Gaussian Risk Models with Financial Constraints

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Abstract: In this paper we investigate Gaussian risk models which include financial elements such as inflation and interest rates. For some general models for inflation and interest rates, we obtain an asymptotic expansion of the finite-time ruin probability for Gaussian risk models. Furthermore, we derive an approximation of the conditional ruin time by an exponential random variable as the initial capital tends to infinity.

Key Words: Finite-time ruin probability; conditional ruin time; exponential approximation; Gaussian risk process; inflation; interest.

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1 Introduction

A central topic in the actuarial literature, inspired by the early contributions of Lundberg (1903) and Cramér (1930), is the computation of the ruin probability over both finite-time and infinite-time horizon; see e.g., Rolski et al. (1999), Mikosch (2008), Asmussen and Albrecher (2010) and the references therein. As mentioned in Mikosch (2008) calculation of the ruin probability is considered as the "jewel" of the actuarial mathematics. In fact, exact formulas for both finite-time and infinite-time ruin probability are known only for few special models. Therefore, asymptotic methods have been developed to derive expansions of the ruin probability as the initial capital/reserve increases to infinity. Following Chapter 11.4 in Rolski et al. (1999) the risk reserve process of an insurance company can be modelled by a stochastic process $\{\tilde{U}(t), t \ge 0\}$ given as

$$\tilde{U}(t) = u + ct - \int_0^t Z(s)\, ds, \quad t \ge 0, \tag{1.1}$$

where $u \ge 0$ is the initial reserve, $c > 0$ is the rate of premium received by the insurance company, and $\{Z(t), t \ge 0\}$ is a centered Gaussian process with almost surely continuous sample paths; the process $\{Z(t), t \ge$

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$0\}$ is frequently referred to as the loss rate of the insurance company. Under the assumption that $\{Z(t), t \geq 0\}$ is stationary the asymptotics of the infinite-time ruin probability of the process (1.1) defined by

$$\psi_\infty(u) = \mathbb{P}\left\{ \inf_{t\in[0,\infty)} \tilde{U}(t) < 0 \right\}, \quad u \geq 0$$

has been investigated in Hüsler and Piterbarg (2004), Dębicki (2002), Dieker (2005) and Kobelkov (2005); see also Hüsler and Piterbarg (1999) and Hashorva et al. (2013). Therein the exact speed of convergence to 0 of $\psi_\infty(u)$ as $u \to \infty$ was dealt with.

In order to account for the financial nature of the risks and thus for the time-value of the money as well as other important economic factors, in this paper we shall consider a more general risk process which includes inflation/deflation effects and interest rates (cf. Chapter 11.4 in Rolski et al. (1999)). Essentially, in case of inflation, a monetary unit at time 0 has the value $e^{-\delta_1(t)}$ at time t, where $\delta_1(t), t \geq 0$ is a positive function with $\delta_1(0) = 0$. In case of interest, a monetary unit invested at time 0 has the value $e^{\delta_2(t)}$ at time t, where $\delta_2(t), t \geq 0$ is another positive function with $\delta_2(0) = 0$.

Assuming first that the premium rate and the loss rate have to be adjusted for inflation, we arrive at the following risk reserve process

$$u + c\int_0^t e^{\delta_1(s)} ds - \int_0^t e^{\delta_1(s)} Z(s) ds, \quad t \geq 0.$$

Since the insurance company invests the surplus and thus accounting for investment effects the resulting risk reserve process is

$$U(t) = e^{\delta_2(t)} \left(u + c\int_0^t e^{\delta_1(s)-\delta_2(s)} ds - \int_0^t e^{\delta_1(s)-\delta_2(s)} Z(s) ds \right), \quad t \geq 0. \tag{1.2}$$

We shall refer to $\{U(t), t \geq 0\}$ as the *risk reserve process in an economic environment*; see Chapter 11.4 in Rolski et al. (1999) for a detailed discussion on the effects of financial factors on the risk reserve processes.

In the case that $\delta_1(t) = 0, \delta_2(t) = \delta t, t \geq 0$, with $\delta > 0$, the random process $\{U(t), t \geq 0\}$ reduces to a *risk reserve process with constant force of interest*. For a class of stationary Gaussian processes $\{Z(t), t \geq 0\}$ with twice differentiable covariance function, the exact asymptotics of the infinite-time ruin probability for the risk reserve process with constant force of interest was obtained in He and Hu (2007). Since therein the authors considered only smooth Gaussian process, the method of proof relied on the well-known Rice method; see e.g., Piterbarg (1996).

Let T be any positive constant. The principal goal of this contribution is the derivation of the exact asymptotics of the finite-time ruin probability of the risk reserve process U given by

$$\begin{aligned} \psi_T(u) &:= \mathbb{P}\left\{ \inf_{t\in[0,T]} U(t) < 0 \right\} \\ &= \mathbb{P}\left\{ \sup_{t\in[0,T]} \left(\int_0^t e^{-\delta(s)} Z(s)\, ds - c\int_0^t e^{-\delta(s)} ds \right) > u \right\} \end{aligned} \tag{1.3}$$

as $u \to \infty$, where $\{Z(t), t \geq 0\}$ is a general centered Gaussian process with almost surely continuous sample paths and $\delta(t) = \delta_2(t) - \delta_1(t), t \geq 0$ is some measurable real-valued function satisfying $\delta(0) = 0$. Note in passing that $\delta(t) > 0$ means that the interest contributes more to the risk reserve process than the inflation at time t, and vice versa.

In Theorem 2.1 below we shall show that $\psi_T(u)$ has asymptotically, as $u \to \infty$, (non-standard) normal distribution. This emphases the qualitative difference between asymptotics in finite- and infinite-time horizon scenario; see He and Hu (2007).

A related, interesting and vastly analyzed quantity is the time of ruin which in our model is defined as

$$\tau(u) = \inf\{t \geq 0 : U(t) < 0\}, \quad u \geq 0. \tag{1.4}$$

Using that $\mathbb{P}\{\tau(u) < T\} = \mathbb{P}\left\{\inf_{t\in[0,T]} U(t) < 0\right\}$, investigation of distributional properties of the time of ruin under the condition that ruin occurs in a certain time period has attracted substantial attention; see e.g., the seminal contribution Segerdahl (1955) and the monographs Embrechts et al. (1997) and Asmussen and Albrecher (2010). Recent results for infinite-time Gaussian and Lévy risk models are derived in Hüsler (2006), Hüsler and Piterbarg (2008), Hüsler and Zhang (2008), Griffin and Maller (2012), Griffin (2013) and Hashorva and Ji (2013).

In Theorem 2.4 we derive a novel result, which shows that as $u \to \infty$, the sequence of random variables $\{\xi_u, u > 0\}$, defined (on the same probability space) by

$$\xi_u \stackrel{d}{=} u^2(T - \tau(u))\Big|(\tau(u) < T) \tag{1.5}$$

converges in distribution to an exponential random variable (here $\stackrel{d}{=}$ stands for the equality of the distribution functions). This, somewhat surprising result, contrasts with the infinite-time case analyzed by Hüsler and Piterbarg (2008) and Hashorva and Ji (2013), where the limiting random variable is normally distributed.

Organization of the paper: The main results concerning the finite-time ruin probability and the approximation of ξ_u are displayed in Section 2, whereas the proofs are relegated to Section 3. We conclude this contribution with a short Appendix.

2 Main Results

Let the loss rate of the insurance company $\{Z(t), t \geq 0\}$ be modelled by a centered Gaussian process with almost surely continuous sample paths and covariance function $\mathbb{C}ov\left(Z(s), Z(t)\right) = R(s, t)$. As mentioned in the Introduction we shall require that $\delta(0) = 0$. For notational simplicity we shall define below

$$Y(t) := \int_0^t e^{-\delta(s)} Z(s) ds, \quad \sigma^2(t) := \mathbb{V}ar(Y(t)), \quad \widetilde{\delta}(t) := \int_0^t e^{-\delta(s)} ds, \quad t \in [0, T]. \tag{2.6}$$

In what follows let $\sigma'(t)$ be the derivative of $\sigma(t)$, and let Ψ denote the survival function of a $N(0,1)$ random variable. We are interested in the asymptotic behavior of (1.3) as the initial reserve u tends to infinity, i.e., we shall investigate the asymptotics of

$$\psi_T(u) = \mathbb{P}\left\{ \sup_{t\in[0,T]} \left(Y(t) - c\widetilde{\delta}(t) \right) > u \right\}$$

as $u \to \infty$. In our first result below we derive an asymptotic expansion of $\psi_T(u)$ in terms of $c, \sigma(T), \widetilde{\delta}(T)$.

Theorem 2.1. *Let $\{Z(t), t \ge 0\}$ be a centered Gaussian process with almost surely continuous sample paths and covariance function $R(s,t), s,t \ge 0$. Further let $\delta(t), t \ge 0$, be some measurable function with $\delta(0) = 0$. If $\sigma(t)$ attains its maximum over $[0,T]$ at the unique point $t = T$ and $\sigma'(T) > 0$, then*

$$\psi_T(u) \ = \mathbb{P}\left\{ \mathcal{N} > (u + c\widetilde{\delta}(T))/\sigma(T) \right\} (1 + o(1)) \tag{2.7}$$

holds as $u \to \infty$, with $\mathcal{N}$ a $N(0,1)$ random variable.

The following result is an immediate consequence of Theorem 2.1.

Corollary 2.2. *Let $\{Z(t), t \ge 0\}$ and $\delta(t), t \ge 0$ be given as in Theorem 2.1. If $R(s,t) > 0$ for any $s,t \in [0,T]$, then* (2.7) *is satisfied.*

Remarks 2.3. *a) It follows from the proof of Theorem 2.1 that* (2.7) *still holds if $Y(t) := \int_0^t e^{-\delta_1(s)} Z(s) ds$ and $\widetilde{\delta}(t) := \int_0^t e^{-\delta_2(s)} ds$ in (2.6).*
b) In the asymptotic behavior of $\psi_T(u)$ the positive constant $\sigma'(T)$ does not appear. It appears however explicitly in the the approximation of the conditional ruin time as shown in our second theorem below.

Along with the analysis of the ruin probability in risk theory an important theoretical topic is the behavior of the ruin time. For infinite-time horizon results in this direction are well-known; see e.g., Asmussen and Albrecher (2010), Hüsler and Piterbarg (2008) and Hashorva and Ji (2013) for the normal approximation of the conditional distribution of the ruin time $\tau(u)$ given that $\tau(u) < \infty$.
In our second result below we show that (appropriately rescaled) ruin time $\tau(u)$ conditioned that $\tau(u) < T$ is asymptotically, as $u \to \infty$, exponentially distributed with parameter $\sigma'(T)/(\sigma(T))^3$.

Theorem 2.4. *Under the conditions of Theorem 2.1, we have*

$$\lim_{u\to\infty} \boldsymbol{P}\left\{ u^2(T - \tau(u)) \le x \middle| \tau(u) < T \right\} = 1 - \exp\left(-\frac{\sigma'(T)}{\sigma^3(T)} x \right), \quad x \ge 0. \tag{2.8}$$

Note in passing that (2.8) means the convergence in distribution

$$\xi_u \stackrel{d}{\to} \xi, \quad u \to \infty, \tag{2.9}$$

where ξ is exponentially distributed such that

$$e_T := \mathbb{E}(\xi) = \frac{\sigma^3(T)}{\sigma'(T)} > 0.$$

We present next three illustrating examples.

Example 2.5. *Let $\{Z(t), t \geq 0\}$ be an Ornstein-Uhlenbeck process with parameter $\lambda > 0$, i.e., Z is a stationary process with covariance function $R(s,t) = \exp(-\lambda|s-t|)$. If $\delta(t) = \delta t, t \geq 0$ with $\delta \in (0, \lambda)$, then*

$$\widetilde{\delta}(t) = \frac{1}{\delta}\left(1 - e^{-\delta t}\right), \quad \sigma^2(t) = \frac{1}{(\lambda-\delta)\delta}\left(1 - e^{-2\delta t}\right) - \frac{2}{\lambda^2 - \delta^2}\left(1 - e^{-(\lambda+\delta)t}\right), \ t \in [0, T].$$

Therefore, from Corollary 2.2, we obtain that

$$\begin{aligned} \psi_T(u) &= \frac{1}{u\sqrt{2\pi}}\sqrt{\frac{1}{(\lambda-\delta)\delta}\left(1 - e^{-2\delta T}\right) - \frac{2}{\lambda^2-\delta^2}\left(1 - e^{-(\lambda+\delta)T}\right)} \\ &\quad \times \exp\left(-\frac{(u + c/\delta\left(1 - e^{-\delta T}\right))^2}{2\left(\frac{1}{(\lambda-\delta)\delta}\left(1 - e^{-2\delta T}\right) - \frac{2}{\lambda^2-\delta^2}\left(1 - e^{-(\lambda+\delta)T}\right)\right)}\right)(1 + o(1)) \end{aligned}$$

as $u \to \infty$. Furthermore, in view of Theorem 2.4 the convergence in (2.9) *holds with*

$$e_T = \frac{((\lambda+\delta)(1 - e^{-2\delta T}) - 2\delta(1 - e^{-(\lambda+\delta)T}))^2}{(\lambda-\delta)(\lambda+\delta)^2\delta^2 e^{-\delta T}(e^{-\delta T} - e^{-\lambda T})}.$$

Example 2.6. *Let $\{Z(t), t \geq 0\}$ be a Slepian process, i.e.,*

$$Z(t) = B(t+1) - B(t), \quad t \geq 0,$$

with B a standard Brownian motion. For this model we have $R(s,t) = \max(1 - |s-t|, 0)$. If further $\delta(t) = \delta t, t \geq 0$ with $\delta \neq 0$, then

$$\widetilde{\delta}(t) = \frac{1}{\delta}\left(1 - e^{-\delta t}\right), \quad \sigma^2(t) = \frac{1}{\delta^2} - \frac{1}{\delta^3} - \frac{2}{\delta^2}e^{-\delta t} + \frac{2t}{\delta^2}e^{-\delta t} + \frac{\delta+1}{\delta^3}e^{-2\delta t}, \ t \in [0, 1].$$

Consequently, Corollary 2.2 implies, as $u \to \infty$

$$\psi_1(u) = \sqrt{\frac{\delta - 1 + (\delta+1)e^{-2\delta}}{2\pi\delta^3}}\, u^{-1}\exp\left(-\frac{\delta\left(\delta u + c\left(1 - e^{-\delta}\right)\right)^2}{2(\delta - 1 + (\delta+1)e^{-2\delta})}\right)(1 + o(1)).$$

Further by Theorem 2.4 the convergence in (2.9) *holds with*

$$e_1 = \frac{(\delta - 1 + (\delta+1)e^{-2\delta})^2}{\delta^4 e^{-\delta} - \delta^4(\delta+1)e^{-2\delta}}.$$

Example 2.7. *Let $\{Z(t), t \geq 0\}$ be a standard Brownian motion and assume that $\delta(t) = t^2/2, t \geq 0$. Since $R(s,t) = \min(s,t)$ we obtain*

$$\widetilde{\delta}(t) = \sqrt{2\pi}(1/2 - \Psi(t)), \quad \sigma^2(t) = (\sqrt{2}-1)\sqrt{\pi} - 2\sqrt{2\pi}\Psi(t) + 2\sqrt{\pi}\Psi(\sqrt{2}t), \ t \in [0, T].$$

Applying once again Corollary 2.2 we obtain

$$\begin{aligned}\psi_T(u) &= \sqrt{\frac{(\sqrt{2}-1)\sqrt{\pi}-2\sqrt{2\pi}\Psi(T)+2\sqrt{\pi}\Psi(\sqrt{2}T)}{2\pi}}u^{-1}\\ &\quad\times\exp\left(-\frac{\left(u+\sqrt{2\pi}c(1/2-\Psi(T))\right)^2}{2\left((\sqrt{2}-1)\sqrt{\pi}-2\sqrt{2\pi}\Psi(T)+2\sqrt{\pi}\Psi(\sqrt{2}T)\right)}\right)(1+o(1))\end{aligned}$$

as $u\to\infty$. Finally, by Theorem 2.4 the convergence in distribution in (2.9) *holds with*

$$e_T=\frac{\left((\sqrt{2}-1)\sqrt{\pi}-2\sqrt{2\pi}\Psi(T)+2\sqrt{\pi}\Psi(\sqrt{2}T)\right)^2}{\sqrt{2\pi}(\varphi(T)-\varphi(\sqrt{2}T))},$$

where $\varphi=-\Psi'$ is the density function of $N(0,1)$ random variable.

3 Proofs

Before presenting proofs of Theorems 2.1 and 2.4, we introduce some notation. Let $g_u(t)=\frac{u+c\tilde{\delta}(t)}{\sigma(t)}$ and define

$$\begin{aligned}X_u(t) &:= \frac{Y(t)}{\sigma(t)}\frac{g_u(T)}{g_u(t)},\quad \sigma^2_{X_u}(t):=\mathbb{V}ar\left(X_u(t)\right),\\ r_{X_u}(s,t) &:= \mathbb{C}ov\left(\frac{X_u(s)}{\sigma_{X_u}(s)},\frac{X_u(t)}{\sigma_{X_u}(t)}\right)=\mathbb{C}ov\left(\frac{Y(s)}{\sigma(s)},\frac{Y(t)}{\sigma(t)}\right).\end{aligned}$$

Then, we can reformulate (1.3) for all large u as

$$\begin{aligned}\psi_T(u) &= \mathbb{P}\left\{\sup_{t\in[0,T]}\left(\frac{Y(t)}{\sigma(t)}\frac{g_u(T)}{g_u(t)}\right) > g_u(T)\right\}\\ &= \mathbb{P}\left\{\sup_{t\in[0,T]}X_u(t) > g_u(T)\right\},\quad u\ge 0. \qquad (3.10)\end{aligned}$$

Proof of Theorem 2.1 We shall derive first a lower bound for $\psi_T(u)$. It follows from (3.10) that

$$\begin{aligned}\psi_T(u) &\ge \mathbb{P}\left\{\frac{Y(T)}{\sigma(T)} > g_u(T)\right\}\\ &= \Psi(g_u(T))\\ &= \frac{\sigma(T)}{\sqrt{2\pi}}u^{-1}\exp\left(-\frac{(u+c\widetilde{\delta}(T))^2}{2\sigma^2(T)}\right)(1+o(1)) \qquad (3.11)\end{aligned}$$

as $u\to\infty$. Next, we derive the upper bound. Since $R(s,t)=R(t,s)$ for any $s,t\in[0,T]$, we have

$$\sigma^2(t):=\mathbb{V}ar\left(Y(t)\right)=2\int_0^t\int_0^w e^{-\delta(v)-\delta(w)}R(v,w)\,dvdw. \qquad (3.12)$$

Further, since by the assumption the function $\sigma(t)$ attains its unique maximum over $[0,T]$ at $t=T$ and that $\sigma'(T)>0$, there exists some $\theta_1\in(0,T)$ such that $\sigma(t)$ is strictly increasing on $[\theta_1,T]$ and

$$\inf_{t\in[\theta_1,T]}\sigma'(t)>0 \qquad (3.13)$$

implying that for u sufficiently large

$$\sigma'_{X_u}(t) = \frac{\sigma'(t)}{\sigma(T)}\frac{u + c\widetilde{\delta}(T)}{u + c\widetilde{\delta}(t)} - \frac{ce^{\delta(t)}\sigma(t)(u + c\widetilde{\delta}(T))}{(u + c\widetilde{\delta}(t))^2\sigma(T)} > 0$$

for all $t \in [\theta_1, T]$. Hence, for sufficiently large u, $\sigma_{X_u}(t)$ is strictly increasing on $[\theta_1, T]$. Furthermore, since

$$\begin{aligned} 1 - \sigma_{X_u}(t) &= 1 - \frac{g_u(T)}{g_u(t)} \\ &= \frac{(\sigma(T) - \sigma(t))\left(u + c\widetilde{\delta}(t)\right) - c\sigma(t)(\widetilde{\delta}(T) - \widetilde{\delta}(t))}{\sigma(T)\left(u + c\widetilde{\delta}(t)\right)}, \end{aligned}$$

then by the definitions of $\widetilde{\delta}(t)$ and $\sigma(t)$ for any $\varepsilon_1 > 0$ there exist some constants $K > 0$ and $\theta_2 \in (0, T)$ such that

$$\begin{aligned} \widetilde{\delta}(T) - \widetilde{\delta}(t) &\leq K\ (T - t), \\ (1 - \varepsilon_1)\sigma'(T)(T - t) &\leq \sigma(T) - \sigma(t) \leq (1 + \varepsilon_1)\sigma'(T)(T - t) \end{aligned}$$

are valid for all $t \in [\theta_2, T]$. Therefore, we conclude that for u sufficiently large

$$(1 - \varepsilon_1)^2\frac{\sigma'(T)}{\sigma(T)}(T - t) \leq 1 - \sigma_{X_u}(t) \leq (1 + \varepsilon_1)\frac{\sigma'(T)}{\sigma(T)}(T - t) \tag{3.14}$$

for $t \in [\theta_2, T]$. For any $s < t$ we have

$$\begin{aligned} 1 - r_{X_u}(s, t) &= 1 - \mathbb{C}ov\left(\frac{Y(s)}{\sigma(s)}, \frac{Y(t)}{\sigma(t)}\right) \\ &= \frac{\mathbb{V}ar\left(Y(t) - Y(s)\right) - (\sigma(t) - \sigma(s))^2}{2\sigma(s)\sigma(t)} \\ &\leq \frac{\mathbb{V}ar\left(Y(t) - Y(s)\right)}{2\sigma(s)\sigma(t)} \\ &= \frac{\int_s^t\int_s^t R(v, w)e^{-\delta(w) - \delta(v)}dwdv}{2\sigma(s)\sigma(t)}. \end{aligned}$$

The above implies that for sufficiently large u and $s, t \in [\theta_2, T]$

$$1 - r_{X_u}(s, t) \leq C(t - s)^2, \tag{3.15}$$

where $C = \max_{w,v \in [\theta_2, T]} \frac{|R(v,w)|e^{-\delta(w) - \delta(v)}}{2\sigma^2(w)}$. Consequently, in the light of (3.14) and (3.15), for any $\varepsilon > 0$ sufficiently small, we have for some $\theta_0 \in (\max(\theta_1, \theta_2), T)$

$$\sigma_{X_u}(t) \leq \frac{1}{1 + (1 - \varepsilon)(1 - \varepsilon_1)^2\frac{\sigma'(T)}{\sigma(T)}(T - t)}$$

and

$$r_{X_u}(s, t) \geq e^{-(1+\varepsilon)C(t-s)^2}$$

for all $s,t\in[\theta_0,T]$. Next, define a centered Gaussian process $\{Y_\varepsilon(t),t\ge 0\}$ as

$$Y_\varepsilon(t)=\frac{\xi_\varepsilon(t)}{1+(1-\varepsilon)(1-\varepsilon_1)^2\frac{\sigma'(T)}{\sigma(T)}(T-t)},$$

where $\{\xi_\varepsilon(t),t\ge 0\}$ is a centered stationary Gaussian process with covariance function $\mathbb{C}ov\,(\xi_\varepsilon(t),\xi_\varepsilon(s))=e^{-(1+\varepsilon)C(t-s)^2}$. In view of Slepian Lemma (cf. Adler and Taylor (2007) or Berman (1992)) we obtain

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[\theta_0,T]}\left(Y(t)-c\widetilde{\delta}(t)\right)\;>\;u\right\}\;&=\;\mathbb{P}\left\{\sup_{t\in[\theta_0,T]}X_u(t)\;>\;g_u(T)\right\}\\&\le\;\mathbb{P}\left\{\sup_{t\in[\theta_0,T]}\left(\frac{X_u(t)/\sigma_{X_u}(t)}{1+(1-\varepsilon)(1-\varepsilon_1)^2\frac{\sigma'(T)}{\sigma(T)}(T-t)}\right)\;>\;g_u(T)\right\}\\&\le\;\mathbb{P}\left\{\sup_{t\in[\theta_0,T]}Y_\varepsilon(t)\;>\;g_u(T)\right\}\\&=\;\Psi(g_u(T))(1+o(1))\end{aligned}\tag{3.16}$$

as $u\to\infty$, where the last asymptotic equivalence follows from iii) of Theorem 4.1 in Appendix. Moreover since for u sufficiently large there exists some $\lambda\in(0,1)$ such that

$$\sup_{t\in[0,\theta_0]}\sigma_{X_u}(t)\;\le\;\sup_{t\in[0,\theta_0]}\frac{(1+\lambda)\sigma(t)}{\sigma(T)}\;\le\;\frac{(1+\lambda)\sigma(\theta_0)}{\sigma(T)}\;<\;1$$

and

$$\mathbb{P}\left\{\sup_{t\in[0,\theta_0]}X_u(t)\;>a\right\}\le\mathbb{P}\left\{\sup_{t\in[0,\theta_0]}\frac{2Y(t)}{\sigma(T)}>a\right\}\le\frac{1}{2}$$

for some positive number a, we get from Borell inequality (e.g., Piterbarg (1996)) that, for u sufficiently large

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,\theta_0]}\left(Y(t)-c\widetilde{\delta}(t)\right)\;>\;u\right\}\;&=\;\mathbb{P}\left\{\sup_{t\in[0,\theta_0]}X_u(t)\;>\;g_u(T)\right\}\\&\le\;2\Psi\left(\frac{(g_u(T)-a)\sigma(T)}{(1+\lambda)\sigma(\theta_0)}\right)=o(\Psi(g_u(T)))\end{aligned}\tag{3.17}$$

as $u\to\infty$. Combining (3.16) and (3.17), we conclude that

$$\begin{aligned}\psi_T(u)\;&\le\;\mathbb{P}\left\{\sup_{t\in[0,\theta_0]}X_u(t)\;>\;g_u(T)\right\}+\mathbb{P}\left\{\sup_{t\in[\theta_0,T]}X_u(t)\;>\;g_u(T)\right\}\\&=\;\Psi(g_u(T))(1+o(1))\\&=\;\frac{\sigma(T)}{\sqrt{2\pi}}u^{-1}\exp\left(-\frac{(u+c\widetilde{\delta}(T))^2}{2\sigma^2(T)}\right)(1+o(1))\end{aligned}$$

as $u\to\infty$, which together with (3.11) establishes the proof. □

Proof of Corollary 2.2 Since $R(s,t)>0$ for any $s,t\in[0,T]$ it follows from (3.12) that $\sigma(t)$ attains its unique maximum over $[0,T]$ at $t=T$ and $\sigma'(T)>0$. Therefore, the claim follows immediately from Theorem 2.1. □

Proof of Theorem 2.4 In the following we shall use the same notation as in the proof of Theorem 2.1. First note that for any $x > 0$

$$\mathbb{P}\left\{u^2(T-\tau(u)) > x|\tau(u) < T\right\} = \frac{\mathbb{P}\left\{\tau(u) < T - xu^{-2}\right\}}{\mathbb{P}\left\{\tau(u) < T\right\}}.$$

With $T_u := T - xu^{-2}$ and $\widetilde{X}_u(t) := \frac{Y(t)}{\sigma(t)}\frac{g_u(T_u)}{g_u(t)}$ the above can be re-written as

$$\mathbb{P}\left\{u^2(T-\tau(u)) > x|\tau(u) < T\right\} = \frac{\mathbb{P}\left\{\sup_{t\in[0,T_u]}\widetilde{X}_u(t) > g_u(T_u)\right\}}{\mathbb{P}\left\{\sup_{t\in[0,T]}X_u(t) > g_u(T)\right\}}.$$

As in the proof of Theorem 2.1 we have

$$\mathbb{P}\left\{\sup_{t\in[0,T_u]}\widetilde{X}_u(t) > g_u(T_u)\right\} \geq \Psi(g_u(T_u)).$$

In order to derive the upper bound we use a time change such that

$$\mathbb{P}\left\{\sup_{t\in[0,T_u]}\widetilde{X}_u(t) > g_u(T_u)\right\} = \mathbb{P}\left\{\sup_{t\in[0,1]}\widetilde{X}_u(T_u t) > g_u(T_u)\right\}.$$

Similar argumentations as in (3.14) and (3.15) yield that, for some $\theta_0 \in (0,1)$

$$\sigma_{\widetilde{X}_u}(T_u t) \leq \frac{1}{1 + \frac{\sigma'(T_u)}{2\sigma(T_u)}T_u(1-t)}$$

and

$$r_{\widetilde{X}_u}(T_u s, T_u t) \geq e^{-2CT_u^2(t-s)^2}$$

hold for all $s, t \in [\theta_0, 1]$ and all u sufficiently large. Consequently, in view of iii) in Theorem 4.1 and similar argumentations as in the proof of Theorem 2.1 we conclude that

$$\mathbb{P}\left\{\sup_{t\in[0,T_u]}\widetilde{X}_u(t) > g_u(T_u)\right\} \leq \Psi(g_u(T_u))(1+o(1))$$

as $u \to \infty$. Hence

$$\begin{aligned}\mathbb{P}\left\{u^2(T-\tau(u)) > x|\tau(u) < T\right\} &= \frac{\Psi(g_u(T_u))}{\Psi(g_u(T))}(1+o(1)) \\ &= \exp\left(\frac{g_u^2(T) - g_u^2(T_u)}{2}\right)(1+o(1)), \quad u \to \infty. \end{aligned} \tag{3.18}$$

After some standard algebra, it follows that

$$g_u^2(T) - g_u^2(T_u) = \frac{(u + c\widetilde{\delta}(T))^2}{\sigma^2(T)} - \frac{(u + c\widetilde{\delta}(T_u))^2}{\sigma^2(T_u)} = -\frac{2\sigma'(T)}{\sigma^3(T)}x(1+o(1)) \tag{3.19}$$

as $u \to \infty$. Consequently, by (3.18)

$$\lim_{u\to\infty}\mathbb{P}\left\{u^2(T-\tau(u)) > x|\tau(u) < T\right\} = \exp\left(-\frac{\sigma'(T)}{\sigma^3(T)}x\right),$$

which completes the proof. □

4 Appendix

We give below an extension of Theorem D.3 in Piterbarg (1996) suitable for a family of Gaussian processes which is in particular useful for the proof of our main results. We first introduce two well-known constants appearing in the asymptotic theory of Gaussian processes. Let $\{B_\alpha(t), t \ge 0\}$ be a standard fractional Brownian motion with Hurst index $\alpha/2 \in (0,1]$ which is a centered Gaussian process with covariance function

$$\mathbb{C}ov(B_\alpha(t), B_\alpha(s)) = \frac{1}{2}(t^\alpha + s^\alpha - \mid t - s \mid^\alpha), \quad s,t \ge 0.$$

The *Pickands constant* is defined by

$$\mathcal{H}_\alpha = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left(\exp\Big(\sup_{t\in[0,T]}\left(\sqrt{2}B_\alpha(t) - t^\alpha\right)\Big)\right) \in (0,\infty)\ \alpha \in (0,2]$$

and the *Piterbarg constant* is given by

$$\mathcal{P}_\alpha^b = \lim_{T\to\infty} \mathbb{E}\left(\exp\Big(\sup_{t\in[0,T]}\left(\sqrt{2}B_\alpha(t) - (1+b)t^\alpha\right)\Big)\right) \in (0,\infty),\ \ \alpha \in (0,2],\ \ b>0.$$

See for instance Piterbarg (1996) and Dębicki and Mandjes (2003) for properties of the above two constants. Assume, in what follows, that θ and T are two positive constants satisfying $\theta < T$. Let $\{\eta_u(t), t \ge 0\}$ be a family of Gaussian processes satisfying the following three assumptions:

A1 : The variance function $\sigma^2_{\eta_u}(t)$ of η_u attains its maximum over $[\theta, T]$ at the unique point $t = T$ for any u large enough, and further there exist two positive constants A, β and a function $A(u)$ satisfying $\lim_{u\to\infty} A(u) = A$ such that $\sigma_{\eta_u}(t)$ has the following expansion around T for all u large enough

$$\sigma_{\eta_u}(t) = 1 - A(u)(T-t)^\beta(1+o(1)), \quad t \uparrow T.$$

A2 : There exist two constants $\alpha \in (0,2], B > 0$ and a function $B(u)$ satisfying $\lim_{u\to\infty} B(u) = B$ such that the correlation function $r_{\eta_u}(s,t)$ of η_u has the following expansion around T for all u large enough

$$r_{\eta_u}(s,t) = 1 - B(u)|t-s|^\alpha(1+o(1)), \quad \min(s,t) \uparrow T.$$

A3 : For some positive constants $\mathbb{Q}$ and γ, and all u large enough

$$\mathbb{E}\left(\eta_u(s) - \eta_u(t)\right)^2 \le \mathbb{Q}|t-s|^\gamma$$

for any $s,t \in [\theta, T]$.

Theorem 4.1. *Let $\{\eta_u(t), t \ge 0\}$ be a family of Gaussian processes satisfying Assumptions* **A1-A3**.
i) If $\beta > \alpha$, then

$$\mathbb{P}\left\{\sup_{t\in[\theta,T]} \eta_u(t) > u\right\} = \frac{B^{\frac{1}{\alpha}}}{\sqrt{2\pi}A^{\frac{1}{\beta}}}\mathcal{H}_\alpha\Gamma\left(\frac{1}{\beta}+1\right)u^{\frac{2}{\alpha}-\frac{2}{\beta}-1}\exp\left(-\frac{u^2}{2}\right)(1+o(1)), \quad \text{as } u \to \infty.$$

ii) For $\beta = \alpha$ we have

$$\mathbb{P}\left\{\sup_{t\in[\theta,T]} \eta_u(t) > u\right\} = \frac{1}{\sqrt{2\pi}} \mathcal{P}_\alpha^{\frac{A}{B}} u^{-1} \exp\left(-\frac{u^2}{2}\right)(1+o(1)), \quad \textit{as } u \to \infty.$$

iii) If $\beta < \alpha$, then

$$\mathbb{P}\left\{\sup_{t\in[\theta,T]} \eta_u(t) > u\right\} = \frac{1}{\sqrt{2\pi}} u^{-1} \exp\left(-\frac{u^2}{2}\right)(1+o(1)), \quad \textit{as } u \to \infty.$$

Proof of Theorem 4.1 Since from assumptions **A1-A2** we have that, for any $\varepsilon > 0$ and u large enough

$$(A-\varepsilon)(T-t)^\beta(1+o(1)) \le 1 - \sigma_{\eta_u}(t) \le (A+\varepsilon)(T-t)^\beta(1+o(1)), \quad t \uparrow T$$

and

$$(B-\varepsilon)|t-s|^\alpha(1+o(1)) \le 1 - r_{\eta_u}(s,t) \le (B+\varepsilon)|t-s|^\alpha(1+o(1)), \quad \min(s,t) \uparrow T$$

Theorem D.3 in Piterbarg (1996) gives tight asymptotic upper and lower bounds, and thus the claims follow by letting $\varepsilon \to 0$. □

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