EXTREMES OF GAUSSIAN RANDOM FIELDS WITH REGULARLY VARYING DEPENDENCE STRUCTURE

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Abstract: Let $X(t), t \in \mathcal{T}$ be a centered Gaussian random field with variance function $\sigma^2(\cdot)$ that attains its maximum at the unique point $t_0 \in \mathcal{T}$, and let $M(\mathcal{T}) = \sup_{t\in\mathcal{T}} X(t)$. For $\mathcal{T}$ a compact subset of $\mathbb{R}$, the current literature explains the asymptotic tail behaviour of $M(\mathcal{T})$ under some regularity conditions including that $1-\sigma(t)$ has a polynomial decrease to 0 as $t \to t_0$. In this contribution we consider more general case that $1-\sigma(t)$ is regularly varying at t_0. We extend our analysis to Gaussian random fields defined on some compact set $\mathcal{T} \subset \mathbb{R}^2$, deriving the exact tail asymptotics of $M(\mathcal{T})$ for the class of Gaussian random fields with variance and correlation functions being regularly varying at t_0. A crucial novel element is the analysis of families of Gaussian random fields that do not possess locally additive dependence structures, which leads to qualitatively new types of asymptotics.

Key Words: Non-stationary Gaussian processes; Gaussian random fields; extremes; fractional Brownian motion; regular variation; uniform approximation

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1. Introduction

Let $X(t), t \geq 0$ be a centered stationary Gaussian processes with continuous trajectories, unit variance and correlation function $r(\cdot)$ satisfying Pickands's condition

$$1 - r(t) \sim a|t|^\alpha, \quad t \downarrow 0, \qquad \text{and } r(t) < 1, \forall t \neq 0, \tag{1}$$

with $a > 0, \alpha \in (0, 2]$. In our notation $\sim$ means asymptotic equivalence when the argument tends to 0 or ∞.
In the seminal contribution [1] it is shown that under (1), for any T positive

$$\mathbb{P}\left(\sup_{t\in[0,T]} X(t) > u\right) \sim T\mathcal{H}_\alpha(\sqrt{a}u)^{2/\alpha}\mathbb{P}(X(0) > u), \qquad u \to \infty, \tag{2}$$

where the Pickands constant $\mathcal{H}_\alpha$ is defined by

$$\mathcal{H}_\alpha = \lim_{S\to\infty} S^{-1}\mathcal{H}_\alpha[0, S] \in (0, \infty) \quad \text{with} \quad \mathcal{H}_\alpha[S_1, S_2] = \mathbb{E}\left\{\sup_{t\in[S_1,S_2]} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha}\right\}, \quad S_1 < S_2,$$

with $B_\alpha(t), t \geq 0$ a standard fractional Brownian motion (fBm) with Hurst index $\alpha/2 \in (0, 1]$, see [1–15] for various properties of $\mathcal{H}_\alpha$ and related constants.
The asymptotics in (2) is extended in various directions, including $\alpha(t)$-locally-stationary Gaussian processes (see [16]), and general non-stationary Gaussian processes and random fields, see e.g., [11]. A particularly important place in this theory is taken by the result of Piterbarg and Prisjažnjuk [17], where the exact tail asymptotics of $\sup_{t\in[0,T]} X(t)$ is derived in the case that the variance function σ^2 of a centered Gaussian process X has a unique point of maximum in $[0, T]$, say t_0. For simplicity assume that $t_0 \in (0, T)$ and $\sigma(t_0) = 1$. Similarly to the stationary case, in [17] it is assumed that the correlation function $r(s, t) = \mathrm{Corr}(X(s), X(t))$ satisfies for some $a > 0, \alpha \in (0, 2]$

$$1 - r(s, t) \sim a|t - s|^\alpha, \quad s, t \to t_0, \tag{3}$$

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whereas the behaviour of the variance function around the unique maximizer t_0 satisfies

$$1-\sigma(t) \quad \sim \quad b|t-t_0|^{\beta}, \quad t \to t_0 \tag{4}$$

for some $b, \beta > 0$. Assume further that for $C > 0, \nu \in (0, 2]$ the following Hölder continuity condition

$$\mathbb{E}\left\{(X(t)-X(t))^2\right\} \le C|t-s|^{\nu}, \quad \forall s,t \in [t_0-\theta, t_0+\theta] \tag{5}$$

is valid for some small positive θ. By [17], if $\alpha < \beta$

$$\mathbb{P}\left(\sup_{t\in[0,T]} X(t) > u\right) \sim 2\mathcal{H}_\alpha \frac{a^{1/\alpha}}{b^{1/\beta}}\Gamma(1/\beta+1)u^{2/\alpha-2/\beta}\mathbb{P}\left(X(0)>u\right), \tag{6}$$

and for $\alpha = \beta$

$$\mathbb{P}\left(\sup_{t\in[0,T]} X(t) > u\right) \sim \mathcal{P}_\alpha^{b/a}\mathbb{P}\left(X(0)>u\right), \tag{7}$$

where $\mathcal{P}_\alpha^d, d > 0$ is the Piterbarg constant defined by

$$\mathcal{P}_\alpha^d = \lim_{S\to\infty} \mathcal{P}_\alpha^d[-S,S] \in (0,\infty), \quad \text{with} \quad \mathcal{P}_\alpha^d[S_1,S_2] = \mathbb{E}\left\{\sup_{t\in[S_1,S_2]} e^{\sqrt{2}B_\alpha(t)-(1+d)|t|^\alpha}\right\}, \quad S_1 < S_2.$$

When $\alpha > \beta$, then (7) holds with 1 instead of $\mathcal{P}_\alpha^{b/a}$; see also Theorem 2.1 in [10] for the case $T = \infty$.
We note in passing that in fact the Hölder continuity (5) is not needed to derive the asymptotics of (2), which will be shown later in our main theorems; necessary and sufficient conditions that guarantee the global Hölder continuity of X are presented in [18].

The original Pickands assumption (1) and its counterpart (3) can be relaxed to $1-r$ being regularly varying at 0 with index $\alpha \in (0,2]$, see [19, 20]. Specifically, in the case of a non-stationary X we shall assume for some non-negative $\rho \in \mathcal{R}_{\alpha/2}, \alpha \in (0,2]$

$$1-r(s,t) \quad \sim \quad \rho^2(|t-s|), \quad s,t \to t_0. \tag{8}$$

Here $f \in \mathcal{R}_\gamma$ means that the function f is regularly varying at 0 with index γ, see [21–23] for details.
The first goal of this contribution is to extend Piterbarg's results to a more general setup, that is to suppose that t_0 is the unique maximizer of $\sigma^2(t)$ over $[0,T]$, $\sigma^2(t_0) = 1$ and

$$1-\sigma(t) \sim v^2(|t-t_0|), \quad t \to t_0, \tag{9}$$

where $v \ge 0$ and $v \in \mathcal{R}_{\beta/2}, \beta > 0$. In Theorem 2.1 we show that the asymptotic tail behaviour of $\sup_{t\in[0,T]} X(t)$ can be determined under the assumption that $1-\sigma$ can be compared with $1-\rho$, namely if further

$$\lim_{t\downarrow 0} \frac{v^2(t)}{\rho^2(t)} = \gamma \in [0,\infty]. \tag{10}$$

Note that, in Piterbarg's result mentioned above the limit γ is assumed to exist.
Our second goal is to analyze the tail distribution asymptotics of supremum of centered Gaussian random field $X(s,t), s \in [-T_1, T_1], t \in [T_2, T_2]$ with unique point that maximizes its variance function, say $(0,0)$. Although extremes of Gaussian random fields with regularly varying correlation function are discussed in [19], see also [24–32] for new developments on extremes of Gaussian random fields, most of the results in the existing literature are focused on the analysis of Gaussian random fields with *locally additive* dependence structure, that is if for $a_i, b_i, i = 1,2$ positive

$$\mathrm{Var}\big(X(0,0)\big) - \mathrm{Var}\big(X(s,t)\big) \sim a_1|s|^{\beta_1} + a_2|t|^{\beta_2}$$

and

$$1-\mathrm{Corr}(X(s,t), X(s_1,t_1)) \sim b_1|s-s_1|^{\alpha_1} + b_2|t-t_1|^{\alpha_2}$$

as $s, s_1 \to 0$, $t, t_1 \to 0$. It turns out that the investigation of Gaussian random fields that do not satisfy this properties is considerably more delicate. In Section 3 we derive several novel results concerned with the exact tail asymptotics of the maximum of centered Gaussian random fields when both the variance and the correlation functions are regularly varying and do not possess a locally additive structure.

Brief outline of the rest of the paper: Our main result concerning extremes of Gaussian processes is displayed in the Section 2, whereas Section 3 covers Gaussian random fields. The proofs of the theorems are presented in Section 4, whereas some technical results and their proofs are relegated to Appendix A and B.

2. Gaussian Processes

Before continuing with our investigation, we mention first that there are indeed important cases of Gaussian processes that satisfy our general setup in Section 1. Indeed, as remarked in [33] and [34], for any function $\rho^2 \in \mathcal{R}_\alpha, \alpha \in (0,2]$ there exists a centered stationary Gaussian process Y with continuous trajectories, unit variance and correlation function r satisfying (8); see the deep contribution [35] for results on sample path properties of Gaussian random fields. Clearly, for any continuous function $\sigma(t), t \geq 0$ the process $X(t) = \sigma(t)Y(t), t \geq 0$ has continuous trajectories and variance function σ^2.
One instance for the properties of σ is to assume that (9) holds with

$$v^2(t) = b|\ln t|^c t^\beta, \quad b > 0, c \in \mathbb{R}, \beta > 0.$$

For such σ, only the case $c = 0$ can be dealt with using Piterbarg's result mentioned in the Introduction. For example, if $\alpha < \beta$, it is tempting to write $v^2(t) = b(|\ln t|^{c/\beta} t)^\beta$. Then, using that in Piterbarg's result condition (4) explains term $u^{-2/\beta}$ in the asymptotic expansion in (6), the above could imply that (6) still holds if we replace $u^{-2/\beta}$ by $|\ln u|^{-2c/\beta^2} u^{-2/\beta}$. Detailed calculations show that these heuristics are misleading, and in fact the problem is much more complicated and complex. Indeed, the tail asymptotics of the supremum is determined in this case in terms of the (unique) asymptotic inverse of v, which is given by (see Example 1.24 in [23] or Lemma 2 in [36])

$$\overleftarrow{v}(t) \sim \left(\frac{\beta}{2}\right)^{c/\beta} b^{-1/\beta} |\ln t|^{-c/\beta} t^{2/\beta}, \quad t \downarrow 0,$$

where $\overleftarrow{f}$ denotes the asymptotic (unique) inverse of $f \in \mathcal{R}_\gamma$ defined by

$$\overleftarrow{f}(x) = \inf\{y \in (0,1] : f(y) > x\}, \quad x > 0.$$

See, e.g., [23] and [37] for the definitions and properties of asymptotic inverse functions.
Hereafter all regularly varying functions at 0 are assumed to be ultimately non-negative as $t \to 0$. Further $\Psi(u) \sim e^{-u^2/2}/(\sqrt{2\pi}u)$ as $u \to \infty$, denotes the survival function of an $N(0,1)$ random variable.

We state next the main result of this section.

Theorem 2.1. *Let $X(t), t \geq 0$ be a centered Gaussian process with continuous trajectories and variance function σ^2 having unique maximum at $t_0 \in [0,T]$ with $\sigma(t_0) = 1$. Suppose that σ satisfies (9) and the correlation function r of X satisfies (8). Assume further that condition (10) is valid for some $\gamma \in [0,\infty]$.*
i) If $\gamma = 0$, then

$$\mathbb{P}\left(\sup_{t\in[0,T]} X(t) > u\right) \sim C\Gamma(1/\beta+1)\frac{\overleftarrow{v}(1/u)}{\overleftarrow{\rho}(1/u)}\Psi(u),$$

with $C = 2\mathcal{H}_\alpha$ for $t_0 \in (0,T)$ and $C = \mathcal{H}_\alpha$ if $t_0 \in \{0, T_0\}$.
ii) If $\gamma \in (0,\infty]$, then

$$\mathbb{P}\left(\sup_{t\in[0,T]} X(t) > u\right) \sim C\Psi(u),$$

where $C = \mathcal{P}_\alpha^\gamma$ if $t_0 \in (0,T)$ and $C = \mathcal{P}_\alpha^\gamma[0,\infty)$ otherwise. Set $C = 1$ if $\gamma = \infty$.

Remarks 2.2. *Since Theorem 2.1 remain valid if we substitute v by an asymptotically equivalent v^*, we can assume that $v^2(t) = \ell_\sigma(t)t^\beta$ with ℓ_σ a normalized slowly varying function (see e.g., [23, 37]). Similarly, let $\rho^2(t) = \ell_\rho(t)t^\alpha$ with ℓ_ρ another normalized slowly varying function. Set next*

$$\ell_{\rho,\alpha}(x) = \ell_\rho(x^{1/\alpha}), \quad \ell_{\sigma,\beta}(x) = \ell_\sigma(x^{1/\beta}).$$

If further $x\ell^{\sharp}_{\sigma,\beta}(x)$ and $x\ell^{\sharp}_{\rho,\alpha}(x)$ denote the asymptotic inverses of $x\ell_{\sigma,\beta}(x)$ and $x\ell_{\rho,\alpha}(x)$ respectively, then we have

$$v(x)=\sqrt{\ell_{\sigma,\beta}(x^\beta)}x^{\beta/2},\quad \rho(x)=\sqrt{\ell_{\rho,\alpha}(x^\alpha)}x^{\alpha/2}$$

and thus by Example 1.24 in [23] as $t\to 0$

$$\overleftarrow{v}(t)\sim[\ell^{\sharp}_{\sigma,\beta}(t^2)]^{1/\beta}t^{2/\beta},\quad \overleftarrow{\rho}(t)\sim[\ell^{\sharp}_{\rho,\beta}(t^2)]^{1/\alpha}t^{2/\alpha}.$$

Consequently,

$$\frac{\overleftarrow{v}(1/u)}{\overleftarrow{\rho}(1/u)}\sim u^{2/\alpha-2/\beta}\frac{[\ell^{\sharp}_{\sigma,\beta}(1/u^2)]^{1/\beta}}{[\ell^{\sharp}_{\rho,\beta}(1/u^2)]^{1/\alpha}},\quad u\to\infty.$$

Theorem 2.1 is useful also when dealing with the additive Gaussian random field. Specifically, assume that for $T_1,T_2>0$

$$X(s,t)=\eta_1(s)+\eta_2(t),\quad s\in[-T_1,T_1],t\in[-T_2,T_2],$$

with η_1,η_2 two independent centered Gaussian processes with continuous trajectories. If both η_1 and η_2 are stationary satisfying (1), or η_1 and η_2 satisfy the conditions of Theorem 2.1, then

$$\mathbb{P}\left(\sup_{t\in[-T_i,T_i]}\eta_i(t)>u\right)\sim\mathcal{L}_i(u)u^{\tau_i}e^{-u^2/2}$$

for some $\tau_i\ge -1$, with $\mathcal{L}_i(x)=L>0, x\ge 0$ if $\tau_i=-1$ and $\mathcal{L}_i$ slowly varying at infinity if $\tau_i>-1$. Hence, since

$$\sup_{s\in[-T_1,T_1],t\in[-T_2,T_2]}X(s,t)=\sup_{s\in[-T_1,T_1]}\eta_1(s)+\sup_{t\in[-T_2,T_2]}\eta_2(t),$$

then Lemma 2.3 in [38] implies

$$\mathbb{P}\left(\sup_{s\in[-T_1,T_1],t\in[-T_2,T_2]}X(s,t)>u\right)\sim\sqrt{2\pi}\mathcal{L}_1(u)\mathcal{L}_2(u)u^{\tau_1+\tau_2-1}e^{-u^2/4},\quad u\to\infty. \tag{11}$$

In the particular case that η_i's satisfy the conditions of Theorem 2.1 with $\rho_i,v_i,i=1,2$ instead of ρ and v, where

$$\lim_{t\downarrow 0}\frac{v_i^2(t)}{\rho_i^2(t)}=\gamma_i\in[0,\infty], i=1,2, \tag{12}$$

then (11) can be stated more explicitly, see Theorem 3.1 below.
As we show in the next section, general Gaussian random fields are much more complex to deal with, and the results cannot be derived from Theorem 2.1.

3. Gaussian Random Fields

Extremes of locally additive Gaussian random fields with regularly varying correlation function are discussed in [19]. However, there are no results in the literature if the variance function is determined in terms of regularly varying functions and the dependence structure is not additive. In order to motivate our study, we consider first the additive Gaussian random field $X(s,t)=\eta_1(s)+\eta_2(t), s\in[-T_1,T_1], t\in[-T_2,T_2]$ introduced in Section 2. Thus, using that the variance function $\sigma^2(s,t)$ of $X(s,t)$ is simply given by

$$\sigma^2(s,t)=\sigma_1^2(s)+\sigma_2^2(t)$$

if η_1,η_2 satisfy the assumptions of Theorem 2.1, then the maximizer of $\sigma(s,t)$ is unique.
In this section we shall discuss an extension of Theorem 2.1 to approximation of

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]}X(s,t)>u\right) \tag{13}$$

as $u\to\infty$, where $X(s,t),(s,t)\in[-T_1,T_1]\times[-T_2,T_2]$ is a centered Gaussian random field, with variance function that is maximal at a unique point but possesses dependence structure that is more complex than the additive one discussed above. In particular, we suppose that the variance function $\sigma^2(s,t)=Var(X(s,t))$ attains its maximum at the unique point $(0,0)$ with $\sigma(0,0)=1$ and further

$$1-\sigma(s,t)\sim v_1^2(|b_{11}s+b_{12}t|)+v_2^2(|b_{21}s+b_{22}t|),\quad |b_{11}s+b_{12}t|+|b_{21}s+b_{22}t|\downarrow 0, \tag{14}$$

where $v_i \ge 0$ and $v_i \in \mathcal{R}_{\beta_i/2}, \beta_i > 0, i = 1, 2$.
For the correlation function we shall assume that

$$1 - r(s, t, s_1, t_1) \sim \rho_1^2(|a_{11}(s - s_1) + a_{12}(t - t_1)|) + \rho_2^2(|a_{21}(s - s_1) + a_{22}(t - t_1)|) \tag{15}$$

as $s, s_1, t, t_1 \to 0$ with $\rho_i \ge 0$ and $\rho_i \in \mathcal{R}_{\alpha_i/2}, \alpha_i \in (0, 2], i = 1, 2$.
We refer to [39] and references therein for important Gaussian fields that possess dependence structure like above, including the class of *incremental* Gaussian fields and its applications to such functionals as Shepp statistics and span.
For further analysis it is useful to introduce the following matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Let us observe that the assumption of uniqueness of the maximizer of $\sigma(\cdot, \cdot)$ and (14) imply that $\mathrm{rank}(B) = 2$.
We shall assume that (12) holds and furthermore the following limits

$$\lim_{t \downarrow 0} \frac{\rho_2^2(t)}{\rho_1^2(t)} = \eta \in [0, \infty], \quad \lim_{t \downarrow 0} \frac{v_2^2(t)}{v_1^2(t)} = \theta \in [0, \infty] \tag{16}$$

exist.
It appears that the rank of matrix A plays the key role for the asymptotics of (13), as $u \to \infty$. Thus, in what follows, we shall distinguish between two scenarios, when $\mathrm{rank}(A) = 2$ and $\mathrm{rank}(A) = 1$. We exclude from further analysis the degenerated case of $\mathrm{rank}(A) = 0$.

3.1. **Scenario I:** $\mathrm{rank}(A) = 2$**.** Suppose that A is invertible and observe that $Y(s, t) = X((A^{-1}(s, t)^\top)^\top)$ has under (15) and (14) correlation function r_Y such that

$$1 - r_Y(s, s_1, t, t_1) \sim \rho_1^2(|s - s_1|) + \rho_2^2(|t - t_1|), \quad s, s_1, t, t_1 \to 0, \tag{17}$$

and variance function σ_Y^2 satisfying

$$1 - \sigma_Y(s, t) \sim v_1^2(|c_{11}s + c_{12}t|) + v_2^2(|c_{21}s + c_{22}t|), \quad s, t \to 0, \tag{18}$$

with

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = BA^{-1}.$$

Therefore, with no loss of generality, in this section we tacitly assume that X satisfies (15) with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =: I.$$

Next, define an additive fractional Brownian field W by

$$W(s, t) = \sqrt{2}B_\alpha(s) + \sqrt{2}\widetilde{B}_\alpha(t) - |s|^\alpha - |t|^\alpha,$$

where $B_\alpha(t)$ and $\widetilde{B}_\alpha(t)$ are independent standard fBm's with Hurst index $\alpha/2 \in (0, 1]$. For a given matrix $D = (d_{ij})_{i,j=1,2}$, we define the generalized Piterbarg constant

$$\mathcal{P}_\alpha^{\gamma_1, \gamma_2, D} = \lim_{S \to \infty} \mathbb{E}\left\{ \sup_{(s,t) \in [-S, S]^2} e^{W(s,t) - \gamma_1 |d_{11}s + d_{12}t|^\alpha - \gamma_2 |d_{21}s + d_{22}t|^\alpha} \right\},$$

where $\gamma_1, \gamma_2 > 0$. Note that if $det(\mathrm{D}) \neq 0$, then there exists $\gamma_3 > 0$ such that

$$\gamma_1 |d_{11}s + d_{12}t|^\alpha + \gamma_2 |d_{21}s + d_{22}t|^\alpha \ge \gamma_3(|s|^\alpha + |t|^\alpha), \quad s, t \in \mathbb{R},$$

which implies that $\mathcal{P}_\alpha^{\gamma_1, \gamma_2, D} \le (\mathcal{P}_\alpha^{\gamma_3})^2 < \infty$. Moreover, for $D = I$ we have

$$\mathcal{P}_\alpha^{\gamma_1, \gamma_2, I} = \mathcal{P}_\alpha^{\gamma_1} \mathcal{P}_\alpha^{\gamma_2}.$$

Let for S_1, S_2 non-negative

$$\mathcal{H}_\alpha^{\gamma_1, \gamma_2, b}(S_1, S_2) := \mathbb{E}\left\{ \sup_{(s+bt, t) \in [-S_1, S_1] \times [0, S_2]} e^{W(s,t) - \gamma_1 |s + bt|^\alpha - \gamma_2 |t|^\alpha} \right\}, \tag{19}$$

and

$$\widetilde{\mathcal{H}}_\alpha^{\gamma_1,\gamma_2,b}(S_1,S_2) := \mathbb{E}\left\{\sup_{(s+bt,t)\in[-S_1,S_1]\times[-S_2,S_2]} e^{W(s,t)-\gamma_1|s+bt|^\alpha-\gamma_2|t|^\alpha}\right\}.$$

In order to simplify the notation we set

$$\mathcal{H}_\alpha^{\gamma_1,\gamma_2,b}(S_1)=\mathcal{H}_\alpha^{\gamma_1,\gamma_2,b}(S_1,S_1),\quad \widetilde{\mathcal{H}}_\alpha^{\gamma_1,\gamma_2,b}(S_1)=\widetilde{\mathcal{H}}_\alpha^{\gamma_1,\gamma_2,b}(S_1,S_1),\quad \mathcal{H}_\alpha^{\gamma_1,b}(S_1)=\mathcal{H}_\alpha^{\gamma_1,0,b}(S_1,S_1),$$

and

$$\mathcal{H}_\alpha^{\gamma_1,b}=\lim_{S\to\infty} S^{-1}\mathcal{H}_\alpha^{\gamma_1,b}(S).$$

Further, we shall set below

$$\mathcal{P}_\alpha^\gamma =: 1,\quad \mathcal{P}_\alpha^\gamma[0,\infty) =: 1$$

if $\gamma=\infty$.

Now let us proceed to the analysis of (13) for four special cases whose proofs are all different, and to which one can reduce all other scenarios (as will be advocated at the end of this section).

$\diamond$ Case 1. We say that X is *locally additive*, if both (15) and (14) hold with $A=B=I$. The result below holds for any $\theta,\eta\in[0,\infty]$ defined in (16).

Theorem 3.1. *Suppose that X is a locally additive Gaussian random field.*
i) If $\gamma_1=\gamma_2=0$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t)>u\right)\sim 4\prod_{i=1}^2\left(\Gamma(1/\beta_i+1)\mathcal{H}_{\alpha_i}\frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}\right)\Psi(u).$$

ii) If $\gamma_1=0,\gamma_2\in(0,\infty]$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t)>u\right)\sim 2\Gamma(1/\beta_1+1)\mathcal{H}_{\alpha_1}\mathcal{P}_{\alpha_2}^{\gamma_2}\frac{\overleftarrow{v}_1(1/u)}{\overleftarrow{\rho}_1(1/u)}\Psi(u).$$

iii) If $\gamma_1,\gamma_2\in(0,\infty]$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t)>u\right)\sim \mathcal{P}_{\alpha_1}^{\gamma_1}\mathcal{P}_{\alpha_2}^{\gamma_2}\Psi(u).$$

Remark 3.2. *i) We note that by the use of change of coordinates Theorem 3.1 covers all the combinations of values of γ_1,γ_2.*
ii) As long as the unique variance maximizer is an inner point of $[-T_1,T_1]\times[-T_2,T_2]$, the asymptotics obtained in Theorem 3.1, under the above assumptions, stays the same. If the point of maximum of variance is at the boundary of $[-T_1,T_1]\times[-T_2,T_2]$, then one has only to modify Pickands-Piterbarg constants that appear in the asymptotics given in Theorem 3.1, respectively as already done in Theorem 2.1. In particular, if the variance maximizer $t_0=(-T_1,-T_2)$, then one has to replace in Theorem 3.1 the constant $2\mathcal{H}_{\alpha_i}$ by $\mathcal{H}_{\alpha_i}$ and $\mathcal{P}_{\alpha_i}^{\gamma_i}$ by $\mathcal{P}_{\alpha_i}^{\gamma_i}[0,\infty)$, respectively. This comment is valid for all the following results below.

$\diamond$ Case 2. Here we shall assume that (14) and (15) are satisfied with

$$A=I,\quad B=\begin{pmatrix}1 & b_{12}\\ 0 & 1\end{pmatrix},\ \text{with } \mathrm{b}_{12}\neq 0. \tag{20}$$

Theorem 3.3. *Suppose that (16) is satisfied with $\eta\in(0,\infty)$, $\theta=0$ and (20) holds.*
i) If $\gamma_1=\gamma_2=0$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t)>u\right)\sim 4\prod_{i=1}^2\left(\Gamma(1/\beta_i+1)\mathcal{H}_{\alpha_i}\frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}\right)\Psi(u).$$

ii) If $\gamma_2=0,\gamma_1\in(0,\infty]$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t)>u\right)\sim 2\Gamma(1/\beta_1+1)\mathcal{H}_{\alpha_1}^{\gamma_1,b_{12}\eta^{-1/\alpha_1}}\frac{\overleftarrow{v}_1(1/u)}{\overleftarrow{\rho}_1(1/u)}\Psi(u).$$

iii) If $\gamma_2 \in (0,\infty], \gamma_1 = \infty$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t) > u\right) \sim \mathcal{P}_{\alpha_1}^{\gamma_2(|b_{12}|^{\alpha_1}\eta^{-1}+1)^{-1}} \Psi(u).$$

Remark 3.4. *The above theorem covers all the possible combinations of values of γ_1, γ_2, since the assumption that $\eta \in (0,\infty)$, $\theta = 0$ excludes cases $\gamma_1 \in [0,\infty), \gamma_2 \in (0,\infty]$.*
Although the same asymptotics are derived in i) of Theorem 3.1 and i) of Theorem 3.3, their proofs require a substantially different approach. Thus we did not combine those cases in one result.

◇ <u>Case 3.</u> The assumptions on A and B are the same as in Case 2 above, however we shall suppose that $\eta = 0$, $\theta \in (0,\infty)$. Since $\theta \in (0,\infty)$, we set $\beta = \beta_1 = \beta_2$. Let $\mu \in (-\infty,\infty)$ be the point at which $|1 + b_{12}t|^\beta + \theta|t|^\beta$ attains its minimum over $(-\infty,\infty)$. We have $\mu \in [-1/|b_{12}|, 1/|b_{12}|]$. Further, set

$$M_\beta = \inf_{t\in(-\infty,\infty)} \left(|1 + b_{12}t|^\beta + \theta|t|^\beta\right) \tag{21}$$

and define

$$\mathcal{P}_\beta^{g_s} = \lim_{S\to\infty} \mathcal{P}_\beta^{g_s}[-S,S], \quad \text{with} \quad \mathcal{P}_\beta^{g_s}[-S,S] = \mathbb{E}\left\{\sup_{t\in[-S,S]} e^{\sqrt{2}B_\beta(t)-|t|^\beta - g_s(t)}\right\}, \quad S>0, s\geq 0,$$

where

$$g_s(t) = \theta^{-1}\gamma_2\left(|s + b_{12}t|^\beta + \theta|t|^\beta - |(1+b_{12}\mu)s|^\beta - \theta|\mu s|^\beta\right), s \geq 0, t\in\mathbb{R}.$$

Further, set

$$\mathcal{I}_\beta = \int_{-\infty}^{\infty} \mathcal{P}_\beta^{g_{|s|}} e^{-\frac{\gamma_2 M_\beta}{\theta}|s|^\beta} ds \in (0,\infty).$$

The finiteness of $\mathcal{I}_\beta$ follows from the fact that for any $\epsilon > 0$, there exists a positive constant $c_\epsilon > 0$ such that

$$g_s(t) + \epsilon|s|^\beta \geq c_\epsilon |t|^\beta, \quad s\geq 0, t\in\mathbb{R}$$

implying that $\mathcal{P}_\beta^{g_s} \leq \mathcal{P}_\beta^{c_\epsilon} e^{\epsilon s^\beta} < \infty$, and thus for $\epsilon \in (0, \theta^{-1}\gamma_2 M_\beta)$

$$\mathcal{I}_\beta \leq 2\int_0^\infty \mathcal{P}_\beta^{g_s} e^{-\frac{\gamma_2 M_\beta}{\theta}s^\beta} ds \leq 2\mathcal{P}_\beta^{c_\epsilon}\int_0^\infty e^{-(\frac{\gamma_2 M_\beta}{\theta}-\epsilon)s^\beta} ds < \infty.$$

Theorem 3.5. *Suppose that (20) holds and (16) is satisfied with $\eta = 0$, $\theta \in (0,\infty)$.*
i) If $\gamma_1 = \gamma_2 = 0$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t) > u\right) \sim 4\prod_{i=1}^{2}\left(\Gamma(1/\beta+1)\mathcal{H}_{\alpha_i}\frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}\right)\Psi(u).$$

ii) If $\gamma_1 = 0, \gamma_2 \in (0,\infty)$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t) > u\right) \sim \mathcal{H}_{\alpha_1}\left(\frac{\gamma_2}{\theta}\right)^{1/\beta}\mathcal{I}_\beta\frac{\overleftarrow{v}_1(1/u)}{\overleftarrow{\rho}_1(1/u)}\Psi(u),$$

iii)) If $\gamma_1 = 0, \gamma_2 = \infty$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t) > u\right) \sim 2\Gamma(1/\beta+1)\left(M_\beta\right)^{-1/\beta}\mathcal{H}_{\alpha_1}\frac{\overleftarrow{v}_1(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})}\Psi(u),$$

iv) If $\gamma_1 \in (0,\infty], \gamma_2 = \infty$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t) > u\right) \sim \mathcal{P}_{\alpha_1}^{\gamma_1 M_\beta}\Psi(u).$$

Remark 3.6. *Analogously to the Case 2, the assumption that $\eta = 0$, $\theta \in (0,\infty)$ excludes case $\gamma_1 \in (0,\infty], \gamma_2 \in [0,\infty)$.*

◇ <u>Case 4.</u> Here we still assume that $A = I$ but there are no restrictions on the invertible B.

Theorem 3.7. *Suppose that (15) and (14) hold with $A = I$ and B an invertible matrix, and (16) is satisfied with $\eta, \theta \in (0, \infty)$.*
i) If $\gamma_1 = \gamma_2 = 0$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t) > u\right) \sim \frac{4}{|\det(B)|}\prod_{i=1}^{2}\left(\Gamma(1/\beta_i+1)\mathcal{H}_{\alpha_i}\frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}\right)\Psi(u).$$

ii) If $\gamma_1, \gamma_2 \in (0, \infty)$ or $\gamma_1 = \gamma_2 = \infty$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t) > u\right) \sim \mathcal{P}_{\alpha_1}^{\gamma_1,\gamma_1\theta,B_{\eta,\alpha_1}}\Psi(u),$$

where $\mathcal{P}_{\alpha_1}^{\gamma_1,\theta\gamma_1,B_{\eta,\alpha_1}} = 1$ if $\gamma_1 = \gamma_2 = \infty$ and $B_{\eta,\alpha_1} = \begin{pmatrix} b_{11} & b_{12}\eta^{-1/\alpha_1} \\ b_{21} & b_{22}\eta^{-1/\alpha_1}\end{pmatrix}$.

3.1.1. *Discussion.* As mentioned above, all other cases for $\mathrm{rank}(A) = 2$ can be reduced to the analysis of the field of one of types covered by Case 1-4. For the sake of transparency, let us first consider $A = I$ and B such that exactly one element b_{ij} equals to 0. With no loss of generality, by a change of variables, we can assume that

$$B = \begin{pmatrix} 1 & b_{12} \\ 0 & 1 \end{pmatrix}, \quad b_{12} \neq 0.$$

Then the following holds:

- ◇ $\underline{\theta = \infty}$: The asymptotics of (13) in this case is covered by Case 1 above, since by Lemma 6.4 we obtain

$$v_1^2(|s+b_{12}t|) + v_2^2(|t|) \sim v_1^2(|s|) + v_2^2(|t|), \quad s,t \to 0.$$

- ◇ $\underline{\eta = \infty}$: Let $Z(s,t) = X(s - b_{12}t, t)$, which is a *locally additive* Gaussian random field. Indeed, it follows from Lemma 6.4 that

$$1 - r_Z(s,t,s_1,t_1) \sim \rho_1^2(|s - s_1 - b_{12}(t-t_1)|) + \rho_2^2(|t-t_1|) \sim \rho_1^2(|s-s_1|) + \rho_2^2(|t-t_1|), \quad s,t,s_1,t_1 \to 0,$$

and

$$1 - \sigma_Z(s,t) \sim v_1^2(|s|) + v_2^2(|t|), \quad s,t \to 0.$$

- ◇ $\underline{\theta = 0, \eta = 0}$: Let $Z(s,t) = X(s, \frac{t-s}{b_{12}})$. Then, again by Lemma 6.4, Z is a *locally additive* Gaussian random field with

$$1 - r_Z(s,t,s_1,t_1) \sim \rho_1^2(|s-s_1|) + |b_{12}|^{-\alpha_2}\rho_2^2(|t-t_1|), \quad s,t,s_1,t_1 \to 0,$$

and $1 - \sigma_Z(s,t) \sim |b_{12}|^{-\beta_2}v_2^2(|s|) + v_1^2(|t|), s, t \to 0$.
- ◇ $\underline{\theta = 0,\ \eta \in (0,\infty)}$: This is covered by Case 2 above.
- ◇ $\underline{\theta \in (0,\infty),\ \eta = 0}$: This is covered by Case 3 above.
- ◇ $\underline{\theta \in (0,\infty),\ \eta \in (0,\infty)}$: This is covered by Case 4 above.

Next, let $A = I$ and $b_{ij} \neq 0$ for $i,j = 1,2$. With no loss of generality we can assume that

$$B = \begin{pmatrix} 1 & b_{12} \\ b_{21} & 1 \end{pmatrix}, \quad b_{12}b_{21} \neq 0.$$

Let us observe that $det(B) = 1 - b_{12}b_{21} \neq 0$, which will be used in several places below. Then the following holds:

- ◇ $\underline{\theta = 0,\ \eta = 0}$: Let $Z(s,t) = X(s, \frac{t-s}{b_{12}})$. Again by Lemma 6.4 Z is a *locally additive* Gaussian random field with

$$1 - r_Z(s,t,s_1,t_1) \sim \rho_1^2(|s-s_1|) + |b_{12}|^{-\alpha_2}\rho_2^2(|t-t_1|), \quad s,t,s_1,t_1 \to 0,$$

and

$$1 - \sigma_Z(s,t) \sim \left|\frac{det(B)}{b_{12}}\right|^{\beta_2} v_2^2(|s|) + v_1^2(|t|), \quad s,t \to 0.$$

◇ $\underline{\theta = 0,\ \eta \in (0,\infty)}$: This is Case 2 with v_2^2 replaced by $|det(B)|^{\beta_2} v_2^2$. Indeed, by Lemma 6.4, we have

$$\begin{aligned} v_1^2(|s+b_{12}t|)+v_2^2(|b_{21}s+t|) &= v_1^2(|s+b_{12}t|)+v_2^2(|b_{21}(s+b_{12}t)+(1-b_{12}b_{21})t|) \\ &\sim v_1^2(|s+b_{12}t|)+|det(B)|^{\beta_2}v_2^2(|t|), \quad s,t\to 0. \end{aligned}$$

◇ $\underline{\theta = 0,\ \eta = \infty}$: Let $Z(s,t) = X(s - b_{12}t, t)$. Again, by Lemma 6.4, Z is a *locally additive* Gaussian random field with

$$1 - r_Z(s,t,s_1,t_1) \sim \rho_1^2(|s-s_1|)+\rho_2^2(|t-t_1|), \quad s,t,s_1,t_1 \to 0,$$

and

$$1-\sigma_Z(s,t)\sim v_1^2(|s|)+v_2^2(|b_{21}s+(1-b_{12}b_{21})t|)\sim v_1^2(|s|)+|det(B)|^{\beta_2}v_2^2(|t|), \quad s,t\to 0.$$

◇ $\underline{\theta \in (0,\infty), \eta = 0}$: Let $Z(s,t) = X(s, t - b_{21}s)$. This is Case 3 with

$$1 - r_Z(s,t,s_1,t_1) \sim \rho_1^2(|s-s_1|)+\rho_2^2(|t-t_1|), \quad s,t,s_1,t_1 \to 0,$$

and $1-\sigma_Z(s,t)\sim |det(B)|^{\beta_1}v_1^2(|s+b_{12}(det(B))^{-1}t|)+v_2^2(|t|), s,t\to 0$.

◇ $\underline{\theta \in (0,\infty),\ \eta \in (0,\infty)}$: This is covered by Case 4.

◇ $\underline{\theta \in (0,\infty),\ \eta = \infty}$: Let $Z(s,t) = X(\frac{t-s}{b_{21}}, s)$. This is Case 3 with

$$1 - r_Z(s,t,s_1,t_1) \sim \rho_2^2(|s-s_1|)+|b_{21}|^{-\alpha_1}\rho_1^2(|t-t_1|), \quad s,t,s_1,t_1 \to 0,$$

and

$$1-\sigma_Z(s,t)\sim \left|\frac{det(B)}{b_{21}}\right|^{\beta_1} v_1^2(|s+(-det(B))^{-1}t|)+v_2^2(|t|), \quad s,t\to 0.$$

◇ $\underline{\theta = \infty,\ \eta = 0}$: Let $Z(s,t) = X(s, t - b_{21}s)$. This is a *locally additive* Gaussian random field with v_1^2 substituted by $|det(B)|^{\beta_1} v_1^2$.

◇ $\underline{\theta = \infty,\ \eta \in (0,\infty)}$: By Lemma 6.4 we have that this is Case 2 with

$$1 - r_X(s,t,s_1,t_1) \sim \rho_1^2(|s-s_1|)+\rho_2^2(|t-t_1|), \quad s,t,s_1,t_1 \to 0,$$

and

$$\begin{aligned} 1-\sigma_X(s,t) &= v_2^2(|b_{21}s+t|)+v_1^2(|b_{21}^{-1}(b_{21}s+t)+(b_{12}-b_{21}^{-1})t|) \\ &\sim v_2^2(|b_{21}s+t|)+v_1^2((b_{12}-b_{21}^{-1})t|) \\ &\sim |b_{21}|^{\beta_2}v_2^2(|s+(b_{21})^{-1}t|)+\left|\frac{det(B)}{b_{21}}\right|^{\beta_1} v_1^2(|t|), \quad s,t\to 0. \end{aligned}$$

◇ $\underline{\theta = \infty,\ \eta = \infty}$: Let $Z(s,t) = X(\frac{s-t}{b_{21}}, t)$. We have that Z is a *locally additive* Gaussian random field with

$$1 - r_Z(s,t,s_1,t_1) \sim |b_{21}|^{-\alpha_1}\rho_1^2(|s-s_1|)+\rho_2^2(|t-t_1|), \quad s,t,s_1,t_1 \to 0,$$

and

$$1-\sigma_Z(s,t)\sim v_2^2(|s|)+\left|\frac{det(B)}{b_{21}}\right|^{\beta_1} v_1^2(|t|), \quad s,t\to 0.$$

3.2. **Scenario II:** $\underline{\text{rank}(A) = 1}$**.** Suppose that $\text{rank}(A) = 1$. Clearly it suffices to consider Gaussian random fields with covariance function that satisfies (15) with $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and variance function satisfying (14). We begin with the analysis of two special cases, to which all other structures of field X can be reduced.

◇ Case 5. Here we shall assume that (15) and (14) are satisfied with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = I. \tag{22}$$

Theorem 3.8. *Suppose that* (22) *holds.*
i) If $\gamma_1 = 0$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t) > u\right) \sim 2\Gamma(1/\beta_1+1)\mathcal{H}_{\alpha_1}\frac{\overleftarrow{v}_1(1/u)}{\overleftarrow{\rho}_1(1/u)}\Psi(u).$$

ii) If $\gamma_1 \in (0,\infty]$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t) > u\right) \sim \mathcal{P}_{\alpha_1}^{\gamma_1}\Psi(u).$$

◇ <u>Case 6.</u> Here we shall assume that (15) and (14) are satisfied with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & b_{12} \\ 0 & 1 \end{pmatrix}, \text{ and } \mathrm{b}_{12} \neq 0. \tag{23}$$

Theorem 3.9. *Suppose that* (23) *holds and* (16) *is satisfied with $\theta \in (0,\infty)$.*
i) If $\gamma_1 = 0$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t) > u\right) \sim 2(M_{\beta_1})^{-1/\beta_1}\Gamma(1/\beta_1+1)\mathcal{H}_{\alpha_1}\frac{\overleftarrow{v}_1(1/u)}{\overleftarrow{\rho}_1(1/u)}\Psi(u).$$

ii) If $\gamma_1 \in (0,\infty]$, then

$$\mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t) > u\right) \sim \mathcal{P}_{\alpha_1}^{\gamma_1 M_{\beta_1}}\Psi(u),$$

with M_β defined in (21).

3.2.1. *Discussion.* Having analyzed the above special cases, we shall proceed with the asymptotics of (13) for a general structure of X. Suppose first, analogously to Scenario I, that X satisfies (15) and (14) with $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and exactly one element of matrix B equals 0. With no loss of generality we can assume that

$$B = \begin{pmatrix} 1 & b_{12} \\ 0 & 1 \end{pmatrix}, \quad b_{12} \neq 0.$$

Then the following holds.

- ◇ <u>$\theta = 0$:</u> Let $Z(s,t) = X(s, \frac{t-s}{b_{12}})$. Then, by Lemma 6.4, this is Case 5 with
$$1 - r_Z(s,t,s_1,t_1) \sim \rho_1^2(|s-s_1|), \quad s,t,s_1,t_1 \to 0, \quad 1-\sigma_Z(s,t) \sim |b_{12}|^{-\beta_2}v_2^2(|s|) + v_1^2(|t|), \quad s,t \to 0.$$
- ◇ <u>$\theta \in (0,\infty)$:</u> This is Case 6.
- ◇ <u>$\theta = \infty$:</u> The asymptotics of (13) in this case is the same as the asymptotis derived in Case 5. Indeed, by Lemma 6.4, we have
$$v_1^2(|s+b_{12}t|) + v_2^2(|t|) \sim v_1^2(|s|) + v_2^2(|t|), \quad s,t \to 0.$$

Finally we discuss the other case where the matrix B is such that $b_{ij} \neq 0$ for $i,j = 1,2$. Again with no loss of generality we can assume that

$$B = \begin{pmatrix} 1 & b_{12} \\ b_{21} & 1 \end{pmatrix}, \quad b_{12}b_{21} \neq 0.$$

Then the following holds with $det(B) = 1 - b_{12}b_{21} \neq 0$:

- ◇ <u>$\theta = 0$:</u> Let $Z(s,t) = X(s, \frac{t-s}{b_{12}})$. This is covered by Case 5.
$$1 - r_Z(s,t,s_1,t_1) \sim \rho_1^2(|s-s_1|), s,t,s_1,t_1 \to 0, \quad 1-\sigma_Z(s,t) \sim \left|\frac{det(B)}{b_{12}}\right|^{\beta_2} v_2^2(|s|) + v_1^2(|t|), \quad s,t \to 0.$$
- ◇ <u>$\theta \in (0,\infty)$:</u> Let $Z(s,t) = X(s, t - b_{21}s)$. Then, by Lemma 6.4, Z is as in Case 6 with
$$1 - r_Z(s,t,s_1,t_1) \sim \rho_1^2(|s-s_1|), \quad s,t,s_1,t_1 \to 0,$$
$$1-\sigma_Z(s,t) \sim |det(B)|^{\beta_1}v_1^2(|s + b_{12}(det(B))^{-1}t|) + v_2^2(|t|), \quad s,t \to 0.$$

◇ $\underline{\theta=\infty}$: Let $Z(s,t)=X(s,t-b_{21}s)$. This is Case 5 with

$$1-r_Z(s,t,s_1,t_1)\sim \rho_1^2(|s-s_1|),\quad s,t,s_1,t_1\to 0,$$

$$1-\sigma_Z(s,t)\sim |det(B)|^{\beta_1}v_1^2(|s|)+v_2^2(|t|),\quad s,t\to 0.$$

4. Proofs

In the rest of this section by $\mathbb{Q},\mathbb{Q}_i, i\geq 1$ we shall denote positive constants that may differ from line to line. Moreover, for a given Gaussian random field Z we write $\overline{Z}$ for the standardised random field.

Proof of Theorem 2.1 Since proofs of all the cases $t_0\in\{0,T\}$ and $t_0\in(0,T)$ follow by the same line of reasoning, we focus only on scenario $t_0=0$. For $S>0, u>1$ we shall set

$$\xi(u)=u^{-1}\ln u,\quad E_u=[0,\overleftarrow{v}(\xi(u))],\quad I_k(u)=\left[kS\overleftarrow{\rho}(u^{-1}),(k+1)S\overleftarrow{\rho}(u^{-1})\right],\quad k\in\mathbb{N}\cup\{0\}.$$

Further for given $\varepsilon\in(0,1/2)$ define

$$u_{k,\epsilon}^-=u(1+(1-\epsilon)\inf_{t\in I_k(u)}v^2(t)),\quad u_{k,\epsilon}^+=u(1+(1+\epsilon)\sup_{t\in I_k(u)}v^2(t)),$$

and

$$N(u)=\left[\frac{\overleftarrow{v}(\xi(u))}{\overleftarrow{\rho}(u^{-1})S}\right]+1.$$

For $L>0$ sufficiently small

$$\mathbb{E}\left\{(\overline{X}(t)-\overline{X}(t))^2\right\}\leq 2(1-r(s,t))\leq 4\rho^2(|t-s|)\leq \mathbb{Q}|t-s|^{\alpha/2},\quad s,t\in[0,L], \tag{24}$$

which ensures the validity of the Hölder continuity condition in a neighborhood of 0. By the fact that for u sufficiently large

$$\sup_{t\in[\overleftarrow{v}(\xi(u)),T]}\sigma(t)\leq 1-\mathbb{Q}(\xi(u))^2,$$

and in light of (24) and [3][Theorem 8.1] (see also [40][Theorem 3]) we have

$$\mathbb{P}\left(\sup_{t\in[\overleftarrow{v}(\xi(u)),L]}X(t)>u\right)\leq \mathbb{P}\left(\sup_{t\in[0,L]}\overline{X}(t)>\frac{u}{1-\mathbb{Q}(\xi(u))^2}\right)\leq \mathbb{Q}Tu^{4/\alpha}\Psi\left(\frac{u}{1-\mathbb{Q}(\xi(u))^2}\right). \tag{25}$$

Moreover, in light of Borell-TIS inequality (see e.g., [41–43]) and the fact that $\sup_{t\in[L,T]}\sigma(t)\leq 1-\delta$ with $\delta>0$

$$\mathbb{P}\left(\sup_{t\in[\overleftarrow{v}(\xi(u)),L]}X(t)>u\right)\leq e^{-\frac{(u-a)^2}{2(1-\delta)}},$$

with $a=\mathbb{E}\left(\sup_{t\in[0,T]}X(t)\right)<\infty$. Consequently, for all large u we have

$$\pi(u)\leq \mathbb{P}\left(\sup_{t\in[0,T]}X(t)>u\right)\leq \pi(u)+\mathbb{Q}Tu^{4/\alpha}\Psi\left(\frac{u}{1-\mathbb{Q}(\xi(u))^2}\right), \tag{26}$$

where

$$\pi(u)=\mathbb{P}\left(\sup_{t\in[0,\overleftarrow{v}(\xi(u))]}X(t)>u\right).$$

Next we give the exact asymptotics of $\pi(u)$ subject to three different scenarios.

<u>Case i) $\gamma=0$</u>. For any u positive we have

$$\sum_{k=0}^{N(u)-1}\mathbb{P}\left(\sup_{t\in I_k(u)}X(t)>u\right)-\sum_{i=1}^{2}\Lambda_i(u)\leq \pi(u)\leq \sum_{k=0}^{N(u)}\mathbb{P}\left(\sup_{t\in I_k(u)}X(t)>u\right), \tag{27}$$

where

$$\Lambda_1(u)=\sum_{k=0}^{N(u)}\mathbb{P}\left(\sup_{t\in I_k(u)}X(t)>u,\sup_{t\in I_{k+1}(u)}X(t)>u\right),$$

and

$$\Lambda_2(u)=\sum_{0\leq k,l\leq N(u),l\geq k+2}\mathbb{P}\left(\sup_{t\in I_k(u)}X(t)>u,\sup_{t\in I_l(u)}X(t)>u\right).$$

The main difference in comparison with the proofs of the classical cases considered in the literature, as e.g., in [3] is contained in the approximation given below.
By Lemma 6.1, we have that for any $0<\epsilon<\min(1,\beta)$ and for u sufficiently large

$$\frac{v^2(s)}{v^2(t)} \geq (1-\epsilon/2)\min\left(\left(\frac{s}{t}\right)^{\beta-\epsilon/2},\left(\frac{s}{t}\right)^{\beta+\epsilon/2}\right) \geq (1-\epsilon/2)\left(\frac{k}{k+1}\right)^{\beta+\epsilon/2}, \quad s,t\in I_k(u), 1\leq k\leq N(u).$$

Consequently, for any $\epsilon>0$, there exists $k_\epsilon\in\mathbb{N}$ such that

$$\inf_{t\in I_k(u)} v^2(t) \geq (1-\epsilon)\sup_{t\in I_k(u)} v^2(t), \quad k_\epsilon\leq k\leq N(u). \tag{28}$$

Let in the following

$$X_{u,k}(t)=\overline{X}(kS\overleftarrow{\rho}(u^{-1})+t), t\in I_0(u), \quad k\in\mathcal{K}_u=\{k, 0\leq k\leq N(u)\}$$

and set $h_k(u)=u^-_{k,\epsilon}$. Applying Lemma 5.1, we obtain

$$\lim_{u\to\infty}\sup_{0\leq k\leq N(u)}\left|(\Psi(u^-_{k,\epsilon}))^{-1}\mathbb{P}\left(\sup_{t\in I_0(u)} X_{u,k}(t)>u^-_{k,\epsilon}\right)-\mathcal{H}_\alpha[0,S]\right|=0. \tag{29}$$

Consequently, as $u\to\infty$,

$$\begin{aligned}
\sum_{k=0}^{N(u)}\mathbb{P}\left(\sup_{t\in I_k(u)} X(t)>u\right) &\leq \sum_{k=0}^{N(u)}\mathbb{P}\left(\sup_{t\in I_k(u)}\overline{X}(t)>u^-_{k,\epsilon}\right)\\
&\leq \sum_{k=0}^{N(u)}\mathbb{P}\left(\sup_{t\in I_0(u)} X_{u,k}(t)>u^-_{k,\epsilon}\right)\\
&\sim \sum_{k=0}^{N(u)}\mathcal{H}_\alpha[0,S]\Psi(u^-_{k,\epsilon})\\
&\sim \mathcal{H}_\alpha[0,S]\Psi(u)\sum_{k=0}^{N(u)} e^{-u^2(1-\epsilon)\inf_{t\in I_k(u)} v^2(t)}.
\end{aligned} \tag{30}$$

Further by Lemma 6.3 (recall $\xi(u)=u^{-1}\ln u$)

$$\begin{aligned}
\sum_{k=0}^{N(u)}\mathbb{P}\left(\sup_{t\in I_k(u)} X(t)>u\right) &\leq \mathcal{H}_\alpha[0,S]\Psi(u)\left(k_\epsilon+\frac{1}{\overleftarrow{\rho}(u^{-1})S}\sum_{k=k_\epsilon}^{N(u)}\int_{t\in I_k(u)} e^{-(1-\epsilon)^2u^2v^2(t)}dt\right)\\
&\sim \mathcal{H}_\alpha[0,S]\left(k_\epsilon+\frac{1}{\overleftarrow{\rho}(u^{-1})S}\int_0^{\overleftarrow{v}(\xi(u))} e^{-(1-\epsilon)^2u^2v^2(t)}dt\right)\Psi(u)\\
&\sim \Gamma(1/\beta+1)\mathcal{H}_\alpha\frac{\overleftarrow{v}(u^{-1})}{\overleftarrow{\rho}(u^{-1})}\Psi(u), \quad u\to\infty, S\to\infty, \epsilon\to 0.
\end{aligned}$$

Similarly, we obtain

$$\sum_{k=0}^{N(u)-1}\mathbb{P}\left(\sup_{t\in I_k(u)} X(t)>u\right) \geq \Gamma(1/\beta+1)\mathcal{H}_\alpha\frac{\overleftarrow{v}(u^{-1})}{\overleftarrow{\rho}(u^{-1})}\Psi(u)(1+o(1)), \quad u\to\infty, S\to\infty. \tag{31}$$

Next we focus on $\Lambda_i(u), i=1,2$. Let $\hat{u}_{k,-\epsilon}=\min(u^-_{k,-\epsilon},u^-_{k+1,-\epsilon})$. Observe that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in I_k(u)} X(t)>u, \sup_{t\in I_{k+1}(u)} X(t)>u\right)\\
&\leq \mathbb{P}\left(\sup_{t\in I_k(u)}\overline{X}(t)>\hat{u}_{k,-\epsilon}\right)+\mathbb{P}\left(\sup_{t\in I_{k+1}(u)}\overline{X}(t)>\hat{u}_{k,-\epsilon}\right)-\mathbb{P}\left(\sup_{t\in I_k(u)\cup I_{k+1}(u)}\overline{X}(t)>\hat{u}_{k,-\epsilon}\right).
\end{aligned}$$

By (29), we have that

$$\lim_{u\to\infty}\sup_{1\leq k\leq N(u)}\left|\frac{\mathbb{P}\left(\sup_{t\in I_k(u)}\overline{X}(t)>\hat{u}_{k,-\epsilon}\right)}{\mathcal{H}_\alpha[0,S]\Psi(\hat{u}_{k,-\epsilon})}-1\right|=\lim_{u\to\infty}\sup_{1\leq k\leq N(u)}\left|\frac{\mathbb{P}\left(\sup_{t\in I_{k+1}(u)}\overline{X}(t)>\hat{u}_{k,-\epsilon}\right)}{\mathcal{H}_\alpha[0,S]\Psi(\hat{u}_{k,-\epsilon})}-1\right|=0$$

and

$$\lim_{u\to\infty}\sup_{1\le k\le N(u)}\left|\frac{\mathbb{P}\left(\sup_{t\in I_k(u)\cup I_{k+1}(u)}\overline{X}(t)>\hat{u}_{k,-\epsilon}\right)}{\mathcal{H}_\alpha[0,2S]\Psi(\hat{u}_{k,-\epsilon})}-1\right|=0,$$

implying that

$$\lim_{u\to\infty}\sup_{0\le k\le N(u)}\frac{\mathbb{P}\left(\sup_{t\in I_k(u)}X(t)>u,\sup_{t\in I_{k+1}(u)}X(t)>u\right)}{\mathcal{H}_\alpha[0,S]\Psi(\hat{u}_{k,-\epsilon})}\le\left(2-\frac{\mathcal{H}_\alpha[0,2S]}{\mathcal{H}_\alpha[0,S]}\right).$$

Since

$$\lim_{S\to\infty}\frac{\mathcal{H}_\alpha[0,2S]}{\mathcal{H}_\alpha[0,S]}=2$$

then, for u sufficiently large, we have

$$\begin{aligned}\Lambda_1(u)&\le 2\left(2-\frac{\mathcal{H}_\alpha[0,2S]}{\mathcal{H}_\alpha[0,S]}\right)\sum_{k=0}^{N(u)}\mathcal{H}_\alpha[0,S]\Psi(\hat{u}_{k,-\epsilon})\le 2\left(2-\frac{\mathcal{H}_\alpha[0,2S]}{\mathcal{H}_\alpha[0,S]}\right)\Gamma(1/\beta+1)\mathcal{H}_\alpha\frac{\overleftarrow{v}(u^{-1})}{\overleftarrow{\rho}(u^{-1})}\Psi(u)\\&= o\left(\frac{\overleftarrow{v}(u^{-1})}{\overleftarrow{\rho}(u^{-1})}\Psi(u)\right),\quad u\to\infty,S\to\infty.\end{aligned}$$

By (8) and applying Lemma 5.4 in Appendix, we have (note that below k,l take values up to $N(u)$, therefore an uniform upper bound for approximating the summands derived in Lemma 5.4 is essential)

$$\begin{aligned}\Lambda_2(u)&\le\sum_{0\le k,l\le N(u),l\ge k+2}\mathbb{P}\left(\sup_{t\in I_k(u)}\overline{X}(t)>u^-_{k,-\epsilon},\sup_{t\in I_l(u)}\overline{X}(t)>u^-_{l,-\epsilon}\right)\\&\le\sum_{0\le k,l\le N(u),l\ge k+2}\mathbb{Q}S^2\Psi\left(\hat{u}_{k,l,-\epsilon}\right)e^{-\mathbb{Q}_1|(l-k)S|^{\alpha/2}}\\&\le\mathbb{Q}S^2\sum_{0\le k\le N(u)}\Psi\left(u^-_{k,-\epsilon}\right)\sum_{l=1}^{\infty}e^{-\mathbb{Q}_1(lS)^{\alpha/2}}\\&\le\mathbb{Q}S^2e^{-\mathbb{Q}_2S^{\alpha/2}}\sum_{k=0}^{N(u)}\Psi\left(u^-_{k,-\epsilon}\right)\\&= o\left(\frac{\overleftarrow{v}(u^{-1})}{\overleftarrow{\rho}(u^{-1})}\Psi(u)\right),\quad u\to\infty,S\to\infty,\end{aligned}\tag{32}$$

with $\hat{u}_{k,l,-\epsilon}=\min(u^-_{k,-\epsilon},u^-_{l,-\epsilon})$. By the above calculations both $\Lambda_1(u)$ and $\Lambda_2(u)$ are negligible. Hence the results displayed in (25)-(32) establish the claim.

Case ii) $\gamma\in(0,\infty)$. For any $u>0$ we have

$$\mathbb{P}\left(\sup_{t\in I_0(u)}X(t)>u\right)\le\pi(u)\le\mathbb{P}\left(\sup_{t\in I_0(u)}X(t)>u\right)+\sum_{k=1}^{N(u)}\mathbb{P}\left(\sup_{t\in I_k(u)}X(t)>u\right).$$

By Lemma 5.1, as $u\to\infty$,

$$\mathbb{P}\left(\sup_{t\in I_0(u)}X(t)>u\right)\sim\mathcal{P}_\alpha^\gamma[0,S]\Psi(u).$$

Moreover, by (29) and (30), for sufficiently small $\epsilon>0$, we have

$$\sum_{k=1}^{N(u)}\mathbb{P}\left(\sup_{t\in I_k(u)}X(t)>u\right)\le\mathcal{H}_\alpha[0,S]\Psi(u)\sum_{k=1}^{N(u)}e^{-u^2(1-\epsilon)\inf_{t\in I_k(u)}v^2(t)}(1+o(1)),\quad u\to\infty.$$

Note that for any $t\in\left[S,2\frac{\overleftarrow{v}(\xi(u))}{\overleftarrow{\rho}(u^{-1})}\right]$ we have $\lim_{u\to\infty}\overleftarrow{\rho}(u^{-1})t=0$. Hence, by Lemma 6.1 we have that for u large enough and $S>1$

$$\frac{v^2(\overleftarrow{\rho}(u^{-1})t)}{v^2(\overleftarrow{\rho}(u^{-1}))}\ge t^{\beta/2},\quad\text{with}\quad t\in\left[S,2\frac{\overleftarrow{v}(\xi(u))}{\overleftarrow{\rho}(u^{-1})}\right].$$

Consequently, for $1\le k\le N(u)$ and S,u sufficiently large

$$u^2\inf_{t\in I_k(u)}v^2(t)\quad>\quad\frac{2}{3}\inf_{t\in[kS,(k+1)S]}\frac{v^2(\overleftarrow{\rho}(u^{-1})t)}{v^2(\overleftarrow{\rho}(u^{-1}))}\frac{v^2(\overleftarrow{\rho}(u^{-1}))}{\rho^2(\overleftarrow{\rho}(u^{-1}))}$$

$$\geq \quad \gamma/2(kS)^{\beta/2}$$

implying

$$\begin{aligned}\sum_{k=1}^{N(u)} \mathbb{P}\left(\sup_{t\in I_k(u)} X(t)>u\right) &\leq \mathcal{H}_\alpha[0,S]\Psi(u)\sum_{k=1}^{N(u)} e^{-(1-\epsilon)\frac{\gamma}{2}(kS)^{\beta/2}}(1+o(1))\\ &\leq \frac{\mathcal{H}_\alpha[0,S]}{S}\Psi(u)Se^{-\mathbb{Q}S^{\beta/2}} = o\left(\Psi(u)\right), \quad u\to\infty, S\to\infty.\end{aligned}$$

Consequently,

$$\pi(u)\sim \lim_{S\to\infty}\mathcal{P}_\alpha^\gamma[0,S]\Psi(u), \quad u\to\infty,$$

establishing the proof.
Case ii) $\gamma=\infty$. Observe that

$$\mathbb{P}\left(X(0)>u\right)\leq \pi(u)\leq \mathbb{P}\left(\sup_{t\in I_0(u)} X(t)>u\right)+\sum_{k=1}^{N(u)}\mathbb{P}\left(\sup_{t\in I_k(u)} X(t)>u\right).$$

In light of Lemma 5.1, we have

$$\mathbb{P}\left(\sup_{t\in I_0(u)} X(t)>u\right)\sim \Psi(u), \quad u\to\infty.$$

With the same arguments as in the proof of case $\gamma\in(0,\infty)$ we obtain

$$\sum_{k=1}^{N(u)}\mathbb{P}\left(\sup_{t\in I_k(u)} X(t)>u\right)=o\left(\Psi(u)\right), \quad u\to\infty, S\to\infty,$$

which completes the proof. □

4.1. **Proofs of Theorems 3.1, 3.3, 3.5, 3.7, 3.8 and 3.9.** Define next for S,u positive

$$I_k(u) = \left[\overleftarrow{\rho_1}(u^{-1})kS, \overleftarrow{\rho_1}(u^{-1})(k+1)S\right], \quad J_k(u)=\left[\overleftarrow{\rho_2}(u^{-1})kS,\overleftarrow{\rho_2}(u^{-1})(k+1)S\right], \quad I_{k,l}(u)=I_k(u)\times J_l(u), \quad k,l\in\mathbb{Z},$$

and (recall $\xi(u)=u^{-1}\ln u$)

$$N_1(u)=\left[\frac{\overleftarrow{v_1}(\xi(u))}{\overleftarrow{\rho_1}(u^{-1})S}\right], \quad N_2(u)=\left[\frac{\overleftarrow{v_2}(\xi(u))}{\overleftarrow{\rho_2}(u^{-1})S}\right].$$

Additionally, let

$$\mathbb{V}_1(u)=\left\{(k,l,k_1,l_1): -N_1(u)-2\leq k\leq k_1\leq N_1(u)+1, -N_2(u)-2\leq l,l_1\leq N_2(u)+1, I_{k,l}\cap I_{k_1,l_1}=\emptyset\right\},$$

$$\mathbb{V}_2(u)=\left\{(k,l,k_1,l_1): -N_1(u)-2\leq k\leq k_1\leq N_1(u)+1, -N_2(u)-2\leq l,l_1\leq N_2(u)+1, (k,l)\neq(k_1,l_1), I_{k,l}\cap I_{k_1,l_1}\neq\emptyset\right\},$$

$$u_{k,l,\epsilon}^- = u(1+(1-\epsilon)\inf_{(s,t)\in I_{k,l}(u)}\left(v_1^2(|b_{11}s+b_{12}t|)+v_2^2(|b_{21}s+b_{22}t|)\right)),$$

$$u_{k,l,\epsilon}^+ = u(1+(1+\epsilon)\sup_{(s,t)\in I_{k,l}(u)}\left(v_1^2(|b_{11}s+b_{12}t|)+v_2^2(|b_{21}s+b_{22}t|)\right)),$$

$$u_{k,\epsilon}^{1,-}=u(1+(1-\epsilon)\inf_{s\in I_k(u)} v_1^2(|s|)), \quad u_{k,\epsilon}^{1,+}=u(1+(1+\epsilon)\sup_{s\in I_k(u)} v_1^2(|s|)),$$

$$u_{l,\epsilon}^{2,-}=u(1+(1-\epsilon)\inf_{s\in J_l(u)} v_2^2(|s|)), \quad u_{l,\epsilon}^{2,+}=u(1+(1+\epsilon)\sup_{s\in J_l(u)} v_2^2(|s|)), k,l\in\mathbb{Z},$$

where $u_{k,l,\epsilon}^{\pm}$ varies according to B.

The general strategy of proofs of Theorems 3.1, 3.3, 3.5, 3.7, 3.8 and 3.9 agrees with the double-sum technique developed for Gaussian random fields in, e.g., [3]. However the variance-covariance structure of some cases substantially differs from the families of Gaussian random fields analyzed in [3] and requires a case-specific approach, on which we focus below. Observe that for all Cases 1-6

$$\pi_1(u)\leq \mathbb{P}\left(\sup_{(s,t)\in[-T_1,T_1]\times[-T_2,T_2]} X(s,t)>u\right)\leq \pi_1(u)+\mathbb{P}\left(\sup_{(s,t)\in E_u} X(s,t)>u\right), \tag{33}$$

where

$$\pi_1(u) = \mathbb{P}\left(\sup_{(s,t)\in D_u} X(s,t) > u \right),$$

and

$$D_u = \left[-\overleftarrow{v_1}(\xi(u)), \overleftarrow{v_1}(\xi(u)) \right] \times \left[-\overleftarrow{v_2}(\xi(u)), \overleftarrow{v_2}(\xi(u)) \right], \quad E_u := ([-T_1,T_1]\times[-T_2,T_2]) \setminus D_u.$$

For Case 1-Case 3 and Case 5-Case 6, by (14) for u sufficiently large we have

$$\sup_{(s,t)\in E_u} \sigma(s,t) \le 1 - \mathbb{Q}u^{-2}\ln^2 u.$$

For Case 4, in light of (14) and Lemma 6.5 we have

$$\sup_{(s,t)\in E_u} \sigma(s,t) \le 1 - \mathbb{Q} \inf_{(s,t)\in E_u} \left(v_1^2(|s|) + v_2^2(|t|) \right) \le 1 - \mathbb{Q}u^{-2}\ln^2 u.$$

Moreover, for ϵ sufficiently small

$$\mathbb{E}\left\{ (\overline{X}(s,t) - \overline{X}(s_1,t_1))^2 \right\} \le \mathbb{Q}_1(|s-s_1|^{\alpha_1/2} + |t-t_1|^{\alpha_2/2}), \quad (s,t),(s_1,t_1) \in [-\epsilon,\epsilon]^2.$$

It follows by the fact that $(0,0)$ is the unique maximizer of σ, Theorem 8.1 in [3] and Borell-TIS inequality that

$$\mathbb{P}\left(\sup_{(s,t)\in E_u} X(s,t) > u \right) \le \mathbb{Q}T_1T_2u^{4/\alpha_1+4/\alpha_2}\Psi\left(\frac{u}{1-\mathbb{Q}u^{-2}\ln^2 u} \right). \tag{34}$$

Consequently, for all Cases 1-6 we focus on the asymptotics of $\pi_1(u)$ as $u\to\infty$, proving that it delivers the asymptotics of (13) as $u \to \infty$.

Proof of Theorem 3.1

Case i). Suppose that $\gamma_1 = \gamma_2 = 0$. For any $0 < \epsilon < 1/2$ and u large enough we have

$$\pi_{1,\epsilon}(u) - \sum_{i=1}^{2} \Lambda_i'(u) \le \pi_1(u) \le \pi_{1,-\epsilon}(u), \tag{35}$$

with

$$\begin{aligned}
\pi_{1,\pm\epsilon}(u) &= \sum_{k=-N_1(u)\pm 2}^{N_1(u)\mp 1} \sum_{l=-N_2(u)\pm 2}^{N_2(u)\mp 1} \mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)} \overline{X}(s,t) > u^{\pm}_{k,l,\epsilon} \right),\\
\Lambda_i'(u) &= \sum_{(k,l,k_1,l_1)\in \mathbb{V}_i(u)} \mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)} \overline{X}(s,t) > u^-_{k,l,\epsilon}, \sup_{(s,t)\in I_{k_1,l_1}(u)} \overline{X}(s,t) > u^-_{k_1,l_1,\epsilon} \right), i=1,2.
\end{aligned}$$

Similarly as given in (28), we have that for any $\epsilon>0$, there exist $k_\epsilon \in \mathbb{N}$ such that

$$\inf_{t\in I_k(u)} v_1^2(|t|) \ge (1-\epsilon) \sup_{t\in I_k(u)} v_1^2(|t|), \quad \inf_{t\in J_l(u)} v_2^2(|t|) \ge (1-\epsilon) \sup_{t\in J_l(u)} v_2^2(|t|) \tag{36}$$

hold for

$$k_\epsilon \le |k| \le N_1(u)+2, \quad k_\epsilon \le |l| \le N_2(u)+2.$$

Let

$$X_{u,k,l}(s,t) = \overline{X}(kS\overleftarrow{\rho}_1(u^{-1})+s, lS\overleftarrow{\rho}_2(u^{-1})+t), \quad \mathcal{K}_u = \{(k,l), |k| \le N_1(u)+2, |l|\le N_2(u)+2\},$$
$$h_{k,l}(u) = u^-_{k,l,\epsilon}, \quad \mathcal{E}_u = I_{0,0}(u), \quad d_u = 0.$$

One can easily check that the conditions of Lemma 5.2 are satisfied, implying that

$$\lim_{u\to\infty} \sup_{(k,l)\in\mathcal{K}_u} \left| \frac{1}{\Psi(u^-_{k,l,\epsilon})} \mathbb{P}\left(\sup_{(s,t)\in I_{0,0}(u)} X_{u,k,l}(s,t) > u^-_{k,l,\epsilon} \right) - \prod_{i=1}^{2} \mathcal{H}_{\alpha_i}[0,S] \right| = 0. \tag{37}$$

Further, using Lemma 6.3 we have

$$\pi_{1,-\epsilon}(u) = \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \mathbb{P}\left(\sup_{(s,t)\in I_{0,0}(u)} X_{u,k,l}(s,t) > u^-_{k,l,\epsilon} \right)$$

$$
\begin{aligned}
&\sim \sum_{k=-N_1(u)-2}^{N_1(u)+1}\sum_{l=-N_2(u)-2}^{N_2(u)+1}\Big(\prod_{i=1}^{2}\mathcal{H}_{\alpha_i}[0,S]\Big)\Psi(u^-_{k,l,\epsilon})\\
&\le \Big(\prod_{i=1}^{2}\mathcal{H}_{\alpha_i}[0,S]\Big)\Psi(u)\left(R_1(u)+R_2(u)+\sum_{k_\epsilon\le |k|\le N_1(u)+2}\frac{1}{\overleftarrow{\rho_1}(u^{-1})S}\int_{s\in I_k(u)}e^{-(1-\epsilon)^2u^2v_1^2(|t|)}dt\right.\\
&\quad\left.\times\sum_{k_\epsilon\le |l|\le N_2(u)+2}\frac{1}{\overleftarrow{\rho_2}(u^{-1})S}\int_{t\in J_l(u)}e^{-(1-\epsilon)^2u^2v_2^2(t)}dt\right)(1+o(1))\\
&\sim \frac{4}{S^2}\Big(\prod_{i=1}^{2}\mathcal{H}_{\alpha_i}[0,S]\frac{1}{\overleftarrow{\rho_i}(u^{-1})}\int_0^{\overleftarrow{v_i}(\xi(u))}e^{-(1-\epsilon)^2u^2v_i^2(t)}dt\Big)\Psi(u)\\
&\sim (1-\epsilon)^{-1/\beta_1-1/\beta_2}\frac{4}{S^2}\Big(\prod_{i=1}^{2}\mathcal{H}_{\alpha_i}[0,S]\Gamma(1/\beta_i+1)\frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}\Big)\Psi(u)
\end{aligned}
$$

$$
\sim 4\Big(\prod_{i=1}^{2}\mathcal{H}_{\alpha_i}\Gamma(1/\beta_i+1)\frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}\Big)\Psi(u),\quad u\to\infty, S\to\infty,\epsilon\to 0, \tag{38}
$$

where

$$
R_1(u)=\sum_{|k|\le k_\epsilon}\sum_{|l|\le N_2(u)+2}e^{-(1-\epsilon)u^2\inf_{s\in I_k(u)}v_1^2(|s|)-(1-\epsilon)u^2\inf_{t\in J_l(u)}v_2^2(|t|)},
$$

$$
R_2(u)=\sum_{|k|\le N_1(u)+2}\sum_{|l|\le k_\epsilon}e^{-(1-\epsilon)u^2\inf_{s\in I_k(u)}v_1^2(|s|)-(1-\epsilon)u^2\inf_{t\in J_l(u)}v_2^2(|t|)}.
$$

Note that (38) holds since in light of Lemma 6.3 we have (recall $\xi(u)=u^{-1}\ln u$)

$$
\begin{aligned}
R_1(u) &\le (2k_\epsilon+1)\left(2k_\epsilon+1+\frac{1}{\overleftarrow{\rho_2}(u^{-1})S}\int_0^{\overleftarrow{v_2}(\xi(u))}e^{-(1-\epsilon)^2u^2v_2^2(t)}dt\right)\\
&\sim (2k_\epsilon+1)(1-\epsilon)^{-1/\beta_2}\frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)S}=o\left(\frac{\overleftarrow{v}_1(1/u)}{\overleftarrow{\rho}_1(1/u)}\frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)}\right),\quad u\to\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
R_2(u) &\le (2k_\epsilon+1)\left(2k_\epsilon+1+\frac{1}{\overleftarrow{\rho_1}(u^{-1})S}\int_0^{\overleftarrow{v_1}(\xi(u))}e^{-(1-\epsilon)^2u^2v_1^2(t)}dt\right)\\
&\sim (2k_\epsilon+1)(1-\epsilon)^{-1/\beta_1}\frac{\overleftarrow{v}_1(1/u)}{\overleftarrow{\rho}_1(1/u)S}\\
&= o\left(\frac{\overleftarrow{v}_1(1/u)}{\overleftarrow{\rho}_1(1/u)}\frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)}\right),\quad u\to\infty.
\end{aligned}
$$

Similarly,

$$
\pi_{1,\epsilon}(u) \ge \sum_{k=-N_1(u)+1}^{N_1(u)-1}\sum_{l=-N_2(u)+1}^{N_2(u)-1}\mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)}\overline{X}(s,t)>u^+_{k,l,\epsilon}\right)
$$

$$
\sim 4\Big(\prod_{i=1}^{2}\mathcal{H}_{\alpha_i}\Gamma(1/\beta_i+1)\frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}\Big)\Psi(u) \tag{39}
$$

as $u\to\infty, S\to\infty,\epsilon\to 0$.

Next we prove that both $\Lambda_1'(u),\Lambda_2'(u)$ are asymptotically negligible. From (15), applying Lemma 5.4 in the Appendix, with

$$
\hat{u}_{k,l,k_1,l_1,\epsilon}=\min(u^-_{k,l,\epsilon},u^-_{k_1,l_1,\epsilon}),\quad \beta^*=\min(\alpha_1,\alpha_2),
$$

we obtain

$$
\Lambda_1'(u) \le \mathbb{Q}S^4\sum_{(k,l,k_1,l_1)\in\mathbb{V}_1(u)}\Psi\left(\hat{u}_{k,l,k_1,l_1,\epsilon}\right)e^{-\mathbb{Q}_1(|k-k_1|^2+|l-l_1|^2)^{\beta^*/4}S^{\beta^*/2}}
$$

$$
\begin{aligned}
&\le \mathbb{Q}S^4 \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \Psi(u_{k,l,\epsilon}^-) \sum_{m+n\ge 1, m,n\ge 0} e^{-\mathbb{Q}_1(|m|^2+|n|^2)^{\beta^*/4}S^{\beta^*/2}}\\
&\le \mathbb{Q}S^4 \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \Psi(u_{k,l,\epsilon}^-) e^{-\mathbb{Q}_2 S^{\beta^*/2}}\\
&= o\left(\pi_{1,\epsilon}(u)\right), \quad u\to\infty, S\to\infty. \qquad (40)
\end{aligned}
$$

Next, we shall focus on $\Lambda_2(u)$. Without loss of generality, we assume that $k_1 = k+1$ and set for $k,l\in\mathbb{Z}$

$$
I_{k,l}^{(1)} = \left[\overleftarrow{\rho_1}(u^{-1})kS, \overleftarrow{\rho_1}(u^{-1})\left(kS+\sqrt{S}\right)\right] \times \left[\overleftarrow{\rho_2}(u^{-1})lS, \overleftarrow{\rho_2}(u^{-1})(l+1)S\right],
$$

$$
I_{k,l}^{(2)} = \left[\overleftarrow{\rho_1}(u^{-1})\left(kS+\sqrt{S}\right), \overleftarrow{\rho_1}(u^{-1})(k+1)S\right] \times \left[\overleftarrow{\rho_2}(u^{-1})lS, \overleftarrow{\rho_2}(u^{-1})(l+1)S\right].
$$

For $(k,l,k_1,l_1)\in\mathbb{V}_2(u), k_1 = k+1$, we have

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^-, \sup_{(s,t)\in I_{k_1,l_1}(u)} \overline{X}(s,t) > u_{k_1,l_1,\epsilon}^-\right)\\
&\le \mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^-, \sup_{(s,t)\in I_{k_1,l_1}^{(1)}(u)} \overline{X}(s,t) > u_{k_1,l_1,\epsilon}^-\right)\\
&+\mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^-, \sup_{(s,t)\in I_{k_1,l_1}^{(2)}(u)} \overline{X}(s,t) > u_{k_1,l_1,\epsilon}^-\right)\\
&:= p_{k,l,k_1,l_1}^{(1)}(u) + p_{k,l,k_1,l_1}^{(2)}(u).
\end{aligned}
$$

It follows from Lemma 5.2 that uniformly with $(k,l,k_1,l_1)\in\mathbb{V}_2(u), k_1 = k+1$

$$
p_{k,l,k_1,l_1}^{(1)}(u) \le \mathbb{P}\left(\sup_{(s,t)\in I_{k_1,l_1}^{(1)}(u)} \overline{X}(s,t) > u_{k_1,l_1,\epsilon}^-\right) \sim \mathcal{H}_{\alpha_1}[0,\sqrt{S}]\mathcal{H}_{\alpha_2}[0,S]\Psi(u_{k,l,\epsilon}^-)
$$

as $u\to\infty$. Further, since each $I_{k,l}(u)\times I_{k_1,l_1}(u)$ has at most 8 neighbours, we have that

$$
\begin{aligned}
\sum_{(k,l,k_1,l_1)\in\mathbb{V}_2(u)} p_{k,l,k_1,l_1}^{(1)}(u) &\le 8 \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \Psi(u_{k,l,\epsilon}^-)\\
&\quad \times \left(\mathcal{H}_{\alpha_1}[0,\sqrt{S}]\mathcal{H}_{\alpha_2}[0,S] + \mathcal{H}_{\alpha_1}[0,S]\mathcal{H}_{\alpha_2}[0,\sqrt{S}]\right)\\
&\le 8\left(\frac{\mathcal{H}_{\alpha_1}[0,\sqrt{S}]}{\mathcal{H}_{\alpha_1}[0,S]} + \frac{\mathcal{H}_{B_{\alpha_2}}[0,\sqrt{S}]}{\mathcal{H}_{\alpha_2}[0,S]}\right)\\
&\quad \times \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \mathcal{H}_{\alpha_1}[0,S]\mathcal{H}_{\alpha_2}[0,S]\Psi(u_{k,l,\epsilon}^-)\\
&= o\left(\pi_{1,\epsilon}(u)\right), \quad u\to\infty, S\to\infty.
\end{aligned}
$$

In light of Lemma 5.4, we have

$$
\begin{aligned}
\sum_{(k,l,k_1,l_1)\in\mathbb{V}_2(u)} p_{k,l,k_1,l_1}^{(2)}(u) &\le \mathbb{Q}S^4 e^{-\mathbb{Q}_1 S^{\beta^*/4}} \sum_{(k,l,k_1,l_1)\in\mathbb{V}_2(u)} \Psi(\hat{u}_{k,l,k_1,l_1,\epsilon})\\
&\le \mathbb{Q}S^4 e^{-\mathbb{Q}_1 S^{\beta^*/4}} \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \Psi(u_{k,l,\epsilon}^-)\\
&= o\left(\pi_{1,\epsilon}(u)\right), \quad u\to\infty, S\to\infty.
\end{aligned}
$$

Consequently,

$$
\Lambda_2'(u) = o\left(\pi_{1,\epsilon}(u)\right), \quad u\to\infty, S\to\infty. \qquad (41)
$$

Combing (35), (38), (39) (40) with (41), we derive that

$$\pi_1(u) \sim 4\Big(\prod_{i=1}^{2} \mathcal{H}_{\alpha_i}\Gamma(1/\beta_i+1)\frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}\Big)\Psi(u), \quad u\to\infty.$$

Hence the claim follows.

Case ii) $\gamma_1=0, \gamma_2\in(0,\infty)$. Let in the sequel

$$\widetilde{I}_{k,0}(u) = I_{k,0}(u)\cup I_{k,-1}(u),$$

$$\mathbb{V}_1^{(1)}(u) = \big\{(k,k_1): -N_1(u)-2\le k<k_1\le N_1(u)+1, k_1-k\ge 2\big\},$$

and

$$\mathbb{V}_2^{(1)}(u) = \big\{(k,k_1): -N_1(u)-2\le k<k_1\le N_1(u)+1, k_1=k+1\big\}.$$

For any $0<\epsilon<1$ and all u large enough

$$\pi_{1,\epsilon}^{(1)}(u) - \sum_{i=1}^{2}\Lambda_i^{(1)}(u) \le \pi_1(u) \le \pi_{1,-\epsilon}^{(1)}(u)+\pi_{1,-\epsilon}^{(2)}(u), \tag{42}$$

with

$$\begin{aligned}
\pi_{1,\pm\epsilon}^{(1)}(u) &= \sum_{k=-N_1(u)\pm 2}^{N_1(u)\mp 1}\mathbb{P}\left(\sup_{(s,t)\in\widetilde{I}_{k,0}(u)}\frac{\overline{X}(s,t)}{1+(1\pm\epsilon)v_2^2(t)} > u_{k,\epsilon}^{1,\pm}\right),\\
\pi_{1,-\epsilon}^{(2)}(u) &= \sum_{k=-N_1(u)-2}^{N_1(u)+1}\sum_{|l|=1,l\neq -1}^{N_2(u)+1}\mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)}\overline{X}(s,t)>u_{k,l,\epsilon}^{-}\right),\\
\Lambda_i^{(1)}(u) &= \sum_{(k,k_1)\in\mathbb{V}_i^{(1)}(u)}\mathbb{P}\left(\sup_{(s,t)\in\widetilde{I}_{k,0}(u)}\overline{X}(s,t)>u_{k,\epsilon}^{1,-}, \sup_{(s,t)\in\widetilde{I}_{k_1,0}(u)}\overline{X}(s,t)>u_{k_1,\epsilon}^{1,-}\right), \quad i=1,2,
\end{aligned}$$

Set further $X_{u,k}(s,t)=\overline{X}(k\overleftarrow{\rho}_1(u^{-1})S+s,t)$ and define

$$\mathcal{K}_u=\{k\colon |k|\le N_1(u)+2\},\quad \mathcal{E}_u=\tilde{I}_{0,0}(u),\quad h_k(u)=u_{k,\epsilon}^{1,-},\quad d_u(s,t)=(1-\epsilon)v_2^2(t).$$

Using Lemma 5.2, we have

$$\lim_{u\to\infty}\sup_{k\in\mathcal{K}_u}\left|(\Psi(u_{k,\epsilon}^{1,-}))^{-1}\mathbb{P}\left(\sup_{(s,t)\in\widetilde{I}_{k,0}(u)}\frac{\overline{X}(s,t)}{1+(1-\epsilon)v_2^2(t)}>u_{k,\epsilon}^{1,-}\right)-\mathcal{H}_{\alpha_1}[0,S]\mathcal{P}_{\alpha_2}^{\gamma_2(1-\epsilon)}[-S,S]\right|=0.$$

Further, by Lemma 6.3, we obtain

$$\begin{aligned}
\pi_{1,-\epsilon}^{(1)}(u) &\sim \sum_{k=-N_1(u)-2}^{N_1(u)+1}\mathcal{H}_{\alpha_1}[0,S]\mathcal{P}_{\alpha_2}^{\gamma_2(1-\epsilon)}[-S,S]\Psi(u_{k,\epsilon}^{1,-})\\
&\le (1+o(1))2\mathcal{H}_{\alpha_1}[0,S]\mathcal{P}_{\alpha_2}^{\gamma_2(1-\epsilon)}[-S,S]\left(2k_\epsilon+1+\frac{\Psi(u)}{\overleftarrow{\rho}_1(u^{-1})S}\int_0^{\overleftarrow{v_1}(u^{-1}\ln u)}e^{-(1-\epsilon)^2u^2v_1^2(t)}dt\right)\\
&\sim 2\Gamma(1/\beta_1+1)\mathcal{H}_{\alpha_1}\mathcal{P}_{\alpha_2}^{\gamma_2}\frac{\overleftarrow{v}_1(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})}\Psi(u), \quad u\to\infty,\epsilon\to 0, S\to\infty.
\end{aligned}\tag{43}$$

Similarly, as $u\to\infty,\varepsilon\to 0, S\to\infty$,

$$\pi_{1,\epsilon}^{(1)}(u)\sim 2\Gamma(1/\beta_1+1)\mathcal{H}_{\alpha_1}\mathcal{P}_{\alpha_2}^{\gamma_2}\frac{\overleftarrow{v}_1(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})}\Psi(u). \tag{44}$$

Moreover, by Lemma 5.2

$$\begin{aligned}
\pi_{1,-\epsilon}^{(2)}(u) &\sim \sum_{k=-N_1(u)-2}^{N_1(u)+2}\sum_{|l|=1,l\neq-1}^{N_2(u)+1}\prod_{i=1}^{2}\mathcal{H}_{\alpha_i}[0,S]\Psi(u_{k,l,\epsilon}^{-})\\
&\le 2\Big(\prod_{i=1}^{2}\mathcal{H}_{\alpha_i}[0,S]\Big)\Psi(u)\sum_{k=-N_1(u)-2}^{N_1(u)+2}e^{-(1-\epsilon)u^2\inf_{s\in I_k(u)}v_1^2(|s|)}\sum_{|l|=1,l\neq-1}^{N_2(u)+1}e^{-(1-\epsilon)u^2v_2^2(\overleftarrow{\rho_2}(u^{-1}))|l^*S|^{\beta_2'}}
\end{aligned}\tag{45}$$

$$
\begin{aligned}
&\leq 2\Big(\prod_{i=1}^{2}\mathcal{H}_{\alpha_i}[0,S]\Big)\Psi(u)\sum_{k=-N_1(u)-2}^{N_1(u)+1} e^{-(1-\epsilon)u^2\inf_{s\in I_k(u)} v_1^2(|s|)}\sum_{|l|=1,l\neq -1}^{N_2(u)+1} e^{-(1-2\epsilon)\gamma_2|l^*S|^{\beta_2'}}\\
&\leq 4\Big(\prod_{i=1}^{2}\mathcal{H}_{\alpha_i}\Big)\frac{\overleftarrow{v}_1(u^{-1})\Psi(u)}{(1-\epsilon)^{2/\beta_1}\overleftarrow{\rho}_1(u^{-1})}S^2 e^{-\mathbb{Q}_4 S^{\beta_2'}}\\
&= o\left(\pi_{1,\epsilon}^{(1)}(u)\right) \qquad (46)
\end{aligned}
$$

as $u\to\infty, S\to\infty$ and $\varepsilon\to 0$, with $l^*=\min(|l|,|l+1|)$ and $0<\beta_2'<\beta_2$. Note that in (45) we use Lemma 6.1 to derive that for u and S large enough

$$
\frac{v_2^2(|t|)}{v_2^2(\overleftarrow{\rho_2}(u^{-1}))}\geq (l^*S)^{\beta_2'}, \quad t\in[l\overleftarrow{\rho_2}(u^{-1})S,(l+1)\overleftarrow{\rho_2}(u^{-1})S] \qquad (47)
$$

holds for $1\leq |l|\leq N_2(u), l\neq -1$. Using Lemma 5.4, we have

$$
\begin{aligned}
\Lambda_1^{(1)}(u) &\leq \mathbb{Q}S^4\sum_{(k,k_1)\in\mathbb{V}_1^{(1)}(u)}\Psi\left(u_{k,k_1,\epsilon}^1\right)e^{-\mathbb{Q}_1(|k-k_1|S)^{\beta^*/2}}\\
&\leq \mathbb{Q}S^4\sum_{k=-N_1(u)-2}^{N_1(u)+1}\Psi(u_{k,\epsilon}^{1,-})\sum_{m\geq 1}e^{-\mathbb{Q}_1 m^{\beta^*/2}S^{\beta^*/2}}\\
&\leq \mathbb{Q}S^4\sum_{k=-N_1(u)-2}^{N_1(u)+1}\Psi(u_{k,\epsilon}^{1,-})e^{-\mathbb{Q}_2 S^{\beta^*/2}}\\
&= o\left(\pi_{1,\epsilon}^{(1)}(u)\right), \quad u\to\infty, S\to\infty, \qquad (48)
\end{aligned}
$$

with $\hat{u}_{k,k_1,\epsilon}=\min(u_{k,\epsilon}^{1,-},u_{k_1,\epsilon}^{1,-})$ and $\beta^*=\min(\alpha_1,\alpha_2)$. Applying again Lemma 5.2 yields that

$$
\begin{aligned}
\Lambda_2^{(1)}(u) &\leq \sum_{k=-N_1(u)-2}^{N_1(u)+1}\Bigg[\mathbb{P}\left(\sup_{(s,t)\in\widetilde{I}_{k,0}(u)}\frac{\overline{X}(s,t)}{1+(1-\epsilon)v_2^2(t)}>u_{k,\epsilon}^{1,-}\right)\\
&\quad +\mathbb{P}\left(\sup_{(s,t)\in\widetilde{I}_{k+1,0}(u)}\frac{\overline{X}(s,t)}{1+(1-\epsilon)v_2^2(t)}>u_{k+1,\epsilon}^{1,-}\right)\\
&\quad -\mathbb{P}\left(\sup_{(s,t)\in\widetilde{I}_{k,0}(u)\cup\widetilde{I}_{k+1,0}(u)}\frac{\overline{X}(s,t)}{1+(1+\epsilon)v_2^2(t)}>\check{u}_{k,k+1,\epsilon}\right)\Bigg]\\
&\leq \left((1+\epsilon)2\mathcal{H}_{B_{\alpha_1}}[0,S]-(1-\epsilon)\mathcal{H}_{B_{\alpha_1}}[0,2S]\right)\\
&\quad \times\mathcal{P}_{\alpha_2}^{\gamma_2(1-\epsilon)}[-S,S]\sum_{k=-N_1(u)-2}^{N_1(u)+1}\Psi(u_{k,\epsilon}^{1,-})\\
&= o\left(\pi_{1,\epsilon}^{(1)}(u)\right), \quad u\to\infty, S\to\infty, \qquad (49)
\end{aligned}
$$

with $\check{u}_{k,k_1,\epsilon}=\max(u_{k,\epsilon}^{1,-},u_{k_1,\epsilon}^{1,-})$. Combining (42), (43), (44), (46), (48) with (49) leads to

$$
\pi_1(u)\sim 2\Gamma(1/\beta_1+1)\mathcal{H}_{\alpha_1}\mathcal{P}_{\alpha_2}^{\gamma_2}\frac{\overleftarrow{v}_1(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})}\Psi(u), \quad u\to\infty,
$$

which together with (33) establishes the proof.

<u>Case ii) $\gamma_1=0,\gamma_2=\infty$.</u> Observe that for u large enough

$$
\mathbb{P}\left(\sup_{|s|\leq\overleftarrow{v}_1(u^{-1}\ln u)}X(s,0)>u\right)\leq\pi_1(u)\leq\pi_{1,-\epsilon}^{(1)}(u)+\pi_{1,-\epsilon}^{(2)}(u),
$$

with $\pi_{1,-\epsilon}^{(1)}(u),\pi_{1,-\epsilon}^{(2)}(u)$ defined in (42). Note that (43), (46) also hold except for the fact that in light of Lemma 5.2, we have to replace both $\mathcal{P}_{\alpha_2}^{\gamma_2(1-\epsilon)}(S)$ and $\mathcal{P}_{\alpha_2}^{\gamma_2}$ by 1 in (43), which gives the upper bound. Moreover, in light of Theorem 2.1

we can derive the asymptotics of

$$\mathbb{P}\left(\sup_{|s|\le \overleftarrow{v}_1(u^{-1}\ln u)} X(s,0)>u\right)$$

as $u\to\infty$, which is the lower bound. One can easily check that the upper bound and lower bound are asymptotically equal, establishing the claim.

<u>Case iii) $\gamma_1,\gamma_2\in(0,\infty)$.</u> Let next

$$\widehat{I}_{0,0}(u)=I_{0,0}(u)\cup I_{-1,0}(u)\cup I_{0,-1}(u)\cup I_{-1,-1}(u).$$

It follows straightforwardly that for any $0<\epsilon<1/2$ and u large enough

$$\pi^{(3)}_{1,\epsilon}(u)\le \pi_1(u)\le \pi^{(3)}_{1,-\epsilon}(u)+\pi^{(4)}_{1,-\epsilon}(u), \tag{50}$$

with

$$\pi^{(3)}_{1,\pm\epsilon}(u)=\mathbb{P}\left(\sup_{(s,t)\in\widehat{I}_{0,0}(u)}\frac{\overline{X}(s,t)}{(1+(1\pm\epsilon)v_1^2(s))(1+(1\pm\epsilon)v_2^2(t))}>u\right),$$

and

$$\pi^{(4)}_{1,-\epsilon}(u)=\sum_{|k|=1,k\neq-1}^{N_1(u)+2}\sum_{|l|=1,l\neq-1}^{N_2(u)+2}\mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)}\overline{X}(s,t)>u^-_{k,l,\epsilon}\right).$$

By Lemma 5.2, it follows that

$$\pi^{(3)}_{1,\pm\epsilon}(u)\sim\prod_{i=1}^{2}\mathcal{P}^{(1\pm\epsilon)\gamma_i}_{\alpha_i}[-S,S]\Psi(u),\quad u\to\infty, S\to\infty. \tag{51}$$

In addition, using Lemma 5.2 and (47), the same argument as given in the derivation of the upper bound for $\pi^{(2)}_{1,-\epsilon}(u)$ yields

$$\pi^{(4)}_{1,-\epsilon}(u)=o(\pi^{(3)}_{1,\pm\epsilon}(u)) \tag{52}$$

as $u\to\infty$ and $S\to\infty$. Combination of (50) and (51) with (52) leads to

$$\pi_1(u)\sim\prod_{i=1}^{2}\mathcal{P}^{\gamma_i}_{\alpha_i}\Psi(u),\quad u\to\infty, S\to\infty,$$

hence the proof of this case is complete.

<u>Case iii) $\gamma_1\in(0,\infty),\gamma_2=\infty$.</u> For u large enough

$$\mathbb{P}\left(\sup_{|s|\le \overleftarrow{v}_1(u^{-1}\ln u)} X(s,0)>u\right)\le\pi_1(u)\le\pi^{(3)}_{1,-\epsilon}(u)+\pi^{(4)}_{1,-\epsilon}(u),$$

with $\pi^{(3)}_{1,-\epsilon}(u)$ and $\pi^{(4)}_{1,-\epsilon}(u)$ defined in (50). By Lemma 5.2, the same arguments as given in previous case shows that

$$\pi_1(u)\le\mathcal{P}^{\gamma_1}_{\alpha_1}\Psi(u)(1+o(1)),\quad u\to\infty.$$

In light of Theorem 2.1, we derive that

$$\mathbb{P}\left(\sup_{|s|\le \overleftarrow{v}_1(u^{-1}\ln u)} X(s,0)>u\right)\sim\mathcal{P}^{\gamma_1}_{\alpha_1}\Psi(u),\quad u\to\infty,$$

which establishes the claim.

<u>Case iii) $\gamma_1,\gamma_2=\infty$.</u> Similarly, (50), (51) and (52) hold with $\pi^{(3)}_{1,\epsilon}(u)$ replaced by $\mathbb{P}(X(0)>u)$ in (50), and $\mathcal{P}^{\gamma_1}_{\alpha_1},\mathcal{P}^{\gamma_2}_{\alpha_2}$ replaced by 1 in (51). $\square$

Proof of Theorem 3.7 Similarly as in (35) for any u positive

$$\pi_1^+(u)-\Lambda^{(1)}(u)\le\pi_1(u)\le\pi_1^-(u), \tag{53}$$

with

$$\pi_1^{\pm}(u)=\sum_{k=-N_1(u)\pm2}^{N_1(u)\mp1}\sum_{l=-N_2(u)\pm2}^{N_2(u)\mp1}\mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)}\overline{X}(s,t)>u^{\pm}_{k,l,\epsilon}\right),$$

$$\Lambda^{(1)}(u) = \sum_{(k,l,k_1,l_1)\in\mathbb{V}_1(u)\cup\mathbb{V}_2(u)} \mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)} X(s,t)>u, \sup_{(s,t)\in I_{k_1,l_1}(u)} X(s,t)>u\right).$$

Since B is a non-singular matrix, then there exists a positive constant $\mu>0$ such that for any s,t,

$$|b_{11}s+b_{12}t|+|b_{21}s+b_{22}t|\ge \mu\left(|s|+|t|\right).$$

Thus, for $(s,t)\in I_{k,l}(u)$ with $|k|,|l|\ge M\ge 2, M\in\mathbb{N}$

$$|b_{11}s+b_{12}t|+|b_{21}s+b_{22}t|\ge \mu S\left((M-1)\overleftarrow{\rho_1}(u^{-1})+(M-1)\overleftarrow{\rho_2}(u^{-1})\right).$$

Set next

$$a(s,t):=v_1^2(|b_{11}s+b_{12}t|)+v_2^2(|b_{21}s+b_{22}t|).$$

By Lemma 6.1, for any $(s,t),(s',t')\in I_{k,l}(u)$ with $M\le |k|\le N_1(u)+2, M\le |l|\le N_2(u)+2$ and $0<\epsilon<\min(1,\beta_1)$, we have that for u sufficiently large

$$\frac{a(s,t)}{v_1^2(|b_{11}s+b_{12}t|+|b_{21}s+b_{22}t|)}\ge (1-\epsilon/3)\left(\nu^{\beta_1+\epsilon}+\theta(1-\nu)^{\beta_1+\epsilon}\right),$$

and

$$\frac{a(s',t')}{v_1^2(|b_{11}s+b_{12}t|+|b_{21}s+b_{22}t|)}\le \frac{(1-\epsilon/3)^2}{1-\epsilon}\left(\max\left((\nu+\delta)^{\beta_1+\epsilon},(\nu+\delta)^{\beta_1-\epsilon}\right)+\theta\max\left((1-\nu+\delta)^{\beta_1+\epsilon},(1-\nu+\delta)^{\beta_1-\epsilon}\right)\right),$$

where

$$\nu=\frac{|b_{11}s+b_{12}t|}{|b_{11}s+b_{12}t|+|b_{21}s+b_{22}t|}\in[0,1],$$

and

$$0\le\delta\le\frac{(|b_{11}|+|b_{12}|+|b_{21}|+|b_{22}|)\left(\overleftarrow{\rho_1}(u^{-1})+\overleftarrow{\rho_2}(u^{-1})\right)S}{|b_{11}s+b_{12}t|+|b_{21}s+b_{22}t|}\le\frac{2\left(\sum_{i,j=1}^2|b_{ij}|\right)(1+\eta^{-1/\alpha_1})}{\mu\left((M-1)+(M-1)\eta^{-1/\alpha_1}\right)}\to 0,$$

as $M\to\infty$. Note that if $\epsilon,\delta>0$ sufficiently small, then

$$\nu^{\beta_1+\epsilon}+\theta(1-\nu)^{\beta_1+\epsilon}\ge(1-\epsilon/3)\left(\max\left((\nu+\delta)^{\beta_1+\epsilon},(\nu+\delta)^{\beta_1-\epsilon}\right)+\theta\max\left((1-\nu+\delta)^{\beta_1+\epsilon},(1-\nu+\delta)^{\beta_1-\epsilon}\right)\right),\quad \nu\in[0,1].$$

Hence, for $\epsilon>0$ sufficiently small, there exists $k_\epsilon\in\mathbb{N}$ such that for u large enough and $k_\epsilon\le|k|\le N_1(u)+2,\quad k_\epsilon\le |l|\le N_2(u)+2$

$$a(s,t)\ge(1-\epsilon)a(s',t'),\quad (s,t),(s',t')\in I_{k,l}(u),$$

which implies that for u large enough

$$\inf_{(s,t)\in I_{k,l}(u)} a(s,t)\ge(1-\epsilon)\sup_{(s,t)\in I_{k,l}(u)} a(s,t),\quad k_\epsilon\le|k|\le N_1(u)+2,\quad k_\epsilon\le|l|\le N_2(u)+2. \tag{54}$$

Case i). Using (37) and by (54), we have

$$\begin{aligned}
\pi_1^-(u) &\sim \sum_{k=-N_1(u)-2}^{N_1(u)+1}\sum_{l=-N_2(u)-2}^{N_2(u)+1}\prod_{i=1}^2\mathcal{H}_{\alpha_i}[0,S]\Psi(u^-_{k,l,\epsilon})\\
&\sim \left(\prod_{i=1}^2\mathcal{H}_{\alpha_i}[0,S]\right)\Psi(u)\sum_{k=-N_1(u)-2}^{N_1(u)+1}\sum_{l=-N_2(u)-2}^{N_2(u)+1} e^{-(1-\epsilon)u^2\inf_{(s,t)\in I_{k,l}(u)}a(s,t)}\\
&\le \left(\prod_{i=1}^2\mathcal{H}_{\alpha_i}[0,S]\right)\Psi(u)\Big(R_3(u)+R_4(u)\\
&\quad+\frac{1}{\overleftarrow{\rho_1}(u^{-1})\overleftarrow{\rho_2}(u^{-1})S^2}\sum_{|k|=k_\epsilon}^{N_1(u)+2}\sum_{|l|=k_\epsilon}^{N_2(u)+2}\int_{(s,t)\in I_{k,l}(u)} e^{-(1-\epsilon)^2u^2a(s,t)}dsdt\Big),
\end{aligned}$$

where

$$R_3(u)=\sum_{|k|\le k_\epsilon}\sum_{l=-N_2(u)-2}^{N_2(u)+1} e^{-(1-\epsilon)u^2\inf_{(s,t)\in I_{k,l}(u)}a(s,t)},$$

and

$$R_4(u)=\sum_{|l|\le k_\epsilon}\sum_{k=-N_1(u)-2}^{N_1(u)+1} e^{-(1-\epsilon)u^2\inf_{(s,t)\in I_{k,l}(u)}a(s,t)}.$$

By linear transformation $(s',t')^\top=B(s,t)^\top$ and Lemma 6.3, we have with $\xi(u)=u^{-1}\ln u$

$$\begin{aligned}
&\frac{1}{\overleftarrow{\rho_1}(u^{-1})\overleftarrow{\rho_2}(u^{-1})}\sum_{|k|=k_\epsilon}^{N_1(u)+2}\sum_{|l|=k_\epsilon}^{N_2(u)+2}\int_{(s,t)\in I_{k,l}(u)} e^{-(1-\epsilon)^2u^2a(s,t)}dsdt\\
&\le \frac{1}{\overleftarrow{\rho_1}(u^{-1})\overleftarrow{\rho_2}(u^{-1})}\int_{-2\overleftarrow{v_1}(\xi(u))}^{2\overleftarrow{v_1}(\xi(u))}\int_{-2\overleftarrow{v_2}(\xi(u))}^{2\overleftarrow{v_2}(\xi(u))} e^{-(1-\epsilon)^2u^2a(s,t)}dsdt\\
&\le \frac{1}{\overleftarrow{\rho_1}(u^{-1})\overleftarrow{\rho_2}(u^{-1})}\frac{1}{|det(B)|}\int_{-\mathbb{Q}\overleftarrow{v_1}(\xi(u))}^{\mathbb{Q}\overleftarrow{v_1}(\xi(u))}\int_{-\mathbb{Q}\overleftarrow{v_2}(\xi(u))}^{\mathbb{Q}\overleftarrow{v_2}(\xi(u))} e^{-(1-\epsilon)^2u^2v_1^2(|s'|)}e^{-(1-\epsilon)^2u^2v_2^2(|t'|)}ds'dt'\\
&= \frac{1}{\overleftarrow{\rho_1}(u^{-1})\overleftarrow{\rho_2}(u^{-1})}\frac{4}{|det(B)|}\int_{0}^{\mathbb{Q}\overleftarrow{v_1}(\xi(u))}\int_{0}^{\mathbb{Q}\overleftarrow{v_2}(\xi(u))} e^{-(1-\epsilon)^2u^2v_1^2(|s'|)}e^{-(1-\epsilon)^2u^2v_2^2(|t'|)}ds'dt'\\
&\sim (1-\epsilon)^{-1/\beta_1-1/\beta_2}\frac{4}{|det(B)|}\left(\prod_{i=1}^d\Gamma(1/\beta_i+1)\frac{\overleftarrow{v_i}(u^{-1})}{\overleftarrow{\rho_i}(u^{-1})}\right)\to\infty,\quad u\to\infty.
\end{aligned}$$

Moreover, by Lemma 6.5 there exists a constant $\kappa_1>0$ such that

$$\kappa_1v_1^2(|s|)+\kappa_1v_2^2(|t|)\le a(s,t),\quad s,t\in\mathbb{R}. \tag{55}$$

Thus we have

$$\begin{aligned}
R_3(u)&\le \sum_{|k|\le k_\epsilon}\sum_{l=-N_2(u)-2}^{N_2(u)+1} e^{-(1-\epsilon)u^2\inf_{(s,t)\in I_{k,l}(u)}\kappa_1\left(v_1^2(|s|)+v_2^2(|t|)\right)}\\
&\le (2k_\epsilon+1)\sum_{l=-N_2(u)-2}^{N_2(u)+1} e^{-(1-\epsilon)u^2\inf_{t\in J_l(u)}\kappa_1v_2^2(|t|)}\\
&\le \mathbb{Q}\frac{\overleftarrow{v_2}(u^{-1})}{\overleftarrow{\rho_2}(u^{-1})}=o\left(\frac{\overleftarrow{v_1}(u^{-1})\overleftarrow{v_2}(u^{-1})}{\overleftarrow{\rho_1}(u^{-1})\overleftarrow{\rho_2}(u^{-1})}\right),\quad u\to\infty.
\end{aligned}$$

Similarly,

$$R_4(u)\le\mathbb{Q}_1\frac{\overleftarrow{v_1}(u^{-1})}{\overleftarrow{\rho_1}(u^{-1})}=o\left(\frac{\overleftarrow{v_1}(u^{-1})\overleftarrow{v_2}(u^{-1})}{\overleftarrow{\rho_1}(u^{-1})\overleftarrow{\rho_2}(u^{-1})}\right),\quad u\to\infty,$$

implying

$$\pi_1^-(u)\le\frac{4}{|det(B)|}\prod_{i=1}^2\left[\Gamma(1/\beta_i+1)\mathcal{H}_{\alpha_i}\frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}\right]\Psi(u)(1+o(1)),\quad u\to\infty,S\to\infty,\epsilon\to0. \tag{56}$$

In the same way we obtain that

$$\pi_1^+(u)\ge\frac{4}{|det(B)|}\prod_{i=1}^2\left[\Gamma(1/\beta_i+1)\mathcal{H}_{\alpha_i}\frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}\right]\Psi(u)(1+o(1)),\quad u\to\infty,S\to\infty. \tag{57}$$

Due to (55), with κ_1 as in Lemma 6.5, letting

$$Y(s,t)=\frac{\overline{X}(s,t)}{1+\frac{\kappa_1}{2}v_1^2(|s|)+\frac{\kappa_1}{2}v_2^2(|t|)},\quad (s,t)\in\mathbb{R}^2 \tag{58}$$

we have

$$\Lambda^{(1)}(u)\le\sum_{(k,l,k_1,l_1)\in\mathbb{V}_1(u)\cup\mathbb{V}_2(u)}\mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)}Y(s,t)>u,\ \sup_{(s,t)\in I_{k_1,l_1}(u)}Y(s,t)>u\right).$$

The same argument as given in the proof of Theorem 3.1 leads to

$$\Lambda^{(1)}(u)=o(\pi_1^-(u)),\quad u\to\infty,S\to\infty. \tag{59}$$

Inserting (56)-(59) into (53) yields

$$\pi_1(u) \sim \frac{4}{|det(B)|} \prod_{i=1}^{2} \Big[\Gamma(1/\beta_i + 1)\mathcal{H}_{\alpha_i} \frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}\Big] \Psi(u),$$

which together with (33) completes the proof.

Case ii) $\gamma_1, \gamma_2 \in (0, \infty)$. Using the same notation for $\widehat{I}_{0,0}(u)$ as that in the proof of Theorem 3.1 for case iii) $\gamma_1, \gamma_2 \in (0, \infty)$, (50) holds with

$$\pi^{(3)}_{1,\pm\epsilon}(u) = \mathbb{P}\left(\sup_{(s,t)\in\widehat{I}_{0,0}(u)} \frac{\overline{X}(s,t)}{1 + (1 \pm \epsilon)a(s,t)} > u \right),$$

and

$$\pi^{(4)}_{1,-\epsilon}(u) = \sum_{|k|=1,k\neq -1}^{N_1(u)+2} \sum_{|l|=1,l\neq -1}^{N_2(u)+2} \mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)} X(s,t) > u \right).$$

Noting that

$$\begin{aligned} & u^2 \left(v_1^2(|b_{11}\overleftarrow{\rho}_1(1/u)s + b_{12}\overleftarrow{\rho}_2(1/u)t|) + v_2^2(|b_{21}\overleftarrow{\rho}_1(1/u)s + b_{22}\overleftarrow{\rho}_2(1/u)t|) \right) \\ & \to \gamma_1 |b_{11}s + b_{12}\eta^{-1/\alpha_1}t|^{\alpha_1} + \theta\gamma_1 |b_{21}s + b_{22}\eta^{-1/\alpha_1}t|^{\alpha_1}, \quad u \to \infty \end{aligned}$$

uniformly with respect to $s, t \in [-S, S]^2$, it follows from Lemma 5.2 that

$$\pi^{(3)}_{1,\pm\epsilon}(u) \sim \mathcal{P}_{\alpha}^{(1\pm\epsilon)\gamma_1,(1\pm\epsilon)\theta\gamma_1,B_{\eta,\alpha}}(S)\Psi(u) \sim \mathcal{P}_{\alpha}^{\gamma_1,\theta\gamma_1,B_{\eta,\alpha}}\Psi(u), \quad u \to \infty, S \to \infty, \epsilon \to 0.$$

Moreover, by Lemma 6.5 and (52), with Y defined by (58)

$$\pi^{(4)}_{1,-\epsilon}(u) \le \sum_{|k|=1,k\neq -1}^{N_1(u)+2} \sum_{|l|=1,l\neq -1}^{N_2(u)+2} \mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)} Y(s,t) > u \right) = o\left(\Psi(u)\right)$$

as $u \to \infty, S \to \infty$. Hence

$$\pi_1(u) \sim \mathcal{P}_{\alpha}^{\gamma_1,\theta\gamma_1,B_{\eta,\alpha}}\Psi(u),$$

which completes the proof.

Case ii) $\gamma_1 = \gamma_2 = \infty$. Observe that for all u large

$$\mathbb{P}\left(X(0,0) > u\right) \le \pi_1(u) \le \pi^{(3)}_{1,-\epsilon}(u) + \pi^{(4)}_{1,-\epsilon}(u),$$

with $\pi^{(3)}_{1,-\epsilon}(u)$ and $\pi^{(4)}_{1,-\epsilon}(u)$ defined as in Case ii) $\gamma_1, \gamma_2 \in (0, \infty)$. Borrowing the arguments in the proof of the aforementioned case we obtain

$$\pi^{(3)}_{1,-\epsilon}(u) + \pi^{(4)}_{1,-\epsilon}(u) \le \Psi(u)(1 + o(1)), \quad u \to \infty,$$

which completes the proof. □

Proof of Theorem 3.3 This scenario requires a modification of the set D_u. Let in the following

$$D^{(1)}_u = \{(s,t), |s + b_{12}t| \le \overleftarrow{v}_1(u^{-1}\ln u), |t| \le \overleftarrow{v}_2(u^{-1}\ln u)\}.$$

We have that (33)-(34) also hold with D_u replaced by $D^{(1)}_u$. In this scenario, denote $\alpha = \alpha_1 = \alpha_2$.

Case i) $\gamma_1 = \gamma_2 = 0$. Let for $u > 0$

$$E^+_l(u) = \{k : I_{k,l}(u) \subset D^{(1)}_u\}, \quad E^-_l(u) = \{k : I_{k,l}(u) \cap D^{(1)}_u \neq \emptyset\},$$

$$E^{(1)}(u) = \Big\{(k,l,k_1,l_1)\colon k \le k_1, I_{k,l}(u) \cap D^{(1)}_u \neq \emptyset, I_{k_1,l_1}(u) \cap D^{(1)}_u \neq \emptyset \text{ and } I_{k,l}(u) \cap I_{k_1,l_1}(u) = \emptyset\Big\},$$

$$E^{(2)}(u) = \Big\{(k,l,k_1,l_1)\colon k \le k_1, I_{k,l}(u) \cap D^{(1)}_u \neq \emptyset, I_{k_1,l_1}(u) \cap D^{(1)}_u \neq \emptyset, (k,l) \neq (k_1,l_1) \text{ and } I_{k,l}(u) \cap I_{k_1,l_1}(u) \neq \emptyset\Big\}.$$

It follows that

$$\pi^+_2(u) - \sum_{i=1}^{2} \Lambda^{(2)}_i(u) \le \pi_1(u) \le \pi^-_2(u),$$

where

$$\pi_2^{\pm}(u) = \sum_{l=-N_2(u)\pm 2}^{N_2(u)\mp 1} \sum_{k\in E_l^{\pm}(u)} \mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^{\pm}\right),$$

$$\Lambda_i^{(2)}(u) = \sum_{(k,l,k_1,l_1)\in E^{(i)}(u)} \mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^{-}, \sup_{(s,t)\in I_{k_1,l_1}(u)} \overline{X}(s,t) > u_{k_1,l_1,\epsilon}^{-}\right).$$

Using (37), we have

$$\begin{aligned}
\pi_2^{-}(u) &= \sum_{l=-N_2(u)-2}^{N_2(u)+1} \sum_{k\in E_0^{-}(u)} \mathbb{P}\left(\sup_{(s,t)\in I_{0,0}(u)} \overline{X}(\overleftarrow{\rho_1}(u^{-1})kS+s, \overleftarrow{\rho_2}(u^{-1})lS+t) > u_{k,l,\epsilon}^{-}\right)\\
&\sim \sum_{l=-N_2(u)-2}^{N_2(u)+1} \sum_{k\in E_l^{-}(u)} \prod_{i=1}^{2}\mathcal{H}_{\alpha_i}[0,S]\Psi(u_{k,l,\epsilon}^{-})\\
&\sim \prod_{i=1}^{2}\mathcal{H}_{\alpha_i}[0,S]\Psi(u) \sum_{l=-N_2(u)-2}^{N_2(u)+1} \sum_{k\in E_l^{-}(u)} e^{-(1-\epsilon)u^2 \inf_{(s,t)\in I_{k,l}(u)}\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right)}.
\end{aligned}$$

We observe that, for u sufficiently large and all $|l|\le N_2(u)+2$ (set $c(u)=\frac{\overleftarrow{\rho}_2(1/u)}{\overleftarrow{\rho}_1(1/u)}$)

$$E_l^{-} \subset \left\{k\in\mathbb{N}:\ |k-[b_{12}lc(u)]| \le N_1(u)+2\left(2+\left[|b_{12}|\eta^{-1/\alpha}\right]\right)\right\},$$

and

$$E_l^{+} \supset \left\{k\in\mathbb{N}:\ |k-[b_{12}lc(u)]| \le N_1(u)-2\left(2+\left[|b_{12}|\eta^{-1/\alpha}\right]\right)\right\}.$$

Similarly as (28), we have that for any $\epsilon>0$ there exists $k_\epsilon\in\mathbb{N}$ such that

$$\inf_{t\in J_l(u)} v_2^2(|t|) \ge (1-\epsilon)\sup_{t\in J_l(u)} v_2^2(|t|)$$

holds for $k_\epsilon \le |l| \le N_2(u)+2$. Moreover,

$$\inf_{(s,t)\in I_{k,l}(u)} v_1^2(|s+b_{12}t|) \ge (1-\epsilon)\sup_{(s,t)\in I_k(u)} v_1^2(|s+b_{12}l\overleftarrow{\rho}_2(u^{-1})S|)$$

hold for $|l|\le N_2(u)+2$ and

$$k\in E_{l,\epsilon}^{-}(u) = \left\{k:\ k_\epsilon \le |k-[b_{12}lc(u)]| \le N_1(u)+2\left(1+\left[|b_{12}|\eta^{-1/\alpha}\right]\right)\right\}.$$

Therefore, in light of Lemma 6.3, we have

$$\begin{aligned}
&\sum_{l=-N_2(u)-2}^{N_2(u)+1} \sum_{k\in E_l^{-}(u)} e^{-(1-\epsilon)u^2\inf_{(s,t)\in I_{k,l}(u)}\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right)}\\
&\le \sum_{l=-N_2(u)-2}^{N_2(u)+1} e^{-(1-\epsilon)u^2\inf_{t\in J_l(u)} v_2^2(|t|)} \sum_{k\in E_l^{-}(u)} e^{-(1-\epsilon)u^2\inf_{(s,t)\in I_{k,l}(u)} v_1^2(|s+b_{12}t|)}\\
&\le \frac{1}{\overleftarrow{\rho}_2(u^{-1})S} \sum_{|l|\ge k_\epsilon}^{N_2(u)+2} \int_{t\in J_l(u)} e^{-(1-\epsilon)^2u^2v_2^2(|t|)}dt\\
&\times\left(2k_\epsilon+1+\frac{1}{\overleftarrow{\rho}_1(1/u)S}\sum_{k\in E_{l,\epsilon}(u)}\int_{s\in I_k(u)} e^{-(1-\epsilon)^2u^2v_1^2(|s+b_{12}l\overleftarrow{\rho}_2(1/u)S|)}ds\right)\\
&+\sum_{|l|=0}^{k_\epsilon}\left(2k_\epsilon+1+\frac{1}{\overleftarrow{\rho}_1(1/u)S}\sum_{k\in E_{l,\epsilon}(u)}\int_{s\in I_k(u)} e^{-(1-\epsilon)^2u^2v_1^2(|s+b_{12}l\overleftarrow{\rho}_2(1/u)S|)}ds\right)
\end{aligned}$$

$$\leq \frac{2(1+o(1))}{\overleftarrow{\rho}_2(u^{-1})S}\int_0^{\mathbb{Q}\overleftarrow{v}_2(u^{-1}\ln u)} e^{-(1-\epsilon)^2u^2v_2^2(|t|)}dt\left(2k_\epsilon+1+\frac{2}{\overleftarrow{\rho}_1(1/u)S}\int_0^{\mathbb{Q}\overleftarrow{v}_1(u^{-1}\ln u)} e^{-(1-\epsilon)^2u^2v_1^2(|s|)}ds\right)$$

$$\sim (1-\epsilon)^{-1/\beta_1-1/\beta_2}\frac{4}{S^2}\prod_{i=1}^{2}\Gamma(1/\beta_i+1)\frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)},\quad u\to\infty.$$

Consequently,

$$\pi_2^-(u) \quad\leq\quad 4\prod_{i=1}^{2}\left[\Gamma(1/\beta_i+1)\mathcal{H}_{\alpha_i}\frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}\right]\Psi(u)(1+o(1)),\quad u\to\infty, S\to\infty,\epsilon\to 0. \tag{60}$$

Let next

$$E_{l,\epsilon}^+(u)=\left\{k\colon k_\epsilon\leq |k-[blc(u)]|\leq N_1(u)-2\Big(1+\Big[|b|\eta^{-1/\alpha}\Big]\Big)\right\}.$$

Similarly,

$$\begin{aligned}\pi_2^+(u) \quad&\geq\quad \prod_{i=1}^{2}\mathcal{H}_{\alpha_i}[0,S]\Psi(u)\sum_{l=-N_2(u)+2}^{N_2(u)-1}\sum_{k\in E_l^+(u)} e^{-(1-\epsilon)u^2\sup_{(s,t)\in I_{k,l}(u)}\left(v_1^2(|s+bt|)+v_2^2(|t|)\right)}\\ &\geq\quad \prod_{i=1}^{2}\mathcal{H}_{\alpha_i}[0,S]\Psi(u)\sum_{|l|=k_\epsilon}^{N_2(u)-2} e^{-(1-\epsilon)u^2\sup_{t\in J_l(u)}v_2^2(|t|)}\sum_{k\in E_{l,\epsilon}^+(u)} e^{-(1-\epsilon)u^2\sup_{(s,t)\in I_{k,l}(u)}v_1^2(|s+bt|)}\\ &\sim\quad 4\prod_{i=1}^{2}\left[\Gamma(1/\beta_i+1)\mathcal{H}_{\alpha_i}\frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}\right]\Psi(u),\quad u\to\infty, S\to\infty,\epsilon\to 0.\end{aligned} \tag{61}$$

Following the same argumentation as given in (40) and (41), we get that $\Lambda_i^{(2)}(u)=o\left(\pi_2^-(u)\right),\quad i=1,2,u\to\infty$, which together with (60) and (61) completes the proof.

Case ii) $\gamma_2=0,\gamma_1\in(0,\infty)$. We first introduce

$$\begin{aligned}L_{0,l}^*(u) &= \left\{(s,t)\colon |s+b_{12}t|\leq \overleftarrow{\rho}_1(1/u)S, t\in[l\overleftarrow{\rho}_2(1/u)S,(l+1)\overleftarrow{\rho}_2(1/u)S]\right\},\\ L_{k,l}(u) &= \left\{(s,t)\colon k\overleftarrow{\rho}_1(1/u)S\leq s+b_{12}t\leq (k+1)\overleftarrow{\rho}_1(1/u)S, t\in[l\overleftarrow{\rho}_2(1/u)S,(l+1)\overleftarrow{\rho}_2(1/u)S]\right\},\end{aligned}$$

$$u_{k,l,\epsilon,*}^-=u\left(1+(1-\epsilon)\inf_{(s,t)\in L_{k,l}(u)}\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right)\right),$$

with $k,l\in\mathbb{Z}$. Then we have

$$\pi_3^+(u)+\sum_{i=1}^{2}\Lambda_i^{(3)}(u)\leq\pi_1(u)\leq\pi_3^-(u)+\pi_4(u),$$

where

$$\begin{aligned}\pi_3^\pm(u) &= \sum_{l=-N_2(u)\pm 2}^{N_2(u)\mp 1}\mathbb{P}\left(\sup_{(s,t)\in L_{0,l}^*(u)}\frac{\overline{X}(s,t)}{1+(1\pm\epsilon)v_1^2(|s+b_{12}t|)}>u_{l,\epsilon}^{2,\pm}\right),\\ \pi_4(u) &= \sum_{|k|\leq N_1(u)+2,k\neq 0,-1}\ \sum_{l=-N_2(u)\pm 2}^{N_2(u)\mp 1}\mathbb{P}\left(\sup_{(s,t)\in L_{k,l}(u)}\overline{X}(s,t)>u_{k,l,\epsilon,*}^-\right),\\ \Lambda_1^{(3)}(u) &= \sum_{-N_2(u)-2\leq l+1<l_1\leq N_2(u)+2}\mathbb{P}\left(\sup_{(s,t)\in L_{0,l}^*(u)}\overline{X}(s,t)>u_{l,\epsilon}^{2,-},\ \sup_{(s,t)\in L_{0,l_1}^*(u)}\overline{X}(s,t)>u_{l_1,\epsilon}^{2,-}\right),\\ \Lambda_2^{(3)}(u) &= \sum_{l=-N_2(u)-2}^{N_2(u)+2}\mathbb{P}\left(\sup_{(s,t)\in L_{0,l}^*(u)}\frac{\overline{X}(s,t)}{1+(1-\epsilon)v_1^2(|s+b_{12}t|)}>u_{l,\epsilon}^{2,-},\ \sup_{(s,t)\in L_{0,l+1}^*(u)}\frac{\overline{X}(s,t)}{1+(1-\epsilon)v_1^2(|s+b_{12}t|)}>u_{l+1,\epsilon}^{2,-}\right).\end{aligned}$$

Let

$$X_{u,l}(s,t)=\overline{X}(-b_{12}l\overleftarrow{\rho}_2(u^{-1})S+s,l\overleftarrow{\rho}_2(u^{-1})S+t),\quad \mathcal{K}_u=\{l,|l|\leq N_2(u)+2\},\quad \mathcal{E}_u=L_{0,0}^*(u),$$

$$h_l(u)=u_{l,\epsilon}^{2,-},\quad d_u(s,t)=(1-\epsilon)v_1^2(|s+b_{12}t|).$$

Since

$$\lim_{u\to\infty}\sup_{l\in\mathcal{K}_u}\left|(u_{l,\epsilon}^{2,-})^2 v_1^2(|\overleftarrow{\rho}_1(1/u)s+b_{12}\overleftarrow{\rho}_2(1/u)t|)-\gamma_1|s+b_{12}\eta^{-1/\alpha}t|\right|=0$$

uniformly for (s,t) over any compact set by Lemma 5.2, we have

$$\lim_{u\to\infty}\sup_{l\in\mathcal{K}_u}\left|\frac{1}{\Psi(u_{l,\epsilon}^{2,-})}\mathbb{P}\left(\sup_{(s,t)\in L_{0,0}^*(u)}\frac{\overline{X}(-b_{12}l\overleftarrow{\rho}_2(1/u)S+s,l\overleftarrow{\rho}_2(1/u)S+t)}{1+(1-\epsilon)v_1^2(|s+b_{12}t|)}>u_{l,\epsilon}^{2,-}\right)-\mathcal{H}_\alpha^{(1-\epsilon)\gamma_1,b_{12}\eta^{-1/\alpha}}(S)\right|=0.$$

Consequently, (recall the definition of the constant in (19))

$$\begin{aligned}
\pi_3^-(u) &= \sum_{l=-N_2(u)-2}^{N_2(u)+1}\mathbb{P}\left(\sup_{(s,t)\in L_{0,0}^*(u)}\frac{\overline{X}(-b_{12}l\overleftarrow{\rho}_2(1/u)S+s,l\overleftarrow{\rho}_2(1/u)S+t)}{1+(1-\epsilon)v_1^2(|s+b_{12}t|)}>u_{l,\epsilon}^{2,-}\right)\\
&\sim \sum_{l=-N_2(u)-2}^{N_2(u)+1}\Psi(u_{l,\epsilon}^{2,-})\mathcal{H}_\alpha^{(1-\epsilon)\gamma_1,b_{12}\eta^{-1/\alpha}}(S)\\
&\sim \mathcal{H}_\alpha^{(1-\epsilon)\gamma_1,b_{12}\eta^{-1/\alpha}}(S)\Psi(u)\sum_{l=-N_2(u)-2}^{N_2(u)+1}e^{-(1-\epsilon)^2u^2\inf_{t\in J_l(u)}v_2^2(|t|)}\\
&\sim \frac{\mathcal{H}_\alpha^{\gamma_1,b_{12}\eta^{-1/\alpha}}(S)}{S}2\Gamma(1/\beta_2+1)\Psi(u)\frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)}(1+o(1)),\quad u\to\infty,\upsilon,\epsilon\to 0.
\end{aligned}\tag{62}$$

Moreover, in light of [39]

$$\lim_{S\to\infty}\frac{\mathcal{H}_\alpha^{\gamma_1,b_{12}\eta^{-1/\alpha}}(S)}{S}=\mathcal{H}_\alpha^{\gamma_1,b_{12}\eta^{-1/\alpha}}\in(0,\infty)$$

implying

$$\pi_3^-(u)\le\mathcal{H}_\alpha^{\gamma_1,b_{12}\eta^{-1/\alpha}}2\Gamma(1/\beta_2+1)\Psi(u)\frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)}(1+o(1)),\quad u\to\infty,S\to\infty,\epsilon\to 0.$$

Similarly,

$$\pi_3^+(u)\ge\mathcal{H}_\alpha^{\gamma_1,b_{12}\eta^{-1/\alpha}}2\Gamma(1/\beta_2+1)\Psi(u)\frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)}(1+o(1)),\quad u\to\infty,S\to\infty,\epsilon\to 0.$$

Note that for u sufficiently large

$$L_{0,0}(u)\subset\left[-(1+2|b_{12}|\eta^{-1/\alpha})\overleftarrow{\rho}_1(1/u)S,(1+2|b_{12}|\eta^{-1/\alpha})\overleftarrow{\rho}_1(1/u)S\right]\times\left[0,\overleftarrow{\rho}_2(1/u)S\right]=:J_{0,0}(u).\tag{63}$$

Thus, with $S_2=(1+2|b|\eta^{-1/\alpha})S$, by (37) with $u_{k,l,\epsilon,*}^-$ instead of $u_{k,l,\epsilon}^-$, we obtain

$$\begin{aligned}
\pi_4(u) &= \sum_{|k|\le N_1(u)+2,k\neq 0,-1}\sum_{l=-N_2(u)-2}^{N_2(u)+1}\mathbb{P}\left(\sup_{(s,t)\in L_{0,0}(u)}\overline{X}(k\overleftarrow{\rho}_1(1/u)S-b_{12}l\overleftarrow{\rho}_2(1/u)S+s,l\overleftarrow{\rho}_2(1/u)S+t)>u_{k,l,\epsilon,*}^-\right)\\
&\le \sum_{|k|\le N_1(u)+2,k\neq 0,-1}\sum_{l=-N_2(u)-2}^{N_2(u)+1}\mathbb{P}\left(\sup_{(s,t)\in J_{0,0}(u)}\overline{X}(k\overleftarrow{\rho}_1(1/u)S-b_{12}l\overleftarrow{\rho}_2(1/u)S+s,l\overleftarrow{\rho}_2(1/u)S+t)>u_{k,l,\epsilon,*}^-\right)\\
&\sim \sum_{|k|\le N_1(u)+2,k\neq 0,-1}\sum_{l=-N_2(u)-2}^{N_2(u)+1}\mathcal{H}_\alpha[-S_2,S_2]\mathcal{H}_\alpha[0,S]\Psi(u)e^{-(1-\epsilon)u^2\inf_{(s,t)\in L_{k,l}(u)}\left(v_1^2(|s+bt|)+v_2^2(|t|)\right)}\\
&\le \mathcal{H}_\alpha[-S_2,S_2]\mathcal{H}_\alpha[0,S]\Psi(u)\sum_{1\le|k|\le N_1(u)+2}e^{-\mathbb{Q}u^2v_1^2(\overleftarrow{\rho}_1(1/u)|k|S)}\sum_{l=-N_2(u)-2}^{N_2(u)+1}e^{-(1-\epsilon)u^2\inf_{t\in J_l(u)}v_2^2(|t|)}\\
&\le 2\Gamma(1/\beta_2+1)(1-\epsilon)^{-2/\beta_2}\frac{\mathcal{H}_\alpha[-S_2,S_2]}{S}\frac{\mathcal{H}_\alpha[0,S]}{S}\frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)}\Psi(u)S\sum_{1\le|k|\le N_1(u)+2}e^{-\mathbb{Q}_1|kS|^{\beta_1/2}}\\
&\le \mathbb{Q}_2\frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)}\Psi(u)e^{-\mathbb{Q}_3S^{\beta_1/2}}=o\left(\pi_3^-(u)\right),\quad u\to\infty,S\to\infty.
\end{aligned}\tag{64}$$

Along the same lines of the proof of Theorem 3.1

$$\Lambda_1^{(3)}(u)+\Lambda_2^{(3)}(u)=o\left(\pi_3^-(u)\right),\quad u\to\infty,$$

which completes the proof of this case.
Case ii) $\gamma_2 = 0, \gamma_1 = \infty$. For any $x > 0$ and u sufficiently large we have

$$\mathbb{P}\left(\sup_{|t|\le \overleftarrow{v}_2(u^{-1}\ln u)} X(-b_{12}t,t) > u\right) \le \pi_1(u) \le \mathbb{P}\left(\sup_{(s,t)\in D_u^{(1)}} \frac{\overline{X}(s,t)}{1+x\rho_1^2(|s+b_{12}t|)+v_2^2(|t|)} > u\right).$$

Since the Gaussian random field on the right hand side of the above satisfies case $\gamma_2 = 0, \gamma_1 = x \in (0,\infty)$, by (62) and (64), for S sufficiently large we obtain

$$\mathbb{P}\left(\sup_{(s,t)\in D_u^{(1)}} \frac{\overline{X}(s,t)}{1+x\rho_1^2(|s+b_{12}t|)+v_2^2(|t|)} > u\right) \le \frac{\mathcal{H}_\alpha^{x,b_{12}\eta^{-1/\alpha}}(S)}{S} 2\Gamma(1/\beta_2+1)\Psi(u)\frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)}(1+o(1)).$$

It follows that for any S positive (recall the definition of the constant in (19))

$$\begin{aligned}\lim_{x\to\infty}\mathcal{H}_\alpha^{x,b_{12}\eta^{-1/\alpha}}(S) &= \lim_{x\to\infty}\mathbb{E}\left\{\sup_{(s+b_{12}\eta^{-1/\alpha}t,t)\in[-S,S]\times[0,S]} e^{W(s,t)-x|s+b_{12}\eta^{-1/\alpha}t|^\alpha}\right\}\\ &= \mathbb{E}\left\{\sup_{(s+b_{12}\eta^{-1/\alpha}t,t)\in\{0\}\times[0,S]} e^{W(s,t)}\right\}\\ &= \mathcal{H}_\alpha[0,LS], \quad L = \left(|b_{12}|^\alpha\eta^{-1}+1\right)^{1/\alpha}.\end{aligned}$$

Hence, as $u\to\infty, x\to\infty, S\to\infty$

$$\mathbb{P}\left(\sup_{(s,t)\in D_u^{(1)}} \frac{\overline{X}(s,t)}{1+x\rho_1^2(|s+b_{12}t|)+v_2^2(|t|)} > u\right) \le 2L\mathcal{H}_\alpha\Gamma(1/\beta_2+1)\Psi(u)\frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)}(1+o(1)).$$

Further, for the random field $X(-b_{12}t,t)$, we have

$$\begin{aligned}1-\sqrt{Var\left(X(-b_{12}t,t)\right)} &\sim v_2^2(|t|), \quad t\to 0,\\ 1-Corr\left(X(-b_{12}t,t),X(-b_{12}s,s)\right) &\sim L^\alpha\rho_2^2(|t-s|), \quad s,t\to 0.\end{aligned} \tag{65}$$

Thus in light of Theorem 2.1, we have

$$\mathbb{P}\left(\sup_{|t|\le\overleftarrow{v}_2(u^{-1}\ln u)} X(-b_{12}t,t) > u\right) \sim 2L\mathcal{H}_\alpha\Gamma(1/\beta_2+1)\Psi(u)\frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)}.$$

Consequently,

$$\pi_1(u) \sim 2L\mathcal{H}_\alpha\Gamma(1/\beta_2+1)\Psi(u)\frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)}, \quad u\to\infty,$$

which completes the proof.
Case iii) $\gamma_2 \in (0,\infty), \gamma_1 = \infty$. Let for $u > 0$

$$\widehat{I}_{0,0}^*(u) = \{(s,t), |s+b_{12}t| \le \overleftarrow{\rho}_1(1/u)S, |t| \le \overleftarrow{\rho}_2(1/u)S\}.$$

For all u sufficiently large, we have

$$\mathbb{P}\left(\sup_{(s,t)\in D_u^{(1)}} X(-b_{12}t,t) > u\right) \le \pi_1(u) \le \pi_{1,-\epsilon}^{(5)}(u) + \pi_{1,-\epsilon}^{(6)}(u), \tag{66}$$

with

$$\pi_{1,-\epsilon}^{(5)}(u) = \mathbb{P}\left(\sup_{(s,t)\in\widehat{I}_{0,0}^*(u)} \frac{\overline{X}(s,t)}{1+x\rho_1^2(|s+b_{12}t|)+(1-\epsilon)v_2^2(|t|)} > u\right),$$

and

$$\pi_{1,-\epsilon}^{(6)}(u) = \sum_{|k|=1,k\neq-1}^{N_1(u)+2}\ \sum_{|l|=1,l\neq-1}^{N_2(u)+2} \mathbb{P}\left(\sup_{(s,t)\in L_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^-\right).$$

Since

$$u^2(x\rho_1^2(|\overleftarrow{\rho}_1(1/u)s+b_{12}\overleftarrow{\rho}_2(1/u)t|)+(1-\epsilon)v_2^2(|\overleftarrow{\rho}_2(1/u)t|)) \to x|s+b_{12}\eta^{-1/\alpha}t|^\alpha+(1-\epsilon)\gamma_2|t|^\alpha, \quad u\to\infty$$

uniformly on any compact set, then, by Lemma 5.2

$$\pi^{(5)}_{1,-\epsilon}(u) \sim \widetilde{\mathcal{H}}_{\alpha}^{x,\gamma_2,b_{12}\eta^{-1/\alpha}}(S)\Psi(u), \quad u\to\infty, \epsilon\to 0.$$

Moreover, by the same argument as given in case ii), we have

$$\lim_{x\to\infty} \widetilde{\mathcal{H}}_{\alpha}^{x,\gamma_2,b_{12}\eta^{-1/\alpha}}(S) = \mathcal{P}_{\alpha}^{\gamma_2\left(|b_{12}|^{\alpha}\eta^{-1}+1\right)^{-1}}[-LS, LS],$$

with $L=\left(|b_{12}|^{\alpha}\eta^{-1}+1\right)^{1/\alpha}$. Hence

$$\pi^{(5)}_{1,-\epsilon}(u) \sim \mathcal{P}_{\alpha}^{\gamma_2\left(|b_{12}|^{\alpha}\eta^{-1}+1\right)^{-1}}\Psi(u), \quad u\to\infty, x\to\infty, \epsilon\to 0, S\to\infty.$$

Using that $L_{0,0}(u)\subset J_{0,0}(u)$, with $J_{0,0}(u)$ defined by (63), and following the same steps as in (64), we get

$$\pi^{(6)}_{1,-\epsilon}(u) = o(\Psi(u)), \quad u\to\infty, S\to\infty.$$

Moreover, it follows from Theorem 2.1 and (65) that

$$\mathbb{P}\left(\sup_{(s,t)\in D_u^{(1)}} X(-b_{12}t,t)>u\right) \sim \mathcal{P}_{\alpha}^{\gamma_2\left(|b_{12}|^{\alpha}\eta^{-1}+1\right)^{-1}}\Psi(u), \quad u\to\infty,$$

which establishes the claim.

Case iv) $\gamma_2=\infty, \gamma_1=\infty$. Clearly, (66) holds with

$$\pi^{(5)}_{1,-\epsilon}(u) = \mathbb{P}\left(\sup_{(s,t)\in \widehat{I}^*_{0,0}(u)} \frac{\overline{X}(s,t)}{1+x\rho_1^2(|s+b_{12}t|)+y\rho_2^2(|t|)}>u\right), \quad x,y>0.$$

Moreover,

$$\pi^{(5)}_{1,-\epsilon}(u) \sim \widetilde{\mathcal{H}}_{\alpha}^{x,y,b_{12}\eta^{-1/\alpha}}(S)\Psi(u)$$

and

$$\lim_{y\to\infty}\lim_{x\to\infty} \widetilde{\mathcal{H}}_{\alpha}^{x,y,b_{12}\eta^{-1/\alpha}}(S) = \lim_{y\to\infty} \mathcal{P}_{\alpha}^{y\left(|b_{12}|^{\alpha}\eta^{-1}+1\right)^{-1}}[-LS, LS]=1.$$

Hence

$$\pi^{(5)}_{1,-\epsilon}(u) \sim \Psi(u), \quad u\to\infty, x\to\infty, y\to\infty.$$

The rest of the proof is the same as for the case $\gamma_2\in(0,\infty), \gamma_1=\infty$. □

Proof of Theorem 3.5 We focus on $\pi_1(u)$ as $u\to\infty$.

Case i) The proof of this case follows line by line the same arguments as given in the proof of Case i) of Theorem 3.9.

Case ii) $\gamma_1=0, \gamma_2\in(0,\infty)$. First we introduce some new notation. Let

$$u^{*-}_{k,\epsilon} = 1+(1-3\epsilon)\inf_{t\in I_k(u)}\left(v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)\right),$$

and

$$\widehat{I}_{k,0}(u)=I_k(u)\times\left(J_{-1}(u)\cup J_0(u)\right), \quad v(s,t)=v_1^2(|s+b_{12}t|)+v_2^2(|t|)-v_1^2(|(1+b_{12}\mu)s|)-v_2^2(|\mu s|), \quad (s,t)\in D_u,$$

where μ is defined right before Theorem 3.5. For any $0<x<y<\frac{S}{2|b_{12}|}$ and $0<\epsilon<1/4$, we have

$$\pi_5^+(u)-\Lambda(u)\le\pi_1(u)\le\pi_5^-(u)+\pi_6(u)+\pi_7(u)+\pi_8(u), \tag{67}$$

where

$$\begin{aligned}
\pi_5^{\pm}(u) &= \sum_{k\in E^{\pm}_{x,y}(u)}\mathbb{P}\left(\sup_{(s,t)\in\widehat{I}_{k,0}(u)}\frac{\overline{X}(s,t)}{1+(1\pm\epsilon)\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right)}>u\right),\\
\pi_6(u) &= \sum_{k\in E_{0,x}(u)}\mathbb{P}\left(\sup_{(s,t)\in\widehat{I}_{k,0}(u)}\frac{\overline{X}(s,t)}{1+(1-\epsilon)\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right)}>u\right),\\
\pi_7(u) &= \sum_{k\in E_{y,\infty}(u)}\mathbb{P}\left(\sup_{(s,t)\in\widehat{I}_{k,0}(u)}\frac{\overline{X}(s,t)}{1+(1-\epsilon)\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right)}>u\right),
\end{aligned}$$

$$\pi_8(u) = \sum_{|k|=0}^{N_1(u)+2} \sum_{|l|=1,l\neq -1}^{N_2(u)+2} \mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)} \frac{\overline{X}(s,t)}{1+(1-\epsilon)\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right)} > u\right),$$

$$\Lambda(u) = \sum_{k<k_1\in E^-_{x,y}(u)} \mathbb{P}\left(\sup_{(s,t)\in \widehat{I}_{k,0}(u)} X(s,t) > u, \sup_{(s,t)\in \widehat{I}_{k_1,0}(u)} X(s,t) > u\right),$$

with

$$\begin{aligned}
E_{0,x} &= \{k, |k|\le N_1(u)+2, I_k(u)\cap[-\overleftarrow{\rho_2}(u^{-1})x, \overleftarrow{\rho_2}(u^{-1})x]\neq\emptyset\},\\
E^-_{x,y} &= \{k, |k|\le N_1(u)+2, I_k(u)\cap\left([-\overleftarrow{\rho_2}(u^{-1})y, -\overleftarrow{\rho_2}(u^{-1})x]\cup[\overleftarrow{\rho_1}(u^{-1})x, \overleftarrow{\rho_1}(u^{-1})y]\right)\neq\emptyset\},\\
E^+_{x,y} &= \{k, |k|\le N_1(u)+2, I_k(u)\subset\left([-\overleftarrow{\rho_2}(u^{-1})y, -\overleftarrow{\rho_2}(u^{-1})x]\cup[\overleftarrow{\rho_1}(u^{-1})x, \overleftarrow{\rho_1}(u^{-1})y]\right)\},\\
E^-_{y,\infty} &= \{k, |k|\le N_1(u)+2, I_k(u)\cap\left([-\infty, -\overleftarrow{\rho_2}(u^{-1})y]\cup[\overleftarrow{\rho_1}(u^{-1})y, \infty]\right)\neq\emptyset\}.
\end{aligned}$$

We observe that for $|s|\in[\frac{i-1}{n}\overleftarrow{\rho_2}(u^{-1}), \frac{i+2}{n}\overleftarrow{\rho_2}(u^{-1})]$ with $x/2\le\frac{i}{n}\le 2y$ and and $|t|\in[0, \overleftarrow{v}_2(u^{-1}\ln u)]$

$$\begin{aligned}
&1+(1-\epsilon)\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right)\\
&\ge\left[1+(1-3\epsilon)\left(v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)\right)\right]\left[1+(1-3\epsilon)v(i\overleftarrow{\rho_2}(u^{-1})/n, t)\right],
\end{aligned}\tag{68}$$

whose proofs is postponed in the Appendix. Let

$$X_{u,k}(s,t)=\overline{X}(k\overleftarrow{\rho}_1(1/u)S+s,t),\quad \mathcal{K}_u=E^-_{i/n,(i+1)/n},\quad \mathcal{E}_u=\widehat{I}_{0,0}(u),$$
$$d_u(s,t)=(1-3\epsilon)u^2v(i\overleftarrow{\rho_2}(u^{-1})/n,t),\quad h_k(u)=u^{*-}_{k,\epsilon}.$$

Note that

$$\lim_{u\to\infty}\sup_{k\in\mathcal{K}_u,t\in[-S,S]}\left|(u^{*-}_{k,\epsilon})^2 v(i\overleftarrow{\rho_2}(u^{-1})/n, \overleftarrow{\rho_2}(u^{-1})t)-g_{i/n}(t)\right|=0.$$

Thus in light of Lemma 5.2

$$\lim_{u\to\infty}\sup_{x/2\le\frac{i}{n}\le 2y}\sup_{k\in E^-_{i/n,(i+1)/n}}\left|(\Psi(u^{*-}_{k,\epsilon}))^{-1}\mathbb{P}\left(\sup_{(s,t)\in\widehat{I}_{0,0}(u)}\frac{\overline{X}(k\overleftarrow{\rho}_1(1/u)S+s,t)}{1+(1-3\epsilon)v(i\overleftarrow{\rho_2}(u^{-1})/n,t)}>u^{*-}_{k,\epsilon}\right)-\mathcal{H}_{\alpha_1}[0,S]\mathcal{P}_\beta^{(1-3\epsilon)g_{i/n}}[-S,S]\right|=0.$$

Consequently, for $[nx]-1\le i\le[ny]$, it follows that

$$\begin{aligned}
&\sum_{k\in E^-_{i/n,(i+1)/n}}\mathbb{P}\left(\sup_{(s,t)\in\widehat{I}_{k,0}(u)}\frac{\overline{X}(s,t)}{1+(1-\epsilon)\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right)}>u\right)\\
&\le\sum_{k\in E^-_{i/n,(i+1)/n}}\mathbb{P}\left(\sup_{(s,t)\in\widehat{I}_{k,0}(u)}\frac{\overline{X}(s,t)}{\left[1+(1-3\epsilon)\left(v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)\right)\right]\left[1+(1-3\epsilon)v(i\overleftarrow{\rho_2}(u^{-1})/n,t)\right]}>u\right)\\
&\le\sum_{k\in E^-_{i/n,(i+1)/n}}\mathbb{P}\left(\sup_{(s,t)\in\widehat{I}_{k,0}(u)}\frac{\overline{X}(s,t)}{1+(1-3\epsilon)v(i\overleftarrow{\rho_2}(u^{-1})/n,t)}>u^{*-}_{k,\epsilon}\right)\\
&=\sum_{k\in E^-_{i/n,(i+1)/n}}\mathbb{P}\left(\sup_{(s,t)\in\widehat{I}_{0,0}(u)}\frac{\overline{X}(k\overleftarrow{\rho}_1(1/u)S+s,t)}{1+(1-3\epsilon)v(i\overleftarrow{\rho_2}(u^{-1})/n,t)}>u^{*-}_{k,\epsilon}\right)\\
&\sim\sum_{k\in E^-_{i/n,(i+1)/n}}\mathcal{H}_{\alpha_1}[0,S]\mathcal{P}_\beta^{(1-3\epsilon)g_{i/n}}[-S,S]\Psi(u^{*-}_{k,\epsilon})\\
&\sim\mathcal{H}_{\alpha_1}[0,S]\mathcal{P}_\beta^{(1-3\epsilon)g_{i/n}}[-S,S]\Psi(u)\sum_{k\in E^-_{i/n,(i+1)/n}}e^{-u^2(1-3\epsilon)\inf_{t\in I_k(u)}\left(v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)\right)}\\
&\le(1+o(1))\frac{\mathcal{H}_{\alpha_1}[0,S]}{S}\mathcal{P}_\beta^{(1-3\epsilon)g_{i/n}}[-S,S]\frac{\Psi(u)}{\overleftarrow{\rho_1}(u^{-1})}2\int_{i\overleftarrow{\rho_2}(u^{-1})/n}^{(i+1)\overleftarrow{\rho_2}(u^{-1})/n}e^{-(1-4\epsilon)\frac{\gamma_2 M_\beta}{\theta}u^2\rho_2^2(|s|)}ds
\end{aligned}$$

as $u\to\infty$, with M_β defined in (21). Using the same arguments as in the proof of Lemma 6.3, we have

$$\int_{i\overleftarrow{\rho_2}(u^{-1})/n}^{(i+1)\overleftarrow{\rho_2}(u^{-1})/n}e^{-(1-4\epsilon)\frac{\gamma_2 M_\beta}{\theta}u^2\rho_2^2(|s|)}ds\sim\frac{2}{\beta}\overleftarrow{\rho_2}(u^{-1})\int_{(i/n)^{\beta/2}}^{((i+1)/n)^{\beta/2}}t^{2/\beta-1}e^{-(1-4\epsilon)\frac{\gamma_2 M_\beta}{\theta}t^2}dt$$

$$\sim \quad \overleftarrow{\rho_2}(u^{-1})\int_{i/n}^{(i+1)/n} e^{-(1-4\epsilon)\frac{\gamma_2 M_\beta}{\theta}t^\beta}dt.$$

Hence

$$\sum_{k\in E^-_{i/n,(i+1)/n}} \mathbb{P}\left(\sup_{(s,t)\in\widehat{I}_{k,0}(u)} \frac{\overline{X}(s,t)}{1+(1-\epsilon)\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right)}>u\right)$$
$$\leq 2\frac{\mathcal{H}_{\alpha_1}[0,S]}{S}\mathcal{P}_\beta^{(1-3\epsilon)g_{i/n}}[-S,S]\frac{\overleftarrow{\rho_2}(u^{-1})}{\overleftarrow{\rho_1}(u^{-1})}\Psi(u)\int_{i/n}^{(i+1)/n} e^{-(1-4\epsilon)\frac{\gamma_2 M_\beta}{\theta}t^\beta}dt(1+o(1))$$
$$\leq 2\mathcal{H}_{\alpha_1}\mathcal{P}_\beta^{g_{i/n}}\frac{\overleftarrow{\rho_2}(u^{-1})}{\overleftarrow{\rho_1}(u^{-1})}\Psi(u)\int_{i/n}^{(i+1)/n} e^{-\frac{\gamma_2 M_\beta}{\theta}t^\beta}dt(1+o(1)), \quad u\to\infty, S\to\infty, \epsilon\to 0.$$

Further, by the continuity of $\mathcal{P}_\beta^{g_s}$ over $s\in[x/2,2y]$, we have

$$\begin{aligned}\pi_5^-(u) &\leq (1+o(1))2\mathcal{H}_{\alpha_1}\frac{\overleftarrow{\rho_2}(u^{-1})}{\overleftarrow{\rho_1}(u^{-1})}\Psi(u)\sum_{i=[nx]-1}^{[ny]+1}\int_{i/n}^{(i+1)/n}\mathcal{P}_\beta^{g_{i/n}}e^{-\frac{\gamma_2 M_\beta}{\theta}t^\beta}dt\\ &\leq (1+o(1))2\mathcal{H}_{\alpha_1}\frac{\overleftarrow{\rho_2}(u^{-1})}{\overleftarrow{\rho_1}(u^{-1})}\Psi(u)\int_x^y \mathcal{P}_\beta^{g_t}e^{-\frac{\gamma_2 M_\beta}{\theta}t^\beta}dt, \quad u\to\infty, S\to\infty, \epsilon\to 0, n\to\infty.\end{aligned}\tag{69}$$

Similarly,

$$\pi_5^+(u)\geq 2\mathcal{H}_{\alpha_1}\frac{\overleftarrow{\rho_2}(u^{-1})}{\overleftarrow{\rho_1}(u^{-1})}\Psi(u)\int_x^y \mathcal{P}_\beta^{g_t}e^{-\frac{\gamma_2 M_\beta}{\theta}t^\beta}dt(1+o(1)), \quad u\to\infty, S\to\infty, \epsilon\to 0, n\to\infty.\tag{70}$$

Next we focus on $\pi_6(u)$. In light of (55) and (58), we have

$$\pi_6(u)\leq \sum_{k\in E_{0,x}(u)}\mathbb{P}\left(\sup_{(s,t)\in\widehat{I}_{k,0}(u)} Y(s,t)>u\right).$$

Hence, following case ii) $\gamma_1=0, \gamma_2\in(0,\infty)$ in Theorem 3.1, we have

$$\begin{aligned}\pi_6(u) &\leq (1+o(1))2\mathcal{H}_{\alpha_1}[0,S]\mathcal{P}_\beta^{\gamma_2\kappa_1/2}[-S,S]\frac{\Psi(u)}{\overleftarrow{\rho}_1(u^{-1})S}\int_0^{x\overleftarrow{\rho_2}(u^{-1})}e^{-\frac{\kappa_1}{2}u^2v_1^2(t)}dt\\ &\leq (1+o(1))2\mathcal{H}_{\alpha_1}\mathcal{P}_\beta^{\gamma_2\kappa_1/2}\frac{\overleftarrow{\rho}_2(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})}\Psi(u)\int_0^x e^{-\frac{\kappa_1}{2}\frac{\gamma_2}{\theta}t^\beta}dt, \quad u\to\infty, S\to\infty.\end{aligned}\tag{71}$$

Similarly,

$$\pi_7(u)\leq (1+o(1))2\mathcal{H}_{\alpha_1}\mathcal{P}_\beta^{\gamma_2\kappa_1/2}\frac{\overleftarrow{\rho}_2(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})}\Psi(u)\int_y^\infty e^{-\frac{\kappa_1}{2}\frac{\gamma_2}{\theta}t^\beta}dt, \quad u\to\infty, S\to\infty.\tag{72}$$

Moreover, by Lemma 6.5, (48) and (49)

$$\Lambda(u)\leq \sum_{k<k_1\in E^-_{x,y}(u)}\mathbb{P}\left(\sup_{(s,t)\in\widehat{I}_{k,0}(u)} Y(s,t)>u, \sup_{(s,t)\in\widehat{I}_{k_1,0}(u)} Y(s,t)>u\right)=o(\pi_5^+(u)), \quad u\to\infty, S\to\infty.\tag{73}$$

Moreover, it follows from Lemma 6.5 and (46) that

$$\pi_8(u)\leq \sum_{|k|=0}^{N_1(u)+2}\sum_{|l|=1, l\neq -1}^{N_2(u)+2}\mathbb{P}\left(\sup_{(s,t)\in I_{k,l}(u)} Y(s,t)>u\right)=o(\pi_5^+(u)), \quad u\to\infty, S\to\infty.\tag{74}$$

Inserting (69)–(74) into (67), we have

$$\pi_1(u)\geq (1+o(1))2\mathcal{H}_{\alpha_1}\frac{\overleftarrow{\rho_2}(u^{-1})}{\overleftarrow{\rho_1}(u^{-1})}\Psi(u)\int_x^y \mathcal{P}_\beta^{g_t}e^{-\frac{\gamma_2 M_\beta}{\theta}t^\beta}dt, \quad u\to\infty, S\to\infty,$$

and

$$\begin{aligned}\pi_1(u) \leq\ & (1+o(1))2\mathcal{H}_{\alpha_1}\frac{\overleftarrow{\rho_2}(u^{-1})}{\overleftarrow{\rho_1}(u^{-1})}\Psi(u)\left(\int_x^y \mathcal{P}_{\alpha_2}^{g_t}e^{-\frac{\gamma_2 M_\beta}{\theta}t^\beta}dt+\mathcal{P}_\beta^{\gamma_2\kappa_1/2}\int_0^x e^{-\frac{\kappa_1}{2}\frac{\gamma_2}{\theta}t^\beta}dt\right.\\ &\left.+\mathcal{P}_\beta^{\gamma_2\kappa_1/2}\int_y^\infty e^{-\frac{\kappa_1}{2}\frac{\gamma_2}{\theta}t^\beta}dt\right)\end{aligned}$$

as $u \to \infty, S \to \infty$. Letting $x \to 0$ and $y \to \infty$ leads to

$$\pi_1(u) \sim (1+o(1))2\mathcal{H}_{\alpha_1} \frac{\overleftarrow{\rho_2}(u^{-1})}{\overleftarrow{\rho_1}(u^{-1})}\Psi(u)\int_0^\infty \mathcal{P}_\beta^{g_t} e^{-\frac{\gamma_2 M_\beta}{\theta}t^\beta}dt, \quad u \to \infty, S \to \infty,$$

which, together with the fact that

$$\overleftarrow{\rho_2}(u^{-1}) \sim \left(\frac{\gamma_2}{\theta}\right)^{1/\beta}\overleftarrow{v_1}(u^{-1})$$

implies the claim, and thus the proof is complete.

Case iii) $\gamma_1 = 0, \gamma_2 = \infty$ Let $X_z^\epsilon(s,t), (s,t) \in \mathbb{R}^2, z > 0, \epsilon > 0$ be homogeneous Gaussian random fields with correlation function

$$1 - Corr\left(X_z^\epsilon(s,t), X_z^\epsilon(s_1,t_1)\right) \sim (1+\epsilon)\rho_1^2(|s-s_1|)(1+o(1)) + \frac{1}{z}v_2^2(|t-t_1|)(1+o(1)), \quad |s-s_1|, |t-t_1| \to 0.$$

Consequently, by Slepian inequality (see e.g., [41]; note in passing that there is a remarkable extension of this inequality for stable processes, see [44])

$$\pi_9^+(u) \le \pi_1(u) \le \pi_9^-(u), \tag{75}$$

where

$$\begin{aligned}\pi_9^+(u) &= \mathbb{P}\left(\sup_{|s|\le \overleftarrow{v_1}(u^{-1}\ln u)} X(s,\mu s) > u\right),\\ \pi_9^-(u) &= \mathbb{P}\left(\sup_{(s,t)\in D_u} \frac{X_z^\epsilon(s,t)}{1+(1-\epsilon)\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right)} > u\right).\end{aligned}$$

It is straightforward to check that $\frac{X_z^\epsilon(s,t)}{1+(1-\epsilon)\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right)}$ satisfies assumptions of Case ii) $\gamma_1 = 0, \gamma_2 = (1-\epsilon)z \in (0,\infty)$. Thus

$$\pi_9^-(u) \le (1+o(1))2\left(\frac{z}{\theta}\right)^{1/\beta}\mathcal{H}_{\alpha_1}\frac{\overleftarrow{v_1}(u^{-1})}{\overleftarrow{\rho_1}(u^{-1})}\Psi(u)\left(\int_0^\infty \mathcal{P}_\beta^{g_{t,z}} e^{-\frac{zM_\beta}{\theta}t^\beta}dt\right), \quad u \to \infty, S \to \infty, \varepsilon \to 0,$$

with

$$g_{s,z}(t) = \frac{z}{\theta}\left(|s+bt|^\beta + \theta|t|^\beta - |(1+b\mu)s|^\beta - \theta|\mu s|^\beta\right), s \ge 0, t \in \mathbb{R}.$$

Replacing t by $z^{-1/\beta}s$ in the above integral yields

$$z^{1/\beta}\int_0^\infty \mathcal{P}_\beta^{g_{t,z}} e^{-\frac{zM_\beta}{\theta}t^\beta}dt = \int_0^\infty \mathcal{P}_\beta^{g_{z^{-1/\beta}s,z}} e^{-\frac{M_\beta}{\theta}s^\beta}ds.$$

Note that for any $\epsilon > 0$, there exists a positive constant $M_\epsilon > 0$ such that for z sufficiently large

$$g_{z^{-1/\beta}s,z}(t) + \epsilon|s|^\beta = \frac{1}{\theta}\left(|s+b_{12}tz^{1/\beta}|^\beta + \theta|tz^{1/\beta}|^\beta - |(1+b_{12}\mu)s|^\beta - \theta|\mu s|^\beta\right) + \epsilon|s|^\beta \ge M_\epsilon z|t|^\beta, \quad t \in \mathbb{R},$$

which implies that

$$\mathcal{P}_\beta^{g_{z^{-1/\beta}s,z}} \le e^{\epsilon|s|^\beta}\mathcal{P}_\beta^{M_\epsilon z}.$$

Since

$$\lim_{z\to\infty}\mathcal{P}_\beta^{M_\epsilon z} = 1,$$

then the dominated convergence theorem implies

$$\begin{aligned}\limsup_{z\to\infty}\int_0^\infty \mathcal{P}_\beta^{g_{z^{-1/\beta}s,z}} e^{-\frac{M_\beta}{\theta}s^\beta}ds &\le \limsup_{z\to\infty}\int_0^\infty \mathcal{P}_\beta^{M_\epsilon z} e^{-\left(\frac{M_\beta}{\theta}-\epsilon\right)s^\beta}ds\\ &\to \left(\frac{M_\beta}{\theta}\right)^{-1/\beta}\Gamma(1/\beta+1), \quad \epsilon \to 0.\end{aligned}$$

Thus we conclude that

$$\pi_9^-(u) \le (1+o(1))2\Gamma(1/\beta+1)\left(M_\beta\right)^{-1/\beta}\mathcal{H}_{\alpha_1}\frac{\overleftarrow{v_1}(u^{-1})}{\overleftarrow{\rho_1}(u^{-1})}\Psi(u), \quad u \to \infty, S \to \infty, \varepsilon \to 0. \tag{76}$$

Next we focus on $\pi_9^+(u)$. One can easily check that the variance and correlation functions of $X(s,\mu s)$ satisfy

$$1 - Var\left(X(s,\mu s)\right) \sim v_1^2(|(1+b_{12}\mu)s|) + v_2^2(|\mu s|) \sim M_\beta v_1^2(|s|), \quad s \to 0,$$

and

$$1 - Corr\left(X(s,\mu s), X(s_1,\mu s_1)\right) \sim \rho_1^2(|s-s_1|) + \rho_2^2(|\mu(s-s_1)|) \sim \rho_1^2(|s-s_1|), \quad s, s_1 \to 0.$$

In light of Theorem 2.1, we have

$$\pi_9^+(u) \sim 2\Gamma(1/\beta+1)\left(M_\beta\right)^{-1/\beta} \mathcal{H}_{\alpha_1} \frac{\overleftarrow{v_1}(u^{-1})}{\overleftarrow{\rho_1}(u^{-1})} \Psi(u), \quad u \to \infty,$$

which combined with (75) and (76) establishes the proof.

Case iv) $\gamma_1 \in (0,\infty), \gamma_2 = \infty$. Let $Z(s,t)$ be a homogeneous Gaussian random field with variance 1 and correlation function satisfying

$$1 - Corr\left(Z(s,t), Z(s_1,t_1)\right) \sim 2\rho_1^2(|s-s_1|) + \rho_1^2(|t-t_1|), \quad |s-s_1| \to 0, |t-t_1| \to 0,$$

and

$$\widehat{I}_{0,0}(u) = \left[-\overleftarrow{\rho_1}(u^{-1})S, \overleftarrow{\rho_1}(u^{-1})S\right] \times \left[-\overleftarrow{\rho_1}(u^{-1})S_1, \overleftarrow{\rho_1}(u^{-1})S_1\right].$$

By Slepian's inequality and Lemma 6.5

$$\pi_{10}^+(u) \le \pi_1(u) \le \pi_{10}^-(u) + \pi_{11}(u),$$

where

$$\begin{aligned} \pi_{10}^{\pm}(u) &= \mathbb{P}\left(\sup_{(s,t)\in \widehat{I}_{0,0}(u)} \frac{\overline{X}(s,t)}{1+(1\pm\epsilon)\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right)} > u \right), \\ \pi_{11}(u) &= \mathbb{P}\left(\sup_{(s,t)\in \left(D_u - \widehat{I}_{0,0}(u)\right)} \frac{Z(s,t)}{1+\frac{\kappa_1}{2}\left(v_1^2(|s|)+v_2^2(|t|)\right)} > u \right). \end{aligned}$$

Note that $\rho_2^2(t) = o(\rho_1^2(t))$ as $t \to 0$ and

$$(1\pm\epsilon)u^2\left(v_1^2(|\overleftarrow{\rho_1}(u^{-1})s + b_{12}\overleftarrow{\rho_1}(u^{-1})t|) + v_2^2(|\overleftarrow{\rho_1}(u^{-1})t|)\right) \to (1\pm\epsilon)\gamma_1\left(|s+b_{12}t|^{\alpha_1} + \theta|t|^{\alpha_1}\right), \quad u \to \infty$$

uniformly with respect to $(s,t) \in [-S,S]\times[-S_1,S_1]$. It follows from Lemma 5.3 that

$$\pi_{10}^{\pm}(u) \sim \Psi(u)\mathbb{E}\left\{ \exp \sup_{(s,t)\in[-S,S]\times[-S_1,S_1]} \left[\sqrt{2}B_{\alpha_1}(s) - |s|^{\alpha_1} - (1\pm\epsilon)\gamma_1\left(|s+b_{12}t|^{\alpha_1}+\theta|t|^{\alpha_1}\right)\right] \right\}.$$

Since

$$\lim_{S_1\to\infty} \mathbb{E}\left\{ \exp \sup_{(s,t)\in[-S,S]\times[-S_1,S_1]} \left[\sqrt{2}B_{\alpha_1}(s) - |s|^{\alpha_1} - (1\pm\epsilon)\gamma_1\left(|s+b_{12}t|^{\alpha_1}+\theta|t|^{\alpha_1}\right)\right] \right\} = \mathcal{P}_{\alpha_1}^{(1\pm\epsilon)\gamma_1 M_{\alpha_1}}[-S,S],$$

we have

$$\pi_{10}^{\pm}(u) \sim \mathcal{P}_{\alpha_1}^{\gamma_1 M_{\alpha_1}} \Psi(u), \quad u \to \infty, S_1 \to \infty, S \to \infty, \varepsilon \to 0.$$

Using that with κ_1 as in Lemma 6.5

$$\frac{Z(s,t)}{1+\frac{\kappa_1}{2}\left(v_1^2(|s|)+v_2^2(|t|)\right)}$$

satisfies the conditions of Case iii) $\gamma_1, \gamma_2 \in (0,\infty)$ of Theorem 3.1, by the same argument as given in the proof of (52), we obtain that $\pi_{11}(u) = o\left(\Psi(u)\right), u \to \infty, S_1 \to \infty, S \to \infty$. Thus the proof is completed.

Case iii) $\gamma_1 = \gamma_2 = \infty$. It follows from (55) and (58) with the specific B in this case that

$$\mathbb{P}\left(X(0,0) > u\right) \le \pi_1(u) \le \mathbb{P}\left(\sup_{(s,t)\in D_u} Y(s,t) > u \right),$$

where κ_1 is defined in Lemma 6.5. The Gaussian random field involved in the right hand side of the above inequality satisfies the assumption of Case iii) $\gamma_1 = \gamma_2 = \infty$ in Theorem 3.1 and therefore it follows that

$$\mathbb{P}\left(\sup_{(s,t)\in D_u} \frac{\overline{X}(s,t)}{1+\frac{\kappa_1}{2}\left(v_1^2(|s|)+v_2^2(|t|)\right)} > u \right) \sim \Psi(u), \quad u \to \infty.$$

This completes the proof. $\square$

Proof of Theorem 3.8 For $\varepsilon > 0$ sufficiently small, let $Z^{\pm\epsilon}$ be a stationary Gaussian process with continuous trajectories, unit variance and correlation function satisfying

$$1 - r_{Z^{\pm\epsilon}}(t) \sim (1 \mp \epsilon)\rho_1^2(|t|), \quad t \to 0.$$

By Slepain's inequality, we have

$$\pi_{12}^{+}(u) \le \pi_1(u) \le \pi_{12}^{-}(u),$$

where

$$\pi_{12}^{\pm}(u) = \mathbb{P}\left(\sup_{(s,t)\in D_u} \frac{Z^{\pm\epsilon}(s)}{1 + (1 \pm \epsilon)\left(v_1^2(|s|) + v_2^2(|t|)\right)} > u \right).$$

By the fact that for any $u > 0$

$$\sup_{(s,t)\in D_u} \frac{Z^{\pm\epsilon}(s)}{1 + (1 \pm \epsilon)\left(v_1^2(|s|) + v_2^2(|t|)\right)} = \sup_{|s| \le \overleftarrow{v_1}(u^{-1}\ln u)} \frac{Z^{\pm\epsilon}(s)}{1 + (1 \pm \epsilon)v_1^2(|s|)}$$

we have

$$\pi_{12}^{\pm}(u) = \mathbb{P}\left(\sup_{|s| \le \overleftarrow{v_1}(u^{-1}\ln u)} \frac{Z^{\pm\epsilon}(s)}{1 + (1 \pm \epsilon)v_1^2(|s|)} > u \right).$$

Hence an application of Theorem 2.1 establishes the claims. $\square$

Proof of Theorem 3.9 Set below for $u > 0$

$$D_u = \left\{ |s| \le \overleftarrow{v_1}(u^{-1}\ln u), |t| \le 2\mu\overleftarrow{v_1}(u^{-1}\ln u) \right\}.$$

Using the same $Z^{\pm\epsilon}$ as in the proof of Theorem 3.8, by Slepian's inequality, we have

$$\pi_{13}^{+}(u) \le \pi_1(u) \le \pi_{13}^{-}(u),$$

where

$$\pi_{13}^{\pm}(u) = \mathbb{P}\left(\sup_{(s,t)\in D_u} \frac{Z^{\pm\epsilon}(s)}{1 + (1 \pm \epsilon)\left(v_1^2(|s + b_{12}t|) + v_2^2(|t|)\right)} > u \right).$$

The same analysis as given between (80) and (81) implies that, for u sufficiently large

$$(1-\epsilon)M_{\beta_1}v_1^2(|s|) \le \inf_{|t| \le 2\mu\overleftarrow{v_1}(u^{-1}\ln u)} v_1^2(|s + b_{12}t|) + v_2^2(|t|) \le (1+\epsilon)M_{\beta_1}v_1^2(|s|)$$

hold for $|s| \le \overleftarrow{v_1}(u^{-1}\ln u)$. Thus we have

$$\pi_{13}^{-}(u) \le \mathbb{P}\left(\sup_{|s| \le \overleftarrow{v_1}(u^{-1}\ln u)} \frac{Z^{-\epsilon}(s)}{1 + (1-\epsilon)^2 M_{\beta_1}v_1^2(|s|)} > u \right),$$

and

$$\pi_{13}^{+}(u) \ge \mathbb{P}\left(\sup_{|s| \le \overleftarrow{v_1}(u^{-1}\ln u)} \frac{Z^{+\epsilon}(s)}{1 + (1+\epsilon)^2 M_{\beta_1}v_1^2(|s|)} > u \right).$$

Hence the claim follows by Theorem 2.1. $\square$

5. Appendix A

In this section we derive some key uniform expansions of the tail of maximum of Gaussian random fields over short intervals. Recall that for any $\gamma \in (0,\infty), S > 0$

$$\mathcal{P}_\alpha^\gamma[0,S] = \mathbb{E}\left\{ \sup_{[0,S]} e^{\sqrt{2}B_\alpha(t) - (1+\gamma)|t|^\alpha} \right\}$$

and we set for any $\alpha \in (0,2]$ and $S > 0$

$$\mathcal{P}_\alpha^\infty[0,S] = 1, \quad \mathcal{P}_\alpha^0[0,S] = \mathcal{H}_\alpha[0,S].$$

The claim of the following three lemmas follows by Theorem 2.1 in [45]; the detailed proofs are omitted here.
In the following $h_k, k \in \mathcal{K}_u$ with $\mathcal{K}_u$ an index set are positive functions such that $\lim_{u\to\infty} h_k(u)/u = 1$ uniformly with respect to $k \in \mathcal{K}_u$.

Lemma 5.1. *Let $X_{u,k}(t), t \in [0, \overleftarrow{\rho}(u^{-1})S], k \in \mathcal{K}_u$ be a sequence of centered Gaussian processes with continuous trajectories, variance 1 and correlation function $r(\cdot,\cdot)$ satisfying (8) uniformly with respect to $k \in \mathcal{K}_u$. Suppose that $\rho \in \mathcal{R}_{\alpha/2}, v \in \mathcal{R}_{\beta/2}$ with $0 < \alpha \le 2, \beta > 0$. If $\lim_{t\to 0} \frac{v^2(t)}{\rho^2(t)} = \gamma \in [0,\infty]$, then*

$$\lim_{u\to\infty} \sup_{k\in\mathcal{K}_u} \left| \frac{1}{\Psi(h_k(u))} \mathbb{P}\left(\sup_{t\in[0,\overleftarrow{\rho}(u^{-1})S]} \frac{X_{u,k}(t)}{1+v^2(t)} > h_k(u) \right) - \mathcal{P}_\alpha^\gamma[0,S] \right| = 0.$$

The case $\gamma = \infty$ in Lemma 5.1 is not covered by Theorem 2.1 in [45], but it straightforwardly follows from the fact that $\lim_{y\to\infty} \mathcal{P}_\alpha^y[0,S] = 1$, and for any $y > 0$ and u sufficiently large

$$\Psi(h_k(u)) \le \mathbb{P}\left(\sup_{t\in[0,\overleftarrow{\rho}(u^{-1})S]} \frac{X_{u,k}(t)}{1+v^2(t)} > h_k(u) \right) \le \mathbb{P}\left(\sup_{t\in[0,\overleftarrow{\rho}(u^{-1})S]} \frac{X_{u,k}(t)}{1+y\rho^2(t)} > h_k(u) \right).$$

Let $\rho_i \in \mathcal{R}_{\alpha_i/2}$, $v_i \in \mathcal{R}_{\beta_i/2}, i = 1,2$ be non-negative functions with $0 < \alpha_i \le 2, \beta_i > 0, i = 1,2$. Let $X_{u,k}(s,t), k \in \mathcal{K}_u$ be centered Gaussian random fields over $\mathcal{E}(u) = \{(\overleftarrow{\rho_1}(u^{-1})s, \overleftarrow{\rho_2}(u^{-1})t), (s,t) \in \mathcal{E}\}$ with $\mathcal{E}$ a compact set containing 0. Suppose further that $X_{u,k}$ has unit variance, continuous trajectories and correlation function $r_k(s,t,s_1,t_1)$ satisfying (17) uniformly with respect to $k \in \mathcal{K}_u$.

Lemma 5.2. *If $d_u(s,t), u > 0$ are continuous functions satisfying*

$$\lim_{u\to\infty} \sup_{(s,t)\in\mathcal{E}, k\in\mathcal{K}_u} \left| h_k^2(u) d_u(\overleftarrow{\rho_1}(u^{-1})s, \overleftarrow{\rho_2}(u^{-1})t) - d(s,t) \right| = 0,$$

then we have

$$\lim_{u\to\infty} \sup_{k\in\mathcal{K}_u} \left| \frac{1}{\Psi(h_k(u))} \mathbb{P}\left(\sup_{(s,t)\in\mathcal{E}(u)} \frac{X_{u,k}(s,t)}{1+d_u(s,t)} > h_k(u) \right) - \mathbb{E}\left\{ \sup_{(s,t)\in\mathcal{E}} e^{W_{\alpha_1,\alpha_2}(s,t) - d(s,t)} \right\} \right| = 0,$$

where $W_{\alpha_1,\alpha_2}(s,t) = \sqrt{2}(B_{\alpha_1}(s) + \widetilde{B}_{\alpha_2}(t)) - |s|^{\alpha_1} - |t|^{\alpha_2}, s,t \in \mathbb{R}$, with B_{α_1} and $\widetilde{B}_{\alpha_2}$ being two independent standard fBm's with Hurst indices $\alpha_1/2 \in (0,1], \alpha_2/2 \in (0,1]$, respectively.

Lemma 5.3. *Suppose that $d_u(s,t), u > 0$ are continuous functions satisfying*

$$\lim_{u\to\infty} \sup_{(s,t)\in\mathcal{E}, k\in\mathcal{K}_u} |h_k^2(u) d_u(\overleftarrow{\rho_1}(u^{-1})s, \overleftarrow{\rho_1}(u^{-1})t) - d(s,t)| = 0.$$

If $\rho_2^2(t) = o(\rho_1^2(t))$ as $t \to 0$, then

$$\lim_{u\to\infty} \sup_{k\in\mathcal{K}_u} \left| \frac{1}{\Psi(h_k(u))} \mathbb{P}\left(\sup_{(s,t)\in\widetilde{\mathcal{E}}(u)} \frac{X_{u,k}(s,t)}{1+d_u(s,t)} > h_k(u) \right) - \mathbb{E}\left\{ \sup_{(s,t)\in\mathcal{E}} e^{\sqrt{2}B_{\alpha_1}(s) - |s|^{\alpha_1} - d(s,t)} \right\} \right| = 0,$$

with B_{α_1} a standard fBm with Hurst index $\alpha_1/2$ and $\widetilde{\mathcal{E}}(u) = \{(\overleftarrow{\rho_1}(u^{-1})s, \overleftarrow{\rho_1}(u^{-1})t) \colon (s,t) \in \mathcal{E}\}$.

Assume now that $X(t), t = (t_1,\ldots,t_d) \in \mathbb{R}^d$ is a Gaussian field with continuous trajectories, unit variance and covariance function satisfying

$$1 - Cov(X(s), X(t)) \sim \sum_{i=1}^d \rho_i^2(|t_i - s_i|), \quad s,t \to 0, \tag{77}$$

with ρ_i positive regularly varying function with index $\alpha_i/2 \in (0,1]$. Set below

$$\overleftarrow{\rho}(u^{-1}) = (\overleftarrow{\rho}_1(u^{-1}), \ldots, \overleftarrow{\rho}_d(u^{-1})), \quad \overleftarrow{\rho}(u^{-1})t = (\overleftarrow{\rho}_1(u^{-1})t_1, \ldots, \overleftarrow{\rho}_d(u^{-1})t_d)$$

and for any $A, B \subset \mathbb{R}^d$ put

$$F(A,B) = \inf_{s\in A, t\in B} \sqrt{\sum_{i=1}^d |s_i - t_i|^2}.$$

Further let

$$D_u = \prod_{i=1}^d \left[-\frac{\delta_u}{\overleftarrow{\rho}_i(u^{-1})}, \quad \frac{\delta_u}{\overleftarrow{\rho}_i(u^{-1})} \right], \quad \mathbb{K} = \{(\lambda_1,\lambda_2) \in D_u \times D_u \colon \lambda_1 + \mathcal{E}_1, \lambda_2 + \mathcal{E}_2 \subset D_u\}.$$

In the following $u_\lambda, \lambda \in D_u$, with $\delta_u \to 0, u \to \infty$ satisfy

$$\lim_{u\to\infty} \sup_{\lambda\in D_u} \left|\frac{u_\lambda}{u} - 1\right| = 0.$$

We state next the result of Corollary 3.2 in [45]. Below $\mathcal{E}_1, \mathcal{E}_2$ are assumed to be compact sets.

Lemma 5.4. *Suppose that $X(t), t = (t_1, \dots, t_d) \in \mathbb{R}^d$ is a Gaussian field with continuous trajectories and unit variance function. If the covariance function satisfying (77), then there exists $\mathcal{C}, \mathcal{C}_1 > 0$ such that for any $S > 1$ and all u large*

$$\sup_{(\lambda_1,\lambda_2)\in\mathbb{K}, \mathcal{E}_1,\mathcal{E}_2\subset[0,S]^d} e^{\mathcal{C}_1 F^{\beta^*/2}(\lambda_1+\mathcal{E}_1,\lambda_2+\mathcal{E}_2)} \frac{\mathbb{P}\left(\sup_{t\in\overleftarrow{\rho}(u^{-1})(\lambda_1+\mathcal{E}_1)} X(t) > u_{\lambda_1}, \sup_{s\in\overleftarrow{\rho}(u^{-1})(\lambda_2+\mathcal{E}_2)} X(s) > u_{\lambda_2}\right)}{S^{2d}\Psi(\min(u_{\lambda_1}, u_{\lambda_2}))} \quad \leq \quad \mathcal{C}$$

with $\beta^* = \min_{i=1,\dots,d} \alpha_i$.

6. Appendix B

Consider a positive function g such that

$$\lim_{u\to\infty} g(u) = \infty, \quad \lim_{u\to\infty} \frac{g(u)}{u} = 0.$$

For $v \in \mathcal{R}_\beta, \beta > 0$ a non-negative function set

$$z_u = \overleftarrow{v}\left(\frac{g(u)}{u}\right), \quad u > 0. \tag{78}$$

We shall investigate first the asymptotic behaviour of an integral determined by g and v. We begin with a useful lemma to demonstrate the upper an lower bound of regularly varying function. We give next the well-knonw Potter's bound for the v, see e.g., [23, 37] for details.

Lemma 6.1. *For any $\epsilon \in (0, \min(1, \beta))$, there exists $t_\epsilon > 0$ such that for any $0 < s, t \le t_\epsilon$*

$$(1-\epsilon)\min\left(\left(\frac{s}{t}\right)^{\beta-\epsilon}, \left(\frac{s}{t}\right)^{\beta+\epsilon}\right) \le \frac{v(s)}{v(t)} \le (1+\epsilon)\max\left(\left(\frac{s}{t}\right)^{\beta-\epsilon}, \left(\frac{s}{t}\right)^{\beta+\epsilon}\right).$$

Lemma 6.2. *i) For any $0 < x \le y < \infty$ and $c > 0$, as $u \to \infty$*

$$\int_0^{xz_u} e^{-cu^2v^2(t)}dt \sim \int_0^{yz_u} e^{-cu^2v^2(t)}dt.$$

ii) If $a \in \mathcal{R}_\beta$ is such that $a(t) \sim v(t)$ as $t \to 0$, then as $u \to \infty$

$$\int_0^{z_u} e^{-u^2v^2(t)}dt \sim \int_0^{\overleftarrow{a}(\frac{g(u)}{u})} e^{-u^2a^2(t)}dt.$$

Proof of Lemma 6.2 i) Using standard properties of regularly varying functions, see e.g., [22], for u sufficiently large and $0 < x < y < \infty$, we have with z_u defined in (78)

$$\begin{aligned}\int_{xz_u}^{yz_u} e^{-cu^2v^2(t)}dt &\le e^{-cu^2v^2((x/2)z_u)}(y-x)z_u\\ &\le e^{-(x/3)^{2\beta}cu^2v^2(z_u)}(y-x)z_u\\ &\le e^{-(x/4)^{2\beta}c(g(u))^2}(y-x)z_u\end{aligned}$$

and

$$\begin{aligned}\int_0^{xz_u} e^{-cu^2v^2(t)}dt &\ge \int_0^{(x/8)z_u} e^{-cu^2v^2(t)}dt\\ &\ge e^{-cu^2v^2((x/7)z_u)}(x/8)z_u\\ &\ge e^{-(x/6)^{2\beta}cu^2v^2(z_u)}(x/8)z_u\\ &\ge e^{-(x/5)^{2\beta}c(g(u))^2}(x/8)z_u,\end{aligned}$$

which imply that, as $u \to \infty$

$$\int_0^{xz_u} e^{-cu^2v^2(t)}dt \sim \int_0^{yz_u} e^{-cu^2v^2(t)}dt.$$

ii) For any $0 < \epsilon < 1/2$

$$(1-\epsilon)a(t) \le v(t) \le (1+\epsilon)a(t)$$

holds for sufficiently small $t > 0$. Consequently, for u sufficiently large

$$\begin{aligned}\int_0^{z_u} e^{-u^2v^2(t)}dt &\le \int_0^{z_u} e^{-(1-\epsilon)^2u^2a^2(t)}dt \le \int_0^{z_u} e^{-u^2a^2((1-2\epsilon)^{1/\beta}t)}dt\\ &= (1-2\epsilon)^{-1/\beta}\int_0^{(1-2\epsilon)^{1/\beta}z_u} e^{-u^2a^2(t)}dt\\ &\le (1-2\epsilon)^{-1/\beta}\int_0^{z_u} e^{-u^2a^2(t)}dt\end{aligned}$$

and

$$\begin{aligned}\int_0^{z_u} e^{-u^2v^2(t)}dt &\ge \int_0^{z_u} e^{-(1+\epsilon)^2u^2a^2(t)}dt \ge \int_0^{z_u} e^{-u^2a^2((1+2\epsilon)^{1/\beta}t)}dt\\ &= (1+2\epsilon)^{-1/\beta}\int_0^{(1+2\epsilon)^{1/\beta}z_u} e^{-u^2a^2(t)}dt\\ &\ge (1+2\epsilon)^{-1/\beta}\int_0^{z_u} e^{-u^2a^2(t)}dt.\end{aligned}$$

Letting $\epsilon \to 0$ and by the fact that $\overleftarrow{a}(\frac{g(u)}{u}) \sim z_u$, we establish the second claim. □

Lemma 6.3. *For any $c > 0$ we have*

$$\int_0^{z_u} e^{-cu^2v^2(t)}dt \sim c^{-1/(2\beta)}\Gamma(1+1/(2\beta))\overleftarrow{v}(1/u), \quad u \to \infty.$$

Proof of Lemma 6.3 By Lemma 6.2, ii) we can assume that $v(x) = \ell(x)x^\beta$ with ℓ normalized slowly varying function at 0. It is well-known that $\ell(x)x^\beta$ is ultimately monotone for any $\beta \neq 0$, ℓ is continuously differentiable and

$$\lim_{x\to 0} \frac{x\ell'(x)}{\ell(x)} = 0. \tag{79}$$

Since v is ultimately monotone, we have with $g(u)$ and z_u defined by (78)

$$\int_0^{z_u} e^{-cu^2v^2(t)}dt \sim u^{-1}\int_0^{g(u)} \frac{1}{v'(\overleftarrow{v}(y/u))}e^{-cy^2}\,dy, \quad u \to \infty.$$

Further, (79) implies

$$\frac{1}{v'(\overleftarrow{v}(y/u))} \sim \frac{1}{\beta}\frac{\overleftarrow{v}(y/u))}{v(\overleftarrow{v}(y/u))} \sim \frac{1}{\beta}\frac{u}{y}\overleftarrow{v}(y/u)$$

Consequently, as $u \to \infty$

$$\begin{aligned}\int_0^{z_u} e^{-cu^2v^2(t)}dt &\sim \frac{1}{\beta}\int_0^{g(u)} \overleftarrow{v}(y/u)y^{-1}e^{-cy^2}\,dy\\ &\sim \frac{1}{\beta}\overleftarrow{v}(1/u)\int_0^{g(u)} \frac{\overleftarrow{v}(y/u)}{\overleftarrow{v}(1/u)}y^{-1}e^{-cy^2}\,dy.\end{aligned}$$

By Lemma 6.1, for any $\epsilon \in (0, \min(1, 1/\beta))$ and all u large

$$\frac{\overleftarrow{v}(y/u)}{\overleftarrow{v}(1/u)} \le (1+\epsilon)\max(y^{1/\beta+\epsilon}, y^{1/\beta-\epsilon}), \quad 0 \le y \le g(u).$$

Since further for any $y > 0$

$$\lim_{u\to\infty} \frac{\overleftarrow{v}(y/u)}{\overleftarrow{v}(1/u)} = y^{1/\beta},$$

then application of the dominated convergence theorem (recall that $\lim_{u\to\infty} g(u) = \infty$) yields

$$\begin{aligned}\int_0^{z_u} e^{-u^2v^2(t)}dt &\sim \frac{1}{\beta}\overleftarrow{v}(1/u)\int_0^{\infty} y^{1/\beta-1}e^{-cy^2}\,dy\\ &\sim c^{-1/(2\beta)}\Gamma(1+1/(2\beta))\overleftarrow{v}(1/u).\end{aligned}$$

Note that alternatively, by [23][Proposition 1.18] it follows that

$$\int_0^{g(u)} \frac{\overleftarrow{v}(y/u)}{\overleftarrow{v}(1/u)} y^{-1} e^{-cy^2}\, dy \sim \int_0^{g(u)} y^{1/\beta-1} e^{-cy^2}\, dy, \quad u\to\infty$$

and thus the claim follows. □

Lemma 6.4. *Suppose that $\rho_1^2 \in \mathcal{R}_{\alpha_1}$ and $\rho_2^2 \in \mathcal{R}_{\alpha_2}$ with $\alpha_1, \alpha_2 > 0$. If $\rho_1^2(|t|) = o(\rho_2^2(|t|))$ as $t\to 0$, then for any $a, b \in \mathbb{R}$*

$$\rho_1^2(|as+bt|) + \rho_2^2(|t|) \sim \rho_1^2(|as|) + \rho_2^2(|t|), \quad s,t\to 0.$$

Proof of Lemma 6.4 The claim follows easily if the product $abst = 0$. Therefore, we suppose next that $abst \neq 0$. It suffices to prove that

$$\lim_{s,t\to 0, st\neq 0} \frac{|\rho_1^2(|as+bt|) - \rho_1^2(|as|)|}{\rho_1^2(|as|) + \rho_2^2(|t|)} = 0.$$

For any $0<\epsilon<\min(1,\alpha_1)$, if $|\frac{as}{bt}| > \frac{4\alpha_1}{\epsilon}$, then

$$1-\frac{\epsilon}{4\alpha_1} \le \frac{|as+bt|}{|as|} \le 1+\frac{\epsilon}{4\alpha_1}.$$

Thus in light of Lemma 6.1, for any s, t sufficiently small we have

$$\begin{aligned}
\frac{|\rho_1^2(|as+bt|) - \rho_1^2(|as|)|}{\rho_1^2(|as|) + \rho_2^2(|t|)} &\le \frac{\rho_1^2(|as|)}{\rho_1^2(|as|) + \rho_2^2(|t|)} \left| \frac{\rho_1^2(|as+bt|)}{\rho_1^2(|as|)} - 1 \right| \\
&\le \left| \frac{\rho_1^2(|as+bt|)}{\rho_1^2(|as|)} - 1 \right| \\
&\le \max\left((1+\epsilon)\left(1+\frac{\epsilon}{4\alpha_1}\right)^{\alpha_1+\epsilon} - 1, 1-(1-\epsilon)\left(1-\frac{\epsilon}{4\alpha_1}\right)^{\alpha_1+\epsilon} \right) \\
&\to 0, \quad s\to 0, \epsilon\to 0.
\end{aligned}$$

For any $\epsilon \in (0,\alpha_1)$, if $|\frac{as}{bt}| \le \frac{4\alpha_1}{\epsilon}$, then

$$\frac{|as+bt|}{|bt|} \le 1+\frac{4\alpha_1}{\epsilon}.$$

Applying Lemma 6.1, we obtain

$$\begin{aligned}
\frac{|\rho_1^2(|as+bt|) - \rho_1^2(|as|)|}{\rho_1^2(|as|) + \rho_2^2(|t|)} &\le \frac{\rho_1^2(|bt|)}{\rho_1^2(|as|) + \rho_2^2(|t|)} \left| \frac{\rho_1^2(|as+bt|)}{\rho_1^2(|bt|)} - \frac{\rho_1^2(|as|)}{\rho_1^2(|bt|)} \right| \\
&\le \frac{\rho_1^2(|bt|)}{\rho_2^2(|t|)}(1+\epsilon)\left[\left(1+\frac{4\alpha_1}{\epsilon}\right)^{\alpha_1+\epsilon} + \left(\frac{4\alpha_1}{\epsilon}\right)^{\alpha_1+\epsilon} \right] \\
&\to 0, \quad t\to 0.
\end{aligned}$$

Hence we complete the proof. □

Lemma 6.5. *Suppose that $v_1^2, v_2^2 \in \mathcal{R}_\beta$ $\beta>0$. If $a_1 v_2^2(|t|) \le v_1^2(|t|) \le a_2 v_2^2(|t|))$ holds for $a_1, a_2 >$ positive and all t sufficiently small, then for any invertible matrix $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, there exist two positive constants κ_1 and κ_2 such that*

$$\kappa_1(v_1^2(|s|) + v_2^2(|t|)) \le v_1^2(|b_{11}s + b_{12}t|) + v_2^2(|b_{21}s+b_{22}t|) \le \kappa_2(v_1^2(|s|) + v_2^2(|t|))$$

is valid in a neighbourhood of 0.

Proof of Lemma 6.5 Without loss of generality, we assume that $|t| \ge |s|$ and $|t|>0$. By Lemma 6.1, we have that for $0<\epsilon<\min(1,\beta)$ and $t>0$ sufficiently small

$$\begin{aligned}
&\frac{v_1^2(|b_{11}s+b_{12}t|) + v_2^2(|b_{21}s+b_{22}t|)}{v_1^2(|s|)+v_2^2(|t|)} \\
&\quad \le \frac{a_2 v_2^2(|t|(|b_{12}| + |b_{11}\frac{s}{t}|)) + v_2^2(|t|(|b_{22}| + |b_{21}\frac{s}{t}|))}{v_2^2(|t|)}
\end{aligned}$$

$$\leq 2\max\left(a_2(|b_{11}|+|b_{12}|)^{\beta+\epsilon}+(|b_{21}|+|b_{22}|)^{\beta+\epsilon},(a_2(|b_{11}|+|b_{12}|)^{\beta-\epsilon}+(|b_{21}|+|b_{22}|)^{\beta-\epsilon}\right).$$

Hence we get the upper bound. For the lower bound, making the following linear transformation

$$(s,t)^\top = B^{-1}(s',t')^\top = \begin{pmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \end{pmatrix}(s',t')^\top$$

and then using the above conclusion, we have

$$\begin{aligned}
&v_1^2(|s|)+v_2^2(|t|)\\
&\quad= v_1^2(|b'_{11}s'+b'_{12}t'|)+v_2^2(|b'_{21}s'+b'_{22}t'|)\\
&\quad\leq 2\max\left(a_2(|b'_{11}|+|b'_{12}|)^{\beta+\epsilon}+(|b'_{21}|+|b'_{22}|)^{\beta+\epsilon},(a_2(|b'_{11}|+|b'_{12}|)^{\beta-\epsilon}+(|b'_{21}|+|b'_{22}|)^{\beta-\epsilon}\right)(v_1^2(|s'|)+v_2^2(|t'|))\\
&\quad\leq 2\max\left(a_2(|b'_{11}|+|b'_{12}|)^{\beta+\epsilon}+(|b'_{21}|+|b'_{22}|)^{\beta+\epsilon},(a_2(|b'_{11}|+|b'_{12}|)^{\beta-\epsilon}+(|b'_{21}|+|b'_{22}|)^{\beta-\epsilon}\right)\\
&\qquad\times\left(v_1^2(|b_{11}s+b_{12}t|)+v_2^2(|b_{21}s+b_{22}t|)\right),
\end{aligned}$$

provided that $|t'|\geq|s'|$ and $|t'|>0$ sufficiently small. This completes the proof. □

Proof of inequality (68). Note that

$$v_1^2(|s+b_{12}t|)+v_2^2(|t|)=v_1^2(|s|)\frac{v_1^2(|s||1+b_{12}t/s|)+v_2^2(|s||t/s|)}{v_1^2(|s|)},\quad (s,t)\in D_u. \tag{80}$$

If $|t/s|\leq M<\infty$, then by uniform convergence theorem (UCT) in [37] we have

$$\sup_{(s,t)\in D_u,|t/s|\leq M}\left|\frac{v_1^2(|s||1+b_{12}t/s|)+v_2^2(|s||t/s|)}{v_1^2(|s|)}-|1+b_{12}t/s|^\beta-\theta|t/s|^\beta\right|\to 0,\quad u\to\infty.$$

If $|t/s|\geq M$, then using Lemma 6.1, for u and M sufficiently large

$$\inf_{(s,t)\in D_u,|t/s|\geq M}\frac{v_1^2(|s||1+b_{12}t/s|)+v_2^2(|s||t/s|)}{v_1^2(|s|)}\geq 1/2\left(\left||b_{12}|M-1\right|^{\beta/2}+\theta M^{\beta/2}\right).$$

Therefore, the minimum of $v_1^2(|s+b_{12}t|)+v_2^2(|t|)$ is attained for $|t/s|\leq M$ with M sufficiently large. Further, the minimum of $|1+b_{12}t/s|^\beta+\theta|t/s|^\beta$ is attained at $\mu=t/s\in[-1/|b_{12}|,1/|b_{12}|]$. Thus, for $(s,t)\in D_u$ and u sufficiently large

$$\begin{aligned}
\frac{v_1^2(|s+b_{12}t|)+v_2^2(|t|)}{v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)} &= \frac{v_1^2(|s+b_{12}t|)+v_2^2(|t|)}{v_1^2(|s|)}\frac{v_1^2(|s|)}{v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)}\\
&\geq \frac{|1+b_{12}\mu|^\beta+\theta|\mu|^\beta}{2}\frac{1}{2(|1+b_{12}\mu|^\beta+\theta|\mu|^\beta)}=1/4,
\end{aligned}\tag{81}$$

Recall that in our notation

$$v(s,t)=v_1^2(|s+b_{12}t|)+v_2^2(|t|)-v_1^2(|(1+b_{12}\mu)s|)-v_2^2(|\mu s|),\qquad (s,t)\in D_u$$

and note that $v(s,t)$ may be negative at some point. For any $\varepsilon>0$ sufficiently small we have

$$\begin{aligned}
&\left[1+(1-2\epsilon)\left(v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)\right)\right][1+(1-2\epsilon)v(s,t)]\\
&\quad=1+(1-2\epsilon)\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right)+(1-2\epsilon)^2\left(v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)\right)v(s,t).
\end{aligned}$$

Moreover (81) yields as $u\to\infty$

$$\left(v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)\right)v(s,t)=o\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right),\quad (s,t)\in D_u,$$

implying for any $0<\epsilon<1/4$ and sufficiently large u

$$\left[1+(1-2\epsilon)\left(v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)\right)\right][1+(1-2\epsilon)v(s,t)]\leq 1+(1-\epsilon)\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right), \tag{82}$$

with $(s,t)\in D_u$. Since for

$$|s|\in[\overleftarrow{\rho_2}(u^{-1})x/2,\overleftarrow{\rho_2}(u^{-1})2y],\quad |t|\in[M\overleftarrow{\rho_2}(u^{-1}),\overleftarrow{v}_2(u^{-1}\ln u)]$$

we have as $u\to\infty$

$$v(s,t) = v_1^2(|t|)\frac{v_1^2(|s+b_{12}t|)+v_2^2(|t|)-v_1^2(|(1+b_{12}\mu)s|)-v_2^2(|\mu s|)}{v_1^2(|t|)}$$

$$\sim \quad v_1^2(|t|)\left(\left|b_{12}+\frac{s}{t}\right|^{\beta}+\theta-|1+b_{12}\mu|^{\beta}\left|\frac{s}{t}\right|^{\beta}-\theta\left|\mu\frac{s}{t}\right|^{\beta}\right),$$

then for M,u sufficiently large

$$v(s,t)\geq\frac{1-3\epsilon}{1-2\epsilon}v(s_1,t_1), \tag{83}$$

with $|s|,|s_1|\in[\overleftarrow{\rho_2}(u^{-1})x/2,\overleftarrow{\rho_2}(u^{-1})2y], |t|,|t_1|\in[M\overleftarrow{\rho_2}(u^{-1}),\overleftarrow{v}_2(u^{-1}\ln u)]$.
Moreover, for any $\epsilon_1>0$, $|s|\in[\overleftarrow{\rho_2}(u^{-1})x/2,\overleftarrow{\rho_2}(u^{-1})2y]$ and $|t|\in[0,M\overleftarrow{\rho_2}(u^{-1})]$, applying again UCT we have

$$\begin{aligned} v(s,t) &\geq v_1^2(|s|)\left[(1-\epsilon_1)\left(|1+b_{12}t/s|^{\beta}+\theta\,|t/s|^{\beta}\right)-(1+\epsilon_1)\left(|1+b_{12}\mu|^{\beta}+\theta|\mu|^{\beta}\right)\right]\\ &\geq -2\epsilon_1\left(|1+b_{12}\mu|^{\beta}+\theta|\mu|^{\beta}\right)v_1^2(|s|),\end{aligned}$$

and for any $|s|,|s_1|\in[\frac{i-1}{n}\overleftarrow{\rho_2}(u^{-1}),\frac{i+2}{n}\overleftarrow{\rho_2}(u^{-1})]$ with $x/2\leq\frac{i}{n}\leq 2y$ and $|t|\in[0,M\overleftarrow{\rho_2}(u^{-1})]$ and u,n sufficiently large

$$\begin{aligned} &|v(s,t)-v(s_1,t)|\leq v_1^2(|s|)\max_{d_1,d_2\in\{\pm\epsilon_1\}}\left|(1+d_1)\left(|1+b_{12}t/s|^{\beta}+|(1+b_{12}\mu)s_1/s|^{\beta}+\theta|\mu s_1/s|^{\beta}\right)\right.\\ &\left.-(1+d_2)\left(|1+b_{12}\mu|^{\beta}+\theta|\mu|^{\beta}+|s_1/s+b_{12}t/s|^{\beta}\right)\right|\\ &\leq v_1^2(|s|)\mathbb{Q}\epsilon_1\left(|1+b_{12}\mu|^{\beta}+\theta|\mu|^{\beta}+|1+2|b_{12}|M/x|^{\beta}\right)+v_1^2(|s|)||s_1/s|^{\beta}-1|\left(|1+b_{12}\mu|^{\beta}+\theta|\mu|^{\beta}\right)\\ &+v_1^2(|s|)\sup_{|z|\in[0,4M/x]}|h_{s_1/s}(z)-h_1(z)|,\end{aligned}$$

where $h_s(z)=|s+b_{12}z|^{\beta}, s,z\in\mathbb{R}$. Therefore, for $|s|,|s_1|\in[\frac{i-1}{n}\overleftarrow{\rho_2}(u^{-1}),\frac{i+2}{n}\overleftarrow{\rho_2}(u^{-1})]$ with $x/2\leq\frac{i}{n}\leq 2y$ and $|t|\in[0,M\overleftarrow{\rho_2}(u^{-1})]$ for any $\epsilon_1>0$ sufficiently small and u,n sufficiently large

$$v(s,t)\geq-\frac{\epsilon}{4}\left(|1+b_{12}\mu|^{\beta}+\theta|\mu|^{\beta}\right)v_1^2(|s|),$$

$$|v(s,t)-v(s_1,t)|\leq\frac{\epsilon}{8}\left(|1+b_{12}\mu|^{\beta}+\theta|\mu|^{\beta}\right)v_1^2(|s|),$$

which implies that (recall that $\lim_{u\to\infty}\sup_{(s,t)\in D_u}|v(s,t)|=0$)

$$\begin{aligned} &\epsilon\left(v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)\right)+\epsilon v(i\overleftarrow{\rho_2}(u^{-1})/n,t)\\ &\quad\geq(1-2\epsilon)\left(v(i\overleftarrow{\rho_2}(u^{-1})/n,t)-v(s,t)\right)-(1-2\epsilon)^2\left(v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)\right)v(s,t)\\ &\quad\quad+(1-3\epsilon)^2\left(v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)\right)v(i\overleftarrow{\rho_2}(u^{-1})/n,t).\end{aligned}$$

Hence, combining the above with (82) and (83) for any $0<\epsilon<1/4$, for n,u sufficiently large we have

$$\begin{aligned} 1+(1-\epsilon)\left(v_1^2(|s+b_{12}t|)+v_2^2(|t|)\right) &\geq \left[1+(1-2\epsilon)\left(v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)\right)\right]\left[1+(1-2\epsilon)v(s,t)\right]\\ &\geq \left[1+(1-3\epsilon)\left(v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)\right)\right]\left[1+(1-3\epsilon)v(i\overleftarrow{\rho_2}(u^{-1})/n,t)\right],\end{aligned}$$

holds for $|s|\in[\frac{i-1}{n}\overleftarrow{\rho_2}(u^{-1}),\frac{i+2}{n}\overleftarrow{\rho_2}(u^{-1})]$ with $x/2\leq\frac{i}{n}\leq 2y$ and $|t|\in[0,\overleftarrow{v}_2(u^{-1}\ln u)]$, which completes the proof. ☐

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