

Efficient simulation of ruin probabilities when claims are mixtures of heavy and light tails

Hansjörg Albrecher *
hansjoerg.albrecher@unil.ch

Martin Bladt *
martin.bladt@unil.ch

Eleni Vatamidou *
eleni.vatamidou@unil.ch

Abstract

We consider the classical Cramér-Lundberg risk model with claim sizes that are mixtures of phase-type and subexponential variables. Exploiting a specific geometric compound representation, we propose control variate techniques to efficiently simulate the ruin probability in this situation. The resulting estimators perform well for both small and large initial capital. We quantify the variance reduction as well as the efficiency gain of our method over another fast standard technique based on the classical Pollaczek-Khinchine formula. We provide a numerical example to illustrate the performance, and show that for more time-consuming conditional Monte Carlo techniques, the new series representation also does not compare unfavorably to the one based on the Pollaczek-Khinchine formula.

Keywords: rare event simulation; ruin probability; Cramér-Lundberg model; insurance risk theory

1. Introduction

The study of ruin probabilities for insurance risk models is a classical topic in applied probability, see e.g. [16]. Explicit formulas for ruin probabilities are available only in specific situations. One such instance is the classical Cramér-Lundberg risk model when claim sizes are of phase-type, see e.g. [3] for more details. However, the tail of such phase-type distributions is exponentially bounded [14], whereas insurance data often suggest heavy tails [1]. In the presence of heavy tails one then typically has to resort to approximations or simulations, and to achieve accuracy for either of the two can be challenging. While highly efficient simulation techniques for ruin probabilities for exponentially bounded claims are available for a long time already (e.g. using Lundberg conjugation [3, Ch.XV]), the field of efficient simulation for heavy tails has only advanced significantly in more recent years and is an active field of research (cf. [5, 11, 12, 15] and [4] for an overview).

Among the many possible modelling approaches for insurance claim sizes, in this paper we will be interested in mixture models, where with a certain probability ϵ a new claim is of a heavy-tailed type and with probability $1 - \epsilon$ it is of a certain light-tailed type. Such a co-existence of heavy and light tails is very intuitive in practice, see e.g. [13, 18]. For small ϵ , [19] used a perturbation approach to devise a numerical approximation scheme for the determination of ruin probabilities in the presence of heavy tails in the spirit of corrected phase-type approximations. Their approach relied on an alternative representation of the Pollaczek-Khinchine (PK) formula that converges more quickly as $\epsilon \rightarrow 0$, see also

*The Faculty of Business and Economics, University of Lausanne, Quartier UNIL-Chamberonne Bâtiment Extranef, 1015 Lausanne, Switzerland

[10]. Inspired by this approach, in this paper we want to study the potential of such an alternative representation for general mixture models and not necessarily small ϵ . The focus here will be to see whether large claim approximations can be used more efficiently as control variates in a simulation procedure than for algorithms based on the classical PK formula. We will show both theoretically and in a numerical implementation that this is indeed the case. The results in principle apply to any situation where claim sizes are a mixture between a tractable light-tailed and a heavy-tailed distribution for which the convolution of the two can be calculated explicitly. Moreover, even if the latter convolution can not be evaluated explicitly, the series representation can be advantageous.

We will also study the performance of the alternative series representation for a conditional Monte Carlo method developed by Asmussen & Kroese [6]. The latter can be applied to the PK formula and leads to a significant reduction of variance for the ruin probability estimator, but at a considerable additional computational cost. It will turn out that for this case, our series representation has no significant advantage over the classical PK approach, but the performance is not worse either.

The rest of the paper is organised as follows. Section 2 describes the risk model based on the mixture of light- and heavy-tailed claims and provides some preliminaries. In Section 3, we then construct a new control variate estimator for the ruin probability based on subexponential properties, which can exploit the advantage of exact ruin probability formulas for the light-tailed component in the mixture. We provide error bounds, investigate the tail behaviour, and quantify the resulting variance reduction when using the control variates, as well as the advantage of our approach to the analogous one based on the PK formula. We also consider the introduction of this alternative series representation for a conditional Monte Carlo framework in the spirit of [6]. In Section 4, we then perform numerical experiments and analyse the results. Finally, we conclude in Section 5.

2. Model description and preliminaries

We start with a short description of phase-type and subexponential distributions in Sections 2.1 and 2.2, as they are building blocks for the risk model of this paper, which is introduced in Section 2.3.

2.1 Phase-type distributions

Consider a state space $E = \{1, 2, \dots, p, p+1\}$ and a Markov jump process $\{X_t\}_{t \geq 0}$ evolving on E . Assume that the first p states are transient and the last remaining state $p+1$ is absorbing. The intensity matrix of this process is given by

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{T} & \mathbf{t} \\ \mathbf{0} & 0 \end{pmatrix},$$

where \mathbf{T} is a $p \times p$ -dimensional sub-intensity matrix, and it consists of the jump rates between the transient states. The initial distribution of $\{X_t\}_{t \geq 0}$ on the transient states $1, \dots, p$ is defined by the vector $\boldsymbol{\pi} = (\pi_1, \dots, \pi_p)$ with $\pi_k = \mathbb{P}(X_0 = k)$ for $k = 1, \dots, p$. Let \mathbf{e} be a p -dimensional column vector of 1's and $\mathbf{t} = -\mathbf{T}\mathbf{e}$. A phase-type distribution is then defined as the absorption time τ of X_t , that is, $\tau := \inf\{t > 0 | X_t = p+1\}$ follows a phase-type distribution with parameters $\boldsymbol{\pi}, \mathbf{T}$.

Phase-type distributions are natural and tractable extensions of the exponential distribution (which is retrieved for $p = 1$), in the sense that their density and distribution functions are explicitly given for

$x > 0$ by the formulae

$$f(x) = \boldsymbol{\pi} \exp(\mathbf{T}x)\mathbf{t} \quad \text{and} \quad F(x) = 1 - \boldsymbol{\pi} \exp(\mathbf{T}x)\mathbf{e},$$

where the exponential of a matrix M is defined as

$$\exp(M) = \sum_{n=0}^{\infty} \frac{M^n}{n!}.$$

The class of phase-type distributions has various attractive properties (it is e.g. closed under mixing, convolutions, exceedances, ordering etc.) and for phase-type distributed insurance claims there exist explicit formulas for ruin probabilities in a number of models (see [3, Ch.IX] for details). In addition, the class is dense (in the sense of weak convergence) among all distributions on the positive real line, so that in principle one may approximate any distribution arbitrarily well with a phase-type distribution. However, by construction phase-type distributions have an exponentially bounded tail, which is often too restrictive in applications.

2.2 Subexponential distributions

In many situations, distributions with a tail heavier than exponential are a better description of the data. Among these, an important subclass is the one of subexponential distributions \mathcal{S} , i.e. for any $n \in \mathbb{N}$,

$$\overline{F^{*n}}(u) \sim n\overline{F}(u), \quad \text{as } u \rightarrow \infty, \quad (1)$$

where $\overline{F}(u) = 1 - F(u)$ is the tail of the underlying distribution function F , see e.g. [17]. This mathematical definition is built around the intuition that the tail behavior of sums of independent such random variables is determined by the largest among them. The class of subexponential distributions comprises most heavy-tailed distributions of practical interest (including the Lognormal, Pareto, and heavy-tailed Weibull distribution). However, for exact calculations this class is not amenable and one typically has to resort to simulation in order to determine quantities like ruin probabilities with subexponential claims, and the latter is known to be challenging due to the rare event character (cf. [3, Ch.XV]). In the sequel, we will need the following well-known asymptotic property of subexponential distributions (see e.g. [9, Cor.3.18] or [3, Cor.X.1.11]):

Property 2.1. Let $F \in \mathcal{S}$ and let A be any distribution with a lighter tail, i.e. $\overline{A}(u) = o(\overline{F}(u))$. Then for the convolution $A * F$ of A and F we have $A * F \in \mathcal{S}$ and $\overline{(A * F)}(u) \sim \overline{F}(u)$.

2.3 The model

Consider the classical Cramér-Lundberg risk model for the surplus process of an insurance portfolio. The premium inflow is assumed at a constant rate (w.l.o.g. 1 per unit time) and claims arrive according to a homogeneous Poisson process $\{N(t)\}_{t \geq 0}$ with rate λ . The claim sizes $U_k \stackrel{\mathcal{D}}{=} U$ are i.i.d. with common distribution function G , and are independent of $\{N(t)\}$. If u is the initial capital, the surplus at time t is then given by

$$R(t) = u + t - \sum_{k=1}^{N(t)} U_k.$$

We also define the claim surplus process $S(t) = u - R(t)$ and its maximum $M = \sup_{0 \leq t < \infty} S(t)$. The probability $\psi(u)$ of ultimate ruin is then

$$\psi(u) = \mathbb{P}(M > u). \quad (2)$$

In addition, we assume that the safety loading condition $\rho = \lambda \mathbb{E}U < 1$ holds and thus the well-known Pollaczek-Khinchine (PK) formula

$$1 - \psi(u) = (1 - \rho) \sum_{k=0}^{\infty} \rho^k (G^e)^{*k}(u) \quad (3)$$

can be used for the evaluation of the ruin probability. Here $G^e(u) = \int_0^u (1 - G(x)) dx / \mathbb{E}U$ is the distribution function of the stationary excess claim size U^e , see e.g. [3].

In this paper, we assume that claim sizes are of a mixture type. Concretely, U is phase-type with probability $1 - \epsilon$ and heavy-tailed (subexponential) with probability ϵ , where $\epsilon \in (0, 1)$. The phase-type claim sizes $B_k \stackrel{\mathfrak{D}}{=} B$ and the subexponential claim sizes $C_k \stackrel{\mathfrak{D}}{=} C$ are both assumed to have finite means μ_B and μ_C , respectively. Denote by $\tilde{G}^e(s)$, $\tilde{F}_p^e(s)$, and $\tilde{F}_h^e(s)$ the Laplace transforms of the stationary excess claim sizes $U_k^e \stackrel{\mathfrak{D}}{=} U^e$, $B_k^e \stackrel{\mathfrak{D}}{=} B^e$, and $C_k^e \stackrel{\mathfrak{D}}{=} C^e$, respectively. Moreover, we set $\delta := \lambda \mu_B$ and $\theta := \lambda \mu_C$, which means that the phase-type and heavy-tailed claims are responsible for expected aggregate claim size $(1 - \epsilon)\delta$ and $\epsilon\theta$ per unit time, respectively. The expected overall aggregate claim size is then given by $\rho = (1 - \epsilon)\delta + \epsilon\theta$. In terms of Laplace transforms, the Pollaczek-Khinchine formula can be written as

$$\mathbb{E}e^{-sM} = (1 - \rho) \sum_{k=0}^{\infty} \rho^k (\tilde{G}^e(s))^k = \frac{1 - \rho}{1 - \rho \tilde{G}^e(s)} = \frac{1 - (1 - \epsilon)\delta - \epsilon\theta}{1 - (1 - \epsilon)\delta \tilde{F}_p^e(s) - \epsilon\theta \tilde{F}_h^e(s)}. \quad (4)$$

Using representation (4), it was shown in [19] that $\psi(u)$ can be expressed as a series expansion involving the ruin probability of a risk process with purely phase-type claim sizes (base model). One easy way to establish a phase-type base model is by simply considering that $G(x) = (1 - \epsilon)F_p(x) + \epsilon$, $x \geq 0$, i.e. discard all heavy-tailed claim sizes. This base model, for which the claim size distribution has an atom at zero, is equivalent to the compound Poisson risk model in which claims arrive at rate $(1 - \epsilon)\lambda$ and follow the distribution of B . We denote by M^\bullet the supremum of its corresponding claim surplus process and we set $\rho^\bullet = (1 - \epsilon)\delta$. The PK formula for this base model takes the form

$$\mathbb{E}e^{-sM^\bullet} = \frac{1 - \rho^\bullet}{1 - \rho^\bullet \tilde{F}_p^e(s)}. \quad (5)$$

We denote by $\psi^\bullet(u)$ the phase-type approximation of $\psi(u)$ that is obtained when we apply Laplace inversion to (5). The following series expansion of $\psi(u)$ for the general risk process was shown in [19, Th.1]. In order to keep this paper self-contained, we repeat the short proof here in the present notation.

Theorem 2.2 ([19]). *We have*

$$\psi(u) = \frac{1 - \rho}{1 - \rho^\bullet} \psi^\bullet(u) + \frac{1 - \rho}{1 - \rho^\bullet} \sum_{k=1}^{\infty} \left(\frac{\epsilon\theta}{1 - \rho^\bullet} \right)^k \mathcal{A}_k(u), \quad (6)$$

where $\mathcal{A}_k(u) = \mathbb{P}(M_0^\bullet + M_1^\bullet + \dots + M_k^\bullet + C_1^e + \dots + C_k^e > u)$ and $M_k^\bullet \stackrel{\mathfrak{D}}{=} M^\bullet$. This expansion converges

for all values of u .

Proof. It can easily be derived that $U^e = IB_k^e + (1-I)C_k^e$, where $I \sim \text{Bernoulli}(\rho^\bullet/(\rho^\bullet + \epsilon\theta))$. Therefore $\tilde{G}^e(s) = \frac{\rho^\bullet}{\rho^\bullet + \epsilon\theta} \tilde{F}_p^e(s) + \frac{\epsilon\theta}{\rho^\bullet + \epsilon\theta} \tilde{F}_h^e(s)$, and we find by virtue of the binomial identity

$$(\tilde{G}^e(s))^\ell = \frac{1}{(\rho^\bullet + \epsilon\theta)^\ell} \sum_{k=0}^{\ell} \binom{\ell}{k} (\rho^\bullet)^{\ell-k} (\tilde{F}_p^e(s))^{\ell-k} (\epsilon\theta)^k (\tilde{F}_h^e(s))^k.$$

Combining [Equations \(4\)](#) and [\(5\)](#), we get

$$\begin{aligned} \mathbb{E}e^{-sM} &= (1 - \rho^\bullet - \epsilon\theta) \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} \binom{\ell}{k} (\rho^\bullet)^{\ell-k} (\tilde{F}_p^e(s))^{\ell-k} (\epsilon\theta)^k (\tilde{F}_h^e(s))^k \\ &= (1 - \rho^\bullet - \epsilon\theta) \sum_{k=0}^{\infty} (\epsilon\theta)^k (\tilde{F}_h^e(s))^k \sum_{\ell=k}^{\infty} \binom{\ell}{k} (\rho^\bullet)^{\ell-k} (\tilde{F}_p^e(s))^{\ell-k} \\ &= (1 - \rho^\bullet - \epsilon\theta) \sum_{k=0}^{\infty} (\epsilon\theta)^k (\tilde{F}_h^e(s))^k \frac{1}{(1 - \rho^\bullet \tilde{F}_p^e(s))^{k+1}} \\ &= (1 - \rho^\bullet - \epsilon\theta) \sum_{k=0}^{\infty} (\epsilon\theta)^k (\tilde{F}_h^e(s))^k \frac{1}{(1 - \rho^\bullet)^{k+1}} (\mathbb{E}e^{-sM^\bullet})^{k+1} \\ &= \frac{1 - \rho}{1 - \rho^\bullet} \sum_{k=0}^{\infty} \left(\frac{\epsilon\theta}{1 - \rho^\bullet} \right)^k (\tilde{F}_h^e(s))^k (\mathbb{E}e^{-sM^\bullet})^{k+1}. \end{aligned}$$

We obtain the provided series expansion for $\psi(u)$ via Laplace inversion and using $\psi^\bullet(u) = \mathbb{P}(M_0^\bullet > u)$. The convergence is granted by $|\mathbb{E}e^{-sM^\bullet}| \leq 1$ and $|\tilde{F}_h^e(s)| \leq 1$, while $\epsilon\theta < 1 - \rho^\bullet$ due to the stability condition $\rho < 1$. \square

[Theorem 2.2](#) provides an alternative interpretation for M , i.e. $M \stackrel{\mathcal{D}}{=} \sum_{k=0}^N (M_k^\bullet + C_k^e)$, where $C_0^e := 0$ and N is a geometric random variable $N \sim \text{Geom}\left(\frac{1-\rho}{1-\rho^\bullet}\right)$. In general, the term corresponding to $k = 0$ is explicit. Note that for various subexponential distributions associated with C_k^e , the term corresponding to $k = 1$ in [\(6\)](#) can be also calculated explicitly, so that

$$\begin{aligned} \psi(u) &= \underbrace{\frac{1 - \rho}{1 - \rho^\bullet} \psi^\bullet(u) + \frac{1 - \rho}{1 - \rho^\bullet} \frac{\epsilon\theta}{1 - \rho^\bullet} \mathbb{P}(M_0^\bullet + M_1^\bullet + C_1^e > u)}_{\text{explicit}} \\ &\quad + \frac{1 - \rho}{1 - \rho^\bullet} \sum_{k=2}^{\infty} \left(\frac{\epsilon\theta}{1 - \rho^\bullet} \right)^k \mathcal{A}_k(u). \end{aligned}$$

Thus, to approximate $\psi(u)$, we only need to have an estimate for

$$\begin{aligned} \varphi(u) &:= \frac{1 - \rho}{1 - \rho^\bullet} \sum_{k=2}^{\infty} \left(\frac{\epsilon\theta}{1 - \rho^\bullet} \right)^k \mathcal{A}_k(u) = \left(\frac{\epsilon\theta}{1 - \rho^\bullet} \right)^2 \mathbb{E} \mathcal{A}_{N+2}(u) \\ &= \left(\frac{\epsilon\theta}{1 - \rho^\bullet} \right)^2 \mathbb{P}(M_0^\bullet + M_1^\bullet + \dots + M_{N+2}^\bullet + C_1^e + \dots + C_{N+2}^e > u), \end{aligned} \tag{7}$$

which we want to approximate by simulating the tail of

$$V \stackrel{\mathcal{D}}{=} M_0^\bullet + M_1^\bullet + C_1^e + \sum_{k=2}^{N+2} (M_k^\bullet + C_k^e), \quad (8)$$

with $N \sim \text{Geom}\left(\frac{1-\rho}{1-\rho^\bullet}\right)$.

Using the above representation, we propose in [Section 3](#) efficient variance reduction techniques for this simulation based on suitably chosen control variates.

3. Control variate techniques

Let $Z(u)$ be the random variable we must simulate in order to calculate its expectation $\varphi(u) = \mathbb{E}Z(u)$. The idea of a control variate is to use another random variable $W(u)$, which has a known expectation $\mathbb{E}W(u)$ and is strongly correlated with $Z(u)$. Thus, the deviation of the simulated from the exact value of $W(u)$ may be used for improving the simulation accuracy for $Z(u)$. If $(Z^{(i)}(u), W^{(i)}(u))$, $i = 1, 2, \dots, \kappa$, are independent copies of $(Z(u), W(u))$, then an efficient control variate estimator is defined as

$$\hat{\varphi}_\kappa(u) := \hat{z}_\kappa(u) + \hat{\alpha}_\kappa(\hat{w}_\kappa(u) - \mathbb{E}W(u)), \quad (9)$$

where

$$\hat{z}_\kappa(u) = \frac{\sum_{i=1}^{\kappa} Z^{(i)}(u)}{\kappa}, \quad \hat{w}_\kappa(u) = \frac{\sum_{i=1}^{\kappa} W^{(i)}(u)}{\kappa}, \quad \hat{\alpha}_\kappa = -\frac{\sum_{i=1}^{\kappa} (Z^{(i)}(u) - \hat{z}_\kappa(u))(W^{(i)}(u) - \hat{w}_\kappa(u))}{\sum_{i=1}^{\kappa} (W^{(i)}(u) - \hat{w}_\kappa(u))^2}. \quad (10)$$

Note that this choice of $\hat{\alpha}_\kappa$ based on the empirical correlation of $Z(u)$ and $W(u)$ optimizes the variance gain, see e.g. [\[1, 4\]](#). We assume now that the distribution of C^e belongs to the class of subexponential distributions satisfying [\(1\)](#). The construction of the concrete $W(u)$ below is inspired by [Property 2.1](#) given in [Section 2.2](#). That is, for sufficiently large u , only the maximum of the subexponential claims will substantially contribute to the probability in [\(7\)](#).

3.1 Max of heavy tails

It is immediately obvious from [Equation \(7\)](#) that we may take

$$Z(u) = \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^2 \mathbb{1}_{\{V>u\}}, \quad (11)$$

and this variable will have the desired mean $\mathbb{E}Z(u) = \varphi(u)$. We also define, for fixed $n \in \mathbb{N}$, the random variable

$$V_n := \max\{C_1^e, \dots, C_{N+2}^e\} \mathbb{1}_{\{N+2 \leq n\}}, \quad (12)$$

which will serve as a component of the control variate of $Z(u)$.

Definition 3.1. For a fixed $n \in \mathbb{N}$, define the control variate

$$W_n(u) = \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^2 \mathbb{1}_{\{V_n>u\}}. \quad (13)$$

The n th order approximation $\varphi_n(u) = \mathbb{E}W_n(u)$ of $\varphi(u)$ is then

$$\varphi_n(u) = \left(\frac{1-\rho}{1-\rho^\bullet}\right) \sum_{k=2}^n \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^k \mathbb{P}(\max\{C_1^e, \dots, C_k^e\} > u). \quad (14)$$

By construction, $\varphi_n(u)$ underestimates $\varphi(u)$. Next we collect some properties of this approximation.

3.1.1 Properties of the approximation

The following lower and upper bounds for the approximation error can be obtained.

Proposition 3.2 (Error bounds). *The error of the approximation $\varphi_n(u)$, $n \in \mathbb{N}$, is bounded from above and below as follows:*

$$\begin{aligned} \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^{n+1} \mathcal{A}_1(u) &\leq \varphi(u) - \varphi_n(u) \leq \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^{n+1} \\ &\quad + \left(1 - \frac{\epsilon\theta}{1-\rho^\bullet}\right) \left(\frac{\epsilon\theta}{1-\rho^\bullet} F_h^e(u)\right)^2 \frac{1 - \left(\frac{\epsilon\theta}{1-\rho^\bullet} F_h^e(u)\right)^{n-1}}{1 - \frac{\epsilon\theta}{1-\rho^\bullet} F_h^e(u)}. \end{aligned}$$

Proof. For simplicity of notation, we set $p := \frac{\epsilon\theta}{1-\rho^\bullet}$. The error of the approximation is equal to

$$\begin{aligned} \varphi(u) - \varphi_n(u) &= (1-p) \sum_{k=2}^{\infty} p^n \mathcal{A}_k(u) - (1-p) \sum_{k=2}^n p^k \mathbb{P}(\max\{C_1^e, \dots, C_k^e\} > u) \\ &= (1-p) \sum_{k=2}^n p^k \left(\mathcal{A}_k(u) - \mathbb{P}(\max\{C_1^e, \dots, C_k^e\} > u)\right) + (1-p) \sum_{k=n+1}^{\infty} p^k \mathcal{A}_k(u). \end{aligned}$$

For the upper bound, we use $\mathbb{P}(\max\{C_1^e, \dots, C_k^e\} > u) = 1 - (F_h^e(u))^k$ and $\mathcal{A}_k(u) \leq 1$ to obtain

$$\varphi(u) - \varphi_n(u) \leq (1-p) \sum_{k=2}^n p^k \mathbb{P}(\max\{C_1^e, \dots, C_k^e\} \leq u) + (1-p) \sum_{k=n+1}^{\infty} p^k = p^{n+1} + (1-p) \sum_{k=2}^n (p F_h^e(u))^k.$$

For the lower bound, we take $\mathcal{A}_k(u) \geq \mathbb{P}(\max\{C_1^e, \dots, C_k^e\} > u)$ when $k \leq n$ and $\mathcal{A}_k(u) \geq \mathcal{A}_1(u)$ otherwise, to calculate

$$\varphi(u) - \varphi_n(u) \geq (1-p) \sum_{k=n+1}^{\infty} p^k \mathcal{A}_1(u) = p^{n+1} \mathcal{A}_1(u),$$

and the proof is complete. \square

Proposition 3.3 (Tail behaviour). *For $C^e \in \mathcal{S}$, the n th approximation*

$$\psi_n(u) := \frac{1-\rho}{1-\rho^\bullet} \psi^\bullet(u) + \frac{1-\rho}{1-\rho^\bullet} \frac{\epsilon\theta}{1-\rho^\bullet} \mathbb{P}(M_0^\bullet + M_1^\bullet + C_1^e > u) + \varphi_n(u)$$

of the target ruin probability $\psi(u)$ has the following tail behaviour:

$$\psi_n(u) \sim \frac{\epsilon\theta}{1-\rho} \left(1 - (n+1) \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^n + n \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^{n+1}\right) \overline{F}_h^e(u), \quad u \rightarrow \infty.$$

Proof. The approximation $\psi^\bullet(u)$ has a phase-type representation; therefore, it is of order $o(\overline{F}_h^e(u))$. The same holds for the tail of the distribution of $M_0^\bullet + M_1^\bullet$. Moreover, since $C^e \in \mathcal{S}$, from [Property 2.1](#) we obtain $\mathbb{P}(M_0^\bullet + M_1^\bullet + C_1^e > u) \sim \overline{F}_h^e(u)$. Finally, from $\mathbb{P}(\max\{C_1^e, \dots, C_n^e\} > u) \leq \mathbb{P}(C_1^e + \dots + C_n^e > u)$ and [\(1\)](#), we deduce that $\mathbb{P}(\max\{C_1^e, \dots, C_n^e\} > u) \sim n\overline{F}_h^e(u)$, which leads to the following result by inserting these asymptotic estimates into [Definition 3.1](#):

$$\begin{aligned} \psi_n(u) &\sim \left(1 - \frac{\epsilon\theta}{1 - \rho^\bullet}\right) \sum_{k=1}^n k \left(\frac{\epsilon\theta}{1 - \rho^\bullet}\right)^k \overline{F}_h^e(u) = \frac{\frac{\epsilon\theta}{1 - \rho^\bullet} \left(1 - (n+1) \left(\frac{\epsilon\theta}{1 - \rho^\bullet}\right)^n + n \left(\frac{\epsilon\theta}{1 - \rho^\bullet}\right)^{n+1}\right)}{1 - \frac{\epsilon\theta}{1 - \rho^\bullet}} \overline{F}_h^e(u) \\ &= \frac{\epsilon\theta}{1 - \rho} \left(1 - (n+1) \left(\frac{\epsilon\theta}{1 - \rho^\bullet}\right)^n + n \left(\frac{\epsilon\theta}{1 - \rho^\bullet}\right)^{n+1}\right) \overline{F}_h^e(u). \end{aligned}$$

□

[Proposition 3.3](#) (in comparison with [Theorem 5](#) in [\[19\]](#)) shows that $\psi_n(u)$ nearly captures the asymptotic behaviour of the exact ruin probability

$$\psi(u) \sim \frac{\epsilon\theta}{1 - \rho} \overline{F}_h^e(u), \quad (15)$$

being off by a factor $\left(1 - (n+1) \left(\frac{\epsilon\theta}{1 - \rho^\bullet}\right)^n + n \left(\frac{\epsilon\theta}{1 - \rho^\bullet}\right)^{n+1}\right) \in (0, 1)$. As expected, the tail of $\psi_n(u)$ underestimates the tail of $\psi(u)$.

3.1.2 Variance reduction

We consider now the bivariate simulation of i.i.d. copies of the random variables V and V_n :

$$(V^{(i)}, V_n^{(i)}), \quad i = 1, 2, \dots, \kappa. \quad (16)$$

For each fixed $n \in \mathbb{N}$, the estimator [\(9\)](#) takes the form

$$\hat{\varphi}_\kappa^n(u) := \hat{z}_\kappa(u) + \hat{\alpha}_\kappa(\hat{w}_\kappa(u) - \varphi_n(u)). \quad (17)$$

We can now establish our main result.

Theorem 3.4 (Variance reduction). *For each fixed $n \in \mathbb{N}$, the variance of the estimator [\(17\)](#) behaves asymptotically as*

$$\text{Var}(\hat{\varphi}_\kappa^n(u)) \sim \left(\frac{\epsilon\theta}{1 - \rho^\bullet}\right)^{n+3} \frac{1 + n \left(\frac{1 - \rho}{1 - \rho^\bullet}\right)}{\frac{1 - \rho}{1 - \rho^\bullet}} \cdot \frac{\overline{F}_h^e(u)}{\kappa}, \quad \text{as } u \rightarrow \infty \quad (18)$$

and satisfies

$$\frac{\text{Var}(\hat{\varphi}_\kappa^n(u))}{\text{Var}(\hat{z}_\kappa(u))} \rightarrow \left(\frac{\epsilon\theta}{1 - \rho^\bullet}\right)^{n-1} \frac{1 + n \left(\frac{1 - \rho}{1 - \rho^\bullet}\right)}{1 + \frac{1 - \rho}{1 - \rho^\bullet}}, \quad \text{as } u \rightarrow \infty. \quad (19)$$

Proof. Since $\mathbb{E}\hat{z}_\kappa(u) = \varphi(u)$, we know from [\[4\]](#) that the proposed estimator has variance

$$\frac{1 - (r_n(u))^2}{\kappa} \text{Var}Z(u), \quad (20)$$

with correlation coefficient $r_n(u) = \text{Corr}(Z(u), W_n(u))$. By the definition of V_n , $\{V_n > u\} \subseteq \{V > u\}$ and consequently $\mathbb{1}_{\{V > u\}} \cdot \mathbb{1}_{\{V_n > u\}} = \mathbb{1}_{\{V_n > u\}}$. We calculate,

$$\begin{aligned} \text{Cov}(Z(u), W_n(u)) &= \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^4 \text{Cov}\left(\mathbb{1}_{\{V > u\}} \cdot \mathbb{1}_{\{V_n > u\}}\right) \\ &= \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^4 \left(\mathbb{E}\left(\mathbb{1}_{\{V > u\}} \mathbb{1}_{\{V_n > u\}}\right) - \mathbb{E}\mathbb{1}_{\{V > u\}} \mathbb{E}\mathbb{1}_{\{V_n > u\}}\right) \\ &= \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^4 \left(\mathbb{P}(V_n > u) - \mathbb{P}(V > u)\mathbb{P}(V_n > u)\right) \\ &= \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^4 \mathbb{P}(V_n > u)\mathbb{P}(V \leq u). \end{aligned}$$

Similarly, we find

$$\begin{aligned} \text{Var}(Z(u)) &= \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^4 \mathbb{P}(V > u)\mathbb{P}(V \leq u), \text{ and} \\ \text{Var}(W_n(u)) &= \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^4 \mathbb{P}(V_n > u)\mathbb{P}(V_n \leq u). \end{aligned}$$

Hence, it is immediate that

$$1 - (r_n(u))^2 = \frac{1 - \mathbb{P}(V_n > u)/\mathbb{P}(V > u)}{1 - \mathbb{P}(V_n > u)}. \quad (21)$$

Following [Proposition 3.3](#), we calculate

$$\begin{aligned} \mathbb{P}(V_n > u) &\sim \left(1 - \frac{\epsilon\theta}{1-\rho^\bullet}\right) \sum_{k=2}^n k \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^{k-2} \overline{F}_h^\epsilon(u) \\ &= \frac{2 - \frac{\epsilon\theta}{1-\rho^\bullet} - (n+1) \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^{n-1} + n \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^n}{1 - \frac{\epsilon\theta}{1-\rho^\bullet}} \overline{F}_h^\epsilon(u) \end{aligned}$$

and

$$\mathbb{P}(V > u) \sim \frac{2 - \frac{\epsilon\theta}{1-\rho^\bullet}}{1 - \frac{\epsilon\theta}{1-\rho^\bullet}} \overline{F}_h^\epsilon(u),$$

as $u \rightarrow \infty$. We finally obtain

$$\frac{\mathbb{P}(V_n > u)}{\mathbb{P}(V > u)} \rightarrow 1 - \frac{(n+1) \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^{n-1} - n \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^n}{2 - \frac{\epsilon\theta}{1-\rho^\bullet}},$$

so that

$$1 - (r_n(u))^2 \rightarrow \frac{(n+1) \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^{n-1} - n \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^n}{2 - \frac{\epsilon\theta}{1-\rho^\bullet}},$$

and the statement of the theorem follows. \square

The above theorem quantifies the asymptotic variance reduction for fixed n as u increases, this reduction being arbitrarily large when n is increased sufficiently.

3.2 Conditional Monte Carlo

While the approach of [Section 3.1](#) is the focus of this paper, for purposes of comparison and completeness we are also interested in the performance of the alternative series representation for the conditional Monte Carlo estimate and its variance reduction proposed in [\[6\]](#). To that end, let us recap here its idea and present its application to our series representation. Define $X_0^* = M_0^\bullet$ and $X_k = M_k^\bullet + C_k^e$, $k = 1, 2, \dots$, so that $V \stackrel{\text{D}}{=} X_0^* + \sum_{k=1}^{N+2} X_k$, where $N \sim \text{Geom}\left(\frac{1-\rho}{1-\rho^\bullet}\right)$ as before. [Equation \(7\)](#) can then be written as

$$\varphi(u) = \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^2 \mathbb{P}(X_0^* + X_1 + \dots + X_{N+2} > u).$$

Note that for fixed $k \geq 1$ and $m_k := \max\{X_1, \dots, X_k\}$, we have

$$\begin{aligned} \mathbb{P}(X_0^* + X_1 + \dots + X_k > u) &= k \mathbb{P}(S_k > u - X_0^*, X_k = m_k) \\ &= k \mathbb{P}(X_k > m_{k-1}, X_k > u - X_0^* - S_{k-1}) = k \mathbb{E}\bar{F}_X(m_{k-1} \vee (u - X_0^* - S_{k-1})), \end{aligned}$$

where \bar{F}_X is the common c.c.d.f. of the X_k 's and $S_\ell = \sum_{k=1}^\ell X_k$, $S_0 = 0$. Consequently, the random variable

$$Z^*(u) = \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^2 (N+2)\bar{F}_X(m_{N+1} \vee (u - X_0^* - S_{N+1})),$$

has the target probability $\varphi(u)$ as its expectation. Notice that this variable plays the same role as $Z(u)$ in the previous approach.

We can further introduce $N\bar{F}_X(u)$ as a control variate for the number of summands (see e.g. [\[11\]](#)).

Definition 3.5. We use the control variate

$$W^*(u) = \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^2 (N+2)\bar{F}_X(u).$$

The resulting approximation $\varphi^*(u) = \mathbb{E}W^*(u)$ of $\varphi(u)$ then is

$$\varphi^*(u) := \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^2 \left(\frac{\epsilon\theta}{1-\rho} + 2\right) \bar{F}_X(u).$$

This control variate leads to the following Asmussen-Kroese (AK)-type estimator:

$$\hat{\psi}_\kappa^*(u) := \hat{z}_\kappa^*(u) + \hat{\alpha}_\kappa^*(\hat{w}_\kappa^*(u) - \varphi^*(u)), \quad (22)$$

where $\hat{z}_\kappa^*(u)$, $\hat{w}_\kappa^*(u)$, and $\hat{\alpha}_\kappa^*$ are calculated via [\(10\)](#) using $Z^*(u)$ and $W^*(u)$.

Remark 3.6. An alternative approach is to set $X_1^* = M_0^\bullet + M_1^\bullet + C_1^e$ and $X_k = M_k^\bullet + C_k^e$, $k = 2, 3, \dots$

and write Equation (7) as

$$\varphi(u) = \left(\frac{\epsilon\theta}{1 - \rho^\bullet} \right)^2 \mathbb{P}(X_1^\star + X_2 + \dots + X_{N+2} > u).$$

Observe that all the random variables on the right hand side of this equation are heavy-tailed and independent, but not identically distributed. Thus, using the AK estimator for non i.i.d. random variables established in [8], we could instead construct a control variate based on the conditional Monte Carlo estimator

$$Z^\star(u) = \left(\frac{\epsilon\theta}{1 - \rho^\bullet} \right)^2 \left(\overline{F}_{X_1^\star}(m_{-1}^\star \vee (u - S_{N+1} + X_1)) + (N+1)\overline{F}_X(m_{-(N+2)}^\star \vee (u - S_N - X_0^\star)) \right),$$

where $m_{-1}^\star = \max\{X_2, \dots, X_{N+2}\}$ and $m_{-k}^\star = \max\{X_1^\star, X_2, \dots, X_{k-1}, X_{k+1}, X_{N+2}\}$.

3.3 Comparison with the Pollaczek-Khinchine expansion

For reference and the purpose of comparison, we also consider the estimators analogous to the ones in Sections 3.1 and 3.2 using the usual PK series expansion of the ruin probability in (3), which we rewrite as

$$\psi(u) = \underbrace{(1 - \rho)\overline{G^e}(u)}_{\text{explicit}} + \underbrace{(1 - \rho) \sum_{k=2}^{\infty} \rho^k (1 - (G^e)^{\ast k}(u))}_{:=\varphi^\circ(u)}.$$

Define the random variables $N^\circ \sim \text{Geom}(1 - \rho)$,

$$V^\circ = \sum_{k=1}^{N^\circ+2} U_k^e,$$

$$V_n^\circ = \max\{U_1^e, \dots, U_{N^\circ+2}^e\} \mathbb{1}_{\{N^\circ \leq n-2\}},$$

and let $S_n^\circ = \sum_{k=1}^n U_k^e$ as well as $m_k^\circ = \max\{U_1^e, \dots, U_k^e\}$. With this notation, the following equations define the analogous control variate estimators of $\varphi^\circ(u)$:

$$Z^\circ(u) = \rho^2 \mathbb{1}_{\{V^\circ > u\}} \quad Z^{\circ,\star}(u) = \rho^2 (N^\circ + 2) \overline{G^e}(m_{N^\circ+1}^\circ \vee (u - S_{N^\circ+1}^\circ))$$

$$W_n^\circ(u) = \rho^2 \mathbb{1}_{\{V_n^\circ > u\}} \quad W_n^{\circ,\star}(u) = \rho^2 (N^\circ + 2) \overline{G^e}(u),$$

and the associated empirical estimator

$$\hat{\varphi}_\kappa^{\circ,n}(u) := \hat{z}_\kappa^\circ(u) + \hat{\alpha}_\kappa^\circ(\hat{w}_\kappa^\circ(u) - \varphi_n^\circ(u)). \quad (23)$$

Observe now that the distributional behaviour of the variable U^e is slightly different from that of C_k^e . Recall that $U^e = IB_k^e + (1 - I)C_k^e$, where $I \sim \text{Bernoulli}(\rho^\bullet/(\rho^\bullet + \epsilon\theta))$. Hence,

$$\mathbb{P}(U^e > u) = \frac{\rho^\bullet}{\rho^\bullet + \epsilon\theta} \mathbb{P}(B^e > u) + \frac{\epsilon\theta}{\rho^\bullet + \epsilon\theta} \mathbb{P}(C^e > u) \sim \frac{\epsilon\theta}{\rho^\bullet + \epsilon\theta} \mathbb{P}(C^e > u),$$

as $u \rightarrow \infty$. Moreover, since C^e is subexponential, the above relation implies that U^e is subexponential

as well. Consequently,

$$\mathbb{P}(m_k^\circ > u) \sim k \frac{\epsilon\theta}{\rho^\bullet + \epsilon\theta} \mathbb{P}(C^e > u) = k \frac{\epsilon\theta}{\rho^\bullet + \epsilon\theta} \overline{F}_h^e(u).$$

Using the above asymptotic expression and following the proof of [Theorem 3.4](#), we obtain the next result.

Theorem 3.7. *For each fixed $n \in \mathbb{N}$, the variance of the estimator (23) behaves asymptotically as*

$$\text{Var}(\hat{\varphi}_\kappa^{\circ,n}(u)) \sim \rho^{n+3} \frac{1+n(1-\rho)}{1-\rho} \cdot \frac{\epsilon\theta}{\rho} \cdot \frac{\overline{F}_h^e(u)}{\kappa}, \quad \text{as } u \rightarrow \infty,$$

and satisfies

$$\frac{\text{Var}(\hat{\varphi}_\kappa^{\circ,n}(u))}{\text{Var}(\hat{z}_\kappa^\circ(u))} \rightarrow \rho^{n-1} \frac{1+n(1-\rho)}{1+(1-\rho)}, \quad \text{as } u \rightarrow \infty. \quad (24)$$

It follows that we can compare the asymptotic effect on the variance between the two different series expansions for the ruin probability, as well as the effect on the proportion of variance reduction due to the use of control variates:

Corollary 3.8. *For each fixed $n \in \mathbb{N}$, the following relations hold:*

$$\frac{\text{Var}(\hat{\varphi}_\kappa^n(u))}{\text{Var}(\hat{\varphi}_\kappa^{\circ,n}(u))} \sim \left[\frac{\epsilon\theta}{1-\rho^\bullet} / \rho \right]^{n+2} \frac{1+n\left(\frac{1-\rho}{1-\rho^\bullet}\right)}{1+n(1-\rho)}, \quad \text{as } u \rightarrow \infty, \quad (25)$$

and

$$\left[\frac{\text{Var}(\hat{\varphi}_\kappa^n(u))}{\text{Var}(\hat{z}_\kappa(u))} \right] \cdot \left[\frac{\text{Var}(\hat{\varphi}_\kappa^{\circ,n}(u))}{\text{Var}(\hat{z}_\kappa^\circ(u))} \right]^{-1} \rightarrow \left[\frac{\epsilon\theta}{1-\rho^\bullet} / \rho \right]^{n-1} \frac{1+n\left(\frac{1-\rho}{1-\rho^\bullet}\right)}{1+n(1-\rho)} \cdot \frac{1+(1-\rho)}{1+\left(\frac{1-\rho}{1-\rho^\bullet}\right)}, \quad \text{as } u \rightarrow \infty.$$

Notice that the inequality $\frac{\epsilon\theta}{1-\rho^\bullet} < \rho$ is actually equivalent to the net profit condition $\rho < 1$. As a consequence, the terms involving powers of $\frac{\epsilon\theta}{1-\rho^\bullet} / \rho < 1$ in the above result guarantee (for large n) a better performance of our new series representation over the classical Pollaczek-Khinchine expansion.

Remark 3.9. Note that the quantity ρ^\bullet depends on ϵ and that $\frac{\epsilon\theta}{1-\rho^\bullet}$ is increasing in ϵ for each fixed aggregate claim rate ρ . Correspondingly, the smaller the proportion ϵ of heavy-tailed claims is, the more our new series representation outperforms the classical Pollaczek-Khinchine expansion. The latter is intuitive, since the largest term will then dominate the others even more strongly, making our approximation even more efficient. The above expressions allow to quantify this effect.

4. Numerical experiments

In this section, we test and numerically illustrate the efficiency of our proposed technique, and compare it to the analogous classical simulation techniques based on the PK representation (3) (see also [3, Ch.XV.2]).

To perform our numerical experiments, we need to specify a mixture claim size distribution for which the distributions of $M_0^\bullet + M_1^\bullet + C_1^e$ and $M_1^\bullet + C_1^e$ can be evaluated explicitly; note that the second convolution is only required for the AK estimator.

4.1 Mixture of exponential and Pareto claim sizes

For the phase-type claim sizes we choose an exponential distribution with rate μ , i.e. $\overline{F}_p(u) = \overline{F}_p^e(u) = e^{-\mu u}$, and $\mu_B = 1/\mu$. For the heavy-tailed claim sizes we consider a shifted Pareto distribution with shape parameter $a > 1$ and scale $b > 0$, i.e. $\overline{F}_h(u) = (1 + u/b)^{-a}$ and $\overline{F}_h^e(u) = (1 + u/b)^{-(a-1)}$, $u \geq 0$, with $\mu_C = b/(a-1)$.

The two tail probabilities of the aforementioned sums of variables are explicitly available. For instance, for $\mu = 3$, $a = 2$, $b = 1$, $\epsilon = 0.1$ and $\rho = 0.99$ they are given by

$$\begin{aligned} \mathbb{P}(M_0^\bullet + M_1^\bullet + C_1^e > u) &= \frac{1}{25.600.000.000(1+u)} \times \left(-41200(223427 + 264627u) \right. \\ &\quad + 297(1+u) \left(400e^{-309u/400}(292973 + 91773u) \right. \\ &\quad \left. \left. + 31827(1691 + 891u)e^{-309(1+u)/400} \left(\text{Ei}\left(\frac{309(1+u)}{400}\right) - \text{Ei}\left(\frac{309}{400}\right) \right) \right) \right) \\ \mathbb{P}(X_1 > u) &= \frac{103}{400(1+u)} + \frac{297}{320000} \times \left(800e^{-309u/400} \right. \\ &\quad \left. + 618e^{-309(1+u)/400} \left(\text{Ei}\left(\frac{309(1+u)}{400}\right) - \text{Ei}\left(\frac{309}{400}\right) \right) \right), \end{aligned} \quad (26)$$

where $\text{Ei}(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt$ is the exponential integral. For all other parameters that we consider, analogous formulas are used. Finally, we calculate $\mathbb{P}(\max\{C_1^e, \dots, C_k^e\} > u) = 1 - (1 - (1 + u/b)^{-(a-1)})^k$.

4.2 Parameters

In all our experiments, we fixed $\mu = 3$ and $b = 1$, while we considered various combinations for the remaining parameters. Motivated by [19], we focused mainly on the cases $\rho \in \{0.9, 0.99, 0.999\}$, where simulations involving heavy tails can be considerably problematic (known as the heavy-traffic regime in the related queueing context, cf. [2]) and where the first two terms of (14) are known to be unable to close the gap between the approximation and the exact ruin probability even for values of $\epsilon = 0.1$. For the remaining parameters we tested $\epsilon \in \{0.1, 0.7\}$ and $a \in \{2, 3, 4\}$.

4.3 Results

In all the presented examples, the order of $\psi_n(u)$ is equal to $n = 100$ and the number of simulations is $\kappa = 10,000$.

We plot in [Figure 1](#) the simulated ruin probability that is obtained using the Monte Carlo estimator (11) together with the heavy-tail approximation (15). The dashed black lines depict the error bounds in [Proposition 3.2](#). We observe in both graphs that the lower bound converges to the heavy-tail approximation (15) as $u \rightarrow \infty$. This behaviour is observed for any n and is in accordance with theory. A similar statement holds for any u as $n \rightarrow \infty$. Further empirical tests show that this convergence in n is remarkably fast. However, one cannot draw a safe conclusion for which choice of parameters the lower bound is below or above the heavy-tail approximation. Finally, we observe in the left graph that

the upper bound is not very tight, as expected by [Proposition 3.2](#), since the chosen parameters give $\epsilon\theta/(1 - \rho^\bullet) = 0.875$. The bound becomes tighter in the right graph, where $\epsilon\theta/(1 - \rho^\bullet) = 0.25$.

From this point on, let us fix the parameters to $a = 2$, $\epsilon = 0.1$, and $\rho = 0.99$ to allow for comparability between [Figures 2](#) and [3](#). Moreover, we use a log-log scale. In [Figure 2](#), we plot the MC estimate [\(11\)](#) (blue solid line) together with the control variate extension [\(17\)](#) (black dashed line) against the heavy-tail approximation [\(15\)](#). We observe that the control variate technique outperforms the crude estimate [\(11\)](#) across the entire range of u (see the variance plot on the right). [Figure 2](#) also compares the simulation results with the ones based on the classical PK formula described in [Section 3.3](#). For the crude version, the latter are competitive for large u , but perform worse for small u . However, for the control variate, our new approach is always significantly and convincingly better. This nicely illustrates the theoretical asymptotic results of [Section 3](#): note that for the present choice of parameters the control variate asymptotically reduces the variance by a factor 0.09 (the constant on the right-hand side of [\(19\)](#)) for our series representation, to be compared with 0.73 for the analogous constant on the right-hand side of [\(24\)](#) for the PK representation. Related to that, the constant on the right-hand side of [\(25\)](#) in [Corollary 3.8](#) is 0.12, which means that our series representation reduces the asymptotic variance by almost 90%, when control variates are used in both cases.

In [Figure 3](#), we plot the simulated ruin probability with the AK estimator (blue solid line) and its control variate extension from [Section 3.2](#) as a function of the heavy-tail approximation [\(15\)](#). We consider in the plot both the PK and our new series expansion. One recognises that the asymptotic behaviour according to [\(15\)](#) (red dotted line) is recovered for all four estimators for sufficiently large u . The right graph illustrates that the introduction of the control variate is a significant improvement in terms of variance reduction for both the PK and our series, and that the two latter approaches perform similarly. The overall variance is much lower than for the method underlying [Figure 2](#). However, one should keep in mind that in terms of computation time the AK estimator in [Figure 3](#) is much more time-consuming (about 20–50 times in our implementations), as the integrals [\(26\)](#) have to be evaluated κ times, whereas for the method in [Figure 2](#) only once for the explicit term in front.

For large ρ , the number of summands tends to be large, and the results of the presented simulations suggest that the approximation

$$\mathbb{P}(M_0^\bullet + M_1^\bullet + \dots + M_k^\bullet + C_1^e + \dots + C_k^e > u) \approx \mathbb{P}(\max\{C_1^e, \dots, C_k^e\} > u) \quad (27)$$

is better than the one employed using the usual PK series expansion

$$\mathbb{P}(U_1^e + \dots + U_k^e > u) \approx \mathbb{P}(\max\{U_1^e, \dots, U_k^e\} > u). \quad (28)$$

Intuitively, the latter is comprised of mixtures of heavy-tailed and light-tailed variables, and hence the number of heavy tailed variables is thinned down, which is a drawback that our new method does not have. This is further supported by the plot in the left panel of [Figure 4](#), where the empirical correlations between the control variates are given. Concretely, when simulating from [\(28\)](#), only $100 \cdot \frac{\epsilon\theta}{\rho}\%$ of our U_k^e 's will actually be heavy-tailed and thus one loses too much information from the original presence of heavy-tailed C_k^e 's, in contrast to [\(27\)](#) where only the light tails are omitted and all heavy tails are kept. Consequently, the new control variate is much more efficient, cf. the factors $\epsilon\theta/\rho$ in [Corollary 3.8](#). In contrast, for the AK estimator the control variate does not significantly differ for the two series

representations, and therefore – while the control variate itself is a huge improvement over the crude estimate (cf. [Figure 3](#) (right)) – there is no improvement from using the alternative representation.

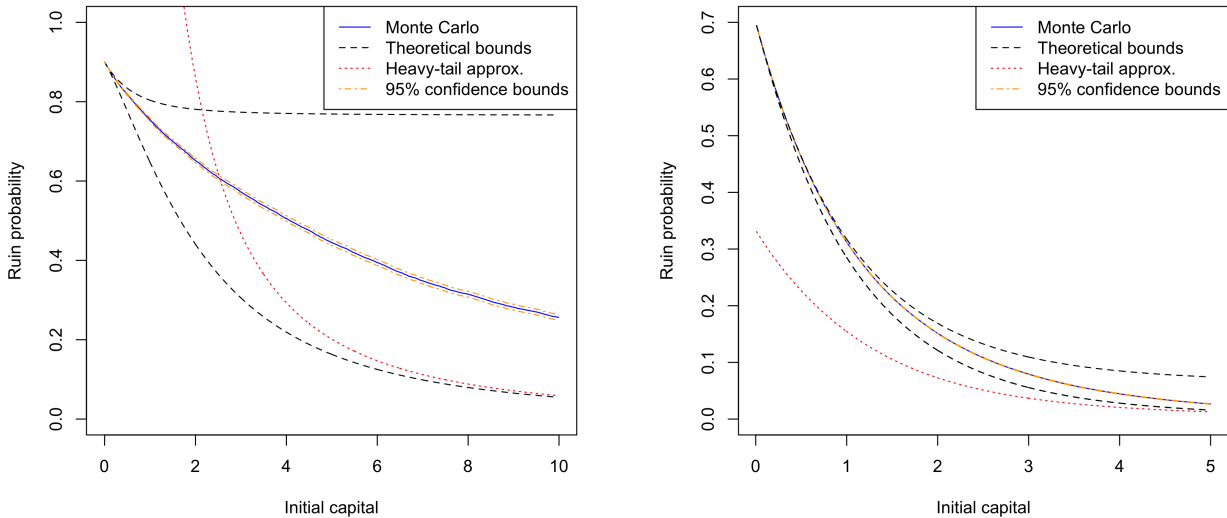


Figure 1: The simulated ruin probability with MC estimator (11) (blue solid line together with its 95% confidence interval in orange) and the heavy-tail approximation (red dotted line), as a function of the initial capital u . The black dashed lines represent the error bounds in [Proposition 3.2](#). Model parameters: $a = 3$ (both) and $\{\epsilon, \rho\} = \{0.7, 0.9\}$ (left) or $\{\epsilon, \rho\} = \{0.1, 0.7\}$ (right).

5. Conclusion

In this paper, we introduced an alternative series expansion for the PK formula in the Cramér-Lundberg model for the case when claims are mixtures of distributions with heavy and light tails. We showed that this can give rise to a significant improvement of simulation algorithms based on this series, both for large and small values of initial capital.

When using the AK conditional Monte Carlo technique, the new series representation performs similarly as the original one based on the PK formula. Both these AK procedures (and particularly their control variate extensions w.r.t. N) have a significantly lower variance for a fixed simulation size when compared to the method of [Section 3.1](#). However, the AK estimator is quite slow because it has to evaluate an improper integral in every iteration for the chosen mixture model. Hence, whenever time matters, the first simulation method based on (17) is preferred, and there our new series is a significant improvement. The latter is particularly the case also in the heavy-traffic regime where simulation is known to be difficult. In addition, the performance is quite convincing also for moderate and low initial capital.

In addition, it is hard or even impossible to use the AK estimator when the distribution of $M_1^\bullet + C_1^e$ is not known explicitly. On the other hand, our estimator can be used even if the probability $\mathbb{P}(M_0^\bullet + M_1^\bullet + C_1^e > u)$ cannot be calculated in a closed form. In such cases, one can simply simulate that latter probability as well and adapt the theoretical results in [Sections 3.1.1](#) and [3.1.2](#) accordingly.

In addition, although we concretely considered a mixture of a phase-type and a subexponential distribution in this paper, the results still hold if we replace $F_p(x)$ by any distribution for which $\psi^\bullet(u) = \mathbb{P}(M_0^\bullet > u)$ has a closed form, e.g. matrix-exponential distributions (cf. [\[7\]](#)). In addition, one

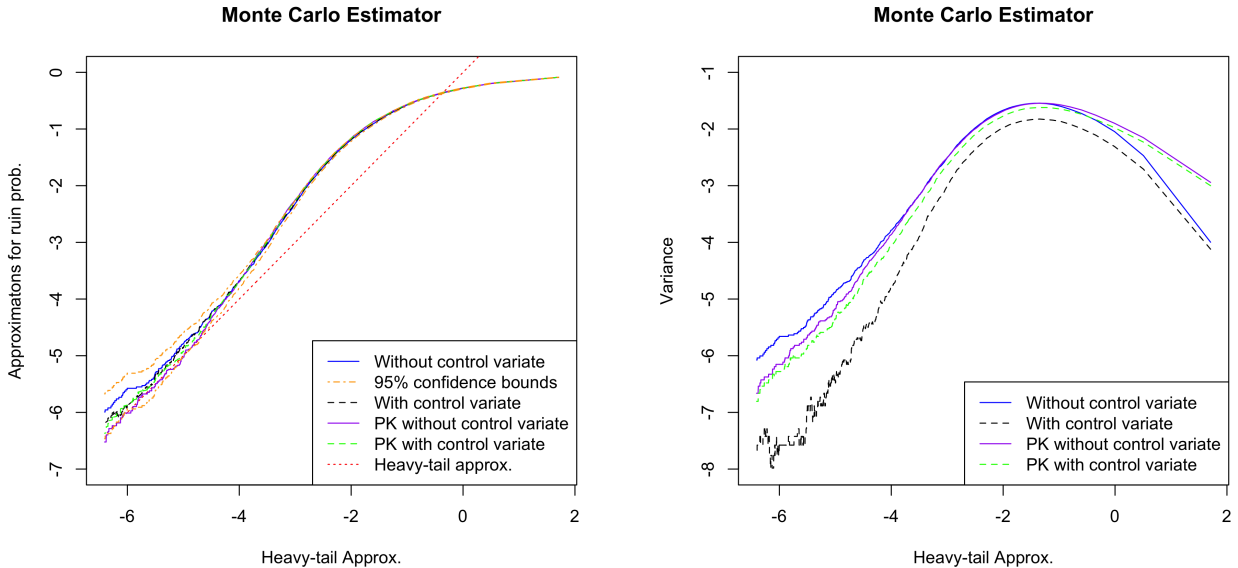


Figure 2: The simulated ruin probability with MC estimator (11) (blue solid line together with its 95% (blue solid line) and the control variate extension (17) (black dashed line) against the heavy-tail approximation (15), plotted in a log-log scale. The corresponding estimates based on the PK formula are depicted in pink and dashed green. Model parameters: $a = 2$, $\epsilon = 0.1$, and $\rho = 0.99$. The respective empirical variances are presented on the right.

can further modify our approach in order to evaluate $\psi^\bullet(u)$ via simulation for any other light-tailed distribution, which is known to produce effortlessly reliable simulation outputs.

Finally, we would like to point out that the ruin probability of the more general Sparre Andersen model also has a Pollaczek-Khinchine-type formula with respect to the ladder height distribution ([3, Ch.VI]). Our estimator is also valid for this model as long as the ladder height distribution can be found explicitly, which is for instance the case when the inter-occurrence times belong to the class of distributions with rational Laplace transform.

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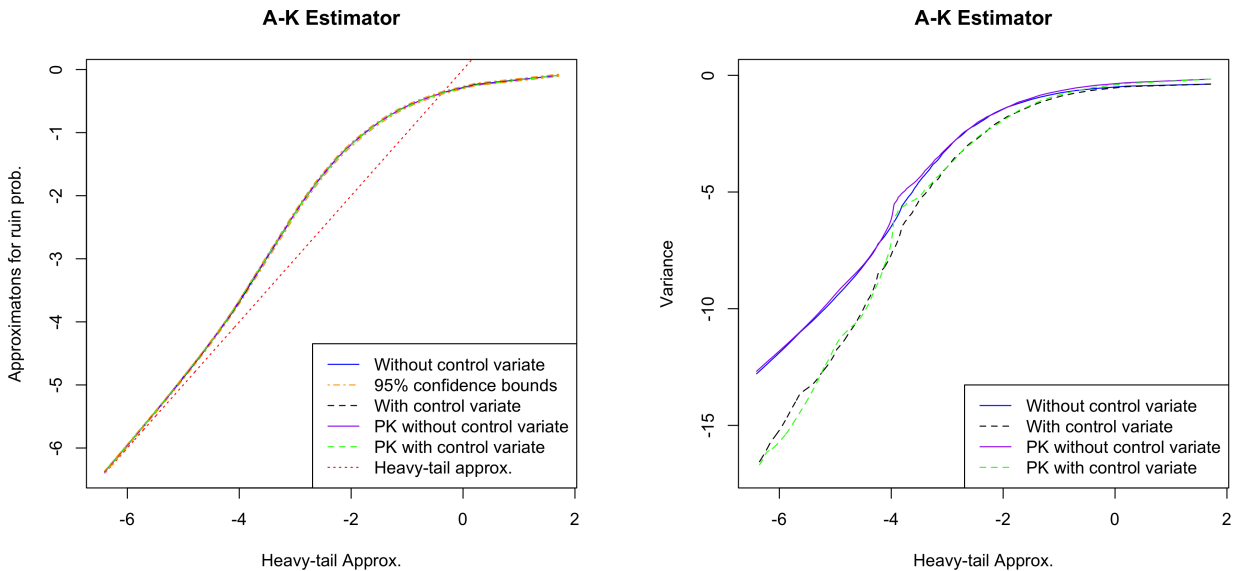


Figure 3: The simulated ruin probability with the AK estimator (blue solid line) and its control variate extension (22) (black dashed line) against the heavy-tail approximation (15), plotted in a log-log scale. The corresponding estimates based on the PK formula are depicted in pink and dashed green. Model parameters: $a = 2$, $\epsilon = 0.1$, and $\rho = 0.99$. The respective empirical variances are presented on the right.

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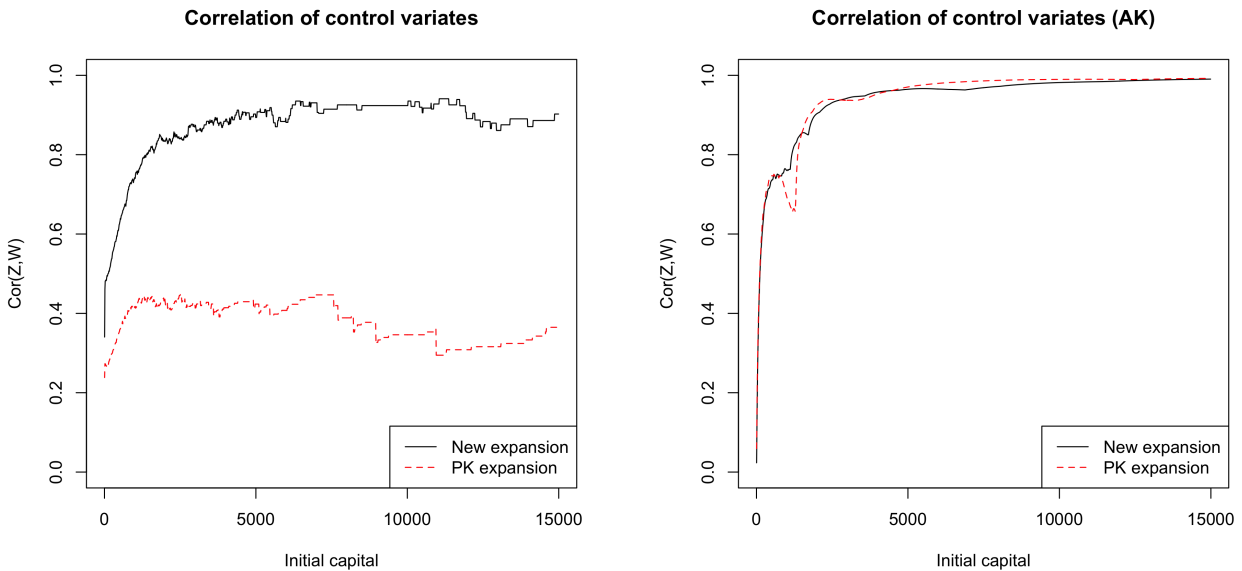


Figure 4: The correlation of the control variate using the new series expansion (black solid line) and the classical one (red dashed line) for the traditional Monte Carlo approach (Section 3.1.2, left panel) and for the AK approach (Section 3.2, right panel). Model parameters: $a = 2$, $\epsilon = 0.1$, and $\rho = 0.99$.

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