CONTEXT-DEPENDENT UTILITIES

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Résumé

We propose context-dependant utility models, where the utility of a multiattribute potential alternative is weighted in a way depending on the multiattribute utility of the present alternative. Such models reflect principles as "satisfaction of a need makes it invisible", or "the need creates the context", as well as other possibilities. They are shown to produce intransitivites as well as context-dependant preference reversals. Relationship with Tversky's additive model as well as Fechnerian and stochastic transitive properties are investigated. A particular class of weights, called NEAR, turns out to have local optimality properties, and reproduce general features of risk attitude in human species.

Keywords : multiattribute aggregation, deteministic and stochastic choice, temperature, entropy, Boltzmannn-Gibbs factor, additive difference model, context-dependent utilities, preference reversals, Fechnerian models, functional equations, strict utility models, risk-attitude, evolution theory.

1 Introduction

Quite common are observations where an individual in situation A expresses preference for situation B, yet retrospectively expresses preference for A after having moved to B. Those cases generally involve multiattribute situations, and common wisdom explains such reversals by invoking latent dimensions, neglected by the individual in A, but suddenly revealed to him when opting for situation B. We find this explanation quite convincing, and attempt here to formalize it by introducing dimensions weighting, with weights dependant on the utility of the current situation. For instance, a decreasing dependence expresses "satisfaction of a need makes it invisible" ; other forms are naturally possible. We call such models *context-dependant*, and show how they generate intransitivities or reversals (theorem 5). More precisely, we introduce two essentially distinct and

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new constructions : unilateral context-dependant utilities, where the utility of a possible situation is evaluated with respect to the present situation, and context-dependent symmetric comparisons, where pairwise preferences are evaluated in a way symmetrically dependant on both alternatives. While those models are not tested in the present article, we suggest how they could be used at analysing psychological or migrational data.

Weighting utilities amounts to break down their translational invariance. Conversely, we show (theorem 1) how invariance properties enjoyed by constant-weight models entail stochastic preference structures formally identical to the Statistical Mechanical framework in Physics. This introduces a quantity, new in this context, the (inverse) temperature, which we interpret as a mesure of the ability and sensitivity at discrimining utilities. Non-constants weights can in turn be interpreted as context dependant temperatures.

Examining conditions under which the context-dependent symmetric comparisons model fits into the "classical" [25] hierarchy (strict, Fechnerian, strongly or weakly stochastic transitive) (theorem 7), we come across Tversky's additive difference model [35] as a particular case of the symmetric model. We also show how "classical" constraints are sometimes sufficient to determine completely the functional form of weights and their aggregation rule in the symmetric case.

The concepts of contentment and frustration are introduced in the unilateral context-dependant framework. Examining the conditions ensuring safeness and efficiency of decisions based upon the latter (theorem 4), we come across two particulary families of weights, called NEAR and PEAR, which turn out to be very similar to weights used in modelling risk attitudes. Although discussions about risk attitudes are outside the scope of the present paper, restricted to certain outcomes, we speculate in the last part on the possibility that NEAR weights could represent an optimal compromise between safe and efficient decision making on one hand, and optimal sequential "editing" of multiattribute situations in a fluctuating and complex environment on the other hand. This optimality, possibly selected by evolution, could in turn shed some light on the prevailing risk attitude in our species.

2 Transitive models

2.1 Additive linear models

Let X_j, $j = 1, ..., m$ denote a set of choice alternatives (situations, bundles of goods, objects or persons relatively to some context...). Let $\alpha = 1, ..., r$ indice different characteristics (features, dimensions, needs...) associated with the alternatives. Then $u_{\alpha j} := u_\alpha(X_j)$ denotes specific utility of alternative X_j w.r.t. the α-th characteristic. Although the latter (playing the role of latent factors in statistical analyses [8]) are less directly observable than alternatives utilities, they arguably might be considered as primitive from the point of view of subject or consumer [22], and provide a rationale for separability in the aggregated utility function [33].

One has to realize that utility aggregation (as well as decision making in an uncertain environnment, not considered here (see e.g. [25] pp. 281-282)) necessitates the consideration of *cardinal* utility.

Specifically, one can consider the simplest aggregation mechanism, namely the additive one [17] :

$$U(X) := \sum_{\alpha=1}^{r} u_\alpha(X) \tag{1}$$

with the interpretation that choice X_j is preferred or indifferent to choice X_k, noted $X_j \ \mathcal{R} \ X_k$, iff $U(X_j) \geq U(X_k)$.

In the economic litterature, additivity of specific utilities is interpreted as independance (as opposed to complementarity or competitivity) between specific dimensions. Some caution has to be exercized here, as simple complementarity / competitivity criteria based upon the sign of the second derivatives of the aggregated utility function are not invariant under monotonic transformations [31] (p.183). Alternatively, (1) can be considered as a *definition* of latent dimensions, given an observed pattern of choices as well as a given utility-choice relationship.

Sufficient conditions for (1) to hold can e.g. be found in [9] [16] [30] [20] [11]. Some direct consequences (see [1][25][12]) are naturally the weak ordering (i.e. transitivity and connectivity) of preferences (satisfied for any utility function) and more specifically independence, i.e. irrelevance of those dimensions for which specific utilities of compared alternatives are identical.

Behavioral consequences follow from some utility-choice relationship. Let A be the *feasible set*, i.e. the set of those alternatives X which could be actually be realized by the subject by a proper allocation of its ressources (money, time, influence on other agents, computations, a.s.o....). Then a deterministic model is obtained when postulating the subject to choose its maximum utility alternative X_0, the (supposedly unique) solution of

$$\max_{X \in A} U(X) = U(X_0) \tag{2}$$

In short, A defines the subject's possibilities and $U(X)$ its wishes. Unicity holds for a convex A and convex $U(X)$ (with strict convexity at least once). In consumer's theory, for instance, $A = \{X = (x_1, ..., x_n) | \sum_{i=1}^{n} p_i x_i \leq M\}$, where alternatives X represent bundles of goods, π_i is the unit price of good i, and M is the total income. The axiom of greed for $U(X)$ (see e.g. [33]) ensures the optimum to lie on the boundary ∂A. In optimization terms, the constraint $X \in A$ is said to be active, which amounts to say that the consumer will spend its entire income.

To admit greater generality, in particular interindividual variations, one considers stochastic models yielding not a single optimum but probability distributions of the form

$$p_0(X) = Z(A)^{-1} \phi(U(X)) \qquad\qquad Z(A) := \sum_{X \in A} \phi(U(X)) \tag{3}$$

where $\phi(U(X))$ is an increasing function. Model (3) has been referred to as the *strict utility* [7] or Bradley-Terry-Luce (BTL) model and holds if the *choice axiom* (the strongest form of the "independance of irrelevant alternatives" axiom) is verified [25]. Also, it generalizes the *strict binary utility model* [25], in which the probability that alternative X is preferred over alternative Y is given by $p(X,Y) = \phi(U(X))/(\phi(U(X)) + \phi(U(Y)))$.

Finally, the BTL model (3) also belongs to the families of *random utility* models ([25] p.338), where the selection of an alternative results from the comparison of random utilities, leading on average to a distribution of the form (3).

Which functional forms $\phi(U(X))$ are relevant ? Suppose first the cardinal utility to have an *interval scale*, i.e. to be defined up to an origin (*relative* utility). As a consequence, the distribution (3) should be invariant w.r.t. uniform translations $U(X) \to U(X) + a$.

Theorem 1

a) *The distribution (3) is invariant w.r.t. uniform translation $U(X) \to U(X) + a$ iff $\phi(U) = c\exp(\beta U)$, where β and $c > 0$ are real parameters.*

b) *The distribution (3) is invariant w.r.t. any specific uniform translation $u_\alpha(X) \to u_\alpha(X) + a_\alpha$ iff [$\phi(U) = c\exp(\beta U)$, where and $c > 0$ are real parameters, and the additive form (1) holds].*

The (elementary) proof, using basic arguments in measurement theory and functional equations [26][10] is left to the reader ; yet, despite its simplicity , theorem 1) do not seem to have been stated as such in the choice and decision litterature. In Physics, more precisely in Statistical Mechanics (see e.g. [18]), $\phi(U) = \exp(\beta U)$ is called Boltzmannn-Gibbs factor, and $\beta > 0$ is called inverse temperature. Actually, the distribution $p_0(X)$ in (3) is the solution to the variational problem

$$\sup_{p(X)\in\mathcal{P}(A)} \sum_{X\in A} [p(X)U(X) - \beta^{-1}p(X)\ln p(X)] \tag{4}$$

where $\mathcal{P}(A)$ is the set of normalized p.d.f. with support in A (see e.g. [5]). Maximization of the first term yields a distribution concentrated on the maximum-utility alternative X_0 (called ground state by physicists), whereas maximization of the second (entropy) term yields a maximum entropy distribution, i.e. uniform in A. Thus temperature β^{-1} arbiters this order-disorder tradeoff : high values of β, i.e. cool sytems, correspond to distributions concentrated around the global maxima, or equivalently to systems sensitive to small utility differences. In the limit $\beta \to \infty$ one recovers the deterministic model (2).

To get some intuition about a possible psychological interpretation of β more intuitive, consider the question of why the subject should choose a distribution (3) with a finite β : after all, his average utility is maximized for $\beta \to \infty$.

As noted, such distributions correspond to infinite sensitivity or discriminating performances. But one can imagine situations where a subject, with perceptual or cognitive limitations, faces the task of evaluating complicated utility functions $U(X)$. He then can choose spending some time and effort in estimating precisely $U(X)$, a task itself of low utility ; or he might opt for a less precise estimate, thus avoiding the hassle of the computation. His optimal tradeoff value β_0 results from a compromise between those two contradictory goals, for instance the value maximizing

$$\sup_{\beta\geq 0}(< U >_\beta - c\beta) \qquad \text{with} \quad < U >_\beta := \frac{\sum_{X\in A} U \exp(\beta U)}{\sum_{X\in A}\exp(\beta U)} = \frac{\partial \ln Z(A,\beta)}{\partial\beta} \tag{5}$$

where $c > 0$ is an unit cost associated to discrimination sensitivity β.

Under the condition $\sup_{X\in A} U(X) < \infty$, and $< (U- < U >_\beta)^2 >_\beta = \text{var}_\beta(U) = \frac{\partial^2 \ln Z(A,\beta)}{\partial\beta^2}$ continuous, (5) has at least a local maximum at some β_0 satisfying $\text{var}_{\beta_0}(U) = c$, provided $c < \sup_\beta \text{var}_\beta(U)$. So at the optimal temperature the utility variance equals the discrimination cost : the rougher the utility landscape (as given by $U(X)$, $X \in A$), the higher the inverse temperature β, i.e. the higher the resources allocated to "discrimination processing" ; in a flat lanscape $U(X) = \text{cst}$, $\text{var}_\beta(U) = 0$ and no advantage whatsoever results from an improved discrimination power.

Also, if $c > \sup_\beta \text{var}_\beta(U)$, the discrimination cost is too high for any discrimination allocation to be worth : uniform random choice ($\beta_0 = 0$) is then optimal in the sense of (5).

The Boltzmannn-Gibbs function $\exp(\beta U)$ can also be met in contingency tables analysis for reasons of convenience(log-linear models) [6], in theoretical statistics due to the central role of exponential families of distribution [4], and naturally in Maximum Entropy formalism [23], itself historically a breed of Statistical Mechanics and Information Theory.

2.2 Additive non linear models

In the classical Bernouillan view, utility refers to a state of the mind and an *absolute* or *ratio scale* therefore seems more appropriate. For instance, if one measures the utility of a situation by the self-reported level of satisfaction (which makes sense, for a subject preferring alternative X_k to X_j is likely to report higher levels of satisfaction in the former situation), one can presume the subject to report satisfaction when its utility lies above a critical value, taken as the origin, and dissatisfaction when its utility lies below this value.

Leaving methodological concerns aside, let us simply assume the existence of a common scale $u_\alpha(X)$ measuring the subject's satisfaction relatively to dimension α in situation X, with $u_\alpha(X) > 0$ coding "rather satisfied (or more)" responses, and $u_\alpha(X) < 0$ coding "rather dissatisfied (or more)" responses. We shall break the translation invariance alluded to in theorem 1b) (and asssociated to the addditive linear model (1)) by considering an *additive non-linear* model. This model tentatively embodies psychological principles such as *"satisfaction of a need makes it invisible"* or *"the strongest need creates the context"* : in this extreme case, the overall satisfaction could be written as $U(X) = \min_{\alpha=1,\ldots,r} u_\alpha(X)$. In a somewhat less abrupt way, we propose the simple form

$$U(X) = \sum_{\alpha=1}^{r} w(u_\alpha(X))\; u_\alpha(X) \tag{6}$$

Definition 1
Model (6) with $w(u) \geq 0$, $w(u)\; u$ increasing is said to be :

• *globally oriented* if $w(u)$ is decreasing (satisfaction of a need makes it less present).

• *neutrally oriented* if $w(u)$ is a positive constant (additive linear model (1)).

• *specifically oriented* if $w(u)$ is increasing (the most satisfied need is favored : "addictive" or more generally "specialization aimed" trends).

In short, additive non linear models (6) models seems to be the simplest ones taking into account a ratio or absolute concept of utility, and contain the additive linear (interval or relative) utility model as a particulary case[1].

As a consequence of theorem 1b), the distribution function (3) with $\phi(U) = \exp(\beta U)$ is not invariant anymore under the translation $u_\alpha(X) \to u_\alpha(X) + a_\alpha$. To first order

$$p_0(X) \to p_0(X)(1 + \beta \sum_{\alpha=1}^{r} a_\alpha(v_\alpha(X) - < v_\alpha >_\beta)) + 0(a^2) \tag{9}$$

with $v_\alpha(X) := \partial_{u_\alpha}[w(u_\alpha(X))\ u_\alpha(X)]$, where ∂_u is the derivative operator. To decide which alternatives will be favored by the translation depends on the sign of the second derivative of $w(u)\ u$. For instance, $w(u) = \pi/2 - \arctan(u)$ defines a globally oriented model with $w(u)\ u$ concave, so alternatives X rarely choosed because of their low specific utilities tend to be relatively more frequently choosen under a uniform increase in utility : intuitively, regions which were too painful to cope with turn out to be comparatively more acceptable under general uniform improvment. An opposite result holds for the specifically oriented model $w(u) = \pi/2 + \arctan(u)$ with $w(u)\ u$ convex. Also, the NEAR, resp. PEAR models introduced in section 3.2 yields an increased probability in the central utility region $u \approx 0$, resp. in the bilateral extreme utility region $|u| >> 0$.

Formally, combinaisons of the form $\beta u_\alpha(X)$ in the linear model are replaced by $\beta_\alpha u_\alpha(X)$ in the non linear model, where $\beta_\alpha(X) := \beta\ w_\alpha(u_\alpha(X))$ plays the role of a *context dependant inverse specific temperature* : the lower the specific utility of a situation, the higher the resulting discrimination sensitivity for globally oriented models : intuitively, suffering increases specific awareness (or induces numbness in specifically oriented models).

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[1]Another way of getting it is to consider the rescaling $w_s(u) := w(su)$, which for sufficiently smooth strictly monotonic weights $w(u)$ yields (with $U_s(X)$ denoting the corresponding rescaled utility (6)) :

$$\lim_{s\to\infty} \frac{U_s(X)}{\sum_{\alpha=1}^{r} w_s(u_\alpha(X))} = \begin{cases} \min_{\alpha=1,\dots,r} u_\alpha(X) & \text{global orientation} \\ \max_{\alpha=1,\dots,r} u_\alpha(X) & \text{specific orientation} \end{cases} \tag{7}$$

and

$$\lim_{s\to 0} \frac{U_s(X)}{\sum_{\alpha=1}^{r} w_s(u_\alpha(X))} = \frac{1}{r} \sum_{\alpha=1}^{r} u_\alpha(X) \tag{8}$$

3 Non-transitive models

3.1 Additive difference model

Transitivity of preferences, an unavoidable consequence of utility theory, has repeatedly been observed by experimental psychologists and economists to be violated [13][19][27] [29] [32][35]. A.Tversky [35] has proposed the following *additive difference* model :

$$X_j \ \mathcal{R} \ X_k \qquad \text{iff} \qquad \sum_{\alpha=1}^{r} W_\alpha(u_\alpha(X_j) - u_\alpha(X_k)) \geq 0 \tag{10}$$

where the increasing odd continuous functions $W_\alpha(.)$ scales relative specific utilities in dimension α. Model (10), of which the additive utility model (1) as well as lexicographic semiorders are both particular or limit cases, can exhibit intransitivities as soon as $r = 2$ (provided $W_2(\delta) \neq W_1(t\delta)$ for δ and some $t > 0$), or $r \geq 3$ (provided $W_\alpha(\delta) \neq t_\alpha \delta$ for all δ and some $t_\alpha > 0$, $\alpha = 1, ..., r$) [35]. In this model (as well as in the context-dependant models below), multiattribute situations are necessary for intransitivities to ocur, a fact repeatedly pointed out in different contexts by others workers, often referring to possible shifting-attention or switching-dimension mechanisms.

Natural extension of additive difference model to stochastic pair comparisons is $p(X_j, X_k) = F(\sum_{\alpha=1}^{r} w_\alpha(u_\alpha(X_j) - u_\alpha(X_k)))$, where $F(.)$ is an increasing function with domain $(0, 1)$. When the $W_\alpha(.)$ are linear, one recovers the so-called *strong (or Fechnerian) utility model*, a particular case of the strict utility model. Further discussion is to be found in section 3.5.1.

3.2 (Unilateral) context-dependant model

We now formalize a mechanism yielding intransitive patterns in multiattribute situations. This mechanism is of weighting-dimensions type rather than of shifting-attention or switching-dimensions type alluded previously.

First we define, in the spirit of section 2.2, the *utility of X_k in context X_j* as

$$U(X_k|X_j) := \sum_{\alpha=1}^{r} w(u_\alpha(X_j)) \ u_\alpha(X_k) = \sum_{\alpha=1}^{r} w(u_{\alpha j}) \ u_{\alpha k} \tag{11}$$

In short, importance of features $\alpha = 1, ...r$ are determined by the corresponding weights in reference alternative or *context* X_j. In Gestaltist terms, X_k is the Figure and X_j the Ground, and (11) is a simple attempt to capture the Ground-depending value of the Figure. Note that the word "context" is already used in a different meaning in the decision theory litterature (see e.g. [13][32]) where is alludes to the possible effect of the *presentation* of a situation ("framing" effects [19]) or the influence of the deterministic [28] or stochastic [14] *feasible set* on the preferences pattern [27]. On the other hand, the possible dependance of the utility of a change upon some(current or expected) reference position [19][27] for *monetary outcomes* has been recurrently recalled at least since Bernouilli, in contrast to the much more discrete explicit recognition of the necessary exclusion of utility interval scales implied by this reference position dependance. Also, the only models we are aware of aimed at capturing this dependance deal with risky monetary outcomes [3][19][27]. By contrast, we are here primarily concerned with non-risky choices and non-monetary dimensions.

By construction, $U(X_j|X_j)$, referred to as the *intrinsic* utility of alternative X_j, is equal to $U(X_j)$ given by (6). We now define choice X_k to be preferred or indifferent to choice X_l *in context* X_j, noted $X_k\ \mathcal{R}_j\ X_l$, iff $U(X_k|X_j) \geq U(X_l|X_j)$. Thus, in presence of m alternatives, there exist m context dependant preferences structures (giving a total of $m^2(m-1)/2$ possible pair comparisons), and in addition the usual context-independant relation $X_k\ \mathcal{R}\ X_k$ iff $U(X_k) \geq U(X_l)$.

A case of special importance occurs when one of the alternatives X_k or X_l to be compared coincides with the reference alternative or current context X_j : here the subject compares its present choice X_j to another potential feasible choice X_k, and strictly prefers the latter iff $U(X_k|X_j) > U(X_j)$. This case will be referred to as the context-dependent preference model, sometimes precised as "unilateral" to distinguish from the symmetric context-dependent models below.

For instance, a globally oriented subject, finding his present life X_1 boring and unexciting, might wish opting for a more adventurous situation X_2, but doing so he might experience lack of security, say, and in this new context X_2 find retrospectively X_1 more attractive. As this example suggests, context-dependant models not only might exhibit intransitivities as soon as $r = 2$ but also context dependant reversals, i.e. cases with $X_k\ \mathcal{R}_j\ X_j$ *and* $X_j\ \mathcal{R}_k\ X_k$! Also, if *indirect* reversals, consisting in the existence of gamble pairs ranked oppositely if converted in juged certainty equivalents or if directly compared have been extensively reported and studied in the decision theory litterature (see e.g. [27] and references therein), let us make quite clear that we deal here with *direct* (context-dependant) reversals.

For instance, consider the following three dimensional hypothetical situation with given specific utilities $u_\alpha(X_j)$ and globally oriented weights ($w(-1) = 3$, $w(0) = 1.5$, $w(1) = 1$) toghether defining intrinsic utilities $U(X_j)$:

	$\alpha = 1$ (excitment)	$\alpha = 2$ (security)	$\alpha = 3$	$U(X_j)$
$j = 1$	-1	1	1	-1
$j = 2$	1	-1	1	-1
$j = 3$	0	0	1	1
$j = 4$	0	1	0	1
$j = 5$	1	1	1	3

Table 1

Then $X_1\ \mathcal{P}_2\ X_2$ and $X_2\ \mathcal{P}_1\ X_1$, where $\mathcal{P}_j$ denotes strict preference (and $\mathcal{I}_j$ indifference, so that $\mathcal{P}_j \cup \mathcal{I}_j = \mathcal{R}_j$).
Also, $X_3\ \mathcal{I}_3\ X_1$ and $X_3\ \mathcal{I}_3\ X_2$ but $X_3\ \mathcal{P}_1\ X_1$ and $X_3\ \mathcal{P}_2\ X_2$.

Also, $X_4\ \mathcal{P}_3\ X_3$, $X_2\ \mathcal{P}_4\ X_4$ and $X_3\ \mathcal{P}_2\ X_2$: this describes a "utility pumping cycle" : at each stage, the subject is wishing to move, ending up at some stage in the initial situation. Note that as far as an alternative dominates along *all* dimensions, it escapes to cycles, as it must ; for instance, $X_5\ \mathcal{P}_j\ X_j$ and $X_5\ \mathcal{P}_5\ X_j$ for all $j \neq 5$.

The general context dependant structure among m alternatives and r dimensions is unquestionably rich (and complex). A simple restriction however holds :

Lemma 2

$$U(X_j) + U(X_k) \leq U(X_j|X_k) + U(X_k|X_j) \qquad \textit{for globally oriented models} \tag{12}$$

$$U(X_j) + U(X_k) \geq U(X_j|X_k) + U(X_k|X_j) \qquad \textit{for specifically oriented models} \tag{13}$$

Proof : for each α, decreasing of $w(.)$ in globally oriented models makes $(w(u_{\alpha j}) - w(u_{\alpha k})(u_{\alpha j} - u_{\alpha k}) \leq 0$; summation over α yields (12). The proof of (13) is similar. □

Corollary 3

a) *For globally oriented models, $X_j \mathcal{R}_j X_k$ implies $X_j \mathcal{R}_k X_k$ (and $X_j \mathcal{P}_j X_k$ implies $X_j \mathcal{P}_k X_k$) : when a subject prefers his actual situation to another possible one, his preference remains when evaluated from the other context.*

b) *For specifically oriented models, $X_j \mathcal{R}_k X_k$ implies $X_j \mathcal{R}_j X_k$ (and $X_j \mathcal{P}_k X_k$ implies $X_j \mathcal{P}_j X_k$) : when a subject prefers another possible situation to his actual one, his preference remains when evaluated from the other context.*

Proof : a straightforward application of lemma 2) to the definitions of strict and weak context-dependent preference relations. □

Let us define *contentment* in X_j w.r.t. X_k as the difference between the (actual) specific utility and the context-dependent utility of another potential alternative, namely $U(X_j) - U(X_k|X_j)$, and *frustration* as its negative, i.e. $U(X_k|X_j) - U(X_j)$. Corollary 3) then says that the positive frustration experienced by a specifically oriented subject is guaranteed not to increase when the subject opts for a change. Also, the positive contentment experienced by a globally oriented subject is guaranteed not to increase when the subject opts for a change. Those are certainly useful guidelines, but one would like more, i.e. to be *guaranteed* not to increase or decrease its *specific utility* $U(X_j)$ (rather than his contentment or frustration) when considering a possible move $X_j \rightarrow X_k$, *given suitable conditions on $U(X_k|X_j) - U(X_j)$, i.e. on contentment or frustration only.* Specifically, one would ensure decision rules guaranting the exclusion of off-digonals possibilities in table 2, where columns denote states of the world, and rows decisions, while X_j refers to the actual situation and X_k to another feasible one, towards which move is considered.

	opportunity $U(X_k) > U(X_j)$	danger $U(X_k) < U(X_j)$
move	OK	worsening
stay	lost opportunity	OK

Table 2

Definition 2 a) weights $w(u)$ are said to be *SIC-safe* ("stay if contended") if positive contentment guarantees intrinsic utility not to increase when opting for a change, i.e. if $U(X_k|X_j) \leq U(X_j)$ implies $U(X_k) \leq U(X_j)$, i.e. if a SIC decision rule never yields lost opportunity.

b) weights $w(u)$ are said to be *MIF-safe* ("move if frustrated") if positive frustration guarantees intrinsic utility not to decrease when opting for a change, i.e. if $U(X_k|X_j) \geq U(X_j)$ implies $U(X_k) \geq U(X_j)$, i.e. if a MIF decision rule never yields worsening.

As we assume $w(u)$ to be positive and $w(u)\ u$ increasing, the previous definitions are the only ones possibly making sense. Suppose weights to be SIC-safe, i.e., with the simplified notations $u_\alpha := u_\alpha(X_j)$ and $u'_\alpha := u_\alpha(X_k)$:

$$\sum_{\alpha=1}^{r} w(u_\alpha)(u'_\alpha - u_\alpha) \leq 0 \quad \text{implies} \quad \sum_{\alpha=1}^{r} w(u'_\alpha)\ u'_\alpha \leq \sum_{\alpha=1}^{r} w(u_\alpha)\ u_\alpha \tag{14}$$

Above restrictions on $w(u)$ make (14) trivially satisfied for $r = 1$. Consider now the multiattribute case $r \geq 2$. Set $\epsilon_\alpha := u'_\alpha - u_\alpha$. Then, up to second order in ϵ, (14) yields

$$\sum_{\alpha=1}^{r} f'(u_\alpha)\epsilon_\alpha + \frac{1}{2}\sum_{\alpha=1}^{r} f''(u_\alpha)\epsilon_\alpha^2 + 0(\epsilon^3) \leq 0 \quad \text{whenever} \quad \sum_{\alpha=1}^{r} w(u_\alpha)\epsilon_\alpha \leq 0 \tag{15}$$

where $f(u) := w(u)\ u$. Relation (15) holds to first order iff $f'(u) = \lambda w(u)$ where $\lambda \geq 0$, i.e. iff $\partial_u \ln w(u) = (\lambda - 1)\partial_u \ln |u|$, iff $w(u) = c|u|^\gamma$ with $c > 0$, $\gamma \geq -1$. Thus $f(u) = c|u|^\gamma u$, $f'(u) = c(\gamma+1)|u|^\gamma \geq 0$ and $f''(u) = c\gamma(\gamma+1)|u|^{\gamma-1}\text{sgn}(u)$. As $f''(u)$ has to be non-positive for relation (15) to hold to second order, and because of the sign-switching implied by sgn(u), the only remaining possibilities are $\gamma = 0$ (and we are back to the neutrally oriented, linear additive, context-independant model (1)) and $\gamma = -1$ for which the intrinsic utility $U(X_j)$ turns out to be simply $\sum_{\alpha=1}^{r} \text{sgn}(u_\alpha)$, i.e. the number of "OK" features minus the number of "not OK" features. So the weights $w(u) = c|u|^{-1}$ are locally, second-order SIC-safe, but are they still "at large"? A simple counter-example ($c = 1$; $r = 2$; $u_1 = -1$; $u_2 = -1$; $u'_1 = 2$ $u'_2 = -5$) shows they are not : $U(X_k|X_j) = -3 < U(X_j) = -2$ but $U(X_k) = 0 > U(X_j) = -2$.

Analogously, if weights are MIF-safe, then

$$\sum_{\alpha=1}^{r} f'(u_\alpha)\epsilon_\alpha + \frac{1}{2}\sum_{\alpha=1}^{r} f''(u_\alpha)\epsilon_\alpha^2 + 0(\epsilon^3) \geq 0 \quad \text{whenever} \quad \sum_{\alpha=1}^{r} w(u_\alpha)\epsilon_\alpha \geq 0 \tag{16}$$

and the reader can check the same reasonning to apply *mutatis mutandis*. We have thus proved the following

Theorem 4
a) *Any weight function with $w(u)$ positive and $w(u)\ u$ increasing is SIC- and MIF-safe in the monoattribute $r = 1$ context-dependent model.*

b) *Only constant positive weights $w(u)$ are SIC- or MIF-safe for the multiattribute $r \geq 2$ context-dependent model. However, safety is locally insured to first order for $w(u) = c|u|^\gamma$ (with $c > 0$, $\gamma \geq -1$) , and for $w(u) = c|u|^{-1}$ (with $c > 0$) up to the second order.*

For sake of continuity, further discussion of theorem 4) is postponed until the conclusion and we instead turn to the question of reversals and intransitivities. First of all, the latter are ruled out by monotonicity of $w(u)u$ in monoattribute situations. For multiattribute situations, transitivity shows $U(X_k|X_j) = U(X_j)$ and $U(X_l|X_k) = U(X_k)$ together to imply $U(X_l|X_j) = U(X_k)$. By an argument essentially similar to the one developped in the proof of theorem 7) below, this can hold iff $w(u) = \text{cst}$. Suppose now reversals to be ruled out. Then $\sum_{\alpha=1}^r w(u_\alpha)\ u'_\alpha = \sum_{\alpha=1}^r w(u_\alpha)\ u_\alpha$ iff $\sum_{\alpha=1}^r w(u'_\alpha)\ u'_\alpha = \sum_{\alpha=1}^r w(u'_\alpha)\ u_\alpha$. Thus $w(u'_\alpha) = \lambda w(u_\alpha)$ for some $\lambda > 0$. As specific utilities can be varied independently in each dimension for $r \geq 2$, this leaves $w(u) = \text{cst}$ as the only possibility. We have thus proved

Theorem 5
a) *Reversals or intransitivities do not occur in the monoattribute ($r = 1$) unilateral context-dependant model.*

b) *Reversals or intransitivities do occur in the multiattribute ($r \geq 2$) unilateral context-dependant model, unless $w(u) = cst$.*

3.3 Application : spatial migrations

Context-dependant preferences of the form $X_j \mathcal{R}_j X_k$ (or $X_k \mathcal{R}_j X_j$) can model preferences pattern for spatial migrations, where X_j is the actual region and X_k a potential destination. Explicitely, let p_{jk} be the conditional probability for a migrant leaving X_j to go to the (distinct) place X_k. This probability should be increasing in $U(X_k|X_j) - U(X_j)$, the expected ameloration. Assuming a Boltzmannn-Gibbs relation, one gets :

$$p_{jk} = Z_j^{-1} \exp(\beta(U(X_k|X_j) - U(X_j))) \qquad \text{with} \qquad Z_j = \sum_{k;k\neq j} \exp(\beta(U(X_k|X_j) - U(X_j))) \tag{17}$$

where $\beta > 0$ and $p_{jj} = 0$ by definition. Note the $U(X_j)$ terms to cancel out. Let X_j, X_k and X_l be three distinct places. In the litterature on spatial migrations, the flows are sometimes said to be transitive if $p_{jk} > p_{kj}$ and $p_{kl} > p_{lk}$ together imply that $p_{jl} > p_{lj}$. Empirically, numerous violations of transitivity, incompatible with the classical gravity model (and related) [34][15], have been observed. Model (17) describes a limit case where distance costs (irrelevant to transitivity violations) are negligible, and is primarily aimed as capturing intransitivities as directly as possible.

To that extent, we define the *intransitive trend* $g(X_j, X_k, X_l)$ associated to the cycle $X_j \to X_k \to X_l \to X_j$ as

$$g(X_j, X_k, X_l) := \ln \frac{p_{jk}\ p_{kl}\ p_{lj}}{p_{kj}\ p_{lk}\ p_{jl}} \tag{18}$$

By construction, $g(\pi(X)) = (-1)^{|\pi|} g(X)$, where $|\pi|$ is the parity of the permutation $\pi \in S_3$. Model (17) yields

$$g(X_j, X_k, X_l) = \beta[U(X_k|X_j) - U(X_j|X_k) + U(X_l|X_k) - U(X_k|X_l) + U(X_j|X_k) - U(X_k|X_j)] \tag{19}$$

Thus intransitivities disappear for context independant utilities, as expected. Notice the form $g(X_j, X_k, X_l) = \beta[G(X_j, X_k, X_l) - G(X_l, X_k, X_j)]$, where $G(X_j, X_k, X_l) := U(X_k|X_j) - U(X_j) + U(X_l|X_k) - U(X_k) + U(X_j|X_l) - U(X_l)$ is the total frustration, or equivalently the cumulative expected improvement when running the cycle $X_j \to X_k \to X_l \to X_j$. $G(X_j, X_k, X_l)$ could also be called "cumulative utility pumping" associated to the cycle, in analogy with "money pumping" situations where a subject is ready to pay for the change $X_j \to X_k$, as well as for $X_k \to X_l$ and $X_l \to X_j$.

Lemma 2) shows the *sum* $G(X_j, X_k, X_l)+G(X_l, X_k, X_j)$ to be positive for globally oriented subjects and negative for specifically oriented subjects. The *difference* can be of any sign. By constrast, binary cumulative frustration $G(X_j, X_k) := U(X_k|X_j) - U(X_j) + U(X_j|X_k) - U(X_k) = G(X_k, X_j)$ is invariant under transpositions, positive for globally oriented subjects, and negative for specifically oriented subjects.

3.4 Deterministic context-dependant symmetric models

Quite common are setups where utilities of two potential alternatives X_j and X_k match well on each dimension, and are yet only loosely related to the actual situation X_l : the categories obviously relevant to both X_j and X_k can fit badly when applied to X_l. In this case it might be preferable to define

$$X_j\ \hat{\mathcal{R}}\ X_k \qquad \text{iff} \qquad \hat{U}(X_j, X_k) := \sum_{\alpha=1}^{r} w(u_{\alpha j}, u_{\alpha k})(u_{\alpha j} - u_{\alpha k}) \geq 0 \tag{20}$$

where $w(u, u')$ is a symmetric function of its arguments, for instance :

$w_a(u, u') := \frac{1}{2}(w(u) + w(u'))$,
$w_b(u, u') := w(\frac{1}{2}(u + u'))$ or
$w_c(u, u') := \max(w(u), w(u'))$

where $w(u)$ is the weight function introduced in (6).

Model (20) could be tested on situations like described by K.May [29], who asked college students to compare hypothetical marriage partners X_1, X_2 and X_3, differing in intelligence, looks and wealth. Let us temptatively assign the following specific utilities[2] :

[2]"... X_1 was described as very intelligent, plain looking, and well off ; X_2 as intelligent, very good looking, and poor ; X_3 as fairly intelligent, good looking, and rich."

	$\alpha = 1$ (intelligence)	$\alpha = 2$ (looks)	$\alpha = 3$ (wealth)
$j = 1$	2	0	1
$j = 2$	1	2	0
$j = 3$	0	1	2

Table 3

The specific utilities are naturally identical ($U(X_1) = U(X_2) = U(X_3) = w(1) + 2w(2)$), but so are the directed pair comparisons : $\hat{U}(X_1, X_2) = \hat{U}(X_2, X_3) = \hat{U}(X_3, X_1) = -\hat{U}(X_2, X_1)$ a.s.o... . In the three cases suggested above, they are :
$\hat{U}_a(X_1, X_2) = w(1) - \frac{1}{2}(w(0) + w(2))$
$\hat{U}_b(X_1, X_2) = w(0.5) + w(1.5) - 2w(1)$
$\hat{U}_c(X_1, X_2) = w(1) - w(0)$
On average, results [29] indicate a circular pattern : $X_1 \hat{\mathcal{R}} X_2$, $X_2 \hat{\mathcal{R}} X_3$ and $X_3 \hat{\mathcal{R}} X_1$, which May interprets as the result of choosing the alternative that is superior in two out of three criteria : this amounts using the additive difference model (10) with $W_\alpha(\delta) = \text{sgn}(\delta)$. In our setup, the observed ordering obtains with for instance $w_a(u)$ convex, $w_b(u)$ concave or $w_c(u)$ increasing.

The present discussion is simply aimed at illustrating the concepts and getting a sense of empirical constraints upon possible models : a real data analysis should start challenging (questionnable) assumptions such as the independence of the three dimensions, their identical scaling, a.s.o... In addition, the observed important variability across the preference patterns of the subjects ruins the deterministic approach : rather, responses should be stochastically expressed in terms of $p(X_j, X_k)$, the probability that alternative X_j is preferred over X_k. The next section is devoted to this approach, as well as models restrictions resulting from "classical" conditions.

3.5 Stochastic context-dependant symmetric models

Consider the stochastic pair comparison model

$$p(X_j, X_k) = G(\hat{U}(X_j, X_k)) \qquad (21)$$

where $G(.)$ is a distribution symmetric around 0, i.e. $G(x) + G(-x) = 1$, and $\hat{U}(X_j, X_k)$ is given by (20).

In what follows, discussions not involving conditions on $G(.)$ clearly apply to deterministic models 3.4 as well.

3.5.1 Fechnerian models

Suppose a Fechnerian [25] condition to hold, i.e. $p(X_j, X_k) = F(\tilde{s}(X_j) - \tilde{s}(X_k))$, for some function $\tilde{s}(X)$ and distribution $F(.)$ symmetric around 0. Defining the increasing, odd function $H := G^{-1} oF$, limiting ourselves temporarily to a *single* dimension ($r = 1$) and requiring $\tilde{s}(X) = s(u(X))$, one gets the condition $w(u', u)(u' - u) = H(s(u') - s(u))$.

Lemma 6

Let $H(.)$ be an odd increasing smooth function, and $w(u',u)$ be a symmetric function with a given diagonal $w(u,u) = w(u) > 0$. Then there exists a smooth stricly increasing transformation $s(u)$ such that $w(u',u)(u'-u) = H(s(u') - s(u))$ holds iff

$$w(u',u) = \frac{1}{(u'-u)} H(\frac{1}{H'(0)} \int_u^{u'} dx \; w(x)) \tag{22}$$

Proof : derivating the condition w.r.t. u' at $u' = u$ yields $H'(0)s'(u) = w(u,u)$, and substituting for $s(u)$ yields (22). Observe the symmetry of $w(u',u)$ as well as diagonal consistency $w(u,u) = w(u)$ to be automatically satisfied. □

Under conditions of lemma 6), $X_j \; \hat{\mathcal{R}} \; X_k$ iff $\sum_{\alpha=1}^r H(s(u_{\alpha j}) - s(u_{\alpha k})) \geq 0$, i.e. iff the additive difference model of section 3.1 holds, with the utilities of the latter related to the present ones by the monotonic tranformation $\tilde{u}_\alpha = s(u_\alpha)$, together with the correspondance $W_\alpha(x) = H(x/H'(0))$: the universality of $W_\alpha(x)$ implied here, i.e. its independance of the dimension α (in which case the additive difference model (10) remains transitive for $r = 2$), is a simple consequence of the assumed universality of the aggregation function $w(u'_\alpha, u_\alpha)$. Had we considered more general, dimension-dependant functions $w_\alpha(u'_\alpha, u_\alpha)$, the additive difference model would then appear as a particulary case of the symmetric context dependant model, characterized by the condition to be separately Fechnerian in each dimension.

Suppose now the model to be jointly Fechnerian in the multiattribute case $r \geq 2$. Then conditions of lemma 6) must in particular hold, together with the condition

$$F(\tilde{s}(X_j) - \tilde{s}(X_k)) = G(\sum_{\alpha=1}^{r} H(s(u_{\alpha j}) - s(u_{\alpha k}))) \tag{23}$$

as well as $H = G^{-1} o F$: thus $H(.)$ has to be linear, and (20) and (22) yield the representation

$$\hat{U}(X_j, X_k) = V(X_j) - V(X_k) \qquad \begin{array}{l} \text{where } V(X) := \sum_{\alpha=1}^r v_\alpha(X) \\ \text{and } v_\alpha(X) := \int_0^{u_\alpha(X)} du \; w(u) + \text{cst.} \end{array} \tag{24}$$

In summary, *multiattribute context dependant models (20) are Fechnerian iff the are effectively context-independant*, as expressed by (24) : Fechnerian (or strong [25]) condition can here be reinterpreted as a condition of effective context-independance in terms of the "renormalized" utilities $v_\alpha(X)$. When existing, the latter make the context-dependant symmetric model (21) effectively additive, with representation (1).

3.5.2 Binary strict utility models

We now investigate conditions which make the stochastic pair comparison model *strict*, i.e. such that $p(X_j, X_k) = \phi(s(X_j))/(\phi(s(X_j)) + \phi(s(X_k)))$ for some functions $\phi(.)$, $s(.)$. Strict models are well known [25] to be Fechnerian too, and in view of (24) the question amounts in the case $r \geq 2$

to conditions on $G(.)$ and $\phi(.)$ ensuring $G(V(X_j) - V(X_k)) = \phi(V(X_j))/(\phi(V(X_j)) + \phi(V(X_k)))$. The translation invariance $V(X) \to V(X) + a$ of the l.h.s. together with theorem 1) yields the unique solution $\phi(V(X)) = \exp(\beta V(X))$ together with $G(x) = 1/(1 + \exp(-\beta x))$. For $r = 1$, the same reasonning yields $G(H(x)) = 1/(1 + \exp(-\beta x))$, for some $H(x)$ increasing, odd, such that $H'(0) = 1$. Putting $H(x) = G^{-1}(1/(1 + \exp(-\beta x)))$ with $\beta = G'(0)$ satisfies all requirements. In a perceptual context rather than in a decision one, the latter condition shows in another way how the inverse temperature expresses the sensitivity $G'(0)$ of discrimination between close stimuli. Actually, the just noticeable difference, defined for instance as the semi-interquartile range in the V scale [10], turns out to be $\Delta V = \ln 3/\beta$ for the Boltzmannn-Gibbs distribution.

In summary, *multiattribute context-dependant models (20) are strict iff they are effectively context independant (as expressed by (24)) and governed by the Boltzmann-Gibbs distribution.*

3.5.3 Weak and strong stochastic transitivity

Let us now suppose *weak stochastic transitivity* (WST) to hold, i.e. the property that $p(X_j, X_k) \geq 1/2$ and $p(X_k, X_l) \geq 1/2$ together imply $p(X_j, X_l) \geq 1/2$. Then WST is equivalent to the property that $\hat{U}(X_j, X_k) = 0$ and $\hat{U}(X_k, X_l) = 0$ together imply $\hat{U}(X_j, X_l) = 0$. Definition (20) (with $w(u, u') > 0$) shows this to hold without addditional requirements for $r = 1$. Suppose now $r \geq 2$, and require WST to hold for the first two dimensions (with constant values on the remaining ones). Let $I(X)$ the set of alternatives indifferent to choice X. Smoothness of $w(u, u')$ makes $I(X)$ a one-dimensional smooth manifold in the utility plane ($r = 2$). Transitivity makes $I(X) = I(X')$ for all $X' \in I(X)$: indifference loci are equivalence classes induced by the indifference relation, and constitute a one-parameter family of curves determined by some representant of the class only.

Equivalently, there exist functions $k(.,.)$ and $K(.)$ such that the representation $f(u, u') := g(u_1, u'_1) + g(u_2, u'_2) = K(k(u_1, u_2) - k(u_1, u_2))$ holds, where $g(u_1, u'_1) := w(u_1, u'_1)(u'_1 - u_1)$. This in turn yields $f_{u'_2} f_{u_1 u'_1} = f_{u_1} f_{u_2 u'_1}$, and thus $g(u, u') = v(u') - v(u)$ for some $v(.)$, and finally, by a the same reasonning as before, $v(u) = \int_0^u dx\ w(x) + \text{cst}$[3]. Thus, for $r \geq 2$, WST is equivalent to Fechnerian property, and thus also to *strong stochastic transitivity* (SST), in view of the well known general hierarchy [25] : [strict utility model] → [Fechnerian property] → [SST] → [WST] (recall SST to be the property that $p(X_j, X_k) \geq 1/2$ and $p(X_k, X_l) \geq 1/2$ together imply $p(X_j, X_l) \geq p(X_j, X_k)$ and $p(X_j, X_l) \geq p(X_k, X_l)$). For $r = 1$, the minimal requirement $w(u, u')(u' - u)$ to be increasing in u' and decreasing in u is sufficient to insure SST.

The next theorem summarizes the results :

Theorem 7

a) *the monoattribute ($r = 1$) stochastic pair comparison model (21) is*

- *weakly transitive*
- *strongly transitive if $w(u, u')(u' - u)$ is increasing in u' and decreasing in u*
- *Fechnerian iff strict iff (22) holds.*

[3] The same reasonning applied to unilateral context-dependant utilities implies $w(u)(u' - u)$ to be expressible as some separable sum $a(u) + b(u')$ thus making $w(u) = \text{cst}$.

b) *the multiattribute ($r \geq 2$) stochastic pair comparison model (21) is*
• *weakly transitive iff strongly transitive iff Fechnerian iff effectively additive linear iff [(22) with $H(x) = cx$ holds].*
• *strict iff in addition $G(x) = 1/(1 + \exp(-\beta x))$ for some real β.*

In some respect, first part theorem 5b) is quite obvious : it simply says that (deterministic or stochastic) multiattibute context-dependent models are transitive iff they can be expressed in a context-independant way in terms of new renormalized, effective specific utilities v_α given in (24).

3.5.4 Application : transitive mutiattribute context-dependant aggregation

As theorem 7) and examples of section 3.4 suggest, picking a "plausible" function $w(u)$ as well as an unrelated "plausible" aggregation context-dependant rule $w(u', u) = g(w(u'), w(u))$ with $g(w(u), w(u)) = w(u)$ is most likely to generate intransitivities as soon as $r \geq 2$. Therefore, requiring transitivity for a given aggregation rule is bound to impose constraints on the functional form of $w(u)$, sometimes sufficient to determine it entirely.

As an example, consider the *quasilinear* aggregation rule $w(u', u) = f^{-1}(\frac{1}{2}(f(w(u') + f(w(u)))$ where $f(w)$ is smooth, stricly increasing. Thus transitivity conditions in theorem 5b) yield the identity

$$f(\frac{1}{u' - u} \int_u^{u'} dx\ w(x)) = \frac{1}{2}(f(w(u') + f(w(u))) \tag{25}$$

Note the context independant case $w(u) = \text{cst}$ to be a trivial solution, as it must. Also, if $w(u)$ is a solution for $f(w)$, so is $w(cu)$ (for $f(w)$ again) and $w(u) + c'$ for $f(w - c')$. Thus it is no wonder that expanding (25) in $(u' - u)$ yields automatically stisfied identities for the first and second order coefficient. The third order condition yields $2f'(w(u))\ w''(u) = -3f''(w(u))\ (w'(u))^2$, thus $w'(u) = A[f'(w(u))]^{-3/2}$. The fourth order condition does not bring anything new.

For instance, conditions above show $w(u) = (au + b)^{2/(3\gamma - 1)}$ when $f(w) = \text{sgn}(\gamma)\ w^\gamma \quad (\gamma \neq 0)$. Note the cases $\gamma = 1$ and $\gamma = -1$ to correspond to the linear, resp. harmonic mean. The geometric mean $f(w) = \ln w$ yields $w(u) = (au + b)^{-2}$. Also, the exponential quasi mean $f(w) = \exp(\gamma w)$ $(\gamma > 0)$ yields $w(u) = \frac{2}{3\gamma} \ln(au + b)$.

4 Conclusion : towards an evolutionary theory of risk attitude ?

1. Nothing forbids the context-dependant model to be be straightforwardly extended to *semi-orders* [24], i.e. typically to relations of the form $X_k \mathcal{P}_j X_l$, iff $U(X_k|X_j) - U(X_l|X_j) \ > \ c \ > \ 0$.

Also, our model, restricted so far to decisons under certainty, can easily be extended to risky situations. For instance, the relation

$$P \ \geq \ Q \qquad \text{iff} \qquad \int\int dP(X)\ dQ(Y)\ \hat{U}(X, Y) \tag{26}$$

between cumulative distributions P and Q supported in the feasible set A, defines a "skew-symmetric bilinear" theory [14][13] with a context-dependent kernel.

2. However, as far as risky situations are considered, we do not resist the pleasure to speculate further on the non-linear utility function $f(u) = c|u|^\gamma u$ appearing in theorem 4b) above. As it is well known, curvature of the utility functions, *in the context of monetary outcomes*, has been advocated for explaining widely observed risk attitudes inconsistent with a linear utility *and* an (objective or subjective) expected utility maximization principle (see [13][19][32] for a general discussion). Empirical evidence shows people to be generally risk-averse for gains and risk-seeking for losses, thus (by Jensens's inequality for expectations) implying the the utility function to be concave for gains and convex for losses. This feature is precisely verified by $f(u) = c|u|^\gamma u$ with $\gamma \in (-1, 0)$ if the identification "$u > 0$ iff gain and $u < 0$ iff loss" is made. Some data [27] suggest widely varying values of γ in the range $(-1, 0)$ for different experiments. Luce and Fishburn [26] have derived a somewhat more general form for $f(u)$ from a system of seven axioms explicitely breaking $u \leftrightarrow -u$ invariance and requiring scaling invariance. By contrast, our derivation uniquely relies upon the general relation (15) or (16).

3. To discuss those matters further, one has to address the question of how a non-constant weight function could offer an advantage for an organism. Suppose for simplicity the feasible set or environnment A to be dimensionnaly separable, i.e. $A = \prod_{\alpha=1}^{r} A_\alpha$ with $\sup_{X_\alpha \in A_\alpha} u(X_\alpha) = a_\alpha$, and suppose the organism to be globally oriented for the whole range of values, i.e. $w(u)$ everywhere decreasing. Then (6) makes the organism most sensitive to the dimension α_0 with lowest specific utility. He then can update his choice relatively to the corresponding A_{α_0}, and repeat the process until no amelioration is possible anymore : decreasing weights then provide a way of "editing" information and decomposing a complex task into r sequential unidimensional operations. This possibility is interesting as far as a_{α_0} is not too low : for if the environnment were particulary hostile regarding dimension α_0, the helpless organism would keep being assaulted by discomfort signals ($w(a_{\alpha_0})$ high). In this case, it would be better for the organism to *reduce* its sensitivity to harsh conditions, i.e. to be specifically oriented ($w(u)$ increasing) for large negative specific utilities. Consider now an organism whose weight function is $w(u) = c|u|^\gamma$ with $\gamma \in [-1, 0)$, refered by virtue of theorem 4b) to as a NEAR (Negative Exponent, Almost Reaching) organism. NEAR organisms are risk-averse for gains and risk-seeking for losses; as $w'(u) = c\gamma|u|^{\gamma-1}\text{sgn}(u)$, they also are globally (specifically) oriented for positive (negative) utilities, with a maximal sensitivity around zero utilities. As far as the environnment is such that a_{α_0} is not too close to zero (in which case sensitivity is uselessly high at the optimum), NEAR organisms manage in reaching optimal intrinsic utility and "serenity" (low weights), for both harsh ($a_{\alpha_0} << 0$) and easy ($a_{\alpha_0} >> 0$) conditions.

By contrast, the PEAR (Positive Exponent, Almost Reaching) organism with weight function $w(u) = c|u|^\gamma$ with $\gamma > 0$ will "edit" dimensions in reverse order, i.e. begin either with the current highest-utility dimensions (in which case specific improvement will still increase its sensitivity and possibly mask signals from others dimensions) or with the current lowest-utility dimensions (where he is threatened to be trapped in low specific maxima with corresponding high and useless sensitivity). As far as a_{α_0} is not too close to zero, PEAR organisms show "concern" (high weights) for both harsh and easy conditions.

It might be argued that if environnement conditions were constant ($a_\alpha(t) = \text{cst}$), then organisms adapting to this environnment would precisely do in a way fixing their zero utilities at that level, i.e. experience $a_\alpha = 0$, in which case PEAR strategies would be more efficient that NEAR ones. But as soon as sufficiently large fluctuations in the environnment or within the species do exist, adaptibility of NEAR organisms proves superior.

Actually, NEAR properties seem more familiar to us. Take for instance $u_{\alpha 0}$ = "air quality". If $a_{\alpha_0} >> 0$, the environnment is in this respect so favorable that we are simply not aware of dimension α_0. Idem if $a_{\alpha_0} << 0$, in which case we are guaranteed a peaceful death. Our concern is actually highest when air quality is mediocre ($a_{\alpha_0} \approx 0$). Similar considerations could apply to quiet versus noisy surroundings, to fulfilled sexual life versus total deprivation, or to physically secure versus insecure environnment. True, particulary high upper utilities a_{α_0} (as largely above average intellectual or athletic performances) as well as particulary low utilities (as a strong mental or pyhsical handicap) are given importance and don't go unnoticed, but the concern in question is the fact of the society (for economic reasons, prestige or compassion) rather than of the organism : for the latter, complete satisfaction *or dissatisfaction* of a need makes it invisible.

4. Another way of looking at NEAR organisms is the following : for "dimension-editing" reasons, i.e. from the necessity of coping with a complex environment A, it is desirable for the weights in the additive function (6) to be non-constant[4]. On the other hand, theorem 4) shows non-constancy incompatible with SIC-safe or MIF-safe decision rules. So a compromise could be reached by satisfying local, first-order safety only. By theorem 4b), NEAR and PEAR organisms are the only non-constant solutions. To discuss their relative merit, let us define the potential situation X_k to be *distinct from actual situation X_j through positive (resp. negative) utilities* if the components $\epsilon_\alpha = u'_\alpha - u_\alpha$ associated with positive (resp. negative) values u_α of the actual situation are of large absolute value compared to the components ϵ_α associated with negative (resp. positive) values. The definition presupposes of course the existence of such positive (resp. negative) utilities. Equations (15) and (16) together with $f''(u) = c\gamma(1+\gamma)|u|^{\gamma-1}\text{sgn}(u)$ and table 2 then show for NEAR organisms possibility of missing opportunity towards situations distinct enough through negative utilities, and possibility of worsening towards situations distinct enough through positive utilities. Thus NEAR organisms, when in bad position (many current negative utilities), are likely to miss opportunities if the potential state is distinct enough from the current situation, but are at least guaranteed not to worsen their condition and conversely. In short, NEAR organisms are likely to be sometimes too conservative when in bad positions and too daring when in good ones. By contrast, PEAR organisms are likely to be sometimes too daring when in bad positions and too conservative when in good ones. Intuitively at least, their strategy appears more hazardous. A more assertive argument would require the introduction of a multiplayer model of struggle for survival and reproduction in the environnment A, as well as a r-variate utility distribution on A.

[4]One could object the additive form (6) to be unduly restrictive ; but recall that the features $\alpha = 1, ..., r$ we have in mind are precisely determined by the property to represent independant dimensions, so additivity is indeed tautological.

5. At it has been said, NEAR organisms are risk-averse for gains and risk-seeking for losses. Such also are individuals in our species in an apparently overwhelming majority : all happens as if NEAR organisms have survived, and PEAR organisms have become extinct. This is a third, evolutionnary argument aimed to assessing the superior fitness for survival and reproduction exhibited by NEAR organisms over PEAR organisms.

Or conversely, the previous arguments 3) and 4) might appear as convincing enough to give a basis for an evolutionnary explanation of the prevailing risk-attitude in our species.

6. Of course, many questions remain unsolved. First of all, our conception of utility has been largely associated with "satisfaction", i.e. "organic states of mind" ; but monetary outcomes, upon which the risk attitude theory has been built, constitute a rather peculiar dimension (rather abstractly related to organic dimensions) whose late appparence in human history cannot have left an evolutionnary print. What is needed here (and beyond the scope of the paper) is a model of the way money can improve organic utilities, thus deriving a monetary utility form an organic one.

Secondly, and related to the first question, the utility $f(u) = w(u)\ u$ defining NEAR and PEAR organisms is the intrinsic utility $U(u)$ (6) rather than the frustration $U(u + u_0|u_0) - U(u_0)$, where u_0 is the initial position. All happen as if the representation of money, only indirectly related to pleasure, were more "cognitively" internalized (compared to other organic dimensions, more "emotionally" encoded), and therefore more transparent to the subject, enabling him to access directly the intrinsic (rather than the context-dependant) utility when dealing with risky monetary outcomes.

Also, although the form of NEAR and PEAR weights has been determined form SIC-safe and MIC-safe considerations explicitely involving concepts of contentment and frustration, the previous discussion has left the u_0 dependance under silence. Such a dependence, apparently empirically small [19] but existing, could of course be implemented in an ad hoc (and unsatisfactory) way in the coefficient c, made decreasing in u_0.

Finally, the exposed theory is essentially symmetric in utility ($U(-u) = -U(u)$), but the living conditions certainly are not, and further refinements are needed at breaking this symmetry, empirically observed in the shape of the utility function, steeper for looses than for gains [19].

7. Expected utility hypothesis has rightly been criticized (see e.g. [2]) for being of arguable relevance to *individuals* generally facing particular choices and decisions only once. However, the very mechanism of evolution relies upon practically unlimited repetitions of choices and decisions, thus ensuring the survival of the best player given the (objective) probabilities of the state of the word : at the *species* level, dealing with expected utilities might reveal itself unavoidable.

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