Extremes of homogeneous Gaussian random fields

Krzysztof Dębicki[1], Enkelejd Hashorva[2], Natalia Soja-Kukieła[3]

1. *Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland,*

2. *Department of Actuarial Science, Faculty of Business and Economics, University of Lausanne, UNIL-Dorigny 1015 Lausanne, Switzerland*

3. *Nicolaus Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland*

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Abstract

Let $\{X(s,t) : s,t \geqslant 0\}$ be a centered homogeneous Gaussian field with a.s. continuous sample paths and correlation function $r(s,t) = Cov(X(s,t), X(0,0))$ such that

$$r(s,t) = 1 - |s|^{\alpha_1} - |t|^{\alpha_2} + o(|s|^{\alpha_1} + |t|^{\alpha_2}), \quad s,t \to 0,$$

with $\alpha_1, \alpha_2 \in (0,2]$, and $r(s,t) < 1$ for $(s,t) \neq (0,0)$. In this contribution we derive an exact asymptotic expansion (as $u \to \infty$) of

$$\mathbb{P}\left(\sup_{(sn_1(u), tn_2(u)) \in [0,x]\times[0,y]} X(s,t) \leqslant u \right),$$

where $n_1(u)n_2(u) = u^{2/\alpha_1 + 2/\alpha_2}\Psi(u)$, which holds uniformly for $(x,y) \in [A,B]^2$ with A, B two positive constants and Ψ the survival function of an $N(0,1)$ random variable. We apply our findings to the analysis of asymptotics of extremes of homogeneous Gaussian fields over more complex parameter sets and a ball of random radius. Additionally we determine the extremal index of the discretised random field determined by $X(s,t)$.

Key words: Gaussian random fields; supremum; tail asymptoticy; extremal index; Berman condition; strong dependence.

1 Introduction

One of the seminal results in extreme value theory of Gaussian processes is the asymptotic behaviour of the distribution of supremum of a centered stationary Gaussian process $\{X(t) : t \geqslant 0\}$ with correlation function satisfying

$$r(t) = Cov(X(t), X(0)) = 1 - |t|^{\alpha} + o(|t|^{\alpha}) \text{ as } t \to 0 \text{ with } \alpha \in (0,2], \tag{1}$$

over intervals of length proportional to

$$\mu(u) = P\left(\sup_{t\in[0,1]} X(t) > u\right)^{-1} (1 + o(1)),$$

see, e.g., Leadbetter et al. [8, Theorem 12.3.4], Arendarczyk and Dębicki [1, Lemma 4.3], Tan and Hashorva [15, Lemma 3.3]. The following theorem gives a preliminary result concerning the aforementioned asymptotics.

Theorem 1. *Let $\{X(t) : t \geqslant 0\}$ be a centered stationary Gaussian process that satisfies (1), and let $0 < A_0 < A_\infty < \infty$ and $x > 0$ be arbitrary constants. If $r(t)\log t \to r \in [0,\infty)$ as $t \to \infty$, then*

$$P\left(\sup_{t\in[0,x\mu(u)]} X(t) \leqslant u\right) \to E\left(\exp\left(-x\exp(-r+\sqrt{2r}\mathcal{W})\right)\right) \in (0,\infty),$$

as $u \to \infty$, uniformly for $x \in [A_0, A_\infty]$, with $\mathcal{W}$ an $N(0,1)$ random variable.

The main goal of this paper is to derive an analogue of the above result for Gaussian random fields; see part (i) of Theorem 2 which constitutes a 2-dimensional counterpart of Theorem 1.

As an application of our findings, in Section 3 we investigate asymptotics of the tail of supremum of a homogeneous Gaussian field over a parameter sets that are approximable by simple sets (part (ii) of Theorem 2) and a ball of random radius. Additionally we analyze the existence of the *extremal index* for discrete-parameter fields associated with homogeneous Gaussian fields with covariance structure satisfying some regularity conditions; see Proposition 2.

2 Preliminaries

Let $\{X(s,t) : s,t \geqslant 0\}$ be a centered homogeneous Gaussian field with a.s. continuous sample paths and correlation function $r(s,t) = Cov(X(s,t), X(0,0))$ such that

A1: $r(s,t) = 1 - |s|^{\alpha_1} - |t|^{\alpha_2} + o(|s|^{\alpha_1} + |t|^{\alpha_2})$ as $s,t \to 0$ with $\alpha_1, \alpha_2 \in (0,2]$;

A2: $r(s,t) < 1$ for $(s,t) \neq (0,0)$;

A3: $\sup_{(s,t)\in\mathcal{S}(0,d)} |r(s,t)\log d - r| \to 0$ as $d \to \infty$, with $r \in [0,\infty)$,

where $\mathcal{S}(0,d)$ denotes the sphere of center $(0,0)$ and radius $d > 0$ in $\mathbb{R}^2$ with Euclidean metric.

We distinguish two separate families of Gaussian fields

- *weakly dependent fields*, satisfying **A3** with $r = 0$,
- *strongly dependent fields*, satisfying **A3** with $r \in (0,\infty)$.

Let $\mathcal{H}_\alpha$ denote the Pickands constant (see [11]), i.e.,

$$\mathcal{H}_\alpha := \lim_{T\to\infty} \frac{E\exp\left(\max_{0\leqslant t\leqslant T}\chi(t)\right)}{T}$$

where $\chi(t) = B_{\alpha/2}(t) - |t|^\alpha$, with $\{B_{\alpha/2}(t) : t \geqslant 0\}$ being a fractional Brownian motion with Hurst parameter $\alpha/2 \in (0,1]$. We note in passing that $\mathcal{H}_\alpha$ appears for the first time in Pickands theorem [11]; a correct proof of that theorem is first given in Piterbarg [12].

For a standard normal random variable $\mathcal{W}$ we write $\Phi(u) = P(\mathcal{W} \leqslant u)$, $\Psi(u) = P(\mathcal{W} > u)$. Recall that

$$\Psi(u) = \frac{1}{\sqrt{2\pi}u}\exp(-u^2/2)(1+o(1)), \quad \text{as } u \to \infty.$$

Following Piterbarg [13, Theorem 7.1] we recall that for a centered stationary Gaussian field $\{X(s,t)\}$ satisfying **A1**, **A2**, for arbitrary $g, h \in (0,\infty)$,

$$P\left(\max_{(s,t)\in[0,g]\times[0,h]} X(s,t) > u\right) = \mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2} ghu^{2/\alpha_1}u^{2/\alpha_2}\Psi(u)(1+o(1)), \tag{2}$$

as $u \to \infty$.

Let $m_1(u) \to \infty$ and $m_2(u) \to \infty$ be functions such that

$$m_1(u) = a_1(u)/\sqrt{\Psi(u)} \quad \text{and} \quad m_2(u) = a_2(u)/\sqrt{\Psi(u)}$$

for some positive functions $a_1(u), a_2(u)$ satisfying $a_1(u)a_2(u) = (\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}u^{2/\alpha_1}u^{2/\alpha_2})^{-1}$, $\log a_1(u) = o(u^2)$ and $\log a_2(u) = o(u^2)$. We note that then

$$m(u) := m_1(u)m_2(u) = P\left(\max_{(s,t)\in[0,1]^2} X(s,t) > u\right)^{-1}(1+o(1)),$$

as $u \to \infty$.

By $\mathcal{B}(0,x)$ we denote a ball in $\mathbb{R}^2$ of center at $(0,0)$ and radius x.

3 Main results

The aim of this section is to prove the following 2-dimensional counterpart of Theorem 1. Recall that $\mathcal{W}$ denotes an $N(0,1)$ random variable. For a given Jordan-measurable set $\mathcal{E} \subset \mathbb{R}^2$ with Lebesgue measure $\mathrm{mes}(\mathcal{E}) > 0$ let $\mathcal{E}_u := \{(x,y) : (x/m_1(u), y/m_2(u)) \in \mathcal{E}\}$. One interesting example is $\mathcal{E}_u = [0, xm_1(u)] \times [0, ym_2(u)]$ for x, y positive, hence $\mathcal{E} = [0,x] \times [0,y]$ and $\mathrm{mes}(\mathcal{E}) = xy$. For such $\mathcal{E}_u$ we shall show below an approximation which holds uniformly on compact intervals of $(0,\infty)^2$. If the structure of the set is not specified, considering thus the supremum of a Gaussian field over some general measurable set $\mathcal{T}_u \subset \mathbb{R}^2$ an ϵ-net $(\mathcal{L}_\varepsilon, \mathcal{U}_\varepsilon)$ approximation of $\mathcal{T}_u$ will be assumed. Specifically, the ϵ-net $(\mathcal{L}_\varepsilon, \mathcal{U}_\varepsilon)$ here means that for any $\varepsilon > 0$ there exist two sets $\mathcal{L}_\varepsilon$ and $\mathcal{U}_\varepsilon$ which are *simple sets* (i.e., finite sums of disjoint rectangles of the form $[a_1, b_1) \times [a_2, b_2)$) such that

$$\lim_{\varepsilon\downarrow 0} \mathrm{mes}(\mathcal{L}_\varepsilon) = \lim_{\varepsilon\downarrow 0} \mathrm{mes}(\mathcal{U}_\varepsilon) = c \in (0,\infty) \tag{3}$$

and

$$\mathcal{L}_{\varepsilon,u} = \{(x,y) : (x/m_1(u), y/m_2(u)) \in \mathcal{L}_\varepsilon\} \subset \mathcal{T}_u \subset \mathcal{U}_{\varepsilon,u} = \{(x,y) : (x/m_1(u), y/m_2(u)) \in \mathcal{U}_\varepsilon\} \subset \mathbb{R}^2.$$

Next we formulate our main results for these two cases.

Theorem 2. *Let $\{X(s,t) : s,t \geqslant 0\}$ be a centered homogeneous Gaussian field with covariance function that satisfies **A1**, **A2** and **A3** with $r \in [0,\infty)$. Then,*
(i) for each $0 < A < B < \infty$,

$$\mathbb{P}\left(\sup_{(s,t)\in[0,xm_1(u)]\times[0,ym_2(u)]} X(s,t) \leqslant u\right) \to \mathbb{E}\big(\exp(-xy\exp(-2r + 2\sqrt{r}\mathcal{W}))\big),$$

as $u \to \infty$, uniformly for $(x,y) \in [A,B]^2$.
(ii) for $\mathcal{T}_u \subset \mathbb{R}^2, u > 0$ such that there exists an ϵ-net $(\mathcal{L}_\varepsilon, \mathcal{U}_\varepsilon)$ satisfying (3)

$$\mathbb{P}\left(\sup_{(s,t)\in\mathcal{T}_u} X(s,t) \leqslant u\right) \to \mathbb{E}\big(\exp(-c\exp(-2r + 2\sqrt{r}\mathcal{W}))\big), \text{ as } u \to \infty.$$

The complete proof of Theorem 2 is given in Section 5.1.

Remark 1. *Following the same reasoning as given in the proof of Theorem 2, assuming that* **A1-A3** *holds, for each $0 < A < B < \infty$, we have*

$$\mathbb{P}\left(\sup_{(s,t)\in\mathcal{B}\left(0,x\sqrt{m(u)}\right)} X(s,t) \leqslant u\right) \to \mathbb{E}\big(\exp(-\pi x^2\exp(-2r + 2\sqrt{r}\mathcal{W}))\big), \tag{4}$$

as $u \to \infty$, uniformly for $x \in [A,B]$; $\mathcal{B}(0,x)$ is a ball in $\mathbb{R}^2$ of center at $(0,0)$ and radius x.

4 Applications

In this section we apply results of Section 3 to the analysis of the asymptotic properties of supremum of a Gaussian field over a random parameter set and to the analysis of dependance structure of homogeneous Gaussian fields.

4.1 Extremes of homogeneous Gaussian fields over a random parameter set

In this section we analyze asymptotic properties of the tail distribution of $\sup_{(s,t)\in\mathcal{B}(0,T)} X(s,t) > u)$, where T is a nonnegative, independent of X random variable. One-dimensional counterpart of this problem was recently analyzed in [1] and [15].

Proposition 1. *Let $\{X(s,t): \ s,t \geqslant 0\}$ be a centered homogeneous Gaussian field with covariance function that satisfies **A1-A3** with $r \in [0,\infty)$, and let T be an independent of X nonnegative random variable.*
(i) If $ET^2 < \infty$, then, as $u \to \infty$,

$$P\left(\sup_{(s,t)\in\mathcal{B}(0,T)} X(s,t) > u\right) = \pi ET^2 \mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2} u^{2/\alpha_1} u^{2/\alpha_2} \Psi(u)(1+o(1)).$$

(ii) If T has a regularly varying survival function at infinity with index $\lambda < 2$, then as $u \to \infty$,

$$P\left(\sup_{(s,t)\in\mathcal{B}(0,T)} X(s,t) > u\right) = 2\pi\mathcal{C}P(T > \sqrt{m(u)})(1+o(1)),$$

where $\mathcal{C} = \int_0^\infty x^{1-\lambda} E\big(\exp(-\pi x^2 \exp(\mathcal{V}_r) + \mathcal{V}_r)\big)dx$ and $\mathcal{V}_r = 2\sqrt{r}\mathcal{W} - 2r$.
(iii) If T is slowly varying at ∞, then, as $u \to \infty$,

$$P\left(\sup_{(s,t)\in\mathcal{B}(0,T)} X(s,t) > u\right) = P(T > \sqrt{m(u)})(1+o(1)).$$

The proof of Proposition 1 is given in Section 5.2.

4.2 Extremal indices for homogeneous Gaussian fields

Following [5], we say that $\theta \in (0,1]$ is the *extremal index* of a homogeneous discrete-parameter stationary random field $\{X_{j,k} : j,k = 1,2,\ldots\}$, if

$$P\left(\max_{j\leqslant a_n,\ k\leqslant b_n} X_{j,k} \leqslant z_n\right) - P(X_{1,1} \leqslant z_n)^{a_n b_n \cdot \theta} \to 0, \tag{5}$$

as $n \to \infty$, for each sequence $(z_n) \subset \mathbb{R}$ and all sequences $(a_n), (b_n) \subset \mathbb{N}$ such that $a_n \to \infty$ and $b_n \to \infty$, as $n \to \infty$, and $1/C \leqslant a_n/b_n \leqslant C$ for some constant $C > 0$. The notion of extremal index θ originated in investigations concerning relationship between the dependence structure of discrete-parameter stationary sequences of random variables and their extremal behaviour [7, 8]; see also [10, 3, 4, 6, 9, 16].

For a given centered homogeneous Gaussian field $\{X(s,t) : s,t \geq 0\}$ that satisfies **A1-A3** introduce a discrete-parameter random field $\{\widetilde{X}_{j,k} : j,k = 1,2,\ldots\}$, with

$$\widetilde{X}_{j,k} := \sup_{(s,t)\in[j-1,j]\times[k-1,k]} X(s,t).$$

The following proposition points out how the difference in the dependance structure between weakly- and strongly-dependant Gaussian fields influences the existence of the extremal index of the associated field $\{\widetilde{X}_{j,k}\}$.

Proposition 2. *Assume that* **A1-A3** *holds for a centered homogeneous Gaussian field* $\{X(s,t): s,t \geqslant 0\}$.
(i) If $r=0$*, then the extremal index of* $\{\widetilde{X}_{j,k} : j,k = 1,2,\ldots\}$ *equals to 1.*
(ii) If $r>0$*, then* $\{\widetilde{X}_{j,k} : j,k = 1,2,\ldots\}$ *does not have an extremal index.*

The proof of Proposition 2 is deferred to Section 5.3.

5 Proofs

Before we prove Theorem 2, we need some auxiliary results. The first one is a 2-dimensional version of Lemma 12.2.11 in [8].

Lemma 1. *Assume that* ***A1***, ***A2*** *hold and* $q_1 = q_1(u) = au^{-2/\alpha_1}$, $q_2 = q_2(u) = au^{-2/\alpha_2}$ *for some* $a>0$*. Then for any* $x,y \geqslant 0$, $g,h>0$ *and rectangle* $I=(x,y)+[0,g]\times[0,h]$*, as* $u\to\infty$,

$$P\left(X(jq_1,kq_2)\leqslant u;\ (jq_1,kq_2)\in I\right) - P\left(X(s,t)\leqslant u;\ (s,t)\in I\right) \leqslant \frac{gh\rho(a)}{m(u)} + o\left(\frac{1}{m(u)}\right),$$

where $\rho(a)\to 0$ *as* $a\to 0$.

Proof. From the homogeneity of the field $\{X(s,t)\}$ we conclude that

$$\begin{aligned} 0 &\leqslant P\left(X(jq_1,kq_2)\leqslant u;\ (jq_1,kq_2)\in I\right) - P\left(X(s,t)\leqslant u;\ (s,t)\in I\right) \\ &\leqslant ([g/q_1]+[h/q_2]+1)P(X(0,0)>u) + P\left(X(jq_1,kq_2)\leqslant u;\ (jq_1,kq_2)\in[0,g]\times[0,h]\right) \\ &\quad - P\left(X(s,t)\leqslant u;\ (s,t)\in[0,g]\times[0,h]\right). \end{aligned}$$

Then there exists a constant K such that

$$([g/q_1]+[h/q_2]+1)P(X(0,0)>u)m(u) \leqslant \frac{K(u^{2/\alpha_1}+u^{2/\alpha_2})\Psi(u)}{\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}u^{2/\alpha_1}u^{2/\alpha_2}\Psi(u)},$$

which implies that $([g/q_1]+[h/q_2]+1)P(X(0,0)>u) = o\left(\frac{1}{m(u)}\right)$, as $u\to\infty$.

Let $T>0$ be given. We divide the set $[0,g]\times[0,h]$ into small rectangles with the side-lengths q_1T and q_2T in the following way

$$\begin{aligned} \Delta_{1,1} &:= [0,q_1T]\times[0,q_2T], \\ \Delta_{l,m} &:= ((l-1)q_1T,(m-1)q_2T)+\Delta_{1,1}, \end{aligned}$$

for $l=1,\ldots,\left\lfloor\frac{g}{q_1T}\right\rfloor$ and $m=1,\ldots,\left\lfloor\frac{h}{q_2T}\right\rfloor$. Then we have that

$$\begin{aligned} &P(X(jq_1,kq_2)\leqslant u;\ (jq_1,kq_2)\in[0,g]\times[0,h]) - P(X(s,t)\leqslant u;\ (s,t)\in[0,g]\times[0,h]) \\ &\leqslant P\left(\sup_{(s,t)\in[0,g]\times[0,h]} X(s,t)>u\right) - \sum_{l=1}^{\left\lfloor\frac{g}{q_1T}\right\rfloor}\sum_{m=1}^{\left\lfloor\frac{h}{q_2T}\right\rfloor} P\left(\max_{(jq_1,kq_2)\in\Delta_{l,m}} X(jq_1,kq_2)>u\right) \\ &\quad + \sum_{(l,m)\neq(l',m')} P\left(\max_{(jq_1,kq_2)\in\Delta_{l,m}} X(jq_1,kq_2)>u,\ \max_{(jq_1,kq_2)\in\Delta_{l',m'}} X(jq_1,kq_2)>u\right). \end{aligned} \tag{6}$$

From [13, Lemma 7.1], as $u\to\infty$,

$$P\left(\sup_{(s,t)\in[0,g]\times[0,h]} X(s,t)>u\right) = \mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}ghu^{2/\alpha_1}u^{2/\alpha_2}\Psi(u)(1+o(1)). \tag{7}$$

Moreover, by homogeneity of $X(\cdot,\cdot)$,

$$\sum_{l=1}^{\left\lfloor\frac{g}{q_1T}\right\rfloor}\sum_{m=1}^{\left\lfloor\frac{h}{q_2T}\right\rfloor} P\left(\max_{(jq_1,kq_2)\in\Delta_{l,m}} X(jq_1,kq_2)>u\right) \sim \frac{ghu^{2/\alpha_1}u^{2/\alpha_2}}{a^2T^2} P\left(\max_{(jq_1,kq_2)\in\Delta_{1,1}} X(jq_1,kq_2)>u\right). \tag{8}$$

We focus on the asymptotics of $P\left(\max_{(jq_1,kq_2)\in\Delta_{1,1}} X(jq_1,kq_2)>u\right)$. Following line-by-line the idea of the proof of Lemma D.1 in [13] we have

$$\begin{aligned}
&P\left(\max_{(jq_1,kq_2)\in\Delta_{1,1}} X(jq_1,kq_2)>u\right)\\
&\quad\sim \Psi(u)\int_{-\infty}^{\infty} e^{w-w^2/(2u^2)}P\left(\max_{(ja,ka)\in[0,aT]^2}\chi_u(ja,ka)>w)\;\Big|\; X(0,0)=u-\frac{w}{u}\right)dw,\\
&\quad\sim \Psi(u)H_{\alpha_1}(T,a)H_{\alpha_2}(T,a),
\end{aligned}$$

where $H_{\alpha_i}(T,a) := E\exp\left(\max_{j\in[0,T]} B_{\alpha_i/2}(ja)-|ja|^{\alpha_i}\right)$, with $B_{\alpha_i/2}(\cdot)$ being a fractional Brownian motion with Hurst parameter $\alpha_i/2$ for $i=1,2$ (see also (12.2.6) in proof of [8, Lemma 12.2.11]).

The above implies that, by (8),

$$\begin{aligned}
&\sum_{l=1}^{\lfloor \frac{g}{q_1T}\rfloor}\sum_{m=1}^{\lfloor \frac{h}{q_2T}\rfloor} P\left(\max_{(jq_1,kq_2)\in\Delta_{l,m}} X(jq_1,kq_2)>u\right)\\
&\quad= ghu^{2/\alpha_1}u^{2/\alpha_2}\Psi(u)\left(\frac{H_{\alpha_1}(T,a)}{aT}\right)\left(\frac{H_{\alpha_2}(T,a)}{aT}\right)(1+o(1)) \qquad (9)
\end{aligned}$$

as $u\to\infty$.

In the next step we prove that the double sum that appears in (6) is negligible, i.e., it is $o\left(\frac{1}{m(u)}\right)$. Indeed, notice that

$$\begin{aligned}
&\sum_{(m,l)\neq(m',l')} P\left(\max_{(jq_1,kq_2)\in\Delta_{m,l}} X(jq_1,kq_2)>u,\ \max_{(jq_1,kq_2)\in\Delta_{m',l'}} X(jq_1,kq_2)>u\right)\\
&\quad\leqslant \sum_{(m,l)\neq(m',l')} P\left(\sup_{(s,t)\in\Delta_{m,l}} X(s,t)>u,\ \sup_{(s,t)\in\Delta_{m',l'}} X(s,t)>u\right)=o\left(\frac{1}{m(u)}\right), \qquad (10)
\end{aligned}$$

where (10) follows from the proof of [13, Lemma 6.1].

Now, combining (7), (9) and (10), we conclude that for any $T>0$ and $a>0$ it holds that

$$\begin{aligned}
&P\left(X(jq_1,kq_2)\leqslant u;\ (jq_1,kq_2)\in[0,g]\times[0,h]\right)-P\left(X(s,t)\leqslant u;\ (s,t)\in[0,g]\times[0,h]\right)\\
&\quad\leqslant ghu^{2/\alpha_1}u^{2/\alpha_2}\Psi(u)\left(\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}-\left(\frac{H_{\alpha_1}(T,a)}{aT}\right)\cdot\left(\frac{H_{\alpha_2}(T,a)}{aT}\right)\right)(1+o(1))\\
&\quad= gh\frac{1-\left(\frac{H_{\alpha_1}(T,a)}{aT}\cdot\frac{H_{\alpha_2}(T,a)}{aT}\right)\mathcal{H}_{\alpha_1}^{-1}\mathcal{H}_{\alpha_2}^{-1}}{m(u)}+o\left(\frac{1}{m(u)}\right).
\end{aligned}$$

Finally, using that

$$\lim_{a\to0}\lim_{T\to\infty}\frac{H_\alpha(T,a)}{aT}=\mathcal{H}_\alpha,$$

see e.g. [8, Lemmas 12.2.4(i),12.2.7(i)], the thesis of the lemma is satisfied with

$$\rho(a):=1-\lim_{T\to\infty}\left(\frac{H_{\alpha_1}(T,a)}{aT}\cdot\frac{H_{\alpha_1}(T,a)}{aT}\right)\mathcal{H}_{\alpha_1}^{-1}\mathcal{H}_{\alpha_2}^{-1}.$$

This completes the proof. ■

Let

$$\rho_T(s,t) := \begin{cases} 1, & 0\leqslant \max(|s|,|t|)<1;\\ |r(s,t)-\frac{r}{\log T}|, & 1\leqslant\max(|s|,|t|)\leqslant T,\end{cases} \qquad (11)$$

$$\varrho_T(s,t) := \begin{cases} |r(s,t)|+(1-r(s,t))\frac{r}{\log T}, & 0\leqslant \max(|s|,|t|)<1;\\ \frac{r}{\log T}, & 1\leqslant\max(|s|,|t|)\leqslant T.\end{cases} \qquad (12)$$

The next lemma combines a 2-dimensional counterpart of Lemma 12.3.1 in [8], for weakly dependent fields, and Lemma 3.1 in [15] for strongly dependent fields.

Lemma 2. *Let $\varepsilon > 0$ be given. Let $q_1 = q_1(u) = au^{-2/\alpha_1}$ and $q_2 = q_2(u) = au^{-2/\alpha_2}$. Suppose that $T_1 = T_1(u) \sim \tau m_1(u)$ and $T_2 = T_2(u) \sim \tau m_2(u)$ for some $\tau > 0$, as $u \to \infty$. Then, providing that conditions **A1**, **A2** and **A3** with $r \in [0, \infty)$ are fulfilled,*

$$\frac{T_1T_2}{q_1q_2} \sum_{(jq_1,kq_2)\in[-T_1,T_1]\times[-T_2,T_2]-(-\varepsilon,\varepsilon)^2} \rho_{T_{\max}}(jq_1,kq_2) \exp\left(-\frac{u^2}{1+\max\left(|r(jq_1,kq_2)|, \varrho_{T_{\max}}(jq_1,kq_2)\right)}\right) \to 0,$$

as $u \to \infty$, where $T_{\max} = \max(T_1, T_2)$.

Proof. Let $T_1(u) \sim \tau m_1(u)$ and $T_2(u) \sim \tau m_2(u)$ for some $\tau > 0$, as $u \to \infty$. Then,

$$\log(T_1T_2) + \log\left(\frac{\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}}{\sqrt{2\pi}}\right) + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - 1\right)\log u - \frac{u^2}{2} \to 2\log\tau.$$

Thus

$$u^2 \sim 2\log(T_1T_2)$$

and

$$\log u = \frac{1}{2}\log 2 + \frac{1}{2}\log\log(T_1T_2) + o(1).$$

Moreover

$$u^2 = 2\log(T_1T_2) + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - 1\right)\log\log(T_1T_2) - 4\log\tau + 2\log\left(\frac{\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}}{2\sqrt{\pi}}2^{1/\alpha_1+1/\alpha_2}\right) + o(1). \tag{13}$$

For $T > 0$ put $\delta_T = \sup_{\varepsilon\leqslant\max(|s|,|t|)\leqslant T} \max(|r(s,t)|, \varrho_T(s,t))$. It is straightforward to see that there exists $\delta < 1$ such that for sufficiently large T we get

$$\delta_T = \sup_{\varepsilon\leqslant\max(|s|,|t|)\leqslant T} \max(|r(s,t)|, \varrho_T(s,t)) < \delta < 1,$$

since δ_T is decreasing in T for large T. Let β be such that $0 < \beta < \frac{1-\delta}{1+\delta}$. Divide $Q := [-T_1, T_1] \times [-T_2, T_2] - (-\varepsilon, \varepsilon)^2$ into two subsets:

$$\begin{aligned} S^* &:= \{(s,t) \in Q : |s| \leqslant T_1^\beta, |t| \leqslant T_2^\beta\}, \\ S &:= Q - S^*. \end{aligned}$$

Firstly, we show that

$$\frac{T_1T_2}{q_1q_2} \sum_{(jq_1,kq_2)\in S^*} \rho_{T_{\max}}(jq,kq) \exp\left(-\frac{u^2}{1+\max(|r(jq,kq)|, \varrho_{T_{\max}}(jq,kq))}\right) \to 0, \tag{14}$$

as $u \to \infty$. By (13) there exists a constant K such that $\exp(-u^2/2) \leqslant \frac{K}{T_1T_2}$. Applying the fact that $u^2 \sim 2\log(T_1T_2)$ and $u^{2/\alpha_1}q_1 = u^{2/\alpha_2}q_2 = a$, for u large enough, we obtain

$$\begin{aligned} &\frac{T_1T_2}{q_1q_2} \sum_{(jq_1,kq_2)\in S^*} \rho_{T_{\max}}(jq,kq) \exp\left(-\frac{u^2}{1+\max(|r(jq,kq)|, \varrho_{T_{\max}}(jq,kq))}\right) \\ &\quad\leqslant \frac{T_1T_2}{q_1q_2}\left(\frac{2T_1^\beta}{q_1}+1\right)\left(\frac{2T_2^\beta}{q_2}+1\right)\exp\left(-\frac{u^2}{1+\delta}\right) \sim 4\frac{(T_1T_2)^{\beta+1}}{q_1^2q_2^2}\left(\exp\left(-\frac{u^2}{2}\right)\right)^{\frac{2}{1+\delta}} \\ &\quad\leqslant 4K^{\frac{2}{1+\delta}}\frac{(T_1T_2)^{\beta+1-\frac{2}{1+\delta}}}{q_1^2q_2^2} \sim \frac{2^{2/\alpha_1+2/\alpha_2+2}K^{\frac{2}{1+\delta}}}{a^4}\left(\log(T_1T_2)\right)^{2/\alpha_1+2/\alpha_2}(T_1T_2)^{\beta-\frac{1-\delta}{1+\delta}}. \end{aligned}$$

Since we choose $\beta < \frac{1-\delta}{1+\delta}$, then (14) holds.

To complete the proof it suffices to show that, as $u \to \infty$,

$$\frac{T_1 T_2}{q_1 q_2} \sum_{(jq_1, kq_2) \in S} \rho_{T_{\max}}(jq_1, kq_2) \exp\left(-\frac{u^2}{1 + \max(|r(jq_1, kq_2)|, \varrho_{T_{\max}}(jq_1, kq_2))}\right) \to 0. \quad (15)$$

In order to do it observe that there exist constants $C > 0$ and $K > 0$ such that

$$\max\left(|r(s,t)|, \varrho_{T_{\max}}(s,t)\right) \cdot \log\left(\sqrt{s^2 + t^2}\right) \leqslant K$$

for all u sufficiently large and (s,t) satisfying $C \leqslant \max(|s|, |t|) \leqslant T_{\max}$. Put $T_{\min} := \min(T_1, T_2)$. Since $T_{\min}^{\beta} > C$ for u large enough, then for (jq_1, kq_2) such that $\max(|jq_1|, |kq_2|) \geqslant T_{\min}^{\beta}$ we have

$$\max\left(|r(jq_1, kq_2)|, \varrho_{T_{\max}}(jq_1, kq_2)\right) \leqslant \frac{K}{\log T_{\min}^{\beta}}.$$

Hence

$$\exp\left(-\frac{u^2}{1 + \max\left(|r(jq_1, kq_2)|, \varrho_{T_{\max}}(jq_1, kq_2)\right)}\right) \leqslant \exp\left(-\frac{u^2}{1 + \frac{K}{\log T_{\min}^{\beta}}}\right) \leqslant \exp\left(-u^2\left(1 - \frac{K}{\log T_{\min}^{\beta}}\right)\right),$$

which implies the following chain of inequalities

$$\begin{aligned}
&\frac{T_1 T_2}{q_1 q_2} \sum_{(jq_1, kq_2) \in S} \rho_{T_{\max}}(jq_1, kq_2) \exp\left(-\frac{u^2}{1 + \max\left(|r(jq_1, kq_2)|, \varrho_{T_{\max}}(jq_1, kq_2)\right)}\right) \\
&\quad \leqslant \frac{T_1 T_2}{q_1 q_2} \sum_{(jq_1, kq_2) \in S} \left| r(jq_1, kq_2) - \frac{r}{\log T_{\max}} \right| \exp\left(-u^2\left(1 - \frac{K}{\log T_{\min}^{\beta}}\right)\right) \\
&\quad \leqslant 4 \frac{T_1^2 T_2^2}{q_1^2 q_2^2} \exp\left(-u^2\left(1 - \frac{K}{\log T_{\min}^{\beta}}\right)\right) \frac{1}{\log T_{\min}^{\beta}} \times \frac{q_1 q_2 \log T_{\min}^{\beta}}{T_1 T_2} \sum_{(jq_1, kq_2) \in S} \left| r(jq_1, kq_2) - \frac{r}{\log T_{\max}} \right| \\
&\quad =: I_1 \times I_2.
\end{aligned}$$

Firstly, we show that factor I_1 is bounded. Indeed, using that

$$u^2 = 2\log(T_1 T_2) + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - 1\right) \log\log(T_1 T_2) + O(1),$$

there exists a constant K' such that for u large enough

$$-u^2\left(1 - \frac{K}{\log T_{\min}^{\beta}}\right) = -u^2 + K \frac{2\log(T_1 T_2) + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - 1\right) \log\log(T_1 T_2) + O(1)}{\log T_{\min}^{\beta}} \leqslant -u^2 + K'.$$

The last inequality follows from the fact that $\frac{\log(T_1 T_2)}{\log T_{\min}^{\beta}} \to 2/\beta$. Moreover,

$$\exp\left(-u^2\left(1 - \frac{K}{\log T_{\min}^{\beta}}\right)\right) \leqslant K'' \exp(-u^2) \leqslant K''' (T_1 T_2)^{-2} (\log(T_1 T_2))^{1 - 2/\alpha_1 - 2/\alpha_2},$$

for some constants K'', K'''. Using that $u^2 \sim 2\log(T_1 T_2)$ and $u^{2/\alpha_1} q_1 = u^{2/\alpha_2} q_2 = a$, we conclude that

$$\begin{aligned}
I_1 &\leqslant 4 \frac{T_1^2 T_2^2}{q_1^2 q_2^2} \exp\left(-u^2\left(1 - \frac{K}{\log T_{\min}^{\beta}}\right)\right) \frac{1}{\log T_{\min}^{\beta}} \\
&\leqslant 4 \frac{T_1^2 T_2^2}{q_1^2 q_2^2} K''' (T_1 T_2)^{-2} (\log(T_1 T_2))^{1 - 2/\alpha_1 - 2/\alpha_2} \frac{1}{\log T_{\min}^{\beta}} \\
&= 4 K''' 2^{2/\alpha_1 + 2/\alpha_2} \frac{1}{a^4} (\log(T_1 T_2))^{2/\alpha_1 + 2/\alpha_2} (\log(T_1 T_2))^{1 - 2/\alpha_1 - 2/\alpha_2} \frac{1}{\log T_{\min}^{\beta}} \sim \frac{K''' 2^{2/\alpha_1 + 2/\alpha_2 + 3}}{a^4 \beta},
\end{aligned}$$

which proves that I_1 is bounded.

In the next step we show that I_2 tends to 0 as $u \to \infty$. Observe that

$$\begin{aligned} I_2 &= \frac{q_1 q_2 \log T_{\min}^{\beta}}{T_1 T_2} \sum_{(jq_1, kq_2)\in S} \left| r(jq_1, kq_2) - \frac{r}{\log T_{\max}} \right| \\ &\leqslant \frac{q_1 q_2}{T_1 T_2} \sum_{(jq_1, kq_2)\in S} \left| r(jq_1, kq_2) \log(\sqrt{(jq_1)^2 + (kq_2)^2} - r \right| \\ &+ \beta r \frac{q_1 q_2}{T_1 T_2} \sum_{(jq_1, kq_2)\in S} \left| 1 - \frac{\log T_{\max}}{\log(\sqrt{(jq_1)^2 + (kq_2)^2}} \right| =: J_1 + J_2. \end{aligned}$$

Combining **A3** with the fact that $a_n \to a$ implies the convergence $(a_1 + a_2 + \ldots + a_n)/n \to a$, as $n \to \infty$ (see [14]), we conclude that J_1 tends to 0, as $u \to \infty$. Additionally, see [8, p. 135],

$$\begin{aligned} J_2 &\leqslant \frac{\beta r}{\log T_{\min}^{\beta}} \frac{q_1 q_2}{T_1 T_2} \sum_{(jq_1, kq_2)\in S} \left| \log \sqrt{(jq_1)^2 + (kq_2)^2} - \log T_{\max} \right| \\ &= \frac{r}{\log T_{\min}} \frac{q_1 q_2}{T_1 T_2} \sum_{(jq_1, kq_2)\in S} \left| \log \left(\frac{\sqrt{(jq_1)^2 + (kq_2)^2}}{T_{\max}} \right) \right| \end{aligned}$$

Suppose that $T_{\max} = T_1$. Then

$$\begin{aligned} \frac{q_1 q_2}{T_1 T_2} \sum_{(jq_1, kq_2)\in S} \left| \log \left(\frac{\sqrt{(jq_1)^2 + (kq_2)^2}}{T_{\max}} \right) \right| &= \frac{q_1 q_2}{T_1 T_2} \sum_{(jq_1, kq_2)\in S} \left| \log \left(\sqrt{\left(\frac{jq_1}{T_1}\right)^2 + \left(\frac{kq_2}{T_2}\right)^2 \left(\frac{T_2}{T_1}\right)^2} \right) \right| \\ &\leqslant \frac{q_1 q_2}{T_1 T_2} \sum_{(jq_1, kq_2)\in S} \left(\left| \log \left(\sqrt{\left(\frac{jq_1}{T_1}\right)^2 + \left(\frac{kq_2}{T_2}\right)^2} \right) \right| + \left| \log \left| \frac{jq_1}{T_1} \right| \right| \right). \end{aligned}$$

Hence

$$J_2 \leqslant \frac{r}{\log T_{\min}} O\left(\int_{-1}^{1} \int_{-1}^{1} \left| \log(\sqrt{x^2 + y^2}) \right| dxdy + \int_{-1}^{1} |\log |x|| dx \right)$$

and (15) holds. The combination of (14) with (15) completes the proof.

■

Lemma 3. *Let $q_1 = q_1(u) = au^{-2/\alpha_1}$, $q_2 = q_2(u) = au^{-2/\alpha_2}$ and suppose that $T = T(u) \to \infty$, as $u \to \infty$. Then, providing that conditions **A1** and **A2** are fulfilled, there exists $\varepsilon > 0$ such that*

$$\begin{aligned} \frac{m(u)}{q_1 q_2} \sum_{0 < \max(|jq_1|, |kq_2|) < \varepsilon} \Bigg[& (1 - r(jq_1, kq_2)) \frac{r}{\log T} \left(1 - \left(r(jq_1, kq_2) + (1 - r(jq_1, kq_2)) \frac{r}{\log T} \right)^2 \right)^{-1/2} \\ & \times \exp\left(-\frac{u^2}{1 + r(jq_1, kq_2) + (1 - r(jq_1, kq_2)) \frac{r}{\log T}} \right) \Bigg] \to 0, \end{aligned}$$

as $u \to \infty$.

Proof. Firstly, note that for $\varepsilon > 0$ small enough

$$\frac{1}{2}(|s|^{\alpha_1} + |t|^{\alpha_2}) \leqslant 1 - r(s, t) \leqslant 2(|s|^{\alpha_1} + |t|^{\alpha_2}), \tag{16}$$

for $0 \leqslant \max(|s|,|t|) < \varepsilon$, due to **A1**. Thus for u large, ε small enough and $0 < \max(|jq_1|, |kq_2|) < \varepsilon$ we have

$$\begin{aligned}
&\left(1-\left(r(jq_1,kq_2)+(1-r(jq_1,kq_2))\frac{r}{\log T}\right)^2\right)^{-1/2} \\
\leqslant\ & \left(1-\left(r(jq_1,kq_2)+(1-r(jq_1,kq_2))\frac{r}{\log T}\right)\right)^{-1/2} = \left((1-r(jq_1,kq_2))\left(1-\frac{r}{\log T}\right)\right)^{-1/2} \\
\leqslant\ & \left(\frac{|jq_1|^{\alpha_1}+|kq_2|^{\alpha_2}}{4}\right)^{-1/2} \leqslant \left(\frac{\max(|jq_1|^{\alpha_1},|kq_2|^{\alpha_2})}{4}\right)^{-1/2} \leqslant \left(\frac{\min(q_1^{\alpha_1},q_2^{\alpha_2})}{4}\right)^{-1/2} = Ku,
\end{aligned}$$

for some constant $K > 0$. Combining the above inequality with (16) and definitions of $m(u)$, q_1 and q_2 we obtain

$$\begin{aligned}
&\frac{m(u)}{q_1q_2}\sum_{0<\max(|jq_1|,|kq_2|)<\varepsilon}\Bigg[(1-r(jq_1,kq_2))\frac{r}{\log T}\left(1-\left(r(jq_1,kq_2)+(1-r(jq_1,kq_2))\frac{r}{\log T}\right)^2\right)^{-1/2} \\
&\qquad\qquad\times\exp\left(-\frac{u^2}{1+r(jq_1,kq_2)+(1-r(jq_1,kq_2))\frac{r}{\log T}}\right)\Bigg] \\
\leqslant\ & K'ue^{u^2/2}\sum_{0<\max(|jq_1|,|kq_2|)<\varepsilon}\Bigg[(|jq_1|^{\alpha_1}+|kq_2|^{\alpha_2})(1+\delta)\frac{ru}{\log T} \\
&\qquad\qquad\times\exp\left(-\frac{u^2}{2-(|jq_1|^{\alpha_1}+|kq_2|^{\alpha_2})(1-\delta-\frac{r(1+\delta)}{\log T})}\right)\Bigg] \\
=\ & K'\frac{ru^2}{\log T}\sum_{0<\max(|jq_1|,|kq_2|)<\varepsilon}\Bigg[(|jq_1|^{\alpha_1}+|kq_2|^{\alpha_2})(1+\delta) \\
&\qquad\qquad\times\exp\left(-\frac{u^2(|jq_1|^{\alpha_1}+|kq_2|^{\alpha_2})(1-\delta-\frac{r(1+\delta)}{\log T})}{4-2(|jq_1|^{\alpha_1}+|kq_2|^{\alpha_2})(1-\delta-\frac{r(1+\delta)}{\log T})}\right)\Bigg] \\
\leqslant\ & K'\frac{ru^2}{\log T}(1+\delta)\frac{8}{u^2}\sum_{0<\max(|jq_1|,|kq_2|)<\varepsilon}\frac{u^2}{8}(|jq_1|^{\alpha_1}+|kq_2|^{\alpha_2})\exp\left(-\frac{u^2(|jq_1|^{\alpha_1}+|kq_2|^{\alpha_2})}{8}\right) \\
=\ & \frac{8rK'(1+\delta)}{\log T}\sum_{0<\max(|jq_1|,|kq_2|)<\varepsilon}\left(\frac{|aj|^{\alpha_1}}{8}+\frac{|ak|^{\alpha_2}}{8}\right)\exp\left(-\left(\frac{|aj|^{\alpha_1}}{8}+\frac{|ak|^{\alpha_2}}{8}\right)\right) \\
=\ & O\left(\frac{K''}{\log T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|x|^{\alpha_1}+|y|^{\alpha_2})e^{-(|x|^{\alpha_1}+|y|^{\alpha_2})}\,dxdy\right),
\end{aligned}$$

as $u \to \infty$. Since $\log T(u) \to \infty$, as $u \to \infty$, and an integral in the last statement is finite, the proof is completed. ∎

5.1 Proof of Theorem 2

Proof of (i). Let $\{X^{(j,k)}(s,t)\}_{j,k}$ be independent copies of $X(s,t)$ and let $\eta(s,t)$ be such that $\eta(s,t) = X^{(j,k)}(s,t)$ for $(s,t) \in [j-1,j) \times [k-1,k)$. For a fixed T we define a Gaussian random field Y_T as follows

$$Y_T(s,t) := \left(1-\frac{r}{\log T}\right)^{1/2}\eta(s,t) + \left(\frac{r}{\log T}\right)^{1/2}\mathcal{W}, \quad \text{for } (s,t)\in[0,T]^2, \tag{17}$$

where $\mathcal{W}$ is an $N(0,1)$ random variable independent of $\eta(s,t)$. Then the covariance of Y_T equals

$$Cov(Y_T(s_0,t_0), Y_T(s_0+s,t_0+t)) = \begin{cases} r(s,t)+(1-r(s,t))\frac{r}{\log T}, & \text{when } [s_0]=[s_0+s],[t_0]=[t_0+t]; \\ \frac{r}{\log T}, & \text{otherwise} \ , \end{cases}$$

for all $s_0, t_0, s, t \geqslant 0$.

Let $n_x := \lfloor x m_1(u) \rfloor$ and $n_y := \lfloor y m_2(u) \rfloor$. Since

$$P\left(\sup_{(s,t)\in[0,n_x+1]\times[0,n_y+1]} X(s,t) \leqslant u\right) \leqslant P\left(\sup_{(s,t)\in[0,xm_1(u)]\times[0,ym_2(u)]} X(s,t) \leqslant u\right) \leqslant P\left(\sup_{(s,t)\in[0,n_x]\times[0,n_y]} X(s,t) \leqslant u\right),$$

we focus on the asymptotics of $P\left(\sup_{(s,t)\in[0,n_x]\times[0,n_y]} X(s,t) \leqslant u\right)$, as $u \to \infty$. Let $\varepsilon > 0$. Divide $[0,n_x]\times[0,n_y]$ into $n_x n_y$ unit squares and then split them into subsets $I^*_{l,m}$ and $I_{l,m}$ as follows

$$\begin{aligned} I_{l,m} &= [(l-1)+\varepsilon, l] \times [(m-1)+\varepsilon, m], \\ I^*_{l,m} &= [l-1,l]\times[m-1,m] \ - \ I_{l,m}, \end{aligned}$$

where $l = 1,\ldots,n_x$, $m = 1,\ldots,n_y$.

Step 1. In the first step we prove that

$$\lim_{u\to\infty} \left| P\left(\sup_{(s,t)\in[0,n_x]\times[0,n_y]} X(s,t) \leqslant u\right) - P\left(\sup_{(s,t)\in\bigcup_{l=1}^{n_x}\bigcup_{m=1}^{n_y} I_{l,m}} X(s,t) \leqslant u\right)\right| \leqslant \rho_1(\varepsilon), \quad (18)$$

uniformly for $(x,y)\in[A_0,A_\infty]^2$ with $\rho_1(\varepsilon)\to 0$ as $\varepsilon \to 0$. This is a consequence of the following sequence of inequalities

$$\begin{aligned} 0 &\leqslant P\left(\sup_{(s,t)\in\bigcup_{l=1}^{n_x}\bigcup_{m=1}^{n_y} I_{l,m}} X(s,t) \leqslant u\right) - P\left(\sup_{(s,t)\in[0,n_x]\times[0,n_y]} X(s,t) \leqslant u\right) \\ &\leqslant n_x n_y P\left(\sup_{(s,t)\in I^*_{1,1}} X(s,t) > u\right) \leqslant A_\infty^2 m(u) P\left(\sup_{(s,t)\in I^*_{1,1}} X(s,t) > u\right) = (2\varepsilon-\varepsilon^2)A_\infty^2(1+o(1)), \end{aligned}$$

as $u\to\infty$, since

$$P\left(\sup_{(s,t)\in I^*_{1,1}} X(s,t) > u\right) = \frac{2\varepsilon-\varepsilon^2}{m(u)}(1+o(1)),$$

as $u \to \infty$, by [13, Theorem 7.1].

Step 2. Let $a>0$ and $q_1 = q_1(u) := au^{-\alpha_1/2}$, $q_2 = q_2(u) := au^{-\alpha_2/2}$. We show that

$$\lim_{u\to\infty}\left| P\left(X(s,t)\leqslant u; (s,t)\in \bigcup_{l=1}^{n_x}\bigcup_{m=1}^{n_y} I_{l,m}\right) - P\left(X(jq_1,kq_2)\leqslant u; (jq_1,kq_2)\in\bigcup_{l=1}^{n_x}\bigcup_{m=1}^{n_y} I_{l,m}\right)\right| \leqslant \rho_2(a), \quad (19)$$

uniformly for $(x,y)\in[A_0,A_\infty]^2$, with $\rho_2(a)\to 0$ as $a\to 0$. Indeed, (19) follows from the fact that

$$
\begin{aligned}
0 \quad &\leqslant \quad P\left(X(s,t) \leqslant u; (s,t) \in \bigcup_{l=1}^{n_x} \bigcup_{m=1}^{n_y} I_{l,m}\right) - P\left(X(jq_1, kq_2) \leqslant u; (jq_1, kq_2) \in \bigcup_{l=1}^{n_x} \bigcup_{m=1}^{n_y} I_{l,m}\right) \\
&\leqslant n_x n_y \max_{l,m} \left[P\left(X(jq_1, kq_2) \leqslant u; (jq_1, kq_2) \in I_{l,m}\right) - P\left(\sup_{(s,t)\in I_{l,m}} X(s,t) \leqslant u\right)\right] \\
&\leqslant n_x n_y (1-\varepsilon)^2 \left(\frac{\rho(a)}{m(u)} + o\left(\frac{1}{m(u)}\right)\right) \qquad (20) \\
&\leqslant A_\infty^2 \rho(a) + A_\infty^2 m(u) o\left(\frac{1}{m(u)}\right) \to A_\infty^2 \rho(a),
\end{aligned}
$$

as $u \to \infty$ with $\rho(a) \to 0$ as $a \to 0$. Inequality (20) is due to Lemma 1.

Step 3. In this step we show that for $T = T(u) := \max(A_\infty m_1(u), A_\infty m_2(u))$ we have

$$
\left|P\left(X(jq_1, kq_2) \leqslant u; (jq_1, kq_2) \in \bigcup_{l=1}^{n_x} \bigcup_{m=1}^{n_y} I_{l,m}\right) - P(Y_T(jq_1, kq_2) \leqslant u; (jq_1, kq_2) \in \bigcup_{l=1}^{n_x} \bigcup_{m=1}^{n_y} I_{l,m})\right| \to 0, \qquad (21)
$$

as $u \to \infty$, uniformly for $(x,y) \in [A_0, A_\infty]^2$.

Indeed, note that for sufficiently large T we have

$$
\begin{aligned}
\left|Cov(X(jq_1, kq_2), X(j'q_1, k'q_2)) - Cov(Y_T(jq_1, kq_2), Y_T(j'q_1, k'q_2))\right| \quad &\leqslant \quad \rho_T((j-j')q_1, (k-k')q_2), \\
\left|Cov(Y_T(jq_1, kq_2), Y_T(j'q_1, k'q_2))\right| \quad &\leqslant \quad \varrho_T((j-j')q_1, (k-k')q_2),
\end{aligned}
$$

for functions ρ_T and ϱ_T defined by (11).

Moreover, for small $\varepsilon > 0$ and $(jq_1, kq_2), (j'q_1, k'q_2) \in \bigcup_{l=1}^{n_x} \bigcup_{m=1}^{n_y} I_{l,m}$ satisfying $\max(|j - j'|q_1, |k-k'|q_2) < \varepsilon$ we get

$$
\left|Cov(X(jq_1, kq_2), X(j'q_1, k'q_2)) - Cov(Y_T(jq_1, kq_2), Y_T(j'q_1, k'q_2))\right| = (1 - r((j-j')q_1, (k-k')q_2)) \frac{r}{\log T}
$$

and

$$
\begin{aligned}
&\max\left(|Cov(X(jq_1, kq_2), X(j'q_1, k'q_2))|, |Cov(Y_T(jq_1, kq_2), Y_T(j'q_1, k'q_2)|\right) \\
&\quad = \quad Cov(Y_T(jq_1, kq_2), Y_T(j'q_1, k'q_2)) \\
&\quad = \quad r((j-j')q_1, (k-k')q_2) + (1 - r((j-j')q_1, (k-k')q_2)) \frac{r}{\log T}.
\end{aligned}
$$

Let $\delta_T = \sup\{\max(|r(s,t)|, \varrho_T(s,t));\ \max(|s|, |t|) \geqslant \varepsilon\}$. Observe that $\delta_T < \delta < 1$ for sufficiently large T. Applying [8, Theorem 4.2.1] we get

$$
\begin{aligned}
&\left| P\left(X(jq_1,kq_2)\leqslant u; (jq_1,kq_2)\in \bigcup_{l=1}^{n_x}\bigcup_{m=1}^{n_y} I_{l,m}\right) - P\left(Y_T(jq_1,kq_2)\leqslant u; (jq_1,kq_2)\in \bigcup_{l,m} I_{l,m}\right)\right| \\
\leqslant\ & \frac{1}{4\pi}\frac{n_x n_y}{q_1 q_2}\sum_{0<\max(|jq_1|,|kq_2|)<\varepsilon}\left[(1-r(jq_1,kq_2))\frac{r}{\log T}\right.\\
& \times\left(1-\left(r(jq_1,kq_2)+(1-r(jq_1,kq_2))\frac{r}{\log T}\right)^2\right)^{-1/2}\exp\left(-\frac{u^2}{1+r(jq_1,kq_2)+(1-r(jq_1,kq_2))\frac{r}{\log T}}\right)\Bigg]\\
+\ & \frac{1}{4\pi}(1-\delta^2)^{-1/2}\frac{n_x n_y}{q_1 q_2}\sum_{(jq_1,kq_2)\in[-n_x,n_x]\times[-n_y,n_y]-(-\varepsilon,\varepsilon)^2}\Bigg[\rho_T(jq_1,kq_2)\\
& \times\exp\left(-\frac{u^2}{1+\max(|r(jq_1,kq_2)|,\varrho_T(jq_1,kq_2))}\right)\Bigg]\\
\leqslant\ & \frac{1}{4\pi}\frac{A_\infty^2 m(u)}{q_1 q_2}\sum_{0<\max(|jq_1|,|kq_2|)<\varepsilon}\left[(1-r(jq_1,kq_2))\frac{r}{\log T}\right.\\
& \times\left(1-\left(r(jq_1,kq_2)+(1-r(jq_1,kq_2))\frac{r}{\log T}\right)^2\right)^{-1/2}\exp\left(-\frac{u^2}{1+r(jq_1,kq_2)+(1-r(jq_1,kq_2))\frac{r}{\log T}}\right)\Bigg]\\
+\ & \frac{1}{4\pi}(1-\delta^2)^{-1/2}\frac{A_\infty^2 m(u)}{q_1 q_2}\times\sum_{(jq_1,kq_2)\in[-A_\infty m_1(u),A_\infty m_1(u)]\times[-A_\infty m_2(u),A_\infty m_2(u)]-(-\varepsilon,\varepsilon)^2}\Bigg[\rho_T(jq_1,kq_2)\\
& \times\exp\left(-\frac{u^2}{1+\max(|r(jq_1,kq_2)|,\varrho_T(jq_1,kq_2))}\right)\Bigg]\\
=:\ & I_1+I_2.
\end{aligned}
$$

Observe that, due to Lemma 3, I_1 tends to 0 as $u\to\infty$. Analogously, by Lemma 2, I_2 tends to 0 as $u\to\infty$. Hence we have shown (21).

Step 4. By definition of the random field Y_T, we have

$$
\begin{aligned}
& P\left(Y_T(jq_1,kq_2)\leqslant u; (jq_1,kq_2)\in\bigcup_{l,m} I_{l,m}\right)\\
&= P\left(\left(1-\frac{r}{\log T}\right)^{1/2}\eta(jq_1,kq_2)+\left(\frac{r}{\log T}\right)^{1/2}\mathcal{W}\leqslant u;\ (jq_1,kq_2)\in\bigcup_{l,m} I_{l,m}\right)\\
&= P\left(\left(1-\frac{r}{\log T}\right)^{1/2}\sup_{(jq_1,kq_2)\in\bigcup_{l,m} I_{l,m}}\eta(jq_1,kq_2)+\left(\frac{r}{\log T}\right)^{1/2}\mathcal{W}\leqslant u\right)\\
&= \int_{-\infty}^{\infty} P\left(\sup_{(jq_1,kq_2)\in\bigcup_{l,m} I_{l,m}}\eta(jq_1,kq_2)\leqslant\frac{u-(r/\log T)^{1/2}z}{(1-r/\log T)^{1/2}}\right)d\Phi(z). \qquad (22)
\end{aligned}
$$

Then for any $z\in\mathbb{R}$

$$
\begin{aligned}
u_z &:= \frac{u-(r/\log T)^{1/2}z}{(1-r/\log T)^{1/2}}\\
&= \left(u-(r/\log T)^{1/2}z\right)\left(1+\frac{1}{2}(r/\log T)+o(r/\log T)\right)\\
&= u+\frac{-2\sqrt{r}z+2r}{u}+o(1/u),
\end{aligned}
$$

as $u \to \infty$, and thus

$$\frac{1}{m(u_z)} = \frac{\exp(-2r + 2\sqrt{r}z)}{m(u)}(1 + o(1)).$$

Hence, we get

$$\begin{aligned} P\left(\sup_{(jq_1,kq_2)\in\bigcup_{l,m} I_{l,m}} \eta(jq_1, kq_2) \leqslant u_z\right) &= \prod_{l,m} P\left(\sup_{(jq_1,kq_2)\in I_{l,m}} X(jq_1, kq_2) \leqslant u_z\right) \\ &= P\left(\sup_{(s,t)\in[0,1]^2} X(s,t) \leqslant u_z\right)^{n_x n_y} (1 + o(1)) \\ &= \left(1 - \frac{1}{m(u_z)}\right)^{xym(u)} (1 + o(1)) \\ &= \exp(-xy\exp(-2r + 2\sqrt{r}z))(1 + o(1)), \qquad (23) \end{aligned}$$

as $u \to \infty$, uniformly for $(x, y) \in [A_0, A_\infty]^2$. Combining (18), (19), (21), (22) and (23) and passing with $\varepsilon \to 0$ and $a \to 0$, we conclude that the proof of (i) is completed.

Proof of (ii). Let $\mathcal{T} \subset \mathbb{R}^2$ be Jordan-measurable with Lebesgue measure $\mathrm{mes}(\mathcal{T}) > 0$. For given $\varepsilon > 0$, let $\mathcal{L}_\varepsilon, \mathcal{U}_\varepsilon \subset \mathbb{R}^2$ be *simple sets* (i.e. finite sums of disjoint rectangles of the form $[a_1, b_1) \times [a_2, b_2)$) such that $\mathcal{L}_\varepsilon \subset \mathcal{T} \subset \mathcal{U}_\varepsilon$ and $\mathrm{mes}(\mathcal{L}_\varepsilon) > \mathrm{mes}(\mathcal{T}) - \varepsilon$, $\mathrm{mes}(\mathcal{U}_\varepsilon) < \mathrm{mes}(\mathcal{T}) + \varepsilon$. Then, following line-by-line the same argument as given in the proof of part (i) of Theorem 2, for $\mathcal{T}_u = \{(x, y) : (x/m_1(u), y/m_2(u)) \in \mathcal{T}\}$, $\mathcal{L}_{\varepsilon,u} = \{(x, y) : (x/m_1(u), y/m_2(u)) \in \mathcal{L}_\varepsilon\}$, $\mathcal{U}_{\varepsilon,u} = \{(x, y) : (x/m_1(u), y/m_2(u)) \in \mathcal{U}_\varepsilon\}$ we have

$$P\left(\sup_{(s,t)\in\mathcal{L}_{\varepsilon,u}} X(s,t) \leqslant u\right) \to E\big(\exp(-\mathrm{mes}(\mathcal{L}_\varepsilon)\exp(-2r + 2\sqrt{r}\mathcal{W}))\big)$$

and

$$P\left(\sup_{(s,t)\in\mathcal{U}_{\varepsilon,u}} X(s,t) \leqslant u\right) \to E\big(\exp(-\mathrm{mes}(\mathcal{U}_\varepsilon)\exp(-2r + 2\sqrt{r}\mathcal{W}))\big),$$

as $u \to \infty$. Thus,

$$P\left(\sup_{(s,t)\in\mathcal{T}_u} X(s,t) \leqslant u\right) \to E\big(\exp(-\mathrm{mes}(\mathcal{T})\exp(-2r + 2\sqrt{r}\mathcal{W}))\big),$$

as $u \to \infty$.

5.2 Proof of Proposition 1

Since the proof of Proposition 1 is analogous to proofs of Theorems 3.1-3.3 in [1], see also Theorem A in [15], we focus only on arguments for (ii).

Let $0 < A_0 < A_\infty$. We have

$$\begin{aligned} &P\left(\sup_{(s,t)\in\mathcal{B}(0,T)} X(s,t) > u\right) = \\ &= \int_0^{A_0\sqrt{m(u)}} P\left(\sup_{(s,t)\in\mathcal{B}(0,x)} X(s,t) > u\right) dF_T(x) + \int_{A_0\sqrt{m(u)}}^{A_\infty\sqrt{m(u)}} P\left(\sup_{(s,t)\in\mathcal{B}(0,x)} X(s,t) > u\right) dF_T(x) \\ &\quad + \int_{A_\infty\sqrt{m(u)}}^{\infty} P\left(\sup_{(s,t)\in\mathcal{B}(0,x)} X(s,t) > u\right) dF_T(x) = I_1 + I_2 + I_3. \end{aligned}$$

Then, for each $\varepsilon > 0$, due to Remark 1, for sufficiently large u, we get

$$\begin{aligned} I_2 &\leq (1+\varepsilon)\int_{A_0}^{A_\infty}(1-E\big(\exp(-\pi x^2\exp(\mathcal{V}_r))\big)dF_T(x\sqrt{m(u)}) \\ &= (1+\varepsilon)\int_{A_0}^{A_\infty} 2\pi x E\big(\exp(-\pi x^2\exp(\mathcal{V}_r)+\mathcal{V}_r)\big)P(T > x\sqrt{m(u)})dx \\ &\quad -(1+\varepsilon)\left(1-E\big(\exp(-\pi A_\infty^2\exp(\mathcal{V}_r))\big)\right)P(T > A_\infty\sqrt{m(u)}) \\ &\quad +(1+\varepsilon)\left(1-E\big(\exp(-\pi A_0^2\exp(\mathcal{V}_r))\big)\right)P(T > A_0\sqrt{m(u)}), \end{aligned}$$

where $\mathcal{V}_r = 2\sqrt{r}\mathcal{W} - 2r$. Hence, using the fact that T is regularly varying,

$$\begin{aligned} \limsup_{u\to\infty}\frac{I_2}{P(T > \sqrt{m(u)})} &\leq (1+\varepsilon)2\pi\int_{A_0}^{A_\infty} x^{1-\lambda}E\big(\exp(-\pi x^2\exp(\mathcal{V}_r)+\mathcal{V}_r)\big)dx \\ &\quad -(1+\varepsilon)\left(1-E\big(\exp(-\pi A_\infty^2\exp(\mathcal{V}_r))\big)\right)A_\infty^{-\lambda} \\ &\quad +(1+\varepsilon)\left(1-E\big(\exp(-\pi A_0^2\exp(\mathcal{V}_r))\big)\right)A_0^{-\lambda}. \end{aligned}$$

In an analogous way we get that

$$\begin{aligned} \liminf_{u\to\infty}\frac{I_2}{P(T > \sqrt{m(u)})} &\geqslant (1-\varepsilon)2\pi\int_{A_0}^{A_\infty} x^{1-\lambda}E\big(\exp(-\pi x^2\exp(\mathcal{V}_r)+\mathcal{V}_r)\big)dx \\ &\quad -(1-\varepsilon)\left(1-E\big(\exp(-\pi A_\infty^2\exp(\mathcal{V}_r))\big)\right)A_\infty^{-\lambda} \\ &\quad +(1-\varepsilon)\left(1-E\big(\exp(-\pi A_0^2\exp(\mathcal{V}_r))\big)\right)A_0^{-\lambda}. \end{aligned}$$

Then, following the same argument as in the proof of Theorem 3.2 in [1], we conclude that $I_1 + I_3 = o(P(T > \sqrt{m(u)}))$ as $u \to \infty$.

Now, passing with $A_0 \to 0$, $A_\infty \to \infty$ and $\varepsilon \to 0$, we conclude that

$$I_2 = 2\pi\int_0^\infty x^{1-\lambda}E\big(\exp(-\pi x^2\exp(\mathcal{V}_r)+\mathcal{V}_r)\big)dx P(T > \sqrt{m(u)})(1+o(1)),$$

as $u \to \infty$.

5.3 Proof of Proposition 2

Proof of (i). Assume that **A3** is satisfied with $r = 0$. Then, by definition of $\{\widetilde{X}_{j,k}\}$, it suffices to show that for the original Gaussian field $\{X(s,t) : s,t \geq 0\}$

$$P\left(\sup_{(s,t)\in[0,f(u)]\times[0,g(u)]} X(s,t) \leqslant z(u)\right) - P\left(\sup_{(s,t)\in[0,1]^2} X(s,t) \leqslant z(u)\right)^{f(u)g(u)} \to 0 \qquad (24)$$

as $u \to \infty$, for each function $z : \mathbb{R}_+ \to \mathbb{R}$ and all pairs of functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(u) \to \infty$ and $g(u) \to \infty$, as $u \to \infty$, and $1/C \leqslant f(u)/g(u) \leqslant C$ for some fixed $C > 0$. Observe that it suffices to consider two cases: continuous $z(u) \nearrow \infty$, as $u \to \infty$, and $z(u) < Const$. We focus on the first case and suppose that $z(u)$ increases to infinity. Then (24) is equivalent to

$$P\left(\sup_{(s,t)\in[0,f^*(u)]\times[0,g^*(u)]} X(s,t) \leqslant u\right) - P\left(\sup_{(s,t)\in[0,1]^2} X(s,t) \leqslant u\right)^{f^*(u)g^*(u)} \to 0, \qquad (25)$$

as $u \to \infty$, with z^{-1} being the inverse function for z and $f^*(u) := f(z^{-1}(u))$, $g^*(u) := g(z^{-1}(u))$.

By (i) of Theorem 2,

$$P\left(\sup_{(s,t)\in\left[0,x\sqrt{m(u)}\right]\times\left[0,y\sqrt{m(u)}\right]} X(s,t) \leqslant u\right) \to e^{-xy}, \qquad (26)$$

as $u \to \infty$, uniformly for $(x,y) \in \mathcal{F}(C) := \left\{(s,t) \in \mathbb{R}_+^2 : 1/C \leqslant s/t \leqslant C\right\} \cup \{0,0\}$, for an arbitrary constant $C > 0$. Moreover the uniform convergence

$$P\left(\sup_{(s,t)\in[0,1]^2} X(s,t) \leqslant u\right)^{xy\cdot m(u)} \to e^{-xy} \tag{27}$$

occurs on the set $\mathcal{F}(C)$.

Let $\bar{f}(u) := f\left(z^{-1}(u)\right)/\sqrt{m(u)}$ and $\bar{g}(u) := g\left(z^{-1}(u)\right)/\sqrt{m(u)}$. The fundamental observation is that it is sufficient to prove (24) for $f(u)$ and $g(u)$ satisfying the additional assumption: $\bar{f}(u) \to a \in [0,\infty]$ and $\bar{g}(u) \to b \in [0,\infty]$, as $u \to \infty$.

Note that $1/C \leqslant f(u)/g(u) \leqslant C$ implies $1/C \leqslant \bar{f}(u)/\bar{g}(u) \leqslant C$. Since the convergence in (26) is uniform, we obtain

$$P\left(\sup_{(s,t)\in[0,f^*(u)]\times[0,g^*(u)]} X(s,t) \leqslant u\right) = P\left(\sup_{(s,t)\in[0,\bar{f}(u)\sqrt{m(u)}]\times[0,\bar{g}(u)\sqrt{m(u)}]} X(s,t) \leqslant u\right) \to e^{-ab},$$

as $u \to \infty$. On the other hand, by (27),

$$P\left(\sup_{(s,t)\in[0,1]^2} X(s,t) \leqslant u\right)^{f^*(u)g^*(u)} = P\left(\sup_{(s,t)\in[0,1]^2} X(s,t) \leqslant u\right)^{\bar{f}(u)\bar{g}(u)\cdot m(u)} \to e^{-ab},$$

as $u \to \infty$, which gives (24).

Proof of (ii). Let us consider the case $r > 0$. Note that for $\mathcal{V}_r = 2\sqrt{r}\mathcal{W} - 2r$ it holds that

$$\begin{aligned} Var\left(\exp(-\exp(\mathcal{V}_r))\right) &= E\left(\exp\left(-2\exp(\mathcal{V}_r)\right)\right) - E\left(\exp(-\exp(\mathcal{V}_r))\right)^2 \\ &= P\left(\max_{j\leqslant 2\left\lfloor\sqrt{m(u)}\right\rfloor, k\leqslant\left\lfloor\sqrt{m(u)}\right\rfloor} \widetilde{X}_{j,k} \leqslant u\right) - P\left(\max_{j,k\leqslant\left\lfloor\sqrt{m(u)}\right\rfloor} \widetilde{X}_{j,k} \leqslant u\right)^2 + o(1), \end{aligned}$$

due to Theorem 2. By contradiction, assume that the extremal index exists and equals $\theta \in (0,1]$. Then for any sequence $(z_n) \subset \mathbb{R}$ we have

$$\begin{aligned} & P\left(\max_{j\leqslant\left\lfloor 2\sqrt{m(z_n)}\right\rfloor, k\leqslant\left\lfloor\sqrt{m(z_n)}\right\rfloor} \widetilde{X}_{j,k} \leqslant z_n\right) - P\left(\max_{j,k\leqslant\left\lfloor\sqrt{m(z_n)}\right\rfloor} \widetilde{X}_{j,k} \leqslant z_n\right)^2 \\ &= \left(P\left(\max_{j\leqslant 2\left\lfloor\sqrt{m(z_n)}\right\rfloor, k\leqslant\left\lfloor\sqrt{m(z_n)}\right\rfloor} \widetilde{X}_{j,k} \leqslant z_n\right) - P\left(\widetilde{X}_{1,1} \leqslant z_n\right)^{2m(z_n)\cdot\theta}\right) \\ &\quad - \left(P\left(\max_{j,k\leqslant\left\lfloor\sqrt{m(z_n)}\right\rfloor} \widetilde{X}_{j,k} \leqslant z_n\right)^2 - \left(P\left(\widetilde{X}_{1,1} \leqslant z_n\right)^{m(z_n)\cdot\theta}\right)^2\right) = o(1), \end{aligned}$$

as $n \to \infty$, which implies that $Var\left(\exp(-\exp(\mathcal{V}_r))\right) = 0$. Keeping in mind that $r > 0$ and $\mathcal{W}$ is an $N(0,1)$ random variable, we obtain a contradiction.

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References

[1] Arendarczyk, M., Dębicki, K. (2012) Exact asymptotics of supremum of a stationary Gaussian process over a random interval. *Statistics & Probability Letters* **82**, 645–652.

[2] Billingsley, P. (1995) Probability and Measure. Wiley-Interscience, New York.

[3] Hsing, T. (1993) Extremal index extimation for a weakly dependent stationary point sequence. *Ann. Statist.* **21** (4) 2043–2071.

[4] Hsing, T., Hüsler, J., Leadbetter, M.R. (1988) On the exceedance point process for stationary sequence. *Probab. Theory Related Fields* **78** 97–112.

[5] Jakubowski, A., Soja-Kukieła, N.. (2013) Managing local dependencies in limit theorems for maxima of stationary random fields, *in preparation.*

[6] Jakubowski, A. (1991) Relative extremal index of two stationary processes. *Stochastic Processes and their Applications* **37** (2), 281–297.

[7] Leadbetter M.R. (1983) Extremes and local dependence in stationary sequences. *Z. Wahr. Ver. Geb.* **65** 291–306.

[8] Leadbetter, M.R., Lindgren, G., Rootzén, H. (1983) Extremes and Related Properties of Random Sequences and Processes. Springer-Verlag.

[9] Leadbetter, M.R., Weissman, I., de Haan, L., Rootzén H. (1989) On clustering of high levels in statistically stationary series. In: J. Samson (Ed.), *Proc. 4th Int. Meet. Statistical Climatology New Zealand Meteorological Service*, Willington.

[10] O'Brien, G. (1987) Extreme values for stationary and Markov sequences. *Ann. Probab.* **15** 281–191.

[11] Pickands, J.III (1969) Asymptotic properties of maximum in a stationary Gaussian process. *Trans. Amer. Soc.* **145**, 75–86.

[12] Piterbarg, V.I. (1972) On the paper by J. Pickands "Upcrosssing probabilities for stationary Gaussian processes". Vestnik Moscow. Univ. Ser. I Mat. Mekh. 27, 25–30. English translation in Moscow Univ. Math. Bull., 27.

[13] Piterbarg, V.I. (1996) Asymptotic Methods in the Theory of Gaussian Processes and Fields. In: Transl. Math. Monographs, vol. 148. AMS, Providence, RI.

[14] Rudin, W. (1976) Principles of Mathematical Analysis, 3rd ed., McGraw-Hill, New York.

[15] Tan Z., Hashorva E. (2013) Limit theorems for extremes of strongly dependent cyclo-stationary χ-processes. Lithuanian Math. J., **53**, 91–102.

[16] Weissman, I., Novak, S.Yu. (1998) On blocks and runs estimators for the extremal index. *J. Stat. Plan. Inference* **66** 281-288.