \infty$ *for all α in some open interval J and that $\mu_g = \int_0^\infty t g(t) dt < \infty$. Then the stationary version of the intensity process λ_t, corresponding to $\nu_t = \sum_{n\in\mathbb{Z}^-} g(t-T_n) Y_n$, fulfills* (8) *for all $\alpha \in J$.*

Proof. For $M_X(\alpha) < 1$, the first condition of Proposition 2.4 and the finiteness of $m(\infty,t)$ trivially hold, so that (8) is fulfilled. In the case $M_X(\alpha) \geq 1$, from $H(\infty,Y) = Y$ the finiteness of

$$\mathbb{E}\left(\overline{H}(t,Y)\, e^{(M_X(\alpha)-1)H(\infty,Y)}\right) \leq \mathbb{E}\left(Y\, e^{(M_X(\alpha)-1)Y}\right)$$

follows by choosing $\alpha^* \in J$ with $\alpha^* > \alpha$ and using $M_Y(M_X(\alpha^*)-1) < \infty$. Further,

$$
\begin{aligned}
\int_0^\infty (\mathbb{E}_Y(e^{(M_X(\alpha)-1)(H(t+w,Y)-H(w,Y))}) - 1)\, dw &\leq \int_0^\infty (\mathbb{E}_Y(e^{(M_X(\alpha)-1)Y\overline{G}(w)}) - 1)\, dw \\
&= \lim_{A\to\infty}\left[w(\mathbb{E}_Y(e^{(M_X(\alpha)-1)Y\overline{G}(w)}) - 1)\right]_0^A + \int_0^\infty w g(w)\mathbb{E}_Y e^{(M_X(\alpha)-1)Y\overline{G}(w)}\, dw \\
&\leq \lim_{A\to\infty}\left[A(\mathbb{E}_Y(e^{(M_X(\alpha)-1)Y\overline{G}(A)}) - 1)\right] + \int_0^\infty w g(w)\mathbb{E}_Y e^{(M_X(\alpha)-1)Y}\, dw \\
&\sim \lim_{A\to\infty}\left[A\,\mathbb{E}Y(M_X(\alpha)-1)\overline{G}(A)\right] + \mu_g M_Y(M_X(\alpha)-1) \\
&= 0 + \mu_g M_Y(M_X(\alpha)-1) < \infty
\end{aligned}
$$

where we used $\mu_g < \infty$ to infer $A\,\overline{G}(A) \to 0$. $\qquad\square$

3 The aggregate claim size S_t

3.1 A general ordering result for S_t

We state a general ordering result on the aggregate claim amount that holds for arbitrary stationary compound Cox processes. Recall that a random variable X is said to dominate a random variable Y in convex order ($X \geq_{\mathrm{cx}} Y$), if $\mathbb{E}(\phi(X)) \geq \mathbb{E}(\phi(Y))$ for all convex functions ϕ.

Proposition 3.1. *Let $S_t = \sum_{i=1}^{N_t} X_i$ for a Cox process N_t with stationary intensity process λ_t and $\mathbb{E}(\lambda_t) = \lambda^*$ for all $t \geq 0$, and $S_t^* = \sum_{j=1}^{N_t^*} X_j$, where N_t^* denotes a homogeneous Poisson process with intensity λ^*, which is independent of N_t. Then*

$$S_t \geq_{cx} S_t^*.$$

Remark 3.1. For arbitrary random variables X, Y with $\mathbb{E}X = \mathbb{E}Y$, the relation $X \geq_{\mathrm{cx}} Y$ is equivalent to

$$\mathbb{E}\big((X-a)^+\big) \; \geq \; \mathbb{E}\big((Y-a)^+\big) \tag{11}$$

for all a (given that the expectation exists), which is the stop-loss or, equivalently, the increasing convex order (see e.g. Theorem 3.A.16 of Shaked & Shanthikumar [27]). Thus the above result immediately entails that the stop-loss premium in the portfolio is larger for the model (1) in stationarity than for the averaged Cramér-Lundberg model.

Proof. Since $\Lambda_t \geq_{\mathrm{cx}} \lambda^* t$, it follows that for any $t > 0$,

$$\mathbb{E}\big(N_t - K\big)^+ \; \geq \; \mathbb{E}\big(N_t^* - K\big)^+ \tag{12}$$

for each $K \geq 0$ (see for instance Kaas *et al.* [18, Example 10.4.6]).
Together with $\mathbb{E}(N_t) = \mathbb{E}(N_t^*)$, this implies $N_t \geq_{\mathrm{cx}} N_t^*$ (cf. Remark 3.1). The assertion then follows by translating the ordering to the compound sums of the i.i.d. individual claims, using for instance Theorem 2.A.7 of [27]. □

3.2 Another representation of S_t

Let us assume (5) for the remainder of this section. Rewriting Corollary 2.2, we can find another interpretation for the aggregate claim amount S_t in the shot-noise Cox model:

Proposition 3.2. *For a fixed time t, the aggregate claim amount S_t in model (1) with (5) is compound Poisson with intensity $\lambda + \rho$ and time-dependent claim size distribution $\widetilde{X}(s)$ with moment-generating function*

$$M_{\widetilde{X}(s)}(\alpha) = \frac{\rho}{\lambda + \rho}\, M_{\widehat{X}(s)}(\alpha) + \frac{\lambda}{\lambda + \rho}\, M_X(\alpha) \tag{13}$$

and

$$M_{\widehat{X}(s)}(\alpha) = \mathbb{E}_Y\big(e^{(M_X(\alpha)-1)H(t-s,Y)}\big). \tag{14}$$

Proof. From (6), the cumulant generating function of S_t is given by

$$
\begin{aligned}
\log \mathbb{E}(e^{\alpha S_t}) &= \lambda t \left(M_X(\alpha) - 1\right) + \rho \int_0^t \left(\mathbb{E}_Y(e^{(M_X(\alpha)-1)H(t-s,Y)}) - 1\right) ds \\
&= (\lambda + \rho) \int_0^t \left(M_{\widetilde{X}(s)}(\alpha) - 1\right) ds. \qquad \square
\end{aligned}
$$

Clearly, $\widetilde{X}(s)$ is a mixture of the original claim size distribution X and the time-dependent $\widehat{X}(s)$. The latter can be interpreted as a random sum $Z = \sum_{i=1}^{N(Y)} X_i$ where $N(Y)$ is Poisson with rate $H(t-s,Y)$ given Y and independent of the X_i. That is, all claims in the time interval $[s,t]$ that are due to the intensity shot at time s are collected in one single *batch* claim and these batch claims are independent of each other (this is the finite-time horizon analogue of the batch process introduced in Section 1). From either this batch interpretation or directly from (14) with $\alpha \to -\infty$, one deduces that $\widehat{X}(s)$ has an atom at zero with weight

$$
\mathbb{P}(\widehat{X}(s) = 0) = \int_0^\infty e^{-H(t-s,Y)} dF_Y(y). \tag{15}
$$

Let us investigate further the structure of the time-dependent random variable $\widehat{X}(s)$ for the multiplicative model (3).

Proposition 3.3. *Suppose $h(t,Y) = g(t)\,Y$.*

(a) *The moments of $\widehat{X}(s)$ are given by*

$$
\mu_{\widehat{X}(s),n} = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{n!}{k_1!\cdots k_n!} \mu_{Y,k} \left(\frac{\mu_{X,1}}{1!}\right)^{k_1} \cdots \left(\frac{\mu_{X,n}}{n!}\right)^{k_n} G(t-s)^k,
$$

where $k_1, \ldots, k_n$ are non-negative integers and $k = k_1 + \cdots + k_n$.

(b) *If $Y \sim Exp(\nu)$, then*

$$
M_{\widehat{X}(s)}(\alpha) = \frac{\nu}{\nu + G(t-s)} + \frac{G(t-s)}{\nu + G(t-s)} M_{Z^*(s)}(\alpha), \tag{16}
$$

with $M_{Z^*(s)}(\alpha) = \frac{\nu}{\frac{\nu + G(t-s)}{M_X(\alpha)} - G(t-s)}$.

Proof. Using

$$
M_{\widehat{X}(s)}(\alpha) = M_Y\left((M_X(\alpha) - 1)G(t-s)\right), \tag{17}
$$

assertion (a) directly follows from Faá di Bruno's formula (see e.g. [15]). For (b), $M_Y(\alpha) = \frac{\nu}{\nu - \alpha}$ leads to

$$
\begin{aligned}
M_{\widehat{X}(s)}(\alpha) &= \frac{\nu}{\nu + G(t-s)} \frac{1}{1 - \frac{G(t-s)}{\nu + G(t-s)} M_X(\alpha)} \\
&= \frac{\nu}{\nu + G(t-s)} \sum_{k=0}^\infty \left(\frac{G(t-s)}{\nu + G(t-s)}\right)^k M_X(\alpha)^k,
\end{aligned}
$$

so that the distribution function $F_{\widehat{X}(s)}$ of $\widehat{X}(s)$ can be expressed through the weighted renewal function

$$F_{\widehat{X}(s)}(\alpha) = \frac{\nu}{\nu + G(t-s)} \sum_{k=0}^{\infty} \left(\frac{G(t-s)}{\nu + G(t-s)} \right)^k F_X^{*k}(\alpha),$$

which is equivalent to (16). □

Equation (16) identifies $\widehat{X}(s)$ as a mixture of a random variable degenerate at 0 (corresponding to no claim in $(s, t]$ due to the shot at s) and a time-dependent random variable $Z^*(s)$, the weights itself depending on time. For instance, $X \sim Exp(\eta)$ leads to $Z^*(s) \sim \text{Exp}(\frac{\nu\eta}{\nu + G(t-s)})$.

The following generalization is at hand. Recall (e.g. [2]) that when X is phase-type (PH) with representation (α, U, E), then α is the initial vector, U is the phase generator and E is the set of non-absorbing states. Write Δ for the absorbing state (for convenience taken to be the same for all PH distributions in question) and let $\gamma_{j\Delta}(s) = 1 - \sum_k \gamma_{jk}(s)$ where the γ_{jk} are defined below, $u_{i\Delta}(s) = -\sum_{i'} u_{ii'}, v_{j\Delta}(s) = -\sum_{j'} v_{jj'}$.

Proposition 3.4. *Suppose that X is PH with representation (α, U, E) and Y is PH with representation (β, V, F). Then $\widehat{X}(s)$ as defined in (17) is PH with representation $\left(\alpha_{\widehat{X}(s)}, U_{\widehat{X}(s)}, E \times F \right)$ where elements i, j, resp. $ij, i'j'$ of $\alpha_{\widehat{X}(s)}, U_{\widehat{X}(s)}$ are given by*

$$\sum_k \beta_k \gamma_{kj}(s)\alpha_i, \quad resp. \quad u_{ii'}\delta_{jj'} + u_{i\Delta}\gamma_{jj'}\alpha_{i'}$$

where $\Gamma(s) = (\gamma_{jk}(s))$ is the matrix

$$\Gamma(s) = \int_0^\infty e^{Vb} G(t-s) e^{-G(t-s)b} db.$$

In particular, $\widehat{X}(s)$ has an atom at zero with weight $M_Y(G(t-s))$.

Proof. The interpretation of $\Gamma(s)$ is as the transition matrix for the Markov process underlying Y observed at the Poisson$(G(t-s))$ epochs only. We think of state ij in $E \times F$ as belonging to an X generated by a Poisson event when the Markov process for Y is in state j. The initial vector for $\widehat{X}(s)$ is $\sum_j \beta_j \gamma_{j\Delta}(s)$ (the probability of no Poisson events before Y) at Δ and $\sum_k \beta_k \gamma_{kj}(s)\alpha_i$ at ij. Finally, the transition rate from ij to $i'j'$ is

$$u_{ii'}\delta_{jj'} + u_{i\Delta}\gamma_{jj'}\alpha_{i'}$$

(the first term represents a change of state for the Markov process underlying the X at a fixed Poisson event and the second a termination at one Poisson event followed by restart at the next, falling before Y), and the absorption rate in ij is $u_{i\Delta}\gamma_{j\Delta}(s)$. □

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Remark 3.2. The above mixing structure indicates a straight-forward way to simulate the aggregate claim size for a given time horizon t. First, one simulates a homogeneous Poisson process with intensity $\lambda + \rho$. For an event at time $s < t$, a claim with distribution function F_X occurs at s with probability $\frac{\lambda}{\lambda+\rho}$. Otherwise, a claim with time-dependent distribution function $F_{\widehat{X}(s)}$ occurs (which has size zero with probability $M_Y(G(t-s))$, see (15)). This can also be interpreted as an independent "thinning" of the compound Poisson$(\lambda + \rho)$-process.

3.3 A saddlepoint approximation for S_t

We briefly mention that the knowledge of the function $\kappa_t(\alpha)$ also allows to apply a saddlepoint approximation for the tail of the aggregate claim size S_t for fixed t in the usual way (see for instance [2, p.317] for details). To that end, consider the (tilted) probability measure

$$\mathbb{P}_\theta(S_t \in dx) = \mathbb{E}(e^{\theta S_t - \kappa_t(\theta)} 1_{\{A \in dx\}}),$$

where $\theta = \theta(x)$ is chosen in such a way that $\mathbb{E}_\theta(S_t) = \kappa'_t(\theta) = x$. As $x \to \infty$, θ approaches the right abscissa of convergence $\alpha_0 = \sup\{\alpha : \kappa_t(\alpha) < \infty\}$, given that $\lim_{\theta \to \alpha_0} \kappa'_t(\theta) = \infty$. From (10), we get $\mathrm{Var}_\theta(S_t) = \kappa''_t(\theta)$ with

$$\kappa''_t(\theta) = \lambda\, t\, M''_X(\theta) + \rho \int_0^t \mathbb{E}_Y \left((H^2(s,Y)\, M'_X(\theta)^2 + H(s,Y)\, M''_X(\theta))\, e^{(M_X(\theta)-1)H(s,Y)} \right) ds$$

$$+ \frac{\mathbb{E}\left(e^{(M_X(\theta)-1)\int_0^t \nu_s\, ds} (M'_X(\theta)^2\, (\int_0^t \nu_s\, ds)^2 + M''_X(\theta)\, \int_0^t \nu_s\, ds) \right)}{\mathbb{E}(e^{(M_X(\theta)-1)\int_0^t \nu_s\, ds})}$$

$$- \frac{\left(M'_X(\theta)\, \mathbb{E}(e^{(M_X(\theta)-1)\int_0^t \nu_s\, ds}\, \int_0^t \nu_s\, ds) \right)^2}{\mathbb{E}^2(e^{(M_X(\theta)-1)\int_0^t \nu_s\, ds})}.$$

Heuristically, for $x \to \infty$ (and thus $\theta \to \alpha_0$) the number of claims under the tilted measure $\mathbb{P}_\theta$ goes to infinity (cf. Section 4.2) and a central limit result holds, namely that the limiting $\mathbb{P}_\theta$-distribution of $(S_t - x)/\sqrt{\kappa''_t(\theta)}$ is standard normal, eventually leading to

$$\mathbb{P}(S_t > x) \sim \frac{e^{-\theta x + \kappa_t(\theta)}}{\theta \sqrt{2\pi\, \kappa''_t(\theta)}} \quad \text{as } x \to \infty.$$

For appropriate conditions to make this argument rigorous, we refer to Jensen [14].

4 Ruin with light-tailed claims

4.1 Infinite time ruin estimates via large deviations

The adjustment coefficient, in the logarithmic sense, is defined by

$$\gamma = -\lim_{u \to \infty} \frac{1}{u} \log \psi(u). \tag{18}$$

In order to show its existence and to determine its value, we make use of the following consequence of the Gärtner-Ellis theorem from large deviations which is due to Glynn & Whitt [13] (see also Nyrhinen [24] and Asmussen [1]):

Theorem 4.1. *Let $\{Z_n\}_{n \in \mathbb{N}}$ be a sequence of real random variables, $S_n = Z_1 + \cdots + Z_n$, $\psi(u) = \mathbb{P}(S_n > u$ for some $n)$ and assume that there exist a finite function κ and positive constants γ, ε such that*

(i) $n^{-1} \log \mathbb{E}(e^{\theta S_n}) \to \kappa(\theta)$ for $|\theta - \gamma| < \varepsilon$,

(ii) $\kappa(\gamma) = 0, \kappa'(\gamma) > 0$,

(iii) $\mathbb{E}(e^{\gamma S_n}) < \infty$ for all n.

Then (18) holds.

Theorem 4.2. *Let the moment-generating function $M_X(\alpha) = \mathbb{E}(e^{\alpha X})$ of the claim size distribution F_X and $\mathbb{E} \exp(\alpha H(\infty, Y))$ exist for all α in a neighborhood of the origin and be steep. Then, for the risk process (1) with claim occurrence according to the shot-noise intensity process (2) satisfying (8), the Cramér-Lundberg approximation*

$$\lim_{u \to \infty} \frac{1}{u} \log \psi(u) = -\gamma$$

holds, where γ is the positive solution of $\kappa(\gamma) = 0$ and κ is given by (9).

Proof. Considering a discrete skeleton $\{S_{nh}\}_{n \in \mathbb{N}}$, Corollary 2.3 implies that $\kappa_{nh}(\alpha)/n$ has a limit of the form

$$\kappa^{(h)}(\alpha) = h \, \kappa(\alpha).$$

Since an easy calculation shows that $(\kappa^{(h)}(\alpha))'' > 0$ for every $\alpha \geq 0$, $\kappa^{(h)}(0) = 0$ and $(\kappa^{(h)})'(0) = h \, (\mu - c) < 0$ by the net profit condition, it follows that $(\kappa^{(h)})'(\gamma) > 0$. Here the required steepness implies that $\kappa(\alpha)$ is unbounded in a neighborhood of its abscissa of convergence and hence guarantees the existence of the solution $\gamma > 0$. Consequently, Theorem 4.1 applies and $\mathbb{P}(\max_n(S_{nh} - c \, n \, h) > u) \sim e^{-\gamma u}$ in the logarithmic sense. Finally, since

$$\max_t(S_t - c \, t) \geq \max_n(S_{nh} - c \, n \, h) \geq \max_t(S_t - c \, t) - c \, h$$

(see also [9]), the maximum over nh can be replaced by the continuous time maximum over t and the theorem follows from $\psi(u) = \mathbb{P}(\max_t(S_t - c \, t) > u)$. $\quad\square$

A refinement of the above result will be given in Theorem 4.5 below.

Proposition 4.3. *Let γ^* denote the adjustment coefficient of the classical compound Poisson model with constant intensity β and otherwise identical parameters. Then $\gamma < \gamma^*$.*

Proof. The defining equation for γ^* in the classical model reads

$$\beta\left(M_X(\gamma^*) - 1\right) = \gamma^* c. \tag{19}$$

From $\mathbb{E}_Y(e^{(M_X(\gamma)-1)H(\infty,Y)} - 1) > \mathbb{E}_Y\big(H(\infty, Y)\big)\left(M_X(\gamma) - 1\right)$ (given that Y is not degenerate at zero), we have from (9)

$$\lambda\big(M_X(\gamma) - 1\big) - \gamma\, c + \rho\, \mathbb{E}_Y\big(H(\infty, Y)\big)\left(M_X(\gamma) - 1\right) < 0,$$

which by (19) and the definition of β gives $\gamma < \gamma^*$. $\qquad\square$

Remark 4.1. For the stationary model with ν_t given by (4), this monotonicity result also follows directly from Theorem 3.1 by choosing $\phi(x) = e^{\alpha x}$ as the convex function in the expectation.

Let $\eta > 0$ denote the security loading defined by

$$c = (1 + \eta)\mu.$$

The following result gives properties of γ and γ^* for the limit $\eta \to 0$:

Proposition 4.4. *As $\eta \to 0$,*

$$\gamma^*(\eta) = \frac{2\mu_X}{\mu_{X,2}}\,\eta + \mathcal{O}(\eta^2), \tag{20}$$

$$\gamma(\eta) = \frac{2\mu_X}{\mu_{X,2}}\,\frac{1}{1 + \frac{\rho\,\mathbb{E}(H^2(\infty,Y))}{\lambda + \rho\,\mathbb{E}(H(\infty,Y))}}\,\eta + \mathcal{O}(\eta^2). \tag{21}$$

Proof. In view of $\kappa(\gamma) = 0$ we consider a Taylor expansion of the equation

$$\exp[H(\infty, y)(e^{\gamma x} - 1)] = 1 + \frac{\gamma\, c}{\rho} - \frac{\lambda}{\rho}\left(e^{\gamma x} - 1\right)$$

with respect to x and $H(\infty, y)$ up to second order, leading to

$$(\gamma\, x + \frac{\gamma^2 x^2}{2})H(\infty, y) + \frac{H^2(\infty, y)\gamma^2 x^2}{2} = \frac{\gamma}{\rho}(1 + \eta)\mu_X\,\beta - \frac{\lambda}{\rho}\left(\gamma\, x + \frac{\gamma^2 x^2}{2}\right) + \mathcal{O}(\gamma^3).$$

Taking expectations with respect to x and y (note that X and Y are independent), (21) follows after a little algebra. The expansion (20) follows similarly; for the latter see also [1]. $\qquad\square$

Remark 4.2. Note that Proposition 4.4 implies

$$\gamma'(\eta)|_{\eta=0} = \frac{2\mu_X}{\mu_{X,2}}\,\frac{1}{1 + \frac{\rho\,\mathbb{E}(H^2(\infty,Y))}{\lambda + \rho\,\mathbb{E}(H(\infty,Y))}} < (\gamma^*)'(\eta)|_{\eta=0}.$$

Remark 4.3. In case the Poisson shot noise process has multiplicative shots (cf. (3)), the above formulas simplify. For instance,

$$\kappa_t(\alpha) = (\lambda(M_X(\alpha) - 1) - \alpha c) t$$
$$+ \rho \int_0^t \left(M_Y\big((M_X(\alpha) - 1)G(s)\big) - 1 \right) ds + \log \mathbb{E} \left(e^{(M_X(\alpha)-1) \int_0^t \nu_s \, ds} \right)$$

and correspondingly the defining equation (9) for the Lundberg coefficient γ reduces to

$$\kappa(\alpha) = \lambda(M_X(\alpha) - 1) - \alpha c + \rho \left(M_Y\big((M_X(\alpha) - 1)\big) - 1 \right).$$

Moreover, from (21) we obtain for $\eta \to 0$

$$\gamma(\eta) = \frac{2\mu_X}{\mu_{X,2}} \frac{1}{1 + \frac{\rho \mu_{Y,2}}{\lambda + \rho \mu_Y}} \eta + \mathcal{O}(\eta^2).$$

Remark 4.4. Considering the shot noise part only (i.e. $\lambda = 0$ and (5)), it follows from (9) that the function $H_a(t, Y) = H(a\,t, Y)$ leads, for any $a > 0$, to the same adjustment coefficient γ. Thus, γ also represents the adjustment coefficient for the limit $a \to \infty$, which is the batch process mentioned in the introduction, where according to a homogeneous Poisson process with intensity ρ, groups of (independent) individual claims occur and the number of claims in each group is determined by a Poisson random variable N_b with parameter $H(\infty, Y)$. Indeed, in this case (9) is just the classical compound Poisson Lundberg equation with individual claim size $\sum_{j=1}^{N_b} X_j$.

4.2   The path to ruin

We now look at the path leading to ruin, given it occurs. For notational convenience, assume (5). From [13] it follows that, for large u, ruin occurs roughly at time $u/\kappa'(\gamma)$ and as if the cumulant generating function of $S_{u/\kappa'(\gamma)}$ is changed from $\kappa_{u/\kappa'(\gamma)}(\alpha)$ to

$$\kappa_{u/\kappa'(\gamma)}(\alpha + \gamma) - \kappa_{u/\kappa'(\gamma)}(\gamma) = (\lambda(M_X(\alpha + \gamma) - M_X(\gamma)) - \alpha c) \frac{u}{\kappa'(\gamma)}$$
$$+ \rho \int_0^{u/\kappa'(\gamma)} \left(\mathbb{E}_Y\big(e^{(M_X(\alpha+\gamma)-1)H(s,Y)}\big) - \mathbb{E}_Y\big(e^{(M_X(\gamma)-1)H(s,Y)}\big) \right) ds.$$

For the multiplicative model (3) this can be rewritten as

$$\kappa_{u/\kappa'(\gamma)}(\alpha + \gamma) - \kappa_{u/\kappa'(\gamma)}(\gamma)$$
$$= \left(\lambda^*(M_{X^*}(\alpha) - 1) - \alpha c \right) \frac{u}{\kappa'(\gamma)} + \int_0^{u/\kappa'(\gamma)} \rho^*(s) \left(M_{Y^*(s)}\big((M_{X^*}(\alpha) - 1)\,G(s)\big) - 1 \right) ds,$$

where

$$\lambda^* = \lambda M_X(\gamma), \quad M_{X^*}(\alpha) = \frac{M_X(\alpha + \gamma)}{M_X(\gamma)}, \quad \rho^*(s) = \rho M_Y(v(\gamma, s)) \quad \text{and}$$

$$M_{Y^*(s)}(\alpha) = \frac{M_Y(M_X(\gamma)\alpha + v(\gamma, s))}{M_Y(v(\gamma, s))}, \quad \text{where } v(\gamma, s) = (M_X(\gamma) - 1)\,G(s).$$

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In other words, the sample path leading to ruin is again a compound Cox process of Poisson shot noise type with the parameters changed in the following way: the intensity process λ_t is multiplied by the constant $M_X(\gamma)$, the jump distribution Y is exponentially tilted by the (time-dependent) factor $v(\gamma, s)$, the claim size distribution is exponentially tilted by the factor γ, and the Poisson process underlying the shot noise is now inhomogeneous with $\rho^*(s)$ increasing from $\rho^*(0) = \rho$ to $\rho^*(u/\kappa'(\gamma)) = \rho M_Y((M_X(\gamma) - 1) G(u/\kappa'(\gamma)))$. This amounts to a change of drift of the surplus process from

$$c - \frac{\partial}{\partial t} \mathbb{E}(S_t) = c - \mu_X \Big(\lambda + \rho \, \mathbb{E}_Y(H(t, Y)) \Big)$$

to

$$c - \frac{\partial}{\partial t} \mathbb{E}(S_t^*) = c - M_X'(\gamma) \Big(\lambda + \rho \, \mathbb{E}_Y \big(H(t, Y) \, e^{(M_X(\gamma) - 1) H(t, Y)} \big) \Big).$$

Hence, as a function of t, the drift of the path to ruin increases from $c - \lambda M_X'(\gamma)$ (for $t = 0$) to the finite limit $c - M_X'(\gamma) \Big(\lambda + \rho \, \mathbb{E}_Y \big(H(u/\kappa'(\gamma), Y) \, e^{(M_X(\gamma) - 1) H(u/\kappa'(\gamma), Y)} \big) \Big) \to c - M_X'(\gamma) \Big(\lambda + \rho \, \mathbb{E}_Y \big(H(\infty, Y) \, e^{(M_X(\gamma) - 1) H(\infty, Y)} \big) \Big)$.

Remark 4.5. For the special case $g(s) = e^{-\delta s}$, the above change of measure was derived in [7] by PDMP techniques in the framework of the Esscher equivalent measure to price insurance-linked contracts on the market. Note that the large deviations approach above is a particularly transparent alternative to derive this result.

4.3 Finite-time ruin probabilities

Denote with $\psi(u, T) = \mathbb{P}(\tau(u) < T)$ the probability of ruin up to time T. Let $a \in \mathbb{R}^+$ and α_a be defined as the unique solution of $\kappa'(\alpha_a) = \frac{1}{a}$, where the convex function $\kappa(\alpha)$ is given by (9). Using the same skeleton argument as in Theorem 4.1, one can apply the large deviation estimate for the finite-time ruin probability of Nyrhinen [23] to our risk process, leading to

$$\lim_{u \to \infty} \frac{1}{u} \log \psi(u, a \, u) = -\gamma_a,$$

with

$$\gamma_a = \begin{cases} \alpha_a - a \, \kappa(\alpha_a), & a < \frac{1}{\kappa'(\gamma)}, \\ \gamma, & a \geq \frac{1}{\kappa'(\gamma)}. \end{cases}$$

4.4 Beyond large deviations

Let us now return to the compound Poisson batch process $\widetilde{R}_t$ introduced in Section 1, which is obtained by moving all arrivals of claims in the batch of claims caused by a catastrophic event at T_n to T_n. This risk process has intensity $\widetilde{\lambda} = \lambda + \rho$ for arrivals of claims and a claim size distribution $F_{\widetilde{X}}$ which is a mixture of F_X and the distribution of a random sum $Z = \sum_{i=1}^{N(Y)} X_i$ where $N(Y)$ is Poisson with rate

$H(\infty, Y)$ given Y and independent of the X_i; the weights are $\lambda/\widetilde{\lambda}$, resp. $\rho/\widetilde{\lambda}$, and the premium rate is c (in the notation of (13), we look at the case $t = \infty$, which makes $\widetilde{X}(s) = \widetilde{X}$ time-independent).

Let us introduce the following random variables associated with a catastrophic event: $N(Y)$, the number of claims triggered by the event; $Z = \sum_1^{N(Y)} X_i'$, the total claim amount caused by the event; and L, the time from the event until the last of the $N(Y)$ claims occurs. Here, again, $N(Y)$ is Poisson with rate $H(\infty, Y)$ given Y and independent of the X_i'. Obviously, $\psi(u) \leq \widetilde{\psi}(u)$.

The following is a refinement of Theorem 4.1:

Theorem 4.5. *For some constant $C_- \leq 1$, $C_- e^{-\gamma u} \leq \psi(u) \leq e^{-\gamma u}$ for all u.*

Proof. The upper inequality is clear from Lundberg's inequality for $\widetilde{\psi}(u)$. For the lower, note that it is well known that $\widetilde{R}_{\widetilde{\tau}(u)-}$ has a limit distribution given $\widetilde{\tau}(u) < \infty$ as $u \to \infty$ (see for instance Theorem 2 of Schmidli [26]). Hence there exists A such that

$$\mathbb{P}\big(\widetilde{\tau}(u) < \infty,\ \widetilde{R}_{\widetilde{\tau}(u)-} \leq A\big)\ \geq\ (1-\epsilon)\widetilde{\psi}(u) \tag{22}$$

for all large u.

Define the pre-$\widetilde{\tau}(u)$ occupation measure $Q^{(u)}$ by

$$Q^{(u)}(F)\ =\ \mathbb{E}\int_0^{\widetilde{\tau}(u)} I\big((\widetilde{S}_t - c\,t) \in F\big)\,\mathrm{d}t\,, \quad F \subseteq (-\infty, u)\,.$$

Then the l.h.s. of (22) is

$$\int_{u-A}^u \widetilde{\lambda}\,\overline{F}_{\widetilde{X}}(u-x)\,Q^{(u)}(\mathrm{d}x)$$

which is bounded above by $\widetilde{\lambda}Q^{(u)}(u-A, u)$. Clearly, we can choose ℓ_1 with $d = \mathbb{P}(Z > A,\ L \leq \ell_1) > 0$. Every ruin event for $\widetilde{R}_t$ will also cause ruin for R_t, if the initial surplus u is lowered by $c\,\ell_1$, given that the variable L corresponding to the batch claim causing ruin does not exceed ℓ_1. Moreover, considering the situation only where the surplus prior to ruin is bounded above by A, we obtain a lower bound for the ruin probability of R_t, getting

$$\psi(u - c\ell_1)\ \geq\ \int_{u-A}^u \widetilde{\lambda}\mathbb{P}(Z > u - x,\ L \leq \ell_1)\,Q^{(u)}(\mathrm{d}x) \geq\ \widetilde{\lambda}\,Q^{(u)}(u-A, u)\,d$$

$$\geq\ d(1-\epsilon)\widetilde{\psi}(u)\,.$$

Appealing to the Cramér-Lundberg asymptotics for $\widetilde{\psi}(u)$, the proof is complete. $\square$

5 Ruin with heavy-tailed claims

Define the integrated tail distribution F_X^I of F_X by

$$F_X^I(x)\ =\ \frac{1}{\mu_X}\int_0^x \overline{F}_X(z)\,\mathrm{d}z.$$

A distribution function F is said to be in the class $\mathcal{S}^*$, if

$$\lim_{x \to \infty} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)} \, \overline{F}(y) \, dy = 2 \int_0^\infty \overline{F}(x) \, dx$$

and $\overline{F}(x) > 0$ for all x (cf. Klüppelberg [16]). In particular, $F \in \mathcal{S}^*$ implies that both $F \in \mathcal{S}$ and $F_I \in \mathcal{S}$ (where $\mathcal{S}$ is the class of subexponential distributions). Lognormal, Weibull and regularly varying distributions with finite mean all belong to $\mathcal{S}^*$.

In addition to $F_X \in \mathcal{S}^*$, we will also throughout need the condition

$$\mathbb{E} \exp \left\{ \alpha \int_0^\infty \nu_s \, ds \right\} \; < \; \infty \quad \text{for some } \alpha > 0 \,. \tag{23}$$

Proposition 5.1. *For the stationary model, (23) is equivalent to*

$$\int_0^\infty \left(\mathbb{E} e^{\alpha \overline{H}(s,Y)} - 1 \right) ds < \infty. \tag{24}$$

In particular, for the multiplicative model a sufficient condition for (23) to hold is $\mathbb{E} e^{\alpha Y} < \infty$ and $\overline{G}(t) = \mathcal{O}(t^{-\beta})$ for some $\beta > 1$.

Proof. By a similar argument as in the proof of Proposition 2.4, one gets

$$\mathbb{E} \exp \left\{ \alpha \int_0^\infty \nu_s \, ds \right\} \; = \; \exp \left\{ \rho \int_0^\infty \left(\mathbb{E} e^{\alpha \overline{H}(s,Y)} - 1 \right) ds \right\},$$

and consequently (24). For $\overline{H}(s,Y) = Y \, \overline{G}(s)$, assume w.l.o.g. $\overline{G}(t) \le t^{-\beta}$, $t \ge 1$. Clearly, the integral from 0 to 1 can be made finite, whereas substituting $z = t^{-\beta}$, the integral from 1 to ∞ is, for $\beta > 1$,

$$\mathbb{E} \int_0^1 \frac{1}{\beta \, z^{1+1/\beta}} \left(e^{\alpha Y z} - 1 \right) dz \; = \; \mathbb{E} \left(1 - e^{\alpha Y} + \alpha Y \int_0^1 e^{\alpha Y z} z^{-1/\beta} \, dz \right)$$

$$\le \; 1 + \frac{\alpha \, \beta}{\beta - 1} \, \mathbb{E}(Y \, e^{\alpha Y}) < \infty. \qquad \square$$

5.1 Infinite time ruin probabilities

Theorem 5.2. *Assume $F_X \in \mathcal{S}^*$, (23) and $\mathbb{E} e^{\alpha H(\infty,Y)} < \infty$ for some $\alpha > 0$. Then*

$$\psi(u) \; \sim \; \frac{\mu}{c - \mu} \, \overline{F}_X^I(u) \,. \tag{25}$$

In the proof, we shall employ coupling with the batch process $\widetilde{R}_t$ discussed in Section 4.4 (together with the notation used there). When $\nu_t \equiv 0$, then clearly $\widetilde{S}_t \ge S_t$ in the sense of sample paths, and so it is trivial that $\psi(u) \le \widetilde{\psi}(u)$. We will see in Lemma 5.3 that $\widetilde{\psi}(u)$ has the claimed asymptotics, establishing the asymptotic upper bound in Theorem 5.2.

Lemma 5.3. *Under the assumptions of Theorem 5.2,* $\widetilde{\psi}(u) \sim \dfrac{\mu}{c-\mu}\overline{F}^I_X(u)$.

Proof. Conditioning upon Y, we get

$$\mathbb{E}z^{N(Y)} = \mathbb{E}\exp\left\{H(\infty, Y)(z-1)\right\}$$

which, under the assumptions of the above theorem, is finite for some $z > 1$ (implying that $\mathbb{P}(N(Y) = n)$ decreases geometrically fast in n). It is well known ([2, p. 259] or [11, p. 45]) that this implies

$$\mathbb{P}\big(Z > x\big) \sim \mathbb{E}H(\infty, Y)\overline{F}_X(x) \tag{26}$$

and hence

$$\overline{F}_{\widetilde{X}}(x) \sim \frac{\lambda}{\widetilde{\lambda}}\overline{F}_X(x) + \frac{\rho\,\mathbb{E}H(\infty, Y)}{\widetilde{\lambda}}\overline{F}_X(x) = \frac{\beta}{\widetilde{\lambda}}\overline{F}_X(x),$$

$$\overline{F}^I_{\widetilde{X}}(x) \sim \frac{\beta}{\widetilde{\lambda}\mu_{\widetilde{X}}}\int_x^\infty \overline{F}_X(z)\,\mathrm{d}z = \frac{\beta\mu_X}{\widetilde{\lambda}\mu_{\widetilde{X}}}\overline{F}^I_X(x) = \frac{\mu}{\widetilde{\lambda}\mu_{\widetilde{X}}}\overline{F}^I_X(x) = \overline{F}^I_X(x)\,;$$

for the last equality, note that

$$\mu_{\widetilde{X}} = \frac{\lambda}{\widetilde{\lambda}}\mu_X + \frac{\rho\,\mathbb{E}H(\infty, Y)}{\widetilde{\lambda}}\mu_X = \frac{\mu}{\widetilde{\lambda}}.$$

Finally, letting $\widetilde{r} = \widetilde{\lambda}\mu_{\widetilde{X}}/c = \mu/c$, we have by standard theory [2, p. 259] or [11, p. 43]) that

$$\widetilde{\psi}(u) \sim \frac{\widetilde{r}}{1-\widetilde{r}}\overline{F}^I_{\widetilde{X}}(x) = \frac{\mu}{c-\mu}\overline{F}^I_X(u). \qquad \square$$

Proof of Theorem 5.2. We first assume $\nu_t \equiv 0$. Consider the aggregate claim process $\breve{S}_t$ obtained from S_t by moving all claims triggered by a catastrophic event and occurring at most ℓ_0 time units later to occur precisely ℓ_0 time units after the catastrophic event, whereas claims occurring more than ℓ_0 time units later are deleted. Then $\psi(u) \geq \breve{\psi}(u)$ for all u. Standard results on translation of Poisson processes imply that the restriction of $\breve{S}_t - ct$ to $t \in [\ell_0, \infty)$ is an ordinary Cramér-Lundberg risk process, and reasoning as in the proof of Lemma 5.3, we obtain

$$\mathbb{P}\Big(\sup_{t\in[\ell_0,\infty)}(\breve{S}_t - \breve{S}_{\ell_0} - c(t-\ell_0)) > u\Big) \sim \frac{\mu(\ell_0)}{c-\mu(\ell_0)}\overline{F}^I_X(u) \tag{27}$$

where $\mu(\ell_0) = \mu_X\big(\lambda + \rho\,\mathbb{E}H(\ell_0, Y)\big)$. Now

$$\sup_{t\in[0,\infty)}(\breve{S}_t - ct) \geq (\breve{S}_{\ell_0} - c\ell_0) + \sup_{t\in[\ell_0,\infty)}\big(\breve{S}_t - \breve{S}_{\ell_0} - c(t-\ell_0)\big). \tag{28}$$

Here the two terms are independent. Since $\breve{S}_{\ell_0}$ is the sum of a Poisson($\lambda\ell_0$) number of X_i's, $\mathbb{P}(\breve{S}_{\ell_0} - c\ell_0 > u) \sim \lambda\ell_0\,\overline{F}_X(u)$, which is dominated by (27). Hence the tail of $\sup_{t\in[0,\infty)}(\breve{S}_t - ct)$ is asymptotically given by (27), and we get

$$\liminf_{u\to\infty}\frac{\psi(u)}{\overline{F}^I_X(u)} \geq \liminf_{u\to\infty}\frac{\breve{\psi}(u)}{\overline{F}^I_X(u)} = \frac{\mu(\ell_0)}{c-\mu(\ell_0)}.$$

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Letting $\ell_0 \to \infty$ and using $\mu(\ell_0) \uparrow \mu$, we obtain

$$\liminf_{u \to \infty} \frac{\psi(u)}{\overline{F}^I_X(u)} \geq \frac{\mu}{c - \mu}.$$

Combining this with the bound $\psi(u) \leq \widetilde{\psi}(u)$ and Lemma 5.3 completes the proof when $\nu_t \equiv 0$.

In the general case, the obtained asymptotics for $\nu_t \equiv 0$ is clearly an asymptotic lower bound. To get an upper, bound the tail of $\sup_{t \in [0,\infty)}(S_t - ct)$ by the tail of the independent sum of $\sum_1^{M(\nu)} X_i$ and the supremum when $\nu_t \equiv 0$. Here $M(\nu)$ denotes the Cox process resulting from ν_t and condition (23) implies that $M(\nu)$ has exponential moments. Consequently (see again [2, p. 259]), the tail of $\sum_1^{M(\nu)} X_i$ is asymptotically proportional to $\overline{F}_X(u) = o(\overline{F}^I_X(u))$, so that the asymptotic behavior of the tail of $\sup_{t \in [0,\infty)}(S_t - ct)$ is given by the one for the case $\nu_t \equiv 0$. $\qquad\square$

Remark 5.1. From Proposition 5.1, we see that Theorem 5.2 is in particular valid for the stationary process with (4), given (24). For that case, (25) can in fact also be deduced directly from Theorem 3.1 of Asmussen et al. [4] through the formulation of λ_t as a stationary ergodic Poisson cluster process, see Theorem 12.6.3 of Rolski *et al.* [25]. Alternatively, in this case the lower bound in Theorem 5.2 also follows from $\psi(u) \geq \psi^*(u)$ (cf. [25, p.513]), where the latter refers to the averaged Cramér-Lundberg process, which has the asymptotics of (25). However, our approach above, which can be extended to general Poisson cluster processes, allows to establish the result beyond the stationary setting (which was a crucial assumption in [25]) and, at least in some cases (like the setting of Proposition 5.1), under weaker conditions. Moreover, our approach leads to ruin estimates for the finite time horizon also (see Section 5.2).

5.2 Finite-time ruin probabilities

Let $e(u) = \mathbb{E}(X - u|X > u)$ denote the mean excess function of the claim size distribution. Note that for $F_X \in \mathcal{S}$, $e(u) \to \infty$ for $u \to \infty$. Recall the definition of the generalized extreme value distribution

$$H_\xi(x) = \begin{cases} \exp(-(1 + \xi x)^{-1/\xi}) & \text{if } \xi \neq 0 \\ \exp(-\exp(-x)) & \text{if } \xi = 0, \end{cases}$$

where $1 + \xi x > 0$, and the usual notation $F_X \in \mathrm{MDA}(H_\xi)$ for F_X being in the maximum domain of attraction of H_ξ. For our purposes, the case $F_X \in \mathrm{MDA}(H_\xi)$ for some $\xi < 0$ is not of interest, since this property implies a finite right end-point of the distribution function (see for instance [11]).

When considering the asymptotics of $\psi(u, t)$ as $u \to \infty$, we will consider the cases $t = e(u)T$ with T fixed, $t/e(u) \to 0$ but $t \to \infty$ and finally t fixed (i.e., independent of u) separately. In the first case:

Theorem 5.4. *Suppose that*

$$\lim_{u \to \infty} \frac{e(u+a)}{e(u)} = 1 \quad for \; any \quad a > 0. \tag{29}$$

If $F_X \in \mathcal{S}^* \cap \mathrm{MDA}(H_\xi)$ for $\xi \geq 0$, (23) and $\mathbb{E}e^{\alpha H(\infty, Y)} < \infty$ for some $\alpha > 0$, then

$$\psi(u, e(u)\,T) \sim \psi(u)\,\bigl(1 + \log H_\xi\bigl((c-\mu)T\bigr)\bigr). \tag{30}$$

In particular, for regularly varying $\overline{F}_X^I$ with index $-1/\xi$, $\xi > 0$,

$$\psi(u, u\,T) \sim \psi(u)\Bigl(1 - \bigl(1 + (c-\mu)T\bigr)^{-1/\xi}\Bigr),$$

and for $F_X \in \mathcal{S}^* \cap \mathrm{MDA}(H_0)$,

$$\psi(u, e(u)\,T) \sim \psi(u)\Bigl(1 - e^{-(c-\mu)T}\Bigr).$$

Proof. Almost identical to the proof of Theorem 5.2. It follows from [3] that the asymptotics of $\widetilde{\psi}(u, e(u)T)$ is as in (30), so the upper bound is clear when $\nu_t \equiv 0$. For the lower bound, note first that $\psi(u,t) \geq \breve{\psi}(u,t)$ and proceed similarly as in the proof of Theorem 5.2 to see that the asymptotics of $\breve{\psi}(u, e(u)T)$ is indeed as in (30). In particular, the tail of $\breve{S}_{\ell_0} - c\ell_0$ is again dominated by the tail of $\sup_{t \in [\ell_0, e(u)T)}\bigl(\breve{S}_t - \breve{S}_{\ell_0} - c(t - \ell_0)\bigr)$. Finally, the case of a general ν_t is dealt with exactly as when $t = \infty$, replacing the sup over $[0, \infty)$ by the sup over $[0, e(u)T)$. The assertion for regularly varying tails follows from $e(u) \sim \xi u$. □

Remark 5.2. Condition (29) on the mean excess function is fulfilled for all heavy-tail distributions of practical interest (cf. [11, p. 296]).

In case of a growth rate $b(u)T$ of the time horizon with $b(u) = o(e(u))$, Theorem 5.4 implies $\psi(u, b(u)T) = o(\psi(u))$. The following theorem gives some more explicit information for this case.

Theorem 5.5. *If $F_X \in \mathcal{S}^* \cap \mathrm{MDA}(H_\xi)$ with $\xi \geq 0$, (23) and $\mathbb{E}e^{\alpha H(\infty, Y)} < \infty$ for some $\alpha > 0$, then*

$$\psi(u, b(u)T) \sim \beta\,\overline{F}_X(u)\,b(u)\,T,$$

when $b(u) \uparrow \infty$ with $b(u) = o(e(u))$.

Lemma 5.6. *If $F_X \in \mathcal{S}^* \cap \mathrm{MDA}(H_\xi)$ with $\xi \geq 0$, the following relation for the finite-time ruin probability of a Cramér-Lundberg process holds:*

$$\psi_{\mathrm{CL}}(u, b(u)T) \sim \lambda_{\mathrm{CL}}\,\overline{F}_X(u)\,b(u)\,T,$$

where λ_{CL} denotes the intensity of the underlying homogeneous Poisson process.

Proof. Consider the random walk $S_n = \sum_{i=1}^n \xi_i$ with $\mathbb{E}(\xi_i) = 0$ and $F_{\xi_i} \in \mathcal{S}^*$. Let $M_\sigma^d := \max_{n \leq \sigma}(S_n - d\,n)$ for some stopping time σ of the random walk and some constant $d \geq 0$. According to a recent result of Foss *et al.* [12], one then has

$$\mathbb{P}(M_\sigma^d > u) \sim \sum_{n \geq 1} \mathbb{P}(\sigma \geq n)\overline{F}_{\xi_i}(u + d\,n)$$

as $u \to \infty$, uniformly over all stopping times σ. Let h be fixed and choose $\xi_i = \sum_{j=1}^{N(h)} X_j - \lambda_{CL}\mu_X h$ (implying $F_{\xi_i} \in \mathcal{S}^*$) and furthermore $\sigma = b(u)\,T$. Denote the ruin probability of the discrete-time process $R_{CL,n}^{(h)}$ $(n \in \mathbb{N})$ (i.e. the Cramér-Lundberg process viewed at time points $n\,h$ only) by $\psi_{CL}^{(h)}$. The above result then translates into

$$
\begin{aligned}
\psi_{CL}^{(h)}(u, b(u)\,T) &\sim \sum_{n \geq 1} \mathbb{P}(b(u)\,T \geq nh)\overline{F}_{\xi_i}(u + (c - \lambda_{CL}\mu_X)nh) \\
&= \sum_{1 \leq n \leq b(u)\,T/h} \overline{F}_{\xi_i}(u + (c - \lambda_{CL}\mu_X)nh).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\psi_{CL}^{(h)}(u, b(u)\,T) &\ll \frac{b(u)\,T}{h}\overline{F}_{\xi_i}(u + (c - \lambda_{CL}\mu_X)h) = \frac{b(u)\,T}{h}\mathbb{P}\left(\sum_{j=1}^{N(h)} X_j > u + c\,h\right) \\
&\sim \lambda_{CL}\,b(u)\,T\,\overline{F}_X(u + c\,h) \sim \lambda_{CL}\,b(u)\,T\,\overline{F}_X(u)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\psi_{CL}^{(h)}(u, b(u)\,T) &\gg \frac{b(u)\,T}{h}\overline{F}_{\xi_i}(u + (c - \lambda_{CL}\mu_X)\,h\,b(u)T) \\
&= \frac{b(u)\,T}{h}\mathbb{P}\left(\sum_{j=1}^{N(h)} X_j > u + (c - \lambda_{CL}\mu_X)h\,b(u)\,T + \lambda_{CL}\mu_X\,h\right) \\
&\sim \lambda_{CL}\,b(u)\,T\,\overline{F}_X(u + (c - \lambda_{CL}\mu_X)\,h\,b(u)\,T + \lambda_{CL}\mu_X\,h) \\
&\sim \lambda_{CL}\,b(u)\,T\,\overline{F}_X(u),
\end{aligned}
$$

where the last asymptotic relation uses

$$
\overline{F}_X(u + b(u)) \sim \overline{F}_X(u),
$$

and since $b(u) = o(e(u))$, the latter is fulfilled for any $F_X \in \mathrm{MDA}(H_\xi)$ (see for instance equation (3.42) of [11]). Thus $\psi_{CL}^{(h)}(u, b(u)\,T) \sim \lambda_{CL}\,b(u)\,T\,\overline{F}_X(u)$. From

$$
\max_{t \leq \lfloor b(u)T/h \rfloor\,h}(S_t - c\,t) \geq \max_{n \leq b(u)T/h}(S_{nh} - c\,n\,h) \geq \max_{t \leq \lfloor b(u)T/h \rfloor\,h}(S_t - c\,t) - c\,h
$$

one finally observes that for $h \to 0$, $\psi_{CL}^{(h)}(u, b(u)\,T)$ can be replaced by $\psi_{CL}(u, b(u)T)$. $\qquad\square$

Lemma 5.7. *Let the random variables* $M, \big(N_1(\lambda)\big)_{\lambda \geq 0}, \big(N_2(\lambda)\big)_{\lambda \geq 0}, X_1, X_2, \ldots$ *all be independent, such that* $N_i(\lambda)$ *is Poisson*(λ), $X_1, X_2, \ldots$ *have distribution* F_X *and* $M \geq 0$ *with* $\mathbb{E}e^{\alpha M} < \infty$ *for some* $\alpha > 0$. *Then*

$$
p_M(u) = \mathbb{P}\bigg(\sum_{i=1}^{N_1(M)+N_2(\lambda)} X_i > u\bigg) \sim \lambda\overline{F}_X(u)
$$

when $u, \lambda = \lambda(u) \to \infty$ *with* $\lambda/e(u) \to 0$.

Proof. The case $M = 0$ a.s. is implicitly contained in Lemma 5.6. Alternatively, for most heavy-tail distributions of interest the statement also follows from Proposition 7.1 of Mikosch & Nagaev [22].

Now consider the case of a general M. What has been noted for $M = 0$, implies that $\lambda \overline{F}_X(u)$ is an asymptotic lower bound for $p_M(u)$. To get an upper, we note that $p_M(u) \leq p'_M(u) + p''_M(u) + p'''_M(u)$ where the three terms are defined as $p_M(u)$ but with the added restrictions $N_1(M) + N_2(\lambda) \leq \lambda(1 + 2\epsilon)$, resp. $N_2(\lambda) > \lambda(1 + \epsilon)$, resp. $N_1(M) > \lambda\epsilon$. Let $z > 1$ (to be specified later) and choose D such that $\overline{F}_X^{*n}(x) \leq Dz^n \overline{F}_X(x)$ for all n and x (which is always possible for $F_X \in \mathcal{S}$). Then

$$
\begin{aligned}
p''_M(u) &\leq D\overline{F}_X(u)\,\mathbb{E}\big[z^{N_1(M)+N_2(\lambda)}\,1_{\{N_2(\lambda)>\lambda(1+\epsilon)\}}\big] \\
&\leq D\overline{F}_X(u)\,\mathbb{E}\big(z^{N_1(M)}\big)\,\mathbb{E}^{1/2}\big(z^{2N_2(\lambda)}\big)\,\mathbb{P}^{1/2}\big(N_2(\lambda)>\lambda(1+\epsilon)\big)
\end{aligned}
$$

(by independence and Cauchy-Schwarz). Inserting the Chernoff bound for $N_2(\lambda)$ yields

$$
\begin{aligned}
p''_M(u) &\leq D\overline{F}_X(u)\,\mathbb{E}\big(z^{N_1(M)}\big)\,\mathbb{E}^{1/2}\big(z^{2N_2(\lambda)}\big)e^{-\lambda\kappa^*(\epsilon)/2} \\
&= D\overline{F}_X(u)\,\mathbb{E}\big(z^{N_1(M)}\big)\,e^{\lambda(z^2-1-\kappa^*(\epsilon))/2}
\end{aligned}
$$

where $\kappa^*(\epsilon) > 0$ (with limit 0 as $\epsilon \downarrow 0$)). Similarly, since $N_1(M)$ has exponential moments by the assumption on M, we have $\mathbb{P}\big(N_1(M) > x\big) < \mathbb{E}(e^{\beta N_1(M)})\,e^{-\beta x}$ for some $\beta > 0$ and get

$$
\begin{aligned}
p'''_M(u) &\leq D\overline{F}_X(u)\,\mathbb{E}\big[z^{N_1(M)+N_2(\lambda)}\,1_{\{N_1(M)>\lambda\epsilon\}}\big] \\
&\leq D\overline{F}_X(u)\,\mathbb{E}\big(z^{N_2(\lambda)}\big)\,\mathbb{E}^{1/2}z^{2N_1(M)}\,\mathbb{P}^{1/2}\big(N_1(M)>\lambda\epsilon\big) \\
&\leq D\overline{F}_X(u)\,e^{\lambda(z-1)}\,\mathbb{E}^{1/2}z^{2N_1(M)}\,\mathbb{E}^{1/2}\big(e^{\beta N_1(M)}\big)e^{-\beta\lambda\epsilon/2}.
\end{aligned}
$$

Choosing z such that

$$
\mathbb{E}z^{2N_1(M)} < \infty, \quad z^2 < 1 + \kappa^*(\epsilon), \quad z < 1 + \beta\epsilon/2,
$$

both $p''_M(u)$ and $p'''_M(u)$ become $o\big(\overline{F}_X(u)\big)$ and hence

$$
\limsup_{u\to\infty}\frac{p_M(u)}{\lambda\overline{F}_X(u)} \;\leq\; \limsup_{u\to\infty}\frac{p'_M(u)}{\lambda\overline{F}_X(u)} \;\leq\; 1 + 2\epsilon\,.
$$

$\square$

Proof of Theorem 5.5. For $\nu_t \equiv 0$, we again have

$$
\breve{\psi}\big(u, b(u)T\big) \leq \psi\big(u, b(u)T\big) \leq \widetilde{\psi}\big(u, b(u)T\big)
$$

for any $\ell_0 > 0$ in the process $\breve{R}_t$. From Lemma 5.6 one deduces $\widetilde{\psi}\big(u, b(u)T\big) \sim \widetilde{\lambda}\,b(u)T\,\overline{F}_{\widetilde{X}}(u) = \beta\,b(u)T\,\overline{F}_X(u)$. Moreover, from (28) it follows that $\breve{\psi}\big(u, b(u)T\big) \sim \widetilde{\psi}\big(u, b(u)T\big)$ for $\ell_0 \uparrow \infty$, since $b(u) \uparrow \infty$ and thus the term $\breve{S}_{\ell_0} - c\ell_0$ in (28) is asymptotically negligible.

For arbitrary ν_t with (23), observe again that the above case $\nu_t \equiv 0$ is an asymptotic lower bound. Furthermore, from

$$\psi\big(u, b(u)T\big) \leq \mathbb{P}(S_{b(u)T} > u) \leq \mathbb{P}\left(\widetilde{S}_{b(u)T} + \sum_{i=1}^{N_1(M)} X_i > u\right),$$

we get the upper bound as a consequence of Lemma 5.7 with $M = \int_0^{b(u)T} \nu_s \, ds$. $\quad\square$

Theorem 5.8. *Assume $F_X \in \mathcal{S}^*$, (23) and $\mathbb{E}e^{\alpha H(\infty, Y)} < \infty$ for some $\alpha > 0$. Then, for a fixed time horizon T,*

$$\psi\big(u, T\big) \sim \left(\lambda T + \rho \int_0^T \mathbb{E}_Y(H(s, Y)) \, ds + \int_0^T \mathbb{E}(\nu_s) \, ds\right) \overline{F}_X(u).$$

Proof. We have
$$\mathbb{P}(S_T - c\,T > u) \leq \psi\big(u, T\big) \leq \mathbb{P}(S_T > u),$$

where $S_T = \sum_{i=1}^{N_T} X_i$ and N_T is the shot-noise process generated by (2). Since under the assumptions of the theorem, $\mathbb{E}e^{\alpha \Lambda_T} < \infty$ for some $\alpha > 0$, it follows that

$$\mathbb{P}(S_T > u) \sim \mathbb{E}(\Lambda_T) \, \overline{F}_X(u),$$

establishing the result. $\quad\square$

6 An adaptive premium rule

Finally, we indicate an extension to a model with reserve-dependent premium rule, introduced in [1] for the classical compound Poisson model. Here, instead of a constant premium rate $c = (1 + \eta)\,\beta\,\mu_X$, the premium rate $c(t)$ at time t is adapted to the claim experience in the portfolio through $c(t) = (1 + \eta)S_{t-}/t$. That is to say, one fixes the security loading η and uses the natural estimator of $\beta\,\mu_X$ based upon the information $\mathcal{F}_{t-}$, where $\mathcal{F}_t = \sigma(S_s : 0 \leq s \leq t)$. Suppose (5). From

$$S_t - \int_0^t c(s) \, ds = \sum_{j=1}^{N_t} X_j - (1 + \eta) \int_0^t \frac{\sum_{j=1}^{N_t} X_j}{s} \, ds = \sum_{j=1}^{N_t} X_j \left(1 - (1 + \eta) \log \frac{t}{T_i}\right),$$

one can reinterpret the risk process as a compound Cox process with time-dependent claims. From (10) and Proposition 2.1 we then arrive at

$$\kappa_t(\alpha) = \log \mathbb{E}(e^{S_t - \int_0^t c(s)\,ds}) = \lambda \int_0^t M_X\left(\alpha\left(1 - (1 + \eta) \log \frac{t}{s}\right)\right) ds - (\lambda + \rho)\,t$$
$$+ \rho \int_0^t \mathbb{E}_Y\left(e^{\int_s^t h(w-s,Y)(M_X(\alpha(1-(1+\eta)\log\frac{t}{w}))-1)\,dw}\right) ds.$$

Consequently,

$$\frac{\kappa_t(\alpha)}{t} = \lambda \int_0^1 M_X\left(\alpha\left(1 + (1+\eta)\log u\right)\right) ds - (\lambda + \rho) + I(t),$$

$$\text{where} \quad I(t) = \frac{\rho}{t} \int_0^t \mathbb{E}_Y\left(e^{\int_s^t h(w-s,Y)(M_X(\alpha(1-(1+\eta)\log\frac{t}{w}))-1)\,dw}\right) ds.$$

It was shown in [1] that the first summand is equivalent to $\lambda\,\mathbb{E}(e^{\alpha X}/(1+(1+\eta)\alpha\,X))$. It then remains to show that the limit $\kappa(\alpha) = \lim_{t\to\infty} \kappa_t(\alpha)/t$ exists and fulfills the assumptions of Theorem 4.1, in which case one again obtains exponential estimates for finite- and infinite-time ruin probabilities along the lines of Section 4. We illustrate this procedure for a specific example:

Example: For the multiplicative model $h(t,Y) = \delta\,e^{-\delta t}\,Y$ $(\delta > 0)$, $X \sim Exp(z_1)$ and $Y \sim Exp(z_2)$, one obtains

$$I(t) = \frac{\rho}{t}\int_0^t \frac{z_1\,ds}{z_1 + 1 - e^{-\delta(t-s)} - \int_s^t \frac{z_2\,\delta\,e^{-\delta(w-s)}\,dw}{z_2 - \alpha\left(1-(1+\eta)\log\frac{t}{w}\right)}} = \rho\,z_1\int_0^1 \frac{du}{k(u,t)}$$

with

$$k(u,t) = z_1 + 1 - e^{-\delta u\,t} - \int_0^{ut} \frac{z_2\,\delta\,e^{-\delta b}\,db}{z_2 - \alpha(1 + (1+\eta)\log(1-u+b/t))}.$$

Define furthermore

$$I = \rho\,z_1 \int_0^1 \frac{du}{k(u)} \quad \text{with} \quad k(u) = z_1 + 1 - \frac{z_2}{z_2 - \alpha(1 + (1+\eta)\log(1-u))}$$

and assume $\alpha < z_1 z_2/(1+z_1)$. We want to show that $I(t) \to I$ as $t \to \infty$. In $k(u,t)$, we have $1 - u + b/t < 1$ in the integral, so a lower bound for the denominator of the integrand is $z_2 - \alpha$, and $k(u,t) \to k(u)$ pointwise is immediate by dominated convergence. We further get

$$k(u,t) \geq z_1 + 1 - e^{-\delta ut} - \int_0^{ut} \frac{z_2\,\delta\,e^{-\delta b}\,db}{z_2 - \alpha}$$

$$= z_1 + 1 - e^{-\delta ut} - z_2\frac{1 - e^{-\delta ut}}{z_2 - \alpha} = \zeta + \frac{\alpha}{z_2 - \alpha}e^{-\delta ut} \geq \zeta,$$

where $\zeta = z_1 + 1 - z_2/(z_2 - \alpha)$ is positive by the assumption $\alpha < z_1 z_2/(1+z_1)$. One more application of dominated convergence with ζ^{-1} as integrable majorant proves

$$I(t) \to I = \rho\left(\frac{z_1}{1+z_1} - z_1\,z_2\,\frac{e^{\frac{z_1 z_2}{(1+z_1)\alpha(1+\eta)} - \frac{1}{1+\eta}}\,\mathrm{Ei}\left(\frac{1}{1+\eta} - \frac{z_1 z_2}{(1+z_1)\alpha(1+\eta)}\right)}{(1+z_1)^2\,\alpha(1+\eta)}\right),$$

where

$$\mathrm{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t}\,dt = 0.577.. + \log|x| + x + \frac{x^2}{4} + \cdots, \quad -\infty < x < 0$$

23

denotes the Exponential Integral. Thus we obtain

$$\kappa(\alpha) = \lambda \, \mathbb{E}\left(\frac{e^{\alpha X}}{1 + (1+\eta)\alpha \, X} \right) - (\lambda + \rho)$$

$$+ \rho \left(\frac{z_1}{1+z_1} - z_1 \, z_2 \, \frac{e^{\frac{z_1 z_2}{(1+z_1)\alpha(1+\eta)} - \frac{1}{1+\eta}} \, \mathrm{Ei}\left(\frac{1}{1+\eta} - \frac{z_1 z_2}{(1+z_1)\alpha(1+\eta)} \right)}{(1+z_1)^2 \, \alpha(1+\eta)} \right),$$

which is a continuous function in $[0, \frac{z_1 z_2}{1+z_1})$. One readily verifies that $\kappa(0) = 0$, $\kappa'(0) = -\eta\,(\lambda + \frac{\rho}{z_1 z_2}) < 0$ and the convexity of $\kappa(\alpha)$. Given that $\kappa(\alpha) \uparrow \infty$ as $\alpha \uparrow \frac{z_1 z_2}{1+z_1}$, it follows that there is a unique adjustment coefficient $\gamma \in (0, \frac{z_1 z_2}{1+z_1})$ with $\kappa(\gamma) = 0$ and $\kappa'(\gamma) > 0$.

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