On Parisian ruin over a finite-time horizon

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Abstract For a risk process $R_u(t) = u + ct - X(t), t \geqslant 0$, where $u \geqslant 0$ is the initial capital, $c > 0$ is the premium rate and $X(t), t \geqslant 0$ is an aggregate claim process, we investigate the probability of the Parisian ruin

$$\mathcal{P}_S(u, T_u) = \mathbb{P}\left\{ \inf_{t\in[0,S]} \sup_{s\in[t,t+T_u]} R_u(s) < 0 \right\}, \quad S, T_u > 0.$$

For X being a general Gaussian processes we derive approximations of $\mathcal{P}_S(u, T_u)$ as $u \to \infty$. As a by-product, we obtain the tail asymptotic behaviour of the infimum of a standard Brownian motion with drift over a finite-time interval.

Keywords Parisian ruin, Gaussian process, Lévy process, fractional Brownian motion, infimum of Brownian motion, generalized Pickands constant, generalized Piterbarg constant

1 Introduction

Consider a random process $\{X(t), t \geqslant 0\}$ which models the aggregate claim process of an insurance portfolio, i.e., $X(t)$ represents the total amount of claims paid up to time t. In a theoretical insurance model the main object of interest is the so-called surplus process R_u, defined by

$$R_u(t) = u + ct - X(t), \quad t \geqslant 0, \tag{1.1}$$

where $c > 0$ models the premium income rate and $u \geqslant 0$ is the initial capital; see e.g., [16]. For any $S \in (0, \infty]$, define the (classical) *probability of ruin* during the time period $[0, S]$ as

$$p_S(u) := \mathbb{P}\left\{ \inf_{t\in[0,S]} R_u(t) < 0 \right\}. \tag{1.2}$$

We refer to [13, 16, 24, 25] and references therein for important investigations of $p_S(u)$.

The contributions [4, 7] introduced and studied the Parisian ruin which allows the surplus process to spend a pre-specified time under level zero before ruin is recognized. Initially, Parisian stopping times have been investigated by [2] in the context of barrier options in mathematical finance.
Let T_u model the pre-specified time which is a positive deterministic function of u. In our setup, the *probability of Pariasian ruin* over the time period $[0, S]$ is defined as

$$\mathcal{P}_S(u, T_u) = \mathbb{P}\left\{ \inf_{t\in[0,S]} \sup_{s\in[t,t+T_u]} R_u(s) < 0 \right\}.$$

Calculation of the probability of Parisian ruin $\mathcal{P}_S(u, T_u)$ is more complex than the calculation of $p_S(u)$. When $S = \infty$ and X is modelled by a specific class of Lévy processes, exact formulas for $\mathcal{P}_\infty(u, T)$, with $T \in (0, \infty)$ are derived in [4, 7, 26]. See also [3, 5, 6, 28] for some recent developments.

In this paper, we shall investigate the probability of Parisian ruin when the initial capital becomes large (tends to infinity) and X is modeled by a Gaussian process. It appears that the qualitative type of the obtained asymptotics is different from that of the corresponding Lévy model. Specifically, if X is a Lévy process such that $X(S)$ has a long-tailed distribution, which in view of [18] means that there exists some function $h(u), u \geqslant 0$ satisfying $\lim_{u\to\infty} \frac{u}{h(u)} = \lim_{u\to\infty} h(u) = \infty$ such that

$$\mathbb{P}\{X(S) > u + h(u)\} = \mathbb{P}\{X(S) > u\} (1 + o(1)), \quad u \to \infty, \tag{1.3}$$

then the following proposition holds.

Proposition 1.1. Let $S > 0$, and $T_u, u \geqslant 0$ be a positive bounded measurable function. If X is a Lévy process such that $X(S)$ has a long-tailed distribution, then

$$\mathcal{P}_S(u, T_u) = \mathbb{P}\left\{X(S) > u\right\}(1 + o(1)), \quad u \to \infty. \tag{1.4}$$

We give the proof of the above proposition in Section 4.
A straightforward application of Proposition 1.1 for X being an α-stable Lévy process with $\alpha \in (1, 2)$, (i.e., $X(t) \overset{d}{=} \mathcal{S}_\alpha(t^{1/\alpha}, \beta, 0), t > 0$, where $\mathcal{S}_\alpha(\sigma, \beta, d)$ denotes a stable random variable with index of stability α, scale parameter σ, skewness parameter β and drift parameter d; see e.g., [32]), implies that

$$\mathcal{P}_S(u, T_u) = \frac{(1-\alpha)}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}\left(\frac{1+\beta}{2}\right) S u^{-\alpha}(1 + o(1)), \quad u \to \infty,$$

where $\Gamma(\cdot)$ denotes the Euler Gamma function.

The above restriction that $X(S)$ is long-tailed excludes the classical case that X is a standard Brownian motion. Given the importance of the Brownian risk process (see e.g., [11, 24, 27]) in this contribution we shall investigate the asymptotics of $\mathcal{P}_S(u, T_u)$ with $S \in (0, \infty)$ for large classes of Gaussian risk processes. It turns out that in contrast to Proposition 1.1, for this model the asymptotics is highly sensitive to T_u. Details are presented in Section 3.

As shown for instance in [11, 21, 23], the calculation of the probability of ruin over an infinite-time horizon for Gaussian risk processes raises interesting theoretical questions for the asymptotic theory of Gaussian processes and related random fields. Similarly, the calculation of the probability of Parisian ruin over finite-time horizon raises several interesting questions as well. For instance, for our investigations it is crucial to obtain certain extensions of Piterbarg lemma, which we shall present in Lemma 5.1 in Appendix. For details on Piterbarg and Pickands lemmas see e.g., [8, 9, 12, 22]. Another interesting problem motivated by this paper is the investigation of the asymptotic behaviour of

$$\mathbb{P}\left\{\inf_{t\in[T_1,T_2]}\Big(X(t) - ct\Big) > u\right\}, \quad T_2 > T_1 > 0$$

as $u \to \infty$ with X a centered non-stationary Gaussian process. This problem seems to be very hard; here we shall deal only with X being a standard Brownian motion $\{B_1(t), t \geqslant 0\}$; see Theorem 2.1 below.

This paper is organized as follows: After some preliminary results given in the next section, in Section 3 we present our main findings. Theorem 3.3 provides the exact asymptotics of $\mathcal{P}_S(u, T_u)$ for T_u converging to 0. When X is a standard Brownian motion our result holds for $u^2 T_u \to T \in [0, \infty)$ as $u \to \infty$. The case of constant or general bounded T_u is investigated in Theorem 3.1, which gives an asymptotic lower bound for $\mathcal{P}_S(u, T)$. Furthermore Theorem 3.2 displays the logarithmic asymptotics of $\mathcal{P}_S(u, T_u)$. All the proofs are relegated to Section 4, followed by an Appendix (Section 5).

2 Preliminaries

Let $\{X(t), t \geqslant 0\}$ be a centered Gaussian process with almost surely (a.s.) continuous sample paths and variance function $\sigma^2(\cdot)$. In our setup, $\sigma(\cdot)$ is not a constant function, and therefore the stationary Gaussian processes are excluded. The theory of extremes of non-stationary Gaussian processes is established in numerous contributions; see e.g., [17, 30]. A key condition in the case of processes with non-constant variance is its local structure at the maximum point of the variance function; for our setup we shall assume the following local condition:
Assumption A1. The standard deviation function $\sigma(\cdot)$ of the Gaussian process X attains its maximum $\widetilde{\sigma}$ on $[0, S]$ at the unique point $t = S$. Further, there exist positive constants β_1, β_2, A, and $A_+ > 0$ (or $A_- < 0$) such that

$$\sigma(t) = \widetilde{\sigma} - A(S-t)^{\beta_1}(1 + o(1)), \quad t \uparrow S$$

and

$$\sigma(t) = \widetilde{\sigma} - A_\pm(t-S)^{\beta_2}(1 + o(1)), \quad t \downarrow S. \tag{2.1}$$

It is worth noting that in our setup the behaviour of $\sigma(\cdot)$ in the right neighborhood of S can be different from that in the left-neighbourhood of S. Specifically, in condition (2.1) the constant $A_\pm$ can be positive or negative, and moreover the index β_2 can be different from the index β_1.

Our next two assumptions are standard, see Chapter 1 in [30].
Assumption A2. There exist some positive constants $\alpha \in (0, 2], D$ such that

$$\mathrm{Cov}\left(\frac{X(t)}{\sigma(t)}, \frac{X(s)}{\sigma(s)}\right) = 1 - D|t-s|^\alpha(1 + o(1)), \quad t, s \to S.$$

Assumption A3. There exist some positive constants Q, γ and $S_1 < S$ such that, for all $s, t \in [S_1, S]$

$$\mathbb{E}\left\{(X(t) - X(s))^2\right\} \leqslant Q|t-s|^{\gamma}. \tag{2.2}$$

Next, we introduce some generalizations of the Pickands and Piterbarg constants. We refer to [29,30,33] for the definitions and properties of the (classical) Pickands and Piterbarg constants. See also [14] for alternative formulas of Pickands constant.
Let $\{B_\alpha(t), t \in \mathbb{R}\}$ be a standard fractional Brownian motion (fBm) with Hurst index $\alpha/2 \in (0,1]$, i.e., it is a centered Gaussian process with a.s. continuous sample paths and covariance function

$$\mathrm{Cov}(B_\alpha(t), B_\alpha(s)) = \frac{1}{2}(|t|^\alpha + |s|^\alpha - |t-s|^\alpha), \quad s, t \in \mathbb{R}.$$

Define the generalized Pickands constant as

$$\widetilde{\mathcal{H}}_\alpha(T) = \lim_{\lambda\to\infty} \frac{1}{\lambda}\widetilde{\mathcal{H}}_\alpha(\lambda, T), \quad T \geqslant 0, , \alpha \in (0,2], \tag{2.3}$$

where

$$\widetilde{\mathcal{H}}_\alpha(\lambda, T) = \mathbb{E}\left\{\exp\left(\sup_{t\in[0,\lambda]}\inf_{s\in[0,T]}\left(\sqrt{2}B_\alpha(t-s) - |t-s|^\alpha\right)\right)\right\} \in (0,\infty), \quad \lambda, T \geqslant 0, \alpha \in (0,2].$$

Further, we define the generalized Piterbarg constant as

$$\widetilde{\mathcal{P}}_{\alpha,\beta}^{b_1,b_2}(T) = \lim_{\lambda\to\infty} \widetilde{\mathcal{P}}_{\alpha,\beta}^{b_1,b_2}(\lambda, T), \quad T \geqslant 0, \ b_1 > 0, b_2 \in \mathbb{R}, \alpha \in (0,2], \beta \geqslant \alpha, \tag{2.4}$$

where, for any positive constants $\lambda, \beta, b_1, T \geqslant 0, \alpha \in (0,2]$ and $b_2 \in \mathbb{R}$

$$\widetilde{\mathcal{P}}_{\alpha,\beta}^{b_1,b_2}(\lambda, T) = \mathbb{E}\left\{\exp\left(\sup_{t\in[0,\lambda]}\inf_{s\in[0,T]}\left(\sqrt{2}B_\alpha(t-s) - |t-s|^\alpha\left[1 + b_1 I_{(t>s)} + b_2 I_{(t\leqslant s, \alpha=\beta)}\right]\right)\right)\right\},$$

with $I_{(\cdot)}$ the indicator function. Note that both $\widetilde{\mathcal{H}}_\alpha(\lambda, T)$ and $\widetilde{\mathcal{P}}_{\alpha,\beta}^{b_1,b_2}(\lambda, T)$ are well defined since

$$\mathbb{E}\left\{\exp\left(\sup_{t\in[0,\lambda]}\sqrt{2}B_\alpha(t)\right)\right\} < \infty, \quad \forall \lambda \geqslant 0,$$

which follows directly from Piterbarg inequality (see Theorem 8.1 in [30]). As it will be seen from the proof of Theorem 3.3 below, both $\widetilde{\mathcal{H}}_\alpha(T)$ and $\widetilde{\mathcal{P}}_{\alpha,\beta}^{b_1,b_2}(T)$ defined above are positive and finite. Note further that the classical Pickands constant $\mathcal{H}_\alpha$ equals $\widetilde{\mathcal{H}}_\alpha(0)$ and the classical Piterbarg constant $\mathcal{H}_\alpha^{b_1}$ equals $\widetilde{\mathcal{P}}_{\alpha,\beta}^{b_1,b_2}(0)$.

Finally, we present a theorem on the asymptotics of the infimum of Brownian motion with linear drift over a finite-time interval, which will be used in the next section and is of some independent interest. Hereafter $\Psi(\cdot)$ denotes the tail distribution function of an $N(0,1)$ random variable and $\varphi(\cdot)$ is its density.

Theorem 2.1. For any $c > 0$ and two constants $T_2 > T_1 > 0$ we have

$$\mathbb{P}\left\{\inf_{t\in[T_1,T_2]}\left(B_1(t) - ct\right) > u\right\} = K_{c,T_2-T_1}\frac{T_1}{u}\Psi\left(\frac{u + cT_1}{\sqrt{T_1}}\right)(1 + o(1)), \quad u \to \infty, \tag{2.5}$$

where

$$K_{c,y} = 2\varphi\left(c\sqrt{y}\right)\frac{1}{\sqrt{y}} - 2c\Psi\left(c\sqrt{y}\right) > 0, \quad y > 0. \tag{2.6}$$

3 Main Results

In this section, we present our main results on the asymptotic behaviour of $\mathcal{P}_S(u, T_u)$ as $u \to \infty$. It turns out that when T_u does not vanish to 0 as $u \to \infty$, the exact asymptotics is very hard to derive. For such cases we shall give a lower asymptotic bound and then the logarithmic asymptotics of $\mathcal{P}_S(u, T_u)$ for X being with stationary increments. Finally, in Theorem 3.3 we show the exact asymptotics of $\mathcal{P}_S(u, T_u)$, under certain restrictions on the speed of convergence of T_u to 0, for X satisfying **A1**–**A3**.

3.1 Logarithmic asymptotics

The following theorem displays an asymptotic lower bound of $\mathcal{P}_S(u,T)$, which is logarithmically exact for all large u. We write below $V'(t)$ for the derivative of the variance function $V(t)=\sigma^2(t)$ if it exists.

Theorem 3.1. Let $\{X(t), t\geqslant 0\}$ be a centered Gaussian process with a.s. continuous sample paths, $X(0)=0$ and stationary increments. If further the variance function $\sigma^2(\cdot)$ is differentiable, strictly increasing and convex, then for any positive constants S,T

$$\mathcal{P}_S(u,T)\geqslant C_{c,\Delta}\frac{\sigma^2(S)}{u}\Psi\left(\frac{u+cS}{\sigma(S)}\right)(1+o(1)),\quad u\to\infty, \tag{3.1}$$

where

$$C_{c,\Delta}=2\varphi\left(\frac{c\sqrt{\Delta}}{V'(S)}\right)\frac{1}{\sqrt{\Delta}}-2\frac{c}{V'(S)}\Psi\left(\frac{c\sqrt{\Delta}}{V'(S)}\right),\quad \Delta=\sigma^2(S+T)-\sigma^2(S).$$

The proof of Theorem 3.1 is given in Section 4.3.

The next result constitutes an LDP counterpart of Proposition 1.1.

Theorem 3.2. Let $\{X(t), t\geqslant 0\}$ be a centered Gaussian process with a.s. continuous sample paths, $X(0)=0$ and stationary increments. If further the variance function $\sigma^2(\cdot)$ is differentiable, strictly increasing and convex, then for any bounded measurable function $T_u>0$ and any $S>0$

$$\lim_{u\to\infty}\frac{\log(\mathcal{P}_S(u,T_u))}{u^2}=-\frac{1}{\sigma^2(S)}. \tag{3.2}$$

Section 4.4 displays the proof of Theorem 3.2.

We note that, the claim in (3.2) matches the logarithmic asymptotics of the classical ruin probability, i.e.

$$\lim_{u\to\infty}\frac{\log(\mathcal{P}_S(u,T_u))}{\log(p_S(u))}=\lim_{u\to\infty}\frac{\log(\mathcal{P}_S(u,T_u))}{\log(\mathbb{P}\{X(S)>u\})}=1$$

and does not depend on the value of the parameter c.

3.2 Exact asymptotics

The problem of finding the exact asymptotics of $\mathcal{P}_S(u,T_u)$ needs much more precise analysis. Next, we discuss the case that T_u is sufficiently small, tending to 0 as $u\to\infty$.

Theorem 3.3. Let $\{X(t), t\geqslant 0\}$ be a centered Gaussian process satisfying assumptions **A1-A3** with the parameters therein, and let T_u be a positive measurable function of u. Assume that $\beta_1\leqslant\beta_2\leqslant 1$. For any positive constant S, we have, as $u\to\infty$:

(i) If $\alpha<\beta_1$ and $\lim_{u\to\infty}T_u u^{2/\alpha}=T\in[0,\infty)$, then

$$\mathcal{P}_S(u,T_u)=\widetilde{\mathcal{H}}_\alpha(D^{\frac{1}{\alpha}}\widetilde{\sigma}^{-\frac{2}{\alpha}}T)\Gamma\left(\frac{1}{\beta_1}+1\right)D^{\frac{1}{\alpha}}A^{-\frac{1}{\beta_1}}\widetilde{\sigma}^{\frac{3}{\beta_1}-\frac{2}{\alpha}}u^{\frac{2}{\alpha}-\frac{2}{\beta_1}}\Psi\left(\frac{u+cS}{\widetilde{\sigma}}\right)(1+o(1)). \tag{3.3}$$

(ii) If $\alpha=\beta_1$ and $\lim_{u\to\infty}T_u u^{2/\alpha}=T\in[0,\infty)$, then

$$\mathcal{P}_S(u,T_u)=\widetilde{\mathcal{P}}_{\alpha,\beta_2}^{A/(D\widetilde{\sigma}),A_\pm/(D\widetilde{\sigma})}(D^{\frac{1}{\alpha}}\widetilde{\sigma}^{-\frac{2}{\alpha}}T)\Psi\left(\frac{u+cS}{\widetilde{\sigma}}\right)(1+o(1)). \tag{3.4}$$

(iii) If $\alpha>\beta_1$ and $\lim_{u\to\infty}T_u u^{2/\alpha}=\lim_{u\to\infty}T_u u^{2/\beta_2}=0$, then

$$\mathcal{P}_S(u,T_u)=\Psi\left(\frac{u+cS}{\widetilde{\sigma}}\right)(1+o(1)). \tag{3.5}$$

Remark 3.1. Clearly, if $T_u=0, u\geqslant 0$, then $\mathcal{P}_S(u,0)$ becomes the classical probability of ruin $p_S(u)$. Since, as mentioned above, $\widetilde{\mathcal{H}}_\alpha(0)=\mathcal{H}_\alpha$ and $\widetilde{\mathcal{P}}_{\alpha,\beta_2}^{A/(D\widetilde{\sigma}),A_\pm/(D\widetilde{\sigma})}(0)=\mathcal{H}_\alpha^{A/(D\widetilde{\sigma})}$, the asymptotics of $p_S(u)$ is retrieved and agrees with findings of [30].

Specialized to the case of the fBm risk process, the above theorem entails the following result.

Corollary 3.4. Let $\{X(t), t \geqslant 0\}$ be a standard fBm with Hurst index $\alpha/2 \in (0,1]$. For any positive constant S, we have, as $u \to \infty$:
(i) If $\alpha \in (0,1)$ and $\lim_{u\to\infty} T_u u^{2/\alpha} = T \in [0,\infty)$, then

$$\mathcal{P}_S(u,T_u) = \widetilde{\mathcal{H}}_\alpha(2^{-\frac{1}{\alpha}}S^{-2}T)\alpha^{-1}2^{1-\frac{1}{\alpha}}S^{\alpha-1}u^{\frac{2}{\alpha}-2}\Psi\left(\frac{u+cS}{S^{\alpha/2}}\right)(1+o(1)).$$

(ii) If $\alpha = 1$ and $\lim_{u\to\infty} T_u u^2 = T \in [0,\infty)$, then

$$\mathcal{P}_S(u,T_u) = \widetilde{\mathcal{P}}_{1,1}^{1,-1}(2^{-1}S^{-2}T)\Psi\left(\frac{u+cS}{S^{1/2}}\right)(1+o(1)).$$

(iii) If $\alpha \in (1,2]$ and $\lim_{u\to\infty} T_u u^2 = 0$, then

$$\mathcal{P}_S(u,T_u) = \Psi\left(\frac{u+cS}{S^{\alpha/2}}\right)(1+o(1)).$$

Remark 3.2. The case that $T_u = T > 0$ for all u large is much more difficult to deal with and most probably needs to develop new techniques that allow derivation of the asymptotics of tail distribution of infimum of a Gaussian process.

Remark 3.3. As in [4,11,26] we define the *Parisian ruin time* of the risk process R_u by

$$\tau_u = \inf\{t \geqslant T_u : t-\kappa_{t,u} \geqslant T_u\}, \quad \text{with } \kappa_{t,u} = \sup\{s\in[0,t] : R_u(s) \geqslant 0\}.$$

Under the assumptions of Corollary 3.4 it follows along the lines of the arguments in [10] that

$$\lim_{u\to\infty} \mathbb{P}\left\{u^2(S-\tau_u) \leqslant x \middle| \tau_u < S\right\} = 1 - \exp\left(-\frac{\alpha}{2}S^{-\alpha-1}x\right) \tag{3.6}$$

holds for any x positive.

4 Proofs

4.1 Proof of Proposition 1.1

First, for any u positive

$$\begin{aligned}\mathcal{P}_S(u,T_u) &= \mathbb{P}\left\{\sup_{t\in[0,S]}\inf_{s\in[t,t+T_u]}\Big(X(s)-cs\Big) > u\right\}\\ &\leqslant \mathbb{P}\left\{\sup_{t\in[0,S]} X(t) > u\right\}.\end{aligned}$$

Further, in view of [1] we have

$$\mathbb{P}\left\{\sup_{t\in[0,S]} X(t) > u\right\} = \mathbb{P}\{X(S)>u\}(1+o(1)), \quad u\to\infty$$

implying thus

$$\mathcal{P}_S(u,T_u) \leqslant \mathbb{P}\{X(S)>u\}(1+o(1)), \quad u\to\infty.$$

We derive next the lower bound. Taking $h(\cdot)$ to be such that (1.3) holds we have

$$\begin{aligned}&\mathbb{P}\left\{\sup_{t\in[0,S]}\inf_{s\in[t,t+T_u]}\Big(X(s)-cs\Big) > u\right\}\\ &\quad\geqslant \mathbb{P}\left\{\inf_{s\in[S,S+T_u]}\Big(X(s)-cs\Big) > u\right\}\\ &\quad\geqslant \mathbb{P}\left\{\inf_{t\in[S,S+T_u]}\Big(X(t)-X(S)-c(t-S)+X(S)-cS\Big) > u, X(S)-cS > u+h(u)\right\}.\end{aligned}$$

Since T_u is bounded, we have $\sup_{u\in[0,\infty)} T_u < M$ for some constant M. By the fact that X has independent and stationary increments we may further write

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in[0,S]}\inf_{s\in[t,t+T_u]}\Big(X(s)-cs\Big)>u\right\}\\
&\geqslant \mathbb{P}\left\{\inf_{t\in[0,M]}\Big(X(t)-ct\Big)>-h(u)\right\}\mathbb{P}\left\{X(S)-cS>u+h(u)\right\}\\
&= \mathbb{P}\left\{X(S)>u\right\}(1+o(1)), \quad u\to\infty
\end{aligned}$$

establishing the proof. □

4.2 Proof of Theorem 2.1

In order to derive the proof of Theorem 2.1, i.e., the exact asymptotic behaviour of the infimum of the standard Brownian motion with drift we shall investigate in Lemma 4.1 the tail asymptotics of the difference $X-Y$ assuming that X has distribution F with unbounded support and $Y\geqslant 0$ almost surely. If for any $\eta>0$

$$\lim_{u\to\infty}\frac{\mathbb{P}\left\{X>u+\eta\right\}}{\mathbb{P}\left\{X>u\right\}}=0, \tag{4.1}$$

then Lemma 2 in [15] entails

$$\lim_{u\to\infty}\frac{\mathbb{P}\left\{X-Y>u\right\}}{\mathbb{P}\left\{X>u\right\}}=\mathbb{P}\left\{Y=0\right\}.$$

If F is in the Gumbel max-domain of attraction with some positive scaling function $w(\cdot)$, i.e.,

$$1-F(u+x/w(u))=\exp(-x)(1-F(u))(1+o(1)), \quad \forall x\in\mathbb{R} \tag{4.2}$$

as $u\to\infty$, then (4.1) is satisfied if additionally $\lim_{u\to\infty} w(u)=\infty$. As shown below, it is possible to derive the exact tail asymptotics of $X-Y$ when $\mathbb{P}\left\{Y=0\right\}=0$ assuming further that for some $\alpha\geqslant 0$

$$\mathbb{P}\left\{Y<x/u\right\}=x^{\alpha}\mathbb{P}\left\{Y<1/u\right\}(1+o(1)), \quad \forall x>0 \tag{4.3}$$

holds as $u\to\infty$.

Lemma 4.1. Let X and Y be two independent random variables. If (4.2) holds for some positive function $w(\cdot)$ such that $\lim_{u\to\infty} w(u)=\infty$ and further $Y\geqslant 0$ satisfies (4.3) with some $\alpha\geqslant 0$, then we have

$$\mathbb{P}\left\{X-Y>u\right\}=\Gamma(\alpha+1)\mathbb{P}\left\{Y<1/w(u)\right\}\mathbb{P}\left\{X>u\right\}(1+o(1)), \quad u\to\infty. \tag{4.4}$$

In particular, if Y possesses a density function $f(\cdot)$ in a neighborhood of 0 such that $f(0)>0$, then

$$\mathbb{P}\left\{X-Y>u\right\}=\frac{f(0)}{w(u)}\mathbb{P}\left\{X>u\right\}(1+o(1)), \quad u\to\infty. \tag{4.5}$$

Proof of Lemma 4.1: The assumption that $\lim_{u\to\infty} w(u)=\infty$ implies that $\exp(X)$ is in the Gumbel MDA with scaling function $w^*(u)=w(\log u)/u$. Further, (4.3) is equivalent with

$$\lim_{u\to\infty}\frac{\mathbb{P}\left\{e^{-Y}>1-x/u\right\}}{\mathbb{P}\left\{e^{-Y}>1-1/u\right\}}=x^{\alpha}, \quad x>0.$$

Since for any positive u we have

$$\mathbb{P}\left\{X-Y>u\right\} = \mathbb{P}\left\{e^{X}e^{-Y}>e^{u}\right\},$$

then by Example 1 in [19] or Theorem 4.2 in [20]

$$\begin{aligned}
\mathbb{P}\left\{X-Y>u\right\} &= \Gamma(\alpha+1)\mathbb{P}\left\{e^{-Y}>1-1/(e^{u}w^*(e^{u}))\right\}\mathbb{P}\left\{e^{X}>e^{u}\right\}(1+o(1))\\
&= \Gamma(\alpha+1)\mathbb{P}\left\{Y<1/w(u)\right\}\mathbb{P}\left\{e^{X}>e^{u}\right\}(1+o(1)), \quad u\to\infty.
\end{aligned}$$

In the special case that Y possesses a density function $f(\cdot)$ with $f(0)>0$, then $\alpha=1$ and

$$\mathbb{P}\left\{Y<1/w(u)\right\}=\frac{f(0)}{w(u)}(1+o(1))$$

as $u\to\infty$ establishing the proof. $\square$

Proof of Theorem 2.1: Let $\mathcal{N}$ be a standard $N(0,1)$ random variable with density function φ which is independent of the Brownian motion $B(\cdot):=B_1(\cdot)$. We have with $\Delta:=T_2-T_1>0$

$$\begin{aligned}\mathbb{P}\left\{\inf_{t\in[T_1,T_2]}\Big(B(t)-ct\Big)>u\right\} &= \mathbb{P}\left\{\inf_{t\in[T_1,T_2]}\Big(B(t)-B(T_1)-c(t-T_1)+B(T_1)-cT_1\Big)>u\right\}\\ &= \mathbb{P}\left\{T_1^{1/2}\mathcal{N}-\sup_{t\in[0,\Delta]}\Big(B(t)+ct\Big)>u+cT_1\right\}.\end{aligned}$$

It is well-known that for any $u>0$ and $c\geqslant 0$

$$\mathbb{P}\left\{\sup_{t\in[0,\Delta]}\Big(B(t)+ct\Big)>u\right\}=\Psi\left(\frac{u-c\Delta}{\sqrt{\Delta}}\right)+e^{2cu}\Psi\left(\frac{u+c\Delta}{\sqrt{\Delta}}\right),$$

hence the density function q of $\sup_{t\in[0,\Delta]}(B(t)+ct)$ is given by

$$q(u)=\varphi\left(\frac{u-c\Delta}{\sqrt{\Delta}}\right)\frac{1}{\sqrt{\Delta}}-2ce^{2cu}\Psi\left(\frac{u+c\Delta}{\sqrt{\Delta}}\right)+e^{2cu}\varphi\left(\frac{u+c\Delta}{\sqrt{\Delta}}\right)\frac{1}{\sqrt{\Delta}},\quad u>0.$$

Since $\sqrt{T_1}\mathcal{N}$ has distribution in the Gumbel MDA with $w(u)=u/T_1$ and

$$q(0)=2\varphi\left(c\sqrt{\Delta}\right)\frac{1}{\sqrt{\Delta}}-2c\Psi\left(c\sqrt{\Delta}\right)>0$$

the claim follows from Lemma 4.1. $\square$

4.3 Proof of Theorem 3.1

For any u positive we have

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,S]}\inf_{s\in[t,t+T]}\Big(X(s)-cs\Big)>u\right\} &\geqslant \mathbb{P}\left\{\inf_{s\in[S,S+T]}\Big(X(s)-cs\Big)>u\right\}\\ &= \mathbb{P}\left\{\sup_{s\in[S,S+T]}\Big(-X(s)+cs\Big)<-u\right\}.\end{aligned}$$

Since we assume that $V(t):=\sigma^2(t)$ is a convex function and $V(0)=0$, then for any $0\leqslant s\leqslant t$

$$V(t)\geqslant V(s)+V(t-s).$$

Therefore, by the Slepian lemma (e.g., [30])

$$\begin{aligned}\mathbb{P}\left\{\sup_{s\in[S,S+T]}\Big(-X(s)+cs\Big)<-u\right\} &\geqslant \mathbb{P}\left\{\sup_{s\in[S,S+T]}\Big(-B(V(s))+cs\Big)<-u\right\}\\ &= \mathbb{P}\left\{\inf_{t\in[S,S+T]}\Big(B(V(t))-ct\Big)>u\right\}\\ &= \mathbb{P}\left\{\inf_{t\in[V(S),V(S+T)]}\Big(B(t)-cg(t)\Big)>u\right\},\end{aligned}$$

where B is a standard Brownian motion and $g(\cdot)$ is the inverse function of $V(\cdot)$. Further, since $g(s),s\geqslant 0$ is differentiable, increasing and concave we have (set $\rho_S=1/V'(S)$ with $V'(t)$ the derivative of $V(t)$)

$$g(s)\leqslant f(s):=\rho_S s+S-\rho_S V(S),\quad s\geqslant 0$$

implying thus

$$
\begin{aligned}
\mathbb{P}\left\{\sup_{t\in[0,S]}\inf_{s\in[t,t+T]}\Big(X(s)-cs\Big)>u\right\} &\geqslant \mathbb{P}\left\{\inf_{t\in[V(S),V(S+T)]}\Big(B(t)-c\rho_S t\Big)>u+c(S-\rho_S V(S))\right\}\\
&= K_{c\rho_S,V(S+T)-V(S)}\frac{V(S)}{u}\Psi\left(\frac{u+cS}{\sqrt{V(S)}}\right)(1+o(1))
\end{aligned}
$$

as $u\to\infty$, where the last equality follows from (2.5), and $K_{c,y}$ is given as in (2.6). □

4.4 Proof of Theorem 3.2

The proof follows straightforwardly from the combination of Theorem 3.1 and the fact that

$$
\begin{aligned}
\mathcal{P}_S(u,T_u) &< \mathcal{P}_S(u)\\
&\leqslant \mathbb{P}\left\{\sup_{t\in[0,S]}X(t)>u\right\}\\
&\leqslant \mathbb{P}\left\{\sup_{t\in[0,S]}B_1(\sigma^2(t))>u\right\} \qquad (4.6)\\
&= 2\Psi\left(\frac{u}{\sigma(S)}\right),
\end{aligned}
$$

where (4.6) follows from the Slepian lemma (recall that $\sigma^2(\cdot)$ is convex). □

4.5 Proof of Theorem 3.3

Let $\delta(u)=(\log u/u)^{2/\beta_1}, u>0$ and set

$$
\Pi(u)=\mathbb{P}\left\{\sup_{t\in[S-\delta(u),S]}\inf_{s\in[t,t+T_u]}\Big(X(s)-cs\Big)>u\right\},\quad u>0.
$$

It follows that

$$
\Pi(u)\leqslant\mathcal{P}_S(u,T_u)=\mathbb{P}\left\{\sup_{t\in[0,S]}\inf_{s\in[t,t+T_u]}\Big(X(s)-cs\Big)>u\right\}\leqslant\Pi(u)+\Pi_o(u),
$$

where $\Pi_o(u)=\mathbb{P}\left\{\sup_{t\in[0,S-\delta(u)]}\Big(X(t)-ct\Big)>u\right\}$. We shall show that

$$
\Pi_o(u)=o(\Pi(u)),\quad u\to\infty, \tag{4.7}
$$

which on the turn implies

$$
\mathcal{P}_S(u,T_u)=\Pi(u)(1+o(1)),\quad u\to\infty.
$$

Next, we derive the exact tail asymptotics of $\Pi(u)$. For notational simplicity we set

$$
g_u(t)=\frac{u+ct}{\sigma(t)},\quad X_u(t)=\frac{X(t)}{\sigma(t)}\frac{g_u(S)}{g_u(t)},\quad \sigma^2_{X_u}(t)=\mathrm{Var}(X_u(t))\quad t\geqslant 0.
$$

By Assumption **A1** for any small $\varepsilon\in(0,1)$, there exists some small $\theta>0$ and $u_0>0$ such that

$$
\begin{aligned}
&(1-\varepsilon)\frac{A}{\widetilde{\sigma}}|t|^{\beta_1}I_{(t>0)}+(1\mp\varepsilon)\frac{A_\pm}{\widetilde{\sigma}}|t|^{\beta_2}I_{(t\leqslant 0)}\\
&\leqslant 1-\frac{g_u(S)}{g_u(S-t)} \qquad (4.8)\\
&\leqslant (1+\varepsilon)\frac{A}{\widetilde{\sigma}}|t|^{\beta_1}I_{(t>0)}+(1\pm\varepsilon)\frac{A_\pm}{\widetilde{\sigma}}|t|^{\beta_2}I_{(t\leqslant 0)}
\end{aligned}
$$

holds for all $t \in [-\theta, \theta]$ and all $u > u_0$. Note that in the derivation of the above inequality we used the fact that $\beta_1 \leqslant 1$ and $\beta_2 \leqslant 1$. By changing the time we obtain

$$\Pi(u) = \mathbb{P}\left\{ \sup_{t \in [0,\delta(u)]} \inf_{s \in [0,T_u]} X_u(S+s-t) > g_u(S) \right\}.$$

The idea for finding the exact asymptotics of $\Pi(u)$ is analogous to the one used in [30]. Let $q = q(u) = u^{-2/\alpha}$ and set for any $\lambda > T$

$$\triangle_k = [k\lambda q, (k+1)\lambda q], \ k \in \mathbb{N}_0, \quad \text{and} \quad N(u) = \lfloor \lambda^{-1}\delta(u)q^{-1} \rfloor + 1,$$

where $\lfloor \cdot \rfloor$ is the ceiling function. We shall investigate separately the following three cases:
(i) $\alpha < \beta_1$, (ii) $\alpha = \beta_1$, (iii) $\alpha > \beta_1$.
Since the case $T = 0$ follows as a limiting result we shall consider for (i) and (ii) only $T \in (0, \infty)$.
(i) $\alpha < \beta_1$: We have by the Bonferroni inequality

$$\sum_{k=0}^{N(u)} \pi_k(u) \geqslant \Pi(u) \geqslant \sum_{k=0}^{N(u)-1} \pi_k(u) - \Sigma(u),$$

where

$$\pi_k(u) = \mathbb{P}\left\{ \sup_{t \in \triangle_k} \inf_{s \in [0,T_u]} X_u(S+s-t) > g_u(S) \right\}, \quad k \in \mathbb{N}_0,$$

$$\Sigma(u) = \sum\sum_{0 \leqslant i<j \leqslant N(u)} \mathbb{P}\left\{ \sup_{t \in \triangle_i} \inf_{s \in [0,T_u]} X_u(S+s-t) > g_u(S), \sup_{t \in \triangle_j} \inf_{s \in [0,T_u]} X_u(S+s-t) > g_u(S) \right\}.$$

In view of (4.8) for any $k = 0, \cdots, N(u)$

$$\begin{aligned} 1 - (1+\varepsilon)\frac{A}{\widetilde{\sigma}} |t-s|^{\beta_1} I_{(t>s)} - (1 \pm \varepsilon)\frac{A_\pm}{\widetilde{\sigma}} |t-s|^{\beta_2} I_{(t \leqslant s)} \\ \leqslant \ \sigma_{X_u}(S+s-t) \\ \leqslant \ 1 - (1-\varepsilon)\frac{A}{\widetilde{\sigma}} |t-s|^{\beta_1} I_{(t>s)} - (1 \mp \varepsilon)\frac{A_\pm}{\widetilde{\sigma}} |t-s|^{\beta_2} I_{(t \leqslant s)} \end{aligned} \tag{4.9}$$

holds for all $(t,s) \in \triangle_k \times [0, T_u]$. Define next

$$Y_u(t,s) = \frac{X_u(S+s-t)}{\sigma_{X_u}(S+s-t)}, \quad t,s \in [0,S].$$

For any small $\varepsilon \in (0,1)$ and $k = 1, \cdots, N(u)$

$$\pi_k(u) \leqslant \mathbb{P}\left\{ \sup_{t \in \triangle_k} \inf_{s \in [0,T_u]} Y_u(t,s) > g_u(S)\left(1 + (1-\varepsilon)^2\frac{A}{\widetilde{\sigma}} |k\lambda q - T_u|^{\beta_1}\right) \right\}$$

and

$$\pi_k(u) \geqslant \mathbb{P}\left\{ \sup_{t \in \triangle_k} \inf_{s \in [0,T_u]} Y_u(t,s) > g_u(S)\left(1 + (1+\varepsilon)^2\frac{A}{\widetilde{\sigma}} |(k+1)\lambda q|^{\beta_1}\right) \right\}$$

are valid for u sufficiently large. Moreover, for u sufficiently large also

$$\pi_0(u) \leqslant \mathbb{P}\left\{ \sup_{t \in \triangle_0} \inf_{s \in [0,T_u]} Y_u(t,s) > g_u(S)\left(1 + (1 \mp \varepsilon)^2\frac{A_\pm}{\widetilde{\sigma}} |f_\pm(u)|^{\beta_2}\right) \right\}$$

and

$$\pi_0(u) \geqslant \mathbb{P}\left\{ \sup_{t \in \triangle_0} \inf_{s \in [0,T_u]} Y_u(t,s) > g_u(S)\left(1 + (1+\varepsilon)^2\frac{A}{\widetilde{\sigma}} |\lambda q|^{\beta_1} + (1 \pm \varepsilon)^2\frac{|A_\pm|}{\widetilde{\sigma}} |h_\pm(u)|^{\beta_2}\right) \right\}$$

are valid, where $f_+(u) = h_-(u) = 0$, $f_-(u) = T_u + \lambda q$ and $h_+(u) = T_u$. Consequently, an application of Lemma 5.1 in Appendix yields that

$$\sum_{k=1}^{N(u)} \pi_k(u) \leqslant \sum_{k=1}^{N(u)} \mathbb{P}\left\{ \sup_{t\in[0,\lambda]} \inf_{s\in[0,T]} Y_u(tq + k\lambda q, sq) > g_u(S) \left(1 + (1-\varepsilon)^2 \frac{A}{\widetilde{\sigma}} \left|k\lambda q - T_u\right|^{\beta_1}\right) \right\}$$

$$= \widetilde{\mathcal{H}}_\alpha(\hat{a}\lambda, \hat{a}T) \frac{1}{\sqrt{2\pi} g_u(S)} \sum_{k=1}^{N(u)} \exp\left(-\frac{(g_u(S))^2 \left(1 + (1-\varepsilon)^2 \frac{A}{\widetilde{\sigma}} \left|k\lambda q - T_u\right|^{\beta_1}\right)^2}{2} \right) (1 + o(1))$$

as $u \to \infty$, where $\hat{a} = D^{1/\alpha} \widetilde{\sigma}^{-2/\alpha}$. Further, since

$$\int_0^\infty \exp(-bx^{\beta_1}) dx = \Gamma\left(\frac{1}{\beta_1} + 1\right) b^{-\frac{1}{\beta_1}}, \quad b > 0, \beta_1 > 0$$

we have

$$\sum_{k=1}^{N(u)} \pi_k(u) \leqslant \frac{1}{\lambda} \widetilde{\mathcal{H}}_\alpha(\hat{a}\lambda, \hat{a}T) \Gamma\left(\frac{1}{\beta_1} + 1\right) \left(\frac{\widetilde{\sigma}^3}{(1-\varepsilon)^2 A}\right)^{\frac{1}{\beta_1}} u^{\frac{2}{\alpha} - \frac{2}{\beta_1}} \Psi(g_u(S))(1 + o(1))$$

as $u \to \infty$. Similarly

$$\sum_{k=1}^{N(u)-1} \pi_k(u) \geqslant \frac{1}{\lambda} \widetilde{\mathcal{H}}_\alpha(\hat{a}\lambda, \hat{a}T) \Gamma\left(\frac{1}{\beta_1} + 1\right) \left(\frac{\widetilde{\sigma}^3}{(1+\varepsilon)^2 A}\right)^{\frac{1}{\beta_1}} u^{\frac{2}{\alpha} - \frac{2}{\beta_1}} \Psi(g_u(S))(1 + o(1))$$

as $u \to \infty$. By Lemma 5.1 and our assumption $\alpha < \beta_1 \leqslant \beta_2$ we obtain

$$\pi_0(u) = \widetilde{\mathcal{H}}_\alpha(\hat{a}\lambda, \hat{a}T) \Psi(g_u(S))(1 + o(1)) = o\left(\sum_{k=1}^{N(u)-1} \pi_k(u) \right)$$

as $u \to \infty$. Further, we have (set $\theta_i(u) := 1 + (1-\varepsilon)^2 \frac{A}{\widetilde{\sigma}} \left|\max(0, i\lambda q - T_u)\right|^{\beta_1} + (1 \mp \varepsilon)^2 \frac{A_\pm}{\widetilde{\sigma}} \left|f_\pm(u)\right|^{\beta_2}$)

$$\Sigma(u) \leqslant \sum\sum_{0 \leqslant i < j \leqslant N(u)} \mathbb{P}\left\{ \sup_{t\in\triangle_i} Y_u(t, 0) > g_u(S)\theta_i(u), \sup_{t\in\triangle_j} Y_u(t, 0) > g_u(S)\theta_i(u) \right\}.$$

Letting $\varepsilon \to 0$ and $\lambda \to \infty$ we conclude by similar arguments as in the proof of Theorem 3.1 in [11] that

$$\Pi(u) = \widetilde{\mathcal{H}}_\alpha(\hat{a}T) \Gamma\left(\frac{1}{\beta_1} + 1\right) D^{\frac{1}{\alpha}} A^{-\frac{1}{\beta_1}} \widetilde{\sigma}^{\frac{3}{\beta_1} - \frac{2}{\alpha}} u^{\frac{2}{\alpha} - \frac{2}{\beta_1}} \Psi(g_u(S))(1 + o(1))$$

as $u \to \infty$, and $\widetilde{\mathcal{H}}_\alpha(T) \in (0, \infty)$.

(ii) $\alpha = \beta_1$: We use the same notation as in Case (i). By the Bonferroni inequality

$$\pi_0(u) \leqslant \Pi(u) \leqslant \pi_0(u) + \sum_{k=1}^{N(u)} \pi_k(u).$$

It follows from Lemma 5.1 that

$$\pi_0(u) = \widetilde{\mathcal{P}}_{\alpha,\beta_2}^{A/(D\widetilde{\sigma}), A_\pm/(D\widetilde{\sigma})}(\hat{a}\lambda, \hat{a}T) \Psi(g_u(S))(1 + o(1)), \ u \to \infty. \tag{4.10}$$

Further, for any small $\varepsilon \in (0, 1)$

$$\sum_{k=1}^{N(u)} \pi_k(u) \leqslant \sum_{k=1}^{N(u)} \mathbb{P}\left\{ \sup_{t\in\triangle_0} Y_u(t + k\lambda q, 0) > g_u(S) \left(1 + (1-\varepsilon)^2 \frac{A}{\widetilde{\sigma}} \left|k\lambda q - T_u\right|^{\beta_1}\right) \right\}$$

is valid for all u sufficiently large. Using Lemma 5.1 (or Lemma 1 in [12]) we have further that

$$\sum_{k=1}^{N(u)} \pi_k(u) \leqslant G\widetilde{\mathcal{H}}_\alpha(\hat{a}\lambda, 0)\Psi(g_u(S)) \sum_{k=1}^{\infty} \exp\left(-\frac{A}{2\widetilde{\sigma}^3}(k\lambda - T)^{\beta_1}\right)(1+o(1))$$

as $u \to \infty$, for some positive constant G. Therefore, we conclude that, for any $\lambda_1, \lambda_2 > T$

$$\begin{aligned}
&\widetilde{\mathcal{P}}_{\alpha,\beta_2}^{A/(D\widetilde{\sigma}),A_\pm/(D\widetilde{\sigma})}(\hat{a}\lambda_2, \hat{a}T) \leqslant \liminf_{u\to\infty} \frac{\Pi(u)}{\Psi(g_u(S))} \leqslant \limsup_{u\to\infty} \frac{\Pi(u)}{\Psi(g_u(S))} \\
&\leqslant \widetilde{\mathcal{P}}_{\alpha,\beta_2}^{A/(D\widetilde{\sigma}),A_\pm/(D\widetilde{\sigma})}(\hat{a}\lambda_1, \hat{a}T) + G\widetilde{\mathcal{H}}_\alpha(\hat{a}\lambda_1, 0)\sum_{k=1}^{\infty} \exp\left(-\frac{A}{2\widetilde{\sigma}^3}(k\lambda_1 - T)^{\beta_1}\right).
\end{aligned}$$

Further, it follows from Corollary D.1 in [30] that $\widetilde{\mathcal{H}}_\alpha(\hat{a}\lambda_1, 0) = \mathcal{H}_\alpha(\hat{a}\lambda_1) \leqslant \lfloor \hat{a}\lambda_1 \rfloor + 1$, and thus

$$\lim_{\lambda_1\to\infty} \widetilde{\mathcal{H}}_\alpha(\hat{a}\lambda_1, 0)\sum_{k=1}^{\infty} \exp\left(-\frac{A}{2\widetilde{\sigma}^3}(k\lambda_1 - T)^{\beta_1}\right) = 0.$$

Consequently, by letting λ_1 and λ_2 tend to infinity we obtain

$$\lim_{u\to\infty} \frac{\Pi(u)}{\Psi(g_u(S))} = \widetilde{\mathcal{P}}_{\alpha,\beta_2}^{A/(D\widetilde{\sigma}),A_\pm/(D\widetilde{\sigma})}(\hat{a}T) \in (0,\infty).$$

(iii) $\alpha > \beta_1$: We use the same notation as in Case (i) and Case (ii). In view of (4.8) and the fact that $\lim_{u\to\infty} T_u u^{2/\alpha} = 0$, for any small $\varepsilon, \varepsilon_1 \in (0,1)$

$$\begin{aligned}
\Pi(u) &\geqslant \mathbb{P}\left\{\inf_{s\in[0,T_u]} X_u(S+s) > g_u(S)\right\} \\
&\geqslant \mathbb{P}\left\{\inf_{s\in[0,T_u]} Y_u(0,s) > g_u(S)\left(1 + (1+\varepsilon)\frac{|A_\pm|}{\widetilde{\sigma}}T_u^{\beta_2}\right)\right\} \\
&\geqslant \mathbb{P}\left\{\inf_{s\in[0,\varepsilon_1]} Y_u(0, su^{-\frac{2}{\alpha}}) > g_u(S)\left(1 + (1+\varepsilon)\frac{|A_\pm|}{\widetilde{\sigma}}T_u^{\beta_2}\right)\right\}
\end{aligned}$$

u sufficiently large. Moreover, it follows from Lemma 5.1 that

$$\begin{aligned}
&\mathbb{P}\left\{\inf_{s\in[0,\varepsilon_1]} Y_u(0, su^{-\frac{2}{\alpha}}) > g_u(S)\left(1 + (1+\varepsilon)\frac{|A_\pm|}{\widetilde{\sigma}}T_u^{\beta_2}\right)\right\} \\
&= \mathcal{H}_\alpha^{\inf}(\hat{a}\varepsilon_1)\Psi\left(g_u(S)\left(1 + (1+\varepsilon)\frac{|A_\pm|}{\widetilde{\sigma}}T_u^{\beta_2}\right)\right)(1+o(1))
\end{aligned}$$

as $u \to \infty$, where

$$\mathcal{H}_\alpha^{\inf}(T) = \mathbb{E}\left\{\exp\left(\inf_{t\in[0,T]}\left(\sqrt{2}B_\alpha(t) - t^\alpha\right)\right)\right\}, \quad T \geqslant 0.$$

Therefore, letting $\varepsilon, \varepsilon_1 \to 0$ we have by the fact that $\lim_{u\to\infty} T_u u^{2/\beta_2} = 0$

$$\begin{aligned}
\Pi(u) &\geqslant \Psi\left(g_u(S)\left(1 + \frac{|A_\pm|}{\widetilde{\sigma}}T_u^{\beta_2}\right)\right)(1+o(1)) \\
&= \Psi(g_u(S))(1+o(1))
\end{aligned}$$

as $u \to \infty$. Next we give the upper bound. Since $\alpha > \beta_1$, we have

$$\Pi(u) \leqslant \mathbb{P}\left\{\sup_{t\in\triangle_0} X_u(S + T_u - t) > g_u(S)\right\}.$$

Further, for any small $\varepsilon \in (0,1)$,

$$\Pi(u) \leqslant \mathbb{P}\left\{\sup_{t\in\triangle_0} Y_u(t, T_u) > g_u(S)\left(1 - (1+\varepsilon)\frac{|A_\pm|}{\widetilde{\sigma}}T_u^{\beta_2}\right)\right\}$$

$$= \mathcal{H}_\alpha^{\sup}(\hat{a}\lambda)\Psi\left(g_u(S)\left(1-(1+\varepsilon)\frac{|A_\pm|}{\widetilde{\sigma}}T_u^{\beta_2}\right)\right)(1+o(1))$$

as $u\to\infty$, where the last equation follows from Lemma 5.1, and

$$\mathcal{H}_\alpha^{\sup}(T) = \mathbb{E}\left\{\exp\left(\sup_{t\in[0,T]}\left(\sqrt{2}B_\alpha(t)-t^\alpha\right)\right)\right\},\quad T\geqslant 0.$$

Consequently, letting $\lambda,\varepsilon\to 0$ and using that $\lim_{u\to\infty} T_u u^{2/\beta_2}=0$, we conclude that, as $u\to\infty$,

$$\Pi(u)\leqslant \Psi(g_u(S))(1+o(1)).$$

Thus the claim follows.

Proof of (4.7). First, for any fixed $\varepsilon\in(0,1)$, by **A1** we can choose some small $\theta_0>0$ such that

$$\sigma(t)\leqslant \widetilde{\sigma}-(1-\varepsilon)A(S-t)^{\beta_1}$$

holds for all $t\in[S-\theta_0,S]$. Additionally, this θ_0 can also be chosen such that

$$\sup_{t\in[0,S-\theta_0)}\sigma(t)<\sigma(S-\theta_0)<\widetilde{\sigma}.$$

Clearly,

$$\Pi_o(u)\leqslant \mathbb{P}\left\{\sup_{t\in[0,S-\theta_0]}\Big(X(t)-ct\Big)>u\right\}+\mathbb{P}\left\{\sup_{t\in[S-\theta_0,S-\delta(u)]}\Big(X(t)-ct\Big)>u\right\}=:\Pi_1(u)+\Pi_2(u).$$

By Borell-TIS inequality (cf. [30])

$$\Pi_1(u)\leqslant \mathbb{P}\left\{\sup_{t\in[0,S-\theta_0]}X(t)>u\right\}\leqslant \exp\left(-\frac{\left(u-\mathbb{E}\left\{\sup_{t\in[0,S]}X(t)\right\}\right)^2}{2\sigma^2(S-\theta_0)}\right)$$

for u sufficiently large. Further, by **A3** we have applying Theorem 8.1 in [30]

$$\begin{aligned}\Pi_2(u) &\leqslant \mathbb{P}\left\{\sup_{t\in[\delta(u),\theta_0]}X(S-t)>u\right\}\\ &\leqslant Gu^{\frac{2}{\gamma}+1}\exp\left(-\frac{u^2}{2\widetilde{\sigma}^2}\left(1+(1-\varepsilon)\frac{A}{\widetilde{\sigma}}(\delta(u))^{\beta_1}\right)\right)\end{aligned}$$

for u sufficiently large, where G is some positive constant independent of u. Consequently, we conclude from the asymptotics of $\Pi(u)$ for all the cases above that $\Pi_o(u)=o(\Pi(u))$, and thus the proof is complete. □

4.6 Proof of Corollary 3.4

Since X is a fBm with Hurst index $\alpha/2$ we have that

$$\sigma(t)=t^{\frac{\alpha}{2}}=S^{\frac{\alpha}{2}}-\frac{\alpha}{2}S^{\frac{\alpha}{2}-1}(S-t)(1+o(1)),\quad t\to S,$$

and

$$\mathrm{Cov}\left(\frac{X(t)}{\sigma(t)},\frac{X(s)}{\sigma(s)}\right)=1-\frac{1}{2S^\alpha}|t-s|^\alpha(1+o(1)),\quad t,s\to S.$$

Moreover, for any $s,t\geqslant 0$

$$\mathbb{E}\left\{(X(t)-X(s))^2\right\}=|t-s|^\alpha.$$

Consequently, the claim follows by an application of Theorem 3.3. □

5 Appendix

Let $\mathbf{D}$ be a compact set in $\mathbb{R}^n$, $n \in \mathbb{N}$ and suppose without loss of generality that $\mathbf{0} \in \mathbf{D}$. Further, let $\{\xi_u(\boldsymbol{t}), \boldsymbol{t} \in \mathbf{D}\}$, $u > 0$ be a family of centered Gaussian random fields with a.s. continuous sample paths and variance function $\sigma^2_{\xi_u}(\cdot)$. Below $||\cdot||$ stands for the Euclidean norm in $\mathbb{R}^n$. We assume that ξ_u satisfies the following conditions:

C1: $\sigma_{\xi_u}(\mathbf{0}) = 1$ for all u large, and there exists some bounded measurable function $d(\cdot)$ on $\mathbf{D}$ such that

$$\lim_{u\to\infty} \sup_{\boldsymbol{t}\in\mathbf{D}} \left| u^2(1-\sigma_{\xi_u}(\boldsymbol{t})) - d(\boldsymbol{t}) \right| = 0.$$

C2: There exist some centered Gaussian random field $\{\eta(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{R}^n\}$ with a.s. continuous sample paths, $\eta(\mathbf{0}) = 0$ and variance function $\sigma^2_\eta(\cdot)$ such that

$$\lim_{u\to\infty} u^2 \mathrm{Var}(\xi_u(\boldsymbol{t}) - \xi_u(\boldsymbol{s})) = 2\mathrm{Var}(\eta(\boldsymbol{t}) - \eta(\boldsymbol{s}))$$

holds for all $\boldsymbol{s}, \boldsymbol{t} \in \mathbf{D}$.

C3: There exist some constants $G, \nu > 0, u_0 > 0$, such that, for any $u > u_0$

$$u^2 \mathrm{Var}(\xi_u(\boldsymbol{t}) - \xi_u(\boldsymbol{s})) \leqslant G\, ||\boldsymbol{t} - \boldsymbol{s}||^\nu$$

holds uniformly with respect to $\boldsymbol{t}, \boldsymbol{s} \in \mathbf{D}$.

As in [12] let $F : C(\mathbf{D}) \to \mathbb{R}$ be a continuous functional acting on $C(\mathbf{D})$, the space of continuous functions on the compact set $\mathbf{D}$. Assume that:

F1: $|F(f)| \leqslant \sup_{\boldsymbol{t}\in\mathbf{D}} |f(\boldsymbol{t})|$ for any $f \in C(\mathbf{D})$.

F2: $F(af + b) = aF(f) + b$ for any $f \in C(\mathbf{D})$ and $a > 0, b \in \mathbb{R}$.

For any bounded measurable function $d(\cdot)$ on $\mathbf{D}$ with $d(\mathbf{0}) = 0$ and F satisfying **F1** we define a constant

$$\mathcal{H}^F_{\eta,d}(\mathbf{D}) = \mathbb{E}\left\{\exp\left(F\left(\sqrt{2}\eta(\boldsymbol{t}) - \sigma^2_\eta(\boldsymbol{t}) - d(\boldsymbol{t})\right)\right)\right\}. \tag{5.1}$$

Along the lines of the proof in [12] we get that $\mathcal{H}^F_{\eta,d}(\mathbf{D}) \in (0,\infty)$. The following result generalizes Lemma 6.1 in [30] and Lemma 1 in [12].

Lemma 5.1. Let $\{\xi_u(\boldsymbol{t}), \boldsymbol{t} \in \mathbf{D}\}$, $u > 0$ be the family of centered Gaussian random fields defined as above satisfying **C1-C3** with some function $d(\cdot)$ and some Gaussian random field η. Let $F : C(\mathbf{D}) \to \mathbb{R}$ be a continuous functional such that **F1-F2** hold. Then, for any positive measurable function $g(\cdot)$ satisfying $\lim_{u\to\infty} g(u)/u = a \in (0,\infty)$

$$\mathbb{P}\{F(\xi_u) > g(u)\} = \mathcal{H}^F_{a\eta,a^2d}(\mathbf{D})\Psi(g(u))(1+o(1)) \tag{5.2}$$

holds as $u \to \infty$, provided that $\mathbb{P}\{F(\xi_u) > g(u)\} > 0$ for all large u.

Proof of Lemma 5.1: The proof is based on the classical approach rooted in the ideas of [29,30]. For all $u > 0$ large

$$\mathbb{P}\{F(\xi_u) > g(u)\} = \Psi(g(u)) \int_{\mathbb{R}} \exp\left(w - \frac{w^2}{2(g(u))^2}\right) \mathbb{P}\left\{F(\xi_u) > g(u) \middle| \xi_u(\mathbf{0}) = g(u) - \frac{w}{g(u)}\right\} dw. \tag{5.3}$$

Let, for any $u > 0, w \in \mathbb{R}$, $\zeta_u = \{\zeta_u(\boldsymbol{t}) = g(u)(\xi_u(\boldsymbol{t}) - g(u)) + w, \boldsymbol{t} \in \mathbf{D}\}$. Using **F2** the conditional probability in the integrand of (5.3) can be written as

$$\mathbb{P}\left\{F(\xi_u) > g(u) \middle| \xi_u(\mathbf{0}) = g(u) - \frac{w}{g(u)}\right\} = \mathbb{P}\{F(\chi_u) > w\},$$

where $\chi_u = \zeta_u | \zeta_u(\mathbf{0}) = 0$. Denote

$$R_{\xi_u}(\boldsymbol{t}, \boldsymbol{s}) = \mathbb{E}\{\xi_u(\boldsymbol{t})\xi_u(\boldsymbol{s})\}, \quad \boldsymbol{s}, \boldsymbol{t} \in \mathbf{D}$$

to be the covariance function of ξ_u. We have that the conditional random field $\chi_u = \{\chi_u(\boldsymbol{t}), \boldsymbol{t} \in \mathbf{D}\}$ has the same finite-dimensional distributions as

$$\left\{g(u)(\xi_u(\boldsymbol{t}) - R_{\xi_u}(\boldsymbol{t},\mathbf{0})\xi_u(\mathbf{0})) - (g(u))^2(1 - R_{\xi_u}(\boldsymbol{t},\mathbf{0})) + w(1 - R_{\xi_u}(\boldsymbol{t},\mathbf{0})), \boldsymbol{t} \in \mathbf{D}\right\}.$$

Therefore, the following convergence

$$\mathbb{E}\left\{\chi_u(\boldsymbol{t})\right\} = -(g(u))^2(1 - R_{\xi_u}(\boldsymbol{t}, \boldsymbol{0})) + w(1 - R_{\xi_u}(\boldsymbol{t}, \boldsymbol{0})) \to -a^2(\sigma_\eta^2(\boldsymbol{t}) + d(\boldsymbol{t})),\ u \to \infty$$

holds, for any $w \in \mathbb{R}$, uniformly with respect to $\boldsymbol{t} \in \mathbf{D}$. Moreover, for any $\boldsymbol{t}, \boldsymbol{s} \in \mathbf{D}$ we have

$$\begin{aligned} \mathrm{Var}\Big(\chi_u(\boldsymbol{t}) - \chi_u(\boldsymbol{s})\Big) &= (g(u))^2 \left(\mathbb{E}\left\{\Big(\xi_u(\boldsymbol{t}) - \xi_u(\boldsymbol{s})\Big)^2\right\} - (R_{\xi_u}(\boldsymbol{t}, \boldsymbol{0}) - R_{\xi_u}(\boldsymbol{s}, \boldsymbol{0}))^2\right) \\ &\to 2a^2 \mathrm{Var}(\eta(\boldsymbol{t}) - \eta(\boldsymbol{s})),\ u \to \infty. \end{aligned}$$

Therefore, the finite-dimensional distributions of χ_u converge to those of $\widetilde{\eta} = \{\sqrt{2}a\eta(\boldsymbol{t}) - \sigma_{a\eta}^2(\boldsymbol{t}) - a^2 d(\boldsymbol{t}), \boldsymbol{t} \in \mathbf{D}\}$, whereas the tightness follows by Proposition 9.7 in [31]. The rest of the proof repeats line-by-line that of Lemma 1 in [12]. $\square$

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References

1 Berman S M. The supremum of a process with stationary independent and symmetric increments. Stochastic Process. Appl., 1986, 23(2): 281–290.

2 Chesney M, Jeanblanc-Picqué M, Yor M. Brownian excursions and Parisian barrier options. Adv. in Appl. Probab., 1997, 29(1): 165–184.

3 Czarna I. Parisian ruin probability with a lower ultimate bankrupt barrier. Scandinavian Actuarial J., in press. DOI: 10.1080/03461238.2014.926288, 2014.

4 Czarna I, Palmowski Z. Ruin probability with Parisian delay for a spectrally negative Lévy risk process. J. Appl. Probab., 2011, 48(4): 984–1002.

5 Czarna I, Palmowski Z. Dividend problem with Parisian delay for a spectrally negative Lévy risk process. J. Optim. Theory Appl., 2014, 161: 239–256.

6 Czarna I, Palmowski Z, Świątek P. Binomial discrete time ruin probability with Parisian delay. http://arxiv.org/abs/1403.7761, 2014.

7 Dassios A, Wu S. Parisian ruin with exponential claims. Preprint, 2008, http://stats.lse.ac.uk/angelos/.

8 Dębicki K. Ruin probability for Gaussian integrated processes. Stochastic Process. Appl., 2002, 98(1): 151–174.

9 Dębicki K, Hashorva E, Ji L. Extremes of a class of non-homogeneous Gaussian random fields. Annals of Probability, accepted, 2014, http://arxiv.org/abs/1405.2952v2.

10 Dębicki K, Hashorva E, Ji L. Gaussian risk models with financial constraints. Scandinavian Actuarial Journal, in press: DOI:10.1080/03461238.2013.850442, 2015.

11 Dębicki K, Hashorva E, Ji L. Parisian ruin of self-similar Gaussian risk processes. Journal of Applied Probability, in press, 2015, http://arxiv.org/abs/1405.2958.

12 Dębicki K, Kosiński K. On the infimum attained by the reflected fractional Brownian motion. Extremes, 2014, 17: 431–446.

13 Dieker A B. Extremes of Gaussian processes over an infinite horizon. Stochastic Process. Appl., 2005, 115(2): 207–248.

14 Dieker A B, Yakir B. On asymptotic constants in the theory of Gaussian processes. Bernoulli, 2014, 20(3): 1600–1619.

15 Embrechts P, Hashorva E, Mikosch T. Aggregation of log-linear risks. J. Appl. Probab., 2014, 51: 203–212.

16 Embrechts P, Klüppelberg C, Mikosch T. Modelling Extremal Events for Insurance and Finance. Springer-Verlag, Berlin, 1997.

17 Fatalov V R. Asymptotics of the probabilities of large deviations for Gaussian fields. Izv. Nats. Akad. Nauk Armenii Mat., 1992, 27(6): 59–85.

18 Foss S, Korshunov D, Zachary S. An Introduction to Heavy-tailed and Subexponential Distributions. Springer-Verlag, New York, second edition, 2013.

19 Hashorva E. Exact tail asymptotics in bivariate scale mixture models. Extremes, 2012, 15(1): 109–128.

20 Hashorva E. Extremes of aggregated Dirichlet risks. Journal of Multivariate Analysis, 2015, 133: 334–345.

21 Hashorva E, Ji L. Approximation of passage times of γ-reflected processes with fBm input. J. Appl. Probab., 2014, 51(3): 713–726.

22 Hashorva E, Ji L. Piterbarg theorems for chi-processes with trend. Extremes, 2015, 18(1): 37–64.

23 Hashorva E, Ji L, Piterbarg V I. On the supremum of γ-reflected processes with fractional Brownian motion as input. Stochastic Process. Appl., 2013, 123: 4111–4127.

24 Hüsler J, Piterbarg V I. Extremes of a certain class of Gaussian processes. Stochastic Process. Appl., 1999, 83(2): 257–271.

25 Hüsler J, Piterbarg V I. A limit theorem for the time of ruin in a Gaussian ruin problem. Stochastic Process. Appl., 2008, 118(11): 2014–2021.

26 Loeffen R, Czarna I, Palmowski Z. Parisian ruin probability for spectrally negative Lévy processes. Bernoulli, 2013, 19(2): 599–609.

27 Michna Z. Self-similar processes in collective risk theory. J. Appl. Math. Stochastic Anal., 1998, 11(4): 429–448.

28 Palmowski Z, Świątek P. A note on first passage probabilities of a Lévy process reflected at a general barrier. Preprint. http://arxiv.org/abs/1403.1025, 2014.

29 Pickands J III. Upcrossing probabilities for stationary Gaussian processes. Trans. Amer. Math. Soc., 1969, 145: 51–73.

30 Piterbarg V I. Asymptotic Methods in the Theory of Gaussian Processes and Fields, volume 148 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996.

31 Piterbarg V I. Twenty Lectures About Gaussian Processes. Atlantic Financial Press, London, New York, 2015.

32 Samorodnitsky G, Taqqu M S. Stable non-Gaussian Random Processes. Stochastic Modeling. Chapman & Hall, New York, 1994.

33 Tan Z, Yang Y. Extremes of Shepp statistics for fractional Brownian motion. Sci China Math, (58):doi: 10.1007/s11425–014–4945–5, 2015.