

ON THE EFFICIENT EVALUATION OF RUIN PROBABILITIES FOR COMPLETELY MONOTONE CLAIM DISTRIBUTIONS

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Abstract

In this paper we propose a highly accurate approximation procedure for ruin probabilities in the classical collective risk model, which is based on a quadrature/rational approximation procedure proposed by Trefethen et al. [12]. For a certain class of claim size distributions (which contains the completely monotone distributions) we give a theoretical justification for the method. We also show that under weaker assumptions on the claim size distribution, the method may still perform reasonably well in some cases. This in particular provides an efficient alternative to a related method proposed by Thorin [10]. A number of numerical illustrations for the performance of this procedure is provided for both completely monotone and other types of random variables.

1 Introduction

Consider the classical compound Poisson model of collective risk theory, where the surplus process $R(t)$ of an insurance portfolio at time t is given by

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i,$$

with $N(t)$ denoting a homogeneous Poisson process with intensity λ , the claim sizes X_i are i.i.d. distributed non-negative random variables with distribution function F and finite mean $\mu = \mathbb{E}(X_i)$, c is a constant premium intensity and the net profit condition $c > \lambda\mu$ holds. The ruin probability for a given initial surplus level u is denoted by

$$\psi(u) = \mathbb{P}(R(t) < 0 \text{ for some } t > 0 \mid R(0) = u)$$

and its properties are a classical object of study in risk theory (see for instance Asmussen [4]). Define $\bar{G}(u) = \mathbb{P}(\sum_{i=1}^{N(t)} X_i > u)$, so $\bar{G}(u)$ is the tail of a compound Poisson distribution with parameter λt . A second crucial quantity for risk management is the probability that the risk process is negative at a prespecified fixed time t , i.e. $\mathbb{P}(R(t) < 0) = \bar{G}(u + ct)$.

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The efficient approximation of $\psi(u)$ and $\overline{G}(u)$, supposing complete knowledge of the claim distribution, are two of the prime objects of study in this research field.

In this paper we focus on the case when the tail of the claim amount distribution $\overline{F}(x) = 1 - F(x)$ is a completely monotone function, and show that under some further conditions and if the Laplace transform of $\psi(u)$ or $\overline{G}(x)$, respectively, can be computed efficiently in the complex plane, then there exists an algorithm with small error bounds for the inversion of these Laplace transforms, that is quite accurate and efficient in terms of computational costs.

Our numerical inversion of the Laplace transform is based on a quadrature rule proposed by Trefethen et al. [12], which is further inspired by the Cody-Meinardus-Varga Chebyshev rational approximation to e^{-x} . This inversion method provides an alternative to another inversion method proposed by Thorin [10] and allows a theoretical justification.

The paper is organized as follows. In Section 2 we review the integration method given in Trefethen et al. [12]. In Section 3 we provide a theoretical justification for the use of this method, by establishing upper bounds for the errors (for determining ruin probabilities and aggregate claim tails), for a certain class of claim size distributions that contains the class of completely monotone distributions. Finally, in Section 4 we give extensive numerical studies of the proposed method and compare it to other approximation techniques. The examples include Pareto, lognormal and Gamma distributed claims. We find that the proposed method turns out to be very competitive in terms of a tradeoff of computation time and accuracy. We also discuss how the method can be used if the underlying distribution is not completely monotone (however, in this case the theoretical upper error bounds are not valid).

2 A method of inversion of Laplace transforms

Define the Laplace transform of a function h by

$$\hat{L}_h(s) = \int_0^\infty e^{-st} h(t) dt. \quad (1)$$

From the usual integro-differential equation for the ruin probability in the Cramér-Lundberg model, it is well known that

$$\hat{L}_\psi(s) = \frac{1}{s} - \frac{c - \lambda\mu}{cs - \lambda(1 - \hat{L}_f(s))} \quad (2)$$

(see e.g. Rolski et al. [9, Equation (5.3.14), Page 165]). On the other hand, the tail $\overline{G}(u) = \mathbb{P}(X_1 + \dots + X_N > u)$ of any compound Poisson sum (with X_i iid random variables with density f and a Poisson random variable N , independent of the X_i) is given by its Laplace transform

$$\hat{L}_{\overline{G}}(s) = \frac{1}{s} - \frac{1}{s} Q_N(\hat{L}_f(s)) = \frac{1}{s} \left(1 - e^{-\lambda(1 - \hat{L}_f(s))} \right), \quad (3)$$

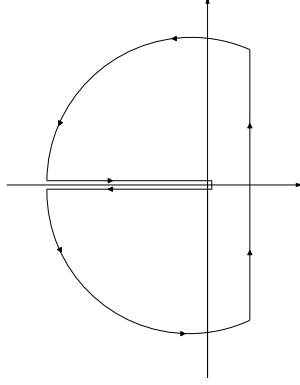


Figure 1: The complex contour

where $Q_N(z) = \mathbb{E}[z^N]$ is the probability generating function of N . A standard inversion formula of (1) gives

$$h(u) = \frac{1}{2\pi\iota} \int_{s_0 - \iota\infty}^{s_0 + \iota\infty} e^{us} \hat{L}_h(s) ds, \quad (4)$$

where s_0 is chosen so that all the singularities of $h(z)$ are to the left of the integration contour $\Gamma := \{\text{Re}(s) = s_0\} = (s_0 - \iota\infty, s_0 + \iota\infty)$.

We summarize now one of the basic ingredients of the method described in Trefethen et al. [12]:

Lemma 2.1. *Assume that $\hat{L}_h(s)$ can be analytically continued to $s \in \mathbb{D} = \mathbb{C} \setminus (-\infty, 0]$, and $\hat{L}_h(s) \rightarrow 0$ as $|s| \rightarrow \infty$ (uniformly for all s bounded away from the negative real axis), then for every $\epsilon > 0$ and $\delta > 0$*

$$h(u) = -\frac{1}{\pi} \int_{-\delta}^{\infty} \text{Im} \left(e^{-u(x-\iota\epsilon)} \hat{L}_h(-x + \iota\epsilon) \right) dx + \frac{1}{\pi} \int_0^{\epsilon} \text{Re} \left(e^{\delta u + \iota x u} \hat{L}_h(\delta + \iota x) \right) dx \quad (5)$$

Proof. For the proof we will use a contour consisting of (i) a segment $[-R, R]$ along the line $\text{Re}(s) = s_0$, (ii) two lines $\{x \pm \iota\epsilon, \text{ for } x \in (-R, \delta)\}$, (iii) the line $\{\delta + \iota x, \text{ for } x \in (-\epsilon, \epsilon)\}$ and (iv) two circle arcs with radius R that connect the segment (i) with the segments in (ii) (cf. Figure 1). At first we let $R \rightarrow \infty$, in which case the integral of the circle arcs tends to zero (cf. [5, p.224]). An application of the Cauchy integral theorem then leads to

$$h(u) = -\frac{1}{2\pi\iota} \left(\int_{-\infty}^{\delta} e^{u(x+\iota\epsilon)} \hat{L}_h(x + \iota\epsilon) dx + \int_{\delta}^{-\infty} e^{u(x-\iota\epsilon)} \hat{L}_h(x - \iota\epsilon) dx \right) - \frac{1}{2\pi} \int_{\epsilon}^{-\epsilon} e^{\delta u + \iota x u} \hat{L}_h(\delta + \iota x) dx.$$

Note now that $\hat{L}_h(\bar{s}) = \overline{\hat{L}_h(s)}$, where \bar{w} is the complex conjugate of w . Indeed, for $\text{Re}(s) > 0$ we have that $\hat{L}_h(s) = \int_0^{\infty} e^{-st} h(t) dt$, from which the observation follows. For $\text{Re}(s) \leq 0$ we may always find an s_0 such that $\text{Re}(s_0) > 0$ and $|s - s_0| < |s_0|$, from which

it follows that s is included in the domain of convergence of the power series around s_0 . The holomorphicity of the analytic continuation of $\hat{L}_h(s)$ on \mathbb{D} yields then

$$\hat{L}_h(s) = \sum_{n=0}^{\infty} \frac{\hat{L}_h^{(n)}(s_0)}{n!} (s - s_0)^n, \quad \hat{L}_h(\bar{s}) = \sum_{n=0}^{\infty} \frac{\hat{L}_h^{(n)}(\bar{s}_0)}{n!} (\overline{s - s_0})^n, \quad \text{and } \hat{L}_h^{(n)}(\bar{s}_0) = \overline{\hat{L}_h^{(n)}(s_0)}$$

implying the observation.

From a substitution in the above integrals it then follows that

$$\begin{aligned} h(u) &= -\frac{1}{2\pi i} \left(\int_{-\delta}^{\infty} e^{-u(x-i\epsilon)} \hat{L}_h(-x+i\epsilon) dx - \int_{-\delta}^{\infty} e^{-u(x+i\epsilon)} \hat{L}_h(-x-i\epsilon) dx \right) \\ &\quad + \frac{1}{2\pi} \int_0^{\epsilon} e^{\delta u + i x u} \hat{L}_h(\delta + i x) dx + \frac{1}{2\pi} \int_0^{\epsilon} e^{\delta u - i x u} \hat{L}_h(\delta - i x) dx \\ &= -\frac{1}{\pi} \int_{-\delta}^{\infty} \text{Im} \left(e^{-u(x-i\epsilon)} \hat{L}_h(-x+i\epsilon) \right) dx + \frac{1}{\pi} \int_0^{\epsilon} \text{Re} \left(e^{\delta u + i x u} \hat{L}_h(\delta + i x) \right) dx \end{aligned}$$

□

If for $\epsilon \rightarrow 0$ we are allowed to interchange limit and integration (for corresponding criteria cf. Section 3), then we get

$$h(u) = -\frac{1}{\pi} \int_0^{\infty} e^{-ux} \text{Im} \left(\hat{L}_h(-x) \right) dx, \quad (6)$$

where $\text{Im}(\hat{L}_h(-x)) := \lim_{\epsilon \rightarrow 0^+} \text{Re} \left(\hat{L}_h(-x+i\epsilon) \right)$. To evaluate the integral in (6), one now uses a rational function of the form

$$r_n(x) = \frac{a_0 + a_1 x + \dots + a_{n-1} x^{n-1}}{b_0 + b_1 x + \dots + b_n x^n}.$$

to approximate e^{-ux} , such that $\sup_{x>0} |e^{-x} - r_n(-x)|$ is small. Denote with z_k and c_k the poles and the residuals, respectively, of $r_n(x)$. If the z_k are distinct, then

$$r_n(x) = \sum_{k=1}^n \frac{c_k}{x - z_k}.$$

By using a contour consisting of the lines that wind around the negative real axis and a circle connecting these lines (see Figure 2) one gets

$$h_n(u) := -\frac{1}{\pi} \int_0^{\infty} r_n(-ux) \text{Im} \left(\hat{L}_h(-x) \right) dx = -\sum_{k=1}^n c_k \hat{L}_h(z_k/u)/u$$

(if $\hat{L}_h(s)$ is not holomorphic, but at least meromorphic, then corresponding residues have to be added to the above expression). The eventual integration error is then bounded by

$$\begin{aligned} |h(u) - h_n(u)| &= \left| \frac{1}{\pi} \int_0^{\infty} (e^{-ux} - r_n(-ux)) \text{Im} \left(\hat{L}_h(-x) \right) dx \right| \\ &\leq \sup_{x>0} |e^{-x} - r_n(-x)| \frac{1}{\pi} \int_0^{\infty} \left| \text{Im} \left(\hat{L}_h(-x) \right) \right| dx. \end{aligned}$$

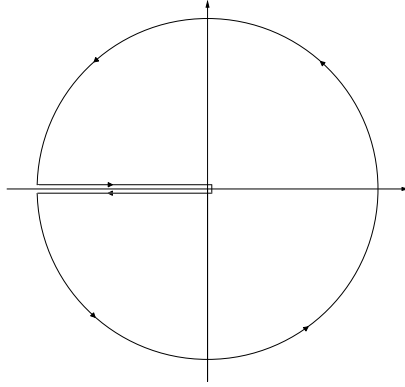


Figure 2: Another complex contour

It is known that one can find an approximation r_n with distinct z_k such that $\sup_{x>0} |e^{-x} - r_n(-x)| = o(9^{-n})$ – see for example [12] for details and references. Furthermore, for this choice of r_n , $c_{2k-1} = \overline{c_{2k}}$ and $z_{2k-1} = \overline{z_{2k}}$, which reduces the number of needed evaluation points by a factor of 2, leading to the simplification

$$h_n(u) = -2 \operatorname{Re} \left(\sum_{k=1}^{n/2} c_{2k-1} \hat{L}_h(z_{2k-1}/u)/u \right), \quad (7)$$

Remark 2.1. *If equation (6) does not hold, then we can approximate e^x in (5) by $r_n(x)$ and use a sufficiently small ϵ to get the same approximation (7). In this case the error bound then has to be adapted accordingly.*

3 Completely monotone distributions

We will now show that if there exists a measure μ such that $h(x) = \int_0^\infty e^{-xu} d\mu(x)$, then the method of Section 2 is applicable. This condition on h is of particular interest since Thorin [10] showed that the ruin probability ($h(x) = \psi(x)$) in the classical risk model with Gamma(α)-distributed claims ($\alpha < 1$) and also with US-Pareto distributed claims fulfills this assumption (in both cases μ is even positive, as the claim distributions are completely monotone).

Proposition 3.1. *Assume that for a function $h(x)$ there exists a (signed) measure μ with*

$$h(x) = \int_0^\infty e^{-xu} d\mu(x).$$

Assume further that

$$\int_0^\infty d|\mu|(t) < \infty,$$

where $|\mu|$ is the total variation measure of μ . Then $\hat{L}_h(s)$ is holomorphic for $s \in \mathbb{D}$ and for every $\epsilon > 0$ $\lim_{|s| \rightarrow \infty, \operatorname{Im}(s) > \epsilon} \hat{L}_h(s) = 0$. Further, for every function $k(s)$ for which there exists an ϵ_0 such that $k(s)$ is holomorphic for all $s \in \{s : \sup_{x \in \mathbb{R}, x \leq 0} |s - x| < \epsilon_0\}$,

$k(\bar{s}) = \overline{k(s)}$ and $k(s) = \mathcal{O}(1/\operatorname{Re}(s))$ as $\operatorname{Re}(s) \rightarrow \infty$ (where $\operatorname{Im}(s)$ stays bounded), there exists a $\delta > 0$ with

$$-\lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{-\delta}^{\infty} \operatorname{Im} \left(k(-x + \iota\epsilon) \hat{L}_h(-x + \iota\epsilon) \right) dx = \int_0^{\infty} k(-x) d\mu(x). \quad (8)$$

Hence in this case representation (6) is applicable.

Proof. At first note that

$$\hat{L}_h(s) = \int_0^{\infty} \frac{1}{x+s} d\mu(x).$$

Hence $\hat{L}_h(s)$ is the Stieltjes transform of μ (cf. [13, Chapter VIII]). It follows that $\hat{L}_h(s)$ is holomorphic for all $s \in \mathbb{D}$ (cf. [13, p. 328, Corollary 2b.1]). We want to show that for every $\epsilon > 0$, $\lim_{|s| \rightarrow \infty, \operatorname{Im}(s) > \epsilon} \hat{L}_h(s) = 0$. At first note that when the distance between s and the negative real axis approaches infinity, then $\hat{L}_h(s) \rightarrow 0$. If s stays near the negative real axis, then for an $\epsilon > 0$ and $\operatorname{Im}(s) > \epsilon$:

$$\begin{aligned} |\hat{L}_h(s)| &\leq \int_{-\operatorname{Re}(s)/2}^{\infty} \left| \frac{1}{x+s} \right| d|\mu|(x) + \int_0^{-\operatorname{Re}(s)/2} \left| \frac{1}{x+s} \right| d|\mu|(x) \\ &\leq \epsilon^{-1} \int_{-\operatorname{Re}(s)/2}^{\infty} d|\mu|(x) + \frac{2}{|-\operatorname{Re}(s)|} \int_0^{\infty} d|\mu|(x), \end{aligned}$$

which tends to zero as $\operatorname{Re}(s) \rightarrow -\infty$.

To prove (8) note that

$$\begin{aligned} \int_{-\delta}^{\infty} \operatorname{Im} \left(k(-x + \iota\epsilon) \hat{L}_h(-x + \iota\epsilon) \right) dx &= \int_{-\delta}^{\infty} \int_0^{\infty} \operatorname{Im} \left(\frac{k(-x + \iota\epsilon)}{t - x + \iota\epsilon} \right) d\mu(t) dx \\ &= \int_0^{\infty} \int_{-\delta}^{\infty} \operatorname{Im} \left(\frac{k(-x + \iota\epsilon)}{t - x + \iota\epsilon} \right) dx d\mu(t) \\ &= \frac{1}{2\iota} \int_0^{\infty} \int_{-\delta}^{\infty} \left(\frac{k(-x + \iota\epsilon)}{t - x + \iota\epsilon} \right) - \left(\frac{k(-x - \iota\epsilon)}{t - x - \iota\epsilon} \right) dx d\mu(t). \end{aligned}$$

We have

$$\int_{-\delta}^{\infty} \operatorname{Im} \left(\frac{k(-x + \iota\epsilon)}{t - x + \iota\epsilon} \right) dx = \lim_{R \rightarrow \infty} \int_{-\delta}^R \left(\frac{k(-x + \iota\epsilon)}{t - x + \iota\epsilon} \right) - \left(\frac{k(-x - \iota\epsilon)}{t - x - \iota\epsilon} \right) dx$$

and by the residual theorem we get

$$\begin{aligned} \int_{-\delta}^R \left(\frac{k(-x + \iota\epsilon)}{t - x + \iota\epsilon} \right) - \left(\frac{k(-x - \iota\epsilon)}{t - x - \iota\epsilon} \right) dx &= -2\pi\iota k(-t) - \int_{\delta - \iota\epsilon}^{\delta + \iota\epsilon} \frac{k(s)}{t+s} ds \\ &\quad - \int_{R + \iota\epsilon}^{R - \iota\epsilon} \frac{k(s)}{t+s} ds = -2\pi\iota k(-t) - \int_{-\epsilon}^{\epsilon} \frac{k(\delta + \iota x)}{t + \delta + \iota x} dx + \int_{-\epsilon}^{\epsilon} \frac{k(-R + \iota x)}{t - R + \iota x} dx. \end{aligned}$$

From

$$\lim_{R \rightarrow \infty} \int_{-\epsilon}^{\epsilon} \frac{k(-R + \iota x)}{t - R + \iota x} dx = 0 \quad \text{and} \quad \left| \int_{-\epsilon}^{\epsilon} \frac{k(\delta + \iota x)}{t + \delta + \iota x} dx \right| \leq 2\epsilon \sup_{-\epsilon \leq x \leq \epsilon} \frac{|k(\delta + \iota x)|}{\delta}$$

it follows that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{-\delta}^{\infty} \operatorname{Im} \left(k(-x + \iota\epsilon) \hat{L}_h(-x + \iota\epsilon) \right) dx &= -\pi \int_0^{\infty} k(-t) d\mu(t) \\ &\quad - \frac{1}{2\iota} \lim_{\epsilon \rightarrow 0} \int_0^{\infty} \int_{-\epsilon}^{\epsilon} \frac{k(\delta + \iota x)}{t + \delta + \iota x} dx d\mu(t) = -\pi \int_0^{\infty} k(-t) d\mu(t). \end{aligned}$$

□

We hence arrive at the following result.

Theorem 3.2. *Assume that for a function $h(x)$ there exists a (signed) measure μ with*

$$h(x) = \int_0^{\infty} e^{-xu} d\mu(x) \quad \text{and} \quad \int_0^{\infty} d|\mu|(t) < \infty.$$

Then for the approximation $h_n(x)$ of equation (7) we have the error bound

$$|h_n(x) - h(x)| \leq \sup_{x>0} |e^{-x} - r_n(-x)| \int_0^{\infty} d|\mu|(t).$$

If further μ is a positive measure (i.e. h is completely monotone), then

$$|h_n(x) - h(x)| \leq \sup_{x>0} |e^{-x} - r_n(-x)| \int_0^{\infty} d\mu(t) = h(0) \sup_{x>0} |e^{-x} - r_n(-x)|.$$

In the following we will show that for completely monotone claim size distributions (which are defined through $\bar{F}(x) = \int_0^{\infty} e^{-xt} d\mu(t)$ for a positive measure μ) and some further restrictions on μ , the ruin probability in the Cramér Lundberg model and the tail of a compound Poisson distribution fulfill the condition of Proposition 3.1. To that end, we will use some results of Thorin [10]. Let us first evaluate $\lim_{\epsilon \rightarrow 0} \hat{L}_f(-x + \iota\epsilon)$.

Lemma 3.3. *Assume that $\bar{F}(x) = \int_0^{\infty} e^{-xt} d\mu(t)$, where μ has a density $f_{\mu}(t)$ which is continuous in x , then*

$$\begin{aligned} \operatorname{Im}(\hat{L}_{\bar{F}}(-x)) &:= \lim_{\epsilon \rightarrow 0^+} \operatorname{Im} \left(\hat{L}_{\bar{F}}(-x + \iota\epsilon) \right) = -2\pi f_{\mu}(x) \quad \text{and} \\ \operatorname{Im}(\hat{L}_f(-x)) &:= \lim_{\epsilon \rightarrow 0^+} \operatorname{Im} \left(\hat{L}_f(-x + \iota\epsilon) \right) = -2\pi x f_{\mu}(x). \end{aligned}$$

If further $f'_{\mu}(t)$ exists and is bounded in a region around x , then

$$\operatorname{Re}(\hat{L}_{\bar{F}}(-x)) := \lim_{\epsilon \rightarrow 0^+} \operatorname{Re} \left(\hat{L}_{\bar{F}}(-x + \iota\epsilon) \right) = \lim_{\delta \rightarrow 0} \left(\int_0^{x-\delta} \frac{1}{(s-x)} f_{\mu}(s) ds + \int_{x+\delta}^{\infty} \frac{1}{(s-x)} f_{\mu}(s) ds \right).$$

Proof. The first statement is proved in [13, Theorem 7b, p. 340], the second statement is proved in Section 2 of [10]. □

Thorin [10] stated for the case of $h(x) = \psi(x)$ that interchanging limit and integration in (6) is feasible if $\hat{L}_f(-x)$ is continuous and some further conditions are fulfilled. In the following we show that in our setup these further conditions and the continuity are in fact fulfilled and in that way provide sufficient general conditions beyond Thorin's explicit examples under which the procedure is applicable.

Lemma 3.4. Assume that $\bar{F}(x) = \int_0^\infty e^{-xt} d\mu(t)$, where μ has a density $f_\mu(t)$ which has a bounded derivative f'_μ for $t > d \geq 0$. If $\int_0^\infty d|\mu|(t) < \infty$, $\lim_{x \rightarrow \infty} x \sup_{t > x} |f_\mu(t)| = 0$ and $\lim_{x \rightarrow \infty} x^2 \sup_{t > x} |f'_\mu(t)| = 0$, then

$$\hat{L}_f(s) \rightarrow 0$$

uniformly for $|s| \rightarrow \infty$.

Proof. We have to show that $\hat{L}_f(-x + \iota\epsilon) \rightarrow 0$ as $x \rightarrow \infty$ uniformly for $\epsilon > 0$. Note that for $(x/2 > d)$ and $\delta > 0$

$$\begin{aligned} \text{Im}(\hat{L}_f(-x + \iota\epsilon)) &= - \int_0^\infty \frac{\epsilon s}{(s-x)^2 + \epsilon^2} d\mu(s) = - \int_0^{x-\delta x} \frac{\epsilon s}{(s-x)^2 + \epsilon^2} d\mu(s) \\ &\quad - \int_{x+\delta x}^\infty \frac{\epsilon s}{(s-x)^2 + \epsilon^2} d\mu(s) - \int_{x-\delta x}^{x+\delta x} \frac{\epsilon s}{(s-x)^2 + \epsilon^2} f_\mu(s) ds. \end{aligned}$$

We have that

$$\int_{x+\delta x}^\infty \frac{\epsilon s}{(s-x)^2 + \epsilon^2} d\mu(s) \leq \int_{x+\delta x}^\infty \frac{s}{2(s-x)} d|\mu|(s) \leq \frac{1+\delta}{2\delta} \int_{x+\delta x}^\infty d|\mu|(s)$$

and

$$\int_0^{x-\delta x} \frac{\epsilon s}{(s-x)^2 + \epsilon^2} d\mu(s) \leq \frac{(1-\delta)\epsilon x}{(\delta x)^2 + \epsilon^2} \int_0^\infty d|\mu|(s) \leq \frac{1-\delta}{2\delta} \int_0^\infty d|\mu|(s).$$

For the last integral we have that

$$\begin{aligned} \int_{x-\delta x}^{x+\delta x} \frac{\epsilon s}{(s-x)^2 + \epsilon^2} f_\mu(s) ds &\leq \sup_{x(1-\delta) < t < x(1+\delta)} |f_\mu(t)| \int_{x-\delta x}^{x+\delta x} \frac{\epsilon s}{(s-x)^2 + \epsilon^2} ds \\ &= 2x \sup_{x(1-\delta) < t < x(1+\delta)} |f_\mu(t)| \arctan\left(\frac{x\delta}{\epsilon}\right). \end{aligned}$$

It follows that for every $0 < \delta < 1$ uniformly for all $\epsilon > 0$

$$\lim_{x \rightarrow \infty} \left| \text{Im}(\hat{L}_f(-x + \iota\epsilon)) \right| \leq \frac{1-\delta}{2\delta}$$

and hence uniformly for $\epsilon > 0$

$$\lim_{x \rightarrow \infty} \left| \text{Im}(\hat{L}_f(-x + \iota\epsilon)) \right| = 0.$$

Next we consider the real part of $\hat{L}_f(-x + \iota\epsilon)$. We have:

$$\begin{aligned} \text{Re}(\hat{L}_f(-x + \iota\epsilon)) &= \int_0^\infty \frac{(s-x)s}{(s-x)^2 + \epsilon^2} d\mu(s) = \int_0^{x-\delta x} \frac{(s-x)s}{(s-x)^2 + \epsilon^2} d\mu(s) \\ &\quad + \int_{x+\delta x}^\infty \frac{(s-x)s}{(s-x)^2 + \epsilon^2} d\mu(s) + \int_{x-\delta x}^{x+\delta x} \frac{(s-x)s}{(s-x)^2 + \epsilon^2} f_\mu(s) ds. \end{aligned}$$

Furthermore

$$\left| \int_{x+\delta x}^{\infty} \frac{(s-x)s}{(s-x)^2 + \epsilon^2} d\mu(s) \right| \leq \int_{x+\delta x}^{\infty} \frac{s}{s-x} d|\mu|(s) \leq \frac{1+\delta}{\delta} \int_{x+\delta x}^{\infty} d|\mu|(x)$$

and

$$\left| \int_0^{x-\delta x} \frac{(s-x)s}{(s-x)^2 + \epsilon^2} d\mu(s) \right| \leq \int_0^{x-\delta x} \frac{s}{x-s} d|\mu|(s) \leq \frac{1-\delta}{\delta} \int_0^{x-\delta x} d|\mu|(x).$$

For the last integral we get for a $\xi_x \in (\inf_{(1-\delta)x < t < (1+\delta)x} f'_\mu(t), \sup_{(1-\delta)x < t < (1+\delta)x} f'_\mu(t))$ and Taylor's formula

$$\int_{x-\delta x}^{x+\delta x} \frac{(s-x)s}{(s-x)^2 + \epsilon^2} f_\mu(s) ds = \int_{x-\delta x}^{x+\delta x} \frac{(s-x)s}{(s-x)^2 + \epsilon^2} f_\mu(x) ds + \int_{x-\delta x}^{x+\delta x} \frac{(s-x)^2 s}{(s-x)^2 + \epsilon^2} \xi_x ds.$$

Evaluating the first integral, we get

$$\left| \int_{x-\delta x}^{x+\delta x} \frac{(s-x)s}{(s-x)^2 + \epsilon^2} f_\mu(x) ds \right| = \left| f_\mu(x) (2\delta x - 2\epsilon \arctan\left(\frac{\delta x}{\epsilon}\right)) \right| \leq 4\delta x f_\mu(x).$$

For the second integral one obtains

$$\left| \int_{x-\delta x}^{x+\delta x} \frac{(s-x)^2 s}{(s-x)^2 + \epsilon^2} \xi_x ds \right| \leq \sup_{(1-\delta)x < t < (1+\delta)x} |f'_\mu(t)| \int_{x-\delta x}^{x+\delta x} s ds \leq \delta x^2 \sup_{(1-\delta)x < t < (1+\delta)x} |f'_\mu(t)|.$$

It follows, as for the imaginary part, that uniformly for $\epsilon > 0$

$$\lim_{x \rightarrow \infty} \left| \operatorname{Re}(\hat{L}_f(-x + \iota\epsilon)) \right| = 0.$$

□

Lemma 3.5. *Assume that $\bar{F}(x) = \int_0^\infty e^{-xt} d\mu(t)$, where μ has a density $f_\mu(t)$ which has a bounded derivative $f'_\mu(t)$ for t in a region around x_0 . Then $\hat{L}_f(-x)$ is continuous in $x = x_0$.*

Proof. From Lemma 3.3 we get that $\operatorname{Im}(\hat{L}_f(-x)) = -2\pi x f_\mu(x)$ and hence $\operatorname{Im}(\hat{L}_f(-x))$ is continuous in $x = x_0$.

For the real part we have to show that:

$$\lim_{x \rightarrow x_0, \epsilon \rightarrow 0} \lim_{\epsilon_0 \rightarrow 0} \int_0^\infty \frac{(s-x)s}{(s-x)^2 + \epsilon^2} - \frac{(s-x_0)s}{(s-x_0)^2 + \epsilon_0^2} d\mu(s) = 0.$$

For $\delta/2 > |x - x_0|$ we can split the integral into three integrals $\int_0^{x_0-\delta} + \int_{x_0-\delta}^{x_0+\delta} + \int_{x_0+\delta}^\infty$. The integrand in the first and third integral can be uniformly bounded for all $|x - x_0| < \delta/2$ and all $\epsilon, \epsilon_0 > 0$. Hence we are allowed to interchange limit and integration and these integrals vanish. For the remaining integral we get for $\xi_{x_0} \in (\inf_{-\delta < t < \delta} f'_\mu(x_0+t), \sup_{-\delta < t < \delta} f'_\mu(x_0+t))$

$$\begin{aligned} \int_{x_0-\delta}^{x_0+\delta} \frac{(s-x_0)s}{(s-x_0)^2 + \epsilon_0^2} f_\mu(s) ds &= f_\mu(x_0) \left(2\delta x_0 - 2\epsilon_0 \arctan\left(\frac{\delta x_0}{\epsilon_0}\right) \right) \\ &\quad + \xi_{x_0} \int_{x_0-\delta}^{x_0+\delta} \frac{(s-x_0)^2 s}{(s-x_0)^2 + \epsilon_0^2} ds \end{aligned}$$

and for $\xi_x \in (\inf_{-\delta < t < \delta} f'_\mu(x_0 + t), \sup_{-\delta < t < \delta} f'_\mu(x_0 + t))$

$$\int_{x_0-\delta}^{x_0+\delta} \frac{(s-x)s}{(s-x)^2 + \epsilon^2} f_\mu(s) \, ds = f_\mu(x) \int_{x_0-\delta}^{x_0+\delta} \frac{(s-x)s}{(s-x)^2 + \epsilon^2} \, ds + \xi_x \int_{x_0-\delta}^{x_0+\delta} \frac{(s-x)^2 s}{(s-x)^2 + \epsilon^2} \, ds.$$

Note that by an integration we get that

$$\lim_{x \rightarrow x_0, \epsilon \rightarrow 0} \left| \int_{x_0-\delta}^{x_0+\delta} \frac{(s-x)s}{(s-x)^2 + \epsilon^2} \, ds \right| = 2\delta.$$

Further we have

$$\lim_{x \rightarrow x_0, \epsilon \rightarrow 0} \int_{x_0-\delta}^{x_0+\delta} \frac{(s-x)^2 s}{(s-x)^2 + \epsilon^2} \, ds = \lim_{\epsilon_0 \rightarrow 0} \int_{x_0-\delta}^{x_0+\delta} \frac{(s-x_0)^2 s}{(s-x_0)^2 + \epsilon_0^2} \, ds = \frac{(x_0 + \delta)^2 - (x_0 - \delta)^2}{2}.$$

It follows that for every $\delta > 0$

$$\lim_{s \rightarrow x_0, \operatorname{Re}(s) > 0} \left| \hat{L}_f(s) - \hat{L}_f(-x_0) \right| < 2\delta + \left(\sup_{x_0-\delta < t < x_0+\delta} f'_\mu(t) - \inf_{x_0-\delta < t < x_0+\delta} f'_\mu(t) \right) \frac{(x_0 + \delta)^2 - (x_0 - \delta)^2}{2}$$

and hence

$$\lim_{s \rightarrow x_0, \operatorname{Re}(s) > 0} \hat{L}_f(s) = \hat{L}_f(-x_0).$$

In the case that $x_0 = 0$ we use the integrals from $\int_0^\delta + \int_\delta^\infty$ and the same proof applies. \square

We can now give alternative general conditions for a result of Thorin [10] and add an error bound:

Theorem 3.6. *Assume that $\bar{F}(x) = \int_0^\infty e^{-xt} \, d\mu(t)$, where for $t > d \geq 0$, μ has a density $f_\mu(t) > 0$ which has a bounded derivative $f'_\mu(t)$. If further $f_\mu(t) = 0$ for $0 \leq t \leq d$, $\lim_{x \rightarrow \infty} x \sup_{t > x} f_\mu(t) = 0$ and $\lim_{x \rightarrow \infty} x^2 \sup_{t > x} |f'_\mu(t)| = 0$, then*

$$\psi(u) = 2 \int_0^\infty \frac{\lambda(c - \lambda\mu)x^{-1} f_\mu(x) e^{-ux}}{\left(c - \lambda \operatorname{Re}(\hat{L}_{\bar{F}}(-x)) \right)^2 + (2\pi f_\mu(x))^2} \, dx + c_n e^{-uR_1}. \quad (9)$$

Here, R_1 is the Lundberg coefficient if it exists, and in this case $c_n e^{-uR_1}$ is the Cramér-Lundberg approximation. If the Lundberg coefficient does not exist, then $c_n = 0$.

For the approximation of equation (7), we get

$$|\psi_n(u) - \psi(u)| < \sup_{x > 0} |e^{-x} - r_n(-x)| \psi(0).$$

Proof. The result follows from Corollary 2 and 3 of [10] and the comments afterwards. The conditions of these corollaries hold due to Lemma 3.4 and Lemma 3.5 given above. \square

Remark 3.1. *It was already mentioned in Thorin [10] that (9) applies for Pareto distributions and the completely monotone Gamma distributions as claim size distributions (cf. also Ramsay [7]). In [10] also an extension to renewal models was considered.*

We now give a similar result for the tail of a compound Poisson distribution, which extends a result for Pareto distributions given in Ramsay [8]:

Theorem 3.7. *Assume that $\bar{F}(x) = \int_0^\infty e^{-xt} d\mu(t)$, where for $t \geq 0$, μ has a density $f_\mu(t)$ which has a bounded derivative $f'_\mu(t)$. If further $\lim_{x \rightarrow \infty} x \sup_{t > x} |f_\mu(t)| = 0$ and $\lim_{x \rightarrow \infty} x^2 \sup_{t > x} |f'_\mu(t)| = 0$, then*

$$\begin{aligned} \bar{G}(u) &= -\frac{1}{\pi} \int_0^\infty e^{-ux} \operatorname{Im} \left(\frac{1}{x} e^{-\lambda(1-\hat{L}_f(-x))} \right) dx = -\frac{1}{\pi} \int_0^\infty e^{-ux} \operatorname{Im} \left(\frac{1}{x} e^{\lambda x \hat{L}_{\bar{F}}(-x)} \right) dx \\ &= \frac{1}{\pi} \int_0^\infty e^{-ux} e^{\lambda x \operatorname{Re}(\hat{L}_{\bar{F}}(-x))} \frac{\sin(2\pi \lambda x f_\mu(x))}{x} dx. \end{aligned}$$

The approximation error of the approximation (7) is then less than

$$\sup_{x>0} |e^{-x} - r(-x)| \leq 2\lambda \sup_{y>0} e^{\lambda y \operatorname{Re}(\hat{L}_{\bar{F}}(-y))} \int_0^\infty |f_\mu(z)| dz. \quad (10)$$

Proof. From Proposition 3.1 and Lemma 3.4 we get that $\hat{L}_f(s)$ is holomorphic for $s \in \mathbb{D}$ and $\lim_{|s| \rightarrow \infty} \hat{L}_f(s) = 0$. Due to Lemma 2.1 we then have to show that we are allowed to interchange limit and integration in

$$-\lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{-\delta}^\infty \operatorname{Im} \left(e^{-u(x-\iota\epsilon)} \frac{1}{-x+\iota\epsilon} \left(1 - e^{-\lambda(1-\hat{L}_f(-x+\iota\epsilon))} \right) \right) dx.$$

By Lemma 3.5 and 3.5, $\operatorname{Re}(\lambda(1-\hat{L}_f(-x+\iota\epsilon)))$ can be uniformly bounded for all $x \geq 0$ and $\epsilon \geq 0$. Hence we get by dominated convergence:

$$\begin{aligned} -\lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{-\delta}^\infty \operatorname{Im} \left(e^{-u(x-\iota\epsilon)} \frac{1}{-x+\iota\epsilon} \left(1 - e^{-\lambda(1-\hat{L}_f(-x+\iota\epsilon))} \right) \right) dx \\ = -\frac{1}{\pi} \int_0^\infty e^{-ux} \operatorname{Im} \left(\frac{1}{x} e^{-\lambda(1-\hat{L}_f(-x))} \right) dx. \end{aligned}$$

Since $\hat{L}_f(s)$ is continuous for $s = 0$, one obtains

$$\lim_{s \rightarrow 0} \left(1 - e^{-\lambda(1-\hat{L}_f(s))} \right) = 0$$

and hence

$$\lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{-\delta}^\delta e^{-u(x-\iota\epsilon)} \frac{1}{-x+\iota\epsilon} \left(1 - e^{-\lambda(1-\hat{L}_f(-x+\iota\epsilon))} \right) dx = 0.$$

For the proof of (10), note that $|\sin(x)| \leq x$ for $x > 0$. □

Remark 3.2. *The Gamma distribution with $f(x) = x^{\alpha-1} e^{-x\gamma} \gamma^\alpha / \Gamma(\alpha)$ and $\hat{L}_f(s) = (1 + s/\gamma)^{-\alpha}$ is completely monotone for $\alpha < 1$ with $f_\mu(x) = \sin(\alpha\pi)(s/\gamma-1)^{-\alpha}/(s\pi)$ for $x > \gamma$. But in this case $f'_\mu(x)$ is not bounded for $x = 1/\gamma$, so that Theorem 3.7 is not applicable. However, Remark 2.1 suggests that we can still use the proposed method.*

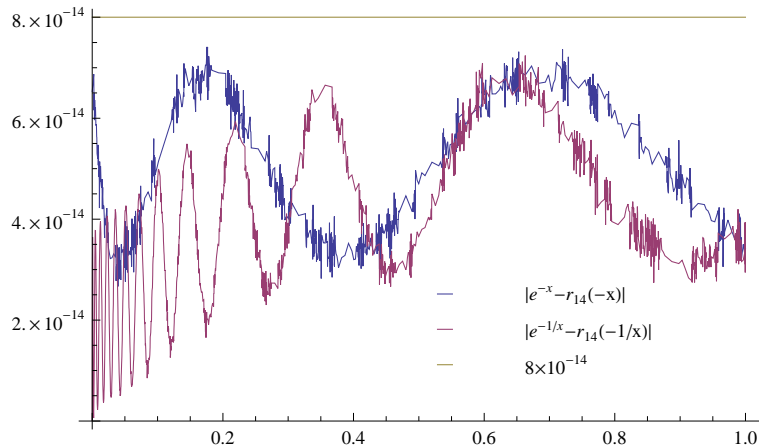


Figure 3: The absolute error of the rational approximation $r_{14}(x)$ for small and large values of x .

4 Some numerical examples

In the following we give some numerical illustrations for the strikingly accurate and fast evaluation of $\psi(u)$ and $\bar{G}(u)$ respectively, according to the procedure suggested in the previous sections (which we call the TWS method).

The simple procedure to obtain the poles z_k and residuals c_k of the rational polynomials r_n is described in Trefethen et al. [12]. As noted there, using usual double precision the algorithm works well for smaller values of n and still reasonably for $n = 14$. Hence we decided to use $n = 14$ for numerical computations (the procedure in [12] actually only gives an approximation of the best approximating rational function r_{14} , but as a numerical evaluation shows, this is sufficiently accurate; Figure 3 suggests that $\sup_{x>0} |e^{-x} - r_{14}(-x)| < 8 \times 10^{-14}$, which is only about twice the bound $o(9^{-n})$ for the best one). From Theorem 3.6 we then know that for the ruin probability the first 13 digits of its approximation are correct, so the numerical calculation of even very small ruin probabilities is then reliable. However, one should keep in mind that the approximation is computed itself numerically, and one has to care that due to truncation of digits this does not lead to a significant further error (for the Pareto distribution this can for instance be achieved by a reformulation of the Laplace transform expression, cf. below).

In addition to examples with completely monotone distributions, below we also test the procedure for distributions that are not completely monotone and show that in this case indeed the simple error bound of Theorem 3.6 is not valid. We will compare our results with two other methods of Laplace transform inversion provided in Abate & Valko [1], namely a fixed Talbot (FT) algorithm and a Gaver-Wynn-Rho (GWR) algorithm that relies on the use of the Post-Widder formula

$$h(x) = \lim_{k \rightarrow \infty} \frac{(-1)^k}{k!} \left(\frac{k}{x}\right)^{k+1} \hat{L}_h\left(\frac{k}{x}\right).$$

The GWR method has the advantage that it is applicable for a broader class of dis-

tributions, since one only needs the values of the Laplace transform for positive real arguments. However, both methods proposed in [1] rely on multi-precision computing (expressed through a parameter M , cf. [1]), whereas our approach also works satisfactorily with double precision, which has a very positive effect on the computation time (in the tables below, we also report a TWS with rational approximation $r_{40}(x)$ for the case that multi-precision (with 60 digits) is used to make it better comparable with the FT method (with $M = 60$) and the GWR method (with $M = 50$)). In principle, all these methods work satisfactorily for reasonable examples and accuracy. However, especially in the completely monotone case, the TWS method described in this paper is most efficient concerning reasonable precision versus computational costs (followed by the FT algorithm, the GWR method is then the least effective).

We use values of u such that $\psi(u)$ ($\bar{G}(u)$, respectively) is approximately 10^{-k} , where $k \in \{1, \dots, 6\}$. The tables show the absolute value of the relative error of the approximations, where we use the result of the GWR method with $M = 200$ as the reference for the exact value.

4.1 US-Pareto distribution

As a first example, we consider the US-Pareto distribution with $\bar{F}(x) = (1+x)^{-\alpha}$, $x \geq 0$ as a claim distribution. This distribution is completely monotone and its Laplace transform is $\hat{L}_f(s) = \alpha s^\alpha e^s \Gamma(-\alpha, s)$. We provide an example for the ruin probability with $\alpha = 1.5$, $\lambda = 1$ and $c = 2.25$ (Table 1), together with the computation time to generate the whole column for each method. Note that the direct evaluation of $1 - \hat{L}_f(s)$ for $|s|$ small is numerically not stable (column TWS14), so we use the simple identity $1 - \hat{L}_f(s) = e^s(1 - \alpha s^\alpha \Gamma(-\alpha, s)) + (1 - e^s)$ and evaluate $(1 - \alpha s^\alpha \Gamma(-\alpha, s))$ with the series of Formula 6.5.29 of Abramowitz & Stegun [2] (truncated after sufficiently many summands, cf. also Albrecher & Kortschak [3]). For the term $(1 - e^s)$ we use a series expansion as well. These minor changes and an implementation in C++ dramatically improve the performance of the TWS method (column TWS14 C++). In particular it outperforms the FT and GWR method in terms of efficiency.

In Table 2 we see the results for the compound Poisson distribution with Pareto claims and parameter $\alpha = 1.5$ and $\lambda = 10$. Since $1 - \hat{L}_f(s)$ is not in the denominator in this case (cf. (3)), there is no need for the above reformulation and the plain Mathematica implementation of TWS is already quite acceptable (column TWS14), but an implementation in C++ speeds up the computation considerably (column TWS14 C++).

4.2 Completely monotone Gamma distribution

The Gamma distribution with Laplace transform $\hat{L}_f(s) = (1+s/\gamma)^{-\alpha}$ is completely monotone for $\alpha < 1$. As mentioned in Remark 3.2, Theorem 3.7 is not applicable and we do not get an error bound (however, as one can see in Table 4, the TWS method still works, as for the numerical integration this single point is not relevant). However, for the ruin probability we can apply Theorem 3.6.

$\psi(u)$	TWS14	TWS14 C++	TWS40	FT	GWR
$1. \times 10^{-1}$	3.3×10^{-10}	3.9×10^{-13}	1.1×10^{-38}	1.6×10^{-36}	1.3×10^{-33}
$1. \times 10^{-2}$	1.4×10^{-6}	4.5×10^{-12}	1.1×10^{-37}	5.2×10^{-36}	$1. \times 10^{-36}$
$1. \times 10^{-3}$	2.4×10^{-3}	6.1×10^{-12}	8.1×10^{-37}	6.1×10^{-36}	$3. \times 10^{-34}$
$1. \times 10^{-4}$	1.2×10^0	2.7×10^{-10}	1.3×10^{-36}	6.2×10^{-36}	$3. \times 10^{-34}$
$1. \times 10^{-5}$	$2. \times 10^1$	5.4×10^{-9}	1.4×10^{-36}	6.2×10^{-36}	$3. \times 10^{-34}$
$1. \times 10^{-6}$	3.6×10^6	1.9×10^{-8}	1.4×10^{-36}	1.3×10^{-35}	$3. \times 10^{-34}$
time	0.0048 sec	0.0001 sec	0.1056 sec	0.4092 sec	0.7916 sec

Table 1: Relative error of the ruin probability for the US-Pareto distribution with $\alpha = 1.5$

$\bar{G}(u)$	TWS14	TWS14 C++	TWS40	FT	GWR
$1. \times 10^{-1}$	7.1×10^{-13}	1.2×10^{-13}	1.3×10^{-38}	1.9×10^{-36}	7.8×10^{-36}
$1. \times 10^{-2}$	9.6×10^{-12}	8.1×10^{-12}	1.3×10^{-37}	1.9×10^{-35}	1.8×10^{-31}
$1. \times 10^{-3}$	2.9×10^{-11}	5.6×10^{-11}	1.3×10^{-36}	2.5×10^{-34}	2.1×10^{-33}
$1. \times 10^{-4}$	$6. \times 10^{-10}$	2.1×10^{-10}	2.5×10^{-36}	2.4×10^{-33}	1.6×10^{-36}
$1. \times 10^{-5}$	1.5×10^{-8}	3.5×10^{-10}	3.5×10^{-34}	6.5×10^{-33}	1.8×10^{-37}
$1. \times 10^{-6}$	1.3×10^{-7}	7.9×10^{-8}	2.8×10^{-33}	1.4×10^{-32}	1.9×10^{-39}
time	0.0044 sec	0.0001 sec	0.1504 sec	0.6052 sec	1.0664 sec

Table 2: Relative error of tail of compound Poisson distribution for the US-Pareto distribution with $\alpha = 1.5$

In Table 3, we depict results for the Gamma distribution with $\alpha = 1/100$ and $\gamma = 1/100$ (which are the parameters used in Grandell & Segerdahl [6]), $c = 11/10$ and $\lambda = 1$. In Table 4 we list the results for the compound Poisson case with parameters $\alpha = \gamma = 1/4$ and $\lambda = 3$.

4.3 A Gamma distribution that is not completely monotone

Consider now a Gamma distribution with $\alpha > 1$ (which is not completely monotone). One can easily check that in this case $|\hat{L}_\psi(s)| \rightarrow 0$ and $|\hat{L}_{\bar{G}}(s)| \rightarrow 0$ as $|s| \rightarrow \infty$ still holds. However, whereas $\hat{L}_{\bar{G}}(s)$ is holomorphic in \mathbb{D} , $\hat{L}_\psi(s)$ now possesses poles in \mathbb{D} . One then has two possibilities. Either one could try to find the poles of $\hat{L}_\psi(s)$ and add the residuals to the approximation, or one could try to ignore the contribution of the residuals and hope that the introduced error is negligible. In Table 5 we use $\alpha = 5/2$, $\gamma = 5/2$ and $c = 11/10$ without considering the residuals, and we see that the contribution of the poles of $\hat{L}_\psi(s)$ is in fact negligible in this case. The results for the compound Poisson case with $\lambda = 1$ can be found in Table 6 (note that the error bound (10) is again not applicable).

4.4 Pareto distribution

Let us now consider a (classical) Pareto claim size distribution, i.e. $\bar{F}(x) = x^{-\alpha}$, $x \geq 1$, which is not (but almost) completely monotone. Note that $\hat{L}_f(s) = \alpha s^\alpha \Gamma(-\alpha, s)$ and for any $a, b > 0$, $\lim_{t \rightarrow \infty} |\hat{L}_f(t(-a + bt))| = \infty$. For the case of evaluating $\psi(u)$, we get that

$\psi(u)$	TWS14	TWS40	FT	GWR
$1. \times 10^{-1}$	7.2×10^{-12}	2.1×10^{-38}	1.8×10^{-36}	1.8×10^{-40}
$1. \times 10^{-2}$	8.1×10^{-11}	7.2×10^{-39}	1.8×10^{-35}	9.3×10^{-38}
$1. \times 10^{-3}$	2.1×10^{-9}	5.6×10^{-37}	1.8×10^{-34}	1.1×10^{-35}
$1. \times 10^{-4}$	1.9×10^{-7}	$2. \times 10^{-35}$	1.8×10^{-33}	1.4×10^{-32}
$1. \times 10^{-5}$	9.3×10^{-7}	1.4×10^{-34}	1.9×10^{-32}	3.1×10^{-29}
$1. \times 10^{-6}$	$3. \times 10^{-5}$	3.2×10^{-34}	1.9×10^{-31}	6.1×10^{-28}
time	0.0012 sec	0.0304 sec	0.1808 sec	0.7724 sec

Table 3: Relative error of the ruin probability for the Gamma distribution with $\alpha = \gamma = 1/100$

$\bar{G}(u)$	TWS14	TWS40	FT	GWR
$1. \times 10^{-1}$	3.6×10^{-13}	1.4×10^{-39}	1.8×10^{-36}	3.8×10^{-42}
$1. \times 10^{-2}$	3.6×10^{-12}	1.3×10^{-37}	1.8×10^{-35}	8.7×10^{-42}
$1. \times 10^{-3}$	3.4×10^{-11}	1.7×10^{-36}	1.8×10^{-34}	1.6×10^{-39}
$1. \times 10^{-4}$	1.4×10^{-10}	1.5×10^{-35}	1.8×10^{-33}	8.5×10^{-40}
$1. \times 10^{-5}$	5.2×10^{-10}	2.3×10^{-34}	1.8×10^{-32}	9.9×10^{-38}
$1. \times 10^{-6}$	4.5×10^{-8}	2.5×10^{-34}	1.8×10^{-31}	1.3×10^{-35}
time	0.0012 sec	0.0292 sec	0.1692 sec	0.7204 sec

Table 4: Relative error of the tail of compound Poisson distribution for the Gamma distribution with $\alpha = \gamma = 1/4$

$\psi(u)$	TWS14	TWS40	FT	GWR
$1. \times 10^{-1}$	3.8×10^{-13}	8.2×10^{-37}	1.8×10^{-36}	$5. \times 10^{-31}$
$1. \times 10^{-2}$	6.7×10^{-12}	6.8×10^{-37}	1.8×10^{-35}	7.6×10^{-27}
$1. \times 10^{-3}$	1.2×10^{-10}	8.1×10^{-37}	1.8×10^{-34}	1.5×10^{-26}
$1. \times 10^{-4}$	3.3×10^{-9}	1.3×10^{-35}	1.9×10^{-33}	7.9×10^{-26}
$1. \times 10^{-5}$	7.7×10^{-8}	1.4×10^{-34}	1.9×10^{-32}	8.1×10^{-26}
$1. \times 10^{-6}$	$9. \times 10^{-7}$	1.2×10^{-34}	1.9×10^{-31}	3.5×10^{-25}
time	0.0016 sec	0.02440 sec	0.1616 sec	0.8144 sec

Table 5: Relative error of the ruin probability for the gamma distribution with $\alpha = \gamma = 5/2$

$\bar{G}(u)$	TWS14	TWS40	FT	GWR
$1. \times 10^{-1}$	3.9×10^{-12}	1.1×10^{-34}	1.2×10^{-36}	$9. \times 10^{-40}$
$1. \times 10^{-2}$	3.5×10^{-10}	2.9×10^{-32}	1.2×10^{-35}	3.5×10^{-38}
$1. \times 10^{-3}$	3.4×10^{-9}	9.4×10^{-31}	1.2×10^{-34}	$2. \times 10^{-35}$
$1. \times 10^{-4}$	3.2×10^{-8}	2.4×10^{-29}	1.2×10^{-33}	4.8×10^{-34}
$1. \times 10^{-5}$	1.9×10^{-7}	$3. \times 10^{-28}$	1.2×10^{-32}	4.4×10^{-33}
$1. \times 10^{-6}$	2.4×10^{-6}	$3. \times 10^{-27}$	1.2×10^{-31}	3.8×10^{-29}
time	0.0016 sec	0.0324 sec	0.1756 sec	0.8276 sec

Table 6: Relative error of the tail of compound Poisson distribution for the gamma distribution with $\alpha = \gamma = 5/2$

$\psi(u)$	TWS14	TWS40	FT	GWR
$1. \times 10^{-1}$	$7. \times 10^{-13}$	$2. \times 10^{-27}$	8.6×10^{-32}	2.9×10^{-19}
$1. \times 10^{-2}$	1.1×10^{-11}	2.5×10^{-31}	1.9×10^{-41}	1.6×10^{-23}
$1. \times 10^{-3}$	6.1×10^{-10}	1.9×10^{-32}	$2. \times 10^{-40}$	1.6×10^{-28}
$1. \times 10^{-4}$	1.4×10^{-8}	3.4×10^{-34}	2.2×10^{-39}	5.3×10^{-28}
$1. \times 10^{-5}$	$2. \times 10^{-6}$	4.2×10^{-35}	2.5×10^{-38}	3.4×10^{-25}
$1. \times 10^{-6}$	9.6×10^{-5}	1.3×10^{-33}	2.2×10^{-37}	2.7×10^{-24}
time	0.0032 sec	0.1232 sec	0.6256 sec	0.7420 sec

Table 7: Relative error of the ruin probability for the Pareto distribution with $\alpha = 3$

at least $\hat{L}_\psi(s) \rightarrow 0$ for $s \rightarrow \infty$ and $\hat{L}_\psi(s)$ is meromorphic with infinitely many poles in \mathbb{D} , whose residuals go to zero exponentially fast as $u \rightarrow \infty$, cf. Albrecher & Kortschak [3]. Also, the limit at the negative real axis exists. Further one can show by carefully selecting the complex contour, that besides the problems of the poles of $\hat{L}_\psi(s)$, the TWS method can still be applied. As we see in Table 7 for the parameters $\alpha = 3$, $\lambda = 1$ and $c = 5/3$, the contribution of the poles of $\hat{L}_\psi(s)$ can indeed be neglected.

The situation is substantially different for the compound Poisson distribution with Pareto claims. In this case $\hat{L}_{\bar{G}}(s)$ is holomorphic in \mathbb{D} , but $\lim_{|s| \rightarrow \infty} |\hat{L}_{\bar{G}}(s)| \neq 0$. Hence one has to expect that the TWS and the FT method do not work well. Further we discovered that also the GWR method does not work as well here as in the other cases. In particular, we had to increase the working precision, which slowed down the evaluation. In Table 8, we used $\alpha = 3.3$ and $\lambda = 2$. Here we used $M = 16$ for the FT method (to save computation time), as in this case a higher value of M does need not necessarily improve the quality of the approximation, especially for small values of u . The same is true for the TWS method and parameter n .

4.5 Lognormal distribution

Finally, we consider a lognormal claim size distribution. As was shown in Thorin & Wikstad [11], in this case

$$\bar{F}(x) = \int_0^\infty e^{-xt} f_\mu(x) dx.$$

$\bar{G}(u)$	TWS14	TWS40	FT	GWR
$1. \times 10^{-1}$	1.6×10^{-3}	1.8×10^{-5}	$4. \times 10^{2866892}$	8.8×10^{-5}
$1. \times 10^{-2}$	9.3×10^{-6}	1.5×10^1	9.4×10^{-6}	4.4×10^{-6}
$1. \times 10^{-3}$	$7. \times 10^{-8}$	9.4×10^{-10}	8.7×10^{-8}	4.3×10^{-7}
$1. \times 10^{-4}$	6.7×10^{-8}	6.1×10^{-20}	2.8×10^{-9}	$6. \times 10^{-10}$
$1. \times 10^{-5}$	3.2×10^{-7}	2.7×10^{-23}	6.3×10^{-7}	5.6×10^{-14}
$1. \times 10^{-6}$	$8. \times 10^{-7}$	5.9×10^{-26}	$2. \times 10^{-5}$	9.1×10^{-17}
time	0.0068 sec	0.2724 sec	0.1224 sec	0.8829 sec

Table 8: Relative error of the tail of compound Poisson distribution with Pareto claims ($\alpha = 3.3$)

$u \setminus c$	1.05	1.1	1.15	1.2	1.25	1.3	2.
100	0.550743	0.343954	0.235726	0.173086	0.133839	0.107647	0.0253454
1000	0.0419949	0.0109919	0.0057413	0.0038406	0.0028796	0.0023021	0.0006037
10000	0.0000812	0.0000376	0.0000244	0.0000181	0.0000144	0.0000119	3.5×10^{-6}

Table 9: Ruin probability for the lognormal distribution with $\mu = -1.62$ and $\sigma = 1.8$ and different c .

where f_μ is a real-valued continuously differentiable function (but not positive, so the lognormal distribution is not completely monotone). In principle, one can apply the theory developed in Section 3 as well, but one can not ensure the absence of poles. Nevertheless, the main problem in this case is that the Laplace transform $\hat{L}_f(s)$ is not known explicitly. In this case we only considered the TWS method with $n = 14$. To evaluate $\hat{L}_f(s)$, we used

$$\hat{L}_f(s) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-e^x} e^{-\frac{(x-\log(s)-\mu)^2}{2\sigma^2}} dx,$$

and numerical evaluation of this integral. Since we need a high accuracy of $\hat{L}_f(s)$, this slows down the procedure significantly. We use the example of [11], with $\mu = -1.62$, $\sigma = 1.8$ and $\lambda = 1$ and in Table 9 it is shown that the results of [11] are replicated by the TWS method. Table 9 shows that indeed in absence of an explicit formula for the Laplace transform, the computation time is much slower than in the previous cases. The TWS method also works quite well in this case and is, at a much lower computational cost, already satisfactorily accurate.

5 Conclusion

Quick numerical evaluations of ruin probabilities and tails of aggregate claims are a challenge in the presence of heavy tails. In this paper we showed that a quadrature method proposed by Trefethen et al. [12] is well-suited for the evaluation of the integrals that appear in the inverse Laplace transform of these quantities. For arbitrary quantities who can be expressed as the Laplace transform of a signed measure μ , we gave a theoretical justification of this approach by establishing useful error bounds. A special case for which

$\psi(u)$	TWS14	TWS40	FT	GWR
$1. \times 10^{-1}$	5.7×10^{-13}	$1. \times 10^{-38}$	1.7×10^{-36}	5.8×10^{-37}
$1. \times 10^{-2}$	2.6×10^{-13}	$1. \times 10^{-37}$	1.7×10^{-35}	1.2×10^{-28}
$1. \times 10^{-3}$	7.3×10^{-12}	1.1×10^{-36}	1.7×10^{-34}	5.4×10^{-29}
$1. \times 10^{-4}$	1.3×10^{-9}	1.1×10^{-35}	1.5×10^{-33}	5.4×10^{-25}
$1. \times 10^{-5}$	1.2×10^{-8}	1.1×10^{-34}	1.1×10^{-32}	3.3×10^{-23}
$1. \times 10^{-6}$	4.8×10^{-8}	1.1×10^{-33}	6.6×10^{-32}	3.6×10^{-27}
time	7 sec	135 sec	406 sec	259 sec

Table 10: Relative error of the ruin probability for the Lognormal distribution with $c = 1.2$

this situation applies is the calculation of ruin probabilities and tails of compound Poisson aggregate claim tails for completely monotone claim distributions. For general claim size distributions, such bounds are not available, but if an explicit formula for the Laplace transform is available, this method can still work, as some of our numerical illustrations indicate.

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