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CONTINUOUS REDUCTIONS ON THE SCOTT DOMAIN AND DECOMPOSABILITY CONJECTURE

Vuilleumier Louis

UNIL | Université de Lausanne

FACULTÉ DES HAUTES ÉTUDES COMMERCIALES

DÉPARTEMENT DES OPÉRATIONS

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THÈSE DE DOCTORAT

présentée à la

Faculté des Hautes Études Commerciales
de l'Université de Lausanne
en cotutelle avec Université de Paris

pour l'obtention des grades de
Docteur ès Sciences en systèmes d'information
et Docteur en Mathématiques

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CONTINUOUS REDUCTIONS ON THE SCOTT DOMAIN AND DECOMPOSABILITY CONJECTURE

Lausanne, le 08 juin 2021

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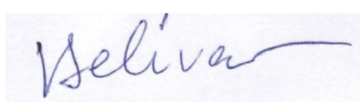

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À mes parents

Résumé

Cette thèse fait partie de la théorie descriptive des ensembles qui est, historiquement, l'étude de la définissabilité dans les espaces polonais [Kec95]. Au cours des dernières décennies, l'avènement de l'informatique fondamentale a provoqué un intérêt grandissant pour les problèmes de définissabilité dans d'autres espaces topologiques plus généraux [GHK$^+$03, GL13, Sco76, Sel06, Wei00]. Dans cette optique, de Brecht a récemment mis en évidence dans [dB13] la classe des espaces quasi-polonais comme étant une classe assez générale puisqu'elle contient de nombreux espaces topologiques impliqués dans le développement de l'informatique fondamentale, mais pas trop générale puisque leur théorie descriptive reste intéressante. Le domaine de Scott $\mathcal{P}\omega$ est l'ensemble $\mathcal{P}(\omega)$ de tous les sous-ensembles d'entiers muni de la topologie de Scott, dont une base est donnée par l'ensemble

$$\{\{x \subseteq \omega : F \subseteq x\} : \ F \subseteq \omega \text{ fini}\}.$$

Il a d'abord été introduit par Scott dans [Sco76] comme une sémantique dénotationnelle du λ-calcul. Le domaine de Scott $\mathcal{P}\omega$ se démarque parmi les quasi-polonais par son universalité [dB13], ce qui en fait un candidat idéal pour la tentative d'extension de la théorie descriptive des ensembles aux quasi-polonais.

Dans la première partie de la thèse, nous adoptons ce point de vue et essayons d'étendre certains outils de la théorie descriptive des ensembles au domaine de Scott $\mathcal{P}\omega$. Plus précisément, nous nous intéressons aux réductions continues sur $\mathcal{P}\omega$. Tout d'abord, nous montrons que l'ordre partiel induit par les réductions par fonctions continues, appelé l'ordre de Wadge et introduit dans [Wad84], sur les boréliens de $\mathcal{P}\omega$ est mal-fondé et contient des antichaines infinies. De plus, nous montrons que ces propriétés, considérées comme mauvaises par la théorie descriptive, se trouvent déjà au niveau de complexité topologique le plus bas possible.

Pour remédier à cela, nous étudions ensuite l'ordre partiel induit par les relations totales et relativement continues introduites dans [BH94]. Cette notion de réductions, étudiée dans [Peq15b], est plus générale que la notion de réductions par fonctions continues et induit une belle hiérarchie sur les sous-ensembles boréliens du domaine de Scott. En effet, l'ordre partiel induit sur ces sous-ensembles est un bel ordre, c'est-à-dire qu'il est bien fondé et ne contient aucune antichaine infinie. Nous caractérisons complètement cet ordre partiel en montrant qu'il est isomorphe à une structure bien connue en théorie descriptive, à savoir la restriction de l'ordre de Wadge sur les boréliens non-auto-duaux de l'espace de Baire ω^ω [Dup01, KLS12], qui est l'ensemble des suites infinies d'entiers muni de la topologie du préfixe dont

une base est donnée par l'ensemble

$$\left\{ \{x \subseteq \omega^\omega : t \sqsubset x\} : \ t \in \omega^{<\omega} \right\}.$$

Dans la deuxième partie de la thèse, nous nous intéressons à un problème de la théorie descriptive des ensembles classique, à savoir le problème de la décomposabilité des fonctions boréliennes sur les espaces polonais. Ce problème est appelé la Conjecture de la Décomposabilité et a récemment été étudié par plusieurs chercheurs [And07, Day19, DKSZ20, GKN21, Kih15, Mar20, MR13, PS12]. En utilisant la machinerie des arbres à questions développée par Duparc dans [Dup01], nous présentons de nouvelles techniques qui permettent d'aborder cette conjecture sous une autre perspective. En particulier, nous isolons une certaine hypothèse qui implique la Conjecture de la Décomposabilité sur les espaces polonais de dimension zéro. Nous prouvons également que cette hypothèse est vérifiée pour un grand nombre de fonctions, ce qui suggère qu'elle est atteignable en toute généralité.

Mots clés : Théorie descriptive des ensembles, Ordre de Wadge, Réductions continues, Domaine de Scott, Conjecture de la Décomposabilité.

Abstract

This thesis belongs to descriptive set theory which is historically the study of definability in Polish spaces [Kec95]. Over the last few decades, the rise of theoretical computer science has led to a growing interest in definability problems over other more general topological spaces [GHK^{+}03, GL13, Sco76, Sel06, Wei00]. For this purpose, de Brecht recently isolated in [dB13] the class of quasi-Polish spaces as a class of spaces general enough for it contains many topological spaces involved in the development of theoretical computer science, but not too general for their descriptive set theory remains interesting. The Scott domain $\mathcal{P}\omega$ is the set $\mathcal{P}(\omega)$ of subsets of integers equipped with the Scott topology whose basis is given by the set

$$\{\{x \subseteq \omega : F \subseteq x\} : \ F \subseteq \omega \text{ fini}\}.$$

It was first introduced by Scott in [Sco76] as a denotational semantic of the λ-calculus. The Scott domain $\mathcal{P}\omega$ stands out among the quasi-Polish spaces for its universality [dB13], which makes it an ideal candidate for the attempt of extending descriptive set theory to the class of quasi-Polish spaces.

In the first part of the thesis, we adopt this point of view and try to extend classical tools of descriptive set theory to the Scott domain $\mathcal{P}\omega$. More precisely, we are interested in continuous reductions on $\mathcal{P}\omega$. First, we show that the partial order induced by reductions via continuous functions, i.e., the Wadge order which was introduced in [Wad84], on the Borel subsets of $\mathcal{P}\omega$ is ill-founded and contains infinite antichains. Moreover, we show that these properties, considered as bad in descriptive set theory, already occur at the lowest possible level of topological complexity.

To remedy this situation, we then study the partial order induced by total and relatively continuous relations that were introduced in [BH94]. This notion of reductions, studied in [Peq15b], is more general than the one induced by continuous functions and it induces a nice hierarchy on the Borel subsets of the Scott domain $\mathcal{P}\omega$. Indeed, the partial order induced on these subsets is a well-quasi-order, i.e., it is well-founded and contains no infinite antichain. We fully characterize this partial order by showing that it is isomorphic to a well-known structure in descriptive set theory, namely the restriction of the Wadge order on the non-self-dual Borel subsets of the Baire space ω^ω [Dup01, KLS12], which is the set of infinite sequences of integers equipped with the prefix topology whose basis is given by the set

$$\left\{\{x \subseteq \omega^\omega : t \sqsubset x\} : \ t \in \omega^{<\omega}\right\}.$$

In the second part of the thesis, we focus on a classical problem of descriptive set theory, namely the one of the decomposability of Borel functions in Polish spaces, also called the Decomposability Conjecture. This problem was extensively studied by several authors [And07, Day19, DKSZ20, GKN21, Kih15, Mar20, MR13, PS12]. Using the question-tree machinery developed by Duparc in [Dup01], we introduce new techniques to tackle this conjecture from a novel perspective. In particular, we are able to isolate a certain hypothesis which implies the Decomposability Conjecture on zero-dimensional Polish spaces. Moreover, we also prove that this hypothesis holds true for a large number of functions, which suggests that it is reachable in full generality.

Keywords : Descriptive set theory, Wadge order, Continuous reductions, Scott domain, Decomposability Conjecture.

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Lausanne, le 10 juin 2021
Louis Vuilleumier

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Résumé substantiel en français

La théorie descriptive des ensembles est l'étude de la définissabilité dans les espaces polonais, i.e., dans les espaces séparables et complètement métrisables [Kec95]. Dans cette théorie, les objets mathématiques sont classés relativement à la complexité topologique de leur définition et les propriétés de chacune de ces classes sont rigoureusement analysées. Cette thèse fait partie de cette théorie et se divise en deux parties.

Dans la première, nous généralisons certains outils classiques de la théorie descriptive des ensembles sur le domaine de Scott $\mathcal{P}\omega$, un espace prépondérant en informatique fondamentale. Plus précisément, nous étudions deux préordres sur les sous-ensembles du domaine de Scott $\mathcal{P}\omega$ qui sont induits par deux notions différentes de réductions continues. Dans la seconde, nous étudions la Conjecture de la Décomposabilité qui est un important problème ouvert de la théorie descriptive des ensembles. En particulier, nous énonçons une hypothèse qui, sous l'axiome de détermination, implique la Conjecture de la Décomposabilité pour les espaces polonais de dimension zéro.

Les espaces quasi-polonais

Durant les dernières décennies, l'avènement de l'informatique fondamentale a injecté un nouveau dynamisme aux mathématiques [GHK^{+}03, GL13, Sco76, Sel06, Wei00]. Naturellement, la définissabilité occupe une place majeure dans ce domaine de recherche. Malheureusement, beaucoup d'espaces topologiques impliqués dans l'informatique fondamentale ne sont pas polonais, voire pas métrisables. Par exemple, la sémantique dénotationnelle étudie les domaines qui ne sont en général pas métrisables. En conséquence, il y a eu un intérêt croissant pour l'extension de la théorie descriptive des ensembles en dehors du contexte des espaces polonais. Le premier obstacle à ce développement aux espaces non-métrisables est la définition classique de la hiérarchie borélienne car elle ne se comporte pas bien dans ces espaces. Par exemple, l'espace de Sierpiński $\mathbb{S} = (2, \{\emptyset, \{1\}, 2\})$ est l'ensemble $2 = \{0, 1\}$ équipé de la topologie qui fait de l'ensemble $\{1\}$ le seul ensemble ouvert non-trivial. Cet espace est non-métrisable et l'ensemble fermé $\{0\}$ n'est pas une intersection dénombrable d'ouverts. En conséquence, l'inclusion $\mathbf{\Sigma}_1^0(\mathbb{S}) \subseteq \mathbf{\Sigma}_2^0(\mathbb{S})$ n'est pas vérifiée avec la définition classique de la hiérarchie borélienne. Ce problème a été surmonté par Selivanov qui a été capable de définir une hiérarchie borélienne qui se comporte bien même pour les espaces non-métrisables [Sel05, Sel06]. Si $\mathcal{X}$ est un espace topologique, nous notons $\mathbf{\Sigma}_1^0(\mathcal{X})$ pour la classe de tous les ensembles ouverts de $\mathcal{X}$, et, pour tout $\alpha \in \omega_1$, nous définissons $\mathbf{\Pi}_\alpha^0(\mathcal{X}) = \left\{A \subseteq \mathcal{X} : A^{\mathrm{c}} \in \mathbf{\Sigma}_\alpha^0(\mathcal{X})\right\}$ et

$$\mathbf{\Sigma}_\alpha^0(\mathcal{X}) = \left\{ \bigcup_{n\in\omega} A_n \cap {A'_n}^{\mathrm{c}} \subseteq \mathcal{X} : \forall n \; \exists \beta_n < \alpha \; A_n, A'_n \in \mathbf{\Sigma}_{\beta_n}^0(\mathcal{X}) \right\}.$$

Cette définition de la hiérarchie borélienne est équivalente à la définition classique dans les espaces métrisables, mais présente l'avantage de bien se comporter dans n'importe quel espace topologique. Plus précisément, si $\mathcal{X}$ est un espace topologique non-dénombrable, nous avons les inclusions suivantes :

$$\begin{array}{ccccccccccc}
\mathbf{\Sigma}_1^0(\mathcal{X}) & \subsetneq & \mathbf{\Sigma}_2^0(\mathcal{X}) & & \subsetneq & \mathbf{\Sigma}_\alpha^0(\mathcal{X}) & & \subsetneq & \mathbf{\Sigma}_\beta^0(\mathcal{X}) & & \\
& \diagdown\!\!\!\!\diagup & & \cdots & \diagup & & \cdots & \diagup & & \cdots & \\
\mathbf{\Pi}_1^0(\mathcal{X}) & \subsetneq & \mathbf{\Pi}_2^0(\mathcal{X}) & & \subsetneq & \mathbf{\Pi}_\alpha^0(\mathcal{X}) & & \subsetneq & \mathbf{\Pi}_\beta^0(\mathcal{X}) & &
\end{array}$$

En se basant sur ce travail, de Brecht a isolé une nouvelle classe d'espaces qui contient les espaces polonais ainsi qu'une importante classe d'espaces non-polonais pour l'informatique fondamentale, à savoir les domaines ω-continus [dB13]. De plus, il a prouvé que la plupart des techniques de théorie descriptive des ensembles sont encore valides pour cette plus grande classe d'espaces, qui sont donc appelés les espaces quasi-polonais. Le domaine de Scott $\mathcal{P}\omega$ est l'ensemble $\mathcal{P}(\omega)$ des sous-ensembles d'entiers équipé de la topologie générée par l'ensemble

$$\{\{x \subseteq \omega : F \subseteq x\} : \; F \subseteq \omega \text{ fini}\}.$$

Le domaine de Scott $\mathcal{P}\omega$ se démarque parmi les espace quasi-polonais par son universalité. En effet, tout espace quasi-polonais est homéomorphe à un sous-ensemble $\mathbf{\Pi}_2^0$ du domaine de Scott $\mathcal{P}\omega$. En d'autres termes, $\mathcal{P}\omega$ est un représentant typique des espaces quasi-polonais mais non-polonais, ce qui justifie notre étude approfondie de cet espace.

Réductions via fonctions continues

Chaque niveau de la hiérarchie borélienne peut être stratifié en ω_1 niveaux par la hiérarchie de la différence de Hausdorff-Kuratowski. Il est intéressant d'observer que les classes des hiérarchies borélienne et de la différence de Hausdorff-Kuratowski sont closes par préimage continue, i.e., si $\Gamma \subseteq \mathcal{P}(\mathcal{X})$ est une telle classe, $A \in \Gamma$ et $f : \mathcal{X} \to \mathcal{X}$ est continue, alors $f^{-1}[A] \in \Gamma$. Cela suggère l'étude des réductions continues comme une stratification plus fine de ces hiérarchies.

Si $\mathcal{X}$ est un espace topologique et $A, B \subseteq \mathcal{X}$, nous disons que A est Wadge réductible à B s'il existe une fonction continue $f : \mathcal{X} \to \mathcal{X}$ telle que $f^{-1}[B] = A$. Dans ce cas, nous écrivons $A \leq_w B$. La relation binaire $\leq_w$ est réflexive et transitive sur $\mathcal{P}(\mathcal{X})$ puisque la fonction identité est continue, tout comme la composition de deux fonctions continues. En d'autres termes, la relation binaire $\leq_w$ est un préordre sur $\mathcal{P}(\mathcal{X})$ qui est appelé le préordre de Wadge de $\mathcal{X}$ en l'honneur de Wadge qui a été le premier à analyser rigoureusement cette relation dans [Wad84]. Il s'agit d'une mesure de complexité topologique naturelle. En effet, si $A \leq_w B$, alors le problème d'appartenance à A peut être réduit via une fonction continue au problème d'appartenance à B. En d'autres termes, A n'est pas plus compliqué que B.

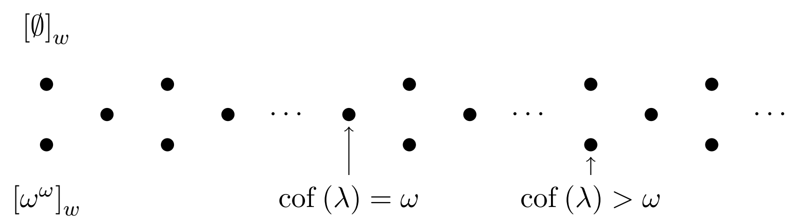
Le préordre de Wadge induit naturellement une structure d'ordre partiel. Si $A \subseteq \mathcal{X}, B \subseteq \mathcal{Y}$, $A \leq_w B$ et $B \leq_w A$, alors nous écrivons $A \equiv_w B$ et disons que A est Wadge équivalent à B. Le degré de Wadge de $A \subseteq \mathcal{X}$ est

$$[A]_w = \{B \subseteq \mathcal{X} : A \equiv_w B\}.$$

L'ensemble de tous les degrés de Wadge $\mathbb{WD}(\mathcal{X})$ de $\mathcal{X}$ hérite de la relation $\leq_w$ qui est un ordre partiel sur $\mathbb{WD}(\mathcal{X})$. L'ordre partiel $(\mathbb{WD}(\mathcal{X}), \leq_w)$ est l'ordre de Wadge sur $\mathcal{X}$. Depuis 50 ans, il a été analysé en profondeur dans le contexte des espaces polonais [Dup01, KLS12, KM19, Lou83, Sch18, Wad84].

Si $\Gamma \subseteq \mathcal{P}(\mathcal{X})$, l'ordre partiel $(\mathbb{WD}_\Gamma(\mathcal{X}), \leq_w)$ est la restriction de l'ordre de Wadge aux degrés générés par Γ, i.e., $\mathbb{WD}_\Gamma(\mathcal{X}) = \{[A]_w : A \in \Gamma\}$. Notons que le préordre de Wadge sur les sous-ensembles Γ admet une antichaine infinie (respectivement une suite strictement $\leq_w$-décroissante) si et seulement si l'ordre de Wadge sur les degrés Γ admet aussi une telle antichaine (respectivement une suite strictement $\leq_w$-décroissante).

Contrairement aux hiérarchies borélienne et de la différence de Hausdorff-Kuratowski, le préordre $\leq_w$ n'induit pas trivialement une structure de bel ordre, i.e., un ordre bien-fondé sans antichaine infinie. En effet, cette propriété des hiérarchies borélienne et de la différence de Hausdorff-Kuratowski découle de leur définition qui dépend d'opérations ensemblistes. Etant donné que la définition du préordre de Wadge ne dépend que de comparaisons par fonctions continues, cette propriété ne se transfère pas directement au préordre de Wadge. En d'autres termes, il n'est pas trivial de savoir si l'ordre de Wadge est un bel ordre, i.e., s'il admet aucune antichaine infinie et aucune suite strictement $\leq_w$-décroissante. Cependant, grâce à la caractérisation des réductions continues via la théorie des jeux et introduite par Wadge dans [Wad84], il est possible de montrer que l'ordre de Wadge sur les boréliens d'un espace polonais de dimension zéro est un bel ordre, où un espace est de dimension zéro s'il admet une base faite d'ensembles à la fois ouverts et fermés. Par exemple, l'espace de Baire ω^ω, qui est homéomorphe aux irrationnels munis de la topologie standard, est polonais de dimension zéro, où ω^ω est l'espace produit de la topologie discrète sur l'ensemble des entiers naturels ω. L'ordre de Wadge sur les boréliens de l'espace de Baire ω^ω est représenté ci-dessous :



Récemment, Schlicht a prouvé que la situation est beaucoup moins satisfaisante dans les espaces polonais qui ne sont pas de dimension zéro. En effet, dans ce cas, il existe une antichaine non-dénombrable dans l'ordre de Wadge des boréliens [Sch18].

L'ordre de Wadge sur le domaine de Scott $\mathcal{P}\omega$

Bien que les hiérarchies borélienne et de la différence de Hausdorff-Kuratowski sont bien connues sur les espaces quasi-polonais, très peu d'informations existent à propos de l'ordre de Wadge sur ces espaces.

La première partie de la thèse est consacrée à l'étude de l'ordre de Wadge sur les sous-ensembles boréliens du domaine de Scott $\mathcal{P}\omega$, i.e., à l'étude de $(\mathbb{WD}_\mathcal{B}(\mathcal{P}\omega), \leq_w)$. Plusieurs résultats ont déjà été obtenus dans la littérature. Dans [Sel05], Selivanov a prouvé l'existence d'une antichaine de taille 4 dans $(\mathbb{WD}_\mathcal{B}(\mathcal{P}\omega), \leq_w)$ ainsi que l'existence de deux degrés minimaux $\leq_w$ à chaque niveau infini $\omega \leq \alpha < \omega_1$ de la hiérarchie de la différence des ouverts. Dans [BG15b], Becher et Grigorieff ont produit, pour chaque niveau infini $\omega \leq \alpha < \omega_1$ de la hiérarchie de la différence des ouverts, des suites strictement $\leq_w$-croissantes de degrés de longueur α, et ils ont également décrit l'unique degré $\leq_w$-maximal pour chacun de ces niveaux.

Dans le premier résultat principal de la thèse, nous prouvons à la fois que $(\mathbb{WD}_{\mathcal{B}}(\mathcal{P}\omega), \leq_w)$ n'est pas bien-fondé et qu'il admet des antichaines infinies. De plus, nous montrons que ces propriétés apparaissent déjà dans les ω-différences d'ouverts, i.e., au plus bas niveau possible de complexité topologique.

Théorème. L'ordre partiel $\left(D_\omega(\mathbf{\Sigma}_1^0)(\mathcal{P}\omega), \leq_w\right)$ n'est pas bien-fondé et admet des antichaines infinies.

Afin d'obtenir ce résultat, nous isolons d'abord une classe $\mathbb{P}_{\text{shr}}$ d'ordres partiels dénombrables appelés *shrubs* qui partagent de nombreuses propriétés avec les arbres bien-fondés. Ensuite, nous définissons la classe $\mathbb{P}_{\text{emb}}$ qui contient des shrubs 2-coloriés ayant de bonnes propriétés. Les éléments de $\mathbb{P}_{\text{emb}}$ sont naturellement comparés via les homomorphismes, i.e., si $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\text{emb}}$, nous écrivons $\mathsf{P} \preccurlyeq_c \mathsf{Q}$ s'il existe un homomorphisme d'ordres partiels 2-coloriés de P dans Q. Clairement, $\preccurlyeq_c$ est un préordre sur $\mathbb{P}_{\text{emb}}$. Nous écrivons $(\mathbb{D}(\mathbb{P}_{\text{emb}}), \preccurlyeq_c)$ pour l'ordre partiel quotient induit. Notre résultat principal est la construction d'un plongement d'ordre de $(\mathbb{D}(\mathbb{P}_{\text{emb}}), \preccurlyeq_c)$ dans l'ordre de Wadge des sous-ensembles $\mathbf{\Delta}_2^0$ du domaine de Scott $\mathcal{P}\omega$, où un ensemble est $\mathbf{\Delta}_2^0$ s'il est à la fois $\mathbf{\Sigma}_2^0$ et $\mathbf{\Pi}_2^0$. Cette construction est une généralisation de [Sel05], où Selivanov définit un tel plongement à partir de la classe des arbres bien-fondés 2-coloriés. Finalement, en utilisant ce plongement, nous prouvons que l'ordre de Wadge sur les sous-ensembles $\mathbf{\Delta}_2^0$ du domaine de Scott $\mathcal{P}\omega$ n'est pas bien-fondé et possède des antichaines infinies.

Ces résultats ont été obtenus avec Jacques Duparc et ont été publiés dans *The Journal of Symbolic Logic* [DV20].

Réductions via relations totales et relativement continues

Le préordre de Wadge ne se comporte pas bien sur les espaces qui ne sont pas polonais de dimension zéro car il y a trop peu de fonctions continues. Pour remédier à ce problème, les mathématiciens ont envisagé des notions de réduction plus générales. Par exemple, ils ont considéré des réductions induites par des classes plus générales de fonctions [MRSS15]. Cependant, de telles réductions ne préservent pas certaines propriétés topologiques importantes car les niveaux inférieurs de la hiérarchie borélienne s'effondrent. Une autre direction consiste à garder la condition de continuité car elle préserve la structure topologique, mais à abandonner la notion de fonctions.

Cela donne la notion de réductions par relation totale et relativement continue qui a été introduite pour la première fois dans [BH94]. Ces réductions sont issues du domaine de la calculabilité qui étudie l'analyse (*computable analysis*) et sont largement étudiées dans [Peq15b].

Soit $\mathcal{X}$ un espace T_0 à base dénombrable. Une représentation admissible de $\mathcal{X}$ est une fonction continue partielle et surjective $\rho : \subseteq \omega^\omega \to \mathcal{X}$ telle que, pour toute fonction continue partielle $f : \subseteq \omega^\omega \to \mathcal{X}$, il existe une fonction continue $h : \mathrm{dom}(f) \to \mathrm{dom}(\rho)$ telle que $\rho\left(h(\alpha)\right) = f(\alpha)$ pour tout $\alpha \in \mathrm{dom}(f)$. Dans [dB13], il est prouvé qu'un espace T_0 à base dénombrable $\mathcal{X}$ admet une représentation admissible *totale* si et seulement si $\mathcal{X}$ est quasi-polonais. Si $\mathcal{X}$ est un espace topologique, une relation totale sur $\mathcal{X}$ est un sous-ensemble $R \subseteq \mathcal{X}^2$ tel que, pour tout $x \in \mathcal{X}$, il existe $x' \in \mathcal{X}$ tel que $(x, x') \in R$. Si $\mathcal{X}$ est un espace T_0 à base dénombrable, une relation totale $R \subseteq \mathcal{X}^2$ est relativement continue s'il existe une (ou de manière équivalente si pour toute) représentation admissible $\rho :\subseteq \omega^\omega \to \mathcal{X}$ de $\mathcal{X}$, il existe un réaliseur continu de R, à savoir une fonction continue $f : \mathrm{dom}(\rho) \to \mathrm{dom}(\rho)$ telle que, pour tout $\alpha \in \mathrm{dom}(\rho)$, nous avons

$$(\rho(\alpha), \rho \circ f(\alpha)) \in R.$$

Si $A, B \subseteq \mathcal{X}$, alors nous écrivons $A \preccurlyeq_w B$ si et seulement s'il existe une relation totale et relativement continue $R \subseteq \mathcal{X}^2$ qui réduit A à B, i.e., si pour tout $x, y \in \mathcal{X}$, si $R(x, y)$ est vrai, alors $(x \in A \leftrightarrow y \in B)$ est aussi vrai. Pour toute fonction continue $f : \mathcal{X} \to \mathcal{X}$, le graphe de la fonction $\{(x, f(x)) : x \in \mathcal{X}\} \subseteq \mathcal{X}^2$ est une relation totale et relativement continue, de sorte que $A \leq_w B$ implique $A \preccurlyeq_w B$. Si $\mathcal{X}$ est un espace quasi-polonais, l'ordre partiel quotient induit par $\preccurlyeq_w$ sur $\mathcal{B}(\mathcal{X})$ est naturellement un bel ordre puisque tout espace quasi-polonais admet une représentation admissible totale. Dans [Peq15b], Pequignot a montré que les deux préordres $\leq_w$ et $\preccurlyeq_w$ coïncident sur tout espace polonais de dimension zéro, de sorte que $\preccurlyeq_w$ peut être considéré comme une généralisation du préordre de Wadge à tous les espaces T_0 à base dénombrable. Si $\mathcal{X}$ est un tel espace, l'ordre partiel quotient induit sur les sous-ensembles boréliens de $\mathcal{X}$ est noté $\left(\mathbb{WD}_{\mathcal{B}}^{\sim_w}(\mathcal{X}), \preccurlyeq_w\right)$.

Le préordre $\preccurlyeq_w$ sur le domaine de Scott $\mathcal{P}\omega$

Dans le deuxième résultat principal de la thèse, nous prouvons que l'ordre partiel $\left(\mathbb{WD}_{\mathcal{B}}^{\sim_w}(\mathcal{X}), \preccurlyeq_w\right)$ est isomorphe à l'ordre de Wadge sur l'espace de Baire ω^ω restreint aux degrés boréliens non-auto-duaux, où un degré $[A]_w$ est non-auto-dual si $[A]_w \neq [A^{\mathrm{c}}]_w$.

Théorème. L'ordre partiel $\left(\mathbb{WD}_{\mathcal{B}}^{\sim_w}(\mathcal{P}\omega), \preccurlyeq_w\right)$ est isomorphe à la restriction de l'ordre partiel $(\mathbb{WD}_{\mathcal{B}}(\omega^\omega), \leq_w)$ aux degrés non-auto-duaux.

Pour montrer ce résultat, nous utilisons les résultats de Fournier qui a prouvé que le même résultat est valable pour l'espace des conciliants Conc [Fou16], où Conc est l'ensemble $\omega^{\leq\omega}$ des suites finies et infinies d'entiers équipé de la topologie du préfixe qui est générée par la base

$$\left\{\left\{x \in \omega^{\leq\omega} : t \sqsubseteq x\right\} : t \in \omega^{<\omega}\right\}.$$

L'espace des conciliants Conc est un espace quasi-polonais. Notre preuve repose essentiellement sur le fait que, $\mathcal{P}\omega$ étant universel parmi les espaces quasi-polonais, il contient une copie de $\omega^{\leq\omega}$.

La Conjecture de la Décomposabilité

La deuxième partie de la thèse est consacrée à l'étude des fonctions boréliennes sur les espaces polonais de dimension zéro. Supposons pour le moment que $\mathcal{X}$ et $\mathcal{Y}$ soient des espaces polonais. Une fonction $f : \mathcal{X} \to \mathcal{Y}$ est borélienne si la préimage de tout sous-ensemble borélien de $\mathcal{Y}$ est un sous-ensemble borélien de $\mathcal{X}$, i.e., si $f^{-1}[B] \in \mathcal{B}(\mathcal{X})$ pour tout $B \in \mathcal{B}(\mathcal{Y})$. De manière équivalente, $f : \mathcal{X} \to \mathcal{Y}$ est borélienne si $f^{-1}[U] \in \mathcal{B}(\mathcal{X})$ pour tout $U \in \mathbf{\Sigma}_1^0(\mathcal{Y})$. Une fonction $f : \mathcal{X} \to \mathcal{Y}$ est continue par morceaux s'il existe une partition dénombrable $\{A_n : n \in \omega\}$ de $\mathcal{X}$ telle que $f \restriction A_n$ est continue pour tout $n \in \omega$. En d'autres termes, une fonction $f : \mathcal{X} \to \mathcal{Y}$ est continue par morceaux si elle est décomposable en un nombre dénombrable de fonctions continues. L'étude de la décomposabilité des fonctions borélienne a commencé il y a plus d'un siècle avec une question posée par Luzin : *Toute fonction borélienne est-elle continue par morceaux ?* Depuis les années 1930, nous savons que la réponse à cette question est négative. Par exemple, soit $(\omega + 1)^\omega$ est équipé du produit de la topologie d'ordre naturel sur $\omega + 1$. Nous définissons la fonction $P : (\omega + 1)^\omega \to \omega^\omega$ comme suit : pour tout $x \in (\omega + 1)^\omega$ et tout $n \in \omega$, $f(x)(n) = 0$ si $x(n) = \omega$ et $f(x)(n) = n + 1$ sinon. La fonction P est appelée fonction de Pawlikowski et constitue un exemple naturel de fonction borélienne qui n'est pas continue par morceaux [Sol98]. Cependant, la question de Luzin a donné naissance à un domaine de recherche encore actif dont fait partie la deuxième partie de cette thèse.

Comme il est d'usage en théorie descriptive des ensembles, la stratégie d'étude des fonctions boréliennes consiste à les stratifier en fonction de leur complexité topologique. Il existe essentiellement trois méthodes de stratification. Soit $\mathcal{F}_{\mathcal{B}}(\mathcal{X}, \mathcal{Y})$ l'ensemble de toutes les fonctions boréliennes de $\mathcal{X}$ dans $\mathcal{Y}$, et $\mathcal{F}_{\mathcal{B}}(\subseteq \mathcal{X},\mathcal{Y})$ l'ensemble de toutes les fonctions partielles boréliennes de $\mathcal{X}$ dans $\mathcal{Y}$.

1. Pour tout $n \in \omega$, soit $f_n : \mathcal{X} \to \mathcal{Y}$ une fonction. Si elle existe, leur limite ponctuelle est notée $\lim_{n\in\omega} f_n : \mathcal{X} \to \mathcal{Y}$. Si nous écrivons $\mathcal{BC}_0(\mathcal{X},\mathcal{Y})$ pour l'ensemble des fonctions continues de $\mathcal{X}$ dans $\mathcal{Y}$ et $\mathcal{BC}_1(\mathcal{X},\mathcal{Y})$ pour l'ensemble des fonctions $\mathbf{\Sigma}^0_2$-mesurables, nous pouvons définir par induction transfinie l'ensemble des fonctions de classe de Baire $\alpha \in \omega_1$ comme
$$\mathcal{BC}_\alpha(\mathcal{X},\mathcal{Y}) = \left\{\lim_{n\in\omega} f_n : f_n \in \mathcal{BC}_{\beta_n}(\mathcal{X},\mathcal{Y}) \text{ pour un certain } \beta_n < \alpha\right\}.$$
Pour tout $\beta < \alpha$, nous avons clairement $\mathcal{BC}_\beta(\mathcal{X},\mathcal{Y}) \subseteq \mathcal{BC}_\alpha(\mathcal{X},\mathcal{Y})$.

2. Soit $f : \mathcal{X} \to \mathcal{Y}$, $\Gamma \subseteq \mathcal{P}(\mathcal{X})$ et $\mathcal{F} \subseteq \mathcal{F}_{\mathcal{B}}(\subseteq \mathcal{X},\mathcal{Y})$. S'il existe une Γ-partition $\{D_n : n \in \omega\} \subseteq \Gamma$ de $\mathcal{X}$ telle que $f \upharpoonright D_n \in \mathcal{F}$, alors f est une $\mathcal{F}$-fonction sur une Γ-partition. Nous écrivons $\mathrm{Dec}(\mathcal{F},\ \Gamma)(\mathcal{X},\mathcal{Y})$ pour l'ensemble des $\mathcal{F}$-fonctions sur une Γ-partition.

3. Soit $\alpha,\beta \in \omega_1$, nous définissons
$$\Lambda_{\alpha,\beta}(\mathcal{X},\mathcal{Y}) = \left\{f : \mathcal{X} \to \mathcal{Y} : \forall B \in \mathbf{\Sigma}^0_\alpha(\mathcal{Y})\ \ f^{-1}[B] \in \mathbf{\Sigma}^0_\beta(\mathcal{X})\right\},$$
et $\Lambda_{\alpha,\beta}(\subseteq \mathcal{X},\mathcal{Y})$ pour l'ensemble de toutes ces fonctions *partielles* de $\mathcal{X}$ dans $\mathcal{Y}$. De plus, si $\alpha,\beta \geq 2$, ou même $\alpha,\beta \geq 1$ si les espaces sont de dimension zéro, alors
$$\Lambda_{\alpha,\beta}(\mathcal{X},\mathcal{Y}) = \left\{f : \mathcal{X} \to \mathcal{Y} : \forall B \in \mathbf{\Delta}^0_\alpha(\mathcal{Y})\ \ f^{-1}[B] \in \mathbf{\Delta}^0_\beta(\mathcal{X})\right\}.$$

Nous considérons souvent $\mathrm{Dec}\left(\Lambda_{\alpha,\beta}(\subseteq \mathcal{X},\mathcal{Y}),\ \mathbf{\Delta}^0_\gamma(\mathcal{X})\right)(\mathcal{X},\mathcal{Y})$ pour des ordinaux $\alpha,\beta,\gamma \in \omega_1$. Dans ce cas, nous écrivons $\mathrm{Dec}\left(\Lambda_{\alpha,\beta},\ \mathbf{\Delta}^0_\gamma\right)(\mathcal{X},\mathcal{Y})$ pour alléger la notation.

Par des résultats classiques de Lebesgue, Hausdorff et Banach, il existe une connexion forte entre deux de ces stratifications (voir [Kec95]).

Théorème. Si $1 \leq \alpha < \omega_1$, alors $f \in \mathcal{BC}_\alpha(\mathcal{X},\mathcal{Y})$ si et seulement si $f \in \Lambda_{1,\alpha+1}(\mathcal{X},\mathcal{Y})$. En particulier, $\mathcal{F}_{\mathcal{B}}(\mathcal{X},\mathcal{Y}) = \bigcup_{\alpha\in\omega_1} \mathcal{BC}_\alpha(\mathcal{X},\mathcal{Y})$.

De plus, si $\mathcal{X}$ est de dimension zéro, alors $f \in \mathcal{BC}_1(\mathcal{X},\mathcal{Y})$ si et seulement si f est la limite ponctuelle d'une suite de fonctions continues.

Dans [JR82], Jayne et Rogers ont prouvé le premier résultat vers une réponse positive à une question du type de celle posée par Luzin. Un espace topologique $\mathcal{X}$ est Suslin s'il est l'image par fonction continue de l'espace de Baire ω^ω. Ainsi, si $\mathcal{X}'$ est un espace polonais, alors $\mathcal{X} \subseteq \mathcal{X}'$ est Suslin si et seulement si $\mathcal{X}$ est analytique.

Théorème ([JR82])**.** Si $\mathcal{X}$ est Suslin, alors

$$\Lambda_{2,2}(\mathcal{X}, \mathcal{Y}) = \mathrm{Dec}\left(\Lambda_{1,1},\ \boldsymbol{\Delta}^0_2\right)(\mathcal{X}, \mathcal{Y}).$$

Même si la preuve donnée dans [JR82] n'utilise que des concepts de topologie générale, elle est, selon ses propres auteurs, *compliquée*. Au fil des années, plusieurs preuves plus simples ont été publiées. Dans [Sol98], Solecki a utilisé des notions de théorie descriptive effective des ensembles pour prouver le résultat. Dans [MRS10, KMRS12], Kačena, Motto Ros et Semmes ont donné une preuve plus simple en utilisant des notions de topologie générale.

La Conjecture de la Décomposabilité est une généralisation du théorème de Jayne-Rogers qui relie les différentes stratifications des fonctions boréliennes mentionnées ci-dessus. Elle apparaît sous différentes formes dans plusieurs articles [And07, MR13, Kih15, GKN21, DKSZ20, PS12].

Conjecture (Conjecture de la Décomposabilité)**.** Si $\mathcal{X}$ est Suslin et $1 \leq m \leq n < \omega$, alors

$$\Lambda_{m,n}(\mathcal{X}, \mathcal{Y}) = \mathrm{Dec}\left(\Lambda_{1,n-m+1},\ \boldsymbol{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y}).$$

En fait, seule une inclusion est difficile à montrer car il est facile de prouver

$$\Lambda_{m,n}(\mathcal{X}, \mathcal{Y}) \supseteq \mathrm{Dec}\left(\Lambda_{1,n-m+1},\ \boldsymbol{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y}).$$

La première extension du théorème de Jayne-Rogers vers la Conjecture de la Décomposabilité apparaît dans [Sem09] pour $\mathcal{X} = \mathcal{Y} = \omega^\omega$. En effet, en utilisant les techniques de théorie des jeux fournies par le fait que ω^ω est de dimension zéro, Semmes a reprouvé le théorème de Jayne-Rogers et a également prouvé les cas $n \leq 3$ de la Conjecture de la Décomposabilité.

Théorème ([Sem09])**.** Si $1 \leq m \leq n \leq 3$, alors

$$\Lambda_{m,n}(\omega^\omega, \omega^\omega) = \mathrm{Dec}\left(\Lambda_{1,n-m+1},\ \boldsymbol{\Delta}^0_n\right)(\omega^\omega, \omega^\omega).$$

Ce résultat a été récemment généralisé à tout espace polonais $\mathcal{X}$ par Ding, Kihara, Semmes et Zhao [DKSZ20].

Théorème ([DKSZ20]). Si $\mathcal{X}$ est polonais et $1 \leq m \leq n = 3$, alors

$$\Lambda_{m,n}(\mathcal{X}, \mathcal{Y}) = \mathrm{Dec}\left(\Lambda_{1,n-m+1},\ \mathbf{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y}).$$

Pour $n > 3$ et à notre connaissance, aucune preuve n'a encore été publiée. Cependant, récemment, Gregoriades, Kihara et Ng ont prouvé que le cas $m = 2$ est suffisant pour prouver la Conjecture de la Décomposabilité.

Théorème ([GKN21]). Soient $\mathcal{X}', \mathcal{Y}$ des espaces polonais, $\mathcal{X} \subseteq \mathcal{X}'$ Suslin et $n \geq 2$. Si

$$\Lambda_{2,n}(\mathcal{X}, \mathcal{Y}) \subseteq \mathrm{Dec}\left(\Lambda_{1,n-1},\ \mathbf{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y}),$$

alors pour tout $2 \leq m \leq n$,

$$\Lambda_{m,n}(\mathcal{X}, \mathcal{Y}) \subseteq \mathrm{Dec}\left(\Lambda_{1,n-m+1},\ \mathbf{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y}).$$

En d'autres termes, le cas $m = 2$ de la Conjecture de la Décomposabilité est la bonne généralisation du théorème de Jayne-Rogers.

Mentionnons également que Day et Marks ont récemment annoncé qu'ils ont prouvé la Conjecture de la Décomposabilité pour $\mathcal{X}$ et $\mathcal{Y}$ polonais en supposant une certaine quantité de détermination. Cependant, ce résultat n'est pas encore publié.

Théorème ([Day19, Mar20]). En supposant la determination des ensembles $\mathbf{\Sigma}^1_2$, la Conjecture de la Décomposabilité est vraie pour $\mathcal{X}$ et $\mathcal{Y}$ polonais.

La Conjecture de la Décomposabilité via les arbres à questions

Dans la dernière partie de la thèse, nous fournissons de nouvelles techniques pour aborder la Conjecture de la Décomposabilité. Elles utilisent le cadre de la théorie des jeux des espaces polonais de dimension zéro ainsi que la machinerie des arbres à questions développée par Duparc dans [Dup01]. Si $\mathcal{X}$ est un espace polonais et $\{A_n : n \in \omega\} \subseteq \mathbf{\Delta}^0_2(\mathcal{X})$, il est bien connu qu'il existe une topologie τ plus fine que la topologie de $\mathcal{X}$ tel que $(\mathcal{X}, \tau)$ est polonais de dimension zéro et $\{A_n : n \in \omega\} \subseteq \mathbf{\Delta}^0_1(\mathcal{X}, \tau)$ [Kec95]. Par conséquent, si $f : \mathcal{X} \to \mathcal{Y} \in \Lambda_{1,3}$, un arbre à questions permet de réduire la complexité topologique de la fonction pour obtenir $f : (\mathcal{X}, \tau) \to \mathcal{Y} \in \Lambda_{1,2}$, et ainsi de procéder à des preuves par induction sur la complexité des fonctions boréliennes.

Dans le troisième résultat principal de la thèse, nous prouvons que si $\mathcal{X}$ et $\mathcal{Y}$ sont polonais de dimension zéro, alors, sous l'axiome de détermination (AD), la Conjecture de la Décomposabilité est une conséquence de l'Hypothèse A suivante.

Hypothèse A. Pour tout $f : [T] \to \omega^\omega \in \Lambda_{1,2}$ où $T \subseteq \omega^{<\omega}$ est un arbre non-vide élagué, il existe un sous-ensemble parfait $\mathcal{P} \subseteq [T]$ tel que:

1. $f : [T] \setminus \mathcal{P} \to \omega^\omega \in \mathrm{Dec}\left(\Lambda_{1,1},\ \mathbf{\Delta}^0_2\right)$.
2. Si $\mathcal{P}$ est non-vide, il existe $\left\{x_n^l : n, l \in \omega\right\} \subseteq \mathcal{T}(\mathcal{P})$, $\{p_n : n \in \omega\} \subseteq \mathcal{T}(\mathcal{P})$ et $\{u_n : n \in \omega\} \subseteq \omega^{<\omega}$ tels que:
   (a) Pour tout $n, l \in \omega$, $p_n \sqsubset x_n^l$ et $u_n \sqsubset f\left(x_n^l\right)$.
   (b) $\{u_n : n \in \omega\} \subseteq \omega^{<\omega}$ est un ensemble d'éléments incompatibles deux-à-deux.
   (c) Pour tout $n \in \omega$, $f^{-1}[u_n] \cap [p_n]$ est propre et non-auto-dual dans $\mathcal{P}$.
   (d) Pour tout $n, l \in \omega$, $x_n^l \in \left[\mathrm{Init}_{\mathcal{P}}\left(f^{-1}[u_n] \cap [p_n]\right)\right]$.
   (e) Si $p \in \mathrm{Init}_{\mathcal{P}}\left(f^{-1}[u_n] \cap [p_n]\right)$, il existe $l \in \omega$ tel que $p \sqsubseteq x_n^l$.
   (f) Si $p \in \mathrm{Init}_{\mathcal{P}}\left(f^{-1}[u_n] \cap [p_n]\right)$, il existe $m \in \omega$ tel que $p \sqsubset p_m \notin \mathrm{Init}_{\mathcal{P}}\left(f^{-1}[u_n] \cap [p_n]\right)$.

Théorème (AD)**.** La Conjecture de la Décomposabilité est une conséquence de l'Hypothèse A.

Nous prouvons également que l'Hypothèse A est atteignable car elle est satisfaite par une grande classe de fonctions. Par exemple, si $f : \mathcal{X} \to \omega^\omega \in \Lambda_{1,2}$, alors la fonction *joint*, dénotée par $id \oplus f$, de f avec la fonction identité $id : \omega^\omega \to \omega^\omega$ satisfait l'Hypothèse A.

De plus, même si l'Hypothèse A n'est pas vérifiée, nous croyons fermement que les nouvelles techniques et constructions introduites dans cette deuxième partie de la thèse offrent une nouvelle perspective pour la résolution de la Conjecture de la Décomposabilité.

Chapter 1

Introduction

Since the dawn of time, mankind has been trying to understand its surrounding world, which has led to organize objects according to their complexity. For such an organization to be meaningful, it should satisfy some desirable properties. One first wishes to isolate the fundamental elements from which the remaining objects are constructed, and then to understand how these fundamental elements combine with each other in order to create more and more involved objects. For mathematicians, this organization process consists in providing a classification of the objects they study according to various notions of complexity in such a way that, the more complex an object is, the higher it appears in the classification.

1.1 The measure problem

We illustrate this idea with an informal presentation of the measure problem because it is at the origin of the mathematical area this thesis belongs to: descriptive set theory. We refer the reader to *The higher infinite* by Kanamori [Kan09] and to *Descriptive set theory* by Moschovakis [Mos09], as well as references therein, for a more detailed historical background on this subject. For the sake of simplicity, we consider the measure problem over the real line $\mathbb{R}$ that aims at defining a notion of size for subsets of $\mathbb{R}$ which generalizes the intuitive geometric notion of length of an interval. First observe that we cannot simply classify subsets with respect to their cardinality, i.e., by counting their members. Indeed, with this notion of size introduced by Cantor in 1872, the interval $[0, 1]$ has the same size — known as the size of the continuum and usually denoted by $\mathfrak{c}$ — as the whole real line $\mathbb{R}$ which contradicts the intuitive geometric notion of length we aim to define [Can72]. At the turn of the twentieth century, Borel and Lebesgue provided an axiomatic approach to the measure problem [Bor98, Leb02] that we also adopt. More precisely, we first state the three different properties that we desire.

1. The size of any interval is its length.
2. The size is invariant under translation.
3. The size of a set which can be decomposed into countably many disjoint pieces is the sum of the size of its pieces.

Thus, mathematically, the measure problem over the real line $\mathbb{R}$ aspires to define a function — called a measure —

$$m : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$$

assigning to any subset $A \subseteq \mathbb{R}$ its size $m(A)$ which is either a positive real number or the infinity and which satisfies[1].

1. For any $a \leq b$, $m([a,b]) = m([a,b[) = m(]a,b]) = m(]a,b[) = b - a$.
2. For any $A \subseteq \mathbb{R}$ and any $r \in \mathbb{R}$, $m(A) = m(\{a + r : a \in A\})$.
3. If $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{P}(\mathbb{R})$ such that $A_n \cap A_m = \emptyset$ holds for all $n \neq m$, then $\sum_{n\in\mathbb{N}} A_n = m\left(\bigcup_{n\in\mathbb{N}} A_n\right)$.

Observe that the last property cannot be extended to arbitrary collections of disjoint subsets because

$$\sum_{r\in\mathbb{R}} m(\{r\}) = 0 \neq \infty = m(\mathbb{R}).$$

A priori, the existence of such a function is far from being obvious. Indeed, it is not clear whether a measure which assigns to *every* subset of the real line $\mathbb{R}$ its intuitive size does exist. Actually, this problem is surprisingly difficult. However, following the work of Borel [Bor98] and Lebesgue [Leb02, Leb05], it is easy to define a measure which assigns to *some* subsets of the real line $\mathbb{R}$, called measurable sets, their intuitive geometric size. First, observe that the three above-mentioned properties imply that it suffices to consider the measure problem over the open interval $]0,1[$ in order to solve it over the whole real line $\mathbb{R}$. Thus, from now on, we consider the measure problem over the unit open interval $\mathcal{X} =]0,1[$.

A σ-algebra $\mathcal{A} \subseteq \mathcal{P}(\mathcal{X})$ on $\mathcal{X}$ is a family of subsets closed under the complementation and the countable union operations, i.e., if we have a set

[1]For any $a, b \in \mathbb{R}$ such that $a \leq b$, we define the intervals $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $[a,b[= \{x \in \mathbb{R} : a \leq x < b\}$, $]a,b] = \{x \in \mathbb{R} : a < x \leq b\}$ and $]a,b[= \{x \in \mathbb{R} : a < x < b\}$.

$\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$, then the complement ${A_0}^{\mathrm{c}} = \mathcal{X} \setminus A_0 = \{x \in \mathcal{X} : x \notin A_0\}$ belongs to $\mathcal{A}$ and the countable union $\bigcup_{n\in\mathbb{N}} A_n = \{x \in \mathcal{X} : \exists n \in \mathbb{N}\ x \in A_n\}$ belongs to $\mathcal{A}$ as well. By the third property, the measurable sets form a σ-algebra on $\mathcal{X}$. Since any interval is measurable, the σ-algebra generated by the intervals, i.e., the closure under the complementation and the countable union operations of the set of intervals of $\mathcal{X}$, only contains measurable sets. This σ-algebra — denoted by $\mathcal{B}(\mathcal{X})$ — was first isolated by Borel and is now called the σ-algebra of Borel sets [Bor98]. Although Borel sets are a primary focus for most mathematicians, they only form a small portion of the family of all subsets of $\mathcal{X}$ for there are only $\mathfrak{c}$ many Borel sets but $2^{\mathfrak{c}}$ many subsets of $\mathcal{X}$.

According to Lebesgue, one classifies the Borel sets according to the complexity of their definition in a transfinite hierarchy containing ω_1 many levels[2], where ω_1 is the first uncountable ordinal [Leb05]. At the bottom level of this classification, one finds the fundamental sets, i.e., the intervals $]a, b[\subseteq \mathcal{X}$, where $0 \leq a \leq b \leq 1$. Following the standard terminology, this bottom level also contains any countable union of these intervals. It is denoted by

$$\mathbf{\Sigma}^0_1(\mathcal{X}) = \left\{ \bigcup_{n\in\mathbb{N}}]a_n, b_n[\ : 0 \leq a_n \leq b_n \leq 1 \text{ for any } n \in \mathbb{N} \right\}.$$

The elements of $\mathbf{\Sigma}^0_1(\mathcal{X})$ are the open subsets of $\mathcal{X}$. Any complement of an open set is a closed set and is also considered as easy to define. Thus we write

$$\mathbf{\Pi}^0_1(\mathcal{X}) = \left\{ A \subseteq \mathcal{X} : A^{\mathrm{c}} \in \mathbf{\Sigma}^0_1(\mathcal{X}) \right\}.$$

Then, one constructs more and more complex sets by recursive applications of the complementation and the countable union operations. More precisely, if $\mathbf{\Pi}^0_\alpha(\mathcal{X})$ is already defined for some countable ordinal α, then we define

$$\mathbf{\Sigma}^0_{\alpha+1}(\mathcal{X}) = \left\{ \bigcup_{n\in\mathbb{N}} A_n : A_n \in \mathbf{\Pi}^0_\alpha(\mathcal{X}) \text{ for any } n \in \mathbb{N} \right\}$$

and

$$\mathbf{\Pi}^0_{\alpha+1}(\mathcal{X}) = \left\{ A : A^{\mathrm{c}} \in \mathbf{\Sigma}^0_{\alpha+1}(\mathcal{X}) \right\}.$$

[2] The levels are indexed by the countable ordinals, namely $1 < 2 < \cdots < \omega < \omega + 1 < \cdots < \omega \cdot 2 < \omega \cdot 2 + 1 < \cdots < \omega^\omega < \omega^\omega + 1 < \cdots < \omega^2 < \omega^2 + 1 < \ldots$

This construction actually extends beyond the finite into the transfinite: if λ is a countable limit ordinal, then

$$\boldsymbol{\Sigma}^0_\lambda(\mathcal{X}) = \left\{ \bigcup_{n\in\mathbb{N}} A_n : A_n \in \bigcup_{\alpha<\lambda} \boldsymbol{\Pi}^0_\alpha(\mathcal{X}) \text{ for any } n \in \mathbb{N} \right\}.$$

Lebesgue proved that the Borel hierarchy is proper, i.e., if α and β are two countable ordinals such that $\alpha < \beta$, then $\boldsymbol{\Sigma}^0_\alpha(\mathcal{X}) \subsetneq \boldsymbol{\Pi}^0_\beta(\mathcal{X})$ [Leb05]. So that the family of all subsets we have defined so far is organized in a proper hierarchy containing ω_1 many levels.

$$\begin{array}{ccccccccc}
\boldsymbol{\Sigma}^0_1(\mathcal{X}) & \subsetneq & \boldsymbol{\Sigma}^0_2(\mathcal{X}) & & \subsetneq & \boldsymbol{\Sigma}^0_\alpha(\mathcal{X}) & & \subsetneq & \boldsymbol{\Sigma}^0_\beta(\mathcal{X}) & \\
 & \nearrow\!\!\!\!\searrow & & \cdots & \nearrow\!\!\!\!\searrow & & \cdots & \nearrow\!\!\!\!\searrow & & \cdots \\
\boldsymbol{\Pi}^0_1(\mathcal{X}) & \subsetneq & \boldsymbol{\Pi}^0_2(\mathcal{X}) & & \subsetneq & \boldsymbol{\Pi}^0_\alpha(\mathcal{X}) & & \subsetneq & \boldsymbol{\Pi}^0_\beta(\mathcal{X}) &
\end{array}$$

Figure 1.1: The Borel hierarchy of $\mathcal{X}$.

In particular, a set $A \subseteq \mathcal{X}$ is Borel if and only if there exists a countable ordinal α such that $A \in \boldsymbol{\Sigma}^0_\alpha(\mathcal{X})$. For any Borel set $A \in \mathcal{B}(\mathcal{X})$, the least such ordinal α measures the number of alternating use of the complementation and countable union operations needed to define A from the open sets. In other words, it measures the complexity of the definition of A and is such that the larger this ordinal is, the more complex the definition of A from the open sets is.

Although the Borel sets fulfill standard closure properties, they somehow fail to satisfy the simple following desirable property: any subset of a set of measure zero also has measure zero. This property is known as the completeness of a measure and was introduced by Lebesgue [Leb02]. Using the notion that a set is null if it is contained in a Borel set of measure zero, Lebesgue extended the Borel sets to the Lebesgue-measurable sets. A set A is Lebesgue-measurable if it is a Borel subset modulo some null set, i.e., if there exists a Borel set $B \in \mathcal{B}(\mathcal{X})$ such that $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is null. The family of Lebesgue-measurable sets is a σ-algebra which contains all the Borel sets. Moreover, Lebesgue proved that it is a strict inclusion, i.e., there exists a Lebesgue-measurable set which is not Borel. This follows from the stronger result that there are $2^{\mathfrak{c}}$ Lebesgue-measurable sets for there exists an uncountable Borel set of measure zero, namely the Cantor set. As a consequence, one wonders if any subset of $\mathcal{X}$ is Lebesgue-measurable.

In 1905, Vitali answered this question negatively [Vit05]. He proved that there exists a non-Lebesgue-measurable set. Unlike all the constructions that we considered so far and which can be formalized in the standard system of axioms of Zermelo-Fraenkel set theory (ZF), Vitali's result makes use of the additional axiom of choice (AC), i.e., its formalization requires the extended set theory ZF+AC (ZFC). The axiom of choice is the statement that any family of non-empty sets admits a choice function assigning to each such non-empty set one of its element. It was first introduced by Zermelo in 1904 [Zer04]. Unlike the other operations we presented so far, the axiom of choice is *controversial* among mathematicians for being non-constructive. Indeed, it allows to consider sets that have no *explicit* definition. As a consequence, Vitali's result is somewhat counterintuitive and it yields the following questions:

(a) Can we characterize the Lebesgue-measurable sets when we assume the axiom of choice?
(b) Is there an alternative axiom to the axiom of choice for which all sets are Lebesgue-measurable?

To deepen these considerations, we shift our focus to the Russian mathematicians.

Luzin got acquainted with the work of the French mathematicians Borel and Lebesgue while he was a student in France. Back to Russia and starting from 1914, he gave a seminar which founded a new tradition of descriptive set theory both in Russia and in Poland because the Polish mathematician Sierpiński was among the audience. Together with his student Suslin, Luzin achieved the next step in the development of descriptive set theory which, as extraordinary as it seems, resulted from a mistake made by Lebesgue and pointed out by Suslin. We say that a set $A \subseteq \mathcal{X}$ is analytic if it is the continuous image of a Borel set, i.e., if there exists a continuous function[3] $f : \mathcal{X} \to \mathcal{X}$ and a Borel set $B \subseteq \mathcal{X}$ such that $f[B] = \{f(x) \in \mathcal{X} : x \in B\} = A$. Following the standard terminology, the family of analytic sets is denoted by $\mathbf{\Sigma}^1_1(\mathcal{X})$. It is the smallest family of subsets of $\mathcal{X}$ containing the Borel sets and closed under continuous images. In [Leb05], Lebesgue wrongly claimed that any analytic set is Borel[4]. As it was previously done with the Borel sets, the definition of the analytic sets yields a new classification of subsets of $\mathcal{X}$ [Luz25b, Sie25]. We denote by $\mathbf{\Pi}^1_1(\mathcal{X})$ the family of co-analytic sets, i.e., the complements of the analytic sets. If $\mathbf{\Pi}^1_n(\mathcal{X})$ is already defined for some $n \in \mathbb{N}$, we denote by $\mathbf{\Sigma}^1_{n+1}(\mathcal{X})$ the family of all continuous images of

[3] A function $f : \mathcal{X} \to \mathcal{X}$ is continuous if $f^{-1}[A] = \{x \in \mathcal{X} : f(x) \in A\} \in \mathbf{\Sigma}^0_1(\mathcal{X})$ for any $A \in \mathbf{\Sigma}^0_1(\mathcal{X})$.

[4] More precisely, Lebesgue claimed that the projection $\{x \in \mathcal{X} : \exists y\ (x, y) \in B\}$ of any Borel set $B \in \mathcal{B}(\mathcal{X} \times \mathcal{X})$ is Borel in $\mathcal{X}$.

$\mathbf{\Pi}_n^1(\mathcal{X})$ sets and by $\mathbf{\Pi}_{n+1}^1(\mathcal{X})$ the family of their complements. We say that a set $A \subseteq \mathcal{X}$ is projective if there exists $n \in \mathbb{N}$ such that $A \in \mathbf{\Sigma}_n^1(\mathcal{X})$. So, as for the Borel sets, the projective sets are organized into a hierarchy — called the projective hierarchy — and the least integer $n \in \mathbb{N}$ such that $A \in \mathbf{\Sigma}_n^1(\mathcal{X})$ stands as a measure of the complexity of the definition of A from the Borel sets. In 1917, Suslin proved the existence of an analytic set which is not Borel and also that the Borel sets are exactly the sets that are both analytic and co-analytic [Sus17]. In particular, Borel sets are characterized from below by the Borel hierarchy and from above by the projective hierarchy, which is also a proper hierarchy as Luzin and Sierpiński showed in 1925 [Luz25a, Luz25c, Sie25].

$$\begin{array}{ccccccccc} & & \mathbf{\Sigma}_1^1(\mathcal{X}) & \subsetneq & \mathbf{\Sigma}_2^1(\mathcal{X}) & & \subsetneq & \mathbf{\Sigma}_n^1(\mathcal{X}) & \\ \mathcal{B}(\mathcal{X}) & \begin{array}{c}\subsetneq\\ \subsetneq\end{array} & & \times & & \cdots & \times & & \cdots \\ & & \mathbf{\Pi}_1^1(\mathcal{X}) & \subsetneq & \mathbf{\Pi}_2^1(\mathcal{X}) & & \subsetneq & \mathbf{\Pi}_n^1(\mathcal{X}) & \end{array}$$

Figure 1.2: The projective hierarchy of $\mathcal{X}$.

As for the Borel sets, the projective sets only form a small portion of all the subsets of $\mathcal{X}$ for there are only $\mathfrak{c}$ many projective sets. In 1917, Luzin proved that every analytic set is Lebesgue-measurable [Luz17]. However, the mathematicians at the time were not able to climb further up inside the projective hierarchy. For instance, it was not clear to them whether all $\mathbf{\Sigma}_2^1(\mathcal{X})$ are Lebesgue-measurable or not.

This dead-end was actually clarified a few years later by the work of Gödel. In 1938, Gödel proved that, if ZF is consistent, then there exists a model of ZFC in which there is a $\mathbf{\Sigma}_2^1(\mathcal{X})$ set which is not Lebesgue-measurable [Göd38]. In particular, Gödel proved the result of Luzin about Lebesgue-measurability to be optimal in the projective hierarchy. Remarkably, the organization of Gödel's article suggests that he considered the measure problem to be as important as his resolution of the consistency of both the continuum hypothesis and the axiom of choice.

In 1963, Cohen introduced the forcing technique to prove that both the continuum hypothesis and the axiom of choice are independent of ZF [Coh63, Coh64, Coh65]. More precisely, assuming that ZF is consistent, then there is a model of ZF where the continuum hypothesis fails, and there is a model of ZF where the axiom of choice fails. Using Cohen's forcing technique, Solovay proved in 1964 that, if we assume the consistency of ZFC together with some large cardinal hypothesis — namely the existence of an inaccessible cardinal — then there is a model of ZFC in which all projective sets are Lebesgue-measurable [Sol64, Sol65, Sol70]. Thus, the Lebesgue-measurability of the $\mathbf{\Sigma}^1_2(\mathcal{X})$ sets is independent of ZFC, which answers (a) mentioned earlier.

For (b), since the axiom of choice provides many pathological sets such as the non-Lebesgue-measurable ones, mathematicians proposed other alternative axioms to avoid such bad behaviors. For our purpose, an important alternative is the axiom of determinacy (AD) whose definition relies on game-theoretical considerations. It was first introduced in 1962 by Mycielski and Steinhaus [MS62]. In this alternative set theory denoted by ZF+AD, every subset of the real line becomes Lebesgue-measurable, as proved by Mycielski and Świerczkowski in 1964 [MS64].

This present work is part of this long-standing enterprise of classifying mathematical objects according to their complexity. In particular, we study various classifications of sets and functions and expect to provide a better understanding of these objects.

1.2 The context

In the remaining part of this introduction, we provide a more formal presentation of the concepts used in this thesis. We also highlight our contributions and describe the organization of the thesis.

Polish space and descriptive set theory

A topological space is Polish if it is separable and completely metrizable, i.e., if there exist a countable dense subset and a complete metric which generates its topology. It follows that such a space admits a countable topological basis. Polish spaces are central in many areas of mathematics. Indeed, all the spaces $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^\omega, \mathbb{C}, \mathbb{C}^n, \mathbb{C}^\omega, [0,1], [0,1]^n, [0,1]^\omega,]0,1[,]0,1[^n,]0,1[^\omega$ as well as all separable Banach spaces are Polish spaces. If we endow ω with the discrete topology, the product space ω^ω is a Polish space called the Baire space. It is homeomorphic to the set of irrational numbers of $\mathbb{R}$ and plays a central role in set theory. Similarly, the set 2^ω is a Polish space called the Cantor space.

A large number of techniques available to study the real line $\mathbb{R}$ also apply in the wider context of Polish spaces. For that reason, mathematicians gladly consider Polish spaces to extend the range of application of their results. Descriptive set theory is the study of definability in the context of

Polish spaces. In this theory, sets are classified according to the topological complexity of their definition and the properties of each of these classes is rigorously analyzed [Kec95].

Quasi-Polish spaces

Over the last few decades, the rise of theoretical computer science gave a major boost to mathematics [GHK$^+$03, GL13, Sco76, Sel06, Wei00]. Naturally, definability is a major concern of theoretical computer science. Unfortunately, several topological spaces involved in theoretical computer science are not Polish and actually not even metrizable. For example, denotational semantic models computation with domains which are topological space that are in general not metrizable. Thus, there has been a growing interest to extend descriptive set theory outside the Polish realm. The first obstacle to the development of descriptive set theory in non-metrizable spaces is the classical definition of the Borel hierarchy since it is not well-behaved for these spaces. For example, with the classical definition of the Borel hierarchy, the inclusion $\mathbf{\Sigma}^0_1(\mathcal{X}) \subseteq \mathbf{\Sigma}^0_2(\mathcal{X})$ does not hold for some non-metrizable spaces $\mathcal{X}$. This difficulty was overcome by Selivanov who was able to define a well-behaved Borel hierarchy even for non-metrizable spaces [Sel05, Sel06]. If $\mathcal{X}$ is a topological space, let $\mathbf{\Sigma}^0_1(\mathcal{X})$ denote the class of all open subsets of $\mathcal{X}$, and, for any $\alpha \in \omega_1$, we define $\mathbf{\Pi}^0_\alpha(\mathcal{X}) = \left\{A \subseteq \mathcal{X} : A^{\mathrm{c}} \in \mathbf{\Sigma}^0_\alpha(\mathcal{X})\right\}$ and

$$\mathbf{\Sigma}^0_\alpha(\mathcal{X}) = \left\{ \bigcup_{n\in\omega} A_n \cap {A'_n}^{\mathrm{c}} \subseteq \mathcal{X} : \forall n \; \exists \beta_n < \alpha \; A_n, A'_n \in \mathbf{\Sigma}^0_{\beta_n}(\mathcal{X}) \right\}.$$

This definition of the Borel hierarchy is actually equivalent to the classical one for metrizable spaces, but offers the advantage of being well-behaved for any topological space. More precisely, if $\mathcal{X}$ is an uncountable topological space, we have the following strict inclusions.

$$\begin{array}{ccccccccccc}
\mathbf{\Sigma}^0_1(\mathcal{X}) & \subsetneq & \mathbf{\Sigma}^0_2(\mathcal{X}) & & \subsetneq & \mathbf{\Sigma}^0_\alpha(\mathcal{X}) & & \subsetneq & \mathbf{\Sigma}^0_\beta(\mathcal{X}) & & \\
& \nearrow\!\!\!\!\!\searrow & & \cdots & \nearrow\!\!\!\!\!\searrow & & \cdots & \nearrow\!\!\!\!\!\searrow & & & \cdots \\
\mathbf{\Pi}^0_1(\mathcal{X}) & \subsetneq & \mathbf{\Pi}^0_2(\mathcal{X}) & & \subsetneq & \mathbf{\Pi}^0_\alpha(\mathcal{X}) & & \subsetneq & \mathbf{\Pi}^0_\beta(\mathcal{X}) & &
\end{array}$$

Figure 1.3: The Borel hierarchy of $\mathcal{X}$.

Building on this work, de Brecht isolated a new class of spaces which contains the Polish spaces as well as an important class of non-Polish spaces for theoretical computer science called the ω-continuous domains [dB13]. Moreover, he showed that the techniques of descriptive set theory still hold for this larger class of spaces, thus called the quasi-Polish spaces (see Definition 2.15 for the formal definition of a quasi-Polish space).

The Scott domain $\mathcal{P}\omega$ is the set $\mathcal{P}(\omega)$ of all subsets of the integers endowed with the topology generated by the basis

$$\{\{x \subseteq \omega : F \subseteq x\} : \ F \subseteq \omega \text{ finite}\}.$$

The Scott domain $\mathcal{P}\omega$ turns out to be a quasi-Polish space [dB13]. It was first introduced by Scott as a denotational semantic for the λ-calculus [Sco76]. If we think of elements $x, y \in \mathcal{P}\omega$ as pieces of data, the inclusion $x \subseteq y$ means that y carries more information than x. Moreover, the topology of $\mathcal{P}\omega$ can be thought of as the topology of finite approximation. Indeed, if $y \in \mathcal{P}\omega$ contains an infinite amount of information — i.e., y is an infinite subset of ω — then y belongs to the basic open set $\{x \subseteq \omega : F \subseteq x\}$ generated by $F \subseteq \omega$ finite if the information contained in F is a finite approximation of the one contained in y. We say that a subset $D \subseteq \mathcal{P}\omega$ is directed if any pair $x, y \in D$ has an upper bound in D, i.e., there exists $z \in D$ such that both $x \subseteq z$ and $y \subseteq z$ hold. In other words, for any pair of elements x, y in a directed subset $D \subseteq \mathcal{P}\omega$, there is a consistent extension in D which gathers the information of both x and y. Importantly, the Scott domain $\mathcal{P}\omega$ is directed-complete, i.e., for any directed subset $D \subseteq \mathcal{P}\omega$, there exists an element $\sup D \in \mathcal{P}\omega$ which contains all the information contained in the elements of D and nothing more, namely

$$\sup D = \{n \in \omega : \exists x \in D \ n \in x\} = \bigcup D \in \mathcal{P}\omega.$$

Moreover, it turns out that a function $f : \mathcal{P}\omega \to \mathcal{P}\omega$ is continuous[5] if and only if it is monotonic and it preserves the supremum of directed subsets, i.e., if $x \subseteq y \in \mathcal{P}\omega$ and $D \subseteq \mathcal{P}\omega$ is directed, then $f(x) \subseteq f(y)$ and $f(\sup D) = \sup f[D]$. In other words, continuity respects the following desired properties of computations.

1. The more information an input holds, the more information its output discloses.
2. If D is directed — all the information gathered in its elements is consistent — then the output of the computation with input $\sup D$ contains exactly all the information gathered in the computation outputs of the elements of D, nothing more, nothing less.

[5] A function $f : \mathcal{P}\omega \to \mathcal{P}\omega$ is continuous if the preimage of every open set in $\mathcal{P}\omega$ is also an open set in $\mathcal{P}\omega$.

To say it otherwise, the Scott domain $\mathcal{P}\omega$ is a suitable model of computation which makes it explicit why it matters that much for theoretical computer science.

The Scott domain $\mathcal{P}\omega$ also stands out among the quasi-Polish spaces for its universality. Indeed, any quasi-Polish space is homeomorphic to a $\mathbf{\Pi}^0_2$-subset of the Scott domain $\mathcal{P}\omega$ [dB13], which further justifies our deep study of this space.

The Wadge preorder of sets

Any level of the Borel hierarchy is further refined into ω_1 many levels by the Hausdorff-Kuratowski difference hierarchy. Interestingly, the classes of both the Borel and the Hausdorff-Kuratowski difference hierarchies of a topological space $\mathcal{X}$ are closed under continuous preimage, i.e., if $\Gamma \subseteq \mathcal{P}(\mathcal{X})$ is such a class, $A \in \Gamma$ and $f : \mathcal{X} \to \mathcal{X}$ is continuous, then $f^{-1}[A] = \{x \in \mathcal{X} : f(x) \in A\} \in \Gamma$. This suggests the study of continuous reducibility as a further refinement of these hierarchies.

If $\mathcal{X}$ is a topological space and $A, B \subseteq \mathcal{X}$, we say that A is continuously reducible — or Wadge-reducible — to B, denoted by $A \leq_w B$, if there exists a continuous function $f : \mathcal{X} \to \mathcal{X}$ such that $f^{-1}[B] = A$. The binary relation $\leq_w$ is reflexive and transitive on $\mathcal{P}(\mathcal{X})$ since both the identity function and the composition of two continuous functions remain continuous. In other words, the binary relation $\leq_w$ is a quasi-order[6] on $\mathcal{P}(\mathcal{X})$. It is called the Wadge preorder of $\mathcal{X}$ in honor of Wadge who was the first to rigorously analyze this relation [Wad72, Wad84]. It is also a natural measure of topological complexity. Indeed, if $A \leq_w B$, then the membership problem for A can be reduced via some continuous function — which is thought of as topologically simple — to the membership problem for B. To say it otherwise, A is not more complex than B.

The Borel hierarchy on a topological space $\mathcal{X}$ naturally associates to any Borel subset $A \subseteq \mathcal{X}$ the least ordinal $\alpha \in \omega_1$ such that $A \in \mathbf{\Sigma}^0_\alpha(\mathcal{X}) \cup \mathbf{\Pi}^0_\alpha(\mathcal{X})$. This ordinal α is the level of A in the Borel hierarchy and it computes the minimal number of alternative use of the union and complementation operations required to construct A from finite intersections of open sets. Moreover, any level $\alpha \in \omega_1$ in the Borel hierarchy is only occupied by two classes, namely $\mathbf{\Sigma}^0_\alpha(\mathcal{X})$ and $\mathbf{\Pi}^0_\alpha(\mathcal{X})$. This yields the stratification of the Borel subsets depicted in Figure 1.3. This stratification of the Borel sub-

[6] In order to avoid any kind of confusion, let us specify that a quasi-order is a standard notion of order theory which is completely independent of the topological notion of a quasi-Polish space.

sets is extremely satisfactory as a notion of complexity since its levels are indexed by ordinals and any level is composed with finitely many classes. In other words, it induces a well-quasi-order on the Borel subsets. This property essentially stems from the fact that the Borel hierarchy relies on two set-theoretical operations. Since both the projective and the Hausdorff-Kuratowski difference hierarchies also rely on set-theoretic operations, they also share this property. As customary in descriptive set theory, the term *hierarchy* is reserved to such satisfactory notions of complexity. In other words, a notion of complexity yields a *hierarchy* if it induces a well-quasi-order, i.e., a well-founded classification in which each level is composed with finitely many classes.

Unlike the Borel, the projective and the Hausdorff-Kuratowski difference hierarchies, the quasi-order $\leq_w$ does not trivially yield a hierarchy since its definition does not rely on set-theoretic operations. To say it otherwise, it is hard to check whether the Wadge preorder is a well-quasi-order, i.e., if it admits no infinite sequence of pairwise $\leq_w$-incompatible sets and no strictly $\leq_w$-decreasing sequence of sets. However, using the game-theoretical techniques introduced by Wadge, one can show that it is a well-quasi-order on the Borel subsets of any zero-dimensional Polish space, where a space is zero-dimensional if it has a basis composed of sets that are both open and closed [Mar75, VW78b, Wad84]. For example, both the Baire space ω^ω and the Cantor space 2^ω are zero-dimensional Polish spaces. In other words, the Wadge preorder yields a hierarchy on the Borel subsets of any zero-dimensional Polish space.

Building on the previous work of Hertling [Her96] and Ikegami [Ike10], Schlicht recently showed that the situation is less satisfactory for non-zero-dimensional Polish spaces for the Wadge preorder on the Borel subsets of any such space admits uncountably many pairwise $\leq_w$-incompatible elements [Sch18].

Another Wadge preorder of sets

The Wadge preorder is not a well-quasi-order in non-zero-dimensional Polish spaces for the main reason that there are not enough continuous functions. To remedy this situation, mathematicians have considered more general reducibility notions. For example, they considered reductions induced by more general classes of functions. However, such reductions fail to preserve several topological properties. In particular, the lower levels of the Borel hierarchy are collapsed. Another direction consists in keeping the continuity condition for it preserves the topological structure, but in relaxing the notion of function. This yields the notion of reducibility via the total relatively continuous relations introduced by Brattka and Hertling [BH94]. This notion of reduction comes from computable analysis and is extensively studied by Pequignot in [Peq15b]. If $\mathcal{X}$ is a countably based T_0-space[7], the quasi-order on $\mathcal{B}(\mathcal{X})$ induced by reductions via total relatively continuous relations is denoted by $\preccurlyeq_w$ and is naturally a well-quasi-order for every quasi-Polish space — and actually for an even much wider class of spaces. Thus, it yields a hierarchy on the Borel subsets of any quasi-Polish space.

Baire class hierarchy of functions

Up to this point, we only considered the topological definability of sets. However, other fundamental objects are functions. As with the open sets, the fundamental functions are the continuous ones for they preserve the topology. Moreover, as for the Borel hierarchy of sets, we consider a hierarchy for functions which was introduced by Baire [Bai99]. To avoid unnecessary technicalities in this introduction, we consider functions $f : \mathcal{X} \to \mathcal{Y}$ for $\mathcal{X}$ and $\mathcal{Y}$ zero-dimensional Polish spaces. The class $\mathcal{BC}(\mathcal{X}, \mathcal{Y})$ of all Baire class functions $f : \mathcal{X} \to \mathcal{Y}$ is the smallest class of functions $f : \mathcal{X} \to \mathcal{Y}$ containing the continuous functions and closed under the pointwise limit operation. It easily stratifies in ω_1 many levels by computing the number of instances of the pointwise limit operation that is needed in order to construct its members from the continuous functions. We write $\mathcal{BC}_0(\mathcal{X}, \mathcal{Y})$ for the class of all continuous functions $f : \mathcal{X} \to \mathcal{Y}$ and, for any $\alpha \in \omega_1$, we write

$$\mathcal{BC}_\alpha(\mathcal{X}, \mathcal{Y}) = \left\{ \lim_{n\in\omega} f_n : \forall n \; \exists \beta_n < \alpha \; f_n \in \mathcal{BC}_{\beta_n}(\mathcal{X}, \mathcal{Y}) \right\}.$$

Clearly, it yields a hierarchy of the Baire class functions for $\mathcal{BC}_\alpha(\mathcal{X}, \mathcal{Y}) \subseteq \mathcal{BC}_\beta(\mathcal{X}, \mathcal{Y})$ holds for all $\alpha < \beta < \omega_1$. As shown by Lebesgue, the Baire class functions are exactly those functions which preserve the Borel subsets, i.e., the preimage of any Borel set of $\mathcal{Y}$ is a Borel set of $\mathcal{X}$ [Leb05]. For this reason, Baire class functions are also called Borel functions.

The Decomposability Conjecture

Let $\mathcal{X}$ and $\mathcal{Y}$ be zero-dimensional Polish spaces. In 1982, Jayne and Rogers showed that a function $f : \mathcal{X} \to \mathcal{Y}$ belongs to $\mathcal{BC}_1(\mathcal{X}, \mathcal{Y})$ if and only if there exists a partition $\{A_i : i \in \omega\} \subseteq \mathbf{\Pi}^0_1(\mathcal{X})$ of $\mathcal{X}$ into countably many

[7] A topological space $\mathcal{X}$ is T_0 if, for any $x, y \in X$ such that $x \neq y$, there exists an open set $U \subseteq \mathcal{X}$ such that $x \in U$ and $y \notin U$ or, $x \notin U$ and $y \in U$.

closed sets such that f is continuous on any element of the partition, i.e., $f : A_i \to \mathcal{Y}$ is continuous [JR82]. This result is now known as the Jayne-Rogers Theorem. One of the main open problem in descriptive set theory is the Decomposability Conjecture, a natural generalization of the Jayne-Rogers Theorem to all finite Borel ranks. If $0 < m, n < \omega$, we define

$$\Lambda_{m,n}(\mathcal{X}, \mathcal{Y}) = \left\{ f : \mathcal{X} \to \mathcal{Y} : \forall A \in \mathbf{\Sigma}^0_m(\mathcal{Y}) \;\; f^{-1}[A] \in \mathbf{\Sigma}^0_n(\mathcal{X}) \right\}.$$

This is a generalization of the continuous functions for we have $\Lambda_{1,1}(\mathcal{X}, \mathcal{Y}) = \mathcal{BC}_0(\mathcal{X}, \mathcal{Y})$. We also define $\operatorname{Dec}\left(\Lambda_{1,m},\ \mathbf{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y})$ for the set of all function $f : \mathcal{X} \to \mathcal{Y}$ such that there exists a partition $\{A_i : i \in \omega\} \subseteq \mathbf{\Delta}^0_n(\mathcal{X})$ of $\mathcal{X}$ into countably many $\mathbf{\Delta}^0_n(\mathcal{X})$ sets[8] such that $f : A_i \to \mathcal{Y} \in \Lambda_{1,m}(A_i, \mathcal{Y})$. The Decomposability Conjecture is the statement that the following equality holds for any $0 < m \leq n < \omega$

$$\Lambda_{m,n}(\mathcal{X}, \mathcal{Y}) = \operatorname{Dec}\left(\Lambda_{1,n-m+1},\ \mathbf{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y}).$$

The Jayne-Rogers Theorem is the instance of the Decomposability Conjecture where $m = n = 2$ because any countable $\mathbf{\Delta}^0_2$-partition of a zero-dimensional Polish space can be refined into a countable closed partition.

1.3 Our contribution

The present work contains three main contributions.

1. Firstly, although the Borel and the Hausdorff-Kuratowski difference hierarchies are well-understood for quasi-Polish spaces, very few was known about the Wadge preorder on these spaces. We prove that the Wadge preorder on the Borel subsets of the Scott domain $\mathcal{P}\omega$ is not a well-quasi-order. Moreover, we prove that such a bad behavior already occurs at the lowest possible topological level.

2. Secondly, we give the complete picture of the quasi-order $\preccurlyeq_w$ on the Borel subsets of the Scott domain $\mathcal{P}\omega$. In particular, we prove it to be equivalent to the restriction of the Wadge preorder on the non-self-dual Borel subsets of ω^ω, where a set is non-self-dual if $A \not\leq_w A^{\mathrm{c}}$ and equivalent means that the induced quotient posets are isomorphic.

3. Thirdly, we introduce a new assumption and prove that, under the axiom of determinacy, it implies the Decomposability Conjecture for

[8] A subset $A \subseteq \mathcal{X}$ of $\mathcal{X}$ belongs to $\mathbf{\Delta}^0_n(\mathcal{X})$ if it belongs to both $\mathbf{\Sigma}^0_n(\mathcal{X})$ and $\mathbf{\Pi}^0_n(\mathcal{X})$.

zero-dimensional Polish spaces. We also prove that this assumption is not too wild for it is fulfilled by many functions. We believe that the techniques introduced are as important as the results. Indeed, they offer a novel viewpoint to tackle the Decomposability Conjecture which fits the game-theoretical setting of zero-dimensional Polish spaces.

1.4 Organization of the thesis

Throughout the thesis, we assume some basic knowledge about general topology [Kec95, Mos09] and set theory [Jec03, Kan09]. As usual, ZF denotes the standard Zermelo-Fraenkel set theory and ZFC denotes the theory ZF together with the axiom of choice (AC).

Chapter 2

We give the necessary background for the thesis. First, we fix the notations for the whole thesis (Section 2.1). We also introduce the Gale-Stewart games which are infinite games with perfect information (Section 2.2). Then, we gently introduce the class of quasi-Polish spaces as well as its universal space, the Scott domain $\mathcal{P}\omega$ (Section 2.3). Finally, we introduce the Wadge order — which is the induced quotient poset of the Wadge preorder mentioned in this introduction — and present some useful related results and techniques (Section 2.4).

Chapter 3

First, we exhibit a class $\mathbb{P}_{\mathrm{shr}}$ of countable posets called shrubs (Definition 3.9). In particular, shrubs share many properties with well-founded trees. Then, we define the class $\mathbb{P}_{\mathrm{emb}}$ which contains well-behaved 2-colored shrubs (Definition 3.11). The elements of $\mathbb{P}_{\mathrm{emb}}$ are naturally compared via homomorphisms, i.e., if $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\mathrm{emb}}$, we write $\mathsf{P} \preccurlyeq_c \mathsf{Q}$ if there exists a homomorphism of 2-colored posets from P to Q. Clearly, $\preccurlyeq_c$ is a quasi-order on $\mathbb{P}_{\mathrm{emb}}$. We write $(\mathbb{D}(\mathbb{P}_{\mathrm{emb}}), \preccurlyeq_c)$ for its induced quotient poset. Our main result is the construction of an order-embedding from $(\mathbb{D}(\mathbb{P}_{\mathrm{emb}}), \preccurlyeq_c)$ into the Wadge order on the $\boldsymbol{\Delta}^0_2$ subsets of the Scott domain $\mathcal{P}\omega$ (Theorem 3.21). Finally, using this order-embedding, we prove that the Wadge order on the $\boldsymbol{\Delta}^0_2$ subsets of the Scott domain $\mathcal{P}\omega$ is ill-founded and has infinite antichains (Theorem 3.31 and Theorem 3.35).

The material of this chapter has been published in *The Journal of Symbolic Logic* [DV20].

Chapter 4

First, we introduce admissible representations which are an important tool to study countably based T_0-spaces from the viewpoint of computable analysis (Definition 4.1). Then, we introduce the quasi-order $\preccurlyeq_w$ on the subset of any such space (Definition 4.5). This quasi-order can also be obtained by considering reductions via total relatively continuous relations (Lemma 4.6). The two quasi-orders $\leq_w$ and $\preccurlyeq_w$ coincide on any zero-dimensional Polish space so that $\preccurlyeq_w$ can be thought of as a generalization of the Wadge preorder to arbitrary countably based T_0-spaces (Theorem 4.8). If $\mathcal{X}$ is such a space, the induced quotient poset on the Borel subsets of $\mathcal{X}$ is denoted by $\left(\mathbb{WD}_{\mathcal{B}}^{\sim w}(\mathcal{X}), \preccurlyeq_w\right)$. Then, we study this poset for the Scott domain $\mathcal{P}\omega$. The main result of this chapter is the proof that $\left(\mathbb{WD}_{\mathcal{B}}^{\sim w}(\mathcal{P}\omega), \preccurlyeq_w\right)$ is isomorphic to the restriction of the Wadge order on the Borel non-self-dual degrees of the Baire space ω^ω (Theorem 4.21), where a degree is an equivalence class for the Wadge preorder.

Chapter 5

We begin with a gentle introduction to the study of Borel functions in which we state the Decomposability Conjecture (Conjecture 5.3) and present some of the more recent development on the subject (Theorems 5.2, 5.5, 5.6, 5.7, 5.9, 5.11, 5.10 and 5.12). The main result of this chapter is the proof that, under the axiom of determinacy, a certain statement (Assumption 5.13) implies the Decomposability Conjecture for zero-dimensional Polish spaces. Since any such space is homeomorphic to a closed subset of the Baire space ω^ω, it suffices to consider functions $f : F \to \omega^\omega$ where F is a closed subset of ω^ω. To prove the main result, we first define the core of a Baire class 1 function $f : \mathcal{X} \to \omega^\omega$ and prove that it topologically characterizes the $\mathrm{Dec}\left(\Lambda_{1,1},\ \mathbf{\Delta}_2^0\right)(\mathcal{X}, \omega^\omega)$-functions (Theorem 5.16). Then, we prove that the core of the function is exactly what is needed to get Assumption 5.13 for a large class of functions (Theorem 5.19). We also prove that the Jayne-Rogers Theorem is an easy consequence of Assumption 5.13 (Theorem 5.20). Then, we prove two generalizations of the Jayne-Rogers Theorem (Theorem 5.34 and Theorem 5.44). More precisely, we prove that, under AD, Assumption 5.13 implies the cases $(m = 2, n = 3)$ and $(m = 2, n = 4)$ of the Decomposability Conjecture. Finally, these two theorems easily yield the main result of this chapter (Theorem 5.45).

Chapter 2

Preliminaries

In Section 2.1, we fix the notations used throughout this thesis. In Section 2.2, we introduce the Gale-Stewart game which is an infinite game with perfect information. This game plays a central role in the development of descriptive set theory. In Section 2.3, we introduce the class of quasi-Polish spaces, a suitable generalization of Polish spaces to some non-metrizable spaces arising in theoretical computer science. In particular, we gather several results suggesting that a reasonable descriptive set theory holds for this class. In this section, we also introduce two quasi-Polish spaces: the conciliatory space Conc and the Scott domain $\mathcal{P}\omega$. The latter is universal among the class of quasi-Polish spaces and is thoroughly studied in Chapter 3 and Chapter 4. Finally in Section 2.4, we introduce the Wadge order. We also gather several useful results about it on Polish spaces. In particular, the question-tree machinery developed by Duparc in [Dup01] is introduced. This machinery is essential to our study of the Decomposability Conjecture in Chapter 5.

2.1 General notations

As usual, we denote by ω or $\mathbb{N}$ the set of all integers and by $\aleph_0$ its cardinality. We also write ω^+ for $\omega \setminus \{0\}$ and ω_1 for the first uncountable ordinal. We use the letters i, j, k, l, m, n for integers and α, β, γ for arbitrary ordinals. Since every ordinal is regarded as the set of its predecessors, if $n \in \omega$, the notation $x \cap n$ stands for $x \cap \{0, 1, \ldots, n-1\}$.

The empty set is denoted by $\emptyset$. If X is a set and $A \subseteq X$, then $A \subseteq X$ is a proper subset of X if $A \notin \{\emptyset, X\}$ and the complement of A in X is denoted by $A^c = X \setminus A = \{x \in X : x \notin A\}$. If X, Y are sets, $f : X \to Y$

is a function, $A \subseteq X$ and $B \subseteq Y$, then we write $f[A] = \{f(x) : x \in A\}$ and $f^{-1}[B] = \{x : f(x) \in B\}$. If f is injective, we write $f^{-1}(y)$ for the unique element $x \in X$ such that $f(x) = y$. We also write $\operatorname{ran}(f) \subseteq Y$ for the range of the function f, $f \restriction A$ for the restriction of the function f to A and $f :\subseteq X \to Y$ for partial functions from X to Y. If $f :\subseteq X \to Y$ is a partial function, we denote its domain by $\operatorname{dom}(f) \subseteq X$. As usual, if $f : X \to Y$ and $g : Y \to Z$, $g \circ f : X \to Z$ denotes the composition of the two functions.

If A is a set, an A-sequence — or simply a sequence in case A is clear from context — is a function $s : \alpha \to A$, also denoted by $(s_\beta)_{\beta<\alpha}$ or $(s(\beta))_{\beta<\alpha}$, from some ordinal $\alpha = \operatorname{lh}(s)$ called the length of the sequence to A. We mainly consider sequences such that $\alpha \in \omega+1 = \omega \cup \{\omega\}$. We use the letters r, s, t, u to denote sequences of finite length and x, y, z to denote arbitrary sequences. The only sequence of length 0 — the empty sequence — is denoted by $\langle\rangle$. If $a \in A$, the sequence (a) of length 1 is simply denoted by a. If s, t are sequences, then t is a prefix of s, written $t \sqsubseteq s$, if $\operatorname{lh}(t) \leq \operatorname{lh}(s)$ and $s_k = t_k$ for all $k < \operatorname{lh}(t)$. If $t \sqsubseteq s$ but $s \not\sqsubseteq t$, we write $t \sqsubset s$. If $s \not\sqsubseteq t$ and $t \not\sqsubseteq s$, then we write $s \perp t$ and say that s and t are incompatible. The concatenation of s and t is defined by $st = (s_0, \dots, s_{\operatorname{lh}(s)-1}, t_0, \dots, t_{\operatorname{lh}(t)-1})$ and if $a \in A$, the concatenation of s and a is simply denoted by $sa = (s_0, \dots, s_{\operatorname{lh}(s)-1}, a)$. The set of all A-sequences of finite length is denoted by $A^{<\omega}$ and the set of all A-sequences of length ω is denoted by A^ω. Naturally, one defines $A^{\leq\omega} = A^{<\omega} \cup A^\omega$. If $x \in A^{\leq\omega}$ and $a, a' \in A$, we denote by $x_{[a/a']}$ the sequence where all occurrences of a' have been replaced by a so that $x_{[\ /a']}$ is the sequence where all occurrences of a' are deleted. If $x \in A^{\leq\omega}$ and $n < \operatorname{lh}(x)$, we write $x_{\restriction n} = (x_0, \dots, x_{n-1})$ for the prefix of x of length n. If $(s_n)_{n\in\omega} \subseteq A^{<\omega}$ is a sequence of finite sequences, the limit $\lim_{n\in\omega} s_n \in A^{\leq\omega}$ is defined as follows: $t \sqsubseteq \lim_{n\in\omega} s_n$ if and only if there exists $l \in \omega$ such that, for any $m \geq l$, one has $t \sqsubseteq s_m$.

A tree $T \subseteq A^{<\omega}$ is a set of finite A-sequences closed under the prefix relation, i.e., if $t \in T$ and $s \sqsubseteq t$, then $s \in T$. The elements of T are called nodes and the $\sqsubseteq$-maximal elements of T are called leaves. A tree is pruned if it has no leaf. It is well-founded if it has no infinite branch — i.e., no function $f : \omega \to T$ such that, if $n < m$, then $f(n) \sqsubset f(m)$. It is ill-founded if it is not well-founded. If it is well-founded, the rank of any $t \in T$ is (well-)defined by $\sqsupseteq$-induction: $\operatorname{rk}_T(t) = 0$ if t is a leaf and $\operatorname{rk}_T(t) = \sup\{\operatorname{rk}_T(s) + 1 : t \sqsubset s\}$ otherwise. The rank $\operatorname{rk}(T)$ of a non empty well-founded tree T is the ordinal $\operatorname{rk}_T(\langle\rangle) \in \omega_1$. Observe that an infinite branch is an element of A^ω. In this thesis, we only consider trees $T \subseteq A^{<\omega}$ with A countable so that we can always consider $T \subseteq \omega^{<\omega}$ via some injection from A into ω.

If $T \subseteq \omega^{<\omega}$ is a tree, we denote by $[T] \subseteq \omega^\omega$ the set of all its infinite branches. If T is a non-empty pruned tree, it is naturally equipped with the prefix topology whose basis is given by $\{[t] : t \in T\}$ where $[t] = \{x \in [T] : t \sqsubset x\}$. This topology coincides with the subspace topology once ω^ω is equipped with the product of the discrete topology on ω. It also coincides with the topology generated by the complete metric:

$$\begin{aligned} d : [T]^2 &\to \mathbb{R}, \\ (x, y) &\mapsto \sup \left\{2^{-n} : x_n \neq y_n\right\}, \end{aligned}$$

where the supremum of the empty set is 0. Unless otherwise stated, any set of the form $[T]$ is equipped with this topology. The space ω^ω of all infinite sequences of integers is called the Baire space and the space 2^ω of all infinite binary sequences is called the Cantor space. The Baire space ω^ω is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$, i.e., the irrational numbers equipped with the standard topology. This topological space is convenient in descriptive set theory for several reasons. It is zero-dimensional, i.e., it has a basis made of clopen sets — sets that are both open and closed. It is also homeomorphic to its double product, i.e., $\omega^\omega \cong \omega^\omega \times \omega^\omega$, so that it is homeomorphic to any finite product of itself. It is also homeomorphic to the countable infinite product of itself, i.e., $\omega^\omega \cong (\omega^\omega)^\omega$. If $T \subseteq \omega^{<\omega}$ is a non-empty pruned tree, the space $[T]$ is homeomorphic to a closed subset of ω^ω. Moreover, $[T]$ is a Polish space, i.e., a separable and completely metrizable space. The set of all non-empty zero-dimensional Polish spaces is, up to homeomorphism, the set of non-empty closed subsets of ω^ω, or, equivalently, the set of all $[T]$ for $T \subseteq \omega^{<\omega}$ any non-empty pruned tree. If $F \in \mathbf{\Pi}^0_1([T])$, we write $\mathcal{T}(F) = \left\{x_{\restriction k} : x \in F \text{ and } k \in \omega\right\} \subseteq T$. It is the unique non-empty pruned tree such that $[\mathcal{T}(F)] = F$.

A binary relation $\leq_Q \subseteq Q \times Q$ on a set Q is reflexive if, for all $q \in Q$, $(q, q) \in \leq_Q$. It is transitive if, for any $q_0, q_1, q_2 \in Q$, $(q_0, q_1), (q_1, q_2) \in \leq_Q$ implies $(q_0, q_2) \in \leq_Q$. A quasi-order on a set Q is any reflexive and transitive binary relation. Whenever $\leq_Q$ is clear from the context, we write Q for the couple $(Q, \leq_Q)$. We use the letters P, Q for quasi-orders and $p \in P, q \in Q$ for their elements. As usual, $q_0 \leq_Q q_1$ stands for $(q_0, q_1) \in \leq_Q$, and $q_0 <_Q q_1$ for $q_0 \leq_Q q_1$ but $q_1 \not\leq_Q q_0$. If $q_0 \not\leq_Q q_1$ and $q_1 \not\leq_Q q_0$, then q_0 and q_1 are said to be incompatible which is denoted by $q_0 \perp_Q q_1$. The set of predecessors of $q \in Q$ is $\mathrm{Pred}(q) = \{q' \in Q : q' \leq_Q q\}$ and the set of its immediate predecessors is $\mathrm{Pred}_{\mathrm{im}(q)} = \{q' \in Q : q' <_Q q \land \neg\exists q''\ q' <_Q q'' <_Q q\}$. Similarly, one defines the set of successors $\mathrm{Succ}(q) = \{q' \in Q : q \leq_Q q'\}$ and the set of immediate successors $\mathrm{Succ}_{\mathrm{im}}(q) = \{q' \in Q : q <_Q q' \land \neg\exists q''\ q <_Q q'' <_Q q'\}$. If Q is a quasi-order and $P \subseteq Q$, then P equipped with the induced relation is also a quasi-order. An antichain is a sequence of pairwise incompatible elements. A strictly $\leq_Q$-increasing (respectively, a strictly $\leq_Q$-decreasing) sequence is a sequence $(q_n)_{n<\omega}$ such that $q_n <_Q q_{n+1}$ (respectively, $q_{n+1} <_Q q_n$) for all $n \in \omega$. A well-quasi-order is a quasi-order Q that has no infinite antichain and no strictly $\leq_Q$-decreasing sequence.

If q and q' are elements of a quasi-order Q such that $q \leq_Q q'$ and $q' \leq_Q q$, then we write $q \sim_Q q'$. The relation $\sim_Q$ is an equivalence relation. The equivalence class of q is called its degree and is denoted by

$$[q]_Q = \{q' \in Q : q \sim_Q q'\}.$$

The set of all degrees is denoted by $\mathbb{D}(Q)$ and $\mathbb{D}(Q)$ inherits the quasi-order $\leq_Q$. More precisely, we set $[q]_Q \leq_Q [q']_Q$ if and only if $q \leq_Q q'$. A quasi-order $\leq_P$ on P is a partial order if $\leq_P$ is antisymmetric, i.e., for any $p_0, p_1 \in P$, $p_0 \leq_P p_1$ and $p_1 \leq_P p_0$ imply $p_0 = p_1$. In that case, P is a partially ordered set, or poset for short. For example, if Q is a quasi-order, the set $\mathbb{D}(Q)$ equipped with $\leq_Q$ is a poset. A homomorphism of quasi-order is a function $\varphi : P \to Q$ where P and Q are quasi-orders and such that, for any $p_0, p_1 \in P$, if $p_0 \leq_P p_1$, then $\varphi(p_0) \leq_Q \varphi(p_1)$. An order-embedding is a homomorphism between two posets $\varphi : P \to Q$ such that for any $p_0, p_1 \in P$, $p_0 \leq_P p_1$ if and only if $\varphi(p_0) \leq_Q \varphi(p_1)$. Thus, order-embeddings are injective.

2.2 Gale-Stewart games

Over the last decades, infinite games have proven to be an invaluable tool in the development of descriptive set theory (see [Tel87, Lar12] for historical background). This thesis is no exception. We use several instances of the Gale-Stewart game which is an infinite game with perfect information introduced in [GS53]. For the whole section, let $T \subseteq \omega^{<\omega}$ be any non-empty pruned tree.

Definition 2.1. If $A \subseteq [T]$, the Gale-Stewart game $G_{[T]}(A)$ with payoff A is a two-player infinite game played in ω rounds. The two players — I and II — play alternatively. At round $2n$, I picks an integer x_{2n} such that $x_{\restriction 2n+1} \in T$ and, at round $2n+1$, II picks an integer x_{2n+1} such that $x_{\restriction 2n+2} \in T$. We say that I wins this run of the game if $x = (x_n)_{n \in \omega} \in A$, otherwise II wins.

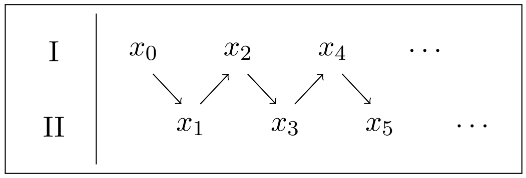


Figure 2.1: A run of the Gale-Stewart game $G_{[T]}(A)$.

For the whole thesis, I is referred to as he and II is referred to as she. In the sequel, we often define instances of the Gale-Stewart game without specifying T or A. However, T is always easily understood via the rules of the game and A via the winning condition of the game. Notice that any two-player infinite game where the two players play alternatively and have a countable set of possible moves at each step is an instance of the above-mentioned Gale-Stewart game. This holds even if one of this move consists in skipping one's turn.

A strategy for I is a non-empty pruned subtree $\sigma \subseteq T$ such that, if $s \in \sigma$ and $\mathrm{lh}\,(s)$ is even, there is a unique immediate successor of s in σ; and if $s \in \sigma$ and $\mathrm{lh}\,(s)$ is odd, then the set of immediate successors of s in σ is the same as the set of immediate successors of s in T. Such a strategy is winning if $[\sigma] \subseteq A$. Similarly, a strategy for II is a non-empty pruned subtree $\tau \subseteq T$ such that, if $t \in \tau$ and $\mathrm{lh}\,(t)$ is odd, there is a unique immediate successor of t in τ; and if $t \in \tau$ and $\mathrm{lh}\,(t)$ is even, then the set of immediate successors of t in τ is the same as the set of immediate successors of t in T. Such a strategy is winning if $[\tau] \subseteq A^{\mathrm{c}}$. Strategies can equivalently be defined as partial functions $\sigma, \tau : \subseteq T \to \omega$ with some auxiliary conditions ensuring that the rules of the game are respected, or even as partial functions $\sigma, \tau : \subseteq T \to T$ with the same kind of auxiliary conditions. We often make use of these alternative viewpoints to describe strategies.

The axiom of dependent choice (DC) is the statement that any non-empty pruned tree has an infinite branch. It is a weak version[1] of AC which is sufficient to prove that at most one of the two players has a winning strategy. Indeed, if σ and τ are winning strategies for I and II, then $\sigma \cap \tau$ is a non-empty pruned tree and thus has an infinite branch that belongs to both A and A^{c}, a contradiction. The game $G_{[T]}(A)$ is determined if either of the two players has a winning strategy. In that case, we also say that $A \subseteq [T]$ is determined. The axiom of determinacy (AD) states that $A \subseteq [T]$ is

[1] The axiom of choice (AC) is the statement that any family of non-empty sets admits a choice function, i.e., a function which assigns to each of these non-empty sets one of its elements.

determined for any T and any A. In [GS53], it is proven that under AC, there exists $A \subseteq \omega^\omega$ which is not determined. In particular, AC and AD contradict each other. However, AD implies the axiom of countable choice and the theory ZF+AD+DC offers the advantage of avoiding several pathological sets whose existence relies on AC. For example, in ZF+AD+DC, any subset of the Baire space ω^ω is Lebesgue-measurable, has the prefect set property and has the Baire property (see Theorem 33.3 in [Jec03]).

Given any $\Gamma \subseteq \mathcal{P}([T])$, we say that Γ-determinacy holds if any $A \in \Gamma$ is determined. By the previous remarks, $\mathcal{P}([\omega^\omega])$-determinacy does not hold in ZFC. In [GS53], it is proven that $\mathbf{\Pi}^0_1([T])$-determinacy holds for any non-empty pruned tree $T \subseteq \omega^{<\omega}$ in ZFC. This result was later generalized to $\mathbf{\Pi}^0_2([T])$-determinacy in [Wol55], to $\mathbf{\Pi}^0_3([T])$-determinacy in [Dav64] and to $\mathbf{\Pi}^0_4([T])$-determinacy in [Par72]. Finally, Martin proved Borel-determinacy in ZFC [Mar75, Mar85].

Theorem 2.2 ([Mar75], ZFC)**.** *If $T \subseteq \omega^{<\omega}$ is a non-empty pruned tree and $A \in \mathcal{B}([T])$, then A is determined.*

In [Har78], Harrington proved this result to be optimal for the determinacy of the analytic sets are independent from ZFC.

2.2.1 Games for functions

As first examples of the Gale-Stewart game, we introduce several game characterizations of partial functions in the Baire space ω^ω. A function $f : A \subseteq \omega^\omega \to \omega^\omega$ is Lipschitz if, for any $t \in \omega^n$ and $x \in A$ such that $t \sqsubset f(x)$, we have $f\left[x_{\restriction n+1}\right] \subseteq [t]$. This corresponds to the 1-Lipschitz functions appearing in any elementary course of calculus. The Lipschitz game characterizes Lipschitz functions. It was introduced by Wadge [Wad84] but first published by Van Wesep [VW78a].

Definition 2.3. Let $f : A \subseteq \omega^\omega \to \omega^\omega$. The Lipschitz game $G_L(f)$ is an instance of the Gale-Stewart game where $T = \omega^{<\omega}$. At round $2n$, I picks an integer x_n and, at round $2n+1$, II picks an integer y_n. At the end of the game, I has produced $x = (x_n)_{n\in\omega} \in \omega^\omega$ and II has produced $y = (y_n)_{n\in\omega} \in \omega^\omega$. II wins this run of the game if and only if either $x \notin A$ or $f(x) = y$.

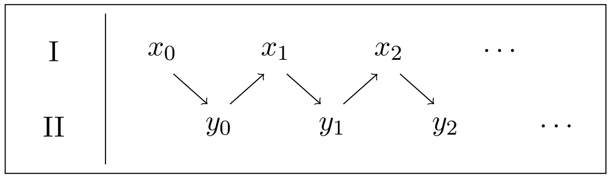


Figure 2.2: A run of the Lipschitz game $G_L(f)$.

Proposition 2.4 ([VW78a, Wad84])**.** *If $f : A \subseteq \omega^\omega \to \omega^\omega$, then f is Lipschitz if and only if* II *has a winning strategy in the game $G_L(f)$.*

The Wadge game is the version of the Lipschitz game where II has the further possibility to skip her turn. It characterizes the continuous functions. It was also introduced by Wadge [Wad72, Wad84].

Definition 2.5. Let $f : A \subseteq \omega^\omega \to \omega^\omega$. The Wadge game $G_w(f)$ is an instance of the Gale-Stewart game with the following rules. At round $2n$, I picks an integer $x_n \in \omega^{<\omega}$ and at round $2n+1$, II picks a sequence $t_n \in \omega^{<\omega}$ such that, for any $n \in \omega$, $\mathrm{lh}(t_{n+1}) \in \{\mathrm{lh}(t_n), \mathrm{lh}(t_n)+1\}$ and $t_n \sqsubseteq t_{n+1}$. At the end of the game, I has produced $x = (x_n)_{n\in\omega} \in \omega^\omega$ and II has produced $y = \lim_{n\in\omega} t_n \in \omega^\omega \cup \omega^{<\omega}$. We say that II wins this run of the game if and only if $x \notin A$ or $f(x) = y$.

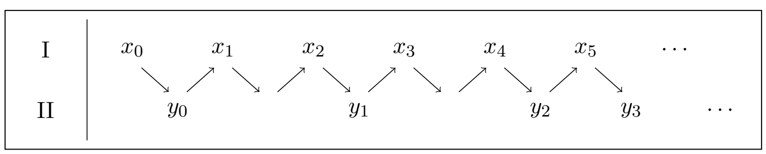


Figure 2.3: A run of the Wadge game $G_w(f)$.

Proposition 2.6 ([Wad84])**.** *If $f : A \subseteq \omega^\omega \to \omega^\omega$, then f is continuous if and only if* II *has a winning strategy in the game $G_w(f)$.*

The eraser game is the version of the Wadge game where II has the further possibility of erasing the last symbols she played. It was introduced by Duparc (implicit in [Dup01]) and characterizes pointwise limits of sequences of continuous functions.

Definition 2.7. Let $f : A \subseteq \omega^\omega \to \omega^\omega$. The eraser game $G_{\twoheadleftarrow}(f)$ is an instance of the Gale-Stewart game with the following rules. At round $2n$, I picks an integer $x_n \in \omega^{<\omega}$ and, at round $2n+1$, II picks a sequence $t_n \in \omega^{<\omega}$ such that, for any $n \in \omega$, $\mathrm{lh}(t_{n+1}) \in \{\mathrm{lh}(t_n)-1, \mathrm{lh}(t_n), \mathrm{lh}(t_n)+1\}$

and $t_n \sqsubseteq t_{n+1}$ or $t_{n+1} \sqsubseteq t_n$. At the end of the game, I has produced $x = (x_n)_{n\in\omega} \in \omega^\omega$ and II has produced $y = \lim_{n\in\omega} t_n \in \omega^\omega \cup \omega^{<\omega}$. We say that II wins this run of the game if and only if $x \notin A$ or $f(x) = y$.

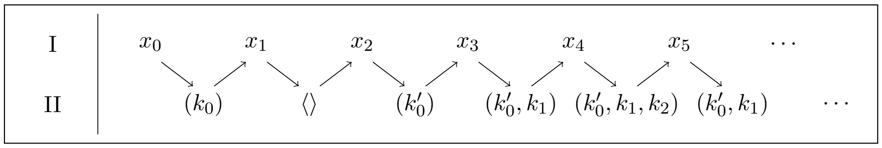


Figure 2.4: A run of the eraser game $G_{\twoheadleftarrow}(f)$.

Proposition 2.8 ([Dup01]). *If $f : A \subseteq \omega^\omega \to \omega^\omega$, then f is the pointwise limit of a sequence of continuous function if and only if* II *has a winning strategy in the game $G_{\twoheadleftarrow}(f)$.*

The backtrack game is the version of the Wadge game where II has the possibility of erasing everything she played so far. A countable partition of a space $\mathcal{X}$ is a family $\{X_n : n \in \omega\} \subseteq \mathcal{P}(\mathcal{X})$ such that $\bigcup_{n\in\omega} X_n = \mathcal{X}$ and, for any $n < m$, one has $X_n \cap X_m = \emptyset$. A function $f : A \subseteq \omega^\omega \to \omega^\omega$ is piecewise continuous on a $\mathbf{\Pi}^0_1(\omega^\omega)$-partition if there exists a countable partition $\{F_n : n \in \omega\} \subset \mathbf{\Pi}^0_1(\omega^\omega)$ of ω^ω into closed subsets such that the restriction $f \restriction A \cap F_n$ is continuous for any $n \in \omega$. The backtrack game was introduced by Van Wesep [VW79] and then used by Andretta [And06] to characterize piecewise continuous functions on a $\mathbf{\Pi}^0_1(\omega^\omega)$-partition.

Definition 2.9. Let $f : A \subseteq \omega^\omega \to \omega^\omega$. The backtrack game $G_{\mathrm{bt}}(f)$ is an instance of the Gale-Stewart game with the following rules. At round $2n$, I picks an integer $x_n \in \omega^{<\omega}$ and, at round $2n+1$, II picks a sequence $t_n \in \omega^{<\omega}$ such that, for any $n \in \omega$, $\mathrm{lh}(t_{n+1}) \in \{0, \mathrm{lh}(t_n), \mathrm{lh}(t_n) + 1\}$, $t_n \sqsubseteq t_{n+1}$ or $t_{n+1} = \langle\rangle$. At the end of the game, I has produced $x = (x_n)_{n\in\omega} \in \omega^\omega$ and II has produced $y = \lim_{n\in\omega} t_n \in \omega^\omega \cup \omega^{<\omega}$. We say that II wins this run of the game if and only if $x \notin A$ or $f(x) = y$.

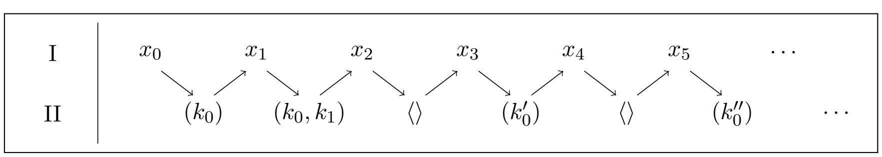


Figure 2.5: A run of the backtrack game $G_{\mathrm{bt}}(f)$.

Proposition 2.10 ([VW79, And06]). *If $f : A \subseteq \omega^\omega \to \omega^\omega$, then f is piecewise continuous on a $\mathbf{\Pi}^0_1(\omega^\omega)$-partition if and only if* II *has a winning strategy in the game $G_{\mathrm{bt}}(f)$.*

Let us also mention that game-theoretical characterizations for other classes of functions — which are not needed in this thesis — were obtained by Semmes, Motto Ros and Nobrega [Nob18, MR11, Sem09].

2.3 Quasi-Polish spaces

As mentioned in the introduction and since descriptive set theory is the study of definability, one naturally wishes to extend it outside the metrizable world to capture topological spaces involved in the development of theoretical computer science [BG15a, BG15b, GHK$^+$03, GL13, Sco72, Sco76, Sco82, Sel05, Sel06, Wei00]. In a series of papers [Sel05, Sel06], Selivanov successfully initiated this idea for ω-algebraic domains. Roughly speaking, an ω-continuous domain is a poset with both a notion of completeness and a notion of smallness. Notice the similarity with a Polish space which is a separable completely metrizable space. The class of ω-algebraic domains is a subclass of the class of ω-continuous domains. Unfortunately, Selivanov's techniques seemed quite different from the ones used on Polish spaces, rising a new question: is there a class of topological spaces which admits a reasonable descriptive set theory and which contains both the Polish spaces and the ω-continuous domains? This question was positively answered by de Brecht with the constitution of the class of quasi-Polish spaces [dB13].

In the literature, descriptive set theory usually begins with the definition of the Borel hierarchy. Unfortunately, the classical definition of this hierarchy does not extend nicely to non-metrizable spaces. For example, the Sierpiński space $\mathbb{S} = (2, \{\emptyset, \{1\}, 2\})$ is the set $2 = \{0, 1\}$ equipped with the topology that makes $\{1\}$ the only non-trivial open set. It is non-metrizable and the closed set $\{0\}$ is not a countable intersection of open sets. To overcome this obstacle and as discussed in the Introduction, Selivanov introduced a new version of the Borel hierarchy that fits arbitrary spaces [Sel05, Sel06]. This generalization is equivalent to the classical one for metrizable spaces and is also well-behaved for non-metrizable spaces. We now recall its definition.

Definition 2.11. Let $\mathcal{X}$ be a topological space and τ its topology. We define $\mathbf{\Sigma}^0_1(\mathcal{X}) = \tau$, and for $1 < \alpha < \omega_1$,

$$\mathbf{\Sigma}^0_\alpha(\mathcal{X}) = \left\{ \bigcup_{n\in\omega} (B_n \setminus B'_n) \ : B_n, B'_n \in \mathbf{\Sigma}^0_{\beta_n}(\mathcal{X}),\ \beta_n < \alpha \right\},$$
$$\mathbf{\Pi}^0_\alpha(\mathcal{X}) = \{A \subseteq \mathcal{X} : A^{\mathrm{c}} \in \mathbf{\Sigma}^0_\alpha(\mathcal{X})\} \text{ and}$$
$$\mathbf{\Delta}^0_\alpha(\mathcal{X}) = \mathbf{\Sigma}^0_\alpha(\mathcal{X}) \cap \mathbf{\Pi}^0_\alpha(\mathcal{X}).$$

The Borel sets of $\mathcal{X}$ are $\mathcal{B}(\mathcal{X}) = \bigcup_{\alpha\in\omega_1} \mathbf{\Sigma}^0_\alpha(\mathcal{X})$ and the Borel hierarchy on $\mathcal{X}$ is the poset

$$\left(\left\{\mathbf{\Delta}^0_\alpha(\mathcal{X}), \mathbf{\Sigma}^0_\alpha(\mathcal{X}), \mathbf{\Pi}^0_\alpha(\mathcal{X})\right\}_{0<\alpha<\omega_1}, \subseteq\right).$$

Proposition 2.12 ([Sel05])**.** *Let $\mathcal{X}$ be a topological space, and $\alpha < \beta < \omega_1$. Then $\mathbf{\Sigma}^0_\alpha(\mathcal{X}) \cup \mathbf{\Pi}^0_\alpha(\mathcal{X}) \subseteq \mathbf{\Delta}^0_\beta(\mathcal{X})$.*

$$\mathbf{\Delta}^0_1(\mathcal{X}) \subsetneq \begin{matrix}\mathbf{\Sigma}^0_1(\mathcal{X}) \\ \mathbf{\Pi}^0_1(\mathcal{X})\end{matrix} \subsetneq \mathbf{\Delta}^0_2(\mathcal{X}) \subsetneq \begin{matrix}\mathbf{\Sigma}^0_2(\mathcal{X}) \\ \mathbf{\Pi}^0_2(\mathcal{X})\end{matrix} \cdots \subseteq \mathbf{\Delta}^0_\alpha(\mathcal{X}) \subsetneq \begin{matrix}\mathbf{\Sigma}^0_\alpha(\mathcal{X}) \\ \mathbf{\Pi}^0_\alpha(\mathcal{X})\end{matrix} \cdots \subseteq \mathbf{\Delta}^0_\beta(\mathcal{X}) \subsetneq \begin{matrix}\mathbf{\Sigma}^0_\beta(\mathcal{X}) \\ \mathbf{\Pi}^0_\beta(\mathcal{X})\end{matrix} \quad \cdots$$

Figure 2.6: The Borel hierarchy of $\mathcal{X}$.

As customary in descriptive set theory, the Hausdorff-Kuratowski difference hierarchy is considered as a first refinement of the Borel hierarchy. Its definition relies on the difference operation.

Definition 2.13. If $0 < \alpha < \omega_1$ and $(A_\beta)_{\beta<\alpha}$ is a sequence of subsets of $\mathcal{X}$, then

$$D_\alpha\big((A_\beta)_{\beta<\alpha}\big) = \bigcup \left\{ A_\beta \setminus \cup_{\gamma<\beta} A_\gamma \ : \ \begin{array}{l}\beta < \alpha, \text{ and} \\ \alpha \text{ and } \beta \text{ have different parities}\end{array} \right\} \subseteq \mathcal{X}.$$

If $0 < \alpha, \beta < \omega_1$, then

$$D_\alpha\big(\mathbf{\Sigma}^0_\beta\big)(\mathcal{X}) = \left\{ D_\alpha\big((A_\gamma)_{\gamma<\alpha}\big) : (A_\gamma)_{\gamma<\alpha} \subseteq \mathbf{\Sigma}^0_\beta(\mathcal{X}) \right\} \subseteq \mathcal{P}(\mathcal{X}).$$

We also set the dual class $\check{D}_\alpha\big(\mathbf{\Sigma}^0_\beta\big)(\mathcal{X}) = \left\{ A \subseteq \mathcal{X} : A^{\mathrm{c}} \in D_\alpha\big(\mathbf{\Sigma}^0_\beta\big)(\mathcal{X}) \right\}$. The Hausdorff-Kuratowski difference hierarchy on $\mathcal{X}$ is the poset

$$\left(\left\{ D_\alpha\big(\mathbf{\Sigma}^0_\beta\big)(\mathcal{X}), \check{D}_\alpha\big(\mathbf{\Sigma}^0_\beta\big)(\mathcal{X}) \right\}_{0<\alpha,\beta<\omega_1}, \subseteq \right).$$

Proposition 2.14 ([Sel05]). *Let $\mathcal{X}$ be a topological space, $\gamma < \omega_1$ and $\alpha < \beta < \omega_1$. Then $D_\alpha\big(\mathbf{\Sigma}^0_\gamma\big)(\mathcal{X}) \subseteq D_\beta\big(\mathbf{\Sigma}^0_\gamma\big)(\mathcal{X}) \cap \check{D}_\beta\big(\mathbf{\Sigma}^0_\gamma\big)(\mathcal{X})$.*

$$
\begin{array}{ccccccccc}
\mathbf{\Sigma}^0_\gamma(\mathcal{X}) & \subseteq & D_2\left(\mathbf{\Sigma}^0_\gamma\right)(\mathcal{X}) & & \subseteq & D_\alpha\left(\mathbf{\Sigma}^0_\gamma\right)(\mathcal{X}) & & \subseteq & D_\beta\left(\mathbf{\Sigma}^0_\gamma\right)(\mathcal{X}) \\
 & \subseteq & & \cdots & \subseteq & & \cdots & \subseteq & & \cdots \\
\mathbf{\Pi}^0_\gamma(\mathcal{X}) & \subseteq & \check{D}_2\left(\mathbf{\Sigma}^0_\gamma\right)(\mathcal{X}) & & \subseteq & \check{D}_\alpha\left(\mathbf{\Sigma}^0_\gamma\right)(\mathcal{X}) & & \subseteq & \check{D}_\beta\left(\mathbf{\Sigma}^0_\gamma\right)(\mathcal{X})
\end{array}
$$

Figure 2.7: The Hausdorff-Kuratowski difference hierarchy on the $\mathbf{\Sigma}^0_\gamma$-subsets of $\mathcal{X}$.

With this new definition of the classical Borel hierarchy, we can generalize descriptive set theory to a wider class of topological spaces.

2.3.1 Quasi-metrics

Polish spaces rely on complete metrics. The idea behind quasi-Polish spaces is to rely on a similar but more general notion. A quasi-metric is a metric whose symmetry condition has been dropped, i.e., a quasi-metric on a set X is a binary function $d : X^2 \to \mathbb{R}$ such that, for any $x, y, z \in X$,

1. $x = y \leftrightarrow d(x, y) = d(y, x) = 0$, and
2. $d(x, z) \leq d(x, y) + d(y, z)$.

The simplest example of a non-symmetric quasi-metric is

$$d_\mathbb{S} : 2^2 \to \mathbb{R},\ d_\mathbb{S}(0, 1) = d_\mathbb{S}(0, 0) = d_\mathbb{S}(1, 1) = 0 \text{ and } d_\mathbb{S}(1, 0) = 1.$$

As in the case of a metric, one defines the open balls $B_d(x, \varepsilon)$. If $x \in X$ and $0 < \varepsilon \in \mathbb{R}$, then $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. The set

$$\{B_d(x, \varepsilon) : x \in X,\ 0 < \varepsilon \in \mathbb{R}\}$$

is a basis of topology on X. The generated topological space is denoted by (X, d) and is not necessarily Hausdorff, but it is a T_0-space. For example, $(2, d_\mathbb{S})$ is the Sierpiński space $\mathbb{S}$. Hence, one thinks of quasi-metrics as a generalization of metrics to T_0-spaces.

From a quasi-metric $d : X^2 \to \mathbb{R}$, we easily construct a metric $\hat{d} : X^2 \to \mathbb{R}$ defined by $\hat{d}(x, y) = \max\{d(x, y),\ d(y, x)\}$. Of course, $B_{\hat{d}}(x, \varepsilon) \subseteq B_d(x, \varepsilon)$. In the literature, there are several notions of completeness for quasi-metrics. The right notion for our purpose is the one chosen in [dB13]. If $d : X^2 \to \mathbb{R}$ is a quasi-metric, a sequence $(x_n)_{n\in\omega} \subseteq X$ is Cauchy if, for all $\varepsilon > 0$, there exists $N \in \omega$ such that for all $N \leq n \leq m$, we have $d(x_n, x_m) < \varepsilon$. The quasi-metric $d : X^2 \to \mathbb{R}$ is complete if any Cauchy sequence in X converges to an element in X with respect to $\hat{d}$. More precisely, if $(x_n)_{n\in\omega}$ is a Cauchy sequence, there exists $x \in X$ such that, for any $0 < \varepsilon \in \mathbb{R}$, there exists $N \in \omega$ such that for any $n \geq N$, one has $\hat{d}(x_n, x) < \varepsilon$. The class of quasi-Polish spaces is the generalization of the Polish spaces to the class of T_0-spaces.

Definition 2.15 (Definition 16 in [dB13]). A topological space $\mathcal{X} = (X, \tau)$ is completely quasi-metrizable if there is a complete quasi-metric on X which generates the same topology as τ. A topological space $\mathcal{X}$ is quasi-Polish if it is countably based and completely quasi-metrizable.

Of course, every Polish space is quasi-Polish, since every metric is a quasi-metric. The simplest example of a quasi-Polish but non-Polish space is the Sierpiński space $\mathbb{S}$. We introduce two more interesting examples of quasi-Polish spaces, the conciliatory space Conc and the Scott domain $\mathcal{P}\omega$.

2.3.2 Domain theory

We first define the conciliatory space Conc and the Scott domain $\mathcal{P}\omega$ through the lens of domain theory because these definitions feel more natural and revealing. For this purpose, let us introduce some terminology (see for example [AJ94, GHK$^+$03] for a more complete introduction to domains).

If $(P, \leq)$ is a poset and $D \subseteq P$, then $x \in P$ is an upper bound of D if, for any $d \in D$, we have $d \leq x$. A subset $D \subseteq P$ of a poset is directed if it is non-empty and any pair of elements in D has an upper bound in D, i.e., for any $x, y \in D$, there exists $z \in D$ such that $x \leq z$ and $y \leq z$. The supremum $\sup D \in P$ of D is an upper bound of D such that, if x is another upper bound of D, then $\sup D \leq x$. If it exists, the supremum of D is necessarily unique. A poset $(P, \leq)$ is directed complete, or dcpo for short, if any of its directed subsets $D \subseteq P$ has a supremum $\sup D \in P$.

On any dcpo, there exists a natural topology called the Scott topology that we now describe. Let $(P, \leq)$ be any dcpo, $x \in P$ is way-below $y \in P$, denoted by $x \ll y$, if for any directed $D \subseteq P$ such that $y \leq \sup D$, there exists $d \in D$ such that $x \leq d$. In particular, if $x \ll y$, then we have $x \leq y$ since $\{y\}$ is directed. We also define $\twoheaduparrow x = \{y \in P : x \ll y\}$ and $\twoheaddownarrow x = \{y \in P : y \ll x\}$ for any $x \in P$. A subset $U \subseteq P$ is Scott open if it is upward closed, i.e., if $x \in U$ and $x \leq y$ implies $y \in U$, and for any directed $D \subseteq P$ such that $\sup D \in U$, we have $D \cap U \neq \emptyset$. The Scott open sets form a topology on P called the Scott topology. A basis for the Scott topology is $\{\twoheaduparrow x : x \in P\}$. A subset $\mathcal{B} \subseteq P$ is a domain theoretic basis if, for any $x \in P$,

$D_x = \mathcal{B} \cap \downarrow\!\!\downarrow x$ is directed and $\sup D_x = x$. An ω-continuous domain is a dcpo that admits a countable domain theoretic basis and an ω-algebraic domain is an ω-continuous domain such that each element of the domain theoretic basis is compact, i.e., it satisfies $x \ll x$. If $\mathcal{B}$ is a domain theoretic basis of an ω-continuous domain P, then it is easy to produce a countable basis for the Scott topology.

Proposition 2.16. *If $(P, \leq)$ is an ω-continuous domain with domain theoretic basis $\mathcal{B}$, then $\{\uparrow\!\!\uparrow x : x \in \mathcal{B}\}$ is a countable topological basis for the Scott topology on $(P, \leq)$.*

Proof. First notice that if $x \leq y$ and $y \ll z$, then $x \ll z$, and that if $x \ll y$ and $y \leq z$, then $x \ll z$.

If $x, y \in \mathcal{B}$ such that there exists $z \in \uparrow\!\!\uparrow x \cap \uparrow\!\!\uparrow y$, then, since $\mathcal{B}$ is a domain theoretic basis, $\mathcal{B} \cap \downarrow\!\!\downarrow z$ is directed and contains both x and y. In particular, there exists an upper bound $z' \in \mathcal{B} \cap \downarrow\!\!\downarrow z$ for the pair (x, y), so that $z \in \uparrow\!\!\uparrow z' \subseteq \uparrow\!\!\uparrow x \cap \uparrow\!\!\uparrow y$. Thus $\{\uparrow\!\!\uparrow x : x \in \mathcal{B}\}$ is a countable basis of a topology.

If U is Scott open and $x \in U$, then $\mathcal{B} \cap \downarrow\!\!\downarrow x$ is directed so that there exists $y \in \mathcal{B} \cap \downarrow\!\!\downarrow x$ such that $y \in U \cap \mathcal{B} \cap \downarrow\!\!\downarrow x$. In particular, $x \in \uparrow\!\!\uparrow y \subseteq U$ so that any Scott open set is open in the topology generated by the set $\{\uparrow\!\!\uparrow x : x \in B\}$. Since any $\uparrow\!\!\uparrow x$ is easily seen to be Scott open, the proof is complete. $\square$

In particular, any ω-continuous domain is naturally countably based once equipped with the Scott topology. It is proven in [dB13] that it is a quasi-Polish space.

Theorem 2.17 (Corollary 45 in [dB13])**.** *Any ω-continuous domain equipped with the Scott topology is a quasi-Polish space.*

Even though the proof of this theorem is quite involved in the general case, it becomes straightforward for the two examples that we consider, namely the conciliatory space Conc and the Scott domain $\mathcal{P}\omega$.

2.3.3 The conciliatory space

The conciliatory space Conc is a natural extension of ω^ω to the set $\omega^{\leq\omega}$ of finite and infinite sequences of integers. More precisely, Conc is the dcpo $(\omega^{\leq\omega}, \sqsubseteq)$ equipped with the Scott topology. One easily characterizes the way-below relation: if x is a finite sequence, then $x \ll y$ if and only if $x \sqsubseteq y$, and if x is an infinite sequence, then $x \ll y$ if and only if $x = y$. In particular, a domain theoretic basis is given by $\omega^{<\omega} \subset \omega^{\leq\omega}$, so that Conc is an ω-algebraic domain. Moreover, a basis for the Scott topology is $\{\uparrow s : s \in \omega^{<\omega}\} = \{\{y : y \sqsupseteq s\} : s \in \omega^{<\omega}\}$, which implies that the Scott topology and the prefix topology coincide on $\omega^{\leq\omega}$.

The quasi-metric definition of Conc is given by:

$$\begin{aligned} d_{\mathsf{Conc}} : \omega^{\leq\omega} \times \omega^{\leq\omega} &\to \mathbb{R} \\ (x,y) &\mapsto 0 && \text{if } x \sqsubseteq y, \\ (x,y) &\mapsto 2^{-\operatorname{lh}(y)-1} && \text{if } y \sqsubset x, \\ (x,y) &\mapsto \sup\left\{2^{-n} : x_n \neq y_n\right\} && \text{otherwise.} \end{aligned}$$

where the supremum of the empty set is 0. The quasi-metric d_{Conc} is complete and $\left(\omega^{\leq\omega}, d_{\mathsf{Conc}}\right)$ admits

$$\left\{B_{d_{\mathsf{Conc}}}\left(s, 2^{-n}\right) : s \in \omega^{<\omega} \text{ and } n \in \omega\right\}$$

as a basis of the topology. In particular, $\left(\omega^{\leq\omega}, d_{\mathsf{Conc}}\right)$ is quasi-Polish. Clearly, the Scott topology and the quasi-metric topology coincide, so that Conc is a quasi-Polish space.

2.3.4 The Scott domain

The Scott domain $\mathcal{P}\omega$ is the dcpo $(\mathcal{P}(\omega), \subseteq)$ equipped with the Scott topology. It was first introduced by Scott as a denotational semantic for the λ-calculus and plays a central role in this thesis [Sco76]. Recall that we already mentioned in the Introduction how this topological space can be thought of as a model of computation.

The supremum in $\mathcal{P}\omega$ is given by the usual set-theoretic union and the way-below relation is easy to characterize: if F is a finite subset of ω, then we have $F \ll y$ if and only if $F \subseteq y$ and if x is infinite, then we have $x \ll y$ if and only if $x = y$. In particular, $\{F \subseteq \omega : F \text{ finite}\}$ is a domain theoretic basis of $(\mathcal{P}(\omega), \subseteq)$, and the set $\{\uparrow F : F \subseteq \omega \text{ finite}\}$ is a basis of the Scott topology. Thus, the dcpo $(\mathcal{P}(\omega), \subseteq)$ endowed with the Scott topology is an ω-algebraic domain.

We now turn to the direct definition of $\mathcal{P}\omega$ through a quasi-metric:

$$\begin{aligned} d_{\mathcal{P}\omega} : \mathcal{P}(\omega) \times \mathcal{P}(\omega) &\to \mathbb{R} \\ (x,y) &\mapsto \sup\left\{2^{-n} : n \in x \setminus y\right\}, \end{aligned}$$

where the supremum of the empty set is 0. The quasi-metric $d_{\mathcal{P}\omega}$ is complete and a basis of $(\mathcal{P}(\omega), d_{\mathcal{P}\omega})$ is given by

$$\left\{B_{d_{\mathcal{P}\omega}}\left(F, 2^{-n}\right) : F \subseteq \omega \text{ finite}, n \in \omega\right\}.$$

In particular, $(\mathcal{P}(\omega), d_{\mathcal{P}\omega})$ is a quasi-Polish space. Clearly, the Scott topology and the quasi-metric topology coincide, so that $\mathcal{P}\omega$ is a quasi-Polish space. The Scott domain $\mathcal{P}\omega$ is also homeomorphic to the product of countably many instances of the Sierpiński space, i.e., $\mathcal{P}\omega \cong \mathbb{S}^\omega$ [Che18].

Finally, we relate another interesting viewpoint on the Scott domain $\mathcal{P}\omega$ that was given in [BG15b, Sco76]. The Cantor space 2^ω and the Scott domain $\mathcal{P}\omega$ are two different topologies on the set $\mathcal{P}(\omega)$. The topology of 2^ω is the topology of *positive and negative* information while the topology of $\mathcal{P}\omega$ is the topology of *positive* information only. Indeed, if we consider a basic open set $[t] \subseteq 2^\omega$, then all elements $x \in [t]$ share both the finite amount of positive information $\{n < \mathrm{lh}(t) : x_n = 1\}$ and the finite amount of negative information $\{n < \mathrm{lh}(t) : x_n = 0\}$. To the contrary, if one considers an open set $\uparrow F \subseteq \mathcal{P}\omega$, then all elements $x \in \uparrow F$ only share the finite amount of positive information $\{n \in \omega : n \in F\}$.

2.3.5 Descriptive set theory for quasi-Polish spaces

We state without proof the main results of [dB13]. These results highlight essentially two facts. Firstly, the class of quasi-Polish spaces is large enough to be interesting: it contains both all Polish spaces and all ω-continuous domains, i.e., the objects of study of the descriptive set theory initiated by Selivanov [Sel05, Sel06]. Secondly, the class of quasi-Polish spaces is small enough to be of interest: lots of results of classical descriptive set theory hold for quasi-Polish spaces.

Theorem 2.18 (Proposition 17, Theorems 18, 23, 41, 58, 70, 74 and Corollaries 26, 45, 52 in [dB13])**.** *Let $\mathcal{X}$ be a quasi-Polish space.*

1. *The class of quasi-Polish spaces contains all Polish spaces and all ω-continuous domain.*
2. *If $\mathcal{X}$ is uncountable, then* $\mathrm{Card}(\mathcal{X}) = 2^{\aleph_0}$.
3. *If $\mathcal{X}$ is uncountable, then the Borel hierarchy of $\mathcal{X}$ does not collapse.*
4. *A subspace $\mathcal{Y} \subseteq \mathcal{X}$ is quasi-Polish if and only if $\mathcal{Y} \in \mathbf{\Pi}^0_2(\mathcal{X})$.*
5. *Any retract of $\mathcal{X}$ is quasi-Polish.*
6. *A non-empty T_0-space $\mathcal{Y}$ is quasi-Polish if and only if there exists a continuous open surjection $f : \omega^\omega \to \mathcal{Y}$.*
7. *The space $\mathcal{X}$ is a Baire space.*
8. *Suslin's theorem holds for $\mathcal{X}$, i.e., $\mathcal{B}(\mathcal{X}) = \mathbf{\Delta}^1_1(\mathcal{X}) = \mathbf{\Sigma}^1_1(\mathcal{X}) \cap \mathbf{\Pi}^1_1(\mathcal{X})$.*
9. *The Hausdorff-Kuratowski Theorem holds for $\mathcal{X}$, i.e., for any $1 \leq \alpha < \omega_1$, $\mathbf{\Delta}^0_{\alpha+1}(\mathcal{X}) = \bigcup_{\beta \in \omega_1} D_\beta(\mathbf{\Sigma}^0_\alpha(\mathcal{X}))$.*

10. *If τ is the topology of $\mathcal{X}$, $1 \leq \alpha < \omega_1$, and $(A_n)_{n\in\omega} \subseteq \mathbf{\Sigma}^0_\alpha(\mathcal{X},\tau)$, then there exists a topology $\tau' \subseteq \mathbf{\Sigma}^0_\alpha(\mathcal{X},\tau)$ such that $\tau \subseteq \tau'$, $(\mathcal{X},\tau')$ is quasi-Polish and $(A_n)_{n\in\omega} \subseteq \mathbf{\Sigma}^0_1(\mathcal{X},\tau')$.*

If $\mathcal{X}$ is a countably based T_0-space with basis $(V_n)_{n\in\omega}$, the map $e : x \mapsto \{n \in \omega : x \in V_n\}$ is an embedding of $\mathcal{X}$ into $\mathcal{P}\omega$, i.e., $\mathcal{X} \cong \operatorname{ran}(e)$. In particular, $\mathcal{P}\omega$ stands out as universal among quasi-Polish spaces which justifies our detailed study of $\mathcal{P}\omega$.

Theorem 2.19 (Theorem 24 in [dB13]). *A countably based T_0-space is quasi-Polish if and only if it is homeomorphic to some $\mathcal{X} \in \mathbf{\Pi}^0_2(\mathcal{P}\omega)$.*

Recently, a number of researchers have shown a growing interest in the descriptive set theory of quasi-Polish spaces [BG15b, Che18, dBP15, dBP17, dB18, dBPS20, dB20, HRSS19, KK17, MRSS15, Sel19, Sel20].

2.4 Wadge theory

If $\mathcal{X}$ is a topological space, a class $\Gamma \subseteq \mathcal{P}(\mathcal{X})$ is a pointclass if it is closed under continuous preimage, i.e., if $A \in \Gamma$ and $f : \mathcal{X} \to \mathcal{X}$ is continuous, then $f^{-1}[A] \in \Gamma$. Any class of the Borel hierarchy and of the Hausdorff-Kuratowski difference hierarchy is a pointclass. This suggests the study of the Wadge theory — the systematic study of continuous reductions — as a further refinement of these hierarchies.

First, we give the general definitions and fix the terminology of the Wadge theory. We then focus on the Wadge order on zero-dimensional Polish spaces with an emphasis on the Baire space ω^ω.

2.4.1 General notations

The systematic study of continuous reductions is due to Wadge [Wad72, Wad84]. He introduced the quasi-order $\leq_w$ — now known as the Wadge preorder — induced by reductions via continuous functions. More precisely, if $\mathcal{X}$ and $\mathcal{Y}$ are two topological spaces, $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$, then $(A, \mathcal{X})$ is Wadge-reducible — or continuously reducible — to $(B, \mathcal{Y})$, denoted by $(A, \mathcal{X}) \leq_w (B, \mathcal{Y})$, if there exists a continuous function $f : \mathcal{X} \to \mathcal{Y}$ such that $f^{-1}[B] = A$, i.e., $x \in A \Leftrightarrow f(x) \in B$ for all $x \in \mathcal{X}$. Clearly, $\leq_w$ is a quasi-order and it measures the topological complexity of subsets. Indeed, $(A, \mathcal{X}) \leq_w (B, \mathcal{Y})$ means that the membership problem of A in such $\mathcal{X}$ can be reduced, via some continuous function, to the membership problem of B in such $\mathcal{Y}$. In other words, A is topologically less complicated in such

$\mathcal{X}$ than B is in such $\mathcal{Y}$. If $\mathcal{X}$ and $\mathcal{Y}$ are clear from the context, we simply write $A \leq_w B$. The Wadge preorder naturally induces a partial order. If $A \subseteq \mathcal{X}, B \subseteq \mathcal{Y}$ and both $A \leq_w B$ and $B \leq_w A$ hold, then we write $A \equiv_w B$ and say that A is Wadge equivalent to B.

The Wadge degree of $A \subseteq \mathcal{X}$ is $[A]_w = \{B \subseteq \mathcal{X} : A \equiv_w B\}$ and the dual degree of $[A]_w$ is

$$\breve{[A]}_w = [A^{\mathrm{c}}]_w = \{B \subseteq \mathcal{X} : A^{\mathrm{c}} \equiv_w B\} = \{B \subseteq \mathcal{X} : B^{\mathrm{c}} \in [A]_w\}.$$

The set of all Wadge degrees $\mathbb{WD}(\mathcal{X})$ of $\mathcal{X}$ inherits the relation $\leq_w$ which is a partial order on $\mathbb{WD}(\mathcal{X})$. The poset $(\mathbb{WD}(\mathcal{X}), \leq_w)$ is the Wadge order on $\mathcal{X}$. Over the last 50 years, this partial order has been extensively studied in the context of Polish spaces [And07, AL12, Dup01, IST19, Kec95, KLS12, KM19, Lou83, LSR88, Sch18, Sel17a, VW78b, Wad72, Wad84, Wad12].

If $\Gamma \subseteq \mathcal{P}(\mathcal{X})$ and $A \subseteq \mathcal{X}$, we say that A is Γ-hard if $B \leq_w A$ for any $B \in \Gamma$, and that A is Γ-complete if it is Γ-hard and $A \in \Gamma$. If $\Gamma \subseteq \mathcal{P}(\mathcal{X})$ is a pointclass, the poset $(\mathbb{WD}_\Gamma(\mathcal{X}), \leq_w)$ is the restriction of the Wadge order on the degrees generated by sets in Γ, i.e., $\mathbb{WD}_\Gamma(\mathcal{X}) = \{[A]_w : A \in \Gamma\}$. Notice that the Wadge preorder on the Γ subsets admits an infinite antichain (respectively, a strictly $\leq_w$-decreasing sequence) if and only if the Wadge order on the Γ degrees also admits an infinite antichain (respectively, a strictly $\leq_w$-decreasing sequence).

2.4.2 Game characterization

In his PhD thesis, Wadge focused on the study of continuous reducibility on non-empty zero-dimensional Polish spaces which are exactly the ones that are homeomorphic to the set $[T]$ of infinite branch of some non-empty pruned tree, or equivalently the non-empty closed subsets of the Baire space ω^ω [Wad84]. In this context, Wadge developed his main tool as a game characterization of the Wadge preorder. Observe that this game is essentially the same as the game $G_w(f)$ for functions $f : A \subseteq \omega^\omega \to \omega^\omega$ (Definition 2.5).

Definition 2.20. Let S and T be two non-empty pruned trees, $A \subseteq [S]$ and $B \subseteq [T]$. The Wadge game $G_w(A, B)([S], [T])$ is an instance of the Gale-Stewart game with the following rules. At round $2n$, I picks an integer x_n such that $x_{\restriction n+1} \in S$ and, at round $2n+1$, II picks a sequence $t_n \in T$ such that, for any $n \in \omega$, $\mathrm{lh}(t_{n+1}) \in \{\mathrm{lh}(t_n), \mathrm{lh}(t_n) + 1\}$ and $t_n \sqsubseteq t_{n+1}$. At the end of the game, I has produced $x = (x_n)_{n\in\omega} \in [S]$ and II has produced $y = \lim_{n\in\omega} t_n \in [T] \cup T$. We say that II wins this run of the game if and only if $y \in [T]$ and $x \in A \leftrightarrow y \in B$.

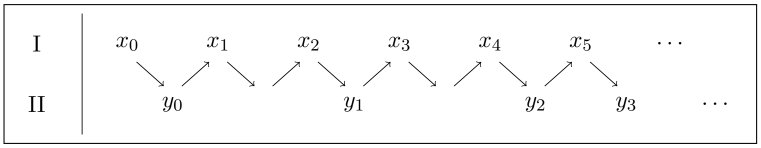


Figure 2.8: A run of the Wadge game $G_w(A, B)([S], [T])$.

To say it otherwise, I plays in S while II plays in T with the further possibility of skipping her turn (in that case we might use the symbol $\mathtt{s}$) and II's goal is to end up in B if and only if I ends up in A. As usual, if $[S]$ and $[T]$ are understood, we simply write $G_w(A, B)$. Clearly, a winning strategy for a player in $G_w(A, B)$ is a winning strategy for the same player in $G_w(A^c, B^c)$. Also, any strategy for II yields a continuous function $f : [S] \to [T]$ and vice-versa. Thus, we obtain the game characterization of the Wadge preorder.

Proposition 2.21 ([Wad84]). *If $A \subseteq [S]$ and $B \subseteq [T]$, then $A \leq_w B$ if and only if* II *has a winning strategy in $G_w(A, B)$.*

Moreover, any winning strategy for I in $G_w(A, B)$ is easily turned into a winning strategy for II in $G_w(B, A^c)$.

Lemma 2.22 ([Wad84]). *If $A \subseteq [S]$, $B \subseteq [T]$ and* I *has a winning strategy in $G_w(A, B)$, then $B \leq_w A^c$.*

Wadge's Lemma is a consequence of the different facts we gathered so far.

Wadge's Lemma 2.23 ([Wad84]). *If $A \subseteq [S]$, $B \subseteq [T]$, and $G_w(A, B)$ is determined, then $A \leq_w B$ or $B^c \leq_w A$.*

Under AD, Wadge's Lemma implies that there is no 3 pairwise $\leq_w$-incompatible subsets of $[T]$ and, if A is $\leq_w$-incompatible with B, then we have $A \equiv_w B^c$. We say that A is self-dual if $A \leq_w A^c$ and non-self-dual if $A \nleq_w A^c$. We use the same terminology for the Wadge degree $[A]_w$ of A. Under AD, the poset $(\mathbb{WD}([T]), \leq_w)$ is almost a linear order. Indeed, it suffices to merge any non-self-dual Wadge degree $[A]_w$ with its dual degree $\breve{[A]}_w$ to get a linear order. This fact is known as the semi-linear ordering principle of the Wadge degree, or SLO for short [AM03, And03]. In 1973, Martin and Monk proved that it is actually a well-order (see Theorem 21.15 in [Kec95]).

Martin-Monk's Theorem 2.24 (AD). *The quasi-order $\leq_w$ on $\mathcal{P}([T])$ is well-founded.*

By Borel-determinacy (Theorem 2.2), any Wadge game on $\mathcal{B}([T])$ is determined in ZFC, so that the previous theorem also holds in ZFC once restricted to the Borel subsets.

Martin-Monk's Theorem 2.25 (ZFC)**.** *The poset* $(\mathbb{WD}_{\mathcal{B}}([T]), \leq_w)$ *is well-founded and has maximal antichains of size 2.*

For this reason, we restrict our presentation to the Wadge order on the Borel degrees $(\mathbb{WD}_{\mathcal{B}}([T]), \leq_w)$. However, we should not forget that, for any class $\Gamma \subseteq \mathcal{P}([T])$ with suitable closure properties, the results of this section still hold provided some determinacy assumption.

The situation in non-zero-dimensional Polish spaces is completely different. For example, Hertling proved in [Her96] that the Wadge order on the Borel degrees of the real line $\mathbb{R}$ — equipped with the standard topology — is ill-founded. This result was then generalized by Ikegami, Schlicht and Tanaka who showed that any poset of size $\aleph_1$ embeds in the Wadge order on the Borel degrees of the real line $\mathbb{R}$ [IST19, Ike10]. In [Sch18], Schlicht proved a similar result for any non-zero-dimensional Polish space.

Theorem 2.26 (Theorem 1.5 in [Sch18])**.** *If $\mathcal{X}$ is a non-zero-dimensional Polish space, then the Wadge order on the Borel degrees of $\mathcal{X}$ has antichains of size* $2^{\aleph_0}$.

It answers the question of the wqoness of the Wadge order on the Borel degrees of any Polish space. However, this problem remains open outside the realm of Polish spaces and in particular for quasi-Polish spaces. We partially answer this question in Chapter 3 where we prove that the Wadge order on the Borel degrees of the Scott domain $\mathcal{P}\omega$ is both ill-founded and contains infinite antichains.

Since $(\mathbb{WD}_{\mathcal{B}}([T]), \leq_w)$ is a well-order with maximal antichain of size 2, we can safely speak about *the Wadge hierarchy* on $\mathbb{WD}_{\mathcal{B}}([T])$. Moreover, there is a natural rank function in the Wadge preorder which transfers to the Wadge order. Let $\mathrm{rk}'_w(\emptyset) = \mathrm{rk}'_w([T]) = 1$, and, for any $A \in \mathcal{B}([T])$ proper,

$$\mathrm{rk}'_w(A) = \sup\left\{\mathrm{rk}'_w(B) + 1 : B \in \mathcal{B}([T]) \wedge B <_w A\right\}.$$

By a compilation of the results due to Martin, Monk, Van Wesep and Wadge, we get a complete picture of the Wadge order on the Borel subsets of ω^ω which is depicted in Figure 2.9.

Theorem 2.27 ([VW78b]). *In $\mathbb{WD}_{\mathcal{B}}(\omega^\omega)$, the minimal Wadge degrees are $[\emptyset]_w$ and $[\omega^\omega]_w$. Moreover, any non-self-dual pair of Wadge degrees has a self-dual successor and any self-dual Wadge degree has a pair of non-self-dual successors. At limit level λ, there is a self-dual Wadge degree if and only if λ is of countable cofinality.*

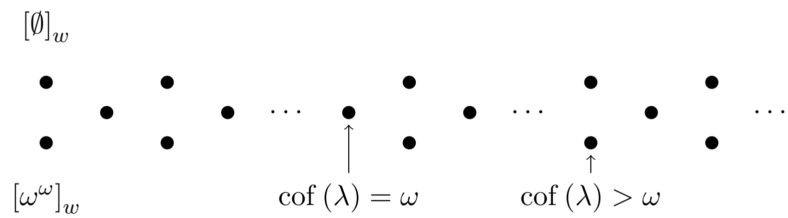


Figure 2.9: The hierarchy of the Borel Wadge degrees of the Baire space ω^ω.

In [VW78b], it is also shown that any self-dual degree in $\mathbb{WD}_{\mathcal{B}}(\omega^\omega)$ can be constructed from the non-self-dual degrees below it. In particular, it suffices to characterize the non-self-dual degrees of the Wadge order on $\mathbb{WD}_{\mathcal{B}}(\omega^\omega)$ to describe the whole poset $(\mathbb{WD}_{\mathcal{B}}(\omega^\omega), \leq_w)$. In the following paragraphs, we describe two different directions towards this characterization which were both introduced in [Wad84].

Clearly, pointclasses are exactly the initial segment of the Wadge preorder. A Wadge pointclass is a pointclass generated by some $A \subseteq \omega^\omega$, namely $\{B \subseteq \omega^\omega : B \leq_w A\}$. As usual, we restrict ourselves to the Wadge pointclasses that are generated by Borel subsets. This yields a natural correspondence between Wadge degrees and Wadge pointclasses:

$$[A]_w \longleftrightarrow \{B \subseteq \omega^\omega : B \leq_w A\}.$$

Any level of the classical Borel hierarchy is a Wadge pointclass, but not vice-versa. Indeed, there are — up to complement — ω_1 many Wadge degrees inside $\mathbf{\Delta}^0_2(\omega^\omega)$, but only one level of the classical Borel hierarchy. Similarly, any level of the Hausdorff-Kuratowski difference hierarchy is a Wadge pointclass, but not vice-versa. Indeed, there are — up to complement — ${\omega_1}^{\omega_1}$ many Wadge degrees inside $\mathbf{\Delta}^0_3(\omega^\omega)$, but only $\omega_1 \cdot 2$ levels of the Hausdorff-Kuratowski difference hierarchy. Thus, by refining the Borel hierarchy with the Hausdorff-Kuratowski difference hierarchy, one captures more Wadge pointclasses. The first direction towards a characterization of the non-self-dual degrees of $\mathbb{WD}_{\mathcal{B}}(\omega^\omega)$ aims at defining ω-ary boolean operations which refine again and again these hierarchies so that any Wadge pointclass is described through these ω-ary boolean operations. This proof was completed by Louveau [Lou83].

The second direction consists in exhibiting, for any non-self-dual Wadge degree $\Gamma \in \mathbb{WD}_{\mathcal{B}}(\omega^\omega)$, a set $A \in \mathcal{B}(\omega^\omega)$ which generates Γ, i.e., $A \in \mathcal{B}(\omega^\omega)$ such that $\Gamma = [A]_w$. This proof was completed by Duparc and relies on the game characterization of the Wadge preorder [Dup01, Dup]. We give more details about this construction in Subsection 2.4.3.

We conclude this subsection by gathering several useful results on the Wadge preorder on the Borel subsets of zero-dimensional Polish spaces with an emphasis on ω^ω. All these results are well-known and considered as *folklore*. We include some proofs to get acquainted with the game-theoretic techniques involved. Let $T \subseteq \omega^{<\omega}$ be a non-empty pruned tree. The following notion of initializable tree is a standard tool to study the Wadge preorder.

Definition 2.28. For any $A \subseteq [T]$, we define the initializable tree of A as

$$\mathrm{Init}_{[T]}(A) = \{t \in T : A \cap [t] \equiv_w A\}.$$

We assume the following fact (see Theorem 13 in [Dup01] for a proof).

Proposition 2.29 (Martin-Monk)**.** *Let $A \subseteq [T]$. The set A is non-self-dual if and only if $\mathrm{Init}_{[T]}(A)$ is ill-founded.*

We describe the first ω_1 many levels of the Wadge order on ω^ω and prove that they exactly corresponds to the set $\mathbb{WD}_{\mathbf{\Delta}^0_2}(\omega^\omega)$. Our description relies on the definition of operations on the subsets of ω^ω. Let $\omega_{>1} = \omega \setminus \{0, 1\}$. For any $x \in {\omega_{>1}}^{\leq\omega}$, let $x^{-2} \in \omega^{\leq\omega}$ be defined as $x^{-2}_k = x_k - 2$ for any $k < \mathrm{lh}(x)$. If $x \notin {\omega_{>1}}^{\leq\omega}$, write $x = u_x m_x x^*$ where $u_x \in {\omega_{>1}}^{\leq\omega}$ and $m_x \in \{0, 1\}$.

Definition 2.30. Let $A_n \subseteq \omega^\omega$ for any $n \in \omega$.

$$\pm A_0 = \{(2n)y \in \omega^\omega : y \in A_0\} \cup \{(2n+1)y \in \omega^\omega : y \notin A_0\},$$

$$\sum_{n\in\omega} A_n = \bigcup_{n\in\omega} \{ny \in \omega^\omega : y \in A_n\},$$

$$\begin{aligned} A_0 + A_1 = &\left\{x \in {\omega_{>1}}^{\omega} : x^{-2} \in A_1\right\} \\ &\cup \{x \notin {\omega_{>1}}^{\omega} : m_x = 0 \ \wedge \ x^* \in A_0\} \\ &\cup \{x \notin {\omega_{>1}}^{\omega} : m_x = 1 \ \wedge \ x^* \notin A_0\}. \end{aligned}$$

There is an obvious game-theoretical interpretation of these operations. If a player — say II — is in charge of $\pm A$, then she can choose to be in charge of A or to be in charge of A^{c}. If she is in charge of $\sum_{n\in\omega} A_n$, then she can choose which A_n she wants to be in charge of. If she is in charge of $A+B$, then she begins the game by being in charge of B and, at any moment, she can decide to be in charge of A or of A^{c}. With these interpretations, we easily have:

1. $A, A^{\mathrm{c}} \leq_w \pm A$ and $(\pm A)^{\mathrm{c}} = \pm A^{\mathrm{c}} \equiv_w \pm A$,
2. $A_n \leq_w \sum_{n\in\omega} A_n$ for any $n \in \omega$ and $\left(\sum_{n\in\omega} A_n\right)^{\mathrm{c}} = \sum_{n\in\omega} {A_n}^{\mathrm{c}}$,
3. $A \leq_w A+\emptyset, A+\omega^\omega$ and $(A+\emptyset)^{\mathrm{c}} = A^{\mathrm{c}} + \omega^\omega \equiv_w A+\omega^\omega$.

Moreover, using Proposition 2.29, we easily obtain the following lemma.

Lemma 2.31. *Let $A_n \subseteq \omega^\omega$ such that $A_n <_w A_{n+1}$ for any $n \in \omega$.*

1. $\pm A_0$ is self-dual.
2. $\sum_{n\in\omega} A_n$ is self-dual.
3. If A_1 is non-self-dual, then $A_0 + A_1$ is also non-self-dual.

The bottom level of the Wadge order on ω^ω is occupied by the non-self-dual degrees $[\emptyset]_w$ and $[\omega^\omega]_w$. Then, one easily describe the successors via the two following lemmas.

Lemma 2.32. *If $A \in \mathcal{B}(\omega^\omega)$ is non-self-dual, the successor of $[A]_w$ in the Wadge order is the self-dual degree $[\pm A]_w$.*

Proof. We already observed that $A \leq_w \pm A$. Since $\pm A$ is self-dual, we have $A <_w \pm A$. Suppose $B <_w \pm A$, then I has a winning strategy in $G_w(\pm A, B)$. If I chooses to be in charge of A, one easily constructs a winning strategy witnessing $A \nleq_w B$. By Wadge's Lemma 2.23, $B \leq_w A^{\mathrm{c}}$. The same reasoning yields $B \leq_w A$ if I chooses to be in charge of A^{c}. □

Lemma 2.33. *If $\pm A \in \mathcal{B}(\omega^\omega)$ is self-dual, the successors of $[\pm A]_w$ in the Wadge order are the non-self-dual degrees $[A+\emptyset]_w$ and $[A+\omega^\omega]_w$.*

Proof. We already observed that $\pm A \leq_w A+\emptyset$. Since $A+\emptyset$ is non-self-dual, $A <_w \pm A$. Suppose $\pm A <_w B$, then I has a winning strategy σ in $G_w(B, \pm A)$. Recall that $\mathtt{s}$ represents the option for II to skip her turn. If $x = \lim_{n\in\omega} \sigma(\mathtt{s}^n)$, then, for any $n \in \omega$, $B \cap \left[x_{\restriction n}\right] \nleq_w \pm A$. Suppose $x \notin B$, one easily constructs a winning strategy witnessing $B \nleq_w A+\omega^\omega$. By Wadge's Lemma 2.23, $A+\emptyset \leq_w B$. If $x \in B$, the same reasoning yields $A+\omega^\omega \leq_w B$, so that $[A+\emptyset]_w$ and $[A+\omega^\omega]_w$ are the successors of $[\pm A]_w$. □

The next lemma describes the limit levels of countable cofinality, so that we completely characterized the first ω_1 many levels of the Wadge order on ω^ω.

Lemma 2.34. *If $A_n \subseteq \omega^\omega$ is non-self-dual and $A_n <_w A_{n+1}$ for any $n \in \omega$, then the least Wadge degree above any $[A_n]_w$ is the self-dual degree $\left[\sum_{n\in\omega} A_n\right]_w$.*

Proof. We already observed that $A_n <_w A_{n+1} \leq_w \sum_{n\in\omega} A_n$ for any $n \in \omega$. Suppose $B <_w \sum_{n\in\omega} A_n$, then I has a winning strategy σ in the game $G_w\left(\sum_{n\in\omega} A_n, B^c\right)$. If $\sigma(\langle\rangle) = n$, then I also has a winning strategy in $G_w(A_n, B^c)$ so that $A_n \nleq_w B^c$. By Wadge's Lemma 2.23, $B \leq_w A_n$, so that $\left[\sum_{n\in\omega} A_n\right]_w$ is the least degree above any $[A_n]_w$. □

By the previous lemmas, any one of the first ω_1 many Wadge degrees can be described from the non-self-dual degrees below it. If $A, B \in \boldsymbol{\Delta}^0_2(\omega^\omega)$, then $\pm A$ and $A + B$ both belongs to $\boldsymbol{\Delta}^0_2(\omega^\omega)$ as finite unions of $\boldsymbol{\Delta}^0_2$ sets. If $A_n \in \boldsymbol{\Delta}^0_2(\omega^\omega)$ for any $n \in \omega$, then $\sum_{n\in\omega} A_n \in \boldsymbol{\Sigma}^0_2(\omega^\omega)$ as a countable union of $\boldsymbol{\Delta}^0_2$ sets. Moreover,

$$\sum_{n\in\omega} A_n = \bigcap_{n\in\omega}\left(nA_n \cup \bigcup_{m\neq n} m\omega^\omega\right) \in \boldsymbol{\Pi}^0_2(\omega^\omega).$$

In particular, if we recursively apply the previous operations to $\emptyset$ and ω^ω, we always get a $\boldsymbol{\Delta}^0_2(\omega^\omega)$ set.

It remains to prove that we exhaust all $\boldsymbol{\Delta}^0_2(\omega^\omega)$ degrees. Let $A_0 = \emptyset \subseteq \omega^\omega$, $A_{\alpha+1} = A_\alpha + \emptyset$ for any $0 < \alpha < \omega_1$, and $A_\lambda = \sum_{n\in\omega} A_{\alpha_n}$ for any $0 < \lambda < \omega_1$ a limit ordinal where $\sup\{\alpha_n : n \in \omega\} = \lambda$. We prove that, for any $A \in \boldsymbol{\Delta}^0_2(\omega^\omega)$, there exists β such that $A \leq_w A_\beta$. Let $\mathcal{S} = \{x \in \omega^\omega : \forall n\ \exists k > n\ x(k) = 0\}$.

Claim 2.35. *The set $\mathcal{S}$ is $\boldsymbol{\Pi}^0_2(\omega^\omega)$-complete.*

Proof of the claim. For any $n \in \omega$, let $S_n \subseteq \omega^{<\omega}$ be the set of all finite sequences containing at least n 0's. Then $A_n = \bigcup_{s\in S_n}[s] \in \boldsymbol{\Sigma}^0_1(\omega^\omega)$ and $\mathcal{S} = \bigcap_{n\in\omega} A_n \in \boldsymbol{\Pi}^0_2(\omega^\omega)$. If $A \in \boldsymbol{\Pi}^0_2(\omega^\omega)$, then $A = \bigcap_{n\in\omega}\bigcup_{k\in\omega}\left[t^k_n\right]$ where $t^k_n \in \omega^{<\omega}$ for any $n, k \in \omega$. We construct a winning strategy for II in the game $G_w(A, \mathcal{S})$. First, as long as I does not play any t^k_0, then II plays 1's. If I does play some t^k_0, then II answers with 0. We go on with this strategy by considering the sets $\left\{t^k_1 : k \in \omega\right\}$, $\left\{t^k_2 : k \in \omega\right\}$, and so on. $\square_{\text{Claim}}$

Let $A \in \mathbf{\Delta}^0_2(\omega^\omega)$ and fix a winning strategy σ_0 for II in $G_w(A, \mathcal{S}^c)$ and a winning strategy σ_1 for II in $G_w(A, \mathcal{S})$. For any $s \in \omega^{<\omega}$, let $h(s) = 0$ if

$$\mathrm{Card}\{n < \mathrm{lh}(s) : \sigma_0(s_n) = 0\} \leq \mathrm{Card}\{n < \mathrm{lh}(s) : \sigma_1(s_n) = 0\}$$

and $h(s) = 1$ otherwise. Let $f_0 : T_0 = \{\langle\rangle\} \to \omega^{<\omega} \times 2,\ \langle\rangle \mapsto (\langle\rangle, 0)$. Suppose that $f_n : T_n \to \omega^{<\omega} \times 2$ is already defined and let $\mathcal{L}(T_n)$ be the set of leaves of T_n. For any $t \in \mathcal{L}(T_n)$, let

$$S_t = \left\{s \in \omega^{<\omega} : \pi_0(f_n(t)) \sqsubset s \text{ and } h(\pi_0(f_n(t))) \neq h(s)\right\}.$$

Let $(s_k)_{k<\kappa(t)}$ be an enumeration of S_t where $\kappa(t) \in \omega + 1$. We define

$$T_{n+1} = T_n \cup \bigcup_{t \in \mathcal{L}(T_n)} \{tk : k < \kappa(t)\},$$

and

$$f_{n+1} = f_n \cup \bigcup_{t \in \mathcal{L}(T_n)} \{(tk, (s_k, h(s_k))) : k < \kappa(t)\}.$$

Finally, let $T_A = \bigcup_{n\in\omega} T_n$ and $f = \bigcup_{n\in\omega} f_n : T_A \to \omega^{<\omega} \times 2$.

Towards a contradiction, suppose that $x \in [T_A]$. Then there exists $y \in \omega^\omega$ such that $\pi_0\left(x_{\restriction n}\right) \sqsubset y$ for any $n \in \omega$. In that case, we must have both

$$\mathrm{Card}\{n \in \omega : \sigma_0(y_n) = 0\} = \mathrm{Card}\{n \in \omega : \sigma_1(y_n) = 0\} = \aleph_0.$$

In particular, $y \in A$ and $y \notin A$, a contradiction. Thus, the tree T_A is well-founded. Let $\beta_A = \mathrm{rk}(T_A)$.

Claim 2.36. *If $A \in \mathbf{\Delta}^0_2(\omega^\omega)$, then $A \leq_w A_{\beta_A}$.*

Proof of the claim. We construct a winning strategy σ for II in the game $G_w\left(A, A_{\beta_A}\right)$. Suppose that, after some round, I already played the sequence $s \in \omega^{<\omega}$ and II the sequence $t \in \omega^{<\omega}$. If I plays n on the next round and $h(sn) = h(s)$, then II answers with $t2$. Otherwise, if $h(sn) = 0$, II answers with tm, $m \in \{0, 1\}$, such that $tm2^\omega \in A_{\beta_A}$, and if $h(sn) = 1$, II answers with tm, $m \in \{0, 1\}$, such that $tm2^\omega \notin A_{\beta_A}$. It remains to prove that this strategy is winning. Suppose that, at the end of the game, I has produced $x \in \omega^\omega$. Since T_A is well-founded, there exists $t \in T_A$ and $n \in \omega$ such that $\pi_0(f(t)) = x_{\restriction n}$ and for any $t \sqsubset t'$ and any $n < m$, we have $\pi_0(f(t')) \neq x_{\restriction m}$. In particular, after I played $x_{\restriction n}$, then II answered only with 2's. If $h\left(x_{\restriction n}\right) = 0$, then $x \in A$ and $\lim_{k\in\omega} \sigma\left(x_{\restriction k}\right) = \sigma\left(x_{\restriction n}\right) 2^\omega \in A_{\beta_A}$. Similarly, if $h\left(x_{\restriction n}\right) = 1$, then $x \notin A$ and $\lim_{k\in\omega} \sigma\left(x_{\restriction k}\right) = \sigma\left(x_{\restriction n}\right) 2^\omega \notin A_{\beta_A}$. $\square_{\text{Claim}}$

To sum up, we proved the following result.

Proposition 2.37. *Let $A \in \mathbf{\Delta}^0_2(\omega^\omega)$.*

1. *A is self-dual if and only if one of the following holds:*
   (a) *$A \equiv_w \pm B$ with B non-self-dual, $B <_w A$.*
   (b) *$A \equiv_w \sum_{n\in\omega} A_n$ with A_n non-self-dual, $A_n <_w A_{n+1} <_w A$ for any $n \in \omega$.*
2. *A is non-self-dual if and only if one of the following holds:*
   (a) *$A = \emptyset$ or $A = \omega^\omega$.*
   (b) *$A \equiv_w B + \emptyset$ or $A \equiv_w B + \omega^\omega$ with B non-self-dual, $B <_w A$.*
   (c) *$A \equiv_w \left(\sum_{n\in\omega} A_n\right) + \omega^\omega$ or $A \equiv_w \left(\sum_{n\in\omega} A_n\right) + \emptyset$ with A_n non-self-dual, $A_n <_w A_{n+1} <_w A$ for any $n \in \omega$.*

The infinite branches of the initializable tree of $A \in \mathbf{\Delta}^0_2(\omega^\omega)$ non-self-dual is either contained in A, or in its complement.

Proposition 2.38. *If $A \in \mathbf{\Delta}^0_2(\omega^\omega)$ is non-self-dual and $A \equiv_w B + \omega^\omega$ for some $B \subseteq \omega^\omega$, then $[\mathrm{Init}_{\omega^\omega}(A)] \subseteq A$. If $A \in \mathbf{\Delta}^0_2(\omega^\omega)$ is non-self-dual and $A \equiv_w B + \emptyset$ for some $B \subseteq \omega^\omega$, then $[\mathrm{Init}_{\omega^\omega}(A)] \subseteq A^{\mathrm{c}}$.*

Proof. We consider the case $A \equiv_w B + \emptyset$. Towards a contradiction, suppose that $x \in [\mathrm{Init}_{\omega^\omega}(A)] \cap A$. Consider a winning strategy σ for II in $G_w(A, B+\emptyset)$ and $y = \sigma(x)$. In that case, $y \notin {\omega_{>1}}^\omega$. Moreover, either $m_y = 0$ and $y^* \in B$, or $m_y = 1$ and $y^* \notin B$. In particular, there exists $n \in \omega$ such that $A \equiv_w A \cap \left[x_{\restriction n}\right] \leq_w B$ or $A \equiv_w A \cap \left[x_{\restriction n}\right] \leq_w B^{\mathrm{c}}$, a contradiction. Thus $[\mathrm{Init}_{\omega^\omega}(A)] \cap A = \emptyset$. □

Moreover, any non-self-dual $A \in \mathbf{\Delta}^0_2(\omega^\omega)$ is complete for some level of the Hausdorff-Kuratowski difference hierarchy.

Proposition 2.39. *If $A \subseteq \omega^\omega$ is non-self-dual, then there exists $\alpha < \omega_1$ such that*

$$A \in D_\alpha\left(\mathbf{\Sigma}^0_1\right) \setminus \check{D}_\alpha\left(\mathbf{\Sigma}^0_1\right)(\omega^\omega),$$

or

$$A^{\mathrm{c}} \in D_\alpha\left(\mathbf{\Sigma}^0_1\right) \setminus \check{D}_\alpha\left(\mathbf{\Sigma}^0_1\right)(\omega^\omega).$$

Moreover, for any ordinal $\alpha < \omega_1$, both sets $D_\alpha\left(\mathbf{\Sigma}^0_1\right) \setminus \check{D}_\alpha\left(\mathbf{\Sigma}^0_1\right)(\omega^\omega)$ and $D_\alpha\left(\mathbf{\Sigma}^0_1\right) \setminus \check{D}_\alpha\left(\mathbf{\Sigma}^0_1\right)(\omega^\omega)$ consist in a single Wadge degree.

Proof. The proof easily follows by induction on the construction of non-self-dual sets. □

In particular, the first level of the Wadge order only consists in the non-self-dual degrees $[\emptyset]_w$ and $[\omega^\omega]_w$. The second level consists in the self-dual degree consisting of all proper clopen subsets of ω^ω. The third level only consists in the non-self-dual degree formed by all closed non-open sets and its dual degree.

The previous results extend to any zero-dimensional Polish space. Indeed, by Proposition 2.8 in [Kec95], if $T \subseteq \omega^{<\omega}$ is a non-empty pruned tree, then $[T]$ is a retract of ω^ω. In particular, if $i : [T] \to \omega^\omega$ is the natural inclusion, there exists $f : \omega^\omega \to [T]$ continuous such that, for any $x \in [T]$, $x = f \circ i(x)$. Thus for any $A \subseteq [T]$, one has $A \equiv_w f^{-1}[A]$.

Proposition 2.40. *If $A \in \mathbf{\Delta}^0_2([T])$ is non-self-dual and $A \equiv_w B + \omega^\omega$ for some $B \subseteq \omega^\omega$, then $\left[\mathrm{Init}_{[T]}(A)\right] \subseteq A$. If $A \in \mathbf{\Delta}^0_2([T])$ is non-self-dual and $A \equiv_w B + \emptyset$ for some $B \subseteq \omega^\omega$, then $\left[\mathrm{Init}_{[T]}(A)\right] \subseteq A^{\mathrm{c}}$.*

Proof. We consider the case $A \equiv_w B + \emptyset$. Towards a contradiction, suppose that $x \in \left[\mathrm{Init}_{[T]}(A)\right] \cap A$. Consider a winning strategy σ for II in $G_w(A, B + \emptyset)$ and $y = \sigma(x)$. In that case, $y \notin {\omega_{>1}}^\omega$, and $m_y = 0$ and $y^* \in B$, or $m_y = 1$ and $y^* \notin B$. In particular, there exists $n \in \omega$ such that $A \equiv_w A \cap \left[x_{\restriction n}\right] \leq_w B$ or $A \equiv_w A \cap \left[x_{\restriction n}\right] \leq_w B^{\mathrm{c}}$, a contradiction. Thus $\left[\mathrm{Init}_{[T]}(A)\right] \cap A = \emptyset$. ☐

By Martin-Monk's Theorem 2.25, the quasi-order $\leq_w$ is well-founded on the Borel subsets of $[T]$. Thus, we obtain the following result.

Proposition 2.41. *If $x \in [T]$ and $A \in \mathcal{B}([T])$, there exists $n \in \omega$ such that, for any $m \geq n$, $A \cap \left[x_{\restriction n}\right] \equiv_w A \cap \left[x_{\restriction m}\right]$.*

The next result is an easy consequence of the two previous propositions.

Proposition 2.42. *If $A \in \mathbf{\Delta}^0_2([T])$, $x \in A$ and $n \in \omega$ such that, for any $m \geq n$, $A \cap \left[x_{\restriction n}\right] \equiv_w A \cap \left[x_{\restriction m}\right]$, then $\left[\mathrm{Init}_{[T]}\left(A \cap \left[x_{\restriction n}\right]\right)\right] \subseteq A$.*
If $x \notin A$, then $\left[\mathrm{Init}_{[T]}\left(A \cap \left[x_{\restriction n}\right]\right)\right] \subseteq A^{\mathrm{c}}$.

To conclude this section, we prove that any $\mathbf{\Delta}^0_2$ subset of $[T]$ of Wadge rank greater than 2 is ultimately open or closed.

Theorem 2.43. *If $A \in \mathbf{\Delta}^0_2 \setminus \mathbf{\Delta}^0_1([T])$, there exists $p \in T$ such that:*
1. *$A \cap [p]$ is non-self-dual in $[T]$,*
2. *$\mathrm{rk}'_w(A \cap [p]) = 2$.*

This theorem follows from a series of lemmas.

Lemma 2.44. *Let $A \in \mathcal{B}([T])$. If $A \equiv_w \pm B$ with $B \subseteq \omega^\omega$ non-self-dual, there exist positions $u_+, u_- \in T$ such that:*

1. $A \cap [u_+] \equiv_w B$,
2. $A \cap [u_-] \equiv_w B^{\mathrm{c}}$.

Proof. Since A is self-dual, $\mathrm{Init}_{[T]}(A)$ is well-founded. Hence, for all $x \in [T]$, there exists a $\sqsubseteq$-minimal $u_x \sqsubset x$ such that $A \cap [u_x] \leq_w B$ or $A \cap [u_x] \leq_w B^{\mathrm{c}}$. Towards a contradiction, suppose that $A \cap [u_x] \equiv_w B$ never occurs, then one always has $A \cap [u_x] \leq_w B^{\mathrm{c}}$ and it is easy to construct a winning strategy for II in $G_w(A, B^{\mathrm{c}})$. Indeed, it suffices for II to wait for I to leave $\mathrm{Init}_{[T]}(A)$ and then use $A \cap [u_x] \leq_w B^{\mathrm{c}}$. This contradicts $A \equiv_w \pm B$. □

Lemma 2.45. *Let $A \in \mathcal{B}([T])$. If $A \equiv_w \sum_{n\in\omega} A_n$ with $A_n \subseteq \omega^\omega$ non-self-dual and $A_n <_w A_{n+1}$ for any $n \in \omega$, then, for any $n \in \omega$, there exists $u_n \in T$ such that:*

1. *$A \cap [u_n]$ is non-self-dual,*
2. $A_n \leq_w A \cap [u_n] <_w A$.

Proof. Since A is self-dual, $\mathrm{Init}_{[T]}(A)$ is well-founded. Hence, for all $x \in [T]$, there exists a $\sqsubseteq$-minimal $u_x \sqsubset x$ such that $A \cap [u_x] <_w A$ or $A \cap [u_x] <_w A^{\mathrm{c}}$. Given $n \in \omega$, consider the set

$$S = \{u_x : A_n \leq_w A \cap [u_x]\}.$$

If $S = \emptyset$, then for any u_x, $A \cap [u_x] \leq_w {A_n}^{\mathrm{c}}$. As in the proof of Lemma 2.44, it would imply that $A \leq_w {A_n}^{\mathrm{c}} < A_{n+1}$, a contradiction. Thus, $S \neq \emptyset$.

Let $u_x \in S$, if $A \cap [u_x]$ is non-self-dual, we are done. If $A \cap [u_x]$ is self-dual and $A \cap [u_x] \equiv_w \pm B$. It suffices to apply Lemma 2.44. Otherwise, we proceed by induction since $A \cap [u_x] <_w A$. □

Lemma 2.46. *Let $A \in \mathbf{\Delta}^0_2([T])$ be non-self-dual. If $A \equiv_w B + \emptyset$ with $B \subseteq \omega^\omega$ non-self-dual, then there exists $u \in T$ such that $A \cap [u]$ is non-self-dual and:*

1. *$A \cap [u] \equiv_w B$, or*
2. $A \cap [u] \equiv_w B^{\mathrm{c}}$.

Moreover, the first case is available as soon as $[\mathrm{Init}_{\omega^\omega}(B)] \cap B \neq \emptyset$ and the second case is available as soon as $[\mathrm{Init}_{\omega^\omega}(B)] \cap B^{\mathrm{c}} \neq \emptyset$.

The same result also holds if $A \equiv_w B + \omega^\omega$.

Proof. Suppose first that $x \in [\mathrm{Init}_{\omega^\omega}(B)] \cap B$. By Proposition 2.42, we have $\left[\mathrm{Init}_{[T]}(A)\right] \cap A = \emptyset$. Now consider a winning strategy τ for II in $G_w(B + \emptyset, A)$. Then, $\tau(0x) \notin \left[\mathrm{Init}_{[T]}(A)\right]$ and there exists $u \sqsubset \tau(0x)$ such that $A \cap [u] <_w A$. Since τ is winning, we get $B \leq_w A \cap [u] <_w A$. If $B \equiv_w A \cap [u]$, we are done. Otherwise, $\pm B \equiv_w A \cap [u]$ and it suffices to apply Lemma 2.44.

The second case is proven similarly, as well as the case $A \equiv_w B + \omega^\omega$. $\square$

Lemma 2.47. *Let $A \in \mathcal{B}([T])$. If $A \equiv_w \left(\sum_{n\in\omega} A_n\right) + \emptyset$ with with $A_n \subseteq \omega^\omega$ non-self-dual and $A_n <_w A_{n+1}$ for any $n \in \omega$, then, for any $n \in \omega$, there exists $u_n \in T$ such that:*

1. *A_{u_n} is non-self-dual,*
2. *$A_n \leq_w A_{u_n} <_w A$.*

The same result also holds if $A \equiv_w \left(\sum_{n\in\omega} A_n\right) + \omega^\omega$.

Proof. First, notice that since $A_n <_w A_{n+1}$ for any $n \in \omega$, we have

$$\left(\sum_{n\in\omega} A_n\right) + \emptyset \equiv_w \left(\sum_{n\in\omega} A_n + \omega^\omega\right) + \emptyset.$$

By Proposition 2.42, we have $\left[\mathrm{Init}_{[T]}(A)\right] \cap A = \emptyset$. Consider a winning strategy τ in $G_w\left(\left(\sum_{n\in\omega} A_n + \omega^\omega\right) + \emptyset, A\right)$ and let $x_n \in \left[\mathrm{Init}_{\omega^\omega}(A_n + \omega^\omega)\right] \subseteq A_n + \omega^\omega$. Then $\tau(0nx_n) \notin \left[\mathrm{Init}_{[T]}(A)\right]$ and there exists $u_n \sqsubset \tau(0nx_n)$ such that $A \cap [u_n] <_w A$. Since τ is winning, we get $A_n \leq_w A \cap [u_n]$. If $A \cap [u_n]$ is non-self-dual, we are done. Otherwise, $A_n <_w A \cap [u_n]$. If $A \cap [u_n] \equiv_w \pm B$ for some non-self-dual $B \subseteq \omega^\omega$, it suffices to use Lemma 2.44. If $A \cap [u_n] \equiv_w \sum_{n\in\omega} A'_n$ for some $\{A'_n : n \in \omega\} \subseteq \mathcal{P}(\omega^\omega)$ such that A'_n is non-self-dual and $A'_n <_w A'_{n+1}$ for any $n \in \omega$, then it suffices to use Lemma 2.45.

The case $A \equiv_w \left(\sum_{n\in\omega} A_n\right) + \omega^\omega$ is proved similarly. $\square$

We are now ready to prove Theorem 2.43.

Proof of Theorem 2.43. The proof goes by induction on $\alpha = \mathrm{rk}'_w(A)$. We distinguish between the non-self-dual case and the self-dual case using Proposition 2.37. Suppose first that A is non-self-dual. If $\mathrm{rk}'_w(A) = 2$, there is nothing to prove. If A is non-self-dual and $A \equiv_w B + \emptyset$ or $A \equiv_w B + \omega^\omega$ for some non-self-dual $B \subseteq \omega^\omega$ such that $B <_w A$, it suffices to use Lemma 2.46 and induction. If A is non-self-dual and $A \equiv_w \left(\sum_{n\in\omega} A_n\right) + \omega^\omega$ or $A \equiv_w \left(\sum_{n\in\omega} A_n\right) + \emptyset$ for some $\{A_n : n \in \omega\} \subseteq \mathcal{P}(\omega^\omega)$ such that A_n is non-self-dual and $A_n <_w A_{n+1}$ for any $n \in \omega$, then it suffices to proceed by induction and use Lemma 2.47. This completes the non-self-dual case.

If A is self-dual and $A \equiv_w \pm B$ for some non-self-dual $B \subseteq \omega^\omega$ such that $B <_w A$, it suffices to use Lemma 2.44 and induction. Finally, if A is self-dual and $A \equiv_w \sum_{n\in\omega} A_n$ for some $\{A_n : n \in \omega\} \subseteq \mathcal{P}(\omega^\omega)$ such that A_n is non-self-dual and $A_n <_w A_{n+1}$ for any $n \in \omega$, then we use Lemma 2.45 and induction. $\square$

2.4.3 Duparc's operations

We introduce the different tools developed by Duparc to exhibit, for any non-self-dual degree $\Gamma \in \mathbb{WD}_{\mathcal{B}}(\omega^\omega)$, a subset $A \subseteq \omega^\omega$ such that $\Gamma = [A]_w$ [Dup01, Dup]. For this purpose, he defined the conciliatory game on the subsets of $\omega^{\leq\omega}$ as a symmetrization of the Wadge game in which both players can skip their turn.

Definition 2.48. Let $A, B \subseteq \omega^{\leq\omega}$. The conciliatory game $G_c(A, B)$ is an instance of the Gale-Stewart game with the following rules. At round $2n$, I picks a sequence $s_n \in \omega^{<\omega}$ and, at round $2n+1$, II picks a sequence $t_n \in \omega^{<\omega}$ such that, for any $n \in \omega$, $\mathrm{lh}(s_{n+1}) \in \{\mathrm{lh}(s_n), \mathrm{lh}(s_n)+1\}$, $s_n \sqsubseteq s_{n+1}$, $\mathrm{lh}(t_{n+1}) \in \{\mathrm{lh}(t_n), \mathrm{lh}(t_n)+1\}$ and $t_n \sqsubseteq t_{n+1}$. At the end of the game, I has produced $x = \lim_{n\in\omega} s_n \in \omega^{\leq\omega}$ and II has produced $y = \lim_{n\in\omega} t_n \in \omega^{\leq\omega}$. We say that II wins this run of the game if and only if $x \in A \leftrightarrow y \in B$.

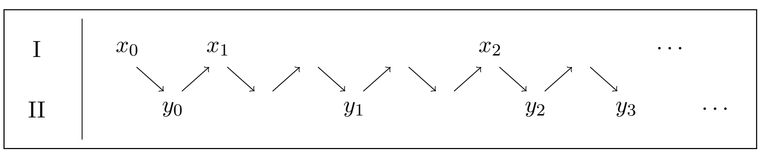


Figure 2.10: A run of the conciliatory game $G_c(A, B)$.

If II has a winning strategy in $G_c(A, B)$, then we write $A \leq_c B$. It is easy to compose winning strategies so that $\leq_c$ is a quasi-order. If both $A \leq_c B$ and $B \leq_c A$ hold, then we write $A \equiv_c B$. The conciliatory degree of $A \subseteq \omega^{\leq\omega}$ is $[A]_c = \left\{B \subseteq \omega^{\leq\omega} : A \equiv_c B\right\}$. The set of all such degrees is denoted by $\mathbb{D}\left({\omega_{\mathtt{b}}}^{\leq\omega}\right)$, so that the couple $\left(\mathbb{D}\left({\omega_{\mathtt{b}}}^{\leq\omega}\right), \leq_c\right)$ is a poset. The conciliatory game is a symmetrization of the Wadge game allowing both players to skip their turn. These two games are intimately connected. Let $\mathtt{b}$ be a symbol not in ω, and $\omega_{\mathtt{b}} = \omega \cup \{\mathtt{b}\}$. Since there is a bijection between ω and $\omega_{\mathtt{b}}$, the two spaces ω^ω and ${\omega_{\mathtt{b}}}^\omega$ are homeomorphic. Consider also the function

$$\rho_{\mathtt{b}} : {\omega_{\mathtt{b}}}^{\omega} \to \omega^{\leq\omega}$$
$$x \mapsto x_{[\ /\mathtt{b}]},$$

i.e., any sequence in ${\omega_{\mathtt{b}}}^{\omega}$ is mapped to the sequence obtained once the symbol $\mathtt{b}$ is omitted. For any $A \subseteq \omega^{\leq\omega}$, we define

$$A^{\mathtt{b}} = \rho_{\mathtt{b}}^{-1}[A] = \left\{x \in {\omega_{\mathtt{b}}}^{\omega} : x_{[\ /\mathtt{b}]} \in A\right\}.$$

Proposition 2.49 ([Dup01])**.** *For any $A, B \subseteq \omega^{\leq\omega}$, $A \leq_c B$ if and only if $A^{\mathtt{b}} \leq_w B^{\mathtt{b}}$.*

Proof. Let the symbol $\mathtt{s}$ represent, for any player, the option to skip his or her turn and let $\tau : (\omega \cup \{\mathtt{s}\})^{<\omega} \to \omega \cup \{\mathtt{s}\}$ be a winning strategy for II in $G_c(A, B)$. The strategy

$$\tau_{\mathtt{b}} : {\omega_{\mathtt{b}}}^{<\omega} \to \omega_{\mathtt{b}} \cup \{\mathtt{s}\}$$
$$s \mapsto \tau\left(s_{[\mathtt{s}/\mathtt{b}]}\right) \text{ if } \tau\left(s_{[\mathtt{b}/\mathtt{s}]}\right) \neq \mathtt{s},$$
$$s \mapsto \mathtt{b} \qquad \text{otherwise},$$

is a winning strategy for II in $G_w\left(A^{\mathtt{b}}, B^{\mathtt{b}}\right)$.

Similarly, if $\sigma : {\omega_{\mathtt{b}}}^{<\omega} \to \omega_{\mathtt{b}} \cup \{\mathtt{s}\}$ is a winning strategy for II in $G_w\left(A^{\mathtt{b}}, B^{\mathtt{b}}\right)$. The strategy

$$\sigma_c : (\omega \cup \{\mathtt{s}\})^{<\omega} \to \omega \cup \{\mathtt{s}\}$$
$$s \mapsto \sigma\left(s_{[\mathtt{b}/\mathtt{s}]}\right) \text{ if } \sigma\left(s_{[\mathtt{b}/\mathtt{s}]}\right) \neq \mathtt{s},$$
$$s \mapsto \mathtt{b} \qquad \text{otherwise},$$

is a winning strategy for II in $G_c(A, B)$. □

To avoid any kind of confusion between $\leq_c$ and $\leq_w$, we think of the set $\omega^{\leq\omega}$ as a set *deprived of any topology.* In [Cam19], Camerlo proved that there is no topology on $\omega^{\leq\omega}$ such that the quasi-orders $\leq_c$ and $\leq_w$ coincide. Although $\omega^{\leq\omega}$ has no topology, we define $\mathbb{D}_{\mathcal{B}}\left({\omega_{\mathtt{b}}}^{\leq\omega}\right)$ as the set of all degrees $[A]_c$ such that $A^{\mathtt{b}} \in \mathcal{B}(\omega^{\omega})$. In particular, the induced mapping

$$\rho_{\mathtt{b}}^{-1} : \left(\mathbb{D}_{\mathcal{B}}\left(\omega^{\leq\omega}\right), \leq_c\right) \to \left(\mathbb{WD}_{\mathcal{B}}(\omega^{\omega}), \leq_w\right)$$
$$[A]_c \mapsto \left[A^{\mathtt{b}}\right]_w$$

is an order-embedding whose range is included in the non-self-dual degrees.

Proposition 2.50 ([Dup01])**.** *If $A \subseteq \omega^{\leq\omega}$, then $A \nleq_c A^{\mathrm{c}}$.*

Proof. We construct a winning strategy for I in the game $G_c(A, A^{\mathrm{c}})$. At round 0, I plays $\langle\rangle$. For the rest of the game, I copies II's previous move. As a consequence, both players produce the same final sequence so that I wins the game. $\square$

It is the main result of [Dup01, Dup] that the order-embedding ρ_{b}^{-1} is actually onto the non-self-dual degrees.

Theorem 2.51 ([Dup01, Dup])**.** *For any non-self-dual $B \in \mathcal{B}(\omega^\omega)$, there exists $A \subseteq \omega^{\leq\omega}$ such that $B \equiv_w A^{\mathsf{b}}$.*

In other words, it suffices to study the poset $\left(\mathbb{D}_{\mathcal{B}}\left({\omega_{\mathsf{b}}}^{\leq\omega}\right), \leq_c\right)$ to describe the Wadge order $(\mathbb{WD}_{\mathcal{B}}(\omega^\omega), \leq_w)$. Theorem 2.51 is obtained through the definitions of several operations on the subsets of $\omega^{\leq\omega}$. Recall that if $x \notin {\omega_{>1}}^{\leq\omega}$, then we uniquely write $x = x^{-2} m_x x^*$ with $x^{-2} \in {\omega_{>1}}^{\leq\omega}$ and $m_x \in \{0, 1\}$. If $A_0, A_1 \subseteq \omega^{\leq\omega}$, we define

$$\begin{aligned} A_0 + A_1 = &\left\{x \in {\omega_{>1}}^{\leq\omega} : x^{-2} \in A_1\right\} \\ &\cup \left\{x \notin {\omega_{>1}}^{\leq\omega} : m_x = 0 \ \wedge \ x^* \in A_0\right\} \\ &\cup \left\{x \notin {\omega_{>1}}^{\leq\omega} : m_x = 1 \ \wedge \ x^* \notin A_0\right\}. \end{aligned}$$

There is a simple game-theoretic interpretation of this operation: if a player — say II — is in charge of the set $A_0 + A_1$, then she starts being in charge of A_1 and, at any moment, she can choose to be in charge of A_0 or of ${A_0}^{\mathrm{c}}$. Observe the similarity with the same operation defined on subsets of ω^ω. The second operation is the supremum operation. If $\omega \leq \alpha < \omega_1$, $A_\beta \subseteq \omega^{\leq\omega}$ for any $\beta < \alpha$, and $b : \omega \to \alpha$ a bijection, we define

$$\sup\left\{A_\beta : \beta < \alpha\right\} = \bigcup_{i\in\omega} \left\{ix \in \omega^{\leq\omega} : x \in A_{b(i)}\right\}.$$

Once again, the game-theoretic interpretation is simple: if a player — say II — is in charge of the set $\sup\left\{A_\beta : \beta < \alpha\right\}$, then she can either stay outside of it by skipping her turn forever or she can decide to be in charge of the set A_β by playing $b^{-1}(\beta) \in \omega$ as her first integer. Using these two operations, we define the multiplication on subsets of $\omega^{\leq\omega}$. Let $A \subseteq \omega^{\leq\omega}$, we define

$A \cdot 1 = A$,

$A \cdot (\alpha + 1) = A \cdot \alpha + A$ for any countable α,

$A \cdot \lambda = \sup\left\{A_\beta : \beta < \lambda\right\}$ for any countable limit ordinal λ.

These operations are the set-theoretic counterpart of addition, supremum and multiplication on the Wadge rank (actually a slightly modified version of rk'_w defined previously). More precisely, if we define the Wadge rank as $\mathrm{rk}_w(\emptyset) = \mathrm{rk}_w(\omega^\omega) = 1$ and, for any non-self-dual proper $A \in \mathcal{B}(\omega^\omega)$,

$$\mathrm{rk}_w(A) = \sup\{\mathrm{rk}_w(B) + 1 : B \subseteq \omega^\omega \text{ non-self-dual such that } B <_w A\},$$

then we get the following result.

Theorem 2.52 (Theorem 4 in [Dup01]). *If $A, B, A_\beta \subseteq \omega^{\leq\omega}$ for any $\beta < \alpha < \omega_1$, then*

1. $\mathrm{rk}_w\left((B + A)^\mathsf{b}\right) = \mathrm{rk}_w\left(B^\mathsf{b}\right) + \mathrm{rk}_w\left(A^\mathsf{b}\right),$
2. $\mathrm{rk}_w\left(\left(\sup\{A_\beta : \beta < \alpha\}\right)^\mathsf{b}\right) = \sup\left\{\mathrm{rk}_w\left(A_\beta^\mathsf{b}\right) : \beta < \alpha\right\},$
3. $\mathrm{rk}_w\left((A \cdot \alpha)^\mathsf{b}\right) = \mathrm{rk}_w\left(A^\mathsf{b}\right) \cdot \alpha.$

In particular, starting with the sets $\emptyset, \omega^{\leq\omega} \subseteq \omega^{\leq\omega}$, it is possible to generate with the operations defined so far any non-self-dual degree in $\mathbb{WD}_{\mathcal{B}}(\omega^\omega)$ of rank $< \omega_1$, i.e., any non-self-dual degree in $\mathbb{WD}_{\boldsymbol{\Delta}^0_2}(\omega^\omega)$.

The operation $A^\sim$

To go further up in the Wadge hierarchy of the Borel sets in ω^ω, we need an operation that allows to jump from the $\boldsymbol{\Delta}^0_2$ sets to a $\boldsymbol{\Sigma}^0_2$-complete set. Let $\twoheadleftarrow$ be a symbol not in ω which stands for *eraser.* Let $\omega_{\twoheadleftarrow} = \omega \cup \{\twoheadleftarrow\}$. The function $\cdot^{\looparrowleft}$ interprets the symbol $\twoheadleftarrow$ as an eraser.

$$\begin{aligned} \cdot^{\looparrowleft} : {\omega_{\twoheadleftarrow}}^{<\omega} &\to \omega^{<\omega} \\ \langle\rangle &\mapsto \langle\rangle \\ sa &\mapsto s^{\looparrowleft}a && \text{if } a \neq \twoheadleftarrow, \\ s\twoheadleftarrow &\mapsto \langle\rangle && \text{if } \mathrm{lh}\left(s^{\looparrowleft}\right) = 0, \\ s\twoheadleftarrow &\mapsto s^{\looparrowleft}{\restriction}_{(\mathrm{lh}(s^{\looparrowleft})-1)} && \text{otherwise.} \end{aligned}$$

This decoding operation naturally extends to infinite sequences. Observe that the image of an infinite sequence might be finite, which justifies the use of the set $\omega^{\leq\omega}$ instead of ω^ω.

$$\begin{aligned} \cdot^{\looparrowleft} : {\omega_{\twoheadleftarrow}}^{\omega} &\to \omega^{\leq\omega} \\ x &\mapsto \lim_{k\in\omega} x{\restriction}_k{}^{\looparrowleft}. \end{aligned}$$

If $A \subseteq \omega^{\leq\omega}$, we define

$$A^{\sim} = A^{\sim_1} = \left\{ x \in {\omega_{\twoheadleftarrow}}^{\leq\omega} : x^{\hookleftarrow} \in A \right\}.$$

Once again, there is a simple game-theoretic interpretation of this operation: if a player — say II — is in charge of $A^{\sim}$, she is actually in charge of A with the further possibility of erasing the last symbol she played so far. Once again, observe the similarity with the eraser game $G_{\twoheadleftarrow}(f)$ for $f : A \subseteq \omega^\omega \to \omega^\omega$ (Definition 2.7). Using a bijection between $\omega_{\twoheadleftarrow}$ and ω, we think of $A^{\sim}$ as a subset of $\omega^{\leq\omega}$. In particular, the previous definition iterates:

$$A^{\sim_{k+1}} = \left\{ x \in {\omega_{\twoheadleftarrow}}^{\leq\omega} : x^{\hookleftarrow} \in A^{\sim_k} \right\}.$$

The operation $\cdot^{\sim}$ allows to climb along the Wadge hierarchy. If $\mathcal{O} = \{\langle\rangle\}^{\mathrm{c}} \subseteq \omega^{\leq\omega}$, it is easy to see that $\mathcal{O}^{\mathsf{b}}$ is $\boldsymbol{\Sigma}^0_1(\omega^\omega)$-complete.

Theorem 2.53 (Lemma 31 in [Dup01])**.** *For any $k \in \omega$, $(\mathcal{O}^{\sim_k})^{\mathsf{b}}$ is $\boldsymbol{\Sigma}^0_{k+1}(\omega^\omega)$-complete.*

Let

$$\varepsilon_0 = \sup \left\{ \underbrace{\omega_1^{\cdot^{\cdot^{\cdot^{\omega_1}}}}}_{n \text{ times}} : n \in \omega \right\}.$$

Given this new operation, it is now possible to generate all the non-self-dual degrees of finite Borel rank, i.e., of

$$\mathbb{WD}_{\bigcup_{n\in\omega} \boldsymbol{\Sigma}^0_n}(\omega^\omega).$$

Indeed, it is proven in [Dup01] that, once restricted to the sets of finite Borel rank, the operation $\cdot^{\sim}$ is the set-theoretic counterpart of the exponentiation of base ω_1 on the rank.

Theorem 2.54 (Theorem 4 in [Dup01])**.** *Suppose $A \subseteq \omega^{\leq\omega}$ such that $\mathrm{rk}_w\left(A^{\mathsf{b}}\right) < \varepsilon_0$, then*

$$\mathrm{rk}_w\left(\left(A^{\sim}\right)^{\mathsf{b}}\right) = \omega_1^{\mathrm{rk}_w\left(A^{\mathsf{b}}\right)}.$$

In particular, granted with the operations defined so far and starting from $\emptyset, \omega^{\leq\omega} \subseteq \omega^{\leq\omega}$, it is now possible to generate the first ε_0 many Wadge degrees, i.e., all the non-self-dual degrees of finite Borel rank.

To climb further in the Wadge hierarchy, a generalization of the $\cdot^{\sim}$ operation is needed which appears in [Dup]. However, this construction is not needed in this thesis.

The operation $A^{\approx}$

In order for the proofs to be conducted by induction, Duparc introduced a kind of inverse to the operation $A^{\sim}$ [Dup01]. The idea comes from the following well-known fact (see Section 13 in [Kec95]): if (X,τ) is a Polish space and $(A_n)_{n\in\omega} \subseteq \mathcal{B}(X,\tau)$, there is a refinement τ' of the topology such that (X,τ') is a zero-dimensional Polish space, $\mathcal{B}(X,\tau) = \mathcal{B}(X,\tau')$ and $(A_n)_{n\in\omega} \subseteq \mathbf{\Delta}^0_1(X,\tau')$. In [Dup01], Duparc defines the notion of *question-trees*, a game-friendly representation of (X,τ') for (X,τ) a zero-dimensional Polish space. We give an alternative — but totally equivalent — definition of this notion.

Definition 2.55 (Definition 25 in [Dup01])**.** Let $T \subseteq \omega^{<\omega}$ be a non-empty pruned tree. A question-tree on $[T]$ is a labeled tree $l : \mathbb{T} \to \mathbf{\Pi}^0_1([T]) \setminus \{\emptyset\}$ satisfying the following properties:

1. $\mathbb{T} \subseteq \omega^{<\omega}$ is a non-empty pruned tree.
2. $l(\langle\rangle) = [T]$.
3. If $s \in \mathbb{T}$, then
$$l(s) = \bigsqcup_{sa\in\mathbb{T}} l(sa).$$
In particular, for any $x \in [T]$, there exists a unique infinite branch $y_x \in [\mathbb{T}]$ such that $x \in \bigcap_{n\in\omega} l\left(y_{x\restriction n}\right)$.
4. If $s \in \mathbb{T}$ and $\mathrm{lh}(s) = n$, there exists $t_s \in T$ such that $\mathrm{lh}(t_s) = n$ and $l(s) \subseteq [t_s]$. In particular, to any $y \in [\mathbb{T}]$, one associates $x_y = \bigcap_{n\in\omega} l\left(y_{\restriction n}\right) \in [T]$.

The set of questions of $\mathbb{T}$ is

$$\mathbb{Q}(\mathbb{T}) = \{l(s) : s \in \mathbb{T}\} \subseteq \mathbf{\Pi}^0_1([T]).$$

As usual for labeled trees, we often ignore the labeling l. For example, if $\mathbb{T}$ is a question-tree, $F \in \mathbb{T}$ stands for $s \in \mathbb{T}$ such that $l(s) = F$. Moreover, any node $s \in \mathbb{T}$ naturally yields the couple $(t_s, l(s))$. By a second abuse of notation, we write $(t,F) \in \mathbb{T}$ to mean $s \in \mathbb{T}$ such that $l(s) = F$ and $t_s = t$ so that we have $\mathbb{T} \subseteq \left(T, \mathbf{\Pi}^0_1([T])\right)^{<\omega}$. In particular, if $(t,F) \in \mathbb{T}$, we use the projections π_0 and π_1 to speak about t and F, i.e., $\pi_0(t,F) = t$ and $\pi_1(t,F) = F$.

The term *question-tree* comes from the following observation: if a player — say II — goes down a question-tree $\mathbb{T}$, she chooses at each step a closed subset in which she will end up. To say it otherwise, if the current position is $s \in \mathbb{T}$, the next move answers the following auxiliary question: *In which closed set among* $\{l(sa) : sa \in \mathbb{T}\}$ *will you end up?*

From the properties of the definition of question-trees, there is a natural one-to-one correspondence between the elements of $[T]$ and the elements of $[\mathbb{T}]$:

$$\begin{aligned} [T] &\to [\mathbb{T}] \\ x &\mapsto y_x \\ x_y &\leftarrow\!\shortmid y. \end{aligned}$$

Since T and $\mathbb{T}$ are non-empty pruned trees, we can equip $[T]$ and $[\mathbb{T}]$ with the prefix topology. In that case, the function $y \mapsto x_y$ is continuous, whereas the function $x \mapsto y_x$ is a pointwise limit of a sequence of continuous functions. However, this set-theoretic correspondence becomes an homeomorphism when one equips the set $[T]$ with the right topology. For this purpose, let τ be the topology on $[T]$ generated by the set $\mathbb{Q}(\mathbb{T})$. For any $F, F' \in \mathbb{Q}(\mathbb{T})$, one has $F \subseteq F', F' \subseteq F$ or $F \cap F' = \emptyset$. In particular, $\mathbb{Q}(\mathbb{T})$ is a basis of τ made of clopen sets. For any $t \in T$, one has $[t] = \bigcup_{s \in S} l(s)$, where $S = \{s \in \mathbb{T} : \mathrm{lh}(s) = \mathrm{lh}(t) \text{ and } l(s) \subseteq [t]\}$. This implies that $\tau \subseteq \mathbf{\Sigma}^0_2([T])$ is a refinement of the prefix topology on $[T]$.

Proposition 2.56. *The function*

$$\begin{aligned} ([T], \tau) &\to [\mathbb{T}] \\ x &\mapsto y_x \\ x_y &\leftarrow\!\shortmid y. \end{aligned}$$

is an homeomorphism.

Proof. It suffices to prove that both functions are open. Since any element of the basis of (T, τ) is $F = l(s) \in \mathbb{Q}(\mathbb{T})$ for some $s \in \mathbb{T}$, one has

$$\{y_x \in [\mathbb{T}] : x \in F\} = [s] \in \mathbf{\Delta}^0_1([\mathbb{T}]).$$

For the other direction, let $s \in \mathbb{T}$. One has

$$\{x_y \in [T] : y \in [s]\} = l(s) \in \mathbf{\Delta}^0_1(T, \tau).$$

□

In particular, since $\mathbb{T}$ is a non-empty pruned tree, $[\mathbb{T}]$ is a game-friendly representation of the zero-dimensional Polish space $([T], \tau)$. Thus, by a slight abuse of notation, we write $[\mathbb{T}]$ to denote the space $([T], \tau)$. As a consequence, if A is a subset of $[T]$, we write $A \subseteq [T]$ when we consider A as a subset of the topological space $[T]$ and $A \subseteq [\mathbb{T}]$ when we consider A as a subset of the topological space $([T], \tau)$. Similarly and throughout the thesis, the two points $x = x_y \in [T]$ and $y = y_x \in [\mathbb{T}]$ are identified. We directly obtain the following important feature of question-trees.

Theorem 2.57 ([Dup01])**.** *Let $n > 0$, $T \subseteq \omega^{<\omega}$, $A \subseteq [T]$ and $\mathbb{T}$ a question-tree on $[T]$. If $A \in \mathbf{\Pi}^0_n([\mathbb{T}])$, then $A \in \mathbf{\Pi}^0_{n+1}([T])$. The same result holds for the $\mathbf{\Sigma}^0_n$-classes, and thus also for the $\mathbf{\Delta}^0_n$-classes.*

Proof. The result easily follows from $\tau \subseteq \mathbf{\Sigma}^0_2([T])$. □

Since the topology on $[\mathbb{T}]$ refines the topology on $[T]$, for any $A \subseteq [T]$, we get $(A, [\mathbb{T}]) \leq_w (A, [T])$. By Remark 26 in [Dup01], there exists a minimal Wadge degree that is reached from A by applying a single question-tree. Moreover, this Wadge degree must be non-self-dual. We denote this question-tree by $\mathbb{T}_A$ and define $A^{\approx} = (A, [\mathbb{T}_A])$. The operation $\cdot^{\approx}$ is almost the inverse of $\cdot^{\sim}$ in the following sense.

Proposition 2.58 (Proposition 30 in [Dup01])**.** *If $A \subseteq \omega^\omega$ and $B \subseteq \omega^{\leq\omega}$, then:*

1. $A^{\approx} \leq_w B^{\mathsf{b}} \Leftrightarrow A \leq_w (B^{\sim})^{\mathsf{b}}$,
2. $B^{\mathsf{b}} \leq_w A^{\approx} \leq_w \Leftrightarrow (B^{\sim})^{\mathsf{b}} \leq_w A$.

We finish this subsection by constructing some specific question-trees. If $\mathcal{F} = \{F_n : n \in \omega\} \subseteq \mathbf{\Pi}^0_1([T])$, we recursively construct the question-tree generated by $\mathcal{F}$ $\mathbb{T}_{\mathcal{F}}$ on $[T]$ such that $\{F_n : n \in \omega\} \subseteq \mathbf{\Delta}^0_1([\mathbb{T}_{\mathcal{F}}])$. The idea is to ask at step $n+1$ if one will end up in the closed set F_n. First, set $l(\langle\rangle) = [T]$. Then, suppose that $s \in \mathbb{T}_{\mathcal{F}}$ and $l(s) \in \mathbf{\Pi}^0_1([T])$ are already defined, and that $\mathrm{lh}(s) = n$. There exists a — possibly empty — sequence $(v_m)_{m\in\alpha} \subseteq T$ for $\alpha \in \omega + 1$ such that $\mathrm{lh}(v_m) = n + 1$ for any $m \in \alpha$ and

$$l(s) \cap F_n = \bigsqcup_{m\in\alpha} [v_m] \cap l(s) \cap F_n.$$

Also, there exists a — possibly empty — sequence $\left(v'_{m'}\right)_{m'\in\alpha'} \subseteq T$ for $\alpha' \in \omega + 1$ such that $\mathrm{lh}\left(v'_{m'}\right) \geq n + 1$ for any $m' \in \alpha'$ and

$$l(s) \setminus F_n = \bigsqcup_{m'\in\alpha'} \left[v'_{m'}\right] \cap l(s).$$

The set $S = \left\{v_m, v'_{m'} : m \in \alpha, m' \in \alpha'\right\}$ is non-empty and countable. Thus, there exist a non-zero ordinal $\beta \in \omega + 1$ and an enumeration of the set S denoted by $(s_k)_{k \in \beta}$. The successors of $s \in \mathbb{T}_{\mathcal{F}}$ are defined as sk for any $k \in \beta$. If $s_k = v_m$ for some m, then $l\,(sk) = [s_k] \cap l\,(s) \cap F_n$ and $t_{sk} = v_m$. Otherwise, $s_k = v'_{m'}$ for some m' and we define $l\,(sk) = [s_k] \cap l\,(s)$ and $t_{sk} = s_{k \restriction n+2}$. Thus, for any countable family of closed sets $\mathcal{F}$, there exists a question-tree $\mathbb{T}_{\mathcal{F}}$ turning them into clopen subsets. One also easily shows that $[\mathbb{T}_{\mathcal{F}}]$ is homeomorphic to $([T], \tau)$, where τ is the topology generated by $\mathcal{F}$ on $[T]$.

A representation of the question-tree $\mathbb{T}_{\mathcal{F}}$ for $\mathcal{F} = \{F_0, F_1\}$ is given in Figure 2.11.

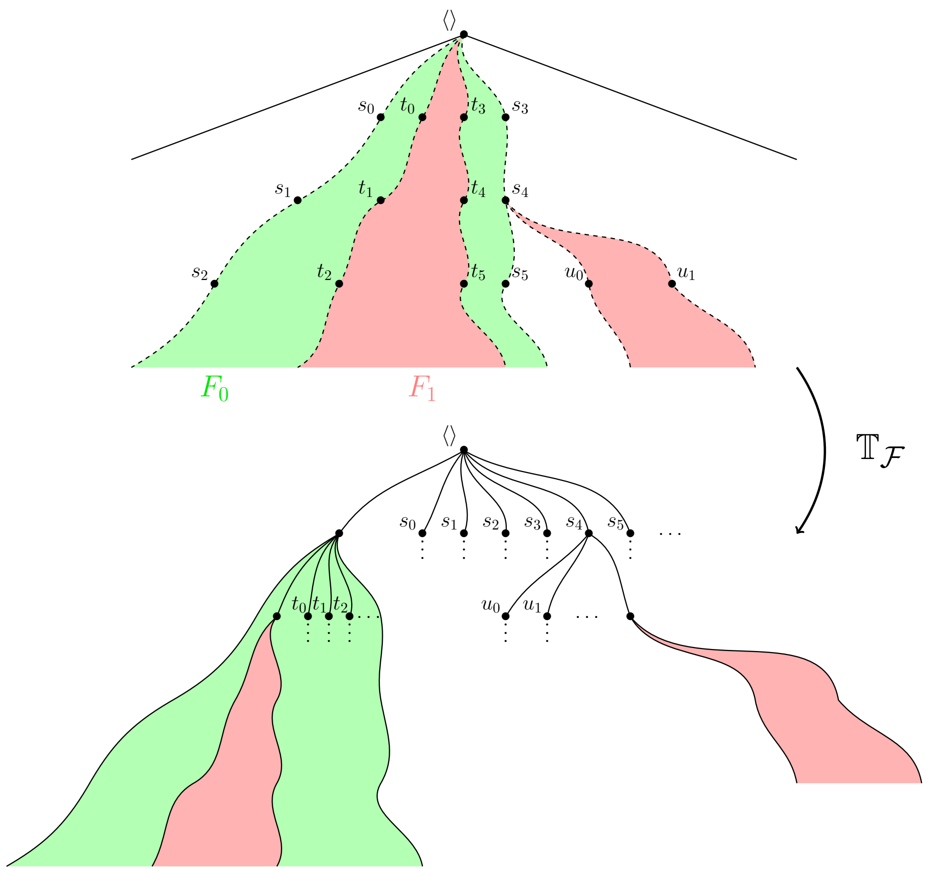


Figure 2.11: The question-tree $\mathbb{T}_{\mathcal{F}}$ generated by $\mathcal{F} = \{F_0, F_1\}$.

We conclude with a second important feature of question-trees.

Theorem 2.59 ([Dup01]). *Let $n > 0$, $T \subseteq \omega^{<\omega}$ non-empty pruned and $A \subseteq [T]$. If $A \in \mathbf{\Pi}^0_{n+1}([T])$, then $A \in \mathbf{\Pi}^0_n([\mathbb{T}_A])$. The same result holds for the $\mathbf{\Sigma}^0_n$-classes, and thus also for the $\mathbf{\Delta}^0_n$-classes.*

Proof. If $A \in \mathbf{\Pi}^0_{n+1}([T])$ with $n+1$ odd, then

$$A = \bigcap_{m_1 \in \omega} \bigcup_{m_2 \in \omega} \cdots \bigcap_{m_{n-1} \in \omega} \bigcup_{m_n \in \omega} F_{m_1, m_2, \ldots, m_{n-1}, m_n},$$

with

$$\mathcal{F} = \left\{ F_{m_1, m_2, \ldots, m_{n-1}, m_n} : m_1, m_2, \ldots, m_{n-1}, m_n \in \omega \right\} \subseteq \mathbf{\Pi}^0_1([T]).$$

Thus, we get $A \in \mathbf{\Pi}^0_n([\mathbb{T}_{\mathcal{F}}])$. By minimality of $(A, [\mathbb{T}_A])$, we also obtain that $A \in \mathbf{\Pi}^0_n([\mathbb{T}_A])$. The same idea also yields the result if $n+1$ is even. □

2.4.4 Another decoding function

We conclude this chapter with the definition of an alternative decoding function for sequences containing erasers which will considerably simplify some later proofs. Although it is different than $\cdot^{\hookleftarrow}$, we prove that both decodings are almost equivalent. Let $\omega_{\{\mathtt{b},\twoheadleftarrow\}} = \omega \cup \{\mathtt{b}, \twoheadleftarrow\}$ where $\mathtt{b}$ and $\twoheadleftarrow$ are disjoint symbols not occurring in ω.

Definition 2.60. We define the function $\widetilde{\cdot} : {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega} \to {\omega_{\mathtt{b}}}^{<\omega}$ by induction on the length $\mathrm{lh}(s)$ of $s \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$. If $t \in {\omega_{\mathtt{b}}}^{<\omega} \setminus \left\{\mathtt{b}^l : l \in \omega\right\}$, we uniquely write $t = u_t m_t \mathtt{b}^l$ with $m_t \in \omega$.

$$\begin{aligned}
\widetilde{\cdot} : {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega} &\to {\omega_{\mathtt{b}}}^{<\omega} \\
\langle\rangle &\mapsto \langle\rangle \\
sa &\mapsto \widetilde{s}a && \text{if } a \neq \twoheadleftarrow, \\
s\twoheadleftarrow &\mapsto \mathtt{b}^{l+1} && \text{if } \widetilde{s} = \mathtt{b}^l \text{ for some } l \in \omega, \\
s\twoheadleftarrow &\mapsto u_{\widetilde{s}}\mathtt{b}^{l+2} && \text{if } \widetilde{s} \notin \left\{\mathtt{b}^l : l \in \omega\right\} \text{ and } \widetilde{s} = u_{\widetilde{s}} m_{\widetilde{s}} \mathtt{b}^l.
\end{aligned}$$

Observe that $\mathrm{lh}(s) = \mathrm{lh}(\widetilde{s})$. If $s\twoheadleftarrow \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$ and $\widetilde{s} = \mathtt{b}^l$, then we say that $s(\mathrm{lh}(s) - 1) = \twoheadleftarrow$ does not erase anything. If $s\twoheadleftarrow \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$ and $\widetilde{s} \notin \left\{\mathtt{b}^l : l \in \omega\right\}$, then we say that $s(\mathrm{lh}(s) - 1) = \twoheadleftarrow$ erases $s(\mathrm{lh}(u_{\widetilde{s}}))$, or that $s(\mathrm{lh}(u_{\widetilde{s}}))$ is erased by $s(\mathrm{lh}(s) - 1) = \twoheadleftarrow$.

The function $\widetilde{\cdot}$ easily extends to infinite sequences. Indeed, if $x \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{\omega}$ and $l \in \omega$, then the sequence $\left(x_{\restriction n}(l)\right)_{n > l} \subseteq \omega_{\mathtt{b}}$ is eventually constant and, say, equal to $\widetilde{x}_l$. This defines the sequence $\widetilde{x} \in {\omega_{\mathtt{b}}}^{\omega}$.

The two decoding functions $\widetilde{\cdot}$ and $\cdot^{\hookleftarrow}$ are almost equivalent for finite sequences in the following sense.

Lemma 2.61. *For any* $s \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$,

$$\widetilde{s}_{[\ /\mathtt{b}]} = {s_{[\ /\mathtt{b}]}}^{\hookleftarrow}.$$

Proof. The proof goes by induction on $\mathrm{lh}\,(s)$. If $\mathrm{lh}\,(s) = 0$, there is nothing to prove. If $sa \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$ and $a \neq \twoheadleftarrow$, then

$$\widetilde{sa}_{[\ /\mathtt{b}]} = \widetilde{s}a_{[\ /\mathtt{b}]} = \widetilde{s}_{[\ /\mathtt{b}]}a_{[\ /\mathtt{b}]} = {s_{[\ /\mathtt{b}]}}^{\hookleftarrow}a_{[\ /\mathtt{b}]} = {sa_{[\ /\mathtt{b}]}}^{\hookleftarrow}.$$

If $s\twoheadleftarrow \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$ and $\widetilde{s}_{[\ /\mathtt{b}]} = {s_{[\ /\mathtt{b}]}}^{\hookleftarrow} = \langle\rangle$, then

$$\widetilde{v\twoheadleftarrow}_{[\ /\mathtt{b}]} = \langle\rangle = {v\twoheadleftarrow_{[\ /\mathtt{b}]}}^{\hookleftarrow}.$$

If $s\twoheadleftarrow \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$ and $\widetilde{s} = u_{\widetilde{s}}m_{\widetilde{s}}\mathtt{b}^l$, then ${s_{[\ /\mathtt{b}]}}^{\hookleftarrow} = \widetilde{s}_{[\ /\mathtt{b}]} = u_{\widetilde{s}[\ /\mathtt{b}]}m_{\widetilde{s}}$ and

$$\widetilde{s\twoheadleftarrow}_{[\ /\mathtt{b}]} = u_{\widetilde{s}[\ /\mathtt{b}]} = {s\twoheadleftarrow_{[\ /\mathtt{b}]}}^{\hookleftarrow}.$$

□

We extend the previous lemma to infinite sequence. To simplify further proofs, we introduce a new definition.

Definition 2.62. Let $x \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{\omega}$. We construct an increasing sequence $(\alpha_i^{\twoheadleftarrow})_{i\in\omega} \subseteq \omega$. Set $\alpha_0^{\twoheadleftarrow} = 0$. Suppose that $\alpha_i^{\twoheadleftarrow} \in \omega$ is already defined. If there exists $m \in \omega$ such that $x_{\alpha_i^{\twoheadleftarrow}}$ is erased by $x_{m-1} = \twoheadleftarrow$, let $\alpha_{i+1}^{\twoheadleftarrow} = m$, otherwise let $\alpha_{i+1}^{\twoheadleftarrow} = \alpha_i^{\twoheadleftarrow} + 1$.

Observe that the sequence $(\alpha_i^{\twoheadleftarrow})_{i\in\omega}$ is constructed *a posteriori* from x. Moreover, it is the sequence of all symbols of x that survive after the decoding of $\twoheadleftarrow$. In particular, for any $i \in \omega$, $x_{\alpha_i^{\twoheadleftarrow}-1}$ is never erased in x. Let us make some observations about this construction. If $x_{\alpha_{i+1}^{\twoheadleftarrow}-1} \neq \twoheadleftarrow$, then $\widetilde{x_{\restriction\alpha_{i+1}^{\twoheadleftarrow}}} = \widetilde{x_{\restriction\alpha_i^{\twoheadleftarrow}}}x_{\alpha_i^{\twoheadleftarrow}}$. If $x_{\alpha_{i+1}^{\twoheadleftarrow}-1} = \twoheadleftarrow$ does not erase anything, then $\widetilde{x_{\restriction\alpha_{i+1}^{\twoheadleftarrow}}} = \widetilde{x_{\restriction\alpha_i^{\twoheadleftarrow}}}\mathtt{b}$. If $x_{\alpha_{i+1}^{\twoheadleftarrow}-1} = \twoheadleftarrow$ does erase something, then $x_{\restriction\alpha_{i+1}^{\twoheadleftarrow}} = x_{\restriction\alpha_i^{\twoheadleftarrow}}g$ where $x_{\alpha_{i+1}^{\twoheadleftarrow}-1}$ erases $x_{\alpha_i^{\twoheadleftarrow}}$. In that case, $\widetilde{x_{\restriction\alpha_{i+1}^{\twoheadleftarrow}}} = \widetilde{x_{\restriction\alpha_i^{\twoheadleftarrow}}}\mathtt{b}^l$, where $l = \mathrm{lh}\,(g)$. In particular, we have the following result.

Lemma 2.63. *If* $x \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{\omega}$ *and* $i \in \omega$, *then*

$$\widetilde{x_{\restriction\alpha_i^{\twoheadleftarrow}}} \sqsubset \widetilde{x_{\restriction\alpha_{i+1}^{\twoheadleftarrow}}}.$$

Moreover, if $x \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{\leq\omega}$, then the elements of $(\alpha_i^{\twoheadleftarrow})_{i\in\omega} \subseteq \omega$ do matter because they yield an initial segment of $\widetilde{x}$.

Lemma 2.64. *If $x \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{\leq\omega}$ and $i \in \omega$, then*

$$\widetilde{x_{\restriction\alpha_i^{\twoheadleftarrow}}} = \widetilde{x}_{\restriction\alpha_i^{\twoheadleftarrow}}.$$

Proof. We prove the result by induction on $i \in \omega$. If $i = 0$, then $\alpha_0^{\twoheadleftarrow} = 0$ and there is nothing to prove. If $x_{\alpha_{i+1}^{\twoheadleftarrow}-1} \neq \twoheadleftarrow$, then $x_{\alpha_{i+1}^{\twoheadleftarrow}-1}$ is never erased in x and

$$\widetilde{x_{\restriction\alpha_{i+1}^{\twoheadleftarrow}}} = \widetilde{x_{\restriction\alpha_i^{\twoheadleftarrow}} x_{\alpha_i^{\twoheadleftarrow}}} = \widetilde{x}_{\restriction\alpha_i^{\twoheadleftarrow}} x_{\alpha_i^{\twoheadleftarrow}} = \widetilde{x}_{\restriction\alpha_{i+1}^{\twoheadleftarrow}}.$$

If $x_{\alpha_{i+1}^{\twoheadleftarrow}-1} = \twoheadleftarrow$ does not erase anything, then

$$\widetilde{x_{\restriction\alpha_{i+1}^{\twoheadleftarrow}}} = \widetilde{x_{\restriction\alpha_i^{\twoheadleftarrow}}}\mathtt{b} = \widetilde{x}_{\restriction\alpha_i^{\twoheadleftarrow}}\mathtt{b} = \widetilde{x}_{\restriction\alpha_{i+1}^{\twoheadleftarrow}}.$$

If $x_{\alpha_{i+1}^{\twoheadleftarrow}-1} = \twoheadleftarrow$ does erase something, then $x_{\restriction\alpha_{i+1}^{\twoheadleftarrow}} = x_{\restriction\alpha_i^{\twoheadleftarrow}} g$ where $x_{\alpha_{i+1}^{\twoheadleftarrow}-1}$ erases $x_{\alpha_i^{\twoheadleftarrow}}$. Thus,

$$\widetilde{x_{\restriction\alpha_{i+1}^{\twoheadleftarrow}}} = \widetilde{x_{\restriction\alpha_i^{\twoheadleftarrow}}}\mathtt{b}^l = \widetilde{x}_{\restriction\alpha_i^{\twoheadleftarrow}}\mathtt{b}^l = \widetilde{x}_{\restriction\alpha_{i+1}^{\twoheadleftarrow}},$$

where $l = \operatorname{lh}(g)$. □

We prove that Lemma 2.61 extends to infinite sequences.

Lemma 2.65. *For any $x \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{\omega}$,*

$$\widetilde{x}_{[\ /\mathtt{b}]} = {x_{[\ /\mathtt{b}]}}^{\hookleftarrow}.$$

Proof. For any $\sqsubseteq$-increasing sequence $(s_n)_{n\in\omega} \subseteq {\omega_{\mathtt{b}}}^{<\omega}$, the following equivalences hold.

$$\begin{aligned} t \sqsubseteq \left(\lim_{n\in\omega} s_n\right)_{[\ /\mathtt{b}]} &\Leftrightarrow \exists t' \in {\omega_{\mathtt{b}}}^{<\omega} \left(t'_{[\ /\mathtt{b}]} = t \ \wedge\ \exists l\ \forall m \geq l\ t' \sqsubseteq s_m\right) \\ &\Leftrightarrow \exists l\ \forall m \geq l\ t \sqsubseteq s_{m[\ /\mathtt{b}]} \\ &\Leftrightarrow t \sqsubseteq \lim_{n\in\omega} s_{n[\ /\mathtt{b}]}. \end{aligned}$$

Also, for any $x \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{\omega}$, we easily have

$$\lim_{n\in\omega} {x_{\restriction n[\ /\mathtt{b}]}}^{\hookleftarrow} = \lim_{n\in\omega} {x_{[\ /\mathtt{b}]\restriction n}}^{\hookleftarrow}.$$

Moreover, for any $\alpha_i^{\twoheadleftarrow} < m < \alpha_{i+1}^{\twoheadleftarrow}$,

$$\widetilde{x_{\restriction\alpha_i^{\twoheadleftarrow}}} \sqsubset \widetilde{x_{\restriction m}} \not\sqsubseteq \widetilde{x_{\restriction\alpha_{i+1}^{\twoheadleftarrow}}} \quad \text{and} \quad {x_{\restriction\alpha_i^{\twoheadleftarrow}[\ /\mathtt{b}]}}^{\hookleftarrow} \sqsubset {x_{\restriction m[\ /\mathtt{b}]}}^{\hookleftarrow} \not\sqsubseteq {x_{\restriction\alpha_{i+1}^{\twoheadleftarrow}[\ /\mathtt{b}]}}^{\hookleftarrow}.$$

These observations together with Lemma 2.63 and Lemma 2.64 yield the following equalities.

$$\begin{aligned}\widetilde{x}_{[\ /\mathtt{b}]} &= \left(\lim_{k\in\omega} \widetilde{x}_{\restriction k}\right)_{[\ /\mathtt{b}]} = \left(\lim_{i\in\omega} \widetilde{x}_{\restriction \alpha_i^{\twoheadleftarrow}}\right)_{[\ /\mathtt{b}]} = \lim_{i\in\omega} \widetilde{x}_{\restriction \alpha_i^{\twoheadleftarrow}\,[\ /\mathtt{b}]} \\ &= \lim_{i\in\omega} \widetilde{x_{\restriction \alpha_i^{\twoheadleftarrow}\,[\ /\mathtt{b}]}} = \lim_{i\in\omega} x_{\restriction \alpha_i^{\twoheadleftarrow}\,[\ /\mathtt{b}]}{}^{\hookleftarrow} = \lim_{k\in\omega} x_{\restriction k [\ /\mathtt{b}]}{}^{\hookleftarrow} \\ &= \lim_{k\in\omega} x_{[\ /\mathtt{b}]\restriction k}{}^{\hookleftarrow} = x_{[\ /\mathtt{b}]}{}^{\hookleftarrow}\end{aligned}$$

□

Let $k \in \omega$ and $\omega_{\{\mathtt{b},\twoheadleftarrow_0,\ldots,\twoheadleftarrow_k\}} = \omega \cup \{\mathtt{b}, \twoheadleftarrow_0, \ldots, \twoheadleftarrow_k\}$ where $\mathtt{b}, \twoheadleftarrow_0, \ldots, \twoheadleftarrow_k$ are disjoint symbols not occurring in ω. We write $\twoheadleftarrow$ for $\twoheadleftarrow_0$. If $x \in \omega_{\{\mathtt{b},\twoheadleftarrow,\ldots,\twoheadleftarrow_k\}}{}^{\omega}$, then $\widetilde{x}^k$ stands for the previously defined function applied to the eraser $\twoheadleftarrow_k$. Also, $\widetilde{x}^{k\to 0}$ stands for the sequence obtained after applying successively the functions $\widetilde{\cdot}^k, \ldots, \widetilde{\cdot}$ to x. In particular, if $A \subseteq \omega^{\leq\omega}$, the previous Lemma yields

$$x \in (A^{\sim_{k+1}})^{\mathtt{b}} \;\Leftrightarrow\; \widetilde{x}^k \in (A^{\sim_k})^{\mathtt{b}} \;\Leftrightarrow\; \ldots \;\Leftrightarrow\; \widetilde{x}^{k\to 0} \in A^{\mathtt{b}}.$$

Let $x \in \omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1,\ldots,\twoheadleftarrow_k\}}{}^{\omega}$. We already know how to define the increasing sequence $\left(\alpha_i^{\twoheadleftarrow_k}\right)_{i\in\omega} \subseteq \omega$. We recursively construct an increasing sequence

$$\left(\alpha_i^{\twoheadleftarrow_l}\right)_{i\in\omega} \subseteq \left\{\alpha_i^{\twoheadleftarrow_{l+1}} : i \in \omega\right\}$$

for any $0 \leq l < k$, where $\twoheadleftarrow\, = \,\twoheadleftarrow_0$. Suppose that we have already defined an increasing sequence $\left\{\alpha_i^{\twoheadleftarrow_{l+1}} : i \in \omega\right\} \subseteq \omega$. Set $\alpha_0^{\twoheadleftarrow_l} = 0$. Suppose now that we have already defined $\alpha_i^{\twoheadleftarrow_l} = \alpha_j^{\twoheadleftarrow_{l+1}}$. If there exists $\alpha_k^{\twoheadleftarrow_{l+1}} \in \omega$ such that $x_{\alpha_i^{\twoheadleftarrow_l}}$ is erased by $x_{\alpha_k^{\twoheadleftarrow_{l+1}}-1} = \twoheadleftarrow_l$, let $\alpha_{i+1}^{\twoheadleftarrow_l} = \alpha_k^{\twoheadleftarrow_{l+1}}$, otherwise let $\alpha_{i+1}^{\twoheadleftarrow_l} = \alpha_{j+1}^{\twoheadleftarrow_{l+1}}$. The sequence $(\alpha_i^{\twoheadleftarrow})_{i\in\omega}$ is called the α-sequence constructed from x. Observe that Definition 2.62 corresponds to the case $k = 0$.

Lemma 2.66. *If $x \in \omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1,\ldots,\twoheadleftarrow_k\}}{}^{\omega}$ and $i \in \omega$, then*

$$\widetilde{x_{\restriction \alpha_i^{\twoheadleftarrow}}}^{k\to 0} = \widetilde{x}^{k\to 0}_{\restriction \alpha_i^{\twoheadleftarrow}}.$$

Proof. We prove the result by induction on k. The case $k = 0$ is Lemma 2.64. Suppose now that the result is proved for $k - 1$, we prove it for k. Since $\{\alpha_i^{\twoheadleftarrow} : i \in \omega\} \subseteq \left\{\alpha_i^{\twoheadleftarrow_k} : i \in \omega\right\}$, Lemma 2.64 yields

$$\widetilde{x_{\restriction \alpha_i^{\twoheadleftarrow}}}^{k} = \widetilde{x}^{k}_{\restriction \alpha_i^{\twoheadleftarrow}}.$$

If $(\beta_i^{\twoheadleftarrow})_{i\in\omega}$ is the α-sequence constructed from $\widetilde{x}^k$, then we clearly have $\{\alpha_i^{\twoheadleftarrow} : i \in \omega\} \subseteq \{\beta_i^{\twoheadleftarrow} : i \in \omega\}$. By induction hypothesis, we obtain

$$\widetilde{x_{\restriction\alpha_i^{\twoheadleftarrow}}}^{k\to 0} = \widetilde{\widetilde{x_{\restriction\alpha_i^{\twoheadleftarrow}}}^{k}}^{k-1\to 0} = \widetilde{\widetilde{x}^k_{\restriction\alpha_i^{\twoheadleftarrow}}}^{k-1\to 0} = \widetilde{\widetilde{x}^k}^{k-1\to 0}_{\restriction\alpha_i^{\twoheadleftarrow}} = \widetilde{x}^{k\to 0}_{\restriction\alpha_i^{\twoheadleftarrow}}.$$

□

Chapter 3

The Wadge order on the Scott domain

We saw that the Scott domain $\mathcal{P}\omega$ plays a central role in the development of descriptive set theory outside the class of Polish spaces for its universality among quasi-Polish spaces. Thus, one naturally wonders about the shape of the Wadge order on the Borel degrees of the Scott domain $\mathcal{P}\omega$. This chapter is devoted to this study. Several results have already been obtained in the literature. In [Sel05], Selivanov proved the existence of an antichain of size 4 in $(\mathbb{WD}_{\mathcal{B}}(\mathcal{P}\omega), \leq_w)$ as well as the existence of two distinct $\leq_w$-minimal Wadge degrees $[Y_\alpha]_w$ and $[Z_\alpha]_w$ at each infinite level $\omega \leq \alpha < \omega_1$ of the Hausdorff-Kuratowski difference hierarchy of open sets. In [BG15b], Becher and Grigorieff exhibited, for each infinite level $\omega \leq \alpha < \omega_1$ of the same hierarchy, some strictly $\leq_w$-increasing chains of Wadge degrees of length α, and also described the unique $\leq_w$-maximal Wadge degree $[C_\alpha]_w$ for each such level. In this chapter, we show both that the Wadge order on the Borel degrees of $\mathcal{P}\omega$ is ill-founded and that it admits infinite antichains. Moreover, we show that these properties occur already within the differences of ω open sets, i.e., at the lowest possible level of topological complexity.

Theorem 3.1. *The quasi-order* $\left(D_\omega(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega), \leq_w\right)$ *is ill-founded and has infinite antichains.*

The results of this chapter have already been published as a joint paper with Duparc in *The Journal of Symbolic Logic* [DV20]. As mentioned in the first paragraph of the article[1], all results except the ones of Section 3.4 are

[1] "With the exception of Section 5, all the results presented in this article — including the main ones — are due to the sole second author" [DV20].

due to the author of this thesis.

3.1 Selivanov's toolbox

The main result of this chapter comes as a generalization of a construction of Selivanov that we recall here [Sel05]. Let $\mathcal{P}_{<\omega}(\omega) = \{x \subseteq \omega : x \text{ finite}\}$.

Definition 3.2. Let T_α be any well-founded tree of rank $\omega \leq \alpha < \omega_1$, $\xi : \omega^{<\omega} \to \omega$ be any injective mapping such that $\xi(\langle\rangle) = 0$, and the function $e : T_\alpha \to \mathcal{P}_{<\omega}(\omega)$ be defined as $e(s) = \{\xi(t) : t \sqsubseteq s\}$. The sets Y_α and Z_α are defined by:

1. $Y_\alpha = e\big[T^1_\alpha\big]$, where $T^1_\alpha = \{s \in T_\alpha : \mathrm{lh}\,(s) \text{ is odd}\}$,
2. $Z_\alpha = B(T_\alpha) \cup Y_\alpha$, where $B(T_\alpha) = \{x \subseteq \omega : \forall s \in T_\alpha \; x \nsubseteq e(s)\}$.

In [Sel05], it is shown that, given any $\omega \leq \alpha < \omega_1$, Y_α and Z_α are differences of α open sets, $\leq_w$-incomparable, and $\leq_w$-minimal among true differences of α open sets.

Theorem 3.3 (Propositions 5.9 and 6.4 in [Sel05])**.** *For $n \in \omega$, $\omega \leq \alpha, \beta < \omega_1$ and $\mathcal{A} \in \mathbf{\Delta}^0_2(\mathcal{P}\omega) \setminus \check{D}_\alpha(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega)$, we have:*

1. *$D_n(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega) \setminus \check{D}_n(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega)$ and $\check{D}_n(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega) \setminus D_n(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega)$ are two $\leq_w$-incomparable degrees,*
2. *$Y_\alpha, Z_\alpha \in D_\alpha(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega) \setminus \check{D}_\alpha(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega)$,*
3. *$Y_\alpha \nleq_w Z_\beta$ and $Z_\beta \nleq_w Y_\alpha$,*
4. *if $\omega \in \mathcal{A}$, then $Z_\alpha \leq_w \mathcal{A}$,*
5. *if $\omega \notin \mathcal{A}$, then $Y_\alpha \leq_w \mathcal{A}$.*

The proof of Theorem 3.3 makes use of Selivanov's characterizations of the $\mathbf{\Delta}^0_2$ subsets and of the $D_\alpha\big(\mathbf{\Sigma}^0_1\big)$ subsets of $\mathcal{P}\omega$. Since our proof will also require these characterizations, we recall them. For this purpose, if $x, y \in \mathcal{P}\omega$ are such that $x \subseteq y$, we introduce the notation

$$[x, y] = \{z \in \mathcal{P}\omega : x \subseteq z \subseteq y\}.$$

Definition 3.4. $\mathcal{A} \subseteq \mathcal{P}\omega$ is approximable if, for all $x \in \mathcal{A}$, there exists $F \in \mathcal{P}_{<\omega}(\omega)$ such that $F \subseteq x$ and $[F, x] \subseteq \mathcal{A}$.

A subset $\mathcal{A}$ of $\mathcal{P}\omega$ is $\mathbf{\Delta}^0_2$ if the membership of any subset $x \subseteq \omega$ to $\mathcal{A}$ can be approximated by a finite subset of x.

Theorem 3.5 (Theorem 3.12 in [Sel05])**.** *Let $\mathcal{A} \subseteq \mathcal{P}\omega$.*

$$\mathcal{A} \in \mathbf{\Delta}^0_2(\mathcal{P}\omega) \iff \mathcal{A} \text{ and } \mathcal{P}\omega \setminus \mathcal{A} \text{ are approximable.}$$

The characterization of $D_\alpha\big(\mathbf{\Sigma}^0_1\big)$ subsets of $\mathcal{P}\omega$ is a stratification of the previous result using the notion of a 1-alternating tree.

Definition 3.6. Let $\mathcal{A} \subseteq \mathcal{P}\omega$ and $0 < \alpha < \omega_1$. A 1-alternating tree for $\mathcal{A}$ of rank α is a homomorphism of quasi-orders

$$f : (T, \sqsubseteq) \to (\mathcal{P}_{<\omega}(\omega), \subseteq)$$

from a well-founded tree $T \subseteq \omega^{<\omega}$ of rank α such that:

1. $f(\langle\rangle) \in \mathcal{A}$, and
2. for all $sn \in T$, we have $(f(s) \in \mathcal{A} \leftrightarrow f(sn) \notin \mathcal{A})$.

Corollary 3.7 (Corollary 3.11 in [Sel05])**.** *Let $\mathcal{A} \subseteq \mathcal{P}\omega$ and $0 < \alpha < \omega_1$.*

$$\mathcal{A} \in D_\alpha(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega) \Longleftrightarrow \begin{cases} \mathcal{A} \in \mathbf{\Delta}^0_2(\mathcal{P}\omega) \text{ and} \\ \text{there is no 1-alternating tree for } \mathcal{A} \text{ of rank } \alpha. \end{cases}$$

3.2 The class $\mathbb{P}_{\mathbf{emb}}$

We define a class $\mathbb{P}_{\text{emb}}$ of 2-colored posets. For this purpose, we introduce several order-theoretic notions. If P is a poset, we denote by

$$\text{Pred}(p) = \{p' \in P : p' \leq_P p\}$$

the set of predecessors of $p \in P$, and by

$$\text{Pred}_{\text{im}}(p) = \big\{p' \in P : (p' <_P p) \wedge \neg\exists p'' \in P \ (p' <_P p'' \wedge p'' <_P p)\big\}$$

the set of its immediate predecessors. A function $\varphi : P \to Q$ preserves immediate predecessors if, for any $p_0, p_1 \in P$, whenever $p_0 \in \text{Pred}_{\text{im}}(p_1)$, then $\varphi(p_0) \in \text{Pred}_{\text{im}}(\varphi(p_1))$. We use homomorphisms in order to compare structures. Let P and Q be two posets. If there exists an injective homomorphism $\varphi : P \to Q$, then we write $P \xrightarrow{\text{1-1 h.}} Q$, and if it is injective and preserves immediate predecessors, then we write $P \rightarrowtail Q$. Notice that $P \rightarrowtail Q$ is more rigid than $P \xrightarrow{\text{1-1 h.}} Q$ since $P \rightarrowtail Q$ implies $P \xrightarrow{\text{1-1 h.}} Q$ but $P \xrightarrow{\text{1-1 h.}} Q$ does not necessarily imply $P \rightarrowtail Q$.

Since we only consider countable posets, let $\mathbb{P}$ denote the class of countable posets. If $P \in \mathbb{P}$, then we can always consider $\leq_P \subseteq \alpha \times \alpha$ where $\alpha \in \omega \cup \{\omega\}$ via any bijection $P \leftrightarrow \alpha$. In particular, all the posets $P \in \mathbb{P}$ that we consider are posets on $P \in \omega \cup \{\omega\}$.

A 2-colored poset is a triple $\mathsf{P} = (P, \leq_P, \text{c}_P)$ where $(P, \leq_P) \in \mathbb{P}$ is a countable poset and $\text{c}_P : P \to 2$ is a 2-coloring. For example, if $T \subseteq \omega^{<\omega}$ is a well-founded tree, then $c_T : T \to 2$ where $c_T(t) = 1$ if and only if $\mathrm{rk}_T(t)$ is even makes $(T, \sqsubseteq, c_T)$ a 2-colored poset. The class of all 2-colored posets is denoted by $\mathbb{P}_c$. We usually use the letters P, Q for 2-colored posets.

As done in [Leh08, Zhu14], we compare them via homomorphisms. A homomorphism between $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_c$ is a homomorphism of posets $\varphi : P \to Q$ such that for all $p \in P$, $\mathrm{c}_P(p) = \mathrm{c}_Q(\varphi(p))$. If there exists a homomorphism from P to Q, then we write $\mathsf{P} \preccurlyeq_c \mathsf{Q}$; if this homomorphism is injective, then we write $\mathsf{P} \xrightarrow{\text{1-1 h.}}_c \mathsf{Q}$; if it is injective and preserves immediate predecessors, then we write $\mathsf{P} \rightarrowtail_c \mathsf{Q}$. The relation $\preccurlyeq_c$ is a quasi-order on $\mathbb{P}_c$. If both $\mathsf{P} \preccurlyeq_c \mathsf{Q}$ and $\mathsf{Q} \preccurlyeq_c \mathsf{P}$ hold, we write $\mathsf{P} \sim_c \mathsf{Q}$. The degree of $\mathsf{P} \in \mathbb{P}_c$ is $[\mathsf{P}]_c = \{\mathsf{Q} \in \mathbb{P}_c : \mathsf{P} \sim_c \mathsf{Q}\}$, and the set of all these degrees is denoted by $\mathbb{D}(\mathbb{P}_c)$. In particular, $\preccurlyeq_c$ induces a partial order on $\mathbb{D}(\mathbb{P}_c)$ so that $(\mathbb{D}(\mathbb{P}_c), \preccurlyeq_c)$ is a poset.

Our next goal is to define a subclass $\mathbb{P}_{\mathrm{emb}} \subseteq \mathbb{P}_c$ of countable 2-colored posets. We begin with the naming of several posets that are useful for the definition of a subclass $\mathbb{P}_{\mathrm{shr}} \subseteq \mathbb{P}$. In Figure 3.1, we represent each poset $(P, \leq_P)$ by its Hasse diagram $G = (P, \to)$. More precisely, if $p, q \in P$, then $p \leq_P q$ if and only if there exists a finite sequence $(p_k)_{k \leq l}$ such that $p_0 = p, p_l = q$ and for all $k < l$, we have $p_k \to p_{k+1}$.

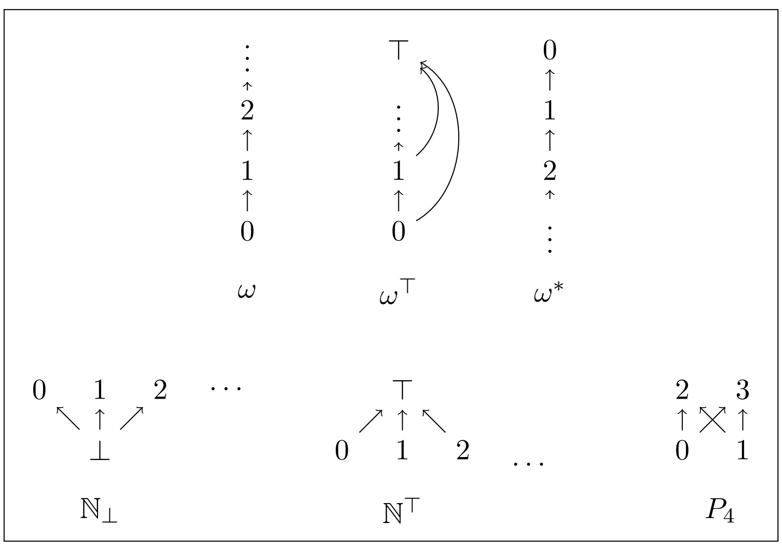


Figure 3.1: Samples of useful countable posets.

In [Sel05], Selivanov worked with well-founded trees in order to construct subsets of $\mathcal{P}\omega$. We generalize this construction to a larger class of posets that we call *shrubs* and that share some of the properties of well-founded trees. For this purpose, we make use of the classical notion of *bounded completeness* that occurs in domain theory [GHK⁺03].

Definition 3.8. Let P be a poset.

1. A subset $B \subseteq P$ is bounded if there exists an element $u \in P$ such that $b \leq_P u$ holds for every $b \in B$. Such a $u \in P$ is called an upper bound of B. The set of all upper bounds of B is denoted by $\mathcal{U}_B$.
2. If the set $\mathcal{U}_B$ of all upper bounds of B has a — necessarily unique — $\leq_P$-minimal element $s_B \in \mathcal{U}_B$ — i.e., $\forall u \in \mathcal{U}_B \; s_B \leq_P u$ — it is called the supremum of B in P.
3. The poset P is bounded complete if every bounded $S \subseteq P$ has a supremum.

A typical example of a poset which is not bounded complete is P_4 as shown in Figure 3.1. All other posets shown in Figure 3.1, as well as $(\mathcal{P}_{<\omega}(\omega), \subseteq)$ and $(\mathcal{P}\omega, \subseteq)$ are bounded complete. Notice that every bounded complete poset P has a unique $\leq_P$-minimal element, namely the supremum of the empty set, usually denoted by $\perp$.

Definition 3.9. The class of all shrubs $\mathbb{P}_{\mathrm{shr}} \subseteq \mathbb{P}$ is the class of all countable posets $P \in \mathbb{P}$ that satisfy:

1. $\omega \overset{\text{1-1 h.}}{\not\longrightarrow} P$.
2. For all $p \in P, \; \mathrm{Card}(\mathrm{Pred}(p)) < \aleph_0$.
3. P is bounded complete.

Well-founded trees, and in particular $\mathbb{N}_\perp$, are typical examples of shrubs. More involved ones will be constructed in the proofs of Theorem 3.31 (Figure 3.5) and Theorem 3.35 (Figure 3.6). To the contrary, ω, $\omega^\top$, ω^*, $\mathbb{N}^\top$, and P_4 are typical examples of posets that are not shrubs.

In the next proposition, we give alternative characterizations to the second item of the previous definition. In particular, we show that the posets we just defined can be embedded into $\mathcal{P}_{<\omega}(\omega)$. We also give an alternative characterization of this second item that exclusively depends on morphisms between posets.

Proposition 3.10. *If $P \in \mathbb{P}$, then the following are equivalent:*

1. *For all $p \in P, \; \mathrm{Card}(\mathrm{Pred}(p)) < \aleph_0$,*
2. *$P \overset{\text{1-1 h.}}{\longrightarrow} \mathcal{P}_{<\omega}(\omega)$,*
3. *$(\omega^\top \overset{\text{1-1 h.}}{\not\longrightarrow} P)$, $(\omega^* \overset{\text{1-1 h.}}{\not\longrightarrow} P)$ and $(\mathbb{N}^\top \overset{\text{1-1 h.}}{\not\longrightarrow} P)$.*

Proof.

(1. ⇒ 2.) We consider $P \in \omega \cup \{\omega\}$ such that $(P, \leq_P)$ is a poset and define a function:

$$\begin{aligned} e : P &\to \mathcal{P}_{<\omega}(\omega) \\ k &\mapsto \{n : n \leq_P k\}. \end{aligned}$$

If $k \leq_P l$, then by transitivity of $\leq_P$, we get $e(k) \subseteq e(l)$. If $k \neq l$, we consider the two cases $k <_P l$ and $k \perp_P l$ (the third case $l <_p k$ is the same as the case $k <_P l$). In both cases, $l \in e(l) \setminus e(k)$. Therefore, we obtain that e witnesses the fact that $P \xrightarrow{\text{1-1 h.}} \mathcal{P}_{<\omega}(\omega)$.

(2. ⇒ 3.) If $\varphi : Q \xrightarrow{\text{1-1 h.}} P$, then for all $q \in Q$, the injectivity of φ implies $\mathrm{Card}\big(\mathrm{Pred}(q)\big) \leq \mathrm{Card}\big(\mathrm{Pred}(\varphi(q))\big)$. Since $\mathrm{Card}\big(\mathrm{Pred}(F)\big) < \aleph_0$ for any $F \in \mathcal{P}_{<\omega}(\omega)$, we get the result by contradiction.

(3. ⇒ 1.) Towards a contradiction, we pick an element $p \in P$ such that $\mathrm{Card}(\mathrm{Pred}(p)) = \aleph_0$. We consider three different cases.

(a) Suppose there exists $q_0 <_P p$ such that there exists no immediate predecessor p' of p satisfying $q_0 \leq_P p'$. Hence, there exists $q_1 <_P p$ such that $q_0 <_P q_1$. We continue the process to construct a sequence $(q_n)_{n\in\omega}$ witnessing $\omega^\top \xrightarrow{\text{1-1 h.}} P$ via the mapping: $\top \mapsto p$, and $n \mapsto q_n$ for any $n \in \omega$.

(b) Suppose there exist infinitely many immediate predecessors $(q_n)_{n\in\omega}$ of $p \in P$, then the mapping: $\top \mapsto p$, and $n \mapsto q_n$ for any $n \in \omega$, witnesses $\mathbb{N}^\top \xrightarrow{\text{1-1 h.}} P$.

(c) Suppose that we are not in the situations (a) and (b); then, by the pigeonhole principle, there exists q_0 an immediate predecessor of p such that $\mathrm{Card}(\mathrm{Pred}(q_0)) = \aleph_0$. If we replace p with q_0 and start the proof again, either we get a contradiction from (a) or (b), or we exhibit q_1 an immediate predecessor of q_0 such that $\mathrm{Card}(\mathrm{Pred}(q_1)) = \aleph_0$. By an infinite iteration of this process, we obtain a sequence $(q_n)_{n\in\omega}$ witnessing $\omega^* \xrightarrow{\text{1-1 h.}} P$ via the mapping: $0 \mapsto p$, and $n \mapsto q_{n-1}$ for any $n \in \omega^+$.

□

In Figure 3.2, we give a name to some specific 2-colored posets that are useful for the next definition: the nodes of the form • and ◦ correspond to color 1 and color 0, respectively.

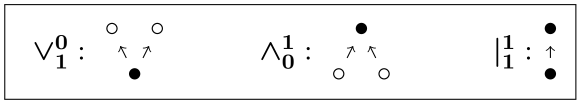


Figure 3.2: Samples of useful 2-colored countable posets.

The next definition introduces the class of *embeddable posets* $\mathbb{P}_{\text{emb}}$. We will later associate a subset $\mathcal{A}_{\mathsf{P}}$ of $\mathcal{P}\omega$ to each such 2-colored poset $\mathsf{P} \in \mathbb{P}_{\text{emb}}$, where the color 1 will correspond to elements inside $\mathcal{A}_{\mathsf{P}}$.

Definition 3.11. The class of embeddable posets $\mathbb{P}_{\text{emb}}$ is the class of countable 2-colored posets $\mathsf{P} = (P, \leq_P, \mathrm{c}_P)$ such that $(P, \leq_P) \in \mathbb{P}_{\text{shr}}$ and whose coloring satisfies:

1. $\mathrm{c}_P(\bot) = 0$,
2. for all $k \in P$ $\leq_P$-maximal, $\mathrm{c}_P(k) = 1$,
3. $(\vee^{\mathbf{0}}_{\mathbf{1}} \not\rightarrowtail_c \mathsf{P})$, $(\wedge^{\mathbf{1}}_{\mathbf{0}} \not\rightarrowtail_c \mathsf{P})$ and $(\,|^{\mathbf{1}}_{\mathbf{1}} \not\rightarrowtail_c \mathsf{P})$.

If P is an embeddable poset, then the nodes of color 1 are isolated. Indeed, if $\mathsf{P} \in \mathbb{P}_{\text{emb}}$, $p \in P$ and $\mathrm{c}_P(p) = 1$, then p has a unique immediate predecessor and has at most one immediate successor[2], depending on whether p is $\leq_P$-maximal or not. Moreover, if they exist, they both have color 0. Thus, we introduce the following notations.

Notation 3.12. For $\mathsf{P} \in \mathbb{P}_{\text{emb}}$, $p \in P$ and $\mathrm{c}_P(p) = 1$, we denote by p^- its unique immediate predecessor; and, if it exists, by p^+ its unique immediate successor. We have $\mathrm{c}_P(p^-) = \mathrm{c}_P(p^+) = 0$.

This means that the direct neighborhood — composed of all immediate predecessors and all immediate successors — of every node of color 1 is of one of the form given in Figure 3.3, depending on whether it is $\leq_P$-maximal or not. The first case occurs when p is $\leq_P$-maximal, and the second one when p is not.

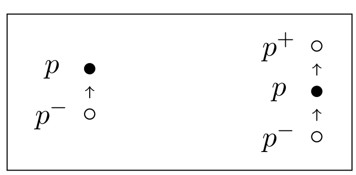


Figure 3.3: The two possible direct neighborhoods of any $p \in P$, where $\mathsf{P} \in \mathbb{P}_{\text{emb}}$ and $\mathrm{c}_P(p) = 1$.

[2]If P is an embeddable poset, then $p \in P$ is an immediate successor of $p' \in P$ if $p' \in \mathrm{Pred}_{\text{im}}(p)$.

3.3 An order-embedding into the Wadge order on the Scott domain

In this section, we associate a subset $\mathcal{A}_{\mathsf{P}} \in \mathbf{\Delta}^0_2(\mathcal{P}\omega)$ to each embeddable poset $\mathsf{P} \in \mathbb{P}_{\text{emb}}$, and show that this association is such that, for any $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\text{emb}}$, $\mathsf{P} \preccurlyeq_c \mathsf{Q}$ if and only if $\mathcal{A}_{\mathsf{P}} \leq_w \mathcal{A}_{\mathsf{Q}}$ (Lemma 3.17). As a consequence, we get our main result that there exists an order-embedding (Theorem 3.21)

$$(\mathbb{D}(\mathbb{P}_{\text{emb}}), \preccurlyeq_c) \to \left(\mathbb{WD}_{\mathbf{\Delta}^0_2}(\mathcal{P}\omega), \leq_w\right),$$

where $\mathbb{D}(\mathbb{P}_{\text{emb}}) = \{[\mathsf{P}]_c : \mathsf{P} \in \mathbb{P}_{\text{emb}}\}$. We first need to label the elements of any embeddable poset.

Definition 3.13. Let $\mathsf{P} \in \mathbb{P}_{\text{emb}}$ so that $P \in \omega \cup \{\omega\}$ has a $\leq_P$-minimal element $m = \bot$ for some $m \in \omega$. The labeling l_P on P is defined by:

$$\begin{aligned} l_P : P &\to \mathcal{P}_{<\omega}(\omega) \\ \bot &\mapsto \emptyset, \\ n &\mapsto \bigcup_{k \leq_P n} \{k\}. \end{aligned}$$

We notice that l_P is an injective homomorphism of posets. Therefore, for every $F \in \mathcal{P}_{<\omega}(\omega)$ in the range of l_P, ${l_P}^{-1}(F)$ is well defined.

In Definition 3.2, Selivanov associated two subsets of the Scott domain to any well-founded tree. We generalize this construction by associating a subset of the Scott domain to any embeddable poset through the labeling given in Definition 3.13.

Definition 3.14. Let $\mathsf{P} \in \mathbb{P}_{\text{emb}}$, we define the subset $\mathcal{A}_{\mathsf{P}} \subseteq \mathcal{P}\omega$ as:

$$\begin{aligned} \mathcal{A}_{\mathsf{P}} &= l_P\left[\mathrm{c}_P^{-1}[\{1\}]\right] \\ &= \left\{x \subseteq \omega : \exists p \in P \ (\mathrm{c}_P(p) = 1 \wedge l_P(p) = x)\right\}. \end{aligned}$$

We also denote by $\mathcal{C}(\mathcal{A}_{\mathsf{P}})$ the set of all finite sets of integers contained in the labeling of an element of P:

$$\mathcal{C}(\mathcal{A}_{\mathsf{P}}) = \left\{F \subseteq \omega : \exists p \in P \ F \subseteq l_P(p)\right\}.$$

The next lemma gathers two crucial observations that arise from the construction given by Definition 3.14.

Lemma 3.15. *Let* $\mathsf{P} \in \mathbb{P}_{\text{emb}}$ *and* $F \in \mathcal{P}_{<\omega}(\omega)$.

1. *If* $F \in \mathcal{C}(\mathcal{A}_{\mathsf{P}})$*, then* $\{p \in P : l_P(p) \subseteq F\}$ *has an upper bound in* P. *By Definition* 3.9*, it has a unique supremum denoted by* $s_F \in P$.
2. $F \in \mathcal{A}_{\mathsf{P}} \Leftrightarrow \big(\mathrm{c}_P(s_F) = 1 \wedge l_P(s_F) = F\big)$.

Proof.

1. Since $F \in \mathcal{C}(\mathcal{A}_{\mathsf{P}}) = \big\{F \subseteq \omega : \exists p \in P\ F \subseteq l_P(p)\big\}$, there exists $p_0 \in P$ such that $F \subseteq l_P(p_0)$. Thus, $p_0 \in P$ is an upper bound of $\{p \in P : l_P(p) \subseteq F\}$.
2. Assume first that $F \in \mathcal{A}_{\mathsf{P}} \subseteq \mathcal{C}(\mathcal{A}_{\mathsf{P}})$. Then, there exists $p_0 \in P$ such that $\mathrm{c}_P(p_0) = 1$ and $l_P(p_0) = F$. It implies that p_0 is the supremum of $\{p \in P : l_P(p) \subseteq F\}$, hence $p_0 = s_F$. We then get $l_P(s_F) = F$ and $\mathrm{c}_P(s_F) = 1$.
   Conversely, from the very definition of $\mathcal{A}_{\mathsf{P}}$, we have $\mathrm{c}_P(s_F) = 1$ and $l_P(s_F) = F$, which implies that $F \in \mathcal{A}_{\mathsf{P}}$.

□

The rest of this section consists in proving that the correspondence $\mathsf{P} \mapsto \mathcal{A}_{\mathsf{P}}$ satisfies that $\mathcal{A}_{\mathsf{P}} \in \mathbf{\Delta}^0_2(\mathcal{P}\omega)$ and for any $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\text{emb}}$, $\mathsf{P} \preccurlyeq_c \mathsf{Q}$ if and only if $\mathcal{A}_{\mathsf{P}} \leq_w \mathcal{A}_{\mathsf{Q}}$. For this, we make use of the well-known result that a continuous mapping from $\mathcal{P}\omega$ to itself is completely determined by its behavior on $\mathcal{P}_{<\omega}(\omega)$.

Lemma 3.16 (Exercice 5.1.62 in [GL13])**.** *Given any homomorphism of posets* $f : \mathcal{P}_{<\omega}(\omega) \to \mathcal{P}\omega$*, there exists a* unique *continuous extension of* f *to the whole Scott domain. This extension is given by*

$$\begin{aligned} \hat{f} : \mathcal{P}\omega &\to \mathcal{P}\omega \\ x &\mapsto \bigcup_{n \in \omega} f\big(x \cap n\big). \end{aligned}$$

Proof. The proof follows from the fact that a function between two dcpos equipped with the Scott topology is continuous if and only if it is monotonic and preserves directed supremums. □

We are now ready for our main proof.

Lemma 3.17. *The following mapping*

$$\begin{aligned} H : (\mathbb{P}_{\text{emb}}, \preccurlyeq_c) &\to (\mathbf{\Delta}^0_2(\mathcal{P}\omega), \leq_w) \\ \mathsf{P} &\mapsto \mathcal{A}_{\mathsf{P}} \end{aligned}$$

satisfies that for any $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\text{emb}}$*, we have*

$$\mathsf{P} \preccurlyeq_c \mathsf{Q} \text{ if and only if } \mathcal{A}_{\mathsf{P}} \leq_w \mathcal{A}_{\mathsf{Q}}.$$

Proof. The proof is divided into the three Claims 3.18, 3.19, and 3.20. The first two claims show that H is a well-defined homomorphism, whereas the third one completes the proof.

Claim 3.18. *If* $\mathsf{P} \in \mathbb{P}_{\mathrm{emb}}$, *then* $\mathcal{A}_{\mathsf{P}} \in \mathbf{\Delta}^0_2(\mathcal{P}\omega)$.

Proof of the claim. We show that the set $\mathcal{A}_{\mathsf{P}}$ is both approximable and co-approximable, i.e., $\mathcal{P}\omega \setminus \mathcal{A}_{\mathsf{P}}$ is approximable. $\mathcal{A}_{\mathsf{P}}$ is approximable because $\mathcal{A}_{\mathsf{P}} \subseteq \mathcal{P}_{<\omega}(\omega)$. For co-approximability, we proceed by contradiction and suppose that $\mathcal{A}_{\mathsf{P}}$ is not co-approximable for some $x \in \mathcal{P}\omega \setminus \mathcal{A}_{\mathsf{P}}$ infinite. So, we fix $F_0 \in [\emptyset, x] \cap \mathcal{A}_{\mathsf{P}}$ and set $p_0 = l_P^{-1}(F_0)$. Assume F_n and p_n are already constructed. Since $\mathcal{A}_{\mathsf{P}}$ is not co-approximable, there exists $F_{n+1} \in \big([F_n, x] \setminus \{F_n\}\big) \cap \mathcal{A}_{\mathsf{P}}$. We set $p_{n+1} = l_P^{-1}(F_{n+1})$. It follows that the function

$$\begin{aligned} \varphi : \omega &\to P \\ n &\mapsto p_n \end{aligned}$$

witnesses $\omega \xrightarrow{\text{1-1 h.}} P$, a contradiction. $\square_{\text{Claim}}$

Claim 3.19. *If* $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\mathrm{emb}}$ *and* $\mathsf{P} \preccurlyeq_c \mathsf{Q}$, *then* $\mathcal{A}_{\mathsf{P}} \leq_w \mathcal{A}_{\mathsf{Q}}$.

Proof of the claim. Suppose that $\mathsf{P} \preccurlyeq_c \mathsf{Q}$ is witnessed by $\varphi : P \to Q$. Consider the function:

$$\begin{aligned} f_\varphi : \mathcal{P}_{<\omega}(\omega) &\to \mathcal{P}\omega \\ F &\mapsto \begin{cases} l_Q\big(\varphi(s_F)\big) & \text{if } F \in \mathcal{C}(\mathcal{A}_{\mathsf{P}}) \wedge \mathrm{c}_P(s_F) = 0, \\ l_Q\big(\varphi(s_F)\big) & \text{if } F \in \mathcal{C}(\mathcal{A}_{\mathsf{P}}) \wedge \mathrm{c}_P(s_F) = 1 \wedge F = l_P(s_F), \\ l_Q\big(\varphi(s_F^-)\big) & \text{if } F \in \mathcal{C}(\mathcal{A}_{\mathsf{P}}) \wedge \mathrm{c}_P(s_F) = 1 \wedge F \subsetneq l_P(s_F), \\ l_Q\big(\varphi(s_F^+)\big) & \text{if } F \in \mathcal{C}(\mathcal{A}_{\mathsf{P}}) \wedge \mathrm{c}_P(s_F) = 1 \wedge F \nsubseteq l_P(s_F), \\ \omega & \text{otherwise,} \end{cases} \end{aligned}$$

where s_F is defined as in Lemma 3.15; s_F^- and s_F^+ are defined as in Notation 3.12; and s_F^+ is replaced by ω whenever s_F is a maximal element in $(P, \leq_P)$.

We show that the function $\hat{f}_\varphi$ given by Lemma 3.16 satisfies $\hat{f}_\varphi^{-1}\big[\mathcal{A}_{\mathsf{Q}}\big] = \mathcal{A}_{\mathsf{P}}$. First, for $\hat{f}_\varphi$ to exist, we need f_φ to be order-preserving. Let $F, G \in \mathcal{P}_{<\omega}(\omega)$ be such that $F \subseteq G$. We have several cases to check:

1. if $G \notin \mathcal{C}(\mathcal{A}_{\mathsf{P}})$, then $f_\varphi(F) \subseteq f_\varphi(G) = \omega$.

Since $G \in \mathcal{C}(\mathcal{A}_{\mathsf{P}})$ implies $F \in \mathcal{C}(\mathcal{A}_{\mathsf{P}})$, we now suppose $F, G \in \mathcal{C}(\mathcal{A}_{\mathsf{P}})$ and thus $s_F \leq_P s_G$.

2. if $\mathrm{c}_P(s_F) = \mathrm{c}_P(s_G) = 0$, then $f_\varphi(F) = l_Q\big(\varphi(s_F)\big) \subseteq l_Q\big(\varphi(s_G)\big) = f_\varphi(G)$,
3. if $\mathrm{c}_P(s_F) = 0$ and $\mathrm{c}_P(s_G) = 1$, then $f_\varphi(F) = l_Q\big(\varphi(s_F)\big) \subseteq l_Q\big(\varphi(s_G^-)\big) \subseteq f_\varphi(G)$,
4. if $\mathrm{c}_P(s_F) = 1$ and $\mathrm{c}_P(s_G) = 0$, then $f_\varphi(F) \subseteq l_Q\big(\varphi(s_F^+)\big) \subseteq l_Q\big(\varphi(s_G)\big) = f_\varphi(G)$,
5. if $\mathrm{c}_P(s_F) = \mathrm{c}_P(s_G) = 1$ and $s_F \neq s_G$, then there exists $p \in P$ such that $s_F <_P p <_P s_G$ holds, because there exist no two consecutive nodes colored by 1. Therefore $f_\varphi(F) \subseteq l_Q\big(\varphi(s_F^+)\big) \subseteq l_Q\big(\varphi(s_G^-)\big) = f_\varphi(G)$.

It only remains to consider the cases where $\mathrm{c}_P(s_F) = \mathrm{c}_P(s_G) = 1$, and $s_F = s_G$:

6. if $F, G \in \mathcal{A}_\mathsf{P}$, then $f_\varphi(F) = l_Q\big(\varphi(s_F)\big) = l_Q\big(\varphi(s_G)\big) = f_\varphi(G)$,
7. if $F \in \mathcal{A}_\mathsf{P}$ and $G \notin \mathcal{A}_\mathsf{P}$, then $f_\varphi(F) = l_Q\big(\varphi(s_F)\big) \subseteq l_Q\big(\varphi(s_F^+)\big) = f_\varphi(G)$,
8. if $F \notin \mathcal{A}_\mathsf{P}$ and $G \in \mathcal{A}_\mathsf{P}$, then $f_\varphi(F) = l_Q\big(\varphi(s_F^-)\big) \subseteq l_Q\big(\varphi(s_F)\big) = f_\varphi(G)$,
9. if $F, G \notin \mathcal{A}_\mathsf{P}$ and $F \subsetneq l_P(s_F)$, then $f_\varphi(F) = l_Q\big(\varphi(s_F^-)\big) \subseteq f_\varphi(G)$,
10. if $F, G \notin \mathcal{A}_\mathsf{P}$ and $F \nsubseteq l_P(s_F)$, then $f_\varphi(F) = l_Q\big(\varphi(s_F^+)\big) = f_\varphi(G)$.

This finishes the proof that $f_\varphi : \mathcal{P}_{<\omega}(\omega) \to \mathcal{P}\omega$ is order-preserving. It follows from Lemma 3.16, that f_φ has a continuous extension $\hat{f}_\varphi : \mathcal{P}\omega \to \mathcal{P}\omega$. We distinguish between three different cases to show that $\hat{f}_\varphi^{-1}\big[\mathcal{A}_\mathsf{Q}\big] = \mathcal{A}_\mathsf{P}$.

$x \in \mathcal{P}_\omega(\omega)$: because $\mathcal{A}_\mathsf{P} \subseteq \mathcal{P}_{<\omega}(\omega)$, we have $x \notin \mathcal{A}_\mathsf{P}$. Suppose, towards a contradiction, that $\hat{f}_\varphi(x) \in \mathcal{A}_\mathsf{Q}$. Since $\mathcal{A}_\mathsf{Q} \subseteq \mathcal{P}_{<\omega}(\omega)$, there exist $F \in \mathcal{P}_{<\omega}(\omega)$ and $n \in \omega$, such that $\hat{f}_\varphi(x) = F \in \mathcal{A}_\mathsf{Q}$ and $f_\varphi\big(x \cap m\big) = F$ both hold for all $m \geq n$. We then notice that, for any $G \in \mathcal{P}_{<\omega}(\omega)$,

$$\begin{aligned} f_\varphi(G) \in \mathcal{A}_\mathsf{Q} &\Rightarrow G \in \mathcal{C}(\mathcal{A}_\mathsf{P}) \wedge \mathrm{c}_P(s_G) = 1 \wedge G = l_P(s_G) \\ &\Rightarrow G \in \mathcal{A}_\mathsf{P}. \end{aligned}$$

Where the first implication comes from the definition of f_φ and the second from Lemma 3.15. We obtain that $x \cap m \in \mathcal{A}_\mathsf{P}$ holds for all $m \geq n$, this implies $\mathrm{c}_P\left(l_P^{-1}(x \cap m)\right) = 1$. Since x is infinite and l_P injective, we can extract a subsequence of $\left(l_P^{-1}(x \cap m)\right)_{m \in \omega}$ witnessing $\omega \xrightarrow{\text{1-1 h.}} P$, a contradiction.

$F \in \mathcal{P}_{<\omega}(\omega) \setminus \mathcal{C}(\mathcal{A}_\mathsf{P})$: $F \notin \mathcal{A}_\mathsf{P}$ holds by the very definition of $\mathcal{C}(\mathcal{A}_\mathsf{P})$. Hence, we have $\omega = f_\varphi(F) = \hat{f}_\varphi(F) \notin \mathcal{A}_\mathsf{Q}$.

$F \in \mathcal{C}(\mathcal{A}_\mathsf{P})$: Suppose first that $F \in \mathcal{A}_\mathsf{P}$. By Lemma 3.15, $\hat{f}_\varphi(F) = l_Q\big(\varphi(s_F)\big)$ is satisfied. Moreover, from $\mathrm{c}_Q\left(\varphi(s_F)\right) = 1$, we get $\hat{f}_\varphi(F) \in \mathcal{A}_\mathsf{Q}$. Suppose now that $F \notin \mathcal{A}_\mathsf{P}$. By Lemma 3.15, there are three cases:

1. if $c_P(s_F) = 0$, then $c_Q\left(\varphi(s_F)\right) = 0$ which implies $\hat{f}_\varphi(F) \notin \mathcal{A}_{\mathsf{Q}}$,
2. if $c_P(s_F) = 1$ and $F \subsetneq l_P(s_F)$, then $c_Q\left(\varphi(s_F^-)\right) = 0$ which implies $\hat{f}_\varphi(F) \notin \mathcal{A}_{\mathsf{Q}}$,
3. if $c_P(s_F) = 1$ and $F \nsubseteq l_P(s_F)$, then $c_Q\left(\varphi(s_F^+)\right) = 0$ which implies $\hat{f}_\varphi(F) \notin \mathcal{A}_{\mathsf{Q}}$.

$\square_{\text{Claim}}$

Claim 3.20. *If* $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\text{emb}}$ *and* $\mathcal{A}_{\mathsf{P}} \leq_w \mathcal{A}_{\mathsf{Q}}$*, then* $\mathsf{P} \preccurlyeq_c \mathsf{Q}$.

Proof of the claim. We assume that $\mathcal{A}_{\mathsf{P}} \leq_w \mathcal{A}_{\mathsf{Q}}$ is witnessed by some continuous function $f : \mathcal{P}\omega \to \mathcal{P}\omega$. We describe a reduction which witnesses $\mathsf{P} \preccurlyeq_c \mathsf{Q}$. First, we need a few observations. Let $p \in P$. Since $\omega \overset{\text{1-1 h.}}{\not\to} P$ and all $\leq_P$-maximal elements have color 1, there exists $p' \in P$ such that both $p \leq_P p'$ and $c_P(p') = 1$ hold. Therefore, $f\big(l_P(p')\big) \in \mathcal{A}_{\mathsf{Q}}$. Hence, for all $p \in P$, we have $f\big(l_P(p)\big) \in \mathcal{C}(\mathcal{A}_{\mathsf{Q}})$. We also define, for all $p \in P$, the set

$$Q_p = \big\{q \in Q : l_Q(q) \subseteq f\big(l_P(p)\big)\big\}.$$

Since $f\big(l_P(p)\big) \in \mathcal{C}(\mathcal{A}_{\mathsf{Q}})$ holds, Lemma 3.15 yields the existence of a unique supremum t_p of Q_p in Q.
We define a mapping:

$$\varphi : P \to Q$$
$$p \mapsto \begin{cases} t_p & \text{if } f\big(l_P(p)\big) \in \mathcal{A}_{\mathsf{Q}}, \\ t_p & \text{if } f\big(l_P(p)\big) \notin \mathcal{A}_{\mathsf{Q}} \wedge c_Q(t_p) = 0, \\ t_p^- & \text{if } f\big(l_P(p)\big) \notin \mathcal{A}_{\mathsf{Q}} \wedge c_Q(t_p) = 1 \wedge l_Q(t_p) \subsetneq f\big(l_P(p)\big), \\ t_p^+ & \text{if } f\big(l_P(p)\big) \notin \mathcal{A}_{\mathsf{Q}} \wedge c_Q(t_p) = 1 \wedge l_Q(t_p) \nsubseteq f\big(l_P(p)\big), \end{cases}$$

where t_p^- and t_p^+ are defined as in Notation 3.12.

For φ to be well-defined, we need t_p^+ not to occur whenever t_p is a $\leq_Q$-maximal element. So, suppose t_p is a $\leq_Q$-maximal element. Since $c_Q(t_p) = 1$, then $t_p \in Q_p$ for it has a unique immediate predecessor. Thus, $l_Q(t_p) \subseteq f\big(l_P(p)\big)$ holds, which shows that t_p^+ does not occur in this case.
Since for every $p \in P$ we have

$$c_P(p) = 1 \Leftrightarrow l_P(p) \in \mathcal{A}_{\mathsf{P}} \Leftrightarrow f\big(l_P(p)\big) \in \mathcal{A}_{\mathsf{Q}},$$

it follows from the definition of φ, that for all $p \in P$ we also have $c_P(p) = c_Q(\varphi(p))$. Therefore, it only remains to show that φ is order-preserving. Suppose $p \leq_P p'$, we get $t_p \leq_Q t_{p'}$. We proceed with cases:

1. if $c_Q(t_p) = c_Q(t_{p'}) = 0$, then $\varphi(p) = t_p \leq_Q t_{p'} = \varphi(p')$,
2. if $c_Q(t_p) = 0$ and $c_Q(t_{p'}) = 1$, then $\varphi(p) = t_p \leq_Q t^-_{p'} \leq_Q \varphi(p')$,
3. if $c_Q(t_p) = 1$ and $c_Q(t_{p'}) = 0$, then $\varphi(p) \leq_Q t^+_p \leq_Q t_{p'} = \varphi(p')$,
4. if $c_Q(t_p) = c_Q(t_{p'}) = 1$ and $t_p \neq t_{p'}$, then there exists some $q \in Q$ that satisfies $t_p <_Q q <_Q t_{p'}$. This finally leads to $\varphi(p) \leq_Q t^+_p \leq_Q t^-_{p'} = \varphi(p')$.

It only remains to consider the cases where $c_Q(t_p) = c_Q(t_{p'}) = 1$, and $t_p = t_{p'}$:

5. if $c_P(p) = c_P(p') = 1$, then $\varphi(p) = t_p = t_{p'} = \varphi(p')$,
6. if $c_P(p) = 1$ and $c_P(p') = 0$, then $\varphi(p) = t_p \leq_Q t^+_p = \varphi(p')$,
7. if $c_P(p) = 0$ and $c_P(p') = 1$, then $\varphi(p) = t^-_p \leq_Q t_p = \varphi(p')$,
8. if $c_P(p) = c_P(p') = 0$ and $l_Q(t_p) \subsetneq f\big(l_P(p)\big)$, then $\varphi(p) = t^-_p \leq_Q \varphi(p')$,
9. if $c_P(p) = c_P(p') = 0$ and $l_Q(t_p) \nsubseteq f\big(l_P(p)\big)$, then $\varphi(p) = t^+_p = \varphi(p')$.

This concludes the proof that φ witnesses $\mathsf{P} \preccurlyeq_c \mathsf{Q}$. $\square_{\text{Claim}}$

So, Claim 3.18 proves that the mapping $H : \mathsf{P} \mapsto \mathcal{A}_\mathsf{P}$ is a well-defined mapping from $(\mathbb{P}_{\text{emb}}, \preccurlyeq_c)$ to $(\boldsymbol{\Delta}^0_2(\mathcal{P}\omega), \leq_w)$, and we conclude from the Claims 3.19 and 3.20 that for any $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\text{emb}}$, $\mathsf{P} \preccurlyeq_c \mathsf{Q}$ if and only if $\mathcal{A}_\mathsf{P} \leq_w \mathcal{A}_\mathsf{Q}$. $\square$

The previous lemma immediately yields the main result.

Theorem 3.21. *The following mapping is an order-embedding:*

$$\begin{aligned}(\mathbb{D}\left(\mathbb{P}_{\text{emb}}\right), \preccurlyeq_c) &\to \left(\mathbb{WD}_{\boldsymbol{\Delta}^0_2}\left(\mathcal{P}\omega\right), \leq_w\right) \\ [\mathsf{P}]_c &\mapsto [\mathcal{A}_\mathsf{P}]_w\,.\end{aligned}$$

3.4 A reduction game on $\mathbb{P}$

This section introduces a game characterization of reductions on 2-colored posets. This characterization and the order-embedding given in Theorem 3.21 are the essential tools that we need in order to study the Wadge order on the Scott domain.

As pointed out by the anonymous referee of [DV20], this game-theoretical approach is not entirely needed in order to obtain the main results of the article (as suggested by Proposition 3.24). However, this version of the Ehrenfeucht-Fraïssé game [Hod93] that we use captures the dynamic viewpoint that was essential — at least for the authors — in obtaining Theorems 3.31 and 3.35.

This game comes as a standard two-player infinite game where the players choose elements of some given posets $\mathsf{P}, \mathsf{Q} \in \mathbb{P}$.

Definition 3.22. Let $\mathsf{P}, \mathsf{Q} \in \mathbb{P}$. The poset game $G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q})$ is defined as a two-player — I and II — game played on ω rounds. At round $2n$, I picks an element $p_n \in P$ and, at round $2n+1$, II picks an element $q_n \in Q$. We further require that there exists $n_0 \in \omega$ such that, for all $n \geq n_0$, $p_n = p_{n_0}$.

We say that II wins the game if and only if the two following conditions are satisfied:

1. $p_n \leq_P p_m \to q_n \leq_Q q_m$ holds for all $n, m \in \omega$,
2. $\mathrm{c}_P(p_n) = \mathrm{c}_Q(q_n)$ for all $n \in \omega$.

Schematically, the game goes as in Figure 3.4.

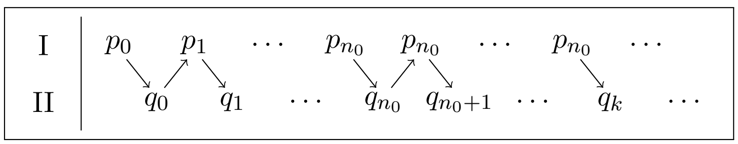


Figure 3.4: A run of the poset game $G_{\mathbb{P}}(P, Q)$ for $\mathsf{P}, \mathsf{Q} \in \mathbb{P}$.

In plain English, I moves inside the 2-colored poset P, whereas II moves inside the 2-colored poset Q. The goal of II is to reproduce (orderwise and colorwise) in Q the run that I is producing in P. Notice that the condition of playing ultimately constant for I is equivalent to requiring that the game stops after finitely many rounds. Related to this game, we introduce the notion of an *ultrapositional strategy* as a strengthening of the usual notion of a strategy.

Definition 3.23. Let $\mathsf{P}, \mathsf{Q} \in \mathbb{P}$. An ultrapositional strategy for II in the game $G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q})$ is a function $\tau : P \to Q$.

Contrary to the usual strategies that rely on the history of the opponent's run, ultrapositional strategies only take into account the last move of the opponent. An ultrapositional strategy is winning if it ensures a win whatever the opponent does. Ultrapositional strategies characterize the reductions inside $\mathbb{P}$ as shown by the next proposition.

Proposition 3.24. *Let* $\mathsf{P}, \mathsf{Q} \in \mathbb{P}$*.*

$$\mathsf{P} \preccurlyeq_c \mathsf{Q} \iff \text{II } \textit{has an ultrapositional winning strategy in } G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q}).$$

Proof. First, suppose that $\mathsf{P} \preccurlyeq_c \mathsf{Q}$ holds and is witnessed by $\varphi : P \to Q$. Observe that φ is also an ultrapositional strategy for II in $G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q})$. From the very definition of a homomorphism between 2-colored posets, it respects the two conditions to be winning for II in $G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q})$.

Conversely, an ultrapositional winning strategy for II in $G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q})$ is a homomorphism $\varphi : P \to Q$ for it respects the two winning conditions. $\square$

We have a reduction between 2-colored posets and their subposets that are closed under the predecessor relation.

Definition 3.25. Let Q be a poset. A subposet $P \subseteq Q$ is an ideal of Q if, for all $p \in P$, we have $\{q \in Q : q \leq_Q p\} \subseteq P$.

Proposition 3.26. *Let* $\mathsf{P}, \mathsf{Q} \in \mathbb{P}$.

$$\textit{If } P \textit{ is an ideal of } Q\textit{, then } \mathsf{P} \preccurlyeq_c \mathsf{Q}.$$

Proof. The inclusion $i : P \to Q,\ p \mapsto p$ witnessing that P is an ideal of Q is an ultrapositional winning strategy for II in $G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q})$. $\square$

3.4.1 On the reduction game on $\mathbb{P}_{\mathbf{fin}}$

In order to simplify some later proofs, we conclude this section with some necessary conditions for an ultrapositional strategy to be winning in a subclass of the embeddable posets.

Definition 3.27. A finite branching poset is an embeddable poset $\mathsf{P} \in \mathbb{P}_{\mathrm{emb}}$ such that every element $p \in P$ which is not $\leq_P$-minimal has finitely many successors, i.e., for all $p \in P$, if $p \neq \bot$, then:

$$\mathrm{Card}\left(\mathrm{Succ}(p)\right) = \mathrm{Card}\left(\{p' \in P : p \leq_P p'\}\right) < \aleph_0.$$

The class of all finite branching posets is denoted by $\mathbb{P}_{\mathrm{fin}}$.

It turns out that the image of a finitely branching poset via the order-embedding of Theorem 3.21 must be topologically reasonably simple.

Proposition 3.28. *If* $\mathsf{P} \in \mathbb{P}_{\mathrm{fin}}$*, then* $\mathcal{A}_{\mathsf{P}} \in D_\omega(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega)$.

Proof. We use the characterization of Corollary 3.7. Since $\mathsf{P} \in \mathbb{P}_{\mathrm{emb}}$ holds, Lemma 3.17 implies that $\mathcal{A}_{\mathsf{P}} \in \mathbf{\Delta}^0_2(\mathcal{P}\omega)$ holds as well. Towards a contradiction, assume that $\mathcal{A}_{\mathsf{P}}$ admits a 1-alternating tree of rank ω, namely:

$$f : T_\omega \to \mathcal{P}_{<\omega}(\omega).$$

This implies that, for every $k \in \omega$, there exists a strictly $\subseteq$-increasing sequence $(F^k_m)_{m<k}$ such that $F^k_0 = f(\langle\rangle)$ and $F^k_m \in \mathcal{A}_{\mathsf{P}}$ both hold for all $m < k$.

Thus, the sequence $\left(l_P^{-1}\big(F_m^k\big)\right)_{l<k}$ is a strictly $\leq_P$-increasing sequence of size k that satisfies

$$\mathrm{c}_P\left(l_P^{-1}\big(F_0^k\big)\right) = \mathrm{c}_P\left(l_P^{-1}\big(f(\langle\rangle)\big)\right) = 1,$$

for every $k \in \omega$. Therefore, we obtain

$$\mathrm{Card}\left(\mathrm{Succ}\left(l_P^{-1}\big(f(\langle\rangle)\big)\right)\right) = \aleph_0.$$

By definition of a finite branching poset, this implies $l_P^{-1}\big(f(\langle\rangle)\big) = \bot$, a contradiction for $\mathrm{c}_P(\bot) = 0$. □

As a corollary, we obtain a somehow more detailed picture of Theorem 3.21.

Corollary 3.29. *The following mapping is an order-embedding:*

$$H : (\mathbb{D}(\mathbb{P}_{\mathrm{fin}}), \preccurlyeq_c) \to \left(\mathbb{WD}_{D_\omega(\mathbf{\Sigma}_1^0)}(\mathcal{P}\omega), \leq_w\right)$$
$$[\mathsf{P}]_c \mapsto [\mathcal{A}_{\mathsf{P}}]_w .$$

Now, we introduce some notations to talk about the game-theoretical strength of a given node in a finite branching poset. Let us fix $\mathsf{P} \in \mathbb{P}_{\mathrm{fin}}$ and $p \in P$. If it exists, let $k_p \in \omega$ be the length of the largest strictly $\leq_P$-increasing sequence $(s_n)_{n<k_p}$ that satisfies $s_0 = p$ and

$$(\mathrm{c}_P(s_n) = \mathrm{c}_P(p) \Leftrightarrow n \text{ is even}).$$

The increasing strength of p in P is

$$\mathrm{Str}_{\mathrm{incr}}(p) = \begin{cases} k_p & \text{if } k_p \in \omega \text{ exists,} \\ \omega & \text{otherwise.} \end{cases}$$

Since $\mathsf{P} \in \mathbb{P}_{\mathrm{fin}}$, the latter case can only occur when $p = \bot$. From a game-theoretical viewpoint, if $p \neq \bot$, then $\mathrm{Str}_{\mathrm{incr}}(p)$ corresponds to the length of the strongest $<_P$-increasing run that a player can take while playing in P.

In a similar manner, we define the decreasing strength of p in P, denoted by $\mathrm{Str}_{\mathrm{decr}}(p) = k \in \omega$, as the length of the largest strictly $\leq_P$-decreasing sequence $(s_n)_{n<k}$ that satisfies $s_0 = p$ and

$$(\mathrm{c}_P(s_n) = \mathrm{c}_P(p) \Leftrightarrow n \text{ is even}).$$

It is well-defined since $\mathrm{Card}(\mathrm{Pred}(p)) < \aleph_0$ holds for every $p \in P$.

The increasing and decreasing strengths of a node are a good indicator of the strength it bears as a position in the game:

Lemma 3.30. *If* $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\text{fin}}$ *and* τ *is a winning ultrapositional strategy for* II *in the game* $G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q})$*, then for all* $p \in P$ *:*

1. $\mathrm{Str}_{\text{incr}}(p) \leq \mathrm{Str}_{\text{incr}}\big(\tau(p)\big)$,
2. $\mathrm{Str}_{\text{decr}}(p) \leq \mathrm{Str}_{\text{decr}}\big(\tau(p)\big)$.

Proof.

1. Towards a contradiction, suppose that $\mathrm{Str}_{\text{incr}}(p) > \mathrm{Str}_{\text{incr}}\big(\tau(p)\big)$. We proceed by cases.

   If $\mathrm{Str}_{\text{incr}}(p) \neq \omega$: assume that $\mathrm{Str}_{\text{incr}}(p) = k$ is witnessed by a sequence $(p_n)_{n<k}$. Since τ is winning, $\big(\tau(p_n)\big)_{n<k}$ is strictly $\leq_Q$-increasing and satisfies $\tau(p_0) = \tau(p)$ and

   $$\Big(\, \mathrm{c}_Q\big(\tau(p_n)\big) = \mathrm{c}_Q\big(\tau(p)\big) \Leftrightarrow n \text{ is even}\Big).$$

   Thus $\mathrm{Str}_{\text{incr}}\big(\tau(p)\big) \geq k$, a contradiction.

   If $\mathrm{Str}_{\text{incr}}(p) = \omega$: for all $k \in \omega$, there exists a strictly $\leq_P$-increasing sequence $(s_n)_{n<k}$ that satisfies $s_0 = p$ and

   $$(\mathrm{c}_P(s_n) = \mathrm{c}_P(p) \Leftrightarrow n \text{ is even}).$$

   Since τ is winning, $\big(\tau(p_n)\big)_{n<k}$ is strictly $\leq_Q$-increasing and satisfies $\tau(p_0) = \tau(p)$ and

   $$\Big(\, \mathrm{c}_Q\big(\tau(p_n)\big) = \mathrm{c}_Q\big(\tau(p)\big) \Leftrightarrow n \text{ is even}\Big).$$

   Therefore, $\mathrm{Str}_{\text{incr}}\big(\tau(p)\big) = \omega$, a contradiction.

2. Towards a contradiction, suppose that $\mathrm{Str}_{\text{decr}}(p) > \mathrm{Str}_{\text{decr}}\big(\tau(p)\big)$. We also suppose that $\mathrm{Str}_{\text{decr}}(p) = k \in \omega$ is witnessed by a sequence $(p_n)_{n<k}$. Since τ is winning, $\big(\tau(p_n)\big)_{n<k}$ is strictly $\leq_Q$-decreasing and satisfies $\tau(p_0) = \tau(p)$ and $\Big(\, \mathrm{c}_Q\big(\tau(p_n)\big) = \mathrm{c}_Q\big(\tau(p)\big) \Leftrightarrow n \text{ is even}\Big)$. Thus $\mathrm{Str}_{\text{decr}}\big(\tau(p)\big) \geq k$, a contradiction.

□

3.5 Ill-foundedness of the Wadge order on the Scott domain

In this section, we prove that the quasi-order $\leq_w$ is already ill-founded within the class of ω-differences of open sets of the Scott domain.

Theorem 3.31. *The quasi-order* $\big(D_\omega(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega), \leq_w\big)$ *is ill-founded.*

Proof. The proof consists in exhibiting a strictly $\preccurlyeq_c$-decreasing sequence of posets $(\mathsf{P}_n)_{n\in\omega^+}$ in $\mathbb{P}_{\text{fin}}$ and making use of Lemma 3.17.

First, let us fix $n \in \omega^+$. We define $\mathsf{P}_n = (P_n, \leq_{P_n}, \mathrm{c}_{P_n})$ as the 2-colored countable poset with colored Hasse diagram given in Figure 3.5.

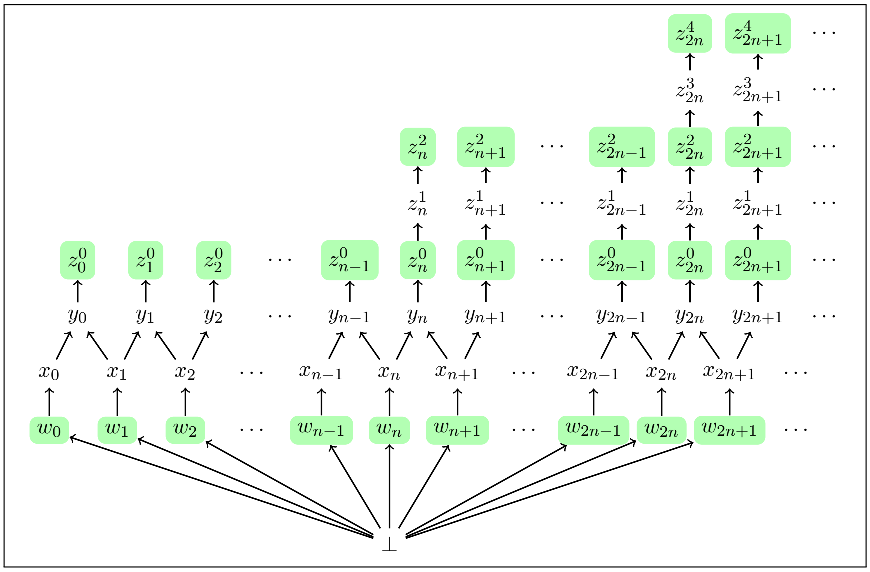


Figure 3.5: The colored Hasse diagram of $\mathsf{P}_n \in \mathbb{P}_{\text{emb}}$ for $n \in \omega^+$.

Formally, the set of nodes is:

$$\begin{aligned} P_n &= \{\perp\} \cup \{w_m, x_m, y_m\}_{m\in\omega} \\ &\quad \cup \big\{z^{2k}_m : k\in\omega, n \geq km\big\} \cup \big\{z^{2k+1}_m : k \in \omega, n \geq (k+1)m\big\}, \end{aligned}$$

the order relation is:

$$\begin{aligned} \leq_{P_n} &= \big\{(\perp, w_m), (w_m, x_m), (x_m, y_m), (x_{m+1}, y_m), (y_m, z^0_m)\big\}_{m\in\omega} \\ &\quad \cup \left\{(z^k_m, z^{k+1}_m) : k \leq \left\lfloor \frac{m}{n} \right\rfloor \cdot 2 - 1\right\}, \end{aligned}$$

where $\left\lfloor \frac{m}{n} \right\rfloor$ denotes the integer part of $\frac{m}{n}$, and the 2-coloring is:

$$\begin{aligned} \mathrm{c}_{P_n} &: P_n \to 2 \\ &p \mapsto 0 \quad \text{if } p \in \{\perp, x_m, y_m\}_{m\in\omega} \cup \bigcup_{m\in\omega} z^{\text{odd}}_m, \\ &p \mapsto 1 \quad \text{if } p \in \{w_m\}_{m\in\omega} \cup \bigcup_{m\in\omega} z^{\text{even}}_m, \end{aligned}$$

where $z_m^{\cdot} = \{z_m^k : k \leq \lfloor \frac{m}{n} \rfloor \cdot 2\}$, $z_m^{\text{even}} = \{z_m^k \in z_k^{\cdot} : k \text{ even}\}$, and $z_m^{\text{odd}} = \{z_m^k \in z_k^{\cdot} : k \text{ odd}\}$.

For all $n \in \omega^+$, it is easy to check that all the requirements that are needed for P_n to belong to $\mathbb{P}_{\text{fin}}$ are fulfilled. Therefore, by Proposition 3.28, we have:

$$\mathcal{A}_{\mathsf{P}_n} \in D_\omega(\mathbf{\Sigma}_1^0)(\mathcal{P}\omega).$$

For the remainder of the proof, we need some notations. For any $k \in \omega$, we call *branch k of P_n* the set of nodes $B_k = \{w_k, x_k, y_k\} \cup z_k^{\cdot}$, and *right-shift in P_n* any sequence of moves of the form (w_k, y_k, w_{k+1}). First, we describe the behavior of an ultrapositional winning strategy facing a right-shift.

Claim 3.32. *Let $n, m \in \omega^+$ and τ be an ultrapositional strategy for* II *in $G_{\mathbb{P}}(\mathsf{P}_n, \mathsf{P}_m)$. If* I*'s moves are a right-shift (w_k, y_k, w_{k+1}) and $\tau(w_k) \in B_l$ for some $l \in \omega$, then $\tau(w_{k+1}) \in B_{l'}$ for some $l' \leq l+1$.*

Proof of the Claim. We split the proof in two different cases.

If $l = 0$ holds: since $w_k \leq_{P_n} y_k$, $\mathrm{c}_{P_n}(y_k) = 0$, τ is winning and $\tau(w_k) \in B_0$, we get $\tau(y_k) \in \{x_0, y_0\}$. Moreover, since $w_{k+1} \leq_{P_n} y_k$, $\mathrm{c}_{P_n}(w_{k+1}) = 1$ and τ is winning, we get:

$$\tau(w_{k+1}) \in \{w_0, w_1\} \subseteq B_0 \cup B_1.$$

If $l \in \omega^+$ holds: once again, since $w_k \leq_{P_n} y_k$, $\mathrm{c}_{P_n}(y_k) = 0$, τ is winning and $\tau(w_k) \in B_l$, we get $\tau(y_k) \in z_{l-1}^{\text{odd}} \cup z_l^{\text{odd}} \cup \{x_l, y_l, y_{l-1}\}$. Moreover, since $w_{k+1} \leq_{P_n} y_k$, $\mathrm{c}_{P_n}(w_{k+1}) = 1$ and τ is winning, we get:

$$\tau(w_{k+1}) \in z_{l-1}^{\text{even}} \cup z_l^{\text{even}} \cup \{w_{l-1}, w_l, w_{l+1}\} \subseteq \bigcup_{l' \leq l+1} B_{l'}.$$

$\square_{\text{Claim}}$

It remains to show that the sequence $(\mathsf{P}_n)_{n \in \omega_+}$ is an infinite strictly $\preccurlyeq_c$-decreasing sequence in $\mathbb{P}_{\text{fin}}$. For this purpose, we prove the two followiong claims.

Claim 3.33. *If $0 < n < m < \omega$, then $\mathsf{P}_m \preccurlyeq_c \mathsf{P}_n$.*

Proof of the Claim. It suffices to observe that P_m is an ideal of P_n and use Proposition 3.26. $\square_{\text{Claim}}$

Claim 3.34. *If* $0 < n < m < \omega$*, then* $\mathsf{P}_n \not\preccurlyeq_c \mathsf{P}_m$.

Proof of the Claim. Towards a contradiction, suppose that $\mathsf{P}_n \preccurlyeq_c \mathsf{P}_m$ holds. By Proposition 3.24, II has a winning ultrapositional strategy τ in the game $G_{\mathbb{P}}(\mathsf{P}_n, \mathsf{P}_m)$.

The idea of the proof is to construct a particular run of the game that τ cannot win. By Claim 3.32, if I plays a sequence of the form $(w_0, y_0, w_1, y_1, w_2, \dots)$ composed with right-shifts, then II's moves are limited. In particular, whenever I shifts from B_k to B_{k+1}, II can only shift from B_l to $B_{l'}$ where $l' \leq l + 1$. Because $n < m$, I can finally reach a node of greater increasing strength than the one reached by II, which leads to a contradiction.

More formally, suppose that I's first move is w_0 so that $\tau(w_0) \in B_{k_0}$ for some $k_0 \in \omega$, and that I plays a run composed with several right-shifts

$$(w_0, y_0, w_1, y_1, w_2, \dots, w_l).$$

By an iteration of Claim 3.32, we get $\tau(w_l) \in B_{l'}$ for some $l' \leq k_0 + l$. Since $n < m$, there exists $n_0 \in \omega$ such that the following inequalities work:

$$\mathrm{Str}_{\mathrm{incr}}(w_{nmn_0}) = 2mn_0 + 3 > 2nn_0 + \mathrm{Str}_{\mathrm{incr}}(w_{k_0}) \geq \mathrm{Str}_{\mathrm{incr}}\left(\tau(w_{nmn_0})\right),$$

which is a contradiction to Lemma 3.30. $\square_{\text{Claim}}$

So, we constructed an infinite strictly $\preccurlyeq_c$-decreasing sequence of embeddable posets, namely

$$\mathsf{P}_1 \succ_c \mathsf{P}_2 \succ_c \mathsf{P}_3 \succ_c \mathsf{P}_4 \succ_c \dots$$

By Lemma 3.17, we obtain an infinite strictly $\leq_w$-decreasing sequence of subsets of $\mathcal{P}\omega$, namely:

$$\mathcal{A}_{\mathsf{P}_1} >_w \mathcal{A}_{\mathsf{P}_2} >_w \mathcal{A}_{\mathsf{P}_3} >_w \mathcal{A}_{\mathsf{P}_4} >_w \dots$$

which were also proved to be differences of ω open sets. $\square$

3.6 Antichains in the Wadge order on the Scott domain

We prove that infinite $\leq_w$-antichains already exist within the class of ω-differences of open subsets of the Scott domain. The proof is nothing but a tailoring of the proof of Theorem 3.31.

Theorem 3.35. *The quasi-order* $\big(D_\omega(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega), \leq_w\big)$ *has infinite antichains.*

Proof. We construct an infinite sequence of embeddable posets $(\mathsf{Q}_n)_{n\in\omega_+}$ that are pairwise $\preccurlyeq_c$-incomparable.

We fix $n \in \omega^+$ and define $\mathsf{Q}_n = (Q_n, \leq_{Q_n}, \mathrm{c}_{Q_n})$ as the 2-colored countable poset with the colored Hasse diagram given in Figure 3.6.

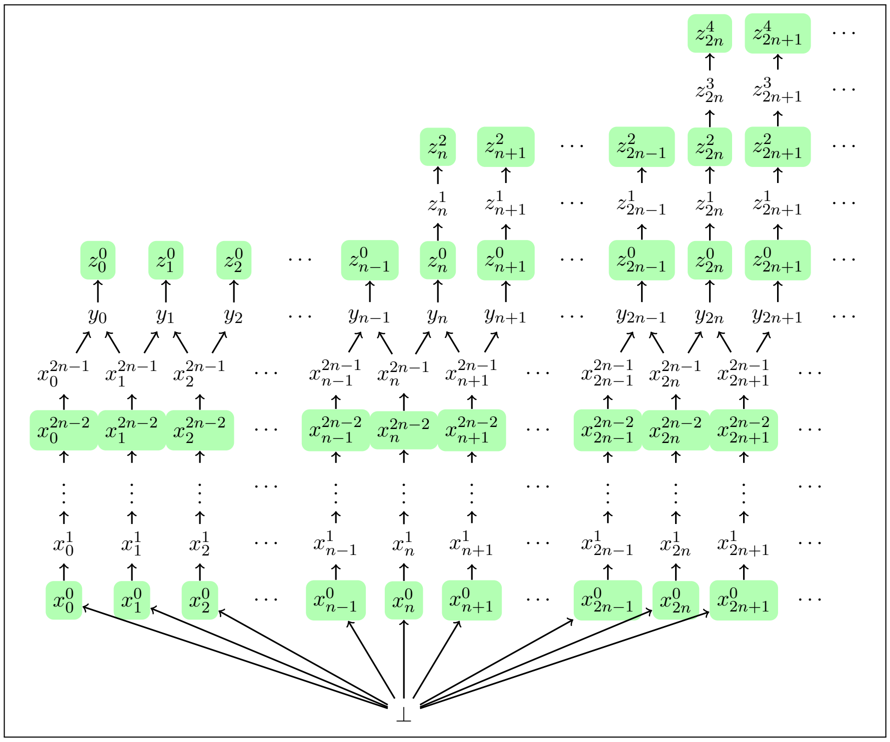


Figure 3.6: The colored Hasse diagram of $\mathsf{Q}_n \in \mathbb{P}_{\mathrm{emb}}$ for $n \in \omega^+$.

Formally, the set of nodes is:

$$\begin{aligned} Q_n = {} & \{\bot\} \cup \{x^k_m, y_m\}_{m\in\omega, k<2n} \\ & \cup \big\{z^{2k}_m : k \in \omega, n \geq km\big\} \cup \big\{z^{2k+1}_m : k \in \omega, n \geq (k+1)m\big\}, \end{aligned}$$

the order relation is:

$$\begin{aligned} \leq_{Q_n} = {} & \big\{(\bot, x^0_m), (x^k_m, x^{k+1}_m), (x^{2n-1}_m, y_m), (x^{2n-1}_{m+1}, y_m), (y_m, z^0_m)\big\}_{m\in\omega, k<2n-1} \\ & \cup \left\{(z^k_m, z^{k+1}_m) : k \leq \left\lfloor \frac{m}{n} \right\rfloor \cdot 2 - 1\right\}, \end{aligned}$$

and the coloring is given by the function:

$$\begin{aligned} c_{Q_n} : Q_n &\to 2 \\ p &\mapsto 0 \quad \text{if } p \in \{\bot, x_m^{2k+1}, y_m\}_{m\in\omega, k<n} \cup \bigcup_{m\in\omega} z_m^{\text{odd}}, \\ p &\mapsto 1 \quad \text{if } p \in \{2_m^{2k}\}_{m\in\omega, k<n} \cup \bigcup_{m\in\omega} z_m^{\text{even}}. \end{aligned}$$

As in the proof of Theorem 3.31, it is easy to see that $\mathsf{Q}_n \in \mathbb{P}_{\text{fin}}$, and thus $\mathcal{A}_{\mathsf{Q}_n} \in D_\omega(\mathbf{\Sigma}_1^0)(\mathcal{P}\omega)$ holds for every $n \in \omega^+$. Now, it remains to show that $(\mathsf{Q}_n)_{n\in\omega+}$ is a sequence of pairwise $\preccurlyeq_c$-incomparable embeddable posets. For this purpose, we define a *right-shift in* Q_n as any sequence of moves of the form $(x_k^{2n-2}, y_k, x_{k+1}^{2n-2})$ for some $k \in \omega$.

Claim 3.36. *If* $0 < n < m < \omega$*, then* $\mathsf{Q}_m \not\preccurlyeq_c \mathsf{Q}_n$.

Proof of the claim. Towards a contradiction, we assume that $\mathsf{Q}_m \preccurlyeq_c \mathsf{Q}_n$ holds. By Proposition 3.24, II has an ultrapositional winning strategy τ in the game $G_\mathbb{P}(\mathsf{Q}_m, \mathsf{Q}_n)$.

The idea of the proof is to exhibit some specific run for I in this game that τ cannot beat. For this purpose, I will use the fact that $n < m$ and several right-shifts to reach an element $q \in \mathsf{Q}_n$ which has a larger increasing strength than $\tau(q)$.

We consider x_0^{2m-2} as I's first move. If II's first move is x_i^{2j} for some $i \in \omega$ and $j < n$, then $\text{Str}_{\text{decr}}\left(x_0^{2m-2}\right) = 2m > 2n \geq \text{Str}_{\text{decr}}\left(x_i^{2j}\right)$, which contradicts Lemma 3.30. Since $c_{Q_m}(x_0^{2m-2}) = 1$, we can assume that $\tau\big(x_0^{2m-2}\big) = z_{l_0}^{2k}$ for some $k, l_0 \in \omega$.

If I's second move is y_0, then II's second move has color 0. Hence, II's second move is of the form $z_{l_0}^{2k'+1}$ for some $k' \in \omega$.

Since $\text{Str}_{\text{decr}}\left(x_1^{2m-2}\right) = 2m > 2n \geq \text{Str}_{\text{decr}}\left(x_i^{2j}\right)$ for all $j < n$, if I's third move is x_1^{2m-2}, then Lemma 3.30 implies that II's third move cannot be of the form x_i^{2j} for some $i, j \in \omega$. So, II's third move is of the form $z_{l_0}^{2k''}$ for some $k'' \in \omega$.

Now, consider the run where I plays right-shifts:

$$\big(x_0^{2m-2}, y_0, x_1^{2m-2}, y_1, x_2^{2m-2}, y_2, \dots\big).$$

By the previous observations, II will only play in $z_{l_0}^{\cdot}$. But there exists $i_0 \in \omega$ such that

$$\text{Str}_{\text{incr}}(y_{i_0}) > \max\{\text{Str}_{\text{incr}}(q) : q \in z_{l_0}^{\cdot}\},$$

which contradicts Lemma 3.30. $\square_{\text{Claim}}$

For the last two claims, we need to introduce the notion of *branches in* Q_n. For any $k \in \omega$, we call branch k of Q_n the set of nodes $B_k = \{x_k^l, y_k\}_{l<2n} \cup z_k^{\cdot}$. The next claim, which concerns the 2-colored countable posets of the form Q_n for some $n \in \omega^+$, is a tailoring of Claim 3.32.

Claim 3.37. *Let $n, m \in \omega^+$ and τ be an ultrapositional strategy for* II *in* $G_{\mathbb{P}}(\mathsf{Q}_n, \mathsf{Q}_m)$. *If* I*'s moves are a right-shift* $(x_k^{2n-2}, y_k, x_{k+1}^{2n-2})$ *and* $\tau\big(x_k^{2n-2}\big) \in B_l$ *holds for some* $l \in \omega$, *then* $\tau\big(x_{k+1}^{2n-2}\big) \in B_{l'}$ *holds for some* $l' \leq l+1$.

Proof of the claim. We proceed as in the proof of Claim 3.32, except that the right-shift (w_k, y_k, w_{k+1}) in P_n is replaced by the right-shift $(x_k^{2n-2}, y_k, x_{k+1}^{2n-2})$ in Q_n. $\square_{\text{Claim}}$

With the help of the previous claim, we finally obtain:

Claim 3.38. *If* $0 < n < m < \omega$, *then* $\mathsf{Q}_n \not\preccurlyeq_c \mathsf{Q}_m$.

Proof of the claim. We proceed as in the proof of Claim 3.34. Towards a contradiction, suppose that $\mathsf{Q}_n \preccurlyeq_c \mathsf{Q}_m$ holds. By Proposition 3.24, II has a winning ultrapositional strategy τ in the game $G_{\mathbb{P}}(\mathsf{Q}_n, \mathsf{Q}_m)$.

Suppose that I's first move is x_0^{2n-2} so that $\tau\big(x_0^{2n-2}\big) \in B_{k_0}$ for some $k_0 \in \omega$, and that I plays a run composed with several right-shifts

$$\big(x_0^{2n-2}, y_0, x_1^{2n-2}, y_1, x_2^{2n-2} \ldots, x_l^{2n-2}\big).$$

By an iteration of Claim 3.37, we get $\tau\big(x_l^{2n-2}\big) \in B_{l'}$ for some $l' \leq k_0 + l$. Since $n < m$, there exists $n_0 \in \omega$ such that the following inequalities work:

$$\mathrm{Str}_{\mathrm{incr}}\left(x_{nmn_0}^{2n-2}\right) = 2mn_0 + 3 > 2nn_0 + \mathrm{Str}_{\mathrm{incr}}\left(x_{k_0}^{0}\right) \geq \mathrm{Str}_{\mathrm{incr}}\Big(\tau\big(x_{nmn_0}^{2n-2}\big)\Big),$$

which contradicts Lemma 3.30. $\square_{\text{Claim}}$

So, we constructed an infinite sequence of pairwise $\preccurlyeq_c$-incomparable embeddable posets, namely $(\mathsf{Q}_n)_{n\in\omega^+}$. By Lemma 3.17, we obtain an infinite sequence of pairwise $\leq_w$-incomparable subsets of $\mathcal{P}\omega$, namely $\big(\mathcal{A}_{\mathsf{Q}_n}\big)_{n\in\omega^+}$. We also proved that all these sets are ω-differences of open sets. $\square$

As a consequence of Theorem 3.31 and Theorem 3.35, we obtain that the Wadge order on $\mathcal{P}\omega$ is both ill-founded and contains infinite antichains.

Theorem 3.39. *The poset*

$$\left(\mathbb{WD}_{D_w\left(\mathbf{\Sigma}^0_\omega\right)}\left(\mathcal{P}\omega\right), \leq_w\right)$$

is ill-founded and contains infinite antichains. In particular, the same holds for the Wadge order on the Borel subsets of $\mathcal{P}\omega$, i.e.,

$$\left(\mathbb{WD}_{\mathcal{B}}\left(\mathcal{P}\omega\right), \leq_w\right)$$

is ill-founded and contains infinite antichains.

The results of this chapter together with the results mentioned at the beginning of this chapter of Selivanov in [Sel05] and of Becher and Grigorieff in [BG15b] yield the following picture for the Wadge order on the ω-differences of open sets in the Scott domain $\mathcal{P}\omega$.

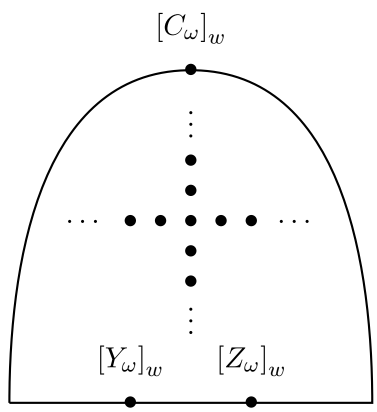


Figure 3.7: The Wadge order on $D_\omega\left(\mathbf{\Sigma}^0_1\right)\left(\mathcal{P}\omega\right)$.

In particular, among the proper ω differences of open sets, there exist two $\leq_w$-minimal Wadge degrees, a unique $\leq_w$-maximal Wadge degree, an infinite strictly $\leq_w$-increasing chain of Wadge degrees, an infinite strictly $\leq_w$-decreasing chain of Wadge degrees as well as an infinite antichain of Wadge degrees. In other words, we encounter a completely opposite situation to the one in the zero-dimensional Polish spaces where this level of the Hausdorff-Kuratowski difference hierarchy is composed of a single Wadge degree. Thus, the Wadge order on the Borel subsets of the Scott domain $\mathcal{P}\omega$ seems to look more like the one of the non-zero-dimensional Polish spaces. Indeed, Schlicht proved in [Sch18] that the Wadge order on the Borel subsets of such spaces contains an antichain of size $2^{\aleph_0}$ (Theorem 2.26).

Chapter 4

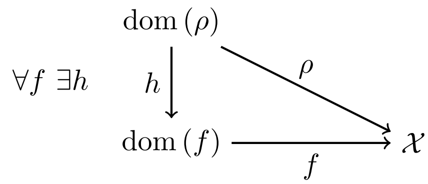
A Wadge hierarchy for countably based T_0-spaces

Computable analysis is the branch of mathematics that studies topological spaces through the lens of computability theory [Wei00]. Following this idea, this chapter introduces a new notion of reducibility $\preccurlyeq_w$ on the subsets of countably based T_0-spaces which naturally yields a hierarchy, i.e., a well-quasi-order, on the Borel subsets of any quasi-Polish space and thus avoids bad behaviors such as Theorem 3.39. It was first studied by Tang in [Tan79, Tan81] on $\mathcal{P}\omega$, then analyze in a more general setting by Selivanov in [Sel17b] and finally thoroughly investigated by Pequignot in [Peq15b]. We prove that this new notion of reducibility yields a partial order on the Borel subsets of the Scott domain $\mathcal{P}\omega$ which is isomorphic to the Wadge order on the Baire space ω^ω restricted to the non-self-dual Borel degrees.

4.1 Admissible representations

Computable analysis relies on encoding the points of a space in the Baire space ω^ω. If $\mathcal{X}$ is a countably based T_0-space, a representation of $\mathcal{X}$ is a partial surjective function $\rho : \subseteq \omega^\omega \to \mathcal{X}$. If $x \in \mathcal{X}$, any element $\alpha \in \mathrm{dom}\,(\rho)$ such that $\rho\,(\alpha) = x$ is a name of x. By definition, any element of $\mathcal{X}$ as at least one name — but potentially much more — in $\mathrm{dom}\,(\rho)$. Not all representations are interesting. For instance, we want representations to preserve the topological structure of $\mathcal{X}$, i.e., we want them to be continuous. An admissible representation is a continuous representation which is able to simulate any partial continuous function from ω^ω to $\mathcal{X}$ via some continuous modification of the names.

Definition 4.1 ([Wei00])**.** Let $\mathcal{X}$ be a countably based T_0-space. An admissible representation of $\mathcal{X}$ is a continuous representation $\rho : \subseteq \omega^\omega \to \mathcal{X}$ such that, for every partial continuous function $f : \subseteq \omega^\omega \to \mathcal{X}$, there exists a continuous function $h : \mathrm{dom}(f) \to \mathrm{dom}(\rho)$ such that $\rho\,(h(\alpha)) = f(\alpha)$ for any $\alpha \in \mathrm{dom}(f)$.



Any countably based T_0-space admits an admissible representation called the standard representation (see Theorem 3.3 in [Peq15b] for a proof).

Theorem 4.2 ([Wei00])**.** *Let $\mathcal{X}$ be a countably based T_0-space with basis $(V_n)_{n\in\omega}$. The partial function $\rho_{\mathrm{st}} : \subseteq \omega^\omega \to \mathcal{X}$ defined as*

$$\rho_{\mathrm{st}}\,(\alpha) = x \leftrightarrow \{n \in \omega : \exists k\ \alpha_k = n\} = \{n \in \omega : x \in V_n\}$$

is an admissible representation called the standard representation. It is open and has Polish fibers, i.e., $\rho_{\mathrm{st}}^{-1}(x)$ is Polish for any $x \in \mathcal{X}$.

4.2 Continuous reducibility via admissible representations

Any admissible representation of $\mathcal{X}$ induces a natural notion of reducibility on the subsets of $\mathcal{X}$, which is nothing but continuous reducibility in the names. It was first studied as a quasi-order by Pequignot [Peq15b].

Definition 4.3. Let ρ be an admissible representation of $\mathcal{X}$, and $A, B \subseteq \mathcal{X}$. The subset A is $\preccurlyeq_w^\rho$-reducible to B, denoted by $A \preccurlyeq_w^\rho B$, if there exists a continuous function $f : \mathrm{dom}(\rho) \to \mathrm{dom}(\rho)$ such that

$$f^{-1}\left[\rho^{-1}\left[B\right]\right] = \rho^{-1}\left[A\right].$$

In other words, $A \preccurlyeq_w^\rho B$ if the set of codes of A is $\leq_w$-reducible to the set of codes of B both inside $\mathrm{dom}(\rho) \subseteq \omega^\omega$.

It is straightforward that $\preccurlyeq_w^\rho$ is a quasi-order. The next proposition shows that the choice of the admissible representation actually does not matter.

Proposition 4.4. *Let ρ and σ be two admissible representations of $\mathcal{X}$, and let $A, B \subseteq \mathcal{X}$. If $A \preccurlyeq_w^\rho B$, then $A \preccurlyeq_w^\sigma B$.*

Proof. Let $\rho : \mathrm{dom}(\rho) \to \mathcal{X}$ and $\sigma : \mathrm{dom}(\sigma) \to \mathcal{X}$ be two admissible representations. There exist two continuous functions $h_0 : \mathrm{dom}(\rho) \to \mathrm{dom}(\sigma)$ such that $\sigma(h_0(\alpha)) = \rho(\alpha)$ for any $\alpha \in \mathrm{dom}(\rho)$, and $h_1 : \mathrm{dom}(\sigma) \to \mathrm{dom}(\rho)$ such that $\rho(h_1(\alpha)) = \sigma(\alpha)$ for any $\alpha \in \mathrm{dom}(\sigma)$. If $A \preccurlyeq_w^\rho B$, there a continuous function $f : \mathrm{dom}(\rho) \to \mathrm{dom}(\rho)$ such that $f^{-1}\left[\rho^{-1}[B]\right] = \rho^{-1}[A]$.

$$\begin{array}{ccc}
\mathrm{dom}(\sigma) & \overset{h_0 \circ f \circ h_1}{\dashrightarrow} & \mathrm{dom}(\sigma) \\
{\scriptstyle h_1}\downarrow & & \uparrow{\scriptstyle h_0} \\
\mathrm{dom}(\rho) & \underset{f}{\longrightarrow} & \mathrm{dom}(\rho)
\end{array}$$

Thus,

$$\begin{aligned}
(h_0 \circ f \circ h_1)^{-1}\left[\sigma^{-1}[B]\right] &= (\sigma \circ h_0 \circ f \circ h_1)^{-1}[B] \\
&= (\rho \circ f \circ h_1)^{-1}[B] \\
&= (\rho \circ h_1)^{-1}[A] \\
&= \sigma^{-1}[A].
\end{aligned}$$

Since $h_0 \circ f \circ h_1 : \mathrm{dom}\,\sigma \to \mathrm{dom}\,\sigma$ is continuous, we get $A \preccurlyeq_w^\sigma B$. □

Since any countably based T_0-space admits an admissible representation, we obtain a new notion of reducibility.

Definition 4.5 (Definition 7.2 in [Peq15b])**.** Let $\mathcal{X}$ be a countably based T_0-space, and let $A, B \subseteq \mathcal{X}$. The subset A is $\preccurlyeq_w$-reducible to B, denoted by $A \preccurlyeq_w B$, if $A \preccurlyeq_w^\rho B$ for some (any) admissible representation ρ of $\mathcal{X}$.

If $A, B \subseteq \mathcal{X}$ and both $A \preccurlyeq_w B$ and $B \preccurlyeq_w A$ hold, then we write $A \sim_w B$. The relation $\sim_w$ is an equivalence relation where the degree of $A \subseteq \mathcal{X}$ is denoted by $[A]_{\sim_w} = \{B \subseteq \mathcal{X} : A \sim_w B\}$. The set of all these degrees is $\mathbb{WD}^{\sim_w}(\mathcal{X})$ and the set of all these degrees generated by Borel sets is $\mathbb{WD}_{\mathcal{B}}^{\sim_w}(\mathcal{X}) = \left\{[A]_{\sim_w} : A \in \mathcal{B}(\mathcal{X})\right\}$. The relation $\preccurlyeq_w$ is a partial order on $\mathbb{WD}_{\mathcal{B}}^{\sim_w}(\mathcal{X})$. If $\rho : \subseteq \omega^\omega \to \mathcal{X}$ is an admissible representation, then the function

$$\begin{aligned}
\rho^{-1} : \left(\mathbb{WD}_{\mathcal{B}}^{\sim_w}(\mathcal{X}), \preccurlyeq_w\right) &\to \left(\mathbb{WD}_{\mathcal{B}}(\mathrm{dom}(\rho)), \leq_w\right) \\
[A]_{\sim_w} &\mapsto \left[\rho^{-1}[A]\right]_w,
\end{aligned}$$

is a well-defined order-embedding. As a consequence, if $\rho : \omega^\omega \to \mathcal{X}$ is a total admissible representation, then $\left(\mathbb{WD}_{\mathcal{B}}^{\sim_w}(\mathcal{X}), \preccurlyeq_w\right)$ is well-founded and admits maximal antichains of size 2.

There is another natural definition of the quasi-order $\preccurlyeq_w$. The Wadge preorder is obtained via reductions by continuous functions. One relaxes this notion of reductions by considering the notion of *total relatively continuous relations* introduced in [BH94] instead of continuous functions. If $\mathcal{X}$ is a topological space, a total relation on $\mathcal{X}$ is a subset $R \subseteq \mathcal{X}^2$ such that, for any $x \in \mathcal{X}$, there exists $x' \in \mathcal{X}$ such that $(x, x') \in R$. If $\mathcal{X}$ is a countably based T_0-space, a total relation $R \subseteq \mathcal{X}^2$ is relatively continuous if for some (any) admissible representation $\rho :\subseteq \omega^\omega \to \mathcal{X}$ of $\mathcal{X}$, there exists a continuous realizer of R, i.e., a continuous function $f : \mathrm{dom}(\rho) \to \mathrm{dom}(\rho)$ such that, for any $\alpha \in \mathrm{dom}(\rho)$, we have

$$(\rho(\alpha), \rho \circ f(\alpha)) \in R.$$

By Lemma 7.3 in [Peq15b], the quasi-order $\preccurlyeq_w$ and the quasi-order induced by total relatively continuous relations coincide.

Lemma 4.6 (Lemma 7.3 in [Peq15b])**.** *If $\mathcal{X}$ is a countably based T_0-space, $A, B \subseteq \mathcal{X}$, then $A \preccurlyeq_w B$ if and only if there exists a total relatively continuous relation $R \subseteq \mathcal{X}^2$ such that for any $x, x' \in \mathcal{X}$, if $R(x, x')$, then $(x \in A \leftrightarrow x' \in B)$.*

For any continuous function $f : \mathcal{X} \to \mathcal{X}$, the graph of the function $\{(x, f(x)) : x \in \mathcal{X}\} \subseteq \mathcal{X}^2$ is a total relatively continuous relations, so that $A \leq_w B$ implies $A \preccurlyeq_w B$.

In [Bra98], Brattka proved that any non-empty Polish space admits a total admissible representation. More recently, de Brecht extended this result and proved that the quasi-Polish spaces are exactly those spaces that admit a total admissible representation [dB13]. This strongly connects computable analysis with descriptive set theory and provides two important and different viewpoints on the class of quasi-Polish spaces.

Theorem 4.7 (Theorem 49 in [dB13])**.** *A countably based space $\mathcal{X}$ is quasi-Polish if and only if it admits a total admissible representation $\rho : \omega^\omega \to \mathcal{X}$.*

In general, the two posets $\left(\mathbb{WD}_{\mathcal{B}}^{\sim_w}(\mathcal{X}), \preccurlyeq_w\right)$ and $\left(\mathbb{WD}_{\mathcal{B}}(\mathcal{X}), \leq_w\right)$ are rather different. For example, let $\mathcal{X}$ be a non-zero-dimensional Polish space. Since $\mathcal{X}$ admits a total admissible representation, the poset $\left(\mathbb{WD}_{\mathcal{B}}^{\sim_w}(\mathcal{X}), \preccurlyeq_w\right)$

is well-founded and has maximal antichains of size 2. On the other hand, Schlicht proved in [Sch18] that, for such spaces, $(\mathbb{WD}_{\mathcal{B}}(\mathcal{X}), \leq_w)$ admits antichains of cardinality $2^{\aleph_0}$ (Theorem 2.26). The zero-dimensional case, however, is completely different.

Theorem 4.8 (Proposition 7.4 in [Peq15b])**.** *If $\mathcal{X}$ is a zero-dimensional Polish space, the two quasi-orders $\preccurlyeq_w$ and $\leq_w$ coincide.*

In particular, the quasi-order $\preccurlyeq_w$ can be thought of as a generalization of the Wadge preorder to the class of countably based T_0-spaces.

4.3 Another Wadge order on the Scott domain

In this section, we are interested in the quasi-order $\preccurlyeq_w$ on the Scott domain $\mathcal{P}\omega$. We prove the poset $\left(\mathbb{WD}_{\mathcal{B}}^{\sim_w}(\mathcal{P}\omega), \preccurlyeq_w\right)$ to be isomorphic to the poset $(\mathbb{WD}_{\mathcal{B}}(\omega^\omega), \leq_w)$ restricted to the non-self-dual degrees. To begin with, we show that the same result holds for the conciliatory space Conc defined in Subsection 2.3.3. For this purpose, we consider the interpretation by Fournier [Fou16] of the work of Duparc [Dup01, Dup]. It relies on the definition of a total admissible representation of Conc.

Let $\mathtt{b}$ be a symbol not in ω, and $\omega_{\mathtt{b}} = \omega \cup \{\mathtt{b}\}$. Clearly, any bijection between $\omega_{\mathtt{b}}$ and ω yields an homeomorphism between the spaces ${\omega_{\mathtt{b}}}^\omega$ and ω^ω. In [Fou16], Fournier considered the following total admissible representation of Conc (Lemma 3.8 in [Fou16])

$$\begin{aligned} \rho_{\mathtt{b}} : {\omega_{\mathtt{b}}}^\omega &\to \mathsf{Conc} \\ x &\mapsto x_{[\ /\mathtt{b}]}, \end{aligned}$$

where $x_{[\ /\mathtt{b}]}$ is the sequence x where all occurrences of $\mathtt{b}$ have been omitted. For any $A \subseteq \mathsf{Conc}$, we write $A^{\mathtt{b}}$ for $\rho_{\mathtt{b}}^{-1}[A]$. As already mentioned, the function $\rho_{\mathtt{b}}^{-1} : \left(\mathbb{WD}_{\mathcal{B}}^{\sim_w}(\mathsf{Conc}), \preccurlyeq_w\right) \to (\mathbb{WD}_{\mathcal{B}}({\omega_{\mathtt{b}}}^\omega), \leq_w)$ is an order-embedding. The range of this embedding is included in the non-self-dual degrees.

Proposition 4.9 ([Dup01])**.** *If $A \in \mathcal{B}(\mathsf{Conc})$, then $A^{\mathtt{b}} \nleq_w {\omega_{\mathtt{b}}}^\omega \setminus A^{\mathtt{b}}$.*

Proof. Let us consider the Wadge game $G_w\left(A^{\mathtt{b}}, {\omega_{\mathtt{b}}}^\omega \setminus A^{\mathtt{b}}\right)$ in ${\omega_{\mathtt{b}}}^\omega$. We define a winning strategy for I. First, I start by playing $\mathtt{b}$. Then, I simply copies the previous move of II if it is not an $\mathtt{s}$, and plays $\mathtt{b}$ if it is an $\mathtt{s}$ — where $\mathtt{s}$ stands for the possibility for II to skip her turn. At the end of the game, if II did not play an infinite sequence, then I wins. Otherwise, I has produced $x \in {\omega_{\mathtt{b}}}^\omega$ and II has produced $y \in {\omega_{\mathtt{b}}}^\omega$. Observe that, by construction, $x_{[\ /\mathtt{b}]} = y_{[\ /\mathtt{b}]}$. In particular, $\rho_{\mathtt{b}}(x) = \rho_{\mathtt{b}}(y)$, which proves that I wins the game. □

The main result of [Dup01, Dup] is the following.

Theorem 4.10 ([Dup01, Dup])**.** *If $B \in \mathcal{B}(\omega^\omega)$ is non-self-dual, there exists $A \subseteq \omega^{\leq\omega}$ such that*

$$B \equiv_w A^{\mathsf{b}}.$$

The following theorem of [dB13] is a tailoring of a previous result of Saint-Raymond (Lemma 17 in [SR07]). As a consequence, the set $A \subseteq \omega^{\leq\omega}$ in the previous theorem is actually Borel in Conc.

Theorem 4.11 (Theorem 68 in [dB13])**.** *Let $\mathcal{X}$ be a countably based T_0-space and $\rho :\subseteq \omega^\omega \to \mathcal{X}$ be an admissible representation. For any $0 < \alpha, \beta \in \omega_1$, if $A \subseteq \mathcal{X}$, then*

$$A \in D_\alpha\left(\boldsymbol{\Sigma}^0_\beta\right)(\mathcal{X}) \Leftrightarrow \rho^{-1}[A] \in D_\alpha\left(\boldsymbol{\Sigma}^0_\beta\right)(\mathrm{dom}(\rho)).$$

In particular, we completely determined the shape of $\left(\mathbb{WD}^{\sim_w}_{\mathcal{B}}(\mathsf{Conc}), \preccurlyeq_w\right)$.

Theorem 4.12 (Theorem 3.10 in [Fou16])**.** *The poset $\left(\mathbb{WD}^{\sim_w}_{\mathcal{B}}(\mathsf{Conc}), \preccurlyeq_w\right)$ is isomorphic to the restriction of the poset $\left(\mathbb{WD}_{\mathcal{B}}(\omega^\omega), \leq_w\right)$ to the non-self-dual degrees.*

The main theorem of this chapter is the fact that same result holds for the Scott domain $\mathcal{P}\omega$. Let us fix a total admissible representation of $\mathcal{P}\omega$.

Proposition 4.13. *The following function is a total admissible representation of $\mathcal{P}\omega$ called the enumeration representation.*

$$\begin{aligned} \rho_{\mathsf{en}} &: \omega^\omega \to \mathcal{P}\omega \\ &x \mapsto \{n \in \omega : \exists k\ x_k = n+1\}. \end{aligned}$$

Proof. Since, for any $\{m_0, \dots, m_k\} \subseteq \omega$ finite subset,

$$\rho_{\mathsf{en}}^{-1}[\Uparrow\{m_0, \dots, m_k\}] = \{x \in \omega^\omega : \exists n_0 \dots \exists n_k\ x_{n_i} = m_i + 1\},$$

the function ρ_{en} is continuous. Let $f :\subseteq \omega^\omega \to \omega^\omega$ be any continuous function. For any $s \in \omega^{<\omega}$, we set $A(s) = \left\{n \in \omega : s \subseteq f^{-1}[\{n\}]\right\}$. Let $h(\langle\rangle) = \emptyset$ and, if $h(s) \subseteq \omega^{<\omega}$ is already defined and $n \in \omega$, we set $h(sn) = h(s)(m+1)$ where $m \in A(sn) \setminus \mathrm{ran}(h(s))$ is minimal if it exists, and $h(sn) = h(s)0$ otherwise. The function $h : \omega^{<\omega} \to \omega^{<\omega}$ easily extends to a continuous function $\tilde{h} : \omega^\omega \to \omega^\omega$. It remains to prove that, for any $x \in \mathrm{dom}(f)$, $\rho_{\mathtt{en}}\left(\tilde{h}(x)\right) = f(x)$:

$$\begin{aligned} m \in f(x) &\Leftrightarrow \exists t \sqsubset x \; t \subseteq f^{-1}\left[\{m\}\right] \\ &\Leftrightarrow \exists n \; m \in A\left(x_{\lceil n}\right) \\ &\Leftrightarrow \exists n \; h\left(x_{\lceil n}\right)(n-1) = m+1 \\ &\Leftrightarrow m \in \rho_{\mathtt{en}}\left(\tilde{h}(x)\right). \end{aligned}$$

□

As previously, if $A \subseteq \mathcal{P}\omega$ holds, we write $A^{\mathtt{en}}$ for $\rho_{\mathtt{en}}^{-1}[A]$ and the preimage of any Borel subset of $\mathcal{P}\omega$ is non-self-dual.

Proposition 4.14. *If $A \in \mathcal{B}(\mathcal{P}\omega)$, then $A^{\mathtt{en}} \not\leq_w \omega^\omega \setminus A^{\mathtt{en}}$.*

Proof. We construct a winning strategy for I in the game $G_w\left(A^{\mathtt{en}}, \omega^\omega \setminus A^{\mathtt{en}}\right)$. First, I starts by playing 0. For the rest of the game, I copies the previous move of II if this move is an integer, and plays 0 otherwise. At the end of a run, if II did not play an infinite sequence, then I wins. Otherwise, I has produced $x \in \omega^\omega$ and II has produced $y \in \omega^\omega$. By construction, $x_{[\ /0]} = y_{[\ /0]}$. In particular, $\rho_{\mathtt{en}}(x) = \rho_{\mathtt{en}}(y)$, which proves that I wins the game. □

In particular, the range of the order-embedding

$$\rho_{\mathtt{en}}^{-1} : \left(\mathbb{WD}_{\mathcal{B}}^{\sim_w}(\mathcal{P}\omega), \preccurlyeq_w\right) \to \left(\mathbb{WD}_{\mathcal{B}}(\omega^\omega), \leq_w\right)$$

is included in the non-self-dual degrees. The rest of this subsection is devoted to the proof that this order-embedding is actually onto the non-self-dual degrees. First, we prove the result for $\mathbf{\Delta}_2^0(\omega^\omega)$ and then for the remaining of $\mathcal{B}(\omega^\omega)$. The $\mathbf{\Delta}_2^0(\omega^\omega)$ case relies on Theorem 3.3 and Theorem 4.11.

Proposition 4.15. *If $B \in \mathbf{\Delta}_2^0(\omega^\omega)$, there exists $A \in \mathbf{\Delta}_2^0(\mathcal{P}\omega)$ such that $B \equiv_w A^{\mathtt{en}}$.*

Proof. If $B \in \mathbf{\Delta}_2^0(\omega^\omega)$ is non-self-dual, by Theorem 2.39, there exists $\alpha \in \omega_1$ such that B or B^c belongs to $D_\alpha\left(\mathbf{\Sigma}_1^0\right)(\omega^\omega) \setminus \check{D}_\alpha\left(\mathbf{\Sigma}_1^0\right)(\omega^\omega)$. Without loss of generality, suppose that $B \in D_\alpha\left(\mathbf{\Sigma}_1^0\right)(\omega^\omega) \setminus \check{D}_\alpha\left(\mathbf{\Sigma}_1^0\right)(\omega^\omega)$. By Theorem 3.3, there exists $A \in D_\alpha\left(\mathbf{\Sigma}_1^0\right)(\mathcal{P}\omega) \setminus \check{D}_\alpha\left(\mathbf{\Sigma}_1^0\right)(\mathcal{P}\omega)$. By Theorem 4.11, we get $A^{\mathtt{en}} \in D_\alpha\left(\mathbf{\Sigma}_1^0\right)(\omega^\omega) \setminus \check{D}_\alpha\left(\mathbf{\Sigma}_1^0\right)(\omega^\omega)$ which is equivalent to $B \equiv_w A^{\mathtt{en}}$. □

The proof outside the realm of the $\mathbf{\Delta}^0_2(\omega^\omega)$ sets is more involved. We actually prove the following result.

Theorem 4.16. *If $A \in \mathcal{B}(\mathsf{Conc}) \setminus \mathbf{\Delta}^0_2(\mathsf{Conc})$, there exists $B \subseteq \mathcal{P}\omega$ such that*

$$A^{\mathsf{b}} \equiv_w B^{\mathsf{en}}.$$

We need the following easy result.

Lemma 4.17. *If $A \in \mathcal{B}(\mathsf{Conc}) \setminus \mathbf{\Delta}^0_2(\mathsf{Conc})$, then $(\emptyset + A)^{\mathsf{b}} \equiv_w A^{\mathsf{b}}$.*

Proof. By Theorem 2.52, $\mathrm{rk}_w\left((\emptyset + A)^{\mathsf{b}}\right) = 1 + \mathrm{rk}_w\left(A^{\mathsf{b}}\right)$. Since we have $\mathrm{rk}_w\left(A^{\mathsf{b}}\right) \geq \omega_1$ and $A^{\mathsf{b}} \leq_w (\emptyset + A)^{\mathsf{b}}$, we obtain $(\emptyset + A)^{\mathsf{b}} \equiv_w A^{\mathsf{b}}$. □

The proof of Theorem 4.16 relies on the idea that $\mathcal{P}\omega$ contains a copy of Conc. Since ω^2 and ω are in bijection, $\mathcal{P}\omega^2$ is homeomorphic to $\mathcal{P}\omega$. In particular, we consider $\mathcal{P}\omega^2$ instead of $\mathcal{P}\omega$. An element $x \in \mathcal{P}\omega^2$ is the graph of a partial function $f_x :\subseteq \omega \to \omega$ if, for any $n \in \omega$, there exists at most one $m \in \omega$ such that $(n, m) \in x$. Let $\mathtt{P} \subseteq \mathcal{P}\omega^2$ be the set of all graphs of partial functions. If $x \in \mathtt{P}$, we define the length of x as $\mathrm{lh}(x) = \min\{k \in \omega : k \notin \mathrm{dom}(f_x)\}$ if it exists, and $\mathrm{lh}(x) = \omega$ if f_x is total. If $x \in \mathtt{P}$, we also define $i_x \in \omega^{\leq\omega}$ as follows: for any $k < \mathrm{lh}(x)$, $i_x(k) \in \omega$ is the unique integer such that $(k, i_x(k)) \in x$. We think of i_x as the initial segment of the function f_x. For any $A \subseteq \mathsf{Conc}$, we define $B_A \subseteq \mathcal{P}\omega^2$ as the sets of all graphs of partial functions such that their initial segment is in A,

$$B_A = \{x \in \mathtt{P} : i_x \in A\}.$$

Since we deal with $\mathcal{P}\omega^2$, the enumeration representation is slightly different than ρ_{en}. Let $(t_n)_{n\in\omega}$ be an enumeration without repetition of ω^2. The enumeration representation becomes

$$\begin{aligned} \rho_{\mathsf{en}^2} : \omega^\omega &\to \mathcal{P}\omega^2 \\ x &\mapsto \left\{t_n \in \omega^2 : \exists k\ x_k = n + 1\right\}. \end{aligned}$$

As previously, if $A \subseteq \mathcal{P}\omega^2$, we write A^{en^2} for $\rho^{-1}_{\mathsf{en}^2}[A]$. The main theorem of this chapter follows from the two following claims.

Claim 4.18. *If $A \subseteq \mathsf{Conc}$, then $A^{\mathsf{b}} \leq_w {B_A}^{\mathsf{en}^2}$.*

Proof of the claim. We construct a winning strategy for II in

$$G_w\left(A^{\mathsf{b}}, {B_A}^{\mathsf{en}^2}\right)\left({\omega_{\mathsf{b}}}^{\omega}, \omega^\omega\right).$$

For any round $n \in \omega$, let α_n be the element of $\omega_{\mathtt{b}}$ played by I, and $u_n = \mathrm{Card}\left\{k \leq n : \alpha_k \neq \mathtt{b}\right\}$.

At round $n \in \omega$, if I plays $\mathtt{b}$, then II answers with 0. Otherwise, I plays $m \in \omega$, and II answers with $k+1 \in \omega$ such that $t_k = (u_n - 1, m)$. To say it otherwise, at each round, II produces the graph of the sequence played by I once every occurrence of the symbol $\mathtt{b}$ has been removed.

At the end of the game, I has produced $x \in {\omega_{\mathtt{b}}}^{\omega}$. Following her strategy, II has produced $y \in \omega^{\omega}$ such that $\rho_{\mathtt{en}^2}(y) = \left\{\left(k, x_{[\ /\mathtt{b}]}(k)\right) : k < \mathrm{lh}\left(x_{[\ /\mathtt{b}]}\right)\right\}$. By definition, $x \in A^{\mathtt{b}}$ if and only if $\rho_{\mathtt{en}^2}(y) \in B_A$, which proves

$$A^{\mathtt{b}} \leq_w {B_A}^{\mathtt{en}^2}.$$

$\square_{\text{Claim}}$

Claim 4.19. *If $A \subseteq \mathsf{Conc}$, then ${B_A}^{\mathtt{en}^2} \leq_w (\emptyset + A)^{\mathtt{b}}$.*

Proof of the claim. We construct a winning strategy for II in

$$G_w\left({B_A}^{\mathtt{en}^2}, (\emptyset + A)^{\mathtt{b}}\right)\left(\omega^{\omega}, {\omega_{\mathtt{b}}}^{\omega}\right).$$

First, observe that $\rho_{\mathtt{en}^2}$ easily extends to a function with domain $\omega^{\leq\omega}$. Indeed, it suffices to define $\rho_{\mathtt{en}^2}(x) = \left\{t_n \in \omega^2 : \exists k < \mathrm{lh}(x)\ \ x_k = n+1\right\}$.

At round 0, if I plays 0, then II answers with $\mathtt{b}$. If I plays $k > 0$, such that $t_{k-1} \neq (0, m)$ for any $m \in \omega$, then II answers with $\mathtt{b}$. Finally, if I plays $k > 0$, such that $t_{k-1} = (0, m)$ for $m \in \omega$, then II answers with $m + 2$.

At round $n+1$, suppose that I has already produced a sequence $\alpha = (\alpha_0, \ldots, \alpha_{n+1}) \in \omega^{n+2}$. If $\rho_{\mathtt{en}^2}(\alpha) \notin \mathtt{P}$, then II choose to be in charge of $\emptyset$. Otherwise, we set $z = \rho_{\mathtt{en}^2}(\alpha) \in \mathtt{P}$. Let $i_z \in \omega^{<\omega}$. By induction, after round n, II already played the sequence $\left(i_z^{+2}\right)_{\restriction l}$ for some $l < \mathrm{lh}(z)$, where $i_z(k) + 2 = i_z^{+2}(k)$ for any $k < \mathrm{lh}(i_z)$. If $l = \mathrm{lh}(z)$, then II plays $\mathtt{b}$. Otherwise, II plays $i_z(l) + 2 \in \omega$.

At the end of the game, I has produced $x \in \omega^{\omega}$. If $\rho_{\mathtt{en}^2}(x) \notin \mathtt{P}$, there exists $k \in \omega$ such that $\rho_{\mathtt{en}^2}\left(x_{\restriction k}\right) \notin \mathtt{P}$, so that II chose to be in charge of $\emptyset$. In particular, $\rho_{\mathtt{en}^2}(x) \notin B_A$ and the sequence $y \in {\omega_{\mathtt{b}}}^{\omega}$ produced by II satsifies $y \notin (\emptyset + A)^{\mathtt{b}}$. Otherwise, let $z = \rho_{\mathtt{en}^2}(x) \in \mathtt{P}$. By induction, one easily proves that, for any $k < \mathrm{lh}(f_z)$, there exists a round n after which II has produced $y_{\restriction n}$ such that $y_{\restriction n [\ /\mathtt{b}]} = \left(i_z^{+2}\right)_{\restriction k}$. In particular,

$$\begin{aligned} z = \rho_{\mathtt{en}^2}(x) \in B_A &\Leftrightarrow i_z \in A \\ &\Leftrightarrow i_z^{+2} \in (\emptyset + A)^{\mathtt{b}} \\ &\Leftrightarrow y_{[\ /\mathtt{b}]} \in (\emptyset + A)^{\mathtt{b}}, \end{aligned}$$

which proves
$$B_A{}^{\mathsf{en}^2} \leq_w (\emptyset + A)^{\mathsf{b}}\,.$$
$\square_{\text{Claim}}$

The general case is an easy consequence of Lemma 4.17, Claim 4.18 and Claim 4.19.

Proposition 4.20. *If $A \in \mathcal{B}(\mathsf{Conc}) \setminus \mathbf{\Delta}^0_2(\mathsf{Conc})$, then $B_A{}^{\mathsf{en}^2} \equiv_w A^{\mathsf{b}}$.*

Proof. The three following inequalities follow from the three preceding results
$$A^{\mathsf{b}} \leq_w B_A{}^{\mathsf{en}^2} \leq_w (\emptyset + A)^{\mathsf{b}} \leq_w A^{\mathsf{b}}.$$
$\square$

As a consequence, we obtain the main result of this chapter.

Theorem 4.21. *The poset $\left(\mathbb{WD}^{\sim_w}_{\mathcal{B}}(\mathcal{P}\omega), \preccurlyeq_w\right)$ is isomorphic to the restriction of the poset $\left(\mathbb{WD}_{\mathcal{B}}(\omega^\omega), \leq_w\right)$ to the non-self-dual degrees.*

We actually proved a stronger result.

Theorem 4.22. *The posets $\left(\mathbb{WD}^{\sim_w}_{\mathcal{B}}(\mathcal{P}\omega), \preccurlyeq_w\right)$ and $\left(\mathbb{WD}^{\sim_w}_{\mathcal{B}}(\mathsf{Conc}), \preccurlyeq_w\right)$ are isomorphic.*

In [Fou16], Fournier proved that, under AD, $\left(\mathbb{WD}^{\sim_w}(\mathsf{Conc}), \preccurlyeq_w\right)$ is isomorphic to the restriction of $\left(\mathbb{WD}(\omega^\omega), \leq_w\right)$ to the non-self-dual degrees (Theorem 3.10 in [Fou16]). Clearly, under AD, our result also extends. Thus, we proved.

Theorem 4.23 (AD)**.** *The poset $\left(\mathbb{WD}^{\sim_w}(\mathcal{P}\omega), \preccurlyeq_w\right)$ is isomorphic to the restriction of the poset $\left(\mathbb{WD}(\omega^\omega), \leq_w\right)$ to the non-self-dual degrees.*

Chapter 5

Towards the Decomposability Conjecture

Throughout the whole chapter, $\mathcal{X}$ and $\mathcal{Y}$ will be separable and metrizable spaces. Nevertheless, some of the mentioned results have a wider range of applications. We refer the reader to the mentioned references for more precise statements. A function $f : \mathcal{X} \to \mathcal{Y}$ is Borel if the preimage of any Borel subset of $\mathcal{Y}$ is a Borel subsets of $\mathcal{X}$, i.e., if $f^{-1}[B] \in \mathcal{B}(\mathcal{X})$ for any $B \in \mathcal{B}(\mathcal{Y})$. Equivalently, $f : \mathcal{X} \to \mathcal{Y}$ is Borel if $f^{-1}[U] \in \mathcal{B}(\mathcal{X})$ for any $U \in \mathbf{\Sigma}^0_1(\mathcal{Y})$. A function $f : \mathcal{X} \to \mathcal{Y}$ is piecewise continuous if there exists a countable partition $\{A_n : n \in \omega\}$ of $\mathcal{X}$ such that $f \restriction A_n$ is continuous for any $n \in \omega$. To say it otherwise, if the function f is decomposable into countably many continuous functions. The study of the decomposability of Borel functions began over a century ago with a question asked by Luzin: *Is every Borel function piecewise continuous?* Since the 1930's, the answer to this question is known to be negative [Kel34, Kur34, Sie37]. Let $(\omega+1)^\omega$ be equipped with the product of the order topology on $\omega+1$. We define the function $P : (\omega+1)^\omega \to \omega^\omega$ as follows: for any $x \in (\omega+1)^\omega$ and any $n \in \omega$, $f(x)(n) = 0$ if $x(n) = \omega$ and $f(x)(n) = n+1$ otherwise. The function P is called Pawlikowski's function and is a natural example of a Borel function — actually even a $\mathbf{\Sigma}^0_2$-measurable functions, i.e., a function satisfying $f^{-1}[U] \in \mathbf{\Sigma}^0_2(\mathcal{X})$ for any $U \in \mathbf{\Sigma}^0_1(\mathcal{Y})$ — which is not piecewise continuous [CMPS91, MR13, Sol98]. However, Luzin's question gave rise to a still active area of research to which the present chapter belongs.

As customary in descriptive set theory, the strategy deployed in order to study Borel functions consists in stratifying them according to their topological complexity. There exist essentially three different methods of stratification. Let $\mathcal{F}_{\mathcal{B}}(\mathcal{X}, \mathcal{Y})$ denote the set of all Borel functions from $\mathcal{X}$ to $\mathcal{Y}$, and $\mathcal{F}_{\mathcal{B}}(\subseteq \mathcal{X}, \mathcal{Y})$ denote the set of all partial Borel functions from $\mathcal{X}$ to $\mathcal{Y}$.

1. For any $n \in \omega$, let $f_n : \mathcal{X} \to \mathcal{Y}$ be any function. If it exists, the pointwise limit of the sequence of functions $(f_n)_{n\in\omega}$ is denoted by $\lim_{n\in\omega} f_n : \mathcal{X} \to \mathcal{Y}$. If we write $\mathcal{BC}_0(\mathcal{X}, \mathcal{Y})$ for the set of continuous functions from $\mathcal{X}$ to $\mathcal{Y}$ and $\mathcal{BC}_1(\mathcal{X}, \mathcal{Y})$ for the set of $\mathbf{\Sigma}^0_2$-measurable functions, we can define by transfinite induction the set of Baire class α functions for $\alpha < \omega_1$ as
$$\mathcal{BC}_\alpha(\mathcal{X}, \mathcal{Y}) = \left\{\lim_{n\in\omega} f_n : f_n \in \mathcal{BC}_{\beta_n}(\mathcal{X}, \mathcal{Y}) \text{ for some } \beta_n < \alpha\right\}.$$
We clearly have $\mathcal{BC}_\beta(\mathcal{X}, \mathcal{Y}) \subseteq \mathcal{BC}_\alpha(\mathcal{X}, \mathcal{Y})$ for any $\beta < \alpha$.

2. Let $f : \mathcal{X} \to \mathcal{Y}$, $\Gamma \subseteq \mathcal{P}(\mathcal{X})$ and $\mathcal{F} \subseteq \mathcal{F}_{\mathcal{B}}(\subseteq \mathcal{X}, \mathcal{Y})$. If there exists a Γ-partition $\{D_n : n \in \omega\} \subseteq \Gamma$ of $\mathcal{X}$ such that $f \restriction D_n \in \mathcal{F}$, then f is an $\mathcal{F}$-function on a Γ-partition. We write $\operatorname{Dec}(\mathcal{F},\ \Gamma)(\mathcal{X}, \mathcal{Y})$ for the set of $\mathcal{F}$-functions on a Γ-partition.

3. Let $\alpha, \beta \in \omega_1$, following Semmes' notation in his PhD thesis [Sem09], we define
$$\Lambda_{\alpha,\beta}(\mathcal{X}, \mathcal{Y}) = \left\{f : \mathcal{X} \to \mathcal{Y} : \forall B \in \mathbf{\Sigma}^0_\alpha(\mathcal{Y})\ \ f^{-1}[B] \in \mathbf{\Sigma}^0_\beta(\mathcal{X})\right\},$$
and $\Lambda_{\alpha,\beta}(\subseteq \mathcal{X}, \mathcal{Y})$ for the set of all such partial functions from $\mathcal{X}$ to $\mathcal{Y}$. Clearly, if $f : \mathcal{X} \to \mathcal{Y} \in \Lambda_{1,n}$ for some $n \in \omega$ and $B \in \mathbf{\Sigma}^0_m(\mathcal{Y})$ for some $m > 0$, then $f^{-1}[B] \in \mathbf{\Sigma}^0_{n+m-1}(\mathcal{X})$. Moreover, if $\alpha, \beta \geq 2$ — or even $\alpha, \beta \geq 1$ if the spaces are zero-dimensional — then
$$\Lambda_{\alpha,\beta}(\mathcal{X}, \mathcal{Y}) = \left\{f : \mathcal{X} \to \mathcal{Y} : \forall B \in \mathbf{\Delta}^0_\alpha(\mathcal{Y})\ \ f^{-1}[B] \in \mathbf{\Delta}^0_\beta(\mathcal{X})\right\}. \tag{5.1}$$

We often consider $\operatorname{Dec}\left(\Lambda_{\alpha,\beta}(\subseteq \mathcal{X}, \mathcal{Y}),\ \mathbf{\Delta}^0_\gamma(\mathcal{X})\right)(\mathcal{X}, \mathcal{Y})$ for some ordinals $\alpha, \beta, \gamma \in \omega_1$. In that case, we write $\operatorname{Dec}\left(\Lambda_{\alpha,\beta},\ \mathbf{\Delta}^0_\gamma\right)(\mathcal{X}, \mathcal{Y})$ to enlighten the notation. Clearly, if $f : \mathcal{X} \to \mathcal{Y} \in \operatorname{Dec}\left(\Lambda_{\alpha,\beta},\ \mathbf{\Delta}^0_\gamma\right)(\mathcal{X}, \mathcal{Y})$ and $\mathcal{X}' \subseteq \mathcal{X}$, then $f \restriction \mathcal{X}' \in \operatorname{Dec}\left(\Lambda_{\alpha,\beta},\ \mathbf{\Delta}^0_\gamma\right)(\mathcal{X}', \mathcal{Y})$. In any of the previous cases, if $\mathcal{X}$ and $\mathcal{Y}$ are clear from context, we might omit them in the notation.

By classical results of Lebesgue, Hausdorff and Banach, there is a strong connection between two of these stratifications (Theorem 24.3 and 24.10 in [Kec95]).

Theorem 5.1 (Lebesgue, Hausdorff, Banach)**.** *If $\mathcal{X}$ and $\mathcal{Y}$ are separable and metrizable, and $1 \leq \alpha < \omega_1$, then we have $f \in \mathcal{BC}_\alpha(\mathcal{X}, \mathcal{Y})$ if and only if $f \in \Lambda_{1,\alpha+1}(\mathcal{X}, \mathcal{Y})$. In particular, $\mathcal{F}_{\mathcal{B}}(\mathcal{X}, \mathcal{Y}) = \bigcup_{\alpha \in \omega_1} \mathcal{BC}_\alpha(\mathcal{X}, \mathcal{Y})$.*

Moreover, if $\mathcal{X}$ is zero-dimensional, then $f \in \mathcal{BC}_1(\mathcal{X}, \mathcal{Y})$ if and only if f is the pointwise limit of a sequence of continuous functions.

In 1982, Jayne and Rogers proved the first result towards a positive answer to a Luzin-like question [JR82]. A topological space $\mathcal{X}$ is Suslin if it is the image of a continuous function from the Baire space ω^ω. Thus, if $\mathcal{X}'$ is a Polish space, then $\mathcal{X} \subseteq \mathcal{X}'$ is Suslin if and only if $\mathcal{X}$ is analytic.

Jayne-Rogers Theorem 5.2 ([JR82])**.** *If $\mathcal{X}$ is Suslin and $\mathcal{Y}$ is separable and metrizable, then*

$$\Lambda_{2,2}(\mathcal{X}, \mathcal{Y}) = \mathrm{Dec}\left(\Lambda_{1,1},\ \boldsymbol{\Delta}^0_2\right)(\mathcal{X}, \mathcal{Y}).$$

Even though the proof given in [JR82] only uses concepts of general topology, it is, according to their authors, *complicated*. Over the years, several simpler proofs have been published. It started in 1998 with Solecki who used notions of effective descriptive set theory to prove the result [Sol98]. In [MRS10, KMRS12], Kačena, Motto Ros and Semmes gave another simpler proof by means of notions from general topology.

The Decomposability Conjecture is a generalization of the Jayne-Rogers Theorem which links the different above-mentioned stratifications of the Borel functions. It appears in different forms in several journal issues [And07, DKSZ20, Kih15, GKN21, MR13, PS12].

The Decomposability Conjecture 5.3. *If $\mathcal{X}$ is Suslin, $\mathcal{Y}$ is separable and metrizable, and $1 \leq m \leq n < \omega$, then*

$$\Lambda_{m,n}(\mathcal{X}, \mathcal{Y}) = \mathrm{Dec}\left(\Lambda_{1,n-m+1},\ \boldsymbol{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y}).$$

Actually, only one inclusion is hard while the other one is easy to prove.

Proposition 5.4. *If $\mathcal{X}$ is Suslin, $\mathcal{Y}$ is separable and metrizable, and $1 \leq m \leq n < \omega$, then*

$$\Lambda_{m,n}(\mathcal{X}, \mathcal{Y}) \supseteq \mathrm{Dec}\left(\Lambda_{1,n-m+1},\ \boldsymbol{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y}).$$

Proof. If $m = 1$, there is nothing to prove. Suppose now that $m \geq 2$ and $f : \mathcal{X} \to \mathcal{Y} \in \mathrm{Dec}\left(\Lambda_{1,n-m+1},\ \boldsymbol{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y})$. There is a countable partition

$\{A_i : i \in \omega\} \subseteq \mathbf{\Delta}^0_n(\mathcal{X})$ such that $f \restriction A_i \in \Lambda_{1,n-m+1}(A_i, \mathcal{Y})$ holds for any $i \in \omega$. Let $B \in \mathbf{\Sigma}^0_m(\mathcal{Y})$, then

$$f^{-1}[B] = \bigcup_{i\in\omega} (f \restriction A_i)^{-1}[B] = \bigcup_{i\in\omega} A_i \cap C_i$$

where $C_i \in \mathbf{\Sigma}^0_n(\mathcal{X})$, so that $f^{-1}[B] \in \mathbf{\Sigma}^0_n(\mathcal{X})$. □

The first extension of the Jayne-Rogers Theorem towards the Decomposability Conjecture appears in [Sem09] for $\mathcal{X} = \mathcal{Y} = \omega^\omega$. Indeed, using game-theoretical techniques provided by the zero-dimensionality of ω^ω, Semmes reproved the Jayne-Rogers Theorem and also proved the cases $n \leq 3$ of the Decomposability Conjecture.

Theorem 5.5 (Theorems 3.4.5, 4.3.7 and 5.2.8 in [Sem09])**.** *If $1 \leq m \leq n \leq 3$, then*

$$\Lambda_{m,n}(\omega^\omega, \omega^\omega) = \mathrm{Dec}\left(\Lambda_{1,n-m+1},\ \mathbf{\Delta}^0_n\right)(\omega^\omega, \omega^\omega).$$

This result was generalized to any Polish space $\mathcal{X}$ by Ding, Kihara, Semmes and Zhao [DKSZ20].

Theorem 5.6 (Theorem 1.2 in [DKSZ20])**.** *If $\mathcal{X}$ is Polish, $\mathcal{Y}$ is separable and metrizable, and $1 \leq m \leq n = 3$, then*

$$\Lambda_{m,n}(\mathcal{X}, \mathcal{Y}) = \mathrm{Dec}\left(\Lambda_{1,n-m+1},\ \mathbf{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y}).$$

In all the above-mentioned articles [DKSZ20, JR82, MRS10, KMRS12, Sem09, Sol98], the strategies adopted in order to prove instances of the Decomposability Conjecture all follow the same guideline. Firstly, the authors suppose that $f : \mathcal{X} \to \mathcal{Y} \in \Lambda_{1,n} \setminus \mathrm{Dec}\left(\Lambda_{1,n-m+1},\ \mathbf{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y})$ for some $1 < m \leq n < \omega$. Secondly, they exhibit a specific $B \in \mathbf{\Sigma}^0_{m-1}(\mathcal{Y})$. Thirdly, they prove that its preimage is complex enough, i.e., $f^{-1}[B] \notin \mathbf{\Delta}^0_n(\mathcal{Y})$. Finally, using Equation (5.1) (page 94), they obtain that $f \notin \Lambda_{m,n}(\mathcal{X}, \mathcal{Y})$, so that

$$\Lambda_{m,n}(\mathcal{X}, \mathcal{Y}) = \mathrm{Dec}\left(\Lambda_{1,n-m+1},\ \mathbf{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y}).$$

In this chapter, we adopt the same strategy.

To the best of our knowledge, no proof has been published for $n > 3$ yet. However, we gather some remarkable results that have been obtained towards the resolution of the Decomposability Conjecture. If $f : \mathcal{X} \to \mathcal{Y}$, let $(id, f) : \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$ denote the function $x \mapsto (x, f(x))$.

Theorem 5.7 (Corollaries 1.2 and 1.3 in [PS12]). *If $\mathcal{X}$ is Suslin, $\mathcal{Y}$ is separable and metrizable, and $n \in \omega_+$, then the function $f : \mathcal{X} \to \mathcal{Y}$ belongs to* $\text{Dec}\left(\Lambda_{1,1},\ \mathbf{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y})$ *if and only if (id, f) belongs to $\Lambda_{n+1,n+1}(\mathcal{X}, \mathcal{X} \times \mathcal{Y})$.*

Moreover, if f is injective and open, or if f belongs to $\Lambda_{1,n}(\mathcal{X}, \mathcal{Y})$ (or $\Lambda_{1,2}(\mathcal{X}, \mathcal{Y})$ if $n = 1$), then $f : \mathcal{X} \to \mathcal{Y}$ belongs to $\text{Dec}\left(\Lambda_{1,1},\ \mathbf{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y})$ *if and only if f belongs to $\Lambda_{n+1,n+1}(\mathcal{X}, \mathcal{Y})$, i.e., the Decomposability Conjecture is true for injective and open functions and for functions whose level of measurability is one below the one of the assumption.*

In [MR13], Motto Ros provided an equivalent statement of the Decomposability Conjecture.

Conjecture 5.8 (Conjecture 6.1 in [MR13]). *If $\mathcal{X}$ is Suslin, $\mathcal{Y}$ is separable and metrizable, $1 < m < n < \omega$ and $f : \mathcal{X} \to \mathcal{Y} \in \Lambda_{m,n}(\mathcal{X}, \mathcal{Y})$, then the topology τ of $\mathcal{X}$ can be refined by a topology $\tau' \subseteq \mathbf{\Sigma}^0_2(\mathcal{X}, \tau)$ such that $(\mathcal{X}, \tau')$ is Suslin and $f : \mathcal{X} \to \mathcal{Y} \in \Lambda_{m,n-1}((\mathcal{X}, \tau'), \mathcal{Y})$.*

Theorem 5.9 (Theorem 6.4 in [MR13]). *Conjecture 5.8 is equivalent to the Decomposability Conjecture.*

Recently, major progress has been made by Gregoriades, Kihara and Ng who proved the Decomposability Conjecture for functions one level of measurability below the one of the assumption [GKN21].

Theorem 5.10 (Theorem 1.1 in [GKN21]). *If $\mathcal{X}'$ and $\mathcal{Y}$ are Polish, $\mathcal{X} \subseteq \mathcal{X}'$ is Suslin and $1 \leq m \leq n < \omega$, then*

$$\Lambda_{m,n}(\mathcal{X}, \mathcal{Y}) = \text{Dec}\left(\Lambda_{1,n-m+1},\ \mathbf{\Delta}^0_{n+1}\right)(\mathcal{X}, \mathcal{Y}).$$

Moreover, if $m \geq 3$ and $f \in \Lambda_{1,n-1}(\mathcal{X}, \mathcal{Y})$, then $f \in \Lambda_{m,n}(\mathcal{X}, \mathcal{Y})$ if and only if $f \in \text{Dec}\left(\Lambda_{1,n-m+1},\ \mathbf{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y})$, i.e., the Decomposability Conjecture holds for all functions $f \in \Lambda_{1,n-1}(\mathcal{X}, \mathcal{Y})$.

The same authors also proved the case $m = 2$ to be sufficient to prove the whole Decomposability Conjecture.

Theorem 5.11 (Corollary 1.2 in [GKN21]). *Let $\mathcal{X}'$ and $\mathcal{Y}$ be Polish, $\mathcal{X} \subseteq \mathcal{X}'$ be Suslin and $n \geq 2$. If*

$$\Lambda_{2,n}(\mathcal{X}, \mathcal{Y}) \subseteq \text{Dec}\left(\Lambda_{1,n-1},\ \mathbf{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y}),$$

then for any $2 \leq m \leq n$,

$$\Lambda_{m,n}(\mathcal{X}, \mathcal{Y}) \subseteq \text{Dec}\left(\Lambda_{1,n-m+1},\ \mathbf{\Delta}^0_n\right)(\mathcal{X}, \mathcal{Y}).$$

To say it otherwise, the case $m = 2$ in the Decomposability Conjecture is the right generalization of the Jayne-Rogers Theorem since it allows to prove the whole Decomposability Conjecture for Polish spaces. This is the reason why we focus on a deep study of this case in the sequel.

Finally, let us mention that Day and Marks recently announced that they proved the Decomposability Conjecture for $\mathcal{X}$ and $\mathcal{Y}$ Polish assuming some determinacy principle. Unfortunately, this result is still unpublished yet.

Theorem 5.12 ([Day19, Mar20]). *Assuming $\mathbf{\Sigma}^1_2$-determinacy, the Decomposability Conjecture is true for $\mathcal{X}$ and $\mathcal{Y}$ Polish.*

In this chapter, we provide new techniques to tackle the Decomposability Conjecture. These techniques make use of the game-theoretical framework of zero-dimensional Polish spaces as well as the question-tree machinery developed by Duparc in [Dup01] (see Subsection 2.4.3). In particular, we prove that if $\mathcal{X}$ and $\mathcal{Y}$ are zero-dimensional Polish spaces, then, under the axiom of determinacy, the Decomposability Conjecture is a consequence of the following Assumption 5.13 (Theorem 5.14).

Assumption 5.13. *For any $f : [T] \to \omega^\omega \in \Lambda_{1,2}$ where $T \subseteq \omega^{<\omega}$ is any non-empty pruned tree, there exists a perfect set $\mathcal{P} \subseteq [T]$ such that:*

1. $f : [T] \setminus \mathcal{P} \to \omega^\omega \in \mathrm{Dec}\left(\Lambda_{1,1},\ \mathbf{\Delta}^0_2\right)$.
2. *If $\mathcal{P}$ is non-empty, there exist three sets $\left\{x^l_n : n, l \in \omega\right\} \subseteq \mathcal{T}(\mathcal{P})$, $\{p_n : n \in \omega\} \subseteq \mathcal{T}(\mathcal{P})$ and $\{u_n : n \in \omega\} \subseteq \omega^{<\omega}$ such that:*
   (a) *For any $n, l \in \omega$, $p_n \sqsubset x^l_n$ and $u_n \sqsubset f\left(x^l_n\right)$.*
   (b) *$\{u_n : n \in \omega\} \subseteq \omega^{<\omega}$ is a set of pairwise incompatible elements.*
   (c) *For any $n \in \omega$, $f^{-1}[u_n] \cap [p_n]$ is proper and non-self-dual in $\mathcal{P}$.*
   (d) *For any $n, l \in \omega$, $x^l_n \in \left[\mathrm{Init}_{\mathcal{P}}\left(f^{-1}[u_n] \cap [p_n]\right)\right]$.*
   (e) *If $p \in \mathrm{Init}_{\mathcal{P}}\left(f^{-1}[u_n] \cap [p_n]\right)$, there exists $l \in \omega$ such that $p \sqsubset x^l_n$.*
   (f) *If $p \in \mathrm{Init}_{\mathcal{P}}\left(f^{-1}[u_n] \cap [p_n]\right)$, there exists $m \in \omega$ such that we have $p \sqsubset p_m \notin \mathrm{Init}_{\mathcal{P}}\left(f^{-1}[u_n] \cap [p_n]\right)$.*

Theorem 5.14 (AD). *The Decomposability Conjecture for zero-dimensional Polish spaces is a consequence of Assumption 5.13.*

Even if Assumption 5.13 happens to fail, we strongly believe that the new techniques and constructions introduced in this chapter offer a novel perspective towards the resolution of the Decomposability Conjecture.

We recall the following standard notation. If $x_0, x_1 \in \omega^\omega$, then $x_0 \oplus x_1 \in \omega^\omega$ is defined as the joint of x_0 and x_1, i.e., $x_0 \oplus x_1 (2n) = x_0 (n)$ and $x_0 \oplus x_1 (2n+1) = x_1 (n)$. If $T_0, T_1 \subseteq \omega^{<\omega}$ are two non-empty pruned tree, let

$$[T_0 \oplus T_1] = \{x_0 \oplus x_1 \in \omega^\omega : x_0 \in [T_0] \text{ and } x_1 \in [T_1]\}.$$

The set $[T_0 \oplus T_1]$ is the set of infinite branches of a tree $T_0 \oplus T_1$. Suppose that $f_0 : [T_0] \to \omega^\omega$ and $f_1 : [T_1] \to \omega^\omega$ are two functions, we define the function $f_0 \oplus f_1 : [T_0 \oplus T_1] \to \omega^\omega$ as follows: $f_0 \oplus f_1 (x)(2n) = f_0 (x)(n)$ and $f_0 \oplus f_1 (x)(2n+1) = f_1 (x)(n)$.

This chapter is organized as follows. In Section 5.1, we characterize the functions $f : A \subseteq \omega^\omega \to \omega^\omega$ that belong to $\text{Dec}\left(\Lambda_{1,1},\ \mathbf{\Delta}^0_2\right)$ through a perfect subset of their domains, called the core of the function. Afterwards, we only consider functions $f : [T] \to \omega^\omega$ for $T \subseteq \omega^{<\omega}$ a non-empty pruned tree. However, since any non-empty zero-dimensional Polish space is homeomorphic to the set of infinite branches $[T]$ of some non-empty pruned tree $T \subseteq \omega^{\leq\omega}$, our results apply for any function between two zero-dimensional Polish spaces. In Section 5.2, we prove that Assumption 5.13 is not too wild for it holds for any function $id \oplus f$, where $f \in \Lambda_{1,2}$. Observe that such functions already appear in Theorem 5.7. Then in Section 5.3, we prove that Assumption 5.13 implies the Jayne-Rogers Theorem. In Sections 5.4 and 5.5, we prove, under AD and Assumption 5.13, two generalizations of the Jayne-Rogers Theorem, namely the cases $(m = 2, n = 3)$ and $(m = 2, n = 4)$ of the Decomposability Conjecture. Finally, Section 5.6 contains the main result of this chapter which is the fact that, under AD and Assumption 5.13, the Decomposability Conjecture holds.

5.1 Characterizing the $\text{Dec}\left(\mathbf{\Lambda_{1,1}},\ \mathbf{\Delta}^0_2\right)$-functions

Let $A \subseteq \omega^\omega$. For any $f : A \to \omega^\omega \in \Lambda_{1,2}$, we exhibit a perfect subset $\mathcal{P}_f \in \mathbf{\Pi}^0_1(A)$ called *the core of the function* f for the topological complexity of f lies in $\mathcal{P}_f$. More precisely, we prove that $f \upharpoonright {\mathcal{P}_f}^{\mathrm{c}} \in \text{Dec}\left(\Lambda_{1,1},\ \mathbf{\Delta}^0_2\right)$ and if $\mathcal{P}_f$ is non-empty, then $f \upharpoonright \mathcal{P}_f \notin \text{Dec}\left(\Lambda_{1,1},\ \mathbf{\Delta}^0_2\right)$. In other words, the function is topologically simple outside its core $\mathcal{P}_f$ and topologically complex inside its core $\mathcal{P}_f$.

We define $\mathcal{P}_f \subseteq A$ by recursively peeling off the continuous parts of f. If $f : A \to \omega^\omega \in \Lambda_{1,2}$, let

$$\text{cont}(f) = \left\{t \in \omega^{<\omega} : f \upharpoonright A \cap [t] \text{ is continuous}\right\}.$$

Clearly, if $t' \in \omega^\omega$, $t \sqsubseteq t'$ and $t \in \operatorname{cont}(f)$, then $t' \in \operatorname{cont}(f)$. We construct a decreasing sequence of closed sets $(\mathcal{C}_\alpha)_{\alpha\in\omega_1}$ in A. Let $\mathcal{C}_0 = A$. For any $\alpha \in \omega_1$, let $f_\alpha = f \upharpoonright \mathcal{C}_\alpha$ and define

$$\mathcal{C}_{\alpha+1} = \mathcal{C}_\alpha \setminus \bigcup_{t \in \operatorname{cont}(f_\alpha)} [t] .$$

If $\lambda \in \omega_1$ is a limit ordinal, let

$$\mathcal{C}_\lambda = \bigcap_{\alpha<\lambda} \mathcal{C}_\alpha .$$

Clearly, $\mathcal{C}_\alpha$ is closed for any $\alpha \in \omega_1$. By Theorem 6.9 in [Kec95], such a decreasing sequence of closed sets eventually stabilizes at some countable ordinal $\beta \in \omega_1$. Let $\mathcal{P}_f = \mathcal{C}_\beta \in \mathbf{\Pi}^0_1(A)$. Observe that $\mathcal{P}_f$ is a perfect set since any isolated point is a continuity point. Moreover

$$\mathcal{P}_f{}^{\mathrm{c}} = \bigsqcup_{\alpha<\beta} \mathcal{C}_\alpha \setminus \mathcal{C}_{\alpha+1},$$

$\mathcal{C}_\alpha \setminus \mathcal{C}_{\alpha+1} \in \mathbf{\Delta}^0_2(A)$ and $f \upharpoonright \mathcal{C}_\alpha \setminus \mathcal{C}_{\alpha+1}$ is continuous so that $f \upharpoonright \mathcal{P}_f{}^{\mathrm{c}}$ is topologically simple. Thus, we proved.

Theorem 5.15. *If $f : A \to \omega^\omega$, then $f \upharpoonright \mathcal{P}_f{}^{\mathrm{c}} \in \operatorname{Dec}\left(\Lambda_{1,1},\ \mathbf{\Delta}^0_2\right)$.*

To the contrary, $f \upharpoonright \mathcal{P}_f$ is topologically complex.

Theorem 5.16. *Let $f : A \to \omega^\omega$. Then $\mathcal{P}_f$ is non-empty if and only if $f \notin \operatorname{Dec}\left(\Lambda_{1,1},\ \mathbf{\Delta}^0_2\right)$.*

Proof. One direction is given by Theorem 5.15. Otherwise, suppose that $\mathcal{P}_f$ is non-empty. We prove $f \notin \operatorname{Dec}\left(\Lambda_{1,1},\ \mathbf{\Delta}^0_2\right)$.

By definition, any $x_0 \in \mathcal{P}_f$ is a discontinuity point of the function $f \upharpoonright \mathcal{P}_f$. In particular, there exists $t_0 \sqsubset f(x_0)$ such that, for any $p \sqsubset x_0$, there exists $x_1 \sqsupset p$ satisfying $t_0 \not\sqsubset f(x_1)$.

We construct a winning strategy for I in the backtrack game $G_{\mathrm{bt}}\left(f \upharpoonright \mathcal{P}_f\right)$. Let $x_0 \in \mathcal{P}_f$. As long as II does not produce t_0, I plays along x_0. If II produces t_0 and I already played $p \sqsubset x_0$, then I decides to play along $x_1 \sqsupset p$ instead of x_0, where x_1 is given by discontinuity to satisfy $t_0 \not\sqsubset f(x_1)$. Then, as long as II does not produce t_1, I plays along x_1, where $t_1 \sqsubset f(x_1)$ is given by discontinuity and satisfies $t_0 \perp t_1$. In order to produce t_1, II has to erase everything she played so far. Suppose she does and I already played $p \sqsubset x_1$, then I decides to play along x_2 instead of x_1 which is also given by discontinuity, and so on. After she plays t_n for some $n \in \omega$, II has to use her eraser to produce t_{n+1} otherwise she looses the game. Thus, to avoid loosing, II has to erase infinitely many times which makes her loose the game. Thus, we designed a winning strategy for I in the game $G_{\mathrm{bt}}\left(f \upharpoonright \mathcal{P}_f\right)$. This implies that $f \upharpoonright \mathcal{P}_f \notin \mathrm{Dec}\left(\Lambda_{1,1},\ \boldsymbol{\Delta}^0_2\right)$ which yields $f \notin \mathrm{Dec}\left(\Lambda_{1,1},\ \boldsymbol{\Delta}^0_2\right)$. □

5.2 On the functions $id \oplus f$

In this section, we show that Assumption 5.13 is not too wild for, if we have $f' : [T'] \to \omega^\omega \in \Lambda_{1,2}$, then $f = id \oplus f' : [T] \to \omega^\omega \in \Lambda_{1,2}$ satisfies Assumption 5.13. Suppose that $\mathcal{P}_f$ is non-empty. We first prove two easy lemmas which hold for any function $f : [T] \to \omega^\omega \in \Lambda_{1,2}$.

Proposition 5.17. *Let $f : [T] \to \omega^\omega \in \Lambda_{1,2}$ such that $\mathcal{P}_f$ is non-empty. If $t \in \mathcal{T}\left(\mathcal{P}_f\right)$, there exist $u_t \in \omega^{<\omega}$ and $p_t \sqsupseteq t$ such that $\mathrm{lh}\left(u_t\right) > \mathrm{lh}\left(t\right) \cdot 2$ and $f^{-1}[u_t] \cap [p_t]$ is $\boldsymbol{\Pi}^0_1\left(\mathcal{P}_f\right)$-complete.*

Proof. Since the function $f \upharpoonright \mathcal{P}_f \cap [t]$ is not continuous, there exists $u \in \omega^{<\omega}$ such that $f^{-1}[u] \notin \boldsymbol{\Delta}^0_1\left(\mathcal{P}_f \cap [t]\right)$. By Theorem 2.43, there exists $p \in \mathcal{T}\left(\mathcal{P}_f\right)$ such that $t \sqsubseteq p$ and $f^{-1}[u] \cap [p]$ is either $\boldsymbol{\Pi}^0_1\left(\mathcal{P}_f \cap [t]\right)$-complete or $\boldsymbol{\Sigma}^0_1\left(\mathcal{P}_f \cap [t]\right)$-complete.

1. If it is $\boldsymbol{\Pi}^0_1\left(\mathcal{P}_f \cap [t]\right)$-complete, let σ be a winning strategy for II in the game $G_w\left(\{0^\omega\}, f^{-1}[u] \cap [p]\right)\left(\omega^\omega, \mathcal{P}_f \cap [t]\right)$. If $x = \sigma\left(0^\omega\right)$, then $u \sqsubset f\left(x\right)$. Let $u \sqsubset u_t \sqsubset f\left(x\right)$ such that $\mathrm{lh}\left(u_t\right) > \mathrm{lh}\left(t\right) \cdot 2$. There exists $k \in \omega$ such that $\sigma\left(0^k\right) \sqsupseteq p$. Let $p_t = \sigma\left(0^k\right)$. Since σ is winning, $u \not\sqsubset f\left(\sigma\left(y\right)\right)$ and $u_t \not\sqsubset f\left(\sigma\left(y\right)\right)$ for any $y \neq 0^\omega$. In particular, the strategy σ witnesses that $f^{-1}[u_t] \cap [p_t]$ is $\boldsymbol{\Pi}^0_1\left(\mathcal{P}_f \cap [t]\right)$-complete. Since $t \sqsubseteq p_t$, we obtain that $f^{-1}[u_t] \cap [p_t]$ is $\boldsymbol{\Pi}^0_1\left(\mathcal{P}_f\right)$-complete.

2. If it is $\boldsymbol{\Sigma}^0_1\left(\mathcal{P}_f \cap [t]\right)$-complete, let σ be a winning strategy for II in the game $G_w\left(\{0^\omega\}^{\mathrm{c}}, f^{-1}[u] \cap [p]\right)\left(\omega^\omega, \mathcal{P}_f \cap [t]\right)$. If $x = \sigma\left(0^\omega\right)$, then $u \not\sqsubset f\left(x\right)$. Let $u_t \sqsubset f(x)$ such that $\mathrm{lh}\left(u_t\right) > \mathrm{lh}\left(t\right) \cdot 2$ and $u_t \perp u$. There exists $k \in \omega$ such that $\sigma\left(0^k\right) \sqsupseteq t$. Let $p_t = \sigma\left(0^k\right)$. Since $u_t \sqsubset f\left(x\right)$ and, for $y \neq 0^\omega$, $u \sqsubset f\left(\sigma\left(y\right)\right)$, we have $u_t \not\sqsubset f\left(\sigma\left(y\right)\right)$. In particular, the strategy σ witnesses that $f^{-1}[u_t] \cap [p_t]$ is $\boldsymbol{\Pi}^0_1\left(\mathcal{P}_f \cap [t]\right)$-complete. Since $t \sqsubseteq p_t$, we obtain that $f^{-1}[u_t] \cap [p_t]$ is $\boldsymbol{\Pi}^0_1\left(\mathcal{P}_f\right)$-complete.

□

Lemma 5.18. *If $t \in T$ and $A \cap [t] \subseteq [T]$ is $\mathbf{\Pi}^0_1([T])$-complete, there exists $e_t \sqsupset t$ such that $A \cap [e_t] = \emptyset$.*

Proof. Consider a winning strategy σ for II in the game

$$G_w(\{0^\omega\}, A \cap [t])(\omega^\omega, [T]).$$

There exists $k \in \omega$ such that $\sigma(0^k) \sqsupseteq t$. Let $x = \sigma(0^k 1^\omega)$. Then $t \sqsubset x \notin A$. Since A^{c} is $\mathbf{\Sigma}^0_1([T] \cap [t])$-complete, there exists $t \sqsubseteq e_t \sqsubset x$ such that $[e_t] \subseteq A^{\mathrm{c}}$ in $[T]$. To say it otherwise, $A \cap [e_t] = \emptyset$. □

We finally come up with the definitions of the sets involved in Assumption 5.13 for functions of the form $f = id \oplus f'$, where $f' : [T] \to \omega^\omega \in \Lambda_{1,2}$.

Theorem 5.19. *Assumption 5.13 is verified for any function of the form $f = id \oplus f'$, where $f' : [T] \to \omega^\omega \in \Lambda_{1,2}$. Moreover, one can take $\mathcal{P}$ to be the core $\mathcal{P}_f$ of the function f.*

Proof. Let $f = id \oplus f' : [T] \to \omega^\omega \in \Lambda_{1,2}$ such that $\mathcal{P}_f$ is non-empty. Let $\mathcal{F}_0 = \{\langle\rangle\}$. We apply Proposition 5.17 to $\langle\rangle \in \mathcal{T}(\mathcal{P}_f)$ to define $u_{\langle\rangle}$ and $p_{\langle\rangle}$. Let also $T_{\langle\rangle} = \mathrm{Init}_{\mathcal{P}_f}\left(f^{-1}\left[u_{\langle\rangle}\right] \cap \left[p_{\langle\rangle}\right]\right)$. For any $t \in T_{\langle\rangle}$ such that $p_{\langle\rangle} \sqsubset t$, there exists $e_t \sqsupset t$ given by Lemma 5.18 such that $f^{-1}\left[u_{\langle\rangle}\right] \cap [e_t] = \emptyset$. We define $\mathcal{F}_1 = \left\{e_t : p_{\langle\rangle} \sqsubset t \in T_{\langle\rangle}\right\}$.

Suppose $s \in \mathcal{F}_k$. We apply Proposition 5.17 to $s \in \mathcal{T}(\mathcal{P}_f)$ to define u_s and p_s. Let also $T_s = \mathrm{Init}_{\mathcal{P}_f}\left(f^{-1}[u_s] \cap [p_s]\right)$. For any $t \in T_s$ such that $p_s \sqsubseteq t$, there exists $e_t \sqsupset t$ given by Lemma 5.18 such that $f^{-1}[u_s] \cap [e_t] = \emptyset$. We define $\mathcal{F}_{k+1} = \bigcup_{s \in \mathcal{F}_k} \{e_t : p_s \sqsubseteq t \in T_s\}$.

Finally, if $\mathcal{F} = \bigcup_{n \in \omega} \mathcal{F}_n \subseteq \omega^{<\omega}$, we define the three sets $\{p_t : t \in \mathcal{F}\} \subseteq \mathcal{T}(\mathcal{P}_f)$, $\{u_t : t \in \mathcal{F}\} \subseteq \omega^{<\omega}$ and a countable dense subset of $[T_t]$ denoted by $\left\{x^l_t : l \in \omega, t \in \mathcal{F}\right\}$.

Since for any $t \in \mathcal{F}$, $\mathrm{lh}(u_t) > \mathrm{lh}(t) \cdot 2$ and $f^{-1}[u_t] \cap [t] \neq \emptyset$, we get $f^{-1}[u_t] \subseteq [t]$. Thus, if $t, t' \in \mathcal{F}$ such that $t \perp t'$, then $f^{-1}[u_t] \cap [t'] = \emptyset$. Also, for any $t \in \mathcal{F}$, we have $f^{-1}[u_t] \cap [e_t] = \emptyset$. □

5.3 A new version of the Jayne-Rogers Theorem

In this section, we prove a new version of the Jayne-Rogers Theorem which states that the Jayne-Rogers Theorem for zero-dimensional Polish spaces is a consequence of Assumption 5.13.

Theorem 5.20 (Jayne-Rogers Theorem)**.** *Assuming Assumption 5.13, if* $f : [T] \to \omega^\omega \in \Lambda_{1,2} \setminus \mathrm{Dec}\left(\Lambda_{1,1},\ \mathbf{\Delta}^0_2\right)$*, then* $f \notin \Lambda_{2,2}$*.*

In particular, if $\mathcal{X}$ *and* $\mathcal{Y}$ *are both zero-dimensional Polish spaces, then Assumption 5.13 implies the Jayne-Rogers Theorem, i.e.,*

$$\Lambda_{2,2}(\mathcal{X},\mathcal{Y}) = \mathrm{Dec}\left(\Lambda_{1,1},\ \mathbf{\Delta}^0_2\right)(\mathcal{X},\mathcal{Y}).$$

Proof. Let $f : [T] \to \omega^\omega \in \Lambda_{1,2} \setminus \mathrm{Dec}\left(\Lambda_{1,1},\ \mathbf{\Delta}^0_2\right)$ and $\tau^{\leftarrow}$ a winning strategy for II in the eraser game $G_{\twoheadleftarrow}(f)$. In this case, $\mathcal{P}$ given by Assumption 5.13 is non-empty. Let $\mathcal{U} = \bigcup_{n\in\omega} [u_n]$. We describe winning strategies for II in the game

$$G_w\left(\left(\mathcal{O}^{\sim_1}\right)^{\mathtt{b}}, f^{-1}[\mathcal{U}]\right)\left({\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{\omega}, \mathcal{P}\right),$$

where $\mathcal{O} = \{\langle\rangle\}^{\mathrm{c}} \subseteq \omega^{\leq\omega}$. By Theorem 2.53, $\left(\mathcal{O}^{\sim_1}\right)^{\mathtt{b}}$ is $\mathbf{\Sigma}^0_2\left({\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{\omega}\right)$-complete, so that Equation (5.1) (page 94) yields $f \restriction \mathcal{P} \notin \Lambda_{2,2}$. We construct these strategies by induction on the length of $s \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$. For the first step, let $\sigma(\langle\rangle) \in \mathcal{T}(\mathcal{P})$ such that $p_0 \sqsubseteq \sigma(\langle\rangle) \sqsubset x^0_0$ and $u_0 \sqsubseteq \tau^{\leftarrow}(\sigma(\langle\rangle))^{\hookleftarrow}$. For the inductive step, let $sa \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$ and suppose that we have already defined $\sigma(s)$ such that there exists $n \in \omega$ satisfying $p_n \sqsubseteq \sigma(s) \sqsubset x^0_n$. We consider two different cases.

1. If $\widetilde{sa}_{[\ /\mathtt{b}]} \neq \langle\rangle$, then let $\sigma(s) \sqsubset \sigma(sa) \sqsubset x^0_n$.
2. Otherwise, $\widetilde{sa}_{[\ /\mathtt{b}]} = \langle\rangle$. In that case, we choose $m \neq n$ such that $\sigma(s) \sqsubset p_m$ and $p_m \notin \mathrm{Init}_{\mathcal{P}}\left(f^{-1}[u_n] \cap [p_n]\right)$. Let $\sigma(sa) \in \mathcal{T}(\mathcal{P})$ such that $p_m \sqsubseteq \sigma(sa) \sqsubset x^0_m$ and $u_m \sqsubseteq \tau^{\leftarrow}(\sigma(sa))^{\hookleftarrow}$.

The strategy σ that we defined is strictly $\sqsubseteq$-increasing. It is schematically depicted in Figure 5.1.

By construction, for any $s \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$, there exists a unique $n \in \omega$ such that $p_n \sqsubseteq \sigma(s) \sqsubset x^0_n$ and $\lim_{k\in\omega} \sigma\left(s0^k\right) \in f^{-1}[u_n]$. By Assumption 5.13, we also have $f^{-1}[u_n] \cap [p_n] \equiv_w f^{-1}[u_n] \cap [\sigma(s)]$ in $\mathcal{P}$. It remains to prove that σ is winning. For this purpose, it suffices to show that, for any $x \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{\omega}$, we have:

$$x \in \left(\mathcal{O}^{\sim_1}\right)^{\mathtt{b}} \ \Leftrightarrow\ \lim_{k\in\omega} \sigma\left(x_{\restriction k}\right) \in f^{-1}[\mathcal{U}].$$

Suppose that I plays $x \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{\omega}$. We consider two different cases.

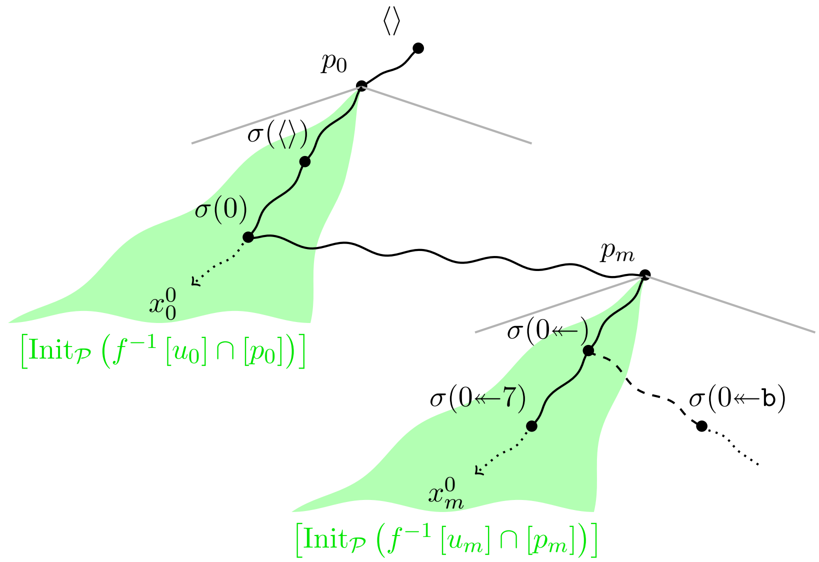


Figure 5.1: The strategy σ.

The case $x \in (\mathcal{O}^{\sim_1})^{\mathtt{b}}$

In that case, there exists a minimal $m \in \omega$ such that, for any $m' > m$, we have

$$\widetilde{x_{\restriction m'}}_{[\ /\mathtt{b}]} \neq \langle\rangle.$$

In particular, after I plays $x_{\restriction m+1}$, II only uses the first case in the construction of the strategy σ. Thus, there exists a unique $n \in \omega$ such that we have $p_n \sqsubseteq \sigma\left(x_{\restriction m+1}\right) \sqsubset x_n^0$, $\lim_{k\in\omega}\sigma\left(x_{\restriction m+1}0^k\right) \in f^{-1}[u_n]$ and, for any $m' > m$, $f^{-1}[u_n] \cap \left[\sigma\left(x_{\restriction m'}\right)\right] \equiv_w f^{-1}[u_n] \cap \left[x_n^0\right]$. The fact that $\lim_{k\in\omega}\sigma\left(x_{\restriction k}\right) \in f^{-1}[u_n]$ then follows from Proposition 2.42.

The case $x \notin (\mathcal{O}^{\sim_1})^{\mathtt{b}}$

We show that, in the case $x \notin (\mathcal{O}^{\sim_1})^{\mathtt{b}}$, we have $\lim_{k\in\omega}\sigma\left(x_{\restriction k}\right) \notin f^{-1}[u_n]$ for each $n \in \omega$. Fix $n \in \omega$. Since $\widetilde{x}_{[\ /\mathtt{b}]} = \langle\rangle$, one has

$$\mathrm{Card}\left\{l \in \omega : \widetilde{x_{\restriction l}}_{[\ /\mathtt{b}]} = \langle\rangle\right\} = \aleph_0.$$

In particular, there exist both a strictly increasing sequence $(n_i)_{i\in\omega} \subseteq \omega$ and an infinite set $\{m_i : i \in \omega\} \subseteq \omega$ such that $u_{m_i} \sqsubseteq \tau^{\leftarrow}\left(\sigma\left(x_{\restriction n_i}\right)\right)^{\looparrowleft}$ and $m_i \neq m_j$ hold for all $i < j$. The result relies on the following proposition.

Proposition 5.21 (Lemma 3.4.2 in [Sem09]). *Let $f : [T] \to \omega^\omega \in \Lambda_{1,2}$ and $\tau^{\leftarrow}$ a winning strategy for* II *in the eraser game $G_{\leftarrow}(f)$. Let $(p_l)_{l\in\omega} \subseteq T$ be a $\sqsubset$-increasing sequence and $z = \lim_{l\in\omega} p_l \in [T]$. Let $\{t_n : n \in \omega\} \subset \omega^{<\omega}$ be a set of pairwise incompatible elements and suppose that for any $l \in \omega$, there exists $t_{n(l)} \in \{t_n : n \in \omega\}$ such that $t_{n(l)} \sqsubseteq \tau^{\leftarrow}(p_l)^{\hookleftarrow}$ and $t_{n(l)} \neq t_{n(l')}$ for any $l < l'$. Then*

$$f(z) \notin \bigcup_{n\in\omega} [t_n] .$$

Proof. Towards a contradiction, suppose $f(z) \in [t_n]$. There exists an infinite subsequence $(p_{l_i})_{i\in\omega}$ such that $t_n \not\sqsubseteq \tau^{\leftarrow}(p_{l_i})^{\hookleftarrow}$ for any $i \in \omega$. In particular, $t_n \not\sqsubseteq \lim_{l\in\omega} \tau^{\leftarrow}\left(z_{\restriction l}\right)^{\hookleftarrow}$. Since $\tau^{\leftarrow}$ is winning, we obtain

$$t_n \not\sqsubseteq \lim_{l\in\omega} \tau^{\leftarrow}\left(z_{\restriction l}\right)^{\hookleftarrow} = f(z).$$

□

This implies that σ is a winning strategy for II in the game

$$G_w\left(\left(\mathcal{O}^{\sim 1}\right)^{\mathtt{b}}, f^{-1}[\mathcal{U}]\right)\left({\omega_{\{\mathtt{b},\leftarrow\}}}^{\omega}, \mathcal{P}\right).$$

By Theorem 2.53 and Equation (5.1) (page 94), we have $f \restriction \mathcal{P} \notin \Lambda_{2,2}$. In particular, $f \notin \Lambda_{2,2}$ which completes the proof of Theorem 5.20. □

We finish this section with some remarks about the strategy σ. Firstly, observe that all along the construction we never made use of x_n^l for $l > 0$. However, II could choose any x_n^l instead of x_n^0 in the construction and obtain another winning strategy σ' in the same game. Thus, Assumption 5.13 actually yields several winning strategies for II in the game

$$G_w\left(\left(\mathcal{O}^{\sim 1}\right)^{\mathtt{b}}, f^{-1}[\mathcal{U}]\right)\left({\omega_{\{\mathtt{b},\leftarrow\}}}^{\omega}, \mathcal{P}\right)$$

for a fixed $\mathcal{U} \in \mathbf{\Sigma}_1^0(\omega^\omega)$. Moreover, for any such strategy σ and any $s \in {\omega_{\{\mathtt{b},\leftarrow\}}}^{<\omega}$, there exist both a unique $n \in \omega$ and some $l \in \omega$ such that $p_n \sqsubseteq \sigma(s) \sqsubset x_n^l$ and $\lim_{k\in\omega} \sigma\left(s0^k\right) \in f^{-1}[u_n]$. These observations do really matter in generalizing the Jayne-Rogers Theorem in the next sections.

Secondly, we notice that our proof of the Jayne-Rogers Theorem is highly inspired from the proof provided by Semmes in [Sem09]. Indeed, the idea in Semmes' construction only differs in that $x_n^0 = x_n^l$ for any $n, l \in \omega$. In particular, II follows some fixed branch $x_n^0 \in \mathcal{P}$ so that Semmes' construction only yields a single — modulo obvious shifts — winning strategy in the game

$$G_w\left(\left(\mathcal{O}^{\sim_1}\right)^{\mathtt{b}}, f^{-1}[\mathcal{U}]\right)\left({\omega_{\{\mathtt{b},\leftarrow\}}}^{\omega}, \mathcal{P}\right).$$

To the contrary, assuming Assumption 5.13, II can follow whatever branch in a dense subset of some $\left[\mathrm{Init}_{\mathcal{P}}\left(f^{-1}[u_n] \cap [p_n]\right)\right]$. To say it otherwise, Assumption 5.13 is more flexible than Semmes' construction. In the next section, this flexibility allows II to somehow undo some of her choices.

5.4 A first generalization of the Jayne-Rogers Theorem

Assuming the axiom of determinacy and Assumption 5.13, we prove a first generalization of the Jayne-Rogers Theorem.

Theorem 5.22 (AD)**.** *Assuming Assumption 5.13, if $f : [T] \to \omega^\omega$ is such that $f \in \Lambda_{1,3} \setminus \mathrm{Dec}\left(\Lambda_{1,2},\ \boldsymbol{\Delta}_3^0\right)$, then we have $f \notin \Lambda_{2,3}$.*

In particular, if $\mathcal{X}$ and $\mathcal{Y}$ are both zero-dimensional Polish spaces, then Assumption 5.13 implies the case $(m = 2, n = 3)$ of the Decomposability Conjecture, i.e.,

$$\Lambda_{2,3}(\mathcal{X}, \mathcal{Y}) = \mathrm{Dec}\left(\Lambda_{1,2},\ \boldsymbol{\Delta}_3^0\right)(\mathcal{X}, \mathcal{Y}).$$

The proof relies on a natural construction which alternatively makes use of Assumption 5.13 and of the question-tree machinery developed in Subsection 2.4.3. Let $f : [T] \to \omega^\omega \in \Lambda_{1,3}$. At the end of the construction, we obtain a perfect subset ${}^0\mathcal{P}$ of $[T]$, a question-tree $\mathbb{T}$ on ${}^0\mathcal{P}$ such that $f : [\mathbb{T}] \to \omega^\omega \in \Lambda_{1,2}$ and a perfect subset ${}^1\mathcal{P} \subseteq [\mathbb{T}]$ given by Assumption 5.13. Moreover, ${}^1\mathcal{P}$ is a dense subset of ${}^0\mathcal{P}$. In other words, we make use of the question-tree machinery in order to decrease the topological complexity of the function so that we can use the construction of the preceding section. The main results of this section state that, under AD, $f \restriction {}^0\mathcal{P}^{\mathrm{c}} \in \mathrm{Dec}\left(\Lambda_{1,2},\ \boldsymbol{\Delta}_3^0\right)$ and, provided ${}^1\mathcal{P}$ — and thus ${}^0\mathcal{P}$ — is non-empty, $f \restriction {}^0\mathcal{P} \notin \Lambda_{2,3}$.

More precisely, if ${}^1\mathcal{P}$ is non-empty, the previous section yields an open set $\mathcal{U} \subseteq \omega^\omega$ and several winning strategies for II in the game

$$G_w\left(\left(\mathcal{O}^{\sim_1}\right)^{\mathtt{b}}, f^{-1}[\mathcal{U}]\right)\left({\omega_{\{\mathtt{b},\leftarrow\}}}^{\omega}, \left({}^1\mathcal{P}, \mathbb{T}\right)\right).$$

We lift these strategies from $\left({}^1\mathcal{P},\mathbb{T}\right)$ to ${}^0\mathcal{P}$ so that II has a winning strategy τ in the game

$$G_w\left(\left(\mathcal{O}^{\sim_2}\right)^{\mathtt{b}}, f^{-1}[\mathcal{U}]\right)\left({\omega_{\{\mathtt{b},\leftarrow,\leftarrow_1\}}}^{\omega}, {}^0\mathcal{P}\right).$$

This lifting is possible under AD and depends on the flexibility given by Assumption 5.13 already mentioned at the end of the preceding section.

We begin with a few remarks on notations. First, observe that ${}^0\mathcal{P}$ has two different natural topologies: the subspace topology inherited from $[T]$ and the question-tree topology inherited from $\mathbb{T}$. To avoid any kind of confusion, we write ${}^0\mathcal{P}$ for the topological space generated by the first one and $[\mathbb{T}]$ for the topological space generated by the second one. The set ${}^1\mathcal{P}$ also has two different natural topologies: the subset topology coming from ${}^0\mathcal{P}$ and the question-tree topology coming from $\mathbb{T}$. Once again and to avoid any kind of confusion, we write ${}^1\mathcal{P}$ for the topological space generated by the first one and $\left({}^1\mathcal{P},\mathbb{T}\right)$ for the topological space generated by the second one. If $\mathbb{T}$ is a question-tree on $[T]$, its elements are denoted by $\mathfrak{s}\in\mathbb{T}$ or $\mathfrak{t}\in\mathbb{T}$. If $\mathfrak{s}\in\mathbb{T}$, then $\mathfrak{s}=(t,C)$ where $t\in T$ and $C\in\mathbf{\Pi}^0_1([T])$. We use the projection functions π_0 and π_1 to speak about t and C, i.e., $\pi_0(\mathfrak{s})=t$ and $\pi_1(\mathfrak{s})=C$.

5.4.1 The construction

In order to exhibit ${}^0\mathcal{P}$, we use a process of peeling off similar as the one of the definition of the core of a $\Lambda_{1,2}$-function (Section 5.1). For the remainder of this construction, let ${}^0\mathcal{P}_{-1}=[T]$ and, if $\lambda\in\omega_1$ is a limit ordinal, let "$\lambda-1$" denote "$<\lambda$". For any $\alpha\in\omega_1$, we define ${}^0\mathcal{P}_{\alpha-1}\in\mathbf{\Pi}^0_1([T])$, ${}^1\mathcal{P}_\alpha\in\mathbf{\Pi}^0_1\left({}^0\mathcal{P}_{\alpha-1}\right)$ and a countable set $\mathcal{F}_\alpha\subseteq\mathbf{\Pi}^0_1\left({}^0\mathcal{P}_{\alpha-1}\right)$. We proceed by induction.

The initial case

Let $f:[T]\to\omega^\omega\in\Lambda_{1,3}$. For any $u\in\omega^{<\omega}$, there exists $\mathcal{F}(u)\subseteq\mathbf{\Pi}^0_1([T])$ countable such that $f^{-1}[u]\in\mathbf{\Delta}^0_2([T],\tau_u)$ where τ_u is the refinement of the prefix topology on $[T]$ obtained by adding the elements of $\mathcal{F}(u)$ as clopen sets (see Theorem 2.59). The set

$$\mathcal{F}_0=\{[t]:t\in T\}\cup\bigcup_{n\in\omega}\mathcal{F}(u)\subseteq\mathbf{\Pi}^0_1([T])$$

is countable and thus generates a question-tree $\mathbb{T}_0$ on $[T]$ which satisfies $f : [\mathbb{T}_0] \to \omega^\omega \in \Lambda_{1,2}$. Let ${}^1\mathcal{P}_0 \subseteq [\mathbb{T}_0]$ be given by Assumption 5.13 and ${}^0\mathcal{P}_0$ be the closure of ${}^1\mathcal{P}_0$ in $[T]$, i.e., the smallest closed subset of $[T]$ which contains ${}^1\mathcal{P}_0$. We make two observations about this construction.

Proposition 5.23. *Let* $f : [T] \to \omega^\omega \in \Lambda_{1,3}$.

1. ${}^0\mathcal{P}_0$ *is perfect.*
2. $f : [T] \setminus {}^0\mathcal{P}_0 \to \omega^\omega \in \mathrm{Dec}\left(\Lambda_{1,2},\ \mathbf{\Delta}^0_3\right)$.

Proof. 1. Let $x \in {}^0\mathcal{P}_0$. For any $n \in \omega$, we show the existence of $x \neq y$ such that $y \in {}^0\mathcal{P}_0 \cap \left[x_{\restriction n}\right]$. By definition of ${}^0\mathcal{P}_0$, there exists $y' \in \left[x_{\restriction n}\right]$ such that $y' \in {}^1\mathcal{P}_0$. Since $\left({}^1\mathcal{P}_0, \mathbb{T}_0\right)$ is perfect and $\mathbb{T}_0$ refines the prefix topology on $[T]$, there exists $y' \neq y$ such that $x_{\restriction n} \sqsubset y \in \left({}^1\mathcal{P}_0, \mathbb{T}_0\right)$. This yields $x_{\restriction n} \sqsubset y \in {}^0\mathcal{P}_0$.

2. By Assumption 5.13, $f : \left([T] \setminus {}^1\mathcal{P}_0, \mathbb{T}_0\right) \to \omega^\omega \in \mathrm{Dec}\left(\Lambda_{1,1},\ \mathbf{\Delta}^0_2\right)$. Thus, there exists a $\mathbf{\Delta}^0_2$-partition $(D_n)_{n\in\omega}$ of $[\mathbb{T}_0]$ such that the function $f : \left(D_n \setminus {}^1\mathcal{P}_0, \mathbb{T}_0\right) \to \omega^\omega$ is continuous for any $n \in \omega$. Since the topology of $[\mathbb{T}_0]$ is a subset of $\mathbf{\Sigma}^0_2\left([T]\right)$, we also get that $(D_n)_{n\in\omega}$ is a $\mathbf{\Delta}^0_3$-partition of $[T]$ by Theorem 2.57.

   Fix $n \in \omega$. Since $f : \left(D_n \setminus {}^1\mathcal{P}_0, \mathbb{T}_0\right) \to \omega^\omega \in \Lambda_{1,1}$, for any $u \in \omega^{<\omega}$, there exists $A_u \in \mathbf{\Delta}^0_1\left([\mathbb{T}_0]\right)$ such that $f^{-1}[u] \cap \left(D_n \setminus {}^1\mathcal{P}_0\right) = A_u \cap \left(D_n \setminus {}^1\mathcal{P}_0\right)$. In particular, $A_u \in \mathbf{\Delta}^0_2\left([T]\right)$ so that $f^{-1}[u] \cap \left(D_n \setminus {}^1\mathcal{P}_0\right) \in \mathbf{\Delta}^0_2\left(D_n \setminus {}^1\mathcal{P}_0\right)$, i.e., $f : D_n \setminus {}^1\mathcal{P}_0 \to \omega^\omega \in \Lambda_{1,2}$.

   Finally, since ${}^1\mathcal{P}_0 \subseteq {}^0\mathcal{P}_0$, we also obtain $f : D_n \setminus {}^0\mathcal{P}_0 \to \omega^\omega \in \Lambda_{1,2}$.

□

The successor case

At successor ordinal $\beta + 1$, the goal is to add some well-chosen closed sets to $\mathcal{F}_\beta$. These closed sets yield new questions in the generated question-tree whose answers are essential for the proof of Theorem 5.22. They come as a generalization of the notion of an initializable tree (Definition 2.28).

Definition 5.24. If $\mathbb{T}$ is a question-tree on $[T]$, $\mathbb{T}' \subseteq \mathbb{T}$ is a non-empty pruned tree and $A \subseteq [\mathbb{T}']$, then we define

$$\mathrm{Init}_{[\mathbb{T}'],[T]}\left(A\right) = \left\{t \in T : \text{there exists } \mathfrak{s} \in \mathrm{Init}_{[\mathbb{T}']}\left(A\right) \text{ such that } t \sqsubseteq \pi_0\left(\mathfrak{s}\right)\right\}.$$

This is clearly a tree. Moreover, if $x \in \left[\mathrm{Init}_{[\mathbb{T}']}\left(A\right)\right]$, then we easily check that $x \in \left[\mathrm{Init}_{[\mathbb{T}'],[T]}\left(A\right)\right]$. Thus, if $A \subseteq [\mathbb{T}']$ is non-self-dual, the set $\left[\mathrm{Init}_{[\mathbb{T}'],[T]}\left(A\right)\right]$ is a non-empty closed subset of $[T]$.

Suppose that we have already defined ${}^{0}\mathcal{P}_{\beta-1} \in \mathbf{\Pi}_1^0([T])$, ${}^{0}\mathcal{P}_{\beta} \in \mathbf{\Pi}_1^0\left({}^{0}\mathcal{P}_{\beta-1}\right)$ and a countable set $\mathcal{F}_{\beta} \subseteq \mathbf{\Pi}_1^0\left({}^{0}\mathcal{P}_{\beta-1}\right)$ for some $\beta \in \omega_1$. Let

$$\mathcal{B}_{\beta} = \left\{{}^{0}\mathcal{P}_{\beta} \cap A_1 \cap \cdots \cap A_m : A_i \in \mathcal{F}_{\beta} \text{ for any } 1 \leq i \leq m\right\} \subseteq \mathbf{\Pi}_1^0\left({}^{0}\mathcal{P}_{\beta}\right)$$

be the countable basis on ${}^{0}\mathcal{P}_{\beta}$ generated by $\mathcal{F}_{\beta}$. We define

$$\mathcal{F}_{\beta+1} = \mathcal{B}_{\beta} \cup \left\{\left[\mathrm{Init}_{\left({}^{1}\mathcal{P}_{\beta}, \mathbb{T}_{\beta}\right), {}^{0}\mathcal{P}_{\beta}}\left(f^{-1}[u] \cap B\right)\right] : u \in \omega^{<\omega}, B \in \mathcal{B}_{\beta}\right\}.$$

In other words, we add questions, i.e., closed sets, which reveal the complexity of the preimage under f of basic open sets in the topology $\left[\mathbb{T}_{\beta}\right]$. Observe that $\mathcal{F}_{\beta+1}$ is a countable subset of $\mathbf{\Pi}_1^0\left({}^{0}\mathcal{P}_{\beta}\right)$. Thus it generates a question-tree $\mathbb{T}_{\beta+1}$ on ${}^{0}\mathcal{P}_{\beta}$. The function $f : \left[\mathbb{T}_{\beta+1}\right] \to \omega^{\omega}$ belongs to $\Lambda_{1,2}$ for the topology of $\left[\mathbb{T}_{\beta+1}\right]$ is a refinement of the topology of $\left[\mathbb{T}_{\beta}\right]$ on the set ${}^{0}\mathcal{P}_{\beta}$. Let ${}^{1}\mathcal{P}_{\beta+1} \subseteq \left[\mathbb{T}_{\beta+1}\right]$ be given by Assumption 5.13 and ${}^{0}\mathcal{P}_{\beta+1}$ be the closure of ${}^{1}\mathcal{P}_{\beta+1}$ in ${}^{0}\mathcal{P}_{\beta}$. We also have the following result.

Proposition 5.25. *Let* $f : [T] \to \omega^{\omega} \in \Lambda_{1,3}$.

1. ${}^{0}\mathcal{P}_{\beta+1}$ *is perfect.*
2. $f : {}^{0}\mathcal{P}_{\beta} \setminus {}^{0}\mathcal{P}_{\beta+1} \to \omega^{\omega} \in \mathrm{Dec}\left(\Lambda_{1,2},\ \mathbf{\Delta}_3^0\right)$.

Proof. It suffices to use the same arguments as the ones in the proof of Proposition 5.23. □

The limit case

Suppose that we have already defined ${}^{0}\mathcal{P}_{\alpha-1} \in \mathbf{\Pi}_1^0([T])$, ${}^{0}\mathcal{P}_{\alpha} \in \mathbf{\Pi}_1^0\left({}^{0}\mathcal{P}_{\alpha-1}\right)$, and $\mathcal{F}_{\alpha} \subseteq \mathbf{\Pi}_1^0\left({}^{0}\mathcal{P}_{\alpha-1}\right)$ countable for any $\alpha < \lambda$, where $0 < \lambda \in \omega_1$ is a limit ordinal. Let ${}^{0}\mathcal{P}_{<\lambda} = \bigcap_{\alpha<\lambda} {}^{0}\mathcal{P}_{\alpha} \in \mathbf{\Pi}_1^0([T])$. The function $f : {}^{0}\mathcal{P}_{<\lambda} \to \omega^{\omega}$ belongs to $\Lambda_{1,3}$. Let also

$$\mathcal{F}_{\lambda} = \left\{A \cap {}^{0}\mathcal{P}_{<\lambda} : A \in \bigcup_{\gamma<\lambda} \mathcal{F}_{\gamma}\right\}.$$

Observe that $\mathcal{F}_{\lambda}$ is a countable subset of $\mathbf{\Pi}_1^0\left({}^{0}\mathcal{P}_{<\lambda}\right)$. Thus it generates a question-tree $\mathbb{T}_{\lambda}$ which satisfies $f : [\mathbb{T}_{\lambda}] \to \omega^{\omega} \in \Lambda_{1,2}$ for the topology of $[\mathbb{T}_{\lambda}]$ refines any topology of $\left({}^{0}\mathcal{P}_{<\lambda}, \mathbb{T}_{\alpha}\right)$, $\alpha < \lambda$. Let ${}^{1}\mathcal{P}_{\lambda} \subseteq [\mathbb{T}_{\lambda}]$ be given by Assumption 5.13 and ${}^{0}\mathcal{P}_{\lambda}$ be the closure of ${}^{1}\mathcal{P}_{\lambda}$ in ${}^{0}\mathcal{P}_{<\lambda}$. Once again, we have the following result.

Proposition 5.26. *Let* $f : [T] \to \omega^\omega \in \Lambda_{1,3}$.
1. ${}^0\mathcal{P}_\lambda$ *is perfect.*
2. $f : {}^0\mathcal{P}_{<\lambda} \setminus {}^0\mathcal{P}_\lambda \to \omega^\omega \in \operatorname{Dec}\left(\Lambda_{1,2},\ \mathbf{\Delta}^0_3\right)$.

Proof. It suffices to use the same arguments as the ones in the proof of Proposition 5.23. □

The construction does terminate

A priori, it is not clear whether this construction reaches a closure point. However, under the axiom of determinacy, Harrington proved the following.

Theorem 5.27 (Theorem 4.5 in [Har78], ZF+AD+DC)**.** *Fix* $\alpha \in \omega_1$*. Any decreasing sequence of* $\mathbf{\Pi}^0_\alpha(\omega^\omega)$ *sets eventually stabilizes.*

The case $\alpha = 1$ is the already mentioned easy Theorem 6.9 in [Kec95]. The preceding Theorem was first noticed useful in this context by Day and Marks [Day19, Mar20]. Moreover, they announced that the proof can easily be modified to obtain.

Theorem 5.28 ([Day19], ZF+AD+DC)**.** *Fix* $\alpha \in \omega_1$*. If* $\left\{\mathcal{F}_\beta : \beta \in \omega_1\right\}$ *such that* $\mathcal{F}_\beta = \{F_n : n \in \omega\} \subseteq \mathbf{\Pi}^0_\alpha(\omega^\omega)$ *and* $\mathcal{F}_\beta \subseteq \mathcal{F}_\gamma$ *for any* $\beta < \gamma$*, then there is* $\mathcal{F}_\delta$ *such that* $\mathcal{F}_\delta = \mathcal{F}_\alpha$ *for any* $\alpha > \delta$.

The case $\alpha = 1$ of Theorem 5.28 is sufficient to ensure our construction to reach a closure point.

The sequence $\left({}^0\mathcal{P}_\alpha\right)_{\alpha\in\omega_1}$ is a $\subseteq$-decreasing sequence of $\mathbf{\Pi}^0_1([T])$ sets so that it stabilizes. Let $\beta \in \omega_1$ be minimal such that ${}^0\mathcal{P}_\beta = {}^0\mathcal{P}_\alpha$ for any $\alpha \geq \beta$. Observe that the set $\{\mathcal{F}_\alpha : \alpha \geq \beta\}$ that we constructed fulfills the hypothesis of Theorem 5.28, so that there exists $\gamma \geq \beta$ such that $\mathcal{F}_\alpha = \mathcal{F}_\gamma$ for any $\alpha \geq \gamma$. By construction, we also have ${}^1\mathcal{P}_\alpha = {}^1\mathcal{P}_\gamma$ for any $\alpha \geq \gamma$.

We finally define, for the remaining of this section, ${}^0\mathcal{P} = {}^0\mathcal{P}_\delta$, ${}^1\mathcal{P} = {}^1\mathcal{P}_\delta$, $\mathcal{F} = \mathcal{F}_{\delta+1}$ and $\mathbb{T} = \mathbb{T}_{\delta+1}$. The following proposition summarizes the important features of this construction.

Proposition 5.29. *Let* $f : [T] \to \omega^\omega \in \Lambda_{1,3}$.
1. ${}^0\mathcal{P}$ *is perfect.*
2. $f : [T] \setminus {}^0\mathcal{P} \to \omega^\omega \in \operatorname{Dec}\left(\Lambda_{1,2},\ \mathbf{\Delta}^0_3\right)$.
3. *For any* $u \in \omega^{<\omega}$ *and any* $\mathfrak{s} \in \mathbb{T}$,

$$\left[\operatorname{Init}_{({}^1\mathcal{P},\mathbb{T}),{}^0\mathcal{P}}\left(f^{-1}[u] \cap [\mathfrak{s}]\right)\right] \in \mathcal{F}.$$

Two useful lemmas

We complete this section by proving two lemmas which stem from the previous construction.

Lemma 5.30. *Let $u \in \omega^{<\omega}$, $\mathfrak{s} \in \mathbb{T}$ and $x \in [\mathbb{T}]$ be such that $f^{-1}[u] \cap [\mathfrak{s}]$ is proper and non-self-dual in $\left({}^1\mathcal{P}, \mathbb{T}\right)$, $\mathfrak{s} \sqsubset x$ and $f^{-1}[u] \cap [\mathfrak{s}] \equiv_w f^{-1}[u] \cap [\mathfrak{s}']$ in $\left({}^1\mathcal{P}, \mathbb{T}\right)$ for any $\mathfrak{s} \sqsubseteq \mathfrak{s}' \sqsubset x$. There exists $\mathfrak{s} \sqsubseteq \mathfrak{s}'' \sqsubset x$ such that*

$$\left[\mathfrak{s}''\right] \subseteq \left[\operatorname{Init}_{\left({}^1\mathcal{P}, \mathbb{T}\right), {}^0\mathcal{P}}\left(f^{-1}[u] \cap [\mathfrak{s}]\right)\right].$$

Proof. Let $F = \left[\operatorname{Init}_{\left({}^1\mathcal{P}, \mathbb{T}\right), {}^0\mathcal{P}}\left(f^{-1}[u] \cap [\mathfrak{s}]\right)\right]$. Since $F \in \boldsymbol{\Delta}^0_1\left({}^0\mathcal{P}, \mathbb{T}\right)$, there exists $\mathfrak{s} \sqsubseteq \mathfrak{s}' \sqsubset x$ such that $[\mathfrak{s}'] \cap F = \emptyset$ or $[\mathfrak{s}'] \subseteq F$. If the first case occurs, then $x \notin F$. In particular, there exists $n \in \omega$ such that $\left[\pi_0\left(x_{\restriction n}\right)\right] \cap F = \emptyset$. Let $\mathfrak{s}'' \sqsupseteq \mathfrak{s}'$ such that $\pi_0(\mathfrak{s}'') \sqsupset \pi_0\left(x_{\restriction n}\right)$. Then $[\pi_0(\mathfrak{s}'')] \cap f^{-1}[u] \cap [\mathfrak{s}] <_w f^{-1}[u] \cap [\mathfrak{s}]$. Since $[\mathfrak{s}''] \subseteq [\mathfrak{s}] \cap [\pi_0(\mathfrak{s}'')]$ and $[\mathfrak{s}''] \in \boldsymbol{\Delta}^0_1\left({}^0\mathcal{P}, \mathbb{T}\right)$, we also get $[\mathfrak{s}''] \cap f^{-1}[u] <_w f^{-1}[u] \cap [\mathfrak{s}]$, a contradiction. □

Lemma 5.31. *Let $u \in \omega^{<\omega}$ and $\mathfrak{s} \in \mathbb{T}$ be such that $f^{-1}[u] \cap [\mathfrak{s}]$ is proper and non-self-dual in $\left({}^1\mathcal{P}, \mathbb{T}\right)$. If $t \in \mathcal{T}\left(\left[\operatorname{Init}_{\left({}^1\mathcal{P}, \mathbb{T}\right), {}^0\mathcal{P}}\left(f^{-1}[u] \cap [\mathfrak{s}]\right)\right]\right)$ and $t \sqsupseteq \pi_0(\mathfrak{s})$, then there exists $\mathfrak{s}' \in \mathcal{T}\left({}^1\mathcal{P}, \mathbb{T}\right)$ such that $t \sqsubseteq \pi_0(\mathfrak{s}')$, $\mathfrak{s} \sqsubseteq \mathfrak{s}'$ and $f^{-1}[u] \cap [\mathfrak{s}'] \equiv_w f^{-1}[u] \cap [\mathfrak{s}]$ in $\left({}^1\mathcal{P}, \mathbb{T}\right)$.*

Proof. By definition, there exists $\mathfrak{s}' \in \operatorname{Init}_{\left({}^1\mathcal{P}, \mathbb{T}\right)}\left(f^{-1}[u] \cap [\mathfrak{s}]\right)$ such that $t \sqsubseteq \pi_0(\mathfrak{s}')$. In particular, $f^{-1}[u] \cap [\mathfrak{s}] \cap [\mathfrak{s}'] \equiv_w f^{-1}[u] \cap [\mathfrak{s}]$ in $\left({}^1\mathcal{P}, \mathbb{T}\right)$. Since $\pi_0(\mathfrak{s}) \sqsubseteq t \sqsubseteq \pi_0(\mathfrak{s}')$, we must have $\mathfrak{s} \sqsubset \mathfrak{s}'$ for otherwise $f^{-1}[u] \cap [\mathfrak{s}] \cap [\mathfrak{s}'] = \emptyset$. □

5.4.2 The strategy τ

Let $f : [T] \to \omega^\omega \in \Lambda_{1,3}$ be such that ${}^0\mathcal{P}$ is non-empty. By construction, ${}^1\mathcal{P}$ is also non-empty. Section 5.3 provides several winning strategies for II in

$$G_w\left(\left(\mathcal{O}^{\sim_1}\right)^{\mathtt{b}}, f^{-1}[\mathcal{U}]\right)\left({\omega_{\{\mathtt{b}, \twoheadleftarrow\}}}^\omega, \left({}^1\mathcal{P}, \mathbb{T}\right)\right),$$

for some fixed $\mathcal{U} \in \boldsymbol{\Sigma}^0_1(\omega^\omega)$, namely $\mathcal{U} = \bigcup_{n \in \omega}[u_n]$ where $\{u_n : n \in \omega\}$ is given by Assumption 5.13. We define a winning strategy τ for II in the game

$$G_w\left(\left(\mathcal{O}^{\sim_2}\right)^{\mathtt{b}}, f^{-1}[\mathcal{U}]\right)\left({\omega_{\{\mathtt{b}, \twoheadleftarrow, \twoheadleftarrow_1\}}}^\omega, {}^0\mathcal{P}\right)$$

by lifting these strategies from $\left({}^1\mathcal{P}, \mathbb{T}\right)$ to ${}^0\mathcal{P}$. We begin with an informal presentation of τ before formally defining it.

Informal description of τ

A representation of the strategy τ seen in $\mathbb{T}$ is given in Figure 5.2, where the notations are specified in the formal definition of τ.

Let $\{u_n : n \in \omega\}$, $\{p_n : n \in \omega\}$ and $\left\{x_n^l : n, l \in \omega\right\}$ be given by Assumption 5.13 for $f : \left({}^1\mathcal{P}, \mathbb{T}\right) \to \omega^\omega \in \Lambda_{1,2}$. Observe that, for any winning strategy σ in

$$G_w\left(\left(\mathcal{O}^{\sim_1}\right)^{\mathtt{b}}, f^{-1}\left[\mathcal{U}\right]\right)\left({\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{\omega}, \left({}^1\mathcal{P}, \mathbb{T}\right)\right)$$

constructed in the previous section and any $s \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$, there exist a unique $n \in \omega$ and some $l \in \omega$ such that $p_n \sqsubseteq \sigma(s) \sqsubset x_n^l$ and $\lim_{k\in\omega} \sigma\left(s0^k\right) \in f^{-1}\left[u_n\right]$. By Assumption 5.13, we also have $\left[p_n\right] \cap f^{-1}\left[u_n\right] \equiv_w \left[\sigma(s)\right] \cap f^{-1}\left[u_n\right]$ in $\left({}^1\mathcal{P}, \mathbb{T}\right)$.

We informally describe the strategy $\tau : {\omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}}^{<\omega} \to \mathcal{T}\left({}^0\mathcal{P}\right)$ in the game

$$G_w\left(\left(\mathcal{O}^{\sim_1}\right)^{\mathtt{b}}, f^{-1}\left[\mathcal{U}\right]\right)\left({\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{\omega}, \left({}^1\mathcal{P}, \mathbb{T}\right)\right).$$

As long as I plays $s \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$, the strategy τ is simply the projection $\pi_0 \circ \sigma$ of a — well-chosen — winning strategy σ in the game

$$G_w\left(\left(\mathcal{O}^{\sim_1}\right)^{\mathtt{b}}, f^{-1}\left[\mathcal{U}\right]\right)\left({\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{\omega}, \left({}^1\mathcal{P}, \mathbb{T}\right)\right).$$

Suppose now that I plays the symbol $\twoheadleftarrow_1$ for the first time. At this moment, I has played a sequence $s\twoheadleftarrow_1$. If the symbol $\twoheadleftarrow_1$ does not erase anything (see Definition 2.60, page 54), then it suffices to continue to play according to $\pi_0 \circ \sigma$ by considering the last symbol to be $\mathtt{b}$ instead of $\twoheadleftarrow_1$. Otherwise, the symbol $\twoheadleftarrow_1$ erases s_n for some $n < \mathrm{lh}(s)$. If $r = s_{\restriction n}$ and $s\twoheadleftarrow_1 = ru$, then $\widetilde{s\twoheadleftarrow_1} = r\mathtt{b}^{\mathrm{lh}(u)}$ with $r \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$. To say it otherwise, after decoding, one interprets what I has played as r. The idea for II is to find a position in $\mathcal{T}\left({}^1\mathcal{P}, \mathbb{T}\right)$ which is *somehow* equivalent to her previous position $\sigma(r)$. As mentioned in the previous paragraph, there exist a unique $n \in \omega$ and some $l \in \omega$ such that $\lim_{k\in\omega} \sigma\left(r0^k\right) \in f^{-1}\left[u_n\right]$, $p_n \sqsubseteq \sigma(r) \sqsubset x_n^l$ and $\left[p_n\right] \cap f^{-1}\left[u_n\right] \equiv_w \left[\sigma(r)\right] \cap f^{-1}\left[u_n\right]$ in $\left({}^1\mathcal{P}, \mathbb{T}\right)$. A position $(t, C) \in \mathcal{T}\left({}^1\mathcal{P}, \mathbb{T}\right)$ is *somehow* equivalent to $\sigma(r)$ if $\sigma(r) \sqsubseteq (t, C)$ and $\left[\sigma(r)\right] \cap f^{-1}\left[u_n\right] \equiv_w \left[(t, C)\right] \cap f^{-1}\left[u_n\right]$. Finally, the goal of II is to find such a position satisfying $\pi_0 \circ \sigma(s) \sqsubseteq t$ in order to have a well-defined strategy in ${}^0\mathcal{P}$. Provided that σ is well-chosen, such a position exists by Lemma 5.31.

From this point on, II follows another strategy σ' instead of σ such that $\sigma(r) \sqsubseteq (t, C) \sqsubseteq \sigma'\left(r\mathtt{b}^{\mathrm{lh}(u)}\right)$. The strategy τ is obtained by repeating the previous process each time I uses the symbol $\twoheadleftarrow_1$.

Formal definition of τ

The strategy τ is formally defined from a set $\left\{\sigma_p : p \in {\omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}}^{<\omega}\right\}$ of winning strategies for II in the game

$$G_w\left(\left(\mathcal{O}^{\sim_1}\right)^{\mathtt{b}}, f^{-1}[\mathcal{U}]\right)\left({\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{\omega}, \left({}^1\mathcal{P}, \mathbb{T}\right)\right).$$

These strategies σ_p are designed so that some well-chosen questions are already answered at each step of the construction.

Consider the function $f : \left({}^1\mathcal{P}, \mathbb{T}\right) \to \omega^\omega \in \Lambda_{1,2}$ and a winning strategy $\tau^{\twoheadleftarrow}$ in the eraser game $G_{\twoheadleftarrow}\left(f : \left({}^1\mathcal{P}, \mathbb{T}\right) \to \omega^\omega\right)$. Since ${}^1\mathcal{P}$ is non-empty, Assumption 5.13 yields the sets $\{u_n : n \in \omega\}$, $\{p_n : n \in \omega\}$ and $\left\{x_n^l : n, l \in \omega\right\}$. We define the strategy $\sigma_{\langle\rangle}$ by induction on the length of $s \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$. First, let $\sigma_{\langle\rangle}(\langle\rangle) \in \mathcal{T}\left({}^1\mathcal{P}, \mathbb{T}\right)$ such that $p_0 \sqsubseteq \sigma_{\langle\rangle}(\langle\rangle) \sqsubset x_0^0$ and $u_0 \sqsubseteq \tau^{\twoheadleftarrow}\left(\sigma_{\langle\rangle}(\langle\rangle)\right)^{\hookleftarrow}$. By Lemma 5.30, there exists $\sigma_{\langle\rangle}(\langle\rangle) \sqsubset \sigma_{\langle\rangle}(\langle\rangle)' \sqsubset x_0^0$ such that $\left[\sigma_{\langle\rangle}(\langle\rangle)'\right] \cap f^{-1}[u_0] \equiv_w [p_0] \cap f^{-1}[u]$ in $\left({}^1\mathcal{P}, \mathbb{T}\right)$ and

$$\left[\sigma_{\langle\rangle}(\langle\rangle)'\right] \subseteq \left[\mathrm{Init}_{({}^1\mathcal{P},\mathbb{T}),{}^0\mathcal{P}}\left(f^{-1}[u_0] \cap [p_0]\right)\right].$$

For the inductive step, let $sa \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$ and suppose that we have already defined $\sigma_{\langle\rangle}(s) \sqsubseteq \sigma_{\langle\rangle}(s)'$ such that there exists a unique $n \in \omega$ satisfying $p_n \sqsubseteq \sigma_{\langle\rangle}(s) \sqsubseteq \sigma_{\langle\rangle}(s)' \sqsubset x_n^0$, both $f^{-1}[u_n] \cap \left[\sigma_{\langle\rangle}(s)\right]$ and $f^{-1}[u_n] \cap \left[\sigma_{\langle\rangle}(s)'\right]$ are Wadge equivalent to $[p_n] \cap f^{-1}[u_n]$ in $\left({}^1\mathcal{P}, \mathbb{T}\right)$ and

$$\left[\sigma_{\langle\rangle}(s)'\right] \subseteq \left[\mathrm{Init}_{({}^1\mathcal{P},\mathbb{T}),{}^0\mathcal{P}}\left(f^{-1}[u_n] \cap \left[\sigma_{\langle\rangle}(s)\right]\right)\right].$$

We consider two different cases.

1. If $\widetilde{sa}_{[\ /\mathtt{b}]} \neq \langle\rangle$, we choose $\sigma_{\langle\rangle}(s)' \sqsubset \sigma_{\langle\rangle}(sa) \sqsubset x_n^0$ such that $f^{-1}[u_n] \cap \left[\sigma_{\langle\rangle}(sa)\right] \equiv_w [p_n] \cap f^{-1}[u_n]$ in $\left({}^1\mathcal{P}, \mathbb{T}\right)$. As previously, Lemma 5.30 yields $\sigma_{\langle\rangle}(sa) \sqsubseteq \sigma_{\langle\rangle}(sa)' \sqsubset x_n^0$ such that $\left[\sigma_{\langle\rangle}(sa)'\right] \cap f^{-1}[u_n] \equiv_w [p_n] \cap f^{-1}[u_n]$ in $\left({}^1\mathcal{P}, \mathbb{T}\right)$ and
$$\left[\sigma_{\langle\rangle}(sa)'\right] \subseteq \left[\mathrm{Init}_{({}^1\mathcal{P},\mathbb{T}),{}^0\mathcal{P}}\left(f^{-1}[u_n] \cap \left[\sigma_{\langle\rangle}(sa)\right]\right)\right].$$
2. Otherwise, $\widetilde{sa}_{[\ /\mathtt{b}]} = \langle\rangle$. In that case, by Assumption 5.13, one chooses $m \in \omega$ such that $\sigma_{\langle\rangle}(s)' \sqsubset p_m$ and $p_m \notin \mathrm{Init}_{({}^1\mathcal{P},\mathbb{T})}\left(f^{-1}[u_n] \cap [p_n]\right)$. Let $\sigma_{\langle\rangle}(sa) \in \mathcal{T}\left({}^1\mathcal{P}, \mathbb{T}\right)$ such that $p_m \sqsubseteq \sigma_{\langle\rangle}(sa) \sqsubset x_m^0$ and $u_m \sqsubseteq \tau^{\twoheadleftarrow}\left(\sigma_{\langle\rangle}(sa)\right)^{\hookleftarrow}$. As previously, Lemma 5.30 yields $\sigma_{\langle\rangle}(sa) \sqsubseteq \sigma_{\langle\rangle}(sa)' \sqsubset x_m^0$ such that $\left[\sigma_{\langle\rangle}(sa)'\right] \cap f^{-1}[u_m] \equiv_w [p_m] \cap f^{-1}[u_m]$ in $\left({}^1\mathcal{P}, \mathbb{T}\right)$ and
$$\left[\sigma_{\langle\rangle}(sa)'\right] \subseteq \left[\mathrm{Init}_{({}^1\mathcal{P},\mathbb{T}),{}^0\mathcal{P}}\left(f^{-1}[u_m] \cap \left[\sigma_{\langle\rangle}(sa)\right]\right)\right].$$

The strategy $\sigma_{\langle\rangle}$ is a winning strategy since its construction follows the guidance of the previous section. Moreover, it is designed to answer some well-chosen questions at each step. More precisely, if $s \in \omega_{\{\mathtt{b},\twoheadleftarrow\}}{}^{<\omega}$, there exists a unique $n \in \omega$ such that $p_n \sqsubseteq \sigma_{\langle\rangle}(s) \sqsubset x_n^0$, $\lim_{k\in\omega} \sigma_{\langle\rangle}(s0^k) \in f^{-1}[u_n]$, $[\sigma_{\langle\rangle}(s)] \cap f^{-1}[u_n] \equiv_w [p_n] \cap f^{-1}[u_n]$ in $({}^1\mathcal{P}, \mathbb{T})$ and, for any $s' \sqsupset s$, one has

$$\left[\sigma_{\langle\rangle}(s')\right] \subseteq \left[\mathrm{Init}_{({}^1\mathcal{P},\mathbb{T}),{}^0\mathcal{P}}\left(f^{-1}[u_n] \cap \left[\sigma_{\langle\rangle}(s)\right]\right)\right] = F.$$

In other words, we make sure that the strategy answers the question F positively.

The other strategies σ_p are defined by induction on the length of $p \in \omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}{}^{<\omega}$. If $p = qa$ with $a \in \omega_{\{\mathtt{b},\twoheadleftarrow\}}$ or if $p = q\twoheadleftarrow_1$ and $p_{\mathrm{lh}(p)-1}$ does not erase anything (see Definition 2.60, page 54), then $\sigma_p = \sigma_q$.

Otherwise, let $p = q\twoheadleftarrow_1 = rg$ be defined such that $p_{\mathrm{lh}(p-1)} = \twoheadleftarrow_1$ erases $p_{\mathrm{lh}(r)}$. The idea is rather simple: the strategy σ_p is defined as a shift of the strategy σ_q which is performed when I plays $\widetilde{r}^1$. This shift is defined via Lemma 5.31 in order to satisfy

$$\pi_0\left(\sigma_q\left(\widetilde{q}^1\right)\right) \sqsubset \pi_0\left(\sigma_p\left(\widetilde{r}^1\right)'\right).$$

More precisely, there exist a unique $n \in \omega$ and some $l \in \omega$ such that $p_n \sqsubseteq \sigma_q\left(\widetilde{r}^1\right) \sqsubset x_n^l$, $\lim_{k\in\omega} \sigma_q\left(\widetilde{r}^1 0^k\right) \in f^{-1}[u_n]$ and $f^{-1}[u_n] \cap \left[\sigma_q\left(\widetilde{r}^1\right)\right] \equiv_w f^{-1}[u_n] \cap [p_n]$ in $({}^1\mathcal{P}, \mathbb{T})$. Since $\widetilde{r}^1 \sqsubset \widetilde{q}^1$, the construction yields

$$\left[\sigma_q\left(\widetilde{q}^1\right)\right] \subseteq \left[\mathrm{Init}_{({}^1\mathcal{P},\mathbb{T}),{}^0\mathcal{P}}\left(f^{-1}[u_n] \cap \left[\sigma_q\left(\widetilde{r}^1\right)\right]\right)\right].$$

By Lemma 5.31, there exists $\mathfrak{t} \in \mathbb{T}$ such that $f^{-1}[u_n] \cap [\mathfrak{t}] \equiv_w f^{-1}[u_n] \cap [p_n]$ in $({}^1\mathcal{P}, \mathbb{T})$, $\pi_0(\mathfrak{t}) \sqsupset \pi_0\left(\sigma_q\left(\widetilde{q}^1\right)\right)$ and $\mathfrak{t} \sqsupset \sigma_q\left(\widetilde{r}^1\right)$. By Assumption 5.13, there exists $l' \in \omega$ such that $p_n \sqsubseteq \mathfrak{t} \sqsubset x_n^{l'}$. Then, by Lemma 5.30, there exists $\mathfrak{t} \sqsubseteq \sigma_p\left(\widetilde{r}^1\right)' \sqsubset x_n^{l'}$ such that $\left[\sigma_p\left(\widetilde{r}^1\right)'\right] \cap f^{-1}[u_n] \equiv_w [p_n] \cap f^{-1}[u_n]$ in $({}^1\mathcal{P}, \mathbb{T})$ and

$$\left[\sigma_p\left(\widetilde{r}^1\right)'\right] \subseteq \left[\mathrm{Init}_{({}^1\mathcal{P},\mathbb{T}),{}^0\mathcal{P}}\left(f^{-1}[u_n] \cap \left[\sigma_q\left(\widetilde{r}^1\right)\right]\right)\right].$$

The strategy σ_p is defined as the strategy σ_q until I plays $\widetilde{r}^1$. At this moment, II considers $\sigma_p\left(\widetilde{r}^1\right) = \sigma_q\left(\widetilde{r}^1\right)$ but replaces $\sigma_q\left(\widetilde{r}^1\right)'$ with $\sigma_p\left(\widetilde{r}^1\right)'$. The remaining of the strategy is defined the same way as $\sigma_{\langle\rangle}$. The strategy σ_p is also defined according to the previous section, thus it is winning.

As previously, some well-chosen questions are positively answered at each step of the strategy. More precisely, if $s \in \omega_{\{\mathtt{b},\twoheadleftarrow\}}{}^{<\omega}$, there exist a unique

$n \in \omega$ and some $l \in \omega$ such that $p_n \sqsubseteq \sigma_p(s) \sqsubset x_n^l$, $\lim_{k\in\omega} \sigma_p\left(s0^k\right) \in f^{-1}[u_n]$, $[\sigma_p(s)] \cap f^{-1}[u_n] \equiv_w [p_n] \cap f^{-1}[u_n]$ in $\left({}^1\mathcal{P}, \mathbb{T}\right)$ and, for any $s' \sqsupset s$, one has

$$\left[\sigma_p\left(s'\right)\right] \subseteq \left[\mathrm{Init}_{\left({}^1\mathcal{P},\mathbb{T}\right),{}^0\mathcal{P}}\left(f^{-1}[u_n] \cap [\sigma_p(s)]\right)\right].$$

By construction, the following strategy τ is well-defined and strictly $\sqsubseteq$-increasing.

$$\begin{aligned}\tau : {\omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}}^{<\omega} &\to {}^0\mathcal{P}\\ p &\mapsto \pi_0\left(\sigma_p\left(\widetilde{p}^1\right)\right).\end{aligned}$$

In Figure 5.2, we schematically represent τ as seen from inside $\mathbb{T}$. For any $p \in {\omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}}^{<\omega}$, we write $\mathtt{t}_p = \sigma_p\left(\widetilde{p}^1\right) \in \mathbb{T}$.

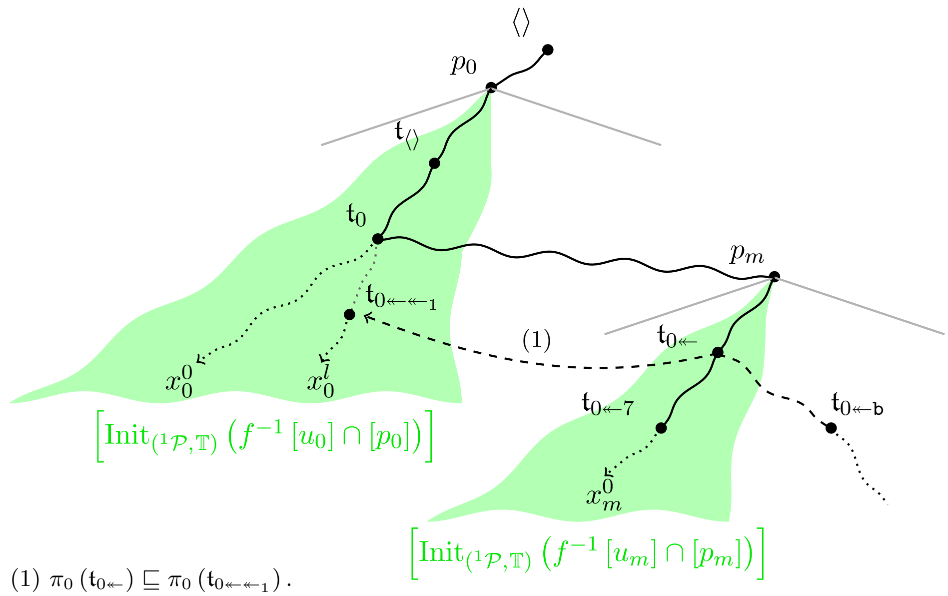


Figure 5.2: The strategy τ as seen from inside $\mathbb{T}$.

It remains to prove that τ is winning, i.e., for any $x \in {\omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}}^{\omega}$,

$$x \in \left(\mathcal{O}^{\sim_2}\right)^{\mathtt{b}} \;\Leftrightarrow\; \lim_{k\in\omega} \tau\left(x_{\restriction k}\right) = \lim_{k\in\omega} \pi_0\left(\sigma_{x_{\restriction k}}\left(\widetilde{x_{\restriction k}}^1\right)\right) \in f^{-1}[\mathcal{U}].$$

For this purpose, we study the sequence

$$\left(\sigma_{x_{\restriction k}}\left(\widetilde{x_{\restriction k}}^1\right)\right)_{k\in\omega} \subseteq \mathbb{T}.$$

Even though this sequence might not be $\sqsubset$-increasing, we prove that it contains an $\sqsubset$-increasing subsequence. Let $\{\alpha_i : i \in \omega\} = \{\alpha_i^{\leftarrow} : i \in \omega\}$ be given by Definition 2.62 (page 55). We consider the sequence

$$\left(\sigma_{x_{\restriction \alpha_i}}\left(\widetilde{x_{\restriction \alpha_i}}^{1}\right)\right)_{i\in\omega}.$$

Claim 5.32. *For any $i \in \omega$,*

$$\sigma_{x_{\restriction \alpha_i}}\left(\widetilde{x_{\restriction \alpha_i}}^{1}\right) \sqsubseteq \sigma_{x_{\restriction \alpha_{i+1}}}\left(\widetilde{x_{\restriction \alpha_{i+1}}}^{1}\right).$$

Proof of the claim. Since $\{\alpha_i : i \in \omega\} \subseteq \left\{\alpha_i^{\twoheadleftarrow_1} : i \in \omega\right\}$, Lemma 2.63 implies

$$\widetilde{x_{\restriction \alpha_i}}^{1} \sqsubset \widetilde{x_{\restriction \alpha_{i+1}}}^{1}. \tag{5.2}$$

If $x_{\alpha_{i+1}-1} \neq \twoheadleftarrow_1$ or if $x_{\alpha_{i+1}-1} = \twoheadleftarrow_1$ does not erase anything, then $\sigma_{x_{\restriction \alpha_i}} = \sigma_{x_{\restriction \alpha_{i+1}}}$ and the fact that $\sigma_{x_{\restriction \alpha_i}}$ is a strategy yields the result. If $x_{\alpha_{i+1}-1} = \twoheadleftarrow_1$, then $x_{\restriction \alpha_{i+1}} = x_{\restriction \alpha_i} g$ such that $x_{\alpha_{i+1}-1}$ erases x_{α_i}. By construction and by Equation (5.2), we get:

$$\sigma_{x_{\restriction \alpha_i}}\left(\widetilde{x_{\restriction \alpha_i}}^{1}\right) = \sigma_{x_{\restriction \alpha_{i+1}}}\left(\widetilde{x_{\restriction \alpha_i}}^{1}\right) \sqsubseteq \sigma_{x_{\restriction \alpha_{i+1}}}\left(\widetilde{x_{\restriction \alpha_{i+1}}}^{1}\right).$$

$\square_{\text{Claim}}$

Since $\mathbb{T}$ is a question-tree on ${}^{0}\mathcal{P}$, we obtain that

$$\left(\pi_0\left(\sigma_{x_{\restriction \alpha_i}}\left(\widetilde{x_{\restriction \alpha_i}}^{1}\right)\right)\right)_{i\in\omega}$$

has a limit in ${}^{0}\mathcal{P}$. By definition, this limit coincides with $\lim_{k\in\omega} \tau\left(x_{\restriction k}\right)$. Thus, it only remains to prove that

$$x \in \left(\mathcal{O}^{\sim_2}\right)^{\mathtt{b}} \Leftrightarrow \lim_{i\in\omega} \sigma_{x_{\restriction \alpha_i}}\left(\widetilde{x_{\restriction \alpha_i}}^{1}\right) \in f^{-1}\left[\mathcal{U}\right].$$

The case $x \in \left(\mathcal{O}^{\sim_2}\right)^{\mathtt{b}}$

Since $x \in \left(\mathcal{O}^{\sim_2}\right)^{\mathtt{b}}$ implies $\widetilde{x}^{1\to 0} \in \mathcal{O}^{\mathtt{b}}$ (see page 57), there exists $m \in \omega$ such that

$$\widetilde{x}^{1\to 0}(m) \neq \mathtt{b}.$$

Let $j \in \omega$ such that $\alpha_j > m$. Using Lemma 2.66, for any $i \geq j$,

$$\widetilde{x_{\restriction \alpha_i}}^{1\to 0}_{[\ /\mathtt{b}]} \neq \langle\rangle. \tag{5.3}$$

By construction, there exist a unique $n \in \omega$ and some $l \in \omega$ such that we have $p_n \sqsubseteq \sigma_{x_{\restriction \alpha_j}}\left(\widetilde{x_{\restriction \alpha_j}}^{1}\right) \sqsubset x_n^l$, $\lim_{k\in\omega} \sigma_{x_{\restriction \alpha_j}}\left(\widetilde{x_{\restriction \alpha_j}}^{1} 0^k\right) \in f^{-1}\left[u_n\right]$ and $\left[\sigma_{x_{\restriction \alpha_j}}\left(\widetilde{x_{\restriction \alpha_j}}^{1}\right)\right] \cap f^{-1}[u_n] \equiv_w f^{-1}\left[u_n\right] \cap \left[p_n\right]$ in $\left({}^{1}\mathcal{P}, \mathbb{T}\right)$.

Claim 5.33. *For any $i \geq j$, $\left[\sigma_{x\restriction\alpha_i}\left(\widetilde{x\restriction\alpha_i}^1\right)\right] \cap f^{-1}[u_n] \equiv_w f^{-1}[u_n] \cap [p_n]$ in $\left({}^1\mathcal{P}, \mathbb{T}\right)$.*

Proof of the claim. Suppose that $\left[\sigma_{x\restriction\alpha_i}\left(\widetilde{x\restriction\alpha_i}^1\right)\right] \cap f^{-1}[u_n] \equiv_w f^{-1}[u_n] \cap [p_n]$ in $\left({}^1\mathcal{P}, \mathbb{T}\right)$. If $\sigma_{x\restriction\alpha_i} = \sigma_{x\restriction\alpha_{i+1}}$, then Equation (5.3) and the construction of the strategy $\sigma_{x\restriction\alpha_i}$ yield the result. Otherwise, $x_{\alpha_{i+1}-1} = \twoheadleftarrow_1$. In this case, $x\restriction\alpha_{i+1} = x\restriction\alpha_i g$ where $x_{\alpha_{i+1}-1}$ erases x_{α_i}. By construction of the strategy $\sigma_{x\restriction\alpha_{i+1}}$, we have

$$\sigma_{x\restriction\alpha_i}\left(\widetilde{x\restriction\alpha_i}^1\right) = \sigma_{x\restriction\alpha_{i+1}}\left(\widetilde{x\restriction\alpha_i}^1\right).$$

Since $\widetilde{x\restriction\alpha_{i+1}}^{1\to 0} = \widetilde{x\restriction\alpha_i}^{1\to 0}\mathtt{b}^{\mathrm{lh}(g)}$, Equation (5.3) and the construction of the strategy $\sigma_{x\restriction\alpha_{i+1}}$ yield the result. $\square_{\text{Claim}}$

By Claim 5.33 and Proposition 2.42, we get

$$\lim_{i\in\omega} \sigma_{x\restriction\alpha_i}\left(\widetilde{x\restriction\alpha_i}^1\right) \in f^{-1}[u_t] \subseteq f^{-1}[\mathcal{U}].$$

The case $x \notin (\mathcal{O}^{\sim 2})^{\mathtt{b}}$

In that case, we show that, for any $n \in \omega$, we have

$$\lim_{i\in\omega} \sigma_{x\restriction\alpha_i}\left(\widetilde{x\restriction\alpha_i}^1\right) \notin f^{-1}[u_n].$$

Fix $n \in \omega$. Since $\widetilde{x}^{1\to 0}_{[\ /\mathtt{b}]} = \langle\rangle$, one has

$$\mathrm{Card}\left\{l \in \omega : \widetilde{x\restriction l}^{1\to 0}_{[\ /\mathtt{b}]} = \langle\rangle\right\} = \aleph_0.$$

In particular, by the construction of the different strategies, there exist a strictly $\sqsubseteq$-increasing sequence $(n_i)_{i\in\omega} \subseteq \{\alpha_i : i \in \omega\}$, an infinite set $\{m_i : i \in \omega\} \subseteq \omega$ and a $\sqsubset$-increasing sequence $(\mathfrak{s}_i)_{i\in\omega} \subseteq \mathcal{T}\left({}^1\mathcal{P}, \mathbb{T}\right)$ such that $\lim_{i\in\omega} \mathfrak{s}_i = \lim_{i\in\omega} \sigma_{x\restriction\alpha_i}\left(\widetilde{x\restriction\alpha_i}^1\right)$, and for any $i < j$, $p_{m_i} \sqsubseteq \sigma_{x\restriction\alpha_i}\left(\widetilde{x\restriction\alpha_i}^1\right) \sqsubset p_{m_{i+1}}$, $u_{m_i} \sqsubseteq \tau^{\leftarrow}(\mathfrak{s}_i)^{\hookleftarrow\mathcal{P}}$ and $m_i \neq m_j$. By Proposition 5.21, we get

$$\lim_{i\in\omega} \sigma_{x\restriction\alpha_i}\left(\widetilde{x\restriction\alpha_i}^1\right) \notin f^{-1}[u_n],$$

which finally implies that τ is winning.

5.4.3 The main result

We prove a first generalization of the Jayne-Rogers Theorem.

Theorem 5.34 (AD)**.** *Assuming Assumption 5.13, if* $f : [T] \to \omega^\omega$ *is such that* $f \in \Lambda_{1,3} \setminus \mathrm{Dec}\left(\Lambda_{1,2},\ \mathbf{\Delta}^0_3\right)$*, then* $f \notin \Lambda_{2,3}$.

In particular, if $\mathcal{X}$ *and* $\mathcal{Y}$ *are both zero-dimensional Polish spaces, then Assumption 5.13 implies the case* $(m = 2, n = 3)$ *of the Decomposability Conjecture, i.e.,*

$$\Lambda_{2,3}\left(\mathcal{X},\mathcal{Y}\right) = \mathrm{Dec}\left(\Lambda_{1,2},\ \mathbf{\Delta}^0_3\right)\left(\mathcal{X},\mathcal{Y}\right).$$

Proof. Since $f \notin \mathrm{Dec}\left(\Lambda_{1,2},\ \mathbf{\Delta}^0_3\right)$, ${}^0\mathcal{P}$ is non-empty. Following our construction, there exists a winning strategy for II in the game

$$G_w\left(\left(\mathcal{O}^{\sim_2}\right)^{\mathtt{b}}, f^{-1}\left[\mathcal{U}\right]\right)\left({\omega_{\{\mathtt{b},\leftarrow,\leftarrow_1\}}}^{\omega}, {}^0\mathcal{P}\right)$$

for some $\mathcal{U} \in \mathbf{\Sigma}^0_1\left(\omega^\omega\right)$. Since $\left(\mathcal{O}^{\sim_2}\right)^{\mathtt{b}}$ is $\mathbf{\Sigma}^0_3$-complete (Theorem 2.53) and this strategy is played in the closed set ${}^0\mathcal{P} \subseteq [T]$, the same strategy yields

$$f^{-1}\left[\mathcal{U}\right] \notin \mathbf{\Delta}^0_3\left([T]\right).$$

This implies that $f : [T] \to \omega^\omega \notin \Lambda_{2,3}$ by Equation (5.1) (page 94). □

5.5 A second generalization of the Jayne-Rogers Theorem

In this section, we use the same strategy as in Section 5.4 to prove a second generalization of the Jayne-Rogers Theorem. As most of the construction is similar, we only highlight the meaningful differences which comes from the stacking of two question-trees. If $\mathcal{X}$ is a zero-dimensional Polish space, we denote by ${}^1\mathbb{T}$ question-trees on $\mathcal{X}$ and by ${}^2\mathbb{T}$ question-trees on $\left[{}^1\mathbb{T}\right]$. The question-trees ${}^2\mathbb{T}$ are, so to speak, question-trees of level 2 on $\mathcal{X}$. In this section, the nodes of ${}^2\mathbb{T}$ are denoted by $\mathfrak{s}$ or $\mathfrak{t}$. If $\mathfrak{s} \in {}^2\mathbb{T}$, then $\mathfrak{s} = (t, C, \mathcal{C})$ where $t \in \omega^{<\omega}$, $C \in \mathbf{\Pi}^0_1\left(\mathcal{X}\right)$ and $\mathcal{C} \in \mathbf{\Pi}^0_1\left(\left[{}^1\mathbb{T}\right]\right) \subseteq \mathbf{\Pi}^0_2\left(\mathcal{X}\right)$. We use the projection functions π_0, π_1 and π_2 to speak about t, C and $\mathcal{C}$, respectively. We also use the notation $\pi_{0,1}\left(\mathfrak{s}\right) = \left(\pi_0\left(\mathfrak{s}\right), \pi_1\left(\mathfrak{s}\right)\right)$.

5.5.1 The construction

For the rest of this construction, let ${}^0\mathcal{P}_{-1} = [T]$ and, if $\lambda \in \omega_1$ is a limit ordinal, "$\lambda - 1$" denotes "$< \lambda$". For any $\alpha \in \omega_1$, we construct by induction

${}^0\mathcal{P}_{\alpha-1} \in \mathbf{\Pi}^0_1([T])$, ${}^0\mathcal{P}_\alpha \in \mathbf{\Pi}^0_1\left({}^0\mathcal{P}_{\alpha-1}\right)$, $\mathcal{F}_\alpha \subseteq \mathbf{\Pi}^0_1\left({}^0\mathcal{P}_{\alpha-1}\right)$ a countable set generating a question-tree ${}^1\mathbb{T}_\alpha$, ${}^1\mathcal{P}_\alpha \in \mathbf{\Pi}^0_1\left(\left[{}^1\mathbb{T}_\alpha\right]\right)$ a perfect subset, a question-tree ${}^2\mathbb{T}_\alpha$ on $\left({}^1\mathcal{P}_\alpha, {}^1\mathbb{T}_\alpha\right)$ generated by a countable set $\mathcal{G}_\alpha \subseteq \mathbf{\Pi}^0_1\left({}^1\mathcal{P}_\alpha, {}^1\mathbb{T}_\alpha\right)$ and ${}^2\mathcal{P}_\alpha \in \mathbf{\Pi}^0_1\left(\left[{}^1\mathbb{T}_\alpha\right]\right)$. Once again, we make use of question-trees in order to decrease the topological complexity of the function so that we can apply the construction of the previous section.

The initial case

Let $f : [T] \to \omega^\omega \in \Lambda_{1,4}$. For any $u \in \omega^{<\omega}$, there exists $\mathcal{F}(u) \subseteq \mathbf{\Pi}^0_1([T])$ countable such that $f^{-1}[u] \in \mathbf{\Delta}^0_3([T], \tau_u)$ where τ_u is the refinement of the prefix topology on $[T]$ obtained by adding the elements of $\mathcal{F}(u)$ as clopen sets. The set

$$\mathcal{F}_0 = \{[t] : t \in T\} \cup \bigcup_{n\in\omega} \mathcal{F}(u) \subseteq \mathbf{\Pi}^0_1([T])$$

is countable and thus generates a question-tree ${}^1\mathbb{T}_0$ on $[T]$ satisfying $f : \left[{}^1\mathbb{T}_0\right] \to \omega^\omega \in \Lambda_{1,3}$. By Proposition 5.29, there exists ${}^1\mathcal{P}_0 \in \mathbf{\Pi}^0_1\left(\left[{}^1\mathbb{T}_0\right]\right)$ perfect, a question-tree ${}^2\mathbb{T}_0$ on $\left({}^1\mathcal{P}_0, {}^1\mathbb{T}_0\right)$ generated by a countable set $\mathcal{G}_0 \subseteq \mathbf{\Pi}^0_1\left({}^1\mathcal{P}_0, {}^1\mathbb{T}_0\right)$ and ${}^2\mathcal{P}_0 \in \mathbf{\Pi}^0_1\left({}^1\mathcal{P}_0, {}^2\mathbb{T}_0\right)$ such that $f : \left[{}^2\mathbb{T}_0\right] \to \omega^\omega \in \Lambda_{1,2}$ and $f : \left([T] \setminus {}^1\mathcal{P}_0, {}^1\mathbb{T}_0\right) \to \omega^\omega \in \mathrm{Dec}\left(\Lambda_{1,2},\ \mathbf{\Delta}^0_3\right)$. We define ${}^0\mathcal{P}_0$ as the closure of ${}^1\mathcal{P}_0$ in $[T]$. We have the following properties.

Proposition 5.35. *Let $f : [T] \to \omega^\omega \in \Lambda_{1,4}$.*

1. *${}^0\mathcal{P}_0$ is perfect.*
2. *$f : [T] \setminus {}^0\mathcal{P}_0 \to \omega^\omega \in \mathrm{Dec}\left(\Lambda_{1,3},\ \mathbf{\Delta}^0_4\right)$.*

Proof. It suffices to use the same arguments as the ones in the proof of Proposition 5.23. □

The successor case

First, we generalize a second time the notion of initializable tree (see Definition 5.24).

Definition 5.36. If $\mathbb{T}$ is a question-tree on $[T]$, $\mathbb{T}' \subseteq \mathbb{T}$ is a non-empty pruned tree, ${}^2\mathbb{T}$ is a question-tree on $\mathbb{T}'$, ${}^2\mathbb{T}' \subseteq {}^2\mathbb{T}$ and $A \subseteq \left[{}^2\mathbb{T}'\right]$, then we define

$$\mathrm{Init}_{[{}^2\mathbb{T}'],[T]}(A) = \left\{t \in T : \text{there exists } \mathfrak{s} \in \mathrm{Init}_{[{}^2\mathbb{T}']}(A) \text{ such that } t \sqsubseteq \pi_0(\mathfrak{s})\right\}.$$

This is clearly a tree. Moreover, if $x \in \left[\mathrm{Init}_{[^2\mathbb{T}']}(A)\right]$, then we easily obtain $x \in \left[\mathrm{Init}_{[^2\mathbb{T}'],[T]}(A)\right]$. Thus, if $A \subseteq \left[^2\mathbb{T}'\right]$ is non-self-dual, the set $\left[\mathrm{Init}_{[^2\mathbb{T}'],[T]}(A)\right]$ is a non-empty closed subset of $[T]$.

Suppose that we have already defined ${}^0\mathcal{P}_{\beta-1} \in \mathbf{\Pi}^0_1([T])$, ${}^0\mathcal{P}_\beta \in \mathbf{\Pi}^0_1\left({}^0\mathcal{P}_{\beta-1}\right)$, $\mathcal{F}_\beta \subseteq \mathbf{\Pi}^0_1\left({}^0\mathcal{P}_{\beta-1}\right)$ a countable set generating a question-tree ${}^1\mathbb{T}_\beta$, a perfect set ${}^1\mathcal{P}_\beta \in \mathbf{\Pi}^0_1\left(\left[{}^1\mathbb{T}_\beta\right]\right)$, a question-tree ${}^2\mathbb{T}_\beta$ on $\left({}^1\mathcal{P}_\beta, {}^1\mathbb{T}_\beta\right)$ generated by a countable set $\mathcal{G}_\beta \subseteq \mathbf{\Pi}^0_1\left({}^1\mathcal{P}_\beta, {}^1\mathbb{T}_\beta\right)$ and ${}^2\mathcal{P}_\beta \in \mathbf{\Pi}^0_1\left(\left[{}^1\mathbb{T}_\beta\right]\right)$ for some $\beta \in \omega_1$. Let

$$\mathcal{B}_\beta = \left\{{}^0\mathcal{P}_\beta \cap A_1 \cap \cdots \cap A_m : A_i \in \mathcal{F}_\beta \text{ for any } 1 \leq i \leq m\right\} \subseteq \mathbf{\Pi}^0_1\left({}^0\mathcal{P}_\beta\right)$$

be the countable basis on ${}^0\mathcal{P}_\beta$ generated by $\mathcal{F}_\beta$ and

$$\mathcal{D}_\beta = \left\{{}^0\mathcal{P}_\beta \cap E_1 \cap \cdots \cap E_m : E_i \in \mathcal{G}_\beta \text{ for any } 1 \leq i \leq m\right\}.$$

We define

$$\mathcal{F}_{\beta+1} = \mathcal{B}_\beta \cup \left\{\left[\mathrm{Init}_{\left({}^2\mathcal{P}_\beta, {}^2\mathbb{T}_\beta\right), {}^0\mathcal{P}_\beta}\left(f^{-1}[u] \cap D\right)\right] : u \in \omega^{<\omega}, D \in \mathcal{D}_\beta\right\}.$$

Once again, we add questions, i.e., closed sets, which reveal the complexity of the preimage under f of basic open sets. However, this time, we do it relatively to the topology $\left[{}^2\mathbb{T}_\beta\right]$ instead of $\left[{}^1\mathbb{T}_\beta\right]$. Observe that $\mathcal{F}_{\beta+1}$ is a countable subset of $\mathbf{\Pi}^0_1\left({}^0\mathcal{P}_\beta\right)$. Let $\mathbb{T}_{\beta+1}$ be the question-tree generated by $\mathcal{F}_{\beta+1}$ on ${}^0\mathcal{P}_\beta$.

The topology of $\left[{}^1\mathbb{T}_{\beta+1}\right]$ is a refinement of the topology of $\left[{}^1\mathbb{T}_\beta\right]$ considered on ${}^0\mathcal{P}_\beta$ so that $f : \left[{}^1\mathbb{T}_{\beta+1}\right] \to \omega^\omega \in \Lambda_{1,3}$. By Proposition 5.29, there exists ${}^1\mathcal{P}_{\beta+1} \in \mathbf{\Pi}^0_1\left(\left[{}^1\mathbb{T}_{\beta+1}\right]\right)$ perfect, a question-tree ${}^2\mathbb{T}_{\beta+1}$ on $\left({}^1\mathcal{P}_{\beta+1}, {}^1\mathbb{T}_{\beta+1}\right)$ generated by a countable $\mathcal{G}_{\beta+1} \subseteq \mathbf{\Pi}^0_1\left({}^1\mathcal{P}_{\beta+1}, {}^1\mathbb{T}_{\beta+1}\right)$ and a perfect set ${}^2\mathcal{P}_{\beta+1} \in \mathbf{\Pi}^0_1\left(\left[{}^2\mathbb{T}_{\beta+1}\right]\right)$ such that both $f : \left[{}^2\mathbb{T}_{\beta+1}\right] \to \omega^\omega \in \Lambda_{1,2}$ and $f : \left({}^0\mathcal{P}_\beta \setminus {}^1\mathcal{P}_{\beta+1}, {}^1\mathbb{T}_{\beta+1}\right) \to \omega^\omega \in \mathrm{Dec}\left(\Lambda_{1,2}, \ \mathbf{\Delta}^0_3\right)$ hold. We define ${}^0\mathcal{P}_{\beta+1}$ as the closure of ${}^1\mathcal{P}_{\beta+1}$ in ${}^0\mathcal{P}_\beta$. We have the following properties.

Proposition 5.37. *Let* $f : [T] \to \omega^\omega \in \Lambda_{1,4}$.

1. ${}^0\mathcal{P}_{\beta+1}$ *is perfect.*
2. $f : {}^0\mathcal{P}_\beta \setminus {}^0\mathcal{P}_{\beta+1} \to \omega^\omega \in \mathrm{Dec}\left(\Lambda_{1,3}, \ \mathbf{\Delta}^0_4\right)$.

Proof. Once again, it suffices to use the same arguments as the ones in the proof of Proposition 5.23. □

The limit case

Suppose that we have already defined ${}^0\mathcal{P}_{\alpha-1} \in \mathbf{\Pi}^0_1([T])$, ${}^0\mathcal{P}_\alpha \in \mathbf{\Pi}^0_1\left({}^0\mathcal{P}_{\alpha-1}\right)$, $\mathcal{F}_\alpha \subseteq \mathbf{\Pi}^0_1\left({}^0\mathcal{P}_{\alpha-1}\right)$ a countable set generating a question-tree ${}^1\mathbb{T}_\alpha$, ${}^1\mathcal{P}_\alpha \in \mathbf{\Pi}^0_1\left(\left[{}^1\mathbb{T}_\alpha\right]\right)$ perfect, a question-tree ${}^2\mathbb{T}_\alpha$ on $\left({}^1\mathcal{P}_\alpha, {}^1\mathbb{T}_\alpha\right)$ generated by a countable set $\mathcal{G}_\alpha \subseteq \mathbf{\Pi}^0_1\left({}^1\mathcal{P}_\alpha, {}^1\mathbb{T}_\alpha\right)$ and ${}^2\mathcal{P}_\alpha \in \mathbf{\Pi}^0_1\left(\left[{}^1\mathbb{T}_\alpha\right]\right)$ for any $\alpha < \lambda$, where $0 < \lambda \in \omega_1$ is a limit ordinal. We define ${}^0\mathcal{P}_{<\lambda} = \bigcap_{\alpha<\lambda} {}^0\mathcal{P}_\alpha \in \mathbf{\Pi}^0_1([T])$. Let $\mathcal{F}_\lambda = \left\{A \cap {}^0\mathcal{P}_{<\lambda} : A \in \bigcup_{\alpha<\lambda} \mathcal{F}_\alpha\right\} \subseteq \mathbf{\Pi}^0_1\left({}^0\mathcal{P}_{<\lambda}\right)$. The set $\mathcal{F}_\lambda$ is countable and thus generates a question-tree ${}^1\mathbb{T}_\lambda$ on ${}^0\mathcal{P}_{<\lambda}$ such that $f : \left[{}^1\mathbb{T}_\lambda\right] \to \omega^\omega \in \Lambda_{1,3}$. Once again, we use Proposition 5.29 to get ${}^1\mathcal{P}_\lambda \in \mathbf{\Pi}^0_1\left(\left[{}^1\mathbb{T}_\lambda\right]\right)$ perfect, a question-tree ${}^2\mathbb{T}_\lambda$ on $\left({}^1\mathcal{P}_\lambda, {}^1\mathbb{T}_\lambda\right)$ generated by a countable $\mathcal{G}_\lambda \subseteq \mathbf{\Pi}^0_1\left({}^1\mathcal{P}_\lambda, {}^1\mathbb{T}_\lambda\right)$ and ${}^2\mathcal{P}_\lambda \in \mathbf{\Pi}^0_1\left(\left[{}^2\mathbb{T}_\lambda\right]\right)$ such that $f : \left[{}^2\mathbb{T}_\lambda\right] \to \omega^\omega \in \Lambda_{1,2}$ and $f : \left({}^0\mathcal{P}_{<\lambda} \setminus {}^1\mathcal{P}_\lambda, {}^1\mathbb{T}_\lambda\right) \to \omega^\omega \in \mathrm{Dec}\left(\Lambda_{1,2},\ \mathbf{\Delta}^0_3\right)$. We define ${}^0\mathcal{P}_\lambda$ as the closure of ${}^1\mathcal{P}_\lambda$ in ${}^0\mathcal{P}_{<\lambda}$. We have the following properties.

Proposition 5.38. *Let* $f : [T] \to \omega^\omega \in \Lambda_{1,4}$.

1. ${}^0\mathcal{P}_\lambda$ *is perfect.*
2. $f : {}^0\mathcal{P}_{<\lambda} \setminus {}^0\mathcal{P}_\lambda \to \omega^\omega \in \mathrm{Dec}\left(\Lambda_{1,3},\ \mathbf{\Delta}^0_4\right)$.

Proof. Once more, it suffices to use the same arguments as the ones in the proof of Proposition 5.23. □

The construction does terminate

By construction, $\left({}^0\mathcal{P}_\alpha\right)_{\alpha\in\omega_1}$ is a $\subseteq$-decreasing sequence of $\mathbf{\Pi}^0_1([T])$ sets so that it stabilizes. Let $\beta \in \omega_1$ be minimal such that ${}^0\mathcal{P}_\beta = {}^0\mathcal{P}_\alpha$ holds for any $\alpha \geq \beta$. Observe that the set $\{\mathcal{F}_\alpha : \alpha \geq \beta\}$ that we constructed fulfills the hypothesis of Theorem 5.28 so that there exists $\gamma \geq \beta$ such that $\mathcal{F}_\alpha = \mathcal{F}_\gamma$ holds for any $\alpha \geq \gamma$. By construction, we also have ${}^1\mathcal{P}_\alpha = {}^1\mathcal{P}_\gamma$ for any $\alpha \geq \gamma$. Observe that the set $\{\mathcal{G}_\alpha : \alpha \geq \gamma\}$ that we constructed also fulfills the hypothesis of Theorem 5.28 so that there exists $\delta \geq \gamma$ such that $\mathcal{G}_\alpha = \mathcal{G}_\delta$ for any $\alpha \geq \delta$. As previously, we have ${}^2\mathcal{P}_\alpha = {}^2\mathcal{P}_\gamma$ for any $\alpha \geq \gamma$.

For the remaining of the section, we set ${}^0\mathcal{P} = {}^0\mathcal{P}_\delta, {}^1\mathcal{P} = {}^1\mathcal{P}_\delta, {}^2\mathcal{P} = {}^2\mathcal{P}_\delta, \mathcal{F} = \mathcal{F}_{\delta+1}, \mathcal{G} = \mathcal{G}_{\delta+1}, {}^1\mathbb{T} = {}^1\mathbb{T}_{\delta+1}$ and ${}^2\mathbb{T} = {}^2\mathbb{T}_{\delta+1}$. We summarize the important features of the construction in the following proposition, which is a natural extension of Proposition 5.29.

Proposition 5.39. *Let* $f : [T] \to \omega^\omega \in \Lambda_{1,4}$.

1. ${}^0\mathcal{P}$ *is perfect.*
2. $f : [T] \setminus {}^0\mathcal{P} \to \omega^\omega \in \mathrm{Dec}\left(\Lambda_{1,3},\ \mathbf{\Delta}^0_4\right)$.
3. *For any* $u \in \omega^{<\omega}$ *and any* $\mathfrak{s} \in {}^2\mathbb{T}$*, one has*

$$\left[\mathrm{Init}_{({}^2\mathcal{P},{}^2\mathbb{T}),{}^0\mathcal{P}}\left(f^{-1}[u] \cap [\mathfrak{s}]\right)\right] \in \mathcal{F}.$$

Two useful lemmas

As previously, we have two lemmas which stem from the previous construction.

Lemma 5.40. *Let $u \in \omega^{<\omega}$, $\mathfrak{s} \in {}^2\mathbb{T}$ and $x \in \left[{}^2\mathbb{T}\right]$ be such that $f^{-1}[u] \cap [\mathfrak{s}]$ is proper and non-self-dual in $\left({}^2\mathcal{P}, {}^2\mathbb{T}\right)$, $\mathfrak{s} \sqsubset x$ and $[\mathfrak{s}'] \cap f^{-1}[u] \equiv_w [\mathfrak{s}] \cap f^{-1}[u]$ in $\left({}^2\mathcal{P}, {}^2\mathbb{T}\right)$ for any $\mathfrak{s} \sqsubseteq \mathfrak{s}' \sqsubset x$. There exists $\mathfrak{s} \sqsubseteq \mathfrak{s}'' \sqsubset x$ such that $[\mathfrak{s}''] \cap f^{-1}[u] \equiv_w [\mathfrak{s}] \cap f^{-1}[u]$ in $\left({}^2\mathcal{P}, {}^2\mathbb{T}\right)$ and*

$$\left[\mathfrak{s}''\right] \subseteq \left[\mathrm{Init}_{\left({}^2\mathcal{P}, {}^2\mathbb{T}\right), {}^0\mathcal{P}}\left(f^{-1}[u] \cap [\mathfrak{s}]\right)\right].$$

Proof. It suffices to use the same arguments as the ones in the proof of Lemma 5.30. □

Lemma 5.41. *Let $u \in \omega^{<\omega}$ and $\mathfrak{s} \in {}^2\mathbb{T}$ be such that $f^{-1}[u] \cap [\mathfrak{s}]$ is proper and non-self-dual in $\left({}^2\mathcal{P}, {}^2\mathbb{T}\right)$. If $t \in \mathcal{T}\left(\left[\mathrm{Init}_{\left({}^2\mathcal{P}, {}^2\mathbb{T}\right), {}^0\mathcal{P}}\left(f^{-1}[u] \cap [\mathfrak{s}]\right)\right]\right)$, then there exists $\mathfrak{s}' \in \mathcal{T}\left({}^2\mathcal{P}, {}^2\mathbb{T}\right)$ such that $t \sqsubseteq \pi_0(\mathfrak{s}')$, $\mathfrak{s} \sqsubseteq \mathfrak{s}'$ and $f^{-1}[u] \cap [\mathfrak{s}'] \equiv_w f^{-1}[u] \cap [\mathfrak{s}]$ in $\left({}^2\mathcal{P}, {}^2\mathbb{T}\right)$.*

Proof. It suffices to use the same arguments as the ones in the proof of Lemma 5.31. □

5.5.2 The strategy $\boldsymbol{\mu}$

Let $f : [T] \to \omega^\omega \in \Lambda_{1,4}$ be such that ${}^0\mathcal{P}$ is non-empty. In particular, both ${}^1\mathcal{P}$ and ${}^2\mathcal{P}$ are also non-empty. The function $f : \left({}^2\mathcal{P}, {}^2\mathbb{T}\right) \to \omega^\omega$ belongs to $\Lambda_{1,2}$ so that Assumption 5.13 yields the sets $\{u_n : n \in \omega\}$, $\{p_n : n \in \omega\}$ and $\left\{x_n^l : n, l \in \omega\right\}$. In Section 5.4, we prove that, if $\mathcal{U} = \bigcup_{n\in\omega}[u_n] \in \boldsymbol{\Sigma}_1^0(\omega^\omega)$, then there exists a winning strategy τ for II in

$$G_w\left(\left(\mathcal{O}^{\sim_2}\right)^{\mathtt{b}}, f^{-1}[\mathcal{U}]\right)\left(\omega_{\{\mathtt{b},\leftarrow,\leftarrow_1\}}{}^\omega, \left({}^1\mathcal{P}, {}^1\mathbb{T}\right)\right).$$

Moreover, τ is lifted from a set $\left\{\sigma_p : p \in \omega_{\{\mathtt{b},\leftarrow,\leftarrow_1\}}{}^{<\omega}\right\}$ of winning strategies for II in

$$G_w\left(\left(\mathcal{O}^{\sim_1}\right)^{\mathtt{b}}, f^{-1}[\mathcal{U}]\right)\left(\omega_{\{\mathtt{b},\leftarrow\}}{}^\omega, \left({}^2\mathcal{P}, {}^2\mathbb{T}\right)\right)$$

such that, for any $s \in \omega_{\{\mathtt{b},\leftarrow\}}{}^{<\omega}$, there exist a unique $n \in \omega$ and some $l \in \omega$ such that $p_n \sqsubseteq \sigma_p(s) \sqsubset x_n^l$, $\lim_{k\in\omega}\sigma_p\left(s0^k\right) \in f^{-1}[u_n]$ and, for any $s' \sqsupset s$, one has

$$\left[\sigma_p\left(s'\right)\right] \subseteq \left[\mathrm{Init}_{\left({}^2\mathcal{P}, {}^2\mathbb{T}\right), {}^1\mathcal{P}}\left(f^{-1}[u_n] \cap [\sigma_p(s)]\right)\right].$$

In this section, we construct a winning strategy μ for II in

$$G_w\left(\left(\mathcal{O}^{\sim_3}\right)^{\mathtt{b}}, f^{-1}\left[\mathcal{U}\right]\right)\left({\omega_{\{\mathtt{b},\twoheadleftarrow_0,\twoheadleftarrow_1,\twoheadleftarrow_2\}}}^{\omega}, {}^0\mathcal{P}\right).$$

The strategy μ is the lift of a set $\left\{\tau_p : p \in {\omega_{\{\mathtt{b},\twoheadleftarrow_0,\twoheadleftarrow_1,\twoheadleftarrow_2\}}}^{<\omega}\right\}$ of winning strategies for II in

$$G_w\left(\left(\mathcal{O}^{\sim_2}\right)^{\mathtt{b}}, f^{-1}\left[\mathcal{U}\right]\right)\left({\omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}}^{\omega}, \left({}^1\mathcal{P}, {}^1\mathbb{T}\right)\right)$$

where each strategy τ_p is itself the lift of a set $\left\{\sigma_{p,p'} : p' \in {\omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}}^{<\omega}\right\}$ of winning strategies for II in

$$G_w\left(\left(\mathcal{O}^{\sim_1}\right)^{\mathtt{b}}, f^{-1}\left[\mathcal{U}\right]\right)\left({\omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}}^{\omega}, \left({}^2\mathcal{P}, {}^2\mathbb{T}\right)\right).$$

Moreover, for any $p \in {\omega_{\{\mathtt{b},\twoheadleftarrow_0,\twoheadleftarrow_1,\twoheadleftarrow_2\}}}^{<\omega}$, any $p' \in {\omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}}^{<\omega}$ and any $s \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$, there exist both a unique $n \in \omega$ and some $l \in \omega$ such that $p_n \sqsubseteq \sigma_{p,p'}(s) \sqsubset x_n^l$, $\lim_{k\in\omega} \sigma_{p,p'}\left(s0^k\right) \in f^{-1}\left[u_n\right]$ and, for any $s' \sqsupset s$, one has

$$\begin{aligned}\left[\sigma_{p,p'}\left(s'\right)\right] \subseteq &\left[\operatorname{Init}_{({}^2\mathcal{P},{}^2\mathbb{T}),{}^1\mathcal{P}}\left(f^{-1}\left[u_n\right] \cap \left[\sigma_{p,p'}(s)\right]\right)\right]\\ &\cap \left[\operatorname{Init}_{({}^2\mathcal{P},{}^2\mathbb{T}),{}^0\mathcal{P}}\left(f^{-1}\left[u_n\right] \cap \left[\sigma_{p,p'}(s)\right]\right)\right].\end{aligned}$$

In other words, at each step of the construction, we make sure to have answered positively two well-chosen questions. A representation of the strategy μ as seen from inside ${}^2\mathbb{T}$ is given in Figure 5.3, where the notations are specified in the following paragraphs.

The construction of these strategies is similar to the one of the previous section. To illustrate this idea, we consider the construction of $\sigma_{\langle\rangle,\langle\rangle}(\langle\rangle)$ and $\sigma_{\langle\rangle,\langle\rangle}(\langle\rangle)'$. Let $\tau^{\twoheadleftarrow}$ be a winning strategy for II in the eraser game

$$G_{\twoheadleftarrow}\left(f : \left({}^2\mathcal{P}, {}^2\mathbb{T}\right) \to \omega^\omega\right).$$

First, let $\sigma_{\langle\rangle,\langle\rangle}(\langle\rangle) \in \mathcal{T}\left({}^2\mathcal{P}, {}^2\mathbb{T}\right)$ be such that $p_0 \sqsubseteq \sigma_{\langle\rangle,\langle\rangle}(\langle\rangle) \sqsubset x_0^0$ and $u_0 \sqsubseteq \tau^{\twoheadleftarrow}\left(\sigma_{\langle\rangle,\langle\rangle}(\langle\rangle)\right)^{\hookleftarrow}$. By Lemma 5.30, there exists $\sigma_{\langle\rangle,\langle\rangle}(\langle\rangle) \sqsubseteq \mathfrak{t} \sqsubset x_0^0$ such that $[\mathfrak{t}] \cap f^{-1}\left[u_0\right] \equiv_w \left[p_0\right] \cap f^{-1}\left[u_0\right]$ in $\left({}^2\mathcal{P}, {}^2\mathbb{T}\right)$ and

$$[\mathfrak{t}] \subseteq \left[\operatorname{Init}_{({}^2\mathcal{P},{}^2\mathbb{T}),{}^1\mathcal{P}}\left(f^{-1}\left[u_n\right] \cap \left[\sigma_{\langle\rangle,\langle\rangle}(\langle\rangle)\right]\right)\right].$$

Also, by Lemma 5.40, there exists $\mathfrak{t} \sqsubseteq \sigma_{\langle\rangle,\langle\rangle}(\langle\rangle)' \sqsubset x_0^0$ such that we have $\left[\sigma_{\langle\rangle,\langle\rangle}(\langle\rangle)'\right] \cap f^{-1}\left[u_n\right] \equiv_w \left[p_0\right] \cap f^{-1}\left[u_0\right]$ in $\left({}^2\mathcal{P}, {}^2\mathbb{T}\right)$ and

$$\left[\sigma_{\langle\rangle,\langle\rangle}(\langle\rangle)'\right] \subseteq \left[\operatorname{Init}_{({}^2\mathcal{P},{}^2\mathbb{T}),{}^0\mathcal{P}}\left(f^{-1}\left[u_n\right] \cap \left[\sigma_{\langle\rangle,\langle\rangle}(\langle\rangle)\right]\right)\right].$$

Using Lemma 5.30 and Lemma 5.40 at each step of the construction, we obtain a strategy $\sigma_{\langle\rangle,\langle\rangle}$ such that, for any $s \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$, there exists a unique $n \in \omega$ such that $p_n \sqsubseteq \sigma_{\langle\rangle,\langle\rangle}(s) \sqsubset x_n^0$, $\lim_{k\in\omega} \sigma_{\langle\rangle,\langle\rangle}\left(s0^k\right) \in f^{-1}[u_n]$ and, for any $s' \sqsupset s$, one has

$$\left[\sigma_{\langle\rangle,\langle\rangle}\left(s'\right)\right] \subseteq \left[\mathrm{Init}_{({}^2\mathcal{P},{}^2\mathbb{T}),{}^1\mathcal{P}}\left(f^{-1}[u_n] \cap \left[\sigma_{\langle\rangle,\langle\rangle}(s)\right]\right)\right] \\ \cap \left[\mathrm{Init}_{({}^2\mathcal{P},{}^2\mathbb{T}),{}^0\mathcal{P}}\left(f^{-1}[u_n] \cap \left[\sigma_{\langle\rangle,\langle\rangle}(s)\right]\right)\right].$$

Clearly, using this technique over and over, it is possible to define the strategy $\sigma_{\langle\rangle,p'}$ for any $p' \in {\omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}}^{<\omega}$ according to the previous section. Moreover, for any $s \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$, there exist a unique $n \in \omega$ and some $l \in \omega$ such that $p_n \sqsubseteq \sigma_{\langle\rangle,p'}(s) \sqsubset x_n^l$, $\lim_{k\in\omega} \sigma_{\langle\rangle,p'}\left(s0^k\right) \in f^{-1}[u_n]$ and, for any $s' \sqsupset s$, one has

$$\left[\sigma_{\langle\rangle,p'}\left(s'\right)\right] \subseteq \left[\mathrm{Init}_{({}^2\mathcal{P},{}^2\mathbb{T}),{}^1\mathcal{P}}\left(f^{-1}[u_n] \cap \left[\sigma_{\langle\rangle,p'}(s)\right]\right)\right] \\ \cap \left[\mathrm{Init}_{({}^2\mathcal{P},{}^2\mathbb{T}),{}^0\mathcal{P}}\left(f^{-1}[u_n] \cap \left[\sigma_{\langle\rangle,p'}(s)\right]\right)\right].$$

This yields a strategy $\tau_{\langle\rangle}$ for II in the game

$$G_w\left(\left(\mathcal{O}^{\sim_2}\right)^{\mathtt{b}}, f^{-1}[\mathcal{U}]\right)\left({\omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}}^{\omega}, \left({}^1\mathcal{P}, {}^1\mathbb{T}\right)\right)$$

which is winning since it is defined accordingly to the guidelines of the previous section.

The other strategies τ_p are defined by induction on the length of $p \in {\omega_{\{\mathtt{b},\twoheadleftarrow_0,\twoheadleftarrow_1,\twoheadleftarrow_2\}}}^{<\omega}$. If $p = qa$ with $a \in \omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}$ or if $p = q\twoheadleftarrow_2$ and $p_{\mathrm{lh}(p)-1}$ does not erase anything (see Definition 2.60, page 54), then $\tau_p = \tau_q$.

Otherwise, let $p = q\twoheadleftarrow_2 = rg$ be defined such that $p_{\mathrm{lh}(p-1)} = \twoheadleftarrow_2$ erases $p_{\mathrm{lh}(r)}$. The idea is rather simple: the strategy τ_p is defined as a shift of the strategy τ_q which is performed when I plays $\widetilde{r}^2$. More precisely, we shift $\sigma_{q,\widetilde{r}^2}\left(\widetilde{r}^{2\to1}\right)'$ to get a newly defined $\sigma_{p,\widetilde{r}^2}\left(\widetilde{r}^{2\to1}\right)'$. This shift is defined via Lemma 5.41 to satisfy

$$\pi_0\left(\tau_q\left(\widetilde{q}^2\right)\right) \sqsubset \pi_0\left(\tau_p\left(\widetilde{r}^2\right)\right).$$

There exist a unique $n \in \omega$ and some $l \in \omega$ such that $p_n \sqsubseteq \sigma_{q,\widetilde{r}^2}\left(\widetilde{r}^{2\to1}\right) \sqsubset x_n^l$ and $\lim_{k\in\omega} \sigma_{q,\widetilde{r}^2}\left(\widetilde{r}^{2\to1}0^k\right) \in f^{-1}[u_n]$. Moreover, by construction, we also have

$$\left[\sigma_{q,\widetilde{q}^2}\left(\widetilde{q}^{2\to1}\right)\right] \subseteq \left[\mathrm{Init}_{({}^2\mathcal{P},{}^2\mathbb{T}),{}^0\mathcal{P}}\left(f^{-1}[u_n] \cap \sigma_{q,\widetilde{r}^2}\left(\widetilde{r}^{2\to1}\right)\right)\right].$$

Using Lemma 5.41, one finds $\mathfrak{t}' \in \mathbb{T}$ such that $[\mathfrak{t}'] \cap f^{-1}[u_n] \equiv_w [p_n] \cap f^{-1}[u_n]$ in $({}^2\mathcal{P}, {}^2\mathbb{T})$, $\pi_0(\mathfrak{t}') \sqsupseteq \pi_0\left(\sigma_{q,\widetilde{q}^2}\left(\widetilde{q}^{2\to 1}\right)\right)$ and $\mathfrak{t}' \sqsupseteq \sigma_{q,\widetilde{r}^2}\left(\widetilde{r}^{2\to 1}\right)$. Then, by Lemma 5.30 and Lemma 5.40, we also choose $\mathfrak{t} \sqsupseteq \mathfrak{t}'$ such that $[\mathfrak{t}] \cap f^{-1}[u_n] \equiv_w [p_n] \cap f^{-1}[u_n]$ in $({}^2\mathcal{P}, {}^2\mathbb{T})$ and

$$[\mathfrak{t}] \subseteq \left[\mathrm{Init}_{({}^2\mathcal{P}, {}^2\mathbb{T}), {}^1\mathcal{P}}\left(f^{-1}[u_n] \cap \left[\sigma_{q,\widetilde{r}^2}\left(\widetilde{r}^{2\to 1}\right)\right]\right)\right] \cap \left[\mathrm{Init}_{({}^2\mathcal{P}, {}^2\mathbb{T}), {}^0\mathcal{P}}\left(f^{-1}[u_n] \cap \left[\sigma_{q,\widetilde{r}^2}\left(\widetilde{r}^{2\to 1}\right)\right]\right)\right].$$

We shift the strategy $\sigma_{q,\widetilde{r}^2}$ by using $\mathfrak{t}$ instead of the previously defined $\sigma_{q,\widetilde{r}^2}\left(\widetilde{r}^{2\to 1}\right)'$ to get a new strategy $\sigma_{p,\widetilde{r}^2}$. Using the previous section, this yields a whole new set of strategies $\left\{\sigma_{p,p'} : p' \in {\omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}}^{<\omega}\right\}$. This set of strategies itself yields a brand new winning strategy τ_p for II in the game

$$G_w\left(\left(\mathcal{O}^{\sim 2}\right)^{\mathtt{b}}, f^{-1}[\mathcal{U}]\right)\left({\omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}}^{\omega}, \left({}^1\mathcal{P}, {}^1\mathbb{T}\right)\right).$$

Moreover, by Assumption 5.13 and by construction, for any $p' \in {\omega_{\{\mathtt{b},\twoheadleftarrow,\twoheadleftarrow_1\}}}^{<\omega}$ and any $s \in {\omega_{\{\mathtt{b},\twoheadleftarrow\}}}^{<\omega}$, there exist a unique $n \in \omega$ and some $l \in \omega$ such that $p_n \sqsubseteq \sigma_{p,p'}(s) \sqsubset x_n^l$, $\lim_{k\in\omega} \sigma_{p,p'}\left(s0^k\right) \in f^{-1}[u_n]$ and, for any $s' \sqsupset s$, one has

$$\left[\sigma_{p,p'}(s')\right] \subseteq \left[\mathrm{Init}_{({}^2\mathcal{P}, {}^2\mathbb{T}), {}^1\mathcal{P}}\left(f^{-1}[u_n] \cap \left[\sigma_{p,p'}(s)\right]\right)\right] \cap \left[\mathrm{Init}_{({}^2\mathcal{P}, {}^2\mathbb{T}), {}^0\mathcal{P}}\left(f^{-1}[u_n] \cap \left[\sigma_{p,p'}(s)\right]\right)\right].$$

The following well-defined strategy μ follows from the construction:

$$\mu : {\omega_{\{\mathtt{b},\twoheadleftarrow_0,\twoheadleftarrow_1,\twoheadleftarrow_2\}}}^{<\omega} \to {}^0\mathcal{P}$$
$$p \mapsto \pi_0\left(\tau_p\left(\widetilde{p}^2\right)\right) = \pi_0\left(\sigma_{p,\widetilde{p}^2}\left(\widetilde{p}^{2\to 1}\right)\right).$$

In Figure 5.3, we schematically represent μ as seen from inside ${}^2\mathbb{T}$. From now on and for the remaining of the section, for any $p \in {\omega_{\{\mathtt{b},\twoheadleftarrow_0,\twoheadleftarrow_1,\twoheadleftarrow_2\}}}^{<\omega}$, we write $\mathfrak{t}_p = \sigma_{p,\widetilde{p}^2}\left(\widetilde{p}^{2\to 1}\right)$ for the element of ${}^2\mathbb{T}$ from which $\mu(p)$ is constructed. The position $\mathfrak{t} \in {}^2\mathbb{T}$ denotes the newly defined

$$\mathfrak{t} = \sigma_{0\twoheadleftarrow\twoheadleftarrow_1\twoheadleftarrow_2, 0\twoheadleftarrow}(0\twoheadleftarrow)'$$

of the previous construction.

It remains to prove that μ is winning, i.e., for any $x \in {\omega_{\{\mathtt{b},\twoheadleftarrow_0,\twoheadleftarrow_1,\twoheadleftarrow_2\}}}^{\omega}$, we have

$$x \in \left(\mathcal{O}^{\sim 3}\right)^{\mathtt{b}} \Leftrightarrow \lim_{k\in\omega} \mu\left(x_{\restriction k}\right) \in f^{-1}[\mathcal{U}].$$

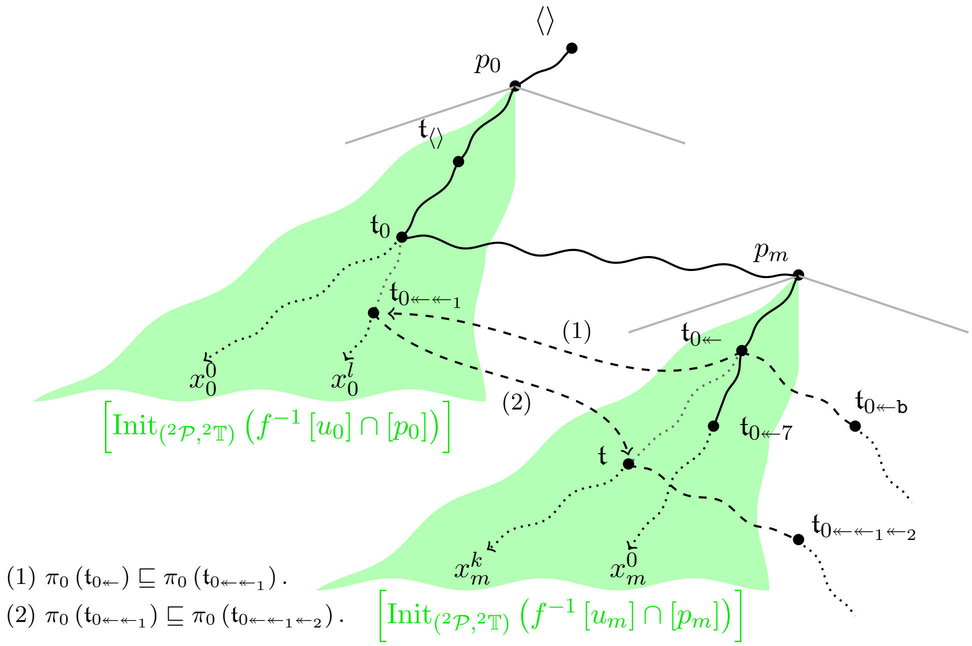


Figure 5.3: The strategy μ as seen from inside ${}^2\mathbb{T}$.

Let $x \in \omega_{\{\mathtt{b},\twoheadleftarrow_0,\twoheadleftarrow_1,\twoheadleftarrow_2\}}{}^{\omega}$, we study the sequence

$$\left(\mathsf{t}_{x\restriction k}\right)_{k\in\omega} \subseteq {}^2\mathbb{T}.$$

Even though this sequence might not be $\sqsubset$-increasing, we prove that it contains an $\sqsubset$-increasing subsequence. Let $\{\alpha_i : i \in \omega\} = \{\alpha_i^{\twoheadleftarrow} : i \in \omega\}$ be given by Definition 2.62 (page 55). We consider the sequence

$$\left(\mathsf{t}_{x\restriction\alpha_i}\right)_{i\in\omega} \subseteq {}^2\mathbb{T}.$$

Claim 5.42. *For any $i \in \omega$,*

$$\mathsf{t}_{x\restriction\alpha_i} \sqsubseteq \mathsf{t}_{x\restriction\alpha_{i+1}}.$$

Proof of the claim. Since $\{\alpha_i : i \in \omega\} \subseteq \left\{\alpha_i^{\twoheadleftarrow_2} : i \in \omega\right\}$, Lemma 2.63 implies

$$\widetilde{x\restriction\alpha_i}^{2} \sqsubset \widetilde{x\restriction\alpha_{i+1}}^{2}. \tag{5.4}$$

If $x_{\alpha_{i+1}-1} \neq \twoheadleftarrow_2$ or if $x_{\alpha_{i+1}-1} = \twoheadleftarrow_2$ does not erase anything, then $\tau_{x\restriction\alpha_i} = \tau_{x\restriction\alpha_{i+1}}$. By Claim 5.32 and $\{\alpha_i : i \in \omega\} \subseteq \left\{\alpha_i^{\twoheadleftarrow_2} : i \in \omega\right\}$, we get the result.

If $x_{\alpha_{i+1}-1} = \twoheadleftarrow_2$, then $x_{\restriction \alpha_{i+1}} = x_{\restriction \alpha_i} g$ such that $x_{\alpha_{i+1}-1}$ erases x_{α_i}. By construction, $\tau_{x_{\restriction \alpha_{i+1}}} \left(\widetilde{x_{\restriction \alpha_i}}^2 \right)$ also comes from $\mathtt{t}_{x_{\restriction \alpha_i}}$. Since we have $\widetilde{x_{\restriction \alpha_{i+1}}}^{2 \to 0} = \widetilde{x_{\restriction \alpha_i}}^{2 \to 0} \mathtt{b}^{\mathrm{lh}(g)}$ and by Equation (5.4), we get the result. $\square_{\text{Claim}}$

Since ${}^2\mathbb{T}$ is a question-tree of level 2 on ${}^0\mathcal{P}$, we obtain that

$$\left(\pi_0 \left(\mathtt{t}_{x_{\restriction \alpha_i}} \right) \right)_{i \in \omega}$$

has a limit in ${}^0\mathcal{P}$. By definition, this limit coincides with $\lim_{k \in \omega} \tau \left(x_{\restriction k} \right)$. Thus, it only remains to prove that

$$x \in \left(\mathcal{O}^{\sim_3} \right)^{\mathtt{b}} \;\Leftrightarrow\; \lim_{i \in \omega} \mathtt{t}_{x_{\restriction \alpha_i}} \in f^{-1} \left[\mathcal{U} \right].$$

The case $x \in \left(\mathcal{O}^{\sim_3} \right)^{\mathtt{b}}$

Since $x \in \left(\mathcal{O}^{\sim_3} \right)^{\mathtt{b}}$ implies $\widetilde{x}^{2 \to 0} \in \mathcal{O}^{\mathtt{b}}$, there exists $m \in \omega$ such that

$$\widetilde{x}^{2 \to 0} (m) \neq \mathtt{b}.$$

Let $j \in \omega$ be such that $\alpha_j > m$. Using Lemma 2.66, for any $i \geq j$, we have

$$\widetilde{x_{\restriction \alpha_i}}^{2 \to 0}_{[\ /\mathtt{b}]} \neq \langle \rangle. \tag{5.5}$$

By construction, there exist a unique $n \in \omega$ and some $l \in \omega$ such that $p_n \sqsubseteq \mathtt{t}_{x_{\restriction \alpha_j}} \sqsubset x_n^l$, $\lim_{k \in \omega} \sigma_{x_{\restriction \alpha_j}, \widetilde{x_{\restriction \alpha_j}}^2} \left(\widetilde{x_{\restriction \alpha_j}}^{2 \to 1} 0^k \right) \in f^{-1} \left[u_n \right]$ and $\left[\mathtt{t}_{x_{\restriction \alpha_j}} \right] \cap f^{-1}[u_n] \equiv_w f^{-1} \left[u_n \right] \cap [p_n]$ in $\left({}^2\mathcal{P}, {}^2\mathbb{T} \right)$.

Claim 5.43. *For any $i \geq j$, $\left[\mathtt{t}_{x_{\restriction \alpha_j}} \right] \cap f^{-1}[u_n] \equiv_w f^{-1} \left[u_n \right] \cap [p_n]$ in $\left({}^2\mathcal{P}, {}^2\mathbb{T} \right)$.*

Proof of the claim. Suppose that $\left[\mathtt{t}_{x_{\restriction \alpha_i}} \right] \cap f^{-1}[u_n] \equiv_w f^{-1} \left[u_n \right] \cap [p_n]$ in $\left({}^2\mathcal{P}, {}^2\mathbb{T} \right)$. If $x_{\alpha_{i+1}-1} \neq \twoheadleftarrow_2$ or if $x_{\alpha_{i+1}-1} = \twoheadleftarrow_2$ does not erase anything, then $\tau_{x_{\restriction \alpha_i}} = \tau_{x_{\restriction \alpha_{i+1}}}$. Equation (5.5), $\{ \alpha_i : i \in \omega \} \subseteq \left\{ \alpha_i^{\twoheadleftarrow_2} : i \in \omega \right\}$ and Claim 5.33 yield the result. If $x_{\alpha_{i+1}-1} = \twoheadleftarrow_2$ erases something, then $x_{\restriction \alpha_{i+1}} = x_{\restriction \alpha_i} g$ such that $x_{\alpha_{i+1}-1}$ erases x_{α_i}. By construction, $\tau_{x_{\restriction \alpha_{i+1}}} \left(\widetilde{x_{\restriction \alpha_i}}^2 \right)$ also comes from $\mathtt{t}_{x_{\restriction \alpha_i}}$. Since $\widetilde{x_{\restriction \alpha_{i+1}}}^{2 \to 0} = \widetilde{x_{\restriction \alpha_i}}^{2 \to 0} \mathtt{b}^{\mathrm{lh}(g)}$, Equation (5.5) and the construction of $\tau_{x_{\restriction \alpha_{i+1}}}$ yield the result. $\square_{\text{Claim}}$

By Claim 5.43 and Proposition 2.42, we get

$$\lim_{i \in \omega} \mathtt{t}_{x_{\restriction \alpha_i}} \in f^{-1}[u_n] \subseteq f^{-1} \left[\mathcal{U} \right].$$

The case $x \notin (\mathcal{O}^{\sim_3})^{\mathtt{b}}$

In that case, we show that, for any $n \in \omega$, we have

$$\lim_{i\in\omega} \mathfrak{t}_{x\restriction\alpha_i} \notin f^{-1}[u_n].$$

Fix $n \in \omega$. Since $\widetilde{x}^{2\to 0}_{[\ /\mathtt{b}]} = \langle\rangle$, one has

$$\operatorname{Card}\left\{l \in \omega : \widetilde{x_{\restriction l}}^{2\to 0}_{[\ /\mathtt{b}]} = \langle\rangle\right\} = \aleph_0.$$

In particular, there exist an increasing sequence $(n_i)_{i\in\omega} \subseteq \{\alpha_i : i \in \omega\}$, an infinite set $\{m_i : i \in \omega\} \subseteq \omega$ and a $\sqsubset$-increasing sequence $(\mathfrak{s}_i)_{i\in\omega} \subseteq \mathcal{T}\left({}^2\mathcal{P}, {}^2\mathbb{T}\right)$ such that $\lim_{i\in\omega} \mathfrak{s}_i = \lim_{i\in\omega} \mathfrak{t}_{x_{\restriction n_i}}$, and for any $i < j$, $p_{m_i} \sqsubseteq \mathfrak{t}_{x_{\restriction n_i}} \sqsubset p_{m_{i+1}}$, $u_{m_i} \sqsubseteq \tau^{\twoheadleftarrow}(\mathfrak{s}_i)^{\hookleftarrow\rho}$ and $m_i \neq m_j$. By Proposition 5.21, we obtain

$$\lim_{i\in\omega} \mathfrak{t}_{x\restriction\alpha_i} \notin f^{-1}[u_n],$$

which finally implies that μ is winning.

5.5.3 The main result

We prove a second generalization of the Jayne-Rogers Theorem.

Theorem 5.44 (AD)**.** *Assuming Assumption 5.13, if $f : [T] \to \omega^\omega$ is such that $f \in \Lambda_{1,4} \setminus \operatorname{Dec}\left(\Lambda_{1,3},\ \boldsymbol{\Delta}^0_4\right)$, then $f \notin \Lambda_{2,4}$.*

In particular, if $\mathcal{X}$ and $\mathcal{Y}$ are both zero-dimensional Polish spaces, then Assumption 5.13 implies the case $(m = 2, n = 4)$ of the Decomposability Conjecture, i.e.,

$$\Lambda_{2,4}(\mathcal{X}, \mathcal{Y}) = \operatorname{Dec}\left(\Lambda_{1,3},\ \boldsymbol{\Delta}^0_4\right)(\mathcal{X}, \mathcal{Y}).$$

Proof. Since $f \notin \operatorname{Dec}\left(\Lambda_{1,3},\ \boldsymbol{\Delta}^0_4\right)$, we have ${}^0\mathcal{P} \neq \emptyset$. We proved that there exists a winning strategy for II in the game

$$G_w\left(\left(\mathcal{O}^{\sim_3}\right)^{\mathtt{b}}, f^{-1}[\mathcal{U}]\right)\left(\omega_{\{\mathtt{b},\twoheadleftarrow_0,\twoheadleftarrow_1,\twoheadleftarrow_2\}}{}^{\omega}, {}^0\mathcal{P}\right),$$

for some open set $\mathcal{U} \in \boldsymbol{\Sigma}^0_1\left({}^0\mathcal{P}\right)$. Since $\left(\mathcal{O}^{\sim_3}\right)^{\mathtt{b}}$ is $\boldsymbol{\Sigma}^0_4$-complete (Theorem 2.53) and this strategy is played in the closed set ${}^0\mathcal{P} \subseteq [T]$, the same strategy yields

$$f^{-1}[\mathcal{U}] \notin \boldsymbol{\Delta}^0_4([T]).$$

This implies that $f : [T] \to \omega^\omega \notin \Lambda_{2,4}$ by Equation (5.1) (page 94). $\square$

5.6 A novel version of the Decomposability Conjecture

Clearly, if we consider question-trees of higher levels, the previous construction can be iterated. As a consequence and under the axiom of determinacy, Assumption 5.13 implies the Decomposability Conjecture.

Theorem 5.45 (AD). *Assuming Assumption 5.13, if we have* $n \geq 2$ *and* $f : [T] \to \omega^\omega$ *such that* $f \in \Lambda_{1,n} \setminus \operatorname{Dec}\left(\Lambda_{1,n-1},\ \boldsymbol{\Delta}^0_n\right)$, *then* $f \notin \Lambda_{2,n}$.

In particular, if $\mathcal{X}$ *and* $\mathcal{Y}$ *are both zero-dimensional Polish spaces, then Assumption 5.13 implies the Decomposability Conjecture, i.e., for any* $1 \leq m \leq n < \omega$, *we have*

$$\Lambda_{m,n}\left(\mathcal{X}, \mathcal{Y}\right) = \operatorname{Dec}\left(\Lambda_{1,n-m+1},\ \boldsymbol{\Delta}^0_n\right)\left(\mathcal{X}, \mathcal{Y}\right).$$

In other words, the game-theoretical framework given by zero-dimensional Polish spaces together with the question-tree machinery on these spaces — which allows to decrease the topological complexity of a function in a game-friendly fashion — yield powerful methods in order to, hopefully, solve the Decomposability Conjecture.

Chapter 6

Open Problems

We conclude by gathering some of the questions that arise from this thesis and which could serve as guidelines for future research.

The Wadge order on quasi-Polish spaces

In Chapter 3, we exhibited a partial order on a class of 2-colored countable posets which embeds in the Wadge order on the $\mathbf{\Delta}^0_2$ degrees of $\mathcal{P}\omega$ (Theorem 3.21, page 71). It would be desirable to find a better description of this partial order, as it was recently done in [KM19] for the Baire space ω^ω. More precisely, the authors showed that the Wadge order on the Borel degrees of ω^ω can be represented by countable joins of countable transfinite nests of 2-colored well-founded trees. Although such a description seems to be out of reach for the moment for the whole Borel degrees of the Scott domain $\mathcal{P}\omega$, a reasonable question is the following.

Question 6.1. Is there any standard order-theoretic structure which is isomorphic to $\left(\mathbb{WD}_{\mathbf{\Delta}^0_2}(\mathcal{P}\omega), \leq_w\right)$?

We also showed that some unwanted properties already occur at a very low topological complexity level in the Wadge order on $\mathcal{P}\omega$. But looking at some more general notions of reducibility may make these bad behaviors disappear as seen in Chapter 4. Following ideas of [AM03, MR09] for Polish spaces, Motto Ros, Schlicht and Selivanov considered in [MRSS15] the class of $\Lambda_{\omega,\omega}$-functions on $\mathcal{P}\omega$

$$\mathcal{F}_0 = \{f : \mathcal{P}\omega \to \mathcal{P}\omega : f^{-1}[\mathcal{A}] \in \mathbf{\Sigma}^0_\omega(\mathcal{P}\omega) \text{ for any } \mathcal{A} \in \mathbf{\Sigma}^0_\omega(\mathcal{P}\omega)\}.$$

They showed that the quasi-order[1] $\leq_{\mathcal{F}_0}$ induces a well-quasi-order on the Borel subsets of $\mathcal{P}\omega$. Thus, the following question seems of interest.

Question 6.2. For which classes of functions $\mathcal{F} \subseteq \mathcal{F}_0$ containing the continuous ones does the induced order $\leq_{\mathcal{F}}$ on the Borel subsets of $\mathcal{P}\omega$ become a well-quasi-order?

Another relevant question concerns the possibility of extending our results to some other quasi-Polish spaces. We essentially focused on $\mathcal{P}\omega$ because it is universal among the quasi-Polish spaces. Since we showed that the Wadge order on $\mathcal{P}\omega$ is not a well-quasi-order, one may ask where the well-behaved quasi-Polish spaces may be found.

Question 6.3. Is there a natural characterization of the quasi-Polish spaces whose Wadge order on the Borel degrees is a well-quasi-order?

In the metrizable setting, Schlicht proved that the Polish spaces for which $\leq_w$ is a well-quasi-order on the Borel degrees are exactly the zero-dimensional ones [Sch18]. It would be interesting to know whether this property somehow extends to the quasi-Polish spaces.

The quasi-order $\preccurlyeq_w$ on quasi-Polish spaces

In Chapter 4, we studied the quasi-order $\preccurlyeq_w$ on the Borel subsets of the Scott domain $\mathcal{P}\omega$, i.e., the quasi-order induced by total relatively continuous relations. We proved that $\left(\mathbb{WD}_{\mathcal{B}}^{\sim w}\left(\mathcal{P}\omega\right), \preccurlyeq_w\right)$ is isomorphic to the Wadge order on the non-self-dual Borel degrees of the Baire space ω^ω (Theorem 4.21, page 92). Since the same result holds for the poset $\left(\mathbb{WD}_{\mathcal{B}}^{\sim w}\left(\mathsf{Conc}\right), \preccurlyeq_w\right)$, it naturally yields the following question.

Question 6.4. Is there a natural characterization of the class $\mathcal{C}$ of countably based T_0-spaces such that $\mathcal{X} \in \mathcal{C}$ if and only if, for any non-self-dual $B \in \mathcal{B}\left(\omega^\omega\right)$, there exists $A \in \mathcal{B}\left(\mathcal{X}\right)$ with $B \sim_w A$?

As conjectured in [Peq15a], a first good candidate is the class of quasi-Polish spaces for they admit a total admissible representation. A second good candidate is the class of Borel representable spaces, where a countably based T_0-space is Borel representable if it admits an admissible representation with Borel domain. Indeed, using game-theoretic techniques, Pequignot proved in [Peq15b] that the quasi-order $\preccurlyeq_w$ on the Borel subsets of any Borel

[1] We write $\mathcal{A} \leq_{\mathcal{F}_0} \mathcal{B}$ if there exists $f \in \mathcal{F}_0$ such that $f^{-1}[\mathcal{B}] = \mathcal{A}$.

representable space $\mathcal{X}$ is well-founded and satisfies Wadge's Lemma, i.e., for any $A, B \in \mathcal{B}(\mathcal{X})$, either $A \preccurlyeq_w B$ or $B^c \preccurlyeq_w A$.

As mentioned in this thesis, it is possible to describe any non-self-dual Wadge pointclass in $\mathcal{B}(\omega^\omega)$ through ω-ary boolean operations [Lou83]. Since $\leq_w$ and $\preccurlyeq_w$ coincide on the Baire space ω^ω, it yields a description of $\{B \subseteq \omega^\omega : B \preccurlyeq_w A\}$ for any non-self-dual $A \in \mathcal{B}(\omega^\omega)$. In [Peq15a, Fou16], the question of extending this result to any quasi-Polish space is asked. Building on the work of Kihara and Montalbán on the Baire space ω^ω [KM19], Selivanov positively answered this question (Theorem 4.11 in [Sel20]) using the notion of infinitary fine hierarchy, an iterated version of the Hausdorff-Kuratowski difference hierarchy. This brings the following question.

Question 6.5. Given a quasi-Polish space $\mathcal{X}$, which are the levels of the infinitary fine hierarchy of $\mathcal{X}$ that contain a complete set?

As for the previous question, an answer to this question would reveal the shape of the quasi-order $\preccurlyeq_w$ on $\mathcal{B}(\mathcal{X})$.

The Decomposability Conjecture

In Chapter 5, we applied the question-tree machinery of [Dup01] to tackle the Decomposability Conjecture. Although we were not able to give a full proof of the conjecture, we strongly believe that these techniques are promising for they fit the game-theoretical framework. In particular, we proved that, under the axiom of determinacy, Assumption 5.13 implies the Decomposability Conjecture for zero-dimensional Polish spaces (Theorem 5.45, page 129). Moreover, we proved that Assumption 5.13 is satisfied for functions of the form $f = id \oplus f' : [T] \to \omega^\omega$, where $f' \in \Lambda_{1,2}$ and $T \subseteq \omega^{<\omega}$ is a non-empty pruned tree (Theorem 5.19, page 102). This provides the following natural question towards the resolution of the Decomposability Conjecture for zero-dimensional Polish spaces.

Question 6.6. Is Assumption 5.13 verified for any function $f : [T] \to \omega^\omega$ such that $f \in \Lambda_{1,2}$?

Also, it seems reasonable that AD is not fully needed. Indeed, in [Day19, Mar20], Day and Marks announced that their proof is performed under $\boldsymbol{\Sigma}^1_2$-determinacy and they even conjectured that it can be performed with less determinacy. As in [Day19, Mar20], we only use determinacy to get Theorem

5.28 (page 110) of which we only make use of case $\alpha = 1$. Thus, we have the same conjecture.

Question 6.7. Is the Decomposability Conjecture for zero-dimensional Polish spaces a consequence of Assumption 5.13 under $\boldsymbol{\Sigma}^1_2$-determinacy?

Finally, in [GKN21], the authors proposed a transfinite version of the Decomposability Conjecture called the Full Decomposability Conjecture.

The Full Decomposability Conjecture 6.8. *If $\mathcal{X}'$ is Polish, $\mathcal{Y}$ is separable and metrizable, $\mathcal{X} \subseteq \mathcal{X}'$ is Suslin, $\alpha \leq \beta < \omega_1$ and $f : \mathcal{X} \to \mathcal{Y}$, then $f \in \Lambda_{1+\alpha,1+\beta}(\mathcal{X}, \mathcal{Y})$ if and only if there exists a countable partition $\{A_n : n \in \omega\} \subseteq \boldsymbol{\Delta}^0_{1+\beta}(\mathcal{X})$ of $\mathcal{X}$ such that $f \restriction A_n \in \Lambda_{1,1+\theta_n}(A_n, \mathcal{Y})$ for some ordinal $\theta_n + \alpha \leq \beta$.*

By [Dup], the question-tree machinery extends outside the realm of the sets of finite Borel ranks. This suggests to apply our construction to the transfinite in the case of $\mathcal{X}$ and $\mathcal{Y}$ Polish and zero-dimensional.

Question 6.9. If $\mathcal{X}$ is a zero-dimensional Polish space and $f : \mathcal{X} \to \omega^\omega$, is it possible to apply our construction to get, for any $\beta \geq \omega$, $f \in \Lambda_{2,1+\beta}(\mathcal{X}, \omega^\omega)$ if and only if there exists a countable partition $\{A_n : n \in \omega\} \subseteq \boldsymbol{\Delta}^0_{1+\beta}(\mathcal{X})$ of $\mathcal{X}$ such that $f \restriction A_n \in \Lambda_{1,1+\theta_n}(A_n, \omega^\omega)$ for some ordinal θ_n satisfying $\theta_n + 1 \leq \beta$?

We strongly believe that the question-tree machinery on zero-dimensional Polish spaces yields the necessary game-theoretical framework in order to solve, under the axiom of determinacy, the Full Decomposability Conjecture for such spaces.

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