Extremes and Products of Multivariate AC-Product Risks

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Abstract: With motivation from Tang et al. (2011), in this paper we consider a tractable multivariate risk structure which includes the Sarmanov dependence structure as a special case. We derive several asymptotic results for both the sum and the product of such risk and then present three applications related to actuarial mathematics.

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1. Introduction

Many modern actuarial tasks such as quantification of large risks and aggregated risk, estimation of ruin probabilities in the presence of financial risks, or reinsurance pricing accounting for both claims and expenses strongly rely on the use of multivariate extreme value theory. Typically, the adequacy of the probabilistic models employed by the actuaries is determined by their flexibility to allow for the dependence among risks. Most of classical insurance models assume independence of risks, a phenomenon which is rarely observed in practical actuarial tasks. The role of the dependence among risks is crucial, especially when modelling the impact of large risks. Dependence modeling and in particular that of large risks has been the topic of several contributions such as Goovaerts et al. (2005), Denuit et al. (2006), Li et al. (2010), Asimit et al. (2011), Chen (2011), Haug et al. (2011), Manner and Segers (2011), Tang et al. (2011), Chen and Yuen (2009,2012) among many others.

Asimit et al. (2011) successfully demonstrates the role of asymptotic dependence and asymptotic independence in actuarial modelling. As shown therein, multivariate risks which exhibit asymptotic dependence imply in general different results compared to multivariate risks which have asymptotic independent components. Tractable multivariate distributions like the Fairlie-Gumbel-Morgenstern (FGM) ones exhibit asymptotic independence.

In various risk models employed by actuaries two related tasks are the asymptotic analysis of aggregated risk, and the asymptotic quantification of the effect random scaling (or deflation) of risks. Since the empirical data always support the fact that risks are stochastically dependent, aggregation of dependent risks has become recently a key topic for insurance, finance, and risk management. Recent results of Mitra and Resnick (2009) and Asimit et al. (2011) pave the way for the analysis of the impact of a single large risk to the aggregated risk.

In a mathematical framework, if $X_0, \ldots, X_n$ are non-negative random variables (rv's) with distribution functions (df's) $F_0, \ldots, F_n$, then the aggregated risk is $S = \sum_{i=0}^n X_i$. In order to avoid triviality, we assume that the risks are all non-degenerate at zero. Large values of S mean large financial risks for the company, and therefore the actuarial interest focusses mainly on the quantification of the probability of such large values, i.e., $\mathbb{P}(S > u)$ where the level u reaches some extreme point.

In another context, X_0 can be considered as the base risk, whereas $X_1, \ldots, X_n$ as random deflators/inflators.

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Of actuarial interest is the asymptotic tail behaviour of the ultimate deflated risk ($u \to \infty$)

$$\mathbb{P}(Z > u), \quad \text{with } Z = X_0 \prod_{i=1}^{n} X_i. \tag{1.1}$$

For independent risks recent results in this direction are derived in Hashorva at al. (2010).
The main goal of this paper is to introduce a tractable class of dependent risks which allows for explicit calculation of various actuarial quantities of interest. The motivation for introducing such a class of risks comes from the simple structure of multivariate FGM df's. By definition, a $(n+1)$-dimensional random vector $\boldsymbol{X} = (X_0, \ldots, X_n)$ has a multivariate FGM df Q with marginal df's $F_0, \ldots, F_n$ if

$$Q(x_0, \ldots, x_n) = \prod_{i=0}^{n} F_i(x)\Big[1 + \sum_{0 \le i < j \le n} \theta_{ij} \overline{F_i}(x_i) \overline{F_j}(x_j)\Big], \quad x_i \in [0, \hat{x}_i],\ 0 \le i \le n, \tag{1.2}$$

where $\overline{F_i} := 1 - F_i$, and θ_{ij}'s are some real constants which satisfy certain restrictions so that Q is a df. Here $\hat{x}_i := \sup\{x \in \mathbb{R} : F_i(x) < 1\}$ stands for the upper endpoint of the marginal df F_i.
Throughout the paper we assume that the risks are non-negative, thus F_i has support on $[0, \infty)$.
The tractability of $\boldsymbol{X}$ with df Q given by (1.2) relates to the fact that Q is obtained by the product distribution $Q^* = \prod_{i=0}^{n} F_i$ (in fact $Q^*(x_0, \ldots, x_n) = \prod_{i=0}^{n} F_i(x_i)$). By a closer inspection, it follows that

$$Q(x_0, \ldots, x_n) = \int_0^{x_0} \cdots \int_0^{x_n} \Big[1 + \sum_{0 \le i < j \le n} \theta_{ij}(1 - 2F_i(s_i))(1 - 2F_j(s_j))\Big] Q^*(\mathrm{d}s_0, \ldots, \mathrm{d}s_n)$$

holds for any $x_i \in [0, \hat{x}_i],\ 0 \le i \le n$. The larger class of multivariate Sarmanov distributions is introduced by substituting above $1 - 2F_i$ by some kernel ϕ_i; some insurance applications of Sarmanov distributions are illustrated in Tang et al. (2011) and Yang and Wang (2012).
Motivated by the underlying relationship between Q and the product df $Q^* = \prod_{i=0}^{n} F_i$, in this paper we consider a wider class of multivariate df's which are absolutely continuous with respect to a product df – we refer to that as AC-product class. Specifically, the members of this class are all absolutely continuous df's with respect to Q^*.
It turns out that under some weak conditions the asymptotic behaviour of the aggregated risk S and the deflated risk Z for risks with an AC-product distribution can be derived explicitly.
Organization of the rest of the paper: In the next section we briefly discuss some basic properties of AC-product distributions. Further, we derive a novel result concerning the Sarmanov distribution, which is the canonical example of the AC-product class. Section 3 shows the asymptotic independence of AC-product risks, whereas Section 4 investigates the asymptotic behaviour of the deflated risk Z under extreme value type conditions on the marginal df's. In Section 5 we present three applications concerning risk aggregation, Value-at-Risk and conditional tail expectation, and the probability of ruin under risky investment. The proofs of all the results are postponed to Section 6.

2. Multivariate AC-Product and Sarmanov Distributions

In this section we present some details on the class of AC-product distributions and Sarmanov distributions. Hereafter $\boldsymbol{X} = (X_0, X_1, \ldots, X_n)$ is a $(n+1)$-dimensional random vector with non-negative univariate marginal df's $F_i,\ 0 \le i \le n$. It is not standard to write the first component of $\boldsymbol{X}$ by X_0; we do this since this component will be a reference one in the part when the products of the components of $\boldsymbol{X}$ are discussed. Clearly, if $\boldsymbol{X}$ possesses the df $Q^* = \prod_{i=0}^{n} F_i$, then the random vector $\boldsymbol{X}$ has independent components, a situation which is often not encountered in practical applications. Starting from this independence setup, a tractable dependence structure is introduced by considering $\boldsymbol{X}$ such that its df Q is absolutely continuous with respect to the product df Q^* i.e.,

$$Q(x_0,\dots,x_n)=\int_0^{x_0}\cdots\int_0^{x_n}\eta(s_0,\dots,s_n)\,Q^*(\mathrm{d}s_0,\dots,\mathrm{d}s_n),\quad x_i\in[0,\hat{x}_i],\ 0\le i\le n, \tag{2.1}$$

where $\eta(\cdot)$ is a non-negative measurable function, i.e., if we write (2.1) as

$$\mathrm{d}Q=\eta\cdot\mathrm{d}Q^*,$$

we see that η is the Radon-Nikodym derivative. Throughout this paper

$$X_0^*,\dots,X_n^*$$

are independent rv's with df's $F_i, 0\le i\le n$, respectively, and thus joint df Q^*. We refer to Q as an AC-product distribution. Since Q is a proper df we shall assume that

$$\mathbb{E}\left\{\eta(X_{x_0}^*,\dots,X_{x_n}^*)\right\}<\infty \tag{2.2}$$

almost surely with respect to Q^* where $X_{x_i}^*=X_i^*$ or $X_{x_i}^*=x_i$ with x_i in the support of F_i. Further, we suppose that

$$\mathbb{E}\left\{\eta(X_0^*,\dots,X_n^*)\right\}=1 \tag{2.3}$$

holds. Clearly, (2.2) is satisfied when $\eta(\cdot)$ is a bounded function.
The Sarmanov distributions mentioned in the Introduction are obtained when

$$\eta(x_0,\dots,x_n)=1+\sum_{0\le k<l\le n}\theta_{kl}\phi_k(x_k)\phi_l(x_l), \tag{2.4}$$

with $\phi_0,\dots,\phi_n$ some given real-valued kernels, and $\theta_{kl},\ 0\le k<l\le n$ non-negative constants.
In order for such $\eta(\cdot)$ to define a proper df, we shall impose the following assumptions on the kernels:
A1. $\phi_i,\ 0\le i\le n$ are not identical to 0 in $[0,\hat{x}_i]$;
A2. for all $x_i\in[0,\hat{x}_i],\ 0\le i\le n$ we have

$$\sum_{0\le k<l\le n}\theta_{kl}\phi_k(x_k)\phi_l(x_l)\ge -1 \tag{2.5}$$

almost surely with respect to Q^*;
A3. for any $0\le i\le n$ we have

$$\mathbb{E}\left\{\phi_i(X_i)\right\}=0. \tag{2.6}$$

Apart form the choice $\phi_i=1-2F_i$ which leads to the FGM distribution, another common specification of the kernels is $\phi_i(s)=g_i(s)-\mathbb{E}\left\{g_i(X_i)\right\}, s>0$, for some function g_i such that $\mathbb{E}\left\{g_i(X_i)\right\}<\infty$.
We may consider for instance $g_i(s)=\exp(-s)$, or $g_i(s)=s^{\alpha_i},\ \alpha_i\in\mathbb{R}$, provided that $\mathbb{E}\left\{X_i^{\alpha_i}\right\}<\infty$ and $\hat{x}_i<\infty$. The next lemma shows that the kernels need to obey certain asymptotic restrictions.

Lemma 2.1. *Let Q be a $(n+1)$-dimensional multivariate Sarmanov distribution of $(X_0,\dots,X_n)$ with η defined by the kernel functions $\phi_i,\ 0\le i\le n$ and non-negative weights $\theta_{kl},\ 0\le k<l\le n$, as in (2.4). Suppose that ϕ_i is continuous at both $0,\ \hat{x}_i$ and bounded on finite intervals of $(0,\hat{x}_i)$. If further A1–A3 hold, then*

$$\sup_{x\in[0,\hat{x}_i]}|\phi_i(x)|<M_i<\infty\quad 0\le i\le n \tag{2.7}$$

holds for some positive constants $M_0,\dots,M_n$.

In the light of Lemma 2.1 if the df $\widetilde{F}_i$ is such that

$$1-\widetilde{F}_i(x)=\int_x^{\hat{x}_i}\Big(1-\frac{\phi_i(u)}{M_i}\Big)F_i(\mathrm{d}u),\quad 0\le x<\hat{x}_i,\ 0\le i\le n, \tag{2.8}$$

then $\widetilde{F}_i$ are proper univariate df's. For each $0\le i\le n$, by (2.6), $\widetilde{F}_i$ is a proper df with the same upper endpoint $\hat{x}_i$ as the df F_i.

Lemma 2.1 motivates the following assumptions on the kernel functions

$$\lim_{x\uparrow\hat{x}_i}\phi_i(x)=\kappa_i\in\mathbb{R},\quad 0\le i\le n, \tag{2.9}$$

which implies that for any $M_i>\kappa_i$

$$1-\widetilde{F}_i(x)\sim\Big(1-\frac{\kappa_i}{M_i}\Big)\overline{F_i}(x),\quad x\uparrow\hat{x}_i. \tag{2.10}$$

In this paper $\sim$ means asymptotic equivalence, i.e., the quotient of both sides tend to 1. A consequence of condition (2.9) is that

$$\lim_{x_i\uparrow\hat{x}_i,i=0,\dots,n}\Big(1+\sum_{0\le k<l\le n}\theta_{kl}\phi_k(x_k)\phi_l(x_l)\Big)=1+\sum_{0\le k<l\le n}\theta_{kl}\kappa_k\kappa_l=:\Lambda_-. \tag{2.11}$$

Since $\eta(\cdot)$ is non-negative, then also Λ_- is non-negative; if we do not explicitly specify the Radon-Nikodyn derivative of Q the natural extension of the above is to require that

$$\lim_{x_i\uparrow\hat{x}_i,i=0,\dots,n}\eta(x_0,\dots,x_n)=\Lambda_-\in[0,\infty). \tag{2.12}$$

3. Asymptotic Independence

In various insurance applications, see e.g., Asimit et al. (2011) it is crucial to find concrete multivariate distributions which possess certain asymptotic dependence properties. As mentioned above, the FGM distribution is included in the class of Sarmanov distributions; Hashorva and Hüsler (1999) shows that these distributions have asymptotically independent marginals (see below for the definition), and therefore the maxima of multivariate random samples with underlying FGM distribution have asymptotically independent components, provided that each marginal distribution is in the max-domain of attraction (MDA) of some univariate df. In order to give more precise statements, we briefly mention that a univariate df F on $\mathbb{R}$ belongs to the MDA of a univariate extreme value df N, denoted by $F\in\mathrm{MDA}(N)$, if

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}|F^n(c_nx+d_n)-N(x)|=0 \tag{3.1}$$

holds for constants $c_n>0$ and $d_n\in\mathbb{R},\ n\ge1$.

Only three candidates for the df N are possible, namely the Fréchet distribution Φ_γ, the Gumbel distribution Λ and the Weibull distribution Ψ_γ, where $\gamma>0$ indexes members of the Fréchet and Weibull families. We mention some basic properties of univariate extreme value distributions and their MDA below; see Resnick (1987) or Embrechts et al. (1997) for more details.

When Q is the df of the bivariate random vector (X_1,X_2) with marginal df's F_1 and F_2 such that $F_i\in\mathrm{MDA}(N_i),\ i=1,2$, then we say that Q has asymptotically independent marginal distributions if for any positive x,y

$$\lim_{n\to\infty}n\mathbb{P}\left(X_1>c_{n1}x+d_{n1},X_2>c_{n2}y+d_{n2}\right)=0,$$

where $c_{ni}>0,\ d_{ni},\ n\ge1,\ i=1,2$ are constants such that $F_i\in\mathrm{MDA}(N_i)$ holds as given by (3.1). In order to simplify the presentation we abbreviate the above as $Q\in\mathrm{MDA}(N_1;N_2)$.

Next, we consider Q a bivariate distribution functions such that it has marginal distributions F_1,F_2 and a positive Radon-Nikodyn density $\eta(\cdot)$ with respect to $F_1\cdot F_2$. The general case of a d-dimensional distribution follows easily since pair-wise asymptotic independence implies asymptotic independence of the multivariate distributions.

Proposition 3.1. *Let Q be a bivariate df as above with marginal df's F_1 and F_2 such that $F_i \in \mathrm{MDA}(N_i),\ i = 1,2$. If further*

$$\limsup_{s\uparrow \hat{x}_1, t\uparrow \hat{x}_2} \eta(s,t) \quad < \quad \infty, \tag{3.2}$$

then $Q \in \mathrm{MDA}(N_1; N_2)$.

By combining the above result with Lemma 2.1 we obtain:

Corollary 3.2. *If Q is a multivariate Sarmanov distribution such that the kernels satisfy the assumptions of Lemma 2.1, then Q has asymptotically independent marginal distributions and it belongs to the max-domain of attraction of a product max-stable distribution provided that $F_i \in MDA(N_i),\ 0 \le i \le n$ with N_i some univariate extreme value distribution.*

4. Extreme Value Risk Models

In various insurance and finance applications the investigation of the tail asymptotics of products is a crucial task, see e.g., Berman (1992), Cline and Samorodnitsky (1994), Jessen and Mikosch (2006), Tang (2006a,b,2008), Hashorva and Pakes (2010), Hashorva et al. (2010,2011,2012), Liu and Tang (2010), Arendarczyk and Dębicki (2011,2012), Constantinescu et al. (2011), Yang et al. (2011), Hashorva (2011,2012), Yang and Wang (2012).
Specifically, if $X_0, \ldots, X_n$ are non-negative rv's modeling some risks, then it is of interest to investigate the tail asymptotics of the deflated risk $Z = \prod_{i=0}^n X_i$ with df H. When $X_i,\ 0 \le i \le n$ are mutually independent, using extreme value theory, it is possible to obtain some explicit results. A classical case is when X_0 has a regularly varying survival function and the other rv's satisfy certain moment conditions which allow to use Breiman's lemma (see Breiman (1965) and the recent results of Yang and Wang (2012)). Recall that a univariate df F has a regularly varying survival function $\overline{F}$ with index $-\gamma \le 0$, if for any $y \in (0, \infty)$

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\gamma}. \tag{4.1}$$

It is well-known that when $\gamma > 0$, then (4.1) is equivalent with $F \in \mathrm{MDA}(\Phi_\gamma)$, see e.g., Mikosch (2009). Note that $\Phi_\gamma(x) = \exp(-x^{-\gamma}),\ x > 0$, and necessarily, for the Fréchet case we have that the upper endpoint of F is infinite. For a univariate df F with a finite upper endpoint $\hat{x} \in (0, \infty)$ we are interested on its asymptotic behaviour at $\hat{x}$. Instead of (4.1) we shall assume for this case that $\overline{F}$ is regularly varying at $\hat{x}$ with index $\gamma \ge 0$ i.e.,

$$\lim_{x\to\infty} \frac{\overline{F}(\hat{x} - y/x)}{\overline{F}(\hat{x} - 1/x)} = y^{\gamma} \tag{4.2}$$

for any $y > 0$. When $\gamma > 0$ the above condition is equivalent with F is in the MDA of the Weibull df Ψ_γ (we recall $\Psi_\gamma(x) = \exp(-|x|^\gamma), x < 0$).
If otherwise specified, in the sequel we assume that $\boldsymbol{X} = (X_0, X_1, \ldots, X_n)$ has the df $dQ = \eta \cdot dQ^*$, where $Q^* = \prod_{i=0}^n F_i$ is a product df with non-degenerate univariate df's $F_0, \ldots, F_n$. We state next the first result of this section, the case of Gumbel MDA is treated in Theorem 4.2 below. Recall that $(X_0^*, \ldots, X_n^*)$ has df Q^*.

Theorem 4.1. *a) Suppose that for each fixed $x_1, \ldots, x_n \ge 0$ and $\overline{\eta}(\cdot)$ some bounded function*

$$\lim_{x_0\to\infty} \eta(x_0, x_1, \ldots, x_n) = \overline{\eta}(x_1, \ldots, x_n). \tag{4.3}$$

If further $\overline{F_0}$ satisfies (4.1) with some $\gamma \ge 0$ and $\mathbb{E}\left\{X_i^{\gamma+\epsilon}\right\} < \infty$ holds for all $1 \le i \le n$ and some $\epsilon > 0$, then the survival function $\overline{H}$ of Z satisfies (4.1) and moreover as $x \to \infty$

$$\overline{H}(x) \sim \overline{F_0}(x)\mathbb{E}\left\{\overline{\eta}(X_1^*, \ldots, X_n^*) \prod_{i=1}^{n} (X_i^*)^{\gamma}\right\}. \tag{4.4}$$

b) Suppose that $F_i,\ 0 \le i \le n$ satisfy (4.2) with $\gamma_i \ge 0,\ 0 \le i \le n$. If (2.12) holds with $\Lambda_- > 0$, then as $x \uparrow \prod_{i=0}^{n} \hat{x}_i$

$$\overline{H}(x) \sim \frac{\Lambda_-}{\Gamma(\sum_{i=0}^{n}\gamma_i + 1)} \prod_{i=0}^{n}\Big(\Gamma(\gamma_i+1)\overline{F_i}(\hat{x}_i\overline{x})\Big), \tag{4.5}$$

where $\overline{x} = x\prod_{i=0}^{n}\hat{x}_i^{-1}$, and $\Gamma(\cdot)$ is the Euler gamma function.

Applied to the case of Sarmanov distributions, Theorem 4.1 a) implies the following result:

Corollary 4.1. *Under the assumptions of statement a) of Theorem 4.1, if further $\boldsymbol{X} = (X_0, X_1, \ldots, X_n)$ follows a multivariate Sarmanov distribution of the form (2.1) such that the assumptions of Lemma 2.1 hold, then as $x \to \infty$*

$$\overline{H}(x) \sim \overline{F_0}(x) \prod_{i=1}^{n} \mathbb{E}\left\{X_i^{\gamma}\right\} \left(1 + \kappa_0 \sum_{1\le l\le n} \theta_{0l} \frac{\mathbb{E}\left\{X_l^{\gamma}\phi_l(X_l)\right\}}{\mathbb{E}\left\{X_l^{\gamma}\right\}} + \sum_{1\le k<l\le n} \theta_{kl} \frac{\mathbb{E}\left\{X_k^{\gamma}\phi_k(X_k)\right\}\mathbb{E}\left\{X_l^{\gamma}\phi_l(X_l)\right\}}{\mathbb{E}\left\{X_k^{\gamma}\right\}\mathbb{E}\left\{X_l^{\gamma}\right\}}\right), \tag{4.6}$$

provided that $\lim_{x\to\infty}\phi_0(x) = \kappa_0 \in \mathbb{R}$.

Remarks: i) For the case of bivariate FGM distributions Jiang and Tang (2011) obtained the tail asymptotic of Z, see also Yang et al. (2011).
ii) Yang and Wang (2012) derived (4.6) under weaker conditions for the bivariate setting of Sarmanov distributions.

In various applications due for instance to different currencies, original risks are linearly transformed. In order to widen the applications to those cases, suppose therefore that $X_i,\ 0 \le i \le n$ has df with lower endpoint equal 0 (i.e., $\inf\{x \in \mathbb{R} : F_i(x) > 0\} = 0$), and let the random vector $\boldsymbol{Y} = (Y_0, \ldots, Y_n)$ be such that

$$Y_i = (a_i + b_i X_i)^{-1}, \quad a_i > 0,\ b_i > 0,\ 0 \le i \le n. \tag{4.7}$$

The df $Q_{\boldsymbol{Y}}$ of $\boldsymbol{Y}$ is related to $Q_{\boldsymbol{Y}}^* := \prod_{i=0}^{n} F_{Y_i}$ by $\mathrm{d}Q_{\boldsymbol{Y}} = \eta_{\boldsymbol{Y}} \cdot \mathrm{d}Q_{\boldsymbol{Y}}^*$, where

$$\eta_{\boldsymbol{Y}}(t_0, \ldots, t_n) = \eta(b_0^{-1}(t_0^{-1} - a_0), \ldots, b_n^{-1}(t_n^{-1} - a_n)).$$

The following result is a consequence of statement b) of Theorem 4.1.

Corollary 4.2. *If $F_i,\ 0 \le i \le n$ is regularly varying at zero with index $\gamma_i \ge 0$, and further*

$$\lim_{x_i\downarrow 0, i=0,\ldots,n} \eta(x_0, \ldots, x_n) = \eta(0+, \ldots, 0+) =: \Lambda_+ \in (0, \infty), \tag{4.8}$$

then as $x \uparrow \prod_{i=0}^{n} a_i^{-1}$

$$\mathbb{P}\Big(\prod_{i=0}^{n} Y_i > x\Big) \sim \frac{\Lambda_+}{\Gamma(\sum_{i=0}^{n}\gamma_i + 1)} \prod_{i=0}^{n}\Big(\Gamma(\gamma_i+1)F_i(b_i^{-1}(y_i^{-1} - a_i))\Big), \quad y_i := x\prod_{j=0, j\ne i}^{n} a_i. \tag{4.9}$$

Next, our main assumption on F_0 is that it belongs to Gumbel MDA; we recall that when in (3.1) $N(x) = \Lambda(x) = \exp(-e^{-x}),\ x \in \mathbb{R}$, then an equivalent condition for (3.1) to hold is

$$\lim_{x \uparrow \hat{x}} \frac{\overline{F}(x + ya(x))}{\overline{F}(x)} = e^{-y}, \quad \forall y \geq 0, \tag{4.10}$$

with some positive scaling function $a(\cdot)$. When (4.10) holds we shall use the abbreviation $F \in \mathrm{MDA}(\Lambda, a(\cdot))$. The scaling function $a(\cdot)$ satisfies

$$\lim_{x \uparrow \hat{x}} \frac{x}{a(x)} = \infty, \text{ and } \lim_{x \uparrow \hat{x}} \frac{\hat{x} - x}{a(x)} = \infty \quad \text{if} \quad \hat{x} < \infty. \tag{4.11}$$

Theorem 4.2. *If $F_0 \in \mathrm{MDA}(\Lambda, a(\cdot))$ and further $F_i,\ 1 \leq i \leq n$ satisfy (4.2) with non-negative constants $\gamma_i,\ 1 \leq i \leq n$, then as $x \uparrow \prod_{i=0}^n \hat{x}_i$*

$$\overline{H}(x) \sim \Lambda_- \overline{F_0}(\widetilde{x}) \prod_{i=1}^n \left(\Gamma(\gamma_i + 1) \overline{F_i}\Big(\hat{x}_i - \frac{\hat{x}_i a(\widetilde{x})}{\widetilde{x}} \Big) \right), \quad \widetilde{x} := x \prod_{i=1}^n \hat{x}_i^{-1}. \tag{4.12}$$

Remarks: i) The Gumbel MDA assumption on F_0 and the assumptions on $F_i,\ 1 \leq i \leq n$ imply that $F_0(\hat{x}_0-) = \cdots = F_n(\hat{x}_n-) = 0$. Consider for simplicity $n = 1$ and $\mathbb{P}(X_1 = \hat{x}_1) = p_1 > 0$. Then under the assumptions of Theorem 4.2 for F_0, we obtain as $x \uparrow \hat{x}_0 \hat{x}_1$

$$\overline{H}(x) \sim \Lambda_- p_1 \overline{F_0}(\hat{x}_1^{-1} x). \tag{4.13}$$

ii) Since uniformly with respect to z in every compact set of $\mathbb{R}$

$$\lim_{u \uparrow \hat{x}} \frac{a(u + za(u))}{a(u)} = 1 \tag{4.14}$$

it follows that H is in the Gumbel MDA with the same scaling function $a(\cdot)$ as F_0.

5. Applications

5.1. **Asymptotics of CTE and VaR.** For this application we regard X_0 as a base rv which models an insurance risk, and $X_1, \ldots, X_n$ as random deflators of the base risk. In various applications, say for instance in risk management, the deflated risk $Z = X_0 \prod_{i=1}^n X_i$ needs to be investigated. We write as above $F_i,\ 1 \leq i \leq n$ for the df of X_i and assume that F_0 is continuous. Due to regulatory restrictions, it is of actuarial interest to quantify the asymptotic behaviour of $\mathrm{VaR}_Z(p)$ (Value-at-Risk) and $\mathrm{CTE}_Z(p)$ (conditional tail expectation) as $p \to 1$. We recall that

$$\mathrm{CTE}_Z(p) := \mathbb{E}\{Z | Z > VaR_Z(p)\} = \mathbb{E}\{Z - VaR_Z(p) | Z > VaR_Z(p)\} + VaR_Z(p), \qquad p \in (0,1),$$

where

$$\mathrm{VaR}_Z(p) := \inf\{x : \mathbb{P}(Z \leq x) \geq p\}, \quad p \in (0,1).$$

See Denuit et al. (2006) for the basic properties of VaR and CTE. When the base risk X_0 is dependent to $X_i,\ 1 \leq i \leq n$ such that the assumptions of Theorem 4.2 hold, then by (4.14) Z has its df in the Gumbel MDA with the same scaling function as X_0. Since the scaling function $a(\cdot)$ is asymptotically equivalent to the mean excess function, we may write (see also Asimit and Badescu (2010))

$$\lim_{p \uparrow 1} \frac{\mathbb{E}\{Z - VaR_Z(p) | Z > VaR_Z(p)\}}{\mathbb{E}\{X_0 - VaR_{X_0}(p) | X_0 > VaR_{X_0}(p)\}} = 1, \tag{5.1}$$

and consequently,

$$\lim_{p \uparrow 1} \frac{\mathrm{CTE}_{X_0}(p)}{VaR_{X_0}(p)} = \lim_{z \uparrow \prod_{i=0}^n \hat{x}_i} \frac{\mathrm{CTE}_Z(z)}{z} = 1. \tag{5.2}$$

It is well-known that for continuous risks CTE is more conservative than Value at Risk (VaR), which is the quantile function of the random variable of interest, if both are evaluated at the same confidence level p (see also the CTE definition from above). In the next result we show that Value-at-Risk for p close to 1 is asymptotically the same for Z and X_0.

Theorem 5.1. *Under the assumptions of Theorem 4.2 we have*

$$\mathrm{CTE}_Z(p) \sim \mathrm{VaR}_Z(p) \sim \mathrm{VaR}_{X_0}(p) \sim \mathrm{CTE}_{X_0}(p) \tag{5.3}$$

as $p \uparrow 1$.

The above theorem shows that both risk measures VaR and CTE for the base risk X_0 and the product risk Z are asymptotically equivalent, provided that the underlying dependence structure of the risks is determined by an AC-product df; see Hashorva et al. (2010) for the case of independent risks.

5.2. **Risk Aggregation.** A key topic in insurance and finance with diverse applications in risk management is the risk aggregation, see e.g., Goovaerts et al. (2005), Geluk and Tang (2009), Asimit et al. (2011), Hashorva (2013a), Kortschak and Hashorva (2013) and the references therein. Asymptotic considerations for the investigation of the total (or aggregated) risk turn out to be quite important when risks are dependent. As already shown in Section 2 the dependence structure of the risks therein exhibit asymptotic independence under a mild condition on the density function $\eta(\cdot)$. For such risks, the Mitra-Resnick methodology developed in Mitra and Resnick (2009) is powerful for deriving exact asymptotic results for the tail of aggregated risk. If $X_0, \ldots, X_n$ are independent, then under various conditions the aggregated risk $S = \sum_{i=0}^n X_i$ has a tractable tail asymptotic behaviour. When X_0 has the df in the Gumbel MDA, under the Mitra-Resnick framework the tail asymptotic behaviour of S is determined by the tail asymptotics of one component, say X_0. In our application below we are able to describe the effect of a single component on the aggregated risk for the risk structures dealt with here. Specifically, let $(X_0, \ldots, X_n)$ has the joint df Q given by

$$\mathrm{d}Q = \eta \cdot \mathrm{d}Q^*, \quad Q^* = \prod_{i=0}^n F_i,$$

with F_i marginal distributions with support on $[0, \infty)$. If Q_{ij} is the df of (X_i, X_j), then by (2.2) we have $\mathrm{d}Q_{ij} = \eta_{ij} \cdot \mathrm{d}(F_i F_j),\ i \neq j$. In view of Proposition 3.1 when

$$\limsup_{s \uparrow \hat{x}_i, t \uparrow \hat{x}_j} \eta_{ij}(s,t) < \infty, \tag{5.4}$$

then Q_{ij} has asymptotically independent marginals, provided that F_i and F_j are in some MDA of a univariate extreme value df. In the next theorem we shall assume that only F_0 is in the Gumbel MDA with some scaling function $a(\cdot)$ and upper endpoint $\hat{x}_0 = \infty$; all the df's $F_i,\ 0 \le i \le n$ satisfy

$$\lim_{x \to \infty} \frac{\overline{F_i}(x)}{\mathbb{P}\left(X_0 > x\right)} = c_i \in [0, \infty), \quad 0 \le i \le n. \tag{5.5}$$

Theorem 5.2. *Let* $(X_0, \ldots, X_n)$ *be as above, and assume that* $F_0 \in \mathrm{MDA}(\Lambda, a(\cdot))$ *and both* (5.4), (5.5) *are satisfied. If further* $\lim_{x \to \infty} a(x) = \infty$ *and for positive constants* $L_{ij}, 1 \le i < j \le n$

$$\lim_{x \to \infty} \max_{0 \le i < j \le n} \frac{\overline{F_i}(a(x)L_{ij})\overline{F_j}(a(x)L_{ij})}{\overline{F_0}(x)} = 0, \tag{5.6}$$

then as $x \to \infty$

$$\mathbb{P}\left(S > x\right) \sim \mathbb{P}\left(X_0 > x\right) \sum_{i=0}^n c_i \tag{5.7}$$

and S has df in the Gumbel MDA with the same scaling function $a(\cdot)$ as F_0.

Clearly, when all marginal distributions are tail equivalent to F_0, then condition (5.6) is the Mitra-Resnick condition for the aggregation of independent risks, see (2.15) in Mitra and Resnick (2009).

5.3. **Ruin in the Presence of Risky Investments.** Our next application concerns the discrete-time insurance risk model discussed in Hashorva et al. (2010) and Tang et al. (2011). Let in the following $R_1, \ldots, R_n$ be independent real-valued rv's with common df F being further independent of $\Delta_1, \ldots, \Delta_n,$ whose support is $(-1, \infty)$. Several authors have considered the asymptotic behaviour as $u \to \infty$ of the following ruin probability

$$\psi(u; n) = \mathbb{P}\Big(\min_{0 \le i \le n} U_i < 0 \;\Big|\; U_0 = u\Big), \quad U_0 = u \ge 0,$$

where

$$\Upsilon_i(U_i + R_i) = U_{i-1}, \qquad 1 \le i \le n,$$

with $\delta_i > 0, p_i \in [0,1),\ 1 \le i \le n$ and

$$\Upsilon_i := (c_i + p_i(1 + \Delta_i))^{-1}, \quad c_i := (1 - p_i)(1 + \delta_i).$$

In the light of Theorem 5.1 in Tang et al. (2011) we have when the df F belongs to the well-known subexponential class

$$\psi(u; n) \sim \sum_{k=1}^{n} \mathbb{P}\Big(R \prod_{i=1}^{k} \Upsilon_i > u\Big), \qquad u \to \infty, \tag{5.8}$$

where $\Upsilon_1, \ldots, \Upsilon_n$ can be arbitrarily dependent. We note in passing that a df F on $[0, \infty)$ is said to be subexponential, written as $F \in \mathcal{S}$, if $\overline{F^{2*}}(x) \sim 2\overline{F}(x)$ as $x \to \infty$, where F^{2*} denotes the two-fold convolution of F; more generally, F on $\mathbb{R}$ is still said to be subexponential if the df $F_+(x) = F(x)\mathbf{1}_{\{x \ge 0\}}$ is subexponential, see e.g., Embrechts et al. (1997).

Hashorva et al. (2010) discussed the case $\Upsilon_i,\ 1 \le i \le n$ are independent, whereas the recent paper Tang et al. (2011) obtained some refinements of (5.8), which allow for dependence assuming a multivariate Sarmanov distribution for $(1 + \Delta_1, \ldots, 1 + \Delta_n)$, see Theorem 4.1 and 4.2 in Tang et al. (2011).

Two recent papers Chen (2011) and Yang and Wang (2012) investigated a similar case where $(R_i, \Upsilon_i),\ 1 \le i \le n$ are independent and for each $1 \le i \le n$, (R_i, Υ_i) follows a bivariate FGM or Sarmanov distribution, respectively, see Theorem 3.1 and Corollary 3.1 in Chen (2011) and Theorem 4.1 in Yang and Wang (2012). As we show below, it is possible to obtain similar results as Theorem 4.1 and 4.2 in Tang et al. (2011) by considering a more general dependence structure.

Next, we assume that $R_1, \ldots, R_n$ are independent real-valued rv's, and $(\Upsilon_1, \ldots, \Upsilon_n)$, independent of $R_1, \ldots, R_n$, has an AC-product distribution. Assume therefore that $\boldsymbol{\Upsilon} = (\Upsilon_1, \ldots, \Upsilon_n)$ has the joint df

$$\mathrm{d}G_{\boldsymbol{\Upsilon}} = \eta_{\boldsymbol{\Upsilon}} \cdot \mathrm{d}G^*, \quad G^* = \prod_{i=1}^{n} G_i,$$

where G_i is the df of $\Upsilon_i,\ 1 \le i \le n$. Thus, for each $1 \le k < n$, $\boldsymbol{\Upsilon}_{\mathbf{k}} = (\Upsilon_1, \ldots, \Upsilon_k)$ has the joint df given by $\eta_{\boldsymbol{\Upsilon}_{\mathbf{k}}} \cdot \prod_{i=1}^{k} G_i$ with

$$\eta_{\boldsymbol{\Upsilon}_{\mathbf{k}}}(x_1, \ldots, x_k) = \int_0^{c_{k+1}^{-1}} \cdots \int_0^{c_n^{-1}} \eta_{\boldsymbol{\Upsilon}}(x_1, \ldots, x_n) \prod_{i=k+1}^{n} G_i(\mathrm{d}x_i). \tag{5.9}$$

Theorem 5.3. *Consider the discrete-time risk model introduced above with* $p_i \in [0,1),\ 1 \le i \le n$. *Assume that* $F \in \mathrm{MDA}(\Lambda,\ a(\cdot)) \cap \mathcal{S}$ *and* $G_i,\ 1 \le i \le n$ *satisfy* (4.2) *at* c_i^{-1} *with non-negative constants* $\gamma_i,\ 1 \le i \le n$. *If for each* $1 \le k \le n$ *the limit* $\Lambda_{\mathbf{\Upsilon_k}-} = \lim_{x_i \uparrow c_i^{-1}, i=1,\ldots,k} \eta_{\mathbf{\Upsilon_k}}(x_1,\ldots,x_k)$ *exists and is finite, then as* $u \to \infty$

$$\psi(u;n) \sim \overline{F}(\widetilde{u}) \sum_{k=1}^{n} \Lambda_{\mathbf{\Upsilon_k}-} \prod_{i=1}^{k} \Big(\Gamma(\gamma_i+1) \overline{G_i}\Big(c_i^{-1} - \frac{c_i^{-1} a(\widetilde{u})}{\widetilde{u}}\Big)\Big), \tag{5.10}$$

where $\widetilde{u} = u \prod_{i=1}^n c_i$.

Note in passing that if $F \in \mathrm{MDA}(\Lambda, a(\cdot))$ with $a(\cdot)$ such that $\lim_{u\to\infty} a(u) = \infty$, then in order to show that $F \in \mathcal{S}$ we can utilise the criteria given in Hashorva et al. (2010).

In the literature, there are several results concerned with the tail asymptotic behaviour of randomly weighted sums with unbounded weights, that is, $p_i,\ 1 \le i \le n$ can be 1, see e.g., Resnick and Willekens (1991). Next, we consider only the case that $\overline{F}$ is regularly varying at infinity.

Theorem 5.4. *Consider the discrete-time risk model introduced above with* $p_i \in [0,1],\ 1 \le i \le n$. *Assume that* F *satisfies* (4.1) *with some* $\gamma \ge 0$ *and for each* $1 \le i \le n$ *and* $\epsilon > 0$ *we have* $\mathbb{E}\left\{\Upsilon_i^{\gamma+\epsilon}\right\} < \infty$. *If the function* $\eta_{\mathbf{\Upsilon}}$ *is bounded, then as* $u \to \infty$

$$\psi(u;n) \sim \mathcal{K}\overline{F}(u), \tag{5.11}$$

where $\mathcal{K} := \sum_{k=1}^n \mathbb{E}\Big\{\eta_{\mathbf{\Upsilon_k}}(\Upsilon_1^*,\ldots,\Upsilon_k^*) \prod_{i=1}^k (\Upsilon_i^*)^\gamma\Big\} \in (0,\infty)$ *and* $\Upsilon_i^*,\ 1 \le i \le n$ *are independent such that* Υ_i^* *and* Υ_i *have the same df for each* $1 \le i \le n$.

6. Proofs

Throughout this section, for two positive functions $u(x)$ and $v(x)$, as x tends to z, we write $u(x) \sim v(x)$ if $\lim_{x\to z} u(x)/v(x) = 1$; write $u(x) = o(v(x))$ if $\lim_{x\to z} u(x)/v(x) = 0$. Further, write $u(x) = O(v(x))$ if $\limsup_{x\to z} u(x)/v(x) < \infty$.

Proof of Lemma 2.1.: We show next that $\limsup_{x\uparrow \hat{x}_1} \phi_1(x) = \infty$ is not possible. Let us suppose for a while that this is possible, say for any given M large, in some left-neighbourhood of $\hat{x}_1$ we have $\phi_1(x) > M$. For any $x \in [0, \hat{x}_1)$ and $y \in [0, \hat{x}_2)$ we obtain

$$\mathbb{P}\left(X_1 > x, X_2 > y\right) = \overline{F_1}(x)\overline{F_2}(y) + \int_x^{\hat{x}_1} \phi_1(s) F_1(\mathrm{d}s) \int_y^{\hat{x}_2} \phi_2(s) F_2(\mathrm{d}s).$$

Since $1 + \phi_1\phi_2$ is the Radon-Nikodyn density of (X_1, X_2) with respect to $F_1 \cdot F_2$ we have

$$1 + \phi_1(x)\phi_2(y) \ge 0, \qquad x \in [0,\hat{x}_1), y \in [0,\hat{x}_2).$$

The above relation and the fact that M can be arbitrarily large imply that ϕ_2 is non-negative (almost surely with respect to the measure generated by F_1F_2). Hence

$$\overline{F_1}(x)\overline{F_2}(y) + \int_x^{\hat{x}_1} \phi_1(s) F_1(\mathrm{d}s) \int_y^{\hat{x}_2} \phi_2(s) F_2(\mathrm{d}s) > \overline{F_1}(x)$$

holds for some x close to $\hat{x}_1$ and y such that $\int_y^{\hat{x}_2} \phi_2(s) F_2(\mathrm{d}s) > 0$. This is a contradiction, since necessarily

$$\mathbb{P}\left(X_1 > x, X_2 > y\right) \le \min(\mathbb{P}\left(X_1 > x\right), \mathbb{P}\left(X_2 > y\right)) = \min(\overline{F_1}(x), \overline{F_2}(y)), \quad x \in [0,\hat{x}_1), y \in [0,\hat{x}_2).$$

If we assume that $\liminf_{x\uparrow\hat{x}_1} \phi_1(x) = -\infty$, then ϕ_2 cannot be positive and the same argument as above can be repeated to show that this is not possible.

Next suppose that $\limsup_{x\downarrow 0}\phi_1(x)=\infty$ is possible. We have

$$\mathbb{P}\left(X_1\le x, X_2\le y\right)=F_1(x)F_1(y)+\int_0^x \phi_1(s)F_1(\mathrm{d}s)\int_0^y \phi_2(s)F_2(\mathrm{d}s),\quad x\in[0,\hat{x}_1), y\in[0,\hat{x}_2).$$

Hence since again it follows that $\phi_2(y)>0$ for some $y\in[0,\hat{x}_2)$, we obtain $F(x,y)>F_1(x)$ which is a contradiction. Hence the proof follows showing with similar arguments that $\liminf_{x\downarrow 0}\phi_1(x)=-\infty$ is also not possible. □

Proof of Proposition 3.1.: By the assumptions we have $\lim_{n\to\infty} c_{ni}x+d_{ni}=\hat{x}_i,\ i=1,2$ for any $x<\hat{x}_i$, hence for some M positive and all n large we have (set $u_{ni}(x)=c_{ni}x+d_{ni}, i=1,2$)

$$\begin{aligned} n\mathbb{P}\left(X_1>u_{n1}(x), X_2>u_{n2}(y)\right) &= n\overline{F_1}(u_{n1}(x))\overline{F_2}(u_{n2}(y))+n\int_{u_{n1}(x)}^{\hat{x}_1}\int_{u_{n2}(y)}^{\hat{x}_2}\eta(s,t)F_1(\mathrm{d}s)F_2(\mathrm{d}t)\\ &\le O(\overline{F_1}(u_{n2}(y)))+nM\int_{u_{n1}(x)}^{\hat{x}_1}\int_{u_{n2}(y)}^{\hat{x}_2}F_1(\mathrm{d}s)F_2(\mathrm{d}t)\\ &\to 0,\quad n\to\infty, \end{aligned}$$

since $F_1\cdot F_2\in \mathrm{MDA}_2(N_1;N_2)$, thus the proof follows. □

Proof of Theorem 4.1.: a) Since F_0 satisfies (4.1), Potter's theorem (see, e.g., Theorem 1.5.6 of Bingham et al. (1987)) implies that for any given $\epsilon>0$ there exist two positive constants C and D such that for all $y>0$ and $x\ge D$

$$\frac{\overline{F_0}(x/y)}{\overline{F_0}(x)}\le Cy^{\gamma\pm\epsilon}. \tag{6.1}$$

For such a constant D, we split the survival function $\overline{H}$ of the deflated risk Z into two parts, namely

$$\begin{aligned} \overline{H}(x) &= \int_{\{x_i>0,i=1\dots n\}}\Big(\int_{x\prod_{i=1}^n x_i^{-1}}^{\infty}\eta(x_0,x_1,\dots,x_n)F_0(\mathrm{d}x_0)\Big)F_1(\mathrm{d}x_1)\cdots F_n(\mathrm{d}x_n)\\ &= \Big(\int_{\{\prod_{i=1}^n x_i\le x/D\}}+\int_{\{\prod_{i=1}^n x_i>x/D\}}\Big)\Big(\int_{x\prod_{i=1}^n x_i^{-1}}^{\infty}\eta(x_0,x_1,\dots,x_n)F_0(\mathrm{d}x_0)\Big)F_1(\mathrm{d}x_1)\cdots F_n(\mathrm{d}x_n)\\ &=: J_1(x)+J_2(x). \end{aligned}\tag{6.2}$$

Since the function $\eta(\cdot)$ is bounded, then by Markov's inequality and $\mathbb{E}\{X_i^{\gamma+\epsilon}\}<\infty, 1\le i\le n$, we have $x\to\infty$

$$J_2=O(1)\mathbb{P}\Big(\prod_{i=1}^n X_i^*>\frac{x}{D}\Big)=O(x^{-(\gamma+\epsilon)})=o(\overline{F_0}(x)),\tag{6.3}$$

where the last equality holds by (6.1). According to the dominated convergence theorem and by (6.1), under the conditions of the theorem we obtain that

$$\begin{aligned} &\lim_{x\to\infty}\frac{J_1(x)}{\overline{F_0}(x)}\\ &=\int_{\{x_i>0,i=1\dots n\}}\Big(\lim_{x\to\infty}\frac{1}{\overline{F_0}(x)}\int_{x\prod_{i=1}^n x_i^{-1}}^{\infty}\eta(x_0,x_1,\dots,x_n)F_0(\mathrm{d}x_0)\mathbf{1}_{\{\prod_{i=1}^n x_i\le x/D\}}\Big)F_1(\mathrm{d}x_1)\cdots F_n(\mathrm{d}x_n)\\ &=\int_{\{x_i>0,i=1\dots n\}}\overline{\eta}(x_1,\dots,x_n)(x_1\cdots x_n)^{\gamma}F_1(\mathrm{d}x_1)\cdots F_n(\mathrm{d}x_n)\\ &=\mathbb{E}\Big\{\overline{\eta}(X_1^*,\dots,X_n^*)\prod_{i=1}^n(X_i^*)^{\gamma}\Big\}. \end{aligned}\tag{6.4}$$

Consequently, the claim follows from (6.2)–(6.4).
b) By the assumptions, since $\hat{x}_i,\ 0 \le i \le n$ is finite, it follows easily that

$$\overline{H}(x) = \mathbb{P}\Big(\prod_{i=0}^{n} X_i > x\Big) \sim \Lambda_{-}\mathbb{P}\Big(\prod_{i=0}^{n} X_i^* > x\Big)$$

as $x \uparrow \prod_{i=0}^{n} \hat{x}_i$, hence the proof is established by a direct application of Lemma 2.1 and Theorem 3.1 in Hashorva et al. (2010). □

Proof of Corollary 4.1.: The proof follows by checking the assumptions of Theorem 4.1 which can then be applied to establish the claim. □

Proof of Theorem 4.2.: We show next the claim for the case $\hat{x}_0 = \infty$ and omit the proof when $\hat{x}_0 < \infty$ since it follows with similar arguments. For notational simplicity we assume that $\hat{x}_i = 1,\ 1 \le i \le n$. In view of the Davis-Resnick property of $\overline{H}_0$ (see Proposition 1.1 of Davis and Resnick (1988), or details in Hashorva (2013b)) we have that

$$\lim_{x\to\infty} \Big(\frac{x}{a(x)}\Big)^{\mu} \frac{\overline{F_0}(\tau x)}{\overline{F_0}(x)} \quad = \quad 0 \tag{6.5}$$

holds for any $\mu \ge 0$ and $\tau > 1$. Hence for any $c > 1$ we have

$$\overline{H}(x) \sim \mathbb{P}\Big(\prod_{i=0}^{n} X_i > x, x \le X_0 \le cx, X_i > 1 - 1/c, i = 1, \ldots, n\Big), \quad x \to \infty.$$

Consequently, by the arbitrariness of c we obtain

$$\overline{H}(x) \sim \Lambda_{-}\mathbb{P}\Big(\prod_{i=0}^{n} X_i^* > x\Big), \quad x \to \infty$$

and hence the proof follows applying again Theorem 3.1 in Hashorva et al. (2010). □

Proof of Theorem 5.1.: In view of (5.2) we need to show that $\mathrm{VaR}_Z(p) \sim \mathrm{VaR}_{X_0}(p)$ as $p \uparrow 1$. Assume for simplicity that $n = 1$ and $\hat{x}_1 = 1$, so we have that $\mathrm{VaR}_Z(p) \le \mathrm{VaR}_{X_0}(p)$ as $p \uparrow 1$. If $G = F_0 F_1$ and G^{-1}, F_0^{-1} are the generalised inverses of G and F_0, respectively, then this can be rewritten as $G^{-1}(1-1/t) \le F_0^{-1}(1-1/t), t > 1$. In view of (6.5) both F_0 and G are in the Gumbel MDA with the same auxiliary function $a(\cdot)$. Consequently (see e.g., Resnick (1987)) $G^{-1}(1 - 1/t)$ and $F_0^{-1}(1 - 1/t)$ are slowly varying functions at infinity. Since further in view of (6.5) for all x large and any $\varepsilon \in (0, 1)$ we have

$$\overline{G}(x) \ge \overline{F}(x(1 - \varepsilon)),$$

then

$$\lim_{t\to\infty} \frac{G^{-1}(1 - 1/t)}{F_0^{-1}(1 - 1/t)} = 1$$

and hence the claim follows. □

Proof of Theorem 5.2.: Since $\lim_{x\to\infty} a(x) = \infty$, for all large x and some $M > 0$ assumption (5.4) implies for any two different indices i, j and $z > 0$

$$\begin{aligned} \mathbb{P}\left(X_i > x, X_j > a(x)z\right) &= \int_x^{\infty} \int_{a(x)z}^{\infty} \eta_{ij}(s,t) F_i(\mathrm{d}s) F_j(\mathrm{d}t) \\ &\le M\overline{F}_i(x)\overline{F}_j(a(x)z). \end{aligned}$$

Consequently by (5.6)

$$\lim_{x\to\infty} \frac{\mathbb{P}\left(X_i > x, X_j > a(x)z\right)}{\mathbb{P}\left(X_0 > x\right)} = 0$$

and thus the proof follows by applying Corollary 3.2 of Mitra and Resnick (2009). □

Proof of Theorem 5.3.: Since $\Upsilon_i,\ 1 \le i \le n$ are upper bounded and F is a subexponential distribution, then (5.8) implies

$$\psi(u;n) \quad \sim \quad \sum_{k=1}^n \mathbb{P}\Big(R_k \prod_{i=1}^k \Upsilon_i > u\Big) = \sum_{k=1}^n \mathbb{P}\Big(R_k^+ \prod_{i=1}^k \Upsilon_i > u\Big), \qquad u \to \infty.$$

By the fact that $R_1, \ldots, R_n$ are independent of $\boldsymbol{\Upsilon}$, then for each $1 \le k \le n$ the random vector $(R_k, \Upsilon_1, \ldots, \Upsilon_k)$ has the joint df

$$\eta_{\boldsymbol{\Upsilon}_\mathbf{k}} \cdot d\Big(F \prod_{i=1}^k G_i\Big),$$

where $\eta_{\boldsymbol{\Upsilon}_\mathbf{k}}$ is defined in (5.9). Consequently, the proof of (5.10) follows by Theorem 4.2. □

Proof of Theorem 5.4.: The proof is similar to that of Theorem 5.3 by using (5.8) and Theorem 4.1 a). □

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