# Asymptotic Results for Renewal Risk Models with Risky Investments

H. Albrecher* C. Constantinescu † E. Thomann‡

**Abstract**

We consider a renewal jump-diffusion process, more specifically a renewal insurance risk model with investments in a stock whose price is modeled by a geometric Brownian motion. Using Laplace transforms and regular variation theory, we introduce a transparent and unifying analytic method for investigating the asymptotic behavior of ruin probabilities and related quantities, in models with light- or heavy-tailed jumps, whenever the distribution of the time between jumps has rational Laplace transform.

## 1 Introduction

For the asymptotic analysis of classical (Poisson) jump-diffusion models, the mathematical tools employed in the risk theory literature are random recurrence equations, large deviations or generators. The versatile random equations approach goes back to Goldie (1991) and is based on the fact that the supremum over all future losses satisfies a random equation, which can be exploited to establish a power-type asymptotic behavior of the tail of that quantity under certain model assumptions. Nyrhinen (1999, 2001) uses this method to establish an asymptotic power decay of the finite and infinite time ruin probability for a discrete-time risk process with stochastic returns on investments and also investigates the transition to the continuous-time model. Gjessing and Paulsen (1997) use a class of random equations, different from those of Goldie (1991), to establish the power decay rate of the ruin probability for a continuous-time risk process with stochastic returns on investments, given that both the surplus process and the investment generating process are Lévy processes. By discretizing the process, Kalashnikov and Norberg (2002) establish upper and lower power-tail bounds for the probability of ultimate ruin, when both the insurance process and the logarithm of the investment are Lévy processes. Using the same approach, Yuen, Wang, and Wu (2006) obtain similar bounds for renewal risk processes with Erlang interarrival times. For several particular cases they show that their upper and lower bounds actually asymptotically match. Collamore (2009) extends the model to a Markov-dependent stochastic economic

*Department of Actuarial Science, Faculty of Business and Economics, University of Lausanne, UNIL-Dorigny, 1015 Lausanne, Switzerland and Swiss Finance Institute, University of Lausanne, UNIL-Dorigny, 1015 Lausanne, Switzerland.

†Corresponding author, Institute for Financial and Actuarial Mathematics, Department of Mathematical Sciences, University of Liverpool, L69 7ZL Liverpool, United Kingdom.

‡Oregon State University, Department of Mathematics, Corvallis, OR, 97331, USA.

environment and develops sharp large deviation asymptotics for the probability of ruin. In all the above papers, the claims (jumps) are assumed to be exponentially bounded.

For heavy-tailed claims, random equations together with large deviation theory are used in the asymptotic analysis of discrete-time risk processes with stochastic returns on investments. For a discrete-time model, in a stochastic environment, Tang and Tsitsiashvili (2003) show that under the assumption that both the financial risk (investments) and the insurance risk (claims) are heavy-tailed, the asymptotic behavior of the finite-time ruin probability is determined by whichever of the two is heavier. In Tang and Tsitsiashvili (2004), they colorblue further derive precise estimates of the probabilities of ruin (both finite- and infinite-time) under the assumption that the claims and the financial risk belong to certain classes of heavy-tailed distributions (namely, extended regularly varying or rapidly varying). For the classical continuous-time Cramér-Lundberg model with the additional feature of investing a constant fraction of the capital in a stock described by a geometric Brownian motion, Gaier and Grandits (2002) first show that when the claim sizes are regularly varying with index $\rho < -1$, then the probability of ruin is also regularly varying with index $\rho < -1$, and then Grandits (2004) shows that among subexponential distributions these are the only ones for which the decay rate of the claim tail carries over to the decay rate of the ruin probability. Gaier and Grandits (2004) extend these results by including a positive interest force, whereas Wei (2009) derives them for extended-regularly varying claims in continuous-time renewal risk models with investments into a Black-Scholes market index.

Paulsen (2002) considers a continuous-time risk process described by two Lévy processes, one regarded as a risk process in a world without economic factors and the other one as return on investments. Using random equations, he shows that for light-tailed claims the probability of ruin decays like a power which depends on the parameters of the investments only. For regularly varying claims, he uses $L_q$ transforms, as in Klüppelberg and Stadtmüller (1998), to conclude that the power decay rate is either a function of the parameters of investments or it behaves like the tail of the claim size distribution.

In Paulsen and Gjessing (1997), functions of the same continuous-time risk process in a stochastic economic environment are introduced as solutions of boundary problems, namely integro-differential equations with regularity conditions. More specifically, when the infinitesimal generator of a function of the risk process equals zero, together with some specific boundary conditions, then the solution of this boundary value problem is the probability of ruin. This approach aligns with the classical approach used in the non-investment case when the probability of ruin is analyzed as a solution of an integro-differential equation, often derived by heuristic methods. In the case of a compound Poisson model with full investment in a risky asset of Black Scholes type, the integro-differential equations for the ruin probability are of order two on the differential side. Conditions on the claim size distribution under which the ruin probability is indeed twice continuously differentiable are identified in Wang and Wu (2001). Note that the case of non-constant investment leads to a stochastic control problem as e.g. studied in Hipp and Plum (2000), Gaier, Grandits, and

Schachermayer (2003) and Kostadinova (2007). For a survey on ruin models under investment, see Paulsen (2008) and Asmussen and Albrecher (2010).

Once we move to non-Poissonian renewal models, we lose the Markov property and therefore formulating the generators of the risk process usually becomes a quite cumbersome (if at all possible) task. However, if instead of considering the continuous non-Markov process, one discretizes the process at renewal times, it is possible to obtain (in many cases) high-order integro-differential equations for the functionals of interest, which can be analyzed using tools from perturbation analysis. Specifically, we first analyze the asymptotic behavior of the Laplace transform of the solutions of these equations in the neighborhood of their singularities and then use Karamata-Tauberian theorems (cf. Bingham, Goldie, and Teugels (1987)) or the Heaviside Principle to draw conclusions about the behavior at infinity of our functions of interest.

We consider investments in a risky asset modeled by a geometric Brownian motion with drift $a$ and volatility $\sigma$. In the small volatility case, namely $2a > \sigma^2$, the striking conclusion is that the probabilities of ruin will have the same asymptotic decay rate for all interarrival time distributions with rational Laplace transforms (i.e. densities satisfying an ordinary differential equation with constant coefficients). For light-tailed claim size distributions, we show that (similarly to the Cramér-Lundberg model with investments) the decay rate of the ruin probability depends either only on the parameters of the risky asset, or only on the tail of the claim sizes. When the claim size distribution is heavy-tailed, the decay rate of the ruin probability is determined by either the parameters of the claim size or the investment distribution, whichever are larger. The large volatility case $2a < \sigma^2$ leads to ruin with probability one, which can be shown by a natural extension of the corresponding result for the Cramér-Lundberg model, as in Frolova, Kabanov, and Pergamenshchikov (2002), Kalashnikov and Norberg (2002), or in the case of premium $c$ being any bounded adapted nonnegative process, as in Pergamenshchikov and Zeitouny (2006), Pergamenshchikov (2009).

The technique can also be applied to other functions of the surplus process, such as the Laplace transform of the time to ruin, finite-time ruin probability and expected discounted penalties. The asymptotic decay rates of these functions of the risk process involve also the discount rate or the Laplace argument. Moreover, for the expected discounted penalty functions, the asymptotic decay rate has an intricate structure, resulting from the interplay between the penalty function, the claim size distribution and the discount factor.

The results of this paper easily translate to the situation when only a (fixed) fraction $\gamma$ $(0 < \gamma < 1)$ of the surplus is invested in the risky asset, since this is equivalent to full investment in a stock with drift $a\gamma$ and volatility $a\sigma$. That is, no matter how small the invested percentage in the risky asset is, its contribution will dominate the asymptotic behavior of the resulting surplus process, which illustrates the importance (and potential danger) such an investment represents.

The rest of the paper is organized as follows. In Section 2 we introduce the model and the method that will be used in the sequel. Section 3 then discusses the asymptotic analysis of the ruin probability in the case that the interarrival densities satisfy ODEs with constant coefficients, for both light- and heavy-tailed claims. Section 4 discusses extensions of the method to more general ruin-related quantities. Finally, Section 5 concludes. Some technical proofs are deferred to an Appendix.

## 2 Renewal Risk Models with Risky Investments

Consider an insurance company that starts with an initial surplus $u$, receives premiums at a constant rate $c$ and continuously invests all its money into a risky asset with a price that follows a geometric Brownian motion with drift $a$ and volatility $\sigma$. The process $Z_t$ representing the value of this portfolio (before considering claims) satisfies the stochastic differential equation

$$dZ_t = (c + aZ_t)dt + \sigma Z_t dB_t$$

where $B_t$ is a standard Brownian motion. The infinitesimal generator of this process is given by

$$A := (c + au)\frac{d}{du} + \frac{\sigma^2}{2}u^2\frac{d^2}{du^2}. \tag{1}$$

Let the claims be independent of the claim occurrence times, and modeled by independent and identically distributed random variables $X_k$ with $\mathbb{E}(X_k) < \infty$, having distribution function $F_X$ with tail $\overline{F}_X = 1 - F_X$ and density function $f_X$. Whenever a claim occurs, the company cashes the corresponding amount of stock in order to pay the claim, so the over-all surplus process of the portfolio is given by

$$U(t) = u + ct + a\int_0^t U(s)ds + \sigma\int_0^t U(s)dB_s - \sum_{k=1}^{N(t)} X_k, \tag{2}$$

where $N(t)$ represents the number of claims occurred up to time $t$. When $N(t)$ is a Poisson process we have the classical jump-diffusion process (see Frolova et al. (2002)). However, in this paper we will allow $N(t)$ to be a renewal process with independent, identical distributed interarrival times $\tau_k$ (between the times $T_k$ of claim arrivals) having a density $f_\tau$ that satisfies an ordinary differential equation with constant coefficients. The latter can always be factorized into first order terms, say

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = \sum_{j=0}^{n}\alpha_j\frac{d^j}{dt^j}f_\tau(t) = \prod_{i=1}^{n}(\frac{d}{dt} + \beta_i)f_\tau(t) = 0, \tag{3}$$

and *homogeneous* initial conditions

$$\begin{aligned} f_\tau^{(k)}(0) &= 0 \quad (k = 0, \dots, n-2), \\ f_\tau^{(n-1)}(0) &= \alpha_0, \end{aligned} \tag{4}$$

or *nonhomogeneous* initial conditions

$$\begin{aligned} f_\tau^{(k)}(0) &= M_k \quad (k = 0, \dots, n-2), \\ f_\tau^{(n-1)}(0) &= \alpha_0. \end{aligned} \tag{5}$$

Here $\alpha_i \in \mathbb{R}$ with $\alpha_n = 1$, and $\beta_i \in \mathbb{C}$ with $\beta_i$ not necessarily all distinct. We call (4) *homogeneous*, because the only non-zero initial value, $f_\tau^{(n-1)}(0) = \alpha_0 = \prod_{i=1}^n \beta_i$ is implied by the fact that $f_\tau$ is a density function (integrating to 1).

Requiring a density to satisfy (3) is equivalent to assuming that its Laplace transform is a rational function. This class of densities contains the class of phase-type distributions, which is popular in the context of ruin and queueing theory (Asmussen and Albrecher, 2010). The properties of the rational Laplace transform class were also utilized in extending exact solutions for ruin problems from the compound Poisson case to the renewal framework, see e.g. Albrecher, Constantinescu, Pirsic, Regensburger, and Rosenkranz (2010).

Moreover, it is easy to see that the boundary conditions (4) for the ODE assumption (3) lead to those densities $f_\tau$, for which their rational Laplace transform has a constant numerator. One can express any density which is a convolution of $n$ exponential densities with parameters $\beta_i$ in the above way, namely $\mathcal{L}(\frac{d}{dt}) = \prod_{i=1}^n (\frac{d}{dt} + \beta_i)$, with homogenous initial conditions (4). Consequently, the Erlang$(n, \beta)$ density

$$f_\tau(t) = \frac{1}{(n-1)!}\beta^n t^{n-1} e^{-\beta t}, \quad \text{for} \quad t > 0, \tag{6}$$

is the special case of equal parameters $\beta_i = \beta$ satisfying equation (3) with operator

$$\mathcal{L}(\frac{d}{dt}) = (\frac{d}{dt} + \beta)^n,$$

and homogenous initial conditions (4).

When the boundary conditions are not homogeneous, the Laplace transform of $f_\tau$ has a *polynomial* numerator of lower degree than that of the polynomial in the denominator. Examples of such distributions are mixtures of exponentials or mixtures of Erlangs. In this paper we will consider as an example in some detail a mixture of two exponentials with density

$$f(t) = \theta\beta_1 e^{-\beta_1 t} + (1-\theta)\beta_2 e^{-\beta_2 t}, \qquad t > 0 \tag{7}$$

(the adaptations for more general members of this class are then in principle possible, but more cumbersome in terms of notation). This will satisfy equation (3) with differential operator

$$\mathcal{L}(\frac{d}{dt}) = (\frac{d}{dt} + \beta_1)(\frac{d}{dt} + \beta_2) \tag{8}$$

and *non-homogeneous* initial conditions

$$\begin{aligned} f_\tau(0) &= \theta\beta_1 + (1-\theta)\beta_2, \\ f_\tau'(0) &= -\theta\beta_1^2 - (1-\theta)\beta_2^2. \end{aligned} \tag{9}$$

Note that for $\theta = \frac{\beta_2}{\beta_2 - \beta_1}$, $\beta_1 \neq \beta_2$, one recovers a *convolution* of two exponentials satisfying (3) with operator (8) and *homogeneous* initial conditions (4), where $\alpha_0 = \beta_1\beta_2$.

The first time the surplus $U(t)$ of the insurance portfolio falls below zero is referred to as the time of ruin

$$T_u = \inf_{t\geq 0}\{U(t) < 0 \mid U(0) = u\}.$$

The probability of ruin is defined as

$$\psi(u) = P(T_u < \infty \mid U(0) = u). \tag{10}$$

Denote by $A^n$ the $n-$ times composition of $A \circ A \ldots \circ A$, with $A$ defined in (1). As shown in Constantinescu and Thomann (2011), whenever the interarrival time density $f_\tau$ satisfies (3) with (5), any function $h$ in the domain $\mathcal{D}(A^n)$ for which the condition $E[h(U(T_1))|U(0) = u] = h(u)$ holds, satisfies the integro-differential equation

$$\mathcal{L}^*(A)h(u) = \sum_{k=0}^{n-1}\left(\sum_{j=k+1}^{n}\alpha_j(-1)^{j-k}f_\tau^{(j-k-1)}(0)\right)\int_0^\infty h^{(k)}(u-x)dF_X(x). \tag{11}$$

Since probability of ruin $\psi(u)$ is in the domain $\mathcal{D}(A^n)$, $\psi(u)$ satisfies the integro-differential equation

$$\mathcal{L}^*(A)\psi(u) = \sum_{k=0}^{n-1}\left(\sum_{j=k+1}^{n}\alpha_j(-1)^{j-k}f_\tau^{(j-k-1)}(0)\right)\int_0^\infty \psi^{(k)}(u-x)dF_X(x).$$

Splitting the integral $\int_0^\infty = \int_0^u + \int_u^\infty$ and then using the definition of $\psi$, this becomes

$$\begin{aligned}\mathcal{L}^*(A)\psi(u) &= \sum_{k=0}^{n-1}\left(\sum_{j=k+1}^{n}\alpha_j(-1)^{j-k}f_\tau^{(j-k-1)}(0)\right)\int_0^u \psi^{(k)}(u-x)dF_X(x)\\ &+ \left(\sum_{j=1}^{n}\alpha_j(-1)^{j}f_\tau^{(j-1)}(0)\right)\int_u^\infty dF_X(x).\end{aligned} \tag{12}$$

In Constantinescu and Thomann (2011), it is also shown that the probability of ruin is the solution of a boundary value problem. Specifically, if a function $h \in \mathcal{D}(A^n)$ satisfies the integro-differential equation (11) together with the regularity condition

$$\lim_{u\to\infty} h(u) = 0,$$

then $h$ is the probability of ruin. This extends the approach developed in Paulsen and Gjessing (1997, Thm. 2.1) for analyzing the probability of ruin in Poisson jump-diffusion processes to renewal-jump diffusion processes. Thus, it is natural to analyze the probability of ruin as a solution of the boundary problem described by the integro-differential equation (12) with the assumption

$$\lim_{u\to\infty} \psi(u) = 0. \tag{13}$$

Here $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$ that describes $f_\tau$. It is explicitly given by

$$\mathcal{L}^*(\frac{d}{dt})f_\tau(t) = \sum_{j=0}^{n}(-1)^j\alpha_j\frac{d^j}{dt^j}f_\tau(t) = \prod_{i=1}^{n}(-\frac{d}{dt}+\beta_i)f_\tau(t), \tag{14}$$

with $\langle \mathcal{L}(\frac{d}{dt})f_\tau, g\rangle = \langle f_\tau, \mathcal{L}^*(\frac{d}{dt})g\rangle$, where the scalar product is defined as $\langle f, g\rangle = \int_0^\infty f(x)g(x)\,dx$ together with homogeneous initial conditions. This adjoint operator plays a major role in the rest of the paper.

Inserting (14) into (12), one obtains that the probability of ruin satisfies the integro-differential equation

$$\prod_{i=1}^{n}(-A+\beta_i)\psi(u) = \sum_{k=0}^{n-1} C_k \int_0^u \psi^{(k)}(u-x)dF_X(x) + C_0 \int_u^\infty dF_X(x), \quad (15)$$

together with the regularity condition (13), where for $k = 0, \ldots, n-1$,

$$C_k = \sum_{j=k+1}^{n} \alpha_j (-1)^{j-k} f_\tau^{(j-k-1)}(0).$$

For homogeneous boundary conditions (4), $C_0 = (-1)^n f_\tau^{(n-1)}(0)$ and for $k = 1, \ldots, n-1$, $C_k = 0$, and thus the right-hand side of (12) simplifies to

$$\mathcal{L}^*(A)\psi(u) = (-1)^n \alpha_0 \left( \int_0^u \psi(u-x)dF_X(x) + \int_u^\infty dF_X(x) \right), \quad (16)$$

which is equivalent to

$$\prod_{i=1}^{n}(-A+\beta_i)\psi(u) = \prod_{i=1}^{n}(-\beta_i) \left( \int_0^u \psi(u-x)dF_X(x) + \int_u^\infty dF_X(x) \right).$$

The latter can be rewritten as

$$\sum_{k=0}^{n}(-1)^k \alpha_k A^k \psi(u) - \prod_{i=1}^{n}(-\beta_i) \int_0^u \psi(u-x)dF_X(x) = \prod_{i=1}^{n}(-\beta_i)\overline{F}_X(u). \quad (17)$$

The structure of equation (12) suggests the use of Laplace transforms, and a natural tool for the asymptotic analysis is regular variation theory (Bingham et al., 1987). Indeed, we will perform an asymptotic analysis at the rightmost singularities in the Laplace domain that through Karamata-Tauberian theorems or the Heaviside Principle will imply the asymptotic behavior at infinity in the real domain. Since the Karamata-Tauberian theorems relate the asymptotic behavior of the Laplace-*Stieltjes* transform of a function at the origin with the asymptotic behavior of this function at infinity, we introduce the auxiliary function

$$\Psi(u) = \begin{cases} 0, & \text{if} \quad u < 0, \\ \int_0^u \psi(x)dx, & \text{if} \quad u \geq 0, \end{cases}$$

and denote its Laplace-*Stieltjes* transform by $\widetilde{\Psi}(s)$. Note that

$$\widetilde{\Psi}(s) = \int_0^\infty e^{-sx} d\Psi(x) = \int_0^\infty e^{-sx}\psi(x)dx := \widehat{\psi}(s),$$

where $\widehat{\psi}(u)$ is the Laplace transform of $\psi$. Since the differential operator $\mathcal{L}^*(A)$ in equation (12) has polynomial coefficients, taking the Laplace transform of

(12) one sees that $\widehat{\psi}(s)$ satisfies a non-homogeneous ordinary differential equation with polynomial coefficients. Furthermore, the homogeneous part of this equation is regular singular at zero, implying that its fundamental solution set has an algebraic behavior at the origin (Fedoryuk, 1993). On the other hand, the particular solution depends on the right-hand side, which is the Laplace transform of the tail of the claim size distribution. Our analysis identifies the asymptotically significant powers among the fundamental and particular solutions of this differential equation for $\widehat{\psi}$, which further determine the asymptotic behavior of $\psi$.

Throughout the paper, we will use the notation

$$\rho = \frac{2a}{\sigma^2} - 1. \tag{18}$$

This is a crucial parameter in determining the asymptotic behavior of the functionals of interest.

# 3 Asymptotic analysis of the probability of ruin

**Theorem 3.1.** *Consider model (2) with positive volatility $\sigma > 0$ and interarrival times having a density $f_\tau$ that satisfies the ODE* (3) *with boundary conditions* (5)*.*

- *If $\rho > 0$, then the ruin probability asymptotically behaves as*

$$\psi(u) \sim C\, u^{-\rho} + k_n\, \overline{F}_X(u), \quad u \to \infty. \tag{19}$$

*Here $C$ is a positive constant and*

$$k_n = \frac{\prod_{i=1}^n \left(-2 + \frac{2a}{\sigma^2} + \frac{2\beta_i}{\sigma^2}\right) - \prod_{i=1}^n \left(\frac{2\beta_i}{\sigma^2}\right)}{\sum_{i=1}^n \left(-3 + \frac{2a}{\sigma^2}\right)\left(-2 + \frac{2a}{\sigma^2} + \frac{2\beta_i}{\sigma^2}\right)}.$$

- *If $\rho \le 0$, $\psi(u) = 1$ for any $u > 0$.*

**Remark 3.1.** *From* (19) *it follows that the asymptotic behavior of $\psi$ is of order $u^{-\rho}$ or $\overline{F}_X(u)$, whichever decays slower. Hence, light-tailed claims do not have any influence on the asymptotic behavior of the ruin probability for this renewal model under investment.*

*Proof.* For $\rho \le 0$, i.e. $2a \le \sigma^2$, the proof of $\psi(u) = 1$ for any $u$ is analogous to the corresponding proof of Frolova et al. (2002), Paulsen (2002) and Pergamenshchikov and Zeitouny (2006). Therefore we consider $\rho > 0$ in the sequel. We consider first the case of homogeneous initial values (Case A) and then present the more general case of nonhomogeneous initial values (Case B).

**Case A.** As discussed in Section 2, under the *homogeneous initial values* (4), the probability of ruin satisfies the integro-differential equation (17). Taking the Laplace transform of (17), and using Lemma A.2 from the appendix (concerning Laplace transform properties), one obtains

$$\sum_{k=0}^{n} (-1)^k \alpha_k \widehat{A}^k \widehat{\psi}(s) - \prod_{i=1}^{n} (-\beta_i) \widehat{\psi}(s) \widehat{f}_X(s) = B_n + \prod_{i=1}^{n} (-\beta_i) \widehat{\overline{F}}_X(s),$$

or equivalently

$$\prod_{i=1}^{n}(-\widehat{A}+\beta_i)\widehat{\psi}(s)-\prod_{i=1}^{n}(-\beta_i)\widehat{\psi}(s)\widehat{f}_X(s)=q(s), \tag{20}$$

where $q(s)=B_n+(-\beta)^n\widehat{\widehat{F}}_X(s)=B_n+(-\beta)^n\left(\frac{1}{s}-\frac{\widehat{f}_X(s)}{s}\right)$. Here $B_1=D_1=c\psi(0), B_2=-2\beta D_1+D_2=-2\beta c\psi(0)+c^2\psi'(0),\ \ldots,$

$$\begin{aligned} B_n &= (-1)^{n-1}\binom{n}{0}c^n\psi^{(n-1)}(0)+(-1)^{n-2}\binom{n}{1}c^{n-1}\beta\psi^{(n-2)}(0)+\ldots \\ &+ (-1)^{n-k}\binom{n}{k-1}c^{n-k+1}\beta^{k-1}\psi^{(n-k+1)}(0)+\ldots+\binom{n}{n-1}c\beta^{n-1}\psi(0). \end{aligned} \tag{21}$$

We now want to analyze the asymptotic behavior at zero of the solutions of the ordinary differential equation (20). For that purpose, insert the operator expression (50) of $\widehat{A}$ into (20) to obtain

$$\begin{aligned} &\prod_{i=1}^{n}\left(-\frac{\sigma^2}{2}s^2\frac{d^2}{ds^2}-(2\sigma^2-a)s\frac{d}{ds}-cs-\sigma^2+a+\beta_i\right)\widehat{\psi}(s) \\ &-\prod_{i=1}^{n}(-\beta_i)\widehat{f}_X(s)\widehat{\psi}(s)=q(s), \end{aligned}$$

which leads to a linear differential equation of order $2n$ with variable coefficients

$$\left(s^{2n}\frac{d^{2n}}{ds^{2n}}+p_{2n-1}(s)s^{2n-1}\frac{d^{2n-1}}{ds^{2n-1}}+...+p_1(s)s\frac{d}{ds}+p_0(s)\right)\widehat{\psi}(s)=\frac{q(s)}{(-\frac{\sigma^2}{2})^n}, \tag{22}$$

where $p_k(s), k=0\ldots 2n-1$, are polynomials of order $2n-k+1$. The homogeneous part of this equation has zero as a regular singular point. By Frobenius' method (see e.g. Fedoryuk (1993)), any set of fundamental solutions of equation (22) can be written as

$$\widehat{\psi}_i(s)=s^{r_i}\gamma_i(s),\quad i=1,\ldots,2n,$$

where $r_i$ $(i=1,\ldots,2n)$ are the solutions of the indicial equation

$$\prod_{i=1}^{n}\left(-(r+1)\left(r+2-\frac{2a}{\sigma^2}\right)+\frac{2\beta_i}{\sigma^2}\right)=\prod_{i=1}^{n}\left(\frac{2\beta_i}{\sigma^2}\right) \tag{23}$$

and $\gamma_i$ are functions that are holomorphic in 0, with $\gamma_i(0)\neq 0$. Normalizing such that $\gamma_i(0)=1$, the general solution of (22) is of the form

$$\widehat{\psi}(s)=\sum_{i=1}^{2n}c_i s^{r_i}\gamma_i(s)+\widehat{\psi}_p(s), \tag{24}$$

where $\widehat{\psi}_p(s)$ is the particular solution. The $2n$ single roots of the indicial equation (23) are obtained as the solutions of the $n$ (uncoupled) quadratic equations

$$(r+1)\left(r+2-\frac{2a}{\sigma^2}\right)=\frac{2\beta_i}{\sigma^2}(1-e^{\frac{2\pi ik}{n}}),\quad k=0,1,2,\ldots,n-1. \tag{25}$$

For $n = 1$, the solutions are $r_1 = -1$ and $r_2 = -2 + \frac{2a}{\sigma^2} = -1 + \rho$, with $r_1 < r_2$, since $\rho > 0$. For $n \geq 2$, Proposition A.1 in the Appendix shows that the real parts of the solutions of (25) can be ordered as

$$\Re(r_{2n-1}) < \cdots < \Re(r_3) < r_1 = -1 < r_2 = -1 + \rho < \Re(r_4) < \cdots < \Re(r_{2n}).$$

The regularity condition (13) now implies that coefficients with odd index in (24) must vanish, $c_{2i+1} = 0$. The decay rate of the remaining homogeneous solutions is driven by the slowest decaying power, the leading term being $r_2 = -2 + \frac{2a}{\sigma^2}$. Using Karamata-Tauberian theorems, this leading term gives, in the real domain, a decay of order

$$u^{1-\frac{2a}{\sigma^2}}, \quad \text{as} \quad u \to \infty.$$

(Note that for $\frac{2a}{\sigma^2} \geq 2$, one can apply an argument of Feller (1971, Ch.XIII, Sec.5).) This is the same asymptotic decay rate as the one derived in Frolova et al. (2002) for exponential interarrival times. We need to consider the possibility of $c_2 = 0$, but it is shown in Proposition A.1 in the Appendix that this leads to a contradiction (concretely, $c_2 = 0$ would imply that the leading term is $s^{r_4}$, itself implying that for Erlang($n$) interarrival times the probability of ruin would decay faster than for exponential interarrival times, which contradicts Proposition A.1 given in the Appendix).

It remains to determine the contribution of the particular solution. With the method of variation of parameters, we can write

$$\widehat{\psi}_p(s) = \left(-\frac{2}{\sigma^2}\right)^n \sum_{i=1}^{2n} \widehat{\psi}_i(s) \int_{\epsilon_0}^{s} \frac{q(t)W_i(t)}{t^{2n}W(t)} dt, \tag{26}$$

where $W$ is the Wronskian determinant of the fundamental system and $W_i$ is the Wronskian determinant obtained from $W$ by replacing the $i$-th column by the column $(0, 0, \ldots, 0, 1)^T \in \mathbb{R}^{2n}$, and $\epsilon_0$ is a very small positive constant. By Lemma A.4 in the Appendix, there exist holomorphic functions $\widetilde{\gamma}$, $\widetilde{\gamma}_m$, $\widetilde{\gamma}(0) \neq 0 \neq \widetilde{\gamma}_m(0)$, for $m = 1, ..., 2n$ such that the Wronskian determinants $W(s)$ and $W_m(s)$ (where $W_m$ is the determinant obtained from $W$ by replacing the $m$-th column by the column $(0, 0, \ldots, 0, 1)^T \in \mathbb{R}^m$) can be written as

$$\frac{W_m(s)}{s^{2n}W(s)} = s^{-r_m-1}\frac{\widetilde{\gamma}_m(s)}{\widetilde{\gamma}(s)}$$

leading to

$$\int_{\epsilon_0}^{s} \frac{q(t)W_i(t)}{t^{2n}W(t)} dt = \int_{\epsilon_0}^{s} q(t)t^{-r_i-1}\frac{\widetilde{\gamma}_i(t)}{\widetilde{\gamma}(t)} dt.$$

Since $q(s) = B_n + (-\beta)^n \widehat{\overline{F}}_X(s)$, (26) translates into

$$\begin{aligned}\widehat{\psi}_p(s) = \prod_{i=1}^{n}\left(-\frac{2}{\sigma^2}\right) B_n \sum_{i=1}^{2n} s^{r_i}\gamma_i(s) \int_{\epsilon_0}^{s} t^{-r_i-1}\frac{\widetilde{\gamma}_i(t)}{\widetilde{\gamma}(t)} dt \\ + \prod_{i=1}^{n}\left(\frac{2\beta_i}{\sigma^2}\right) \sum_{i=1}^{2n} s^{r_i}\gamma_i(s) \int_{\epsilon_0}^{s} \widehat{\overline{F}}_X(t)t^{-r_i-1}\frac{\widetilde{\gamma}_i(t)}{\widetilde{\gamma}(t)} dt. \end{aligned} \tag{27}$$

Depending on $\widehat{\overline{F}}_X$ one can identify two cases:

A1 light-tailed claims with exponentially bounded tails (assume $\widehat{\overline{F}}_X$ has a singularity at $-\mu < 0$ and $\widehat{\overline{F}}_X(-\mu) = \infty$).

A2 heavy-tailed claims ($\widehat{\overline{F}}_X(-\epsilon) = \infty$ for all $\epsilon > 0$).

**Case A1. Light-tailed claims.** Using de l'Hospital rule, one can show that

$$\int_{\epsilon_0}^{s} \widehat{\overline{F}}_X(t)t^{-r_i-1}dt \sim \frac{1}{-r_i}s^{-r_i}\widehat{\overline{F}}_X(s), \quad \text{as} \quad s \to -\mu, \tag{28}$$

since

$$\begin{aligned}\lim_{s\to-\mu}\frac{\int_{\epsilon_0}^{s}\widehat{\overline{F}}_X(t)t^{-r_i-1}dt}{s^{-r_i}\widehat{\overline{F}}_X(s)} &= \lim_{s\to-\mu}\frac{s^{-r_i-1}\widehat{\overline{F}}_X(s)}{-r_is^{-r_i-1}\widehat{\overline{F}}_X(s)+s^{-r_i}\frac{d}{ds}\widehat{\overline{F}}_X(s)}\\ &= \frac{1}{-r_i+\lim_{s\to-\mu}\frac{s}{\widehat{\overline{F}}_X(s)}\frac{d}{ds}\widehat{\overline{F}}_X(s)} = \frac{1}{-r_i}.\end{aligned}$$

Thus,

$$\begin{aligned}\widehat{\psi}_p(s) \sim \prod_{i=1}^{n}\left(-\frac{2}{\sigma^2}\right)B_n\sum_{i=1}^{2n}\frac{1}{-r_i}\gamma_i(-\mu)\frac{\widetilde{\gamma}_i(-\mu)}{\widetilde{\gamma}(-\mu)}\\ +\prod_{i=1}^{n}\left(\frac{2\beta_i}{\sigma^2}\right)\sum_{i=1}^{2n}\frac{1}{-r_i}\gamma_i(-\mu)\frac{\widetilde{\gamma}_i(-\mu)}{\widetilde{\gamma}(-\mu)}\widehat{\overline{F}}_X(s), \quad s\to-\mu.\end{aligned} \tag{29}$$

Normalize $\gamma_i$ such that $\gamma_i(-\mu)\frac{\widetilde{\gamma}_i(-\mu)}{\widetilde{\gamma}(-\mu)} = 1$, for all $i$. Since $-\mu$ is the rightmost singularity of $\widehat{\psi}_p(s)$ and the first term of the sum is analytic in $-\mu$ (can be written as $\sum_{k=0}^{\infty} b_k(s+\mu)^k$), one can apply the Heaviside Operational Principle (see e.g. Abate and Whitt (1997, p.188)) to deduce that

$$\psi_p(u) \sim \prod_{i=1}^{n}\left(\frac{2\beta_i}{\sigma^2}\right)\sum_{i=1}^{2n}\left(\frac{1}{-r_i}\right)\overline{F}_X(u), \quad \text{as} \quad u\to\infty.$$

**Case A2. Heavy-tailed claims.** Using de l'Hospital rule and other limit properties one can show

$$\int_{\epsilon_0}^{s} \widehat{\overline{F}}_X(t)t^{-r_i-1}dt \sim \frac{1}{-r_i}s^{-r_i}\widehat{\overline{F}}_X(s), \quad \text{as} \quad s \to 0, \tag{30}$$

since

$$\begin{aligned}\lim_{s\to 0}\frac{\int_{\epsilon_0}^{s}\widehat{\overline{F}}_X(t)t^{-r_i-1}dt}{s^{-r_i}\widehat{\overline{F}}_X(s)} &= \lim_{s\to 0}\frac{s^{-r_i-1}\widehat{\overline{F}}_X(s)}{-r_is^{-r_i-1}\widehat{\overline{F}}_X(s)+s^{-r_i}\frac{d}{ds}\widehat{\overline{F}}_X(s)}\\ &= \frac{1}{-r_i+\lim_{s\to 0}\frac{s}{\widehat{\overline{F}}_X(s)}\frac{d}{ds}\widehat{\overline{F}}_X(s)} = \frac{1}{-r_i}.\end{aligned}$$

As $\gamma$ and $\gamma_i$, $i = 1, \ldots, n$, are holomorphic at 0, and $r_i \neq 0$, we have ( after normalizing $\gamma_i(0)\frac{\widetilde{\gamma}_i(0)}{\widetilde{\gamma}(0)} = 1$, for all $i$),

$$\gamma_i(s)\int_{\epsilon_0}^{s} t^{-r_i-1}\frac{\widetilde{\gamma}_i(t)}{\widetilde{\gamma}(t)}dt \sim \int_{\epsilon_0}^{s} t^{-r_i-1}dt, \quad \text{as} \quad s\to 0, \tag{31}$$

and thus, as $s \to 0$,

$$\widehat{\psi}_p(s) \sim \left(-\frac{2}{\sigma^2}\right)^n B_n \sum_{i=1}^{2n}\left(-\frac{1}{r_i}\right) + \prod_{i=1}^{n}\left(\frac{2\beta_i}{\sigma^2}\right)\sum_{i=1}^{2n}\left(\frac{1}{-r_i}\right)\widehat{\overline{F}}_X(s), \tag{32}$$

where the first term of the sum is a constant. Since zero is the rightmost singularity of $\widehat{\psi}_p(s)$ (and the first term of the sum is constant) it is analytic in zero (can be written as $\sum_{k=0}^{\infty} b_k s^k$), and one can again apply the Heaviside Operational Principle to deduce that

$$\psi_p(u) \sim \prod_{i=1}^{n}\left(\frac{2\beta_i}{\sigma^2}\right)\sum_{i=1}^{2n}\left(\frac{1}{-r_i}\right)\overline{F}_X(u), \quad \text{as} \quad u \to \infty,$$

Hence, the particular solution does not represent a significant asymptotic term in the case of light-tailed claims. On the other hand, in the case of heavy-tailed claim sizes, one has to compare the power decay $u^{-\rho}$ and the tail of the claim size distribution $\overline{F}_X$, and determine which one is slower. Now applying Vieta's rule on the indicial equation (23),

$$r^{2n} + a_1 r^{2n-1} + \ldots + a_{2n-1}r + a_{2n} = 0, \tag{33}$$

we have that

$$\sum_{i=1}^{2n}\left(\frac{1}{-r_i}\right) = -\sum_{i=1}^{2n}\left(\frac{1}{r_i}\right) = \frac{a_{2n-1}}{a_{2n}}.$$

**Case B.** For the case of interarrival time densities with *nonhomogeneous* initial values, one has to analyze the solutions of the integro-differential equation (15), equivalent to

$$\sum_{k=0}^{n}(-1)^k \alpha_k A^k \psi(u) + \sum_{k=0}^{n-1} C_k \int_0^u \psi^{(k)}(u-x)dF_X(x) = C_0\overline{F}_X(u). \tag{34}$$

Taking the Laplace transform of (34), and using the Laplace transform properties given in the Appendix, one obtains

$$\begin{aligned} &\textstyle\sum_{k=0}^{n}(-1)^k \alpha_k \widehat{A}^k \widehat{\psi}(s) + \left(\sum_{k=0}^{n-1} C_k s^k\right)\widehat{\psi}(s)\widehat{f}_X(s) \\ &= B_n + C_0\widehat{\overline{F}}_X(s) + \left(\textstyle\sum_{k=0}^{n-1} C_k \sum_{j=0}^{k-1} s^{k-j-1}\psi^{(k-j-1)}(0)\right)\widehat{f}_X(s), \end{aligned}$$

where $B_n$ is given by (21). This is equivalent to

$$\prod_{i=1}^{n}(-\widehat{A} + \beta_i)\widehat{\psi}(s) + \left(\sum_{k=0}^{n-1} C_k s^k\right)\widehat{\psi}(s)\widehat{f}_X(s) = q(s), \tag{35}$$

after denoting again the right-hand side by $q(s)$. One can see that insertion of the operator expression (50) of $\widehat{A}$ and division by $(-\frac{\sigma^2}{2})^n$ leads to a linear differential equation of order $2n$. Its homogeneous part is again singular regular at zero. Frobenius' method will produce the same indicial equation (23), which then leads to the same homogeneous solutions for equation (35) as for (20) (see the Appendix for details). However, the particular solution will be different:

$$
\begin{aligned}
\widehat{\psi}_p(s) &= B_n\left(-\frac{2}{\sigma^2}\right)^n \sum_{i=1}^{2n} s^{r_i}\gamma_i(s)\int_{\epsilon_0}^{s} t^{-r_i-1}\frac{\widetilde{\gamma}_i(t)}{\widetilde{\gamma}(t)}dt \\
&+ C_0\left(-\frac{2}{\sigma^2}\right)^n \sum_{i=1}^{2n} s^{r_i}\gamma_i(s)\int_{\epsilon_0}^{s} \widehat{\overline{F}}_X(t)t^{-r_i-1}\frac{\widetilde{\gamma}_i(t)}{\widetilde{\gamma}(t)}dt \\
&+ \left(-\frac{2}{\sigma^2}\right)^n \sum_{k=0}^{n-1} C_k \sum_{i=1}^{2n} s^{r_i}\gamma_i(s)\int_{\epsilon_0}^{s}\left(\sum_{j=0}^{k-1} t^{k-j-1}\psi^{(k-j-1)}(0)\right) \\
&\times\ \widehat{f}_X(t)t^{-r_i-1}\frac{\widetilde{\gamma}_i(t)}{\widetilde{\gamma}(t)}dt.
\end{aligned}
$$

Similarly to (30) derived in the previous section, one can show that for heavy-tailed claims

$$
\int_{\epsilon_0}^{s}\widehat{f}_X(t)t^{-r_i-1}dt \sim \frac{1}{-r_i}s^{-r_i}\widehat{f}_X(s), \quad \text{as} \quad s\to 0, \tag{36}
$$

through

$$
\lim_{s\to 0}\frac{\int_{\epsilon_0}^{s}\widehat{f}_X(t)t^{-r_i-1}dt}{s^{-r_i}\widehat{f}_X(s)} = \lim_{s\to 0}\frac{s^{-r_i-1}\widehat{f}_X(s)}{-r_i s^{-r_i-1}\widehat{f}_X(s)+s^{-r_i}\frac{d}{ds}\widehat{f}_X(s)} = \frac{1}{-r_i},
$$

whereas for light-tailed claims the same is true at the rightmost singularity $-\mu$. According to (30), (31), (36) and the asymptotic relation

$$
\sum_{k=0}^{n-1} C_k\widehat{f}_X(s)\sum_{i=1}^{2n}\left(\frac{1}{-r_i}\right)\sum_{j=0}^{k-1} s^{k-j-1}\psi^{(k-j-1)}(0) \sim \psi(0)\sum_{i=1}^{2n}\left(\frac{1}{-r_i}\right)\sum_{k=0}^{n-1} C_k\widehat{f}_X(s)
$$

as $s\to 0$ for heavy-tailed claims and as $s\to -\mu$ for light-tailed claims, respectively, the particular solution asymptotically behaves as

$$
\begin{aligned}
\widehat{\psi}_p(s) &\sim B_n\left(-\frac{2}{\sigma^2}\right)^n\sum_{i=1}^{2n}\left(\frac{1}{-r_i}\right) \\
&+ C_0\left(-\frac{2}{\sigma^2}\right)^n\sum_{i=1}^{2n}\left(\frac{1}{-r_i}\right)\widehat{\overline{F}}_X(s) \\
&+ \left(-\frac{2}{\sigma^2}\right)^n\psi(0)\sum_{i=1}^{2n}\left(\frac{1}{-r_i}\right)\sum_{k=0}^{n-1} C_k\widehat{f}_X(s).
\end{aligned}
$$

This is the sum of an analytic function $B_n\left(-\frac{2}{\sigma^2}\right)^n\sum_{i=1}^{2n}\left(\frac{1}{-r_i}\right)$ and

$$
\hat{H}(s) = C_0\left(-\frac{2}{\sigma^2}\right)^n\sum_{i=1}^{2n}\left(\frac{1}{-r_i+2}\right)\widehat{\overline{F}}_X(s)+\left(-\frac{2}{\sigma^2}\right)^n\psi(0)\sum_{i=1}^{2n}\left(\frac{1}{-r_i}\right)\sum_{k=0}^{n-1} C_k\widehat{f}_X(s),
$$

the Laplace transform of a function $H$. According to the Heaviside principle, the contribution of the particular solution to the decay will then be given by the inverse Laplace transform of $\hat{H}(s)$, leading to

$$\begin{aligned}\psi_p(u) \quad &\sim \quad C_0\left(-\frac{2}{\sigma^2}\right)^n\sum_{i=1}^{2n}\left(\frac{1}{-r_i}\right)\overline{F}_X(u)\\ &+\left(-\frac{2}{\sigma^2}\right)^n\psi(0)\sum_{i=1}^{2n}\left(\frac{1}{-r_i}\right)\sum_{k=0}^{n-1}C_k f_X(u), \quad \text{as} \quad u\to\infty.\end{aligned}$$

It remains to show that $\psi_p$ behaves asymptotically as $\overline{F}_X$. To establish the dominant term among the two, one needs to analyze the limit $\lim_{u\to\infty}\frac{f_X(u)}{\overline{F}_X(u)}$, which is just the limit of the hazard rate function. For exponentially bounded claims sizes, this limit is constant (see e.g. Asmussen and Albrecher (2010)), implying that the density $f_X$ and the tail $\overline{F}_X$ have the same behavior at infinity. For heavy-tailed claims, the hazard rate function is decreasing to zero, implying that the term with $\overline{F}_X$ decays slower. □

**Remark 3.2.** *In order to determine the constant $C$ in Theorem 3.1 explicitly, one would have to determine the value of the constants $c_i$ in* (**??**) *explictly (by finding the value of the derivatives of $\psi(u)$ in 0, potentially through a study of analyticity properties of $\hat{\psi}(s)$ in the right half-plane). But this basically amounts to determine the exact solution of $\psi(u)$, which is not feasible in general. In particular, the focus of the present approach is to illustrate the simplicity and power of the asymptotic method.*

**Corollary 3.1.** *If $\overline{F}_X(x)$ are regularly varying with index $\alpha$, as $x\to\infty$, the ruin probability behaves asymptotically as*

$$\psi(u)\sim\alpha_n u^{-\min(\rho,\alpha)}, \quad u\to\infty,$$

*for some strictly positive constant $\alpha_n$. For $n=1$, this coincides with Proposition 4.1 of Paulsen (2002). For general interarrival times it coincides with Corollary 3.1 of Wei (2009).*

**Example 3.1.** *For Model (2) with positive volatility $\sigma>0$ and Erlang$(n,\beta)$ distributed interarrival times, Theorem 3.1 applies, with*

$$k_n=\frac{\left(-2+\frac{2a}{\sigma^2}+\frac{2\beta}{\sigma^2}\right)^n-\left(\frac{2\beta}{\sigma^2}\right)^n}{n\left(-3+\frac{2a}{\sigma^2}\right)\left(-2+\frac{2a}{\sigma^2}+\frac{2\beta}{\sigma^2}\right)}.$$

**Example 3.2.** *For Model* (2) *with positive volatility $\sigma>0$ and interarrival times distributed as a mixture of $n$ exponentials with density*

$$f_\tau(t)=\sum_{i=1}^n\theta_i\beta_i e^{-\beta_i t}, \tag{37}$$

*where $\beta_i>0$, $\theta_i\in(0,1)$ and $\sum_{i=1}^n\theta_i=1$, Theorem 3.1 applies as well, with*

$$k_n=\frac{\prod_{i=1}^n\left(-2+\frac{2a}{\sigma^2}+\frac{2\beta_i}{\sigma^2}\right)-\prod_{i=1}^n\left(\frac{2\beta_i}{\sigma^2}\right)}{\sum_{i=1}^n\left(-3+\frac{2a}{\sigma^2}\right)\left(-2+\frac{2a}{\sigma^2}+\frac{2\beta_i}{\sigma^2}\right)}.$$

## 4 Asymptotic analysis of further quantities

The expected discounted penalty function (also called the Gerber-Shiu function) contains the time to ruin $T_u$, and penalizes the surplus immediately before ruin $U(T_u-)$ and the deficit at ruin $\mid U(T_u) \mid$,

$$m_\delta(u) = E\left(e^{-\delta T_u} w(U(T_u-), \mid U(T_u) \mid) 1(T_u < \infty) \mid U(0) = u\right),$$

with $\delta$ being the discount factor. For a penalty function $w \equiv 1$, one retrieves the Laplace transform of the time to ruin and if further $\delta = 0$, the ruin probability $\psi(u)$.

Whenever the interarrival time density $f_\tau$ satisfies an ordinary differential equation $\mathcal{L}(\frac{d}{dt}) f_\tau(t) = 0$, with homogeneous boundary conditions and the investment return process defined by (1) has infinitesimal generator $A$, the corresponding Gerber-Shiu function $m_\delta(u)$ satisfies the integro-dfferential equation

$$\mathcal{L}^*(A-\delta) m_\delta(u) = \alpha_0 \left( \int_0^u m_\delta(u-x) f_X(x) dx + \omega(u) \right). \tag{38}$$

Here $\omega(u) = \int_u^\infty w(u, x-u) f_X(x) dx$ (see e.g. Asmussen and Albrecher (2010)).

**Theorem 4.1.** *Consider Model (2) with positive volatility $\sigma > 0$ and $\rho > 0$ (i.e. assume $2a > \sigma^2$) and assume that $\widehat{\omega}(s)$ exists and $|\frac{d}{ds} \ln \hat{\omega}(s)|_{s=0}| < \infty$. For interarrival time densities satisfying* (3)*, the Gerber-Shiu function $m_\delta$ behaves asymptotically as*

$$m_\delta(u) \sim K\, u^{-1+\frac{\rho}{2}-\sqrt{\left(-\frac{\rho}{2}\right)^2+\delta}} + K_n\, \omega(u), \quad u \to \infty, \tag{39}$$

*for some strictly positive constants $K$ and $K_n$.*

*Proof.* For interarrival times with a density satisfying (3), equation (38) is equivalent to

$$\begin{aligned}
\prod_{i=1}^n (-A+\beta_i+\delta) m_\delta(u) &= \sum_{k=0}^{n-1} \left( \sum_{j=k+1}^n \alpha_j (-1)^{j-k} f_\tau^{(j-k-1)}(0) \right) \\
&\times \int_0^u m_\delta^{(k)}(u-x) dF_X(x) \\
&+ \left( \sum_{j=1}^n \alpha_j (-1)^j f_\tau^{(j-1)}(0) \right) \omega(u).
\end{aligned}$$

Expanding the left-hand side, taking the Laplace transform and then using (49) and (50) from the Appendix, we get a $2n-$th order linear differential equation

$$\prod_{i=1}^n (-\widehat{A}+\beta_i+\delta) \widehat{m}_\delta(s) - \sum_{k=0}^{n-1} \left( \sum_{j=k+1}^n \alpha_j (-1)^{j-k} f_\tau^{(j-k-1)}(0) \right) s^k \widehat{m_\delta}(s) \widehat{f_X}(s) = q(s), \tag{40}$$

where

$$\begin{aligned} q(s) &= B_n + \left(\sum_{j=1}^{n} \alpha_j (-1)^j f_\tau^{(j-1)}(0)\right) \hat{\omega}(s) \\ &- \sum_{k=0}^{n-1} \left(\sum_{j=k+1}^{n} \alpha_j (-1)^{j-k} f_\tau^{(j-k-1)}(0)\right) \sum_{j=1}^{k} s^{j-1} m_\delta^{(k-j)}(0) \widehat{f_X}(s), \end{aligned}$$

with $B_n$ a linear combination of derivatives of $m_\delta(0)$, as in (21). Its homogeneous part has zero as a regular singular point, implying that the solution is a power function. After dividing the equation by $(-\frac{\sigma}{2})^n$, the Frobenius method leads to the indicial equation

$$\prod_{i=1}^{n} \left(-(r+1)\left(r+2-\frac{2a}{\sigma^2}\right) + \frac{2(\beta_i+\delta)}{\sigma^2}\right) = \prod_{i=1}^{n} \left(\frac{2\beta_i}{\sigma^2}\right), \tag{41}$$

with solutions depending on $\delta$. One can show that the solutions relevant for the asymptotic decay for the indicial equation (41) are the solutions of the quadratic equation

$$-(r+1)\left(r+2-\frac{2a}{\sigma^2}\right) + \frac{2\delta}{\sigma^2} = 0. \tag{42}$$

The remaining solutions have real parts outside the interval $(r_1, r_2)$, where $r_1$ and $r_2$ are the solutions of quadratic equation (42),

$$r_{1,2} = -\frac{2-\rho}{2} \pm \sqrt{\left(\frac{-\rho}{2}\right)^2 + \frac{2\delta}{\sigma^2}}.$$

Obviously, for $\delta = 0$, one recovers the ruin probability result. As before, the power $-\frac{2-\rho}{2} + \sqrt{\left(\frac{-\rho}{2}\right)^2 + \frac{2\delta}{\sigma^2}}$ on the Laplace side would not produce a decay to zero at infinity, thus the first candidate for the decay rate is

$$-\frac{2-\rho}{2} - \sqrt{\left(\frac{-\rho}{2}\right)^2 + \frac{2\delta}{\sigma^2}}.$$

Consequently, the slowest asymptotic behavior of the solutions of the homogeneous part on the Laplace transform side is given by

$$\widehat{m}_{\delta_h}(s) \sim K\, s^{-\frac{\eta+1}{2} - \sqrt{\left(\frac{\eta-1}{2}\right)^2 + \frac{2\delta}{\sigma^2}}}, \quad s \to 0,$$

which by the Karamata Tauberian Theorem implies that

$$m_\delta(u) \sim K\, u^{\frac{\eta-1}{2} + \sqrt{\left(\frac{\eta-1}{2}\right)^2 + \frac{2\delta}{\sigma^2}}}, \quad u \to \infty. \tag{43}$$

For the particular solution coming from the non-homogeneous term of equation

(40), one uses again variation of parameters, to obtain

$$\begin{aligned}
\widehat{m}_{\delta_p}(s) &= \left(-\frac{2}{\sigma^2}\right)^n B_n \sum_{i=1}^{2n} s^{r_i}\gamma_i(s)\int_{\epsilon_0}^{s} t^{-r_i-1}\frac{\gamma_i(t)}{\gamma(t)}dt \\
&+ \prod_{i=1}^{n}\left(\frac{2\beta_i}{\sigma^2}\right)\sum_{i=1}^{2n} s^{r_i}\gamma_i(s)\int_{\epsilon_0}^{s}\widehat{\omega}(t)t^{-r_i-1}\frac{\gamma_i(t)}{\gamma(t)}\,dt \\
&+ \left(-\frac{2}{\sigma^2}\right)^n \sum_{k=0}^{n-1} C_k \sum_{i=1}^{n} s^{r_i}\gamma_i(s)\int_{\epsilon_0}^{s}\left(\sum_{j=0}^{k-1} t^{k-j-1} m_\delta^{(k-j-1)}(0)\right) \\
&\times\ \widehat{f}_X(t)t^{-r_i-1}\frac{\widetilde{\gamma}_i(t)}{\widetilde{\gamma}(t)}dt.
\end{aligned}$$

Again, the cases of light-tailed and heavy-tailed claims have to be dealt with separately. Since for a rightmost singularity $s^*$, where $s^*$ is either $-\mu$ or $0$,

$$\begin{aligned}
\lim_{s\to s^*}\frac{\int_{\epsilon_0}^{s}\widehat{\omega}(t)t^{-r_i-1}dt}{s^{-r_i}\widehat{\omega}(s)} &= \lim_{s\to s^*}\frac{s^{-r_i-1}\widehat{\omega}(s)}{-r_i s^{-r_i-1}\widehat{\omega}(s)+s^{-r_i}\frac{d}{ds}\widehat{\omega}(s)} \\
&= \frac{1}{-r_i+\lim_{s\to s^*} s\frac{\frac{d}{ds}\widehat{\omega}(s)}{\widehat{\omega}(s)}} \\
&= \frac{1}{-r_i+\lim_{s\to s^*} s\frac{d}{ds}\ln\widehat{\omega}(s)}
\end{aligned}$$

equals $\frac{1}{-r_i}$ as long as $-\infty < \frac{d}{ds}\ln\hat{\omega}(s)\,|_{s=s^*} < \infty$,

$$\begin{aligned}
\widehat{m}_{\delta_p}(s) \sim&\ \left(-\frac{2}{\sigma^2}\right)^n B_n\sum_{i=1}^{n}\left(\frac{1}{-r_i}\right)+\prod_{i=1}^{n}\left(\frac{2\beta_i}{\sigma^2}\right)\sum_{i=1}^{n}\left(\frac{1}{-r_i}\right)\widehat{\omega}(s), \\
&+\left(-\frac{2}{\sigma^2}\right)^n m_\delta(0)\sum_{i=1}^{n}\left(\frac{1}{-r_i}\right)\sum_{k=0}^{n-1}C_k\widehat{f}_X(s), \quad s\to s^*.
\end{aligned}$$

The Heaviside Principle then gives

$$m_{\delta_p}(u)\sim K_n\,\omega(u), \quad u\to\infty, \tag{44}$$

where $K_n=\prod_{i=1}^{n}\left(\frac{2\beta_i}{\sigma^2}\right)\sum_{i=1}^{n}\left(\frac{1}{-r_i}\right)$. Thus, one may conclude that the decay will be given by the slower among (43) or (44). □

From (39) it is clear that the asymptotic behavior is the result of an interplay between the penalty function and the claim size distribution. This generalizes earlier results that were available for heavy-tailed claims in risk models *without* investments (see e.g. Teugels and Willmot (1987) for the expected deficit at ruin in a compound Poisson risk model, or Tang and Wei (2010) for convolution-equivalent tail penalties and claims in renewal risk models).

# 5 Conclusion

In this paper we show that the asymptotic behavior of the ruin probability and related quantities of the renewal risk model under investment is quite insensitive to the particular interarrival time distribution. We show that the ruin probability asymptotically decays like a power or it equals one, depending solely on the parameters of the risky asset and the tail of the claim size distribution. It is also shown that the asymptotic behavior of the Laplace transform of the time to ruin has a power decay rate. For expected discounted penalty functions, the asymptotic behavior is an interplay between the chosen penalty function and the claim size distribution. In other words, in this framework the financial risk asymptotically dominates the insurance risk stemming from the frequency of claims. The employed method shows the influence of the various factors in an analytic and transparent fashion and may be useful for other asymptotic studies of level-crossing.

# 6 Acknowledgements

We would like to thank Florin Avram, Vicky Fasen, Dominik Kortschak, Zbigniew Palmowski, Jostein Paulsen and the two referees for their fruitful comments on earlier versions of this paper. H.A. and C.C. gratefully acknowledge financial support from the Swiss National Science Foundation Project 200021-124635/1.

# A Appendix

**Proposition A.1.** *Let $\psi_n(u)$ denote the ruin probability of the risk process $U^{(n)}$ defined as*

$$U^{(n)}(t) = u + ct + a\int_0^t U(s)ds + \sigma\int_0^t U(s)dW_s - \sum_{k=1}^{N^{(n)}(t)} X_k, \qquad (45)$$

*with the claim number process $N^{(n)}$ being a renewal process with Erlang$(n)$ interarrival times. Then $\psi_m(u) \geq \psi_n(u)$, for any $m < n$.*

For the proof of Proposition A.1, we use sample path-wise domination. Let

$$U^{(1)}(t) = u + ct + a\int_0^t U(s)ds + \sigma\int_0^t U(s)dW_s - \sum_{k=1}^{N^{(1)}(t)} X_k,$$

be a Cramér-Lundberg risk model with investments in a risky asset with a price which follows a geometric Brownian motion. The interarrival times $\{\tau_k^{(1)}\}_k$ are independent, exponentially distributed random variables, with parameter $\beta$. The claims arrival process $N^{(1)}(t)$ is a Poisson process. Recall that the sum $\tau_1^{(1)}+\tau_2^{(1)}$ of two random variables which are exponentially distributed with parameter $\beta$ is $Erlang(2,\beta)$. In order to not affect the net profit condition, in the following we will be comparing interarrival distributions with the same mean. For instance, when we compare exponential interarrivals $\tau_k^{(1)}$ with $Erlang(2)$ interarrivals $\tau_k^{(2)}$, we choose $Erlang(2, 2\beta)$, such that $\mathbb{E}\tau^{(1)} = \mathbb{E}\tau^{(2)} = \frac{1}{\beta}$. The common underlying Brownian motion, permits a comparison of the surplus processes, with different interarrival times distributions, through a coupling argument. To be precise, one uses

$$Z(t) = Z(0)\exp\{(a-\frac{\sigma^2}{2})t+\sigma W_t\}+c\int_0^t \exp\{(a-\frac{\sigma^2}{2})(t-u)+\sigma(W_t-W_u)\}du, \tag{46}$$

the explicit representation in terms of the Brownian motion of the solution of the stochastic differential equation governing the investment process

$$dZ = (aZ + c)dt + \sigma Z dW_t, \tag{47}$$

given in e.g. Thomann and Waymire (2003). This can be thought of as a type of stochastic Duhamel principle which can be verified using Itô's lemma.

The following lemma introduces a technique later used in the proof of Proposition A.1 .

**Lemma A.1.** *If $Z(t)$ satisfies the equation (46), then for any $0 \le s \le t$,*

$$Z(t) = Z(s)\exp\{(a-\frac{\sigma^2}{2})(t-s)+\sigma(W_t-W_s)\}+c\int_s^t \exp\{(a-\frac{\sigma^2}{2})(t-u)+\sigma(W_t-W_u)\}du.$$

*Proof.* Note that adding and subtracting $s$ and $W_s$, we have

$$Z(0)\exp\{(a-\frac{\sigma^2}{2})t+\sigma W_t\} = Z(0)\exp\{(a-\frac{\sigma^2}{2})(t-s+s)+\sigma(W_t-W_s+W_s)\}.$$

Adding and subtracting $s-u$ and $W_s - W_u$, we get

$$\begin{aligned} c\int_0^t \exp\{(a-\frac{\sigma^2}{2})(t-u)+\sigma(W_t-W_u)\}du &= c\int_0^s \exp\{(a-\frac{\sigma^2}{2})[(t-s)+(s-u)] \\ &+ \sigma\big((W_t-W_s)+(W_s-W_u)\big)\}du \\ &+ c\int_s^t \exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &+ \sigma(W_t-W_u)\}du. \end{aligned}$$

Therefore, we can write $Z(t)$ as a function of an earlier state $Z(s)$ through

$$\begin{aligned} Z(t) &= \Big(Z(0)\exp\{(a-\frac{\sigma^2}{2})s+\sigma W_s\}+c\int_0^s \exp\{(a-\frac{\sigma^2}{2})(s-u) \\ &+ \sigma(W_s-W_u)\}du\Big) \\ &\times \exp\{(a-\frac{\sigma^2}{2})(t-s)+\sigma(W_t-W_s)\}+c\int_s^t \exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &+ \sigma(W_t-W_u)\}du \\ &= Z(s)\exp\{(a-\frac{\sigma^2}{2})(t-s)+\sigma(W_t-W_s)\}+c\int_s^t \exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &+ \sigma(W_t-W_u)\}du. \end{aligned}$$

□

*Proof of Proposition A.1.* Consider first the case $m = 1$, $n = 2$. In order to compare the two ruin probabilities $\psi_1(u)$ and $\psi_2(u)$, one can compare the two surplus processes $U^{(1)}$ and $U^{(2)}$ along each sample path of the Brownian motion. Both start with the same initial surplus $u$ and have the same underlying Brownian motion $W$. Let $T_1^{(1)}$ denote the time of the first claim in the $U^{(1)}$ process. Then for any $0 \leq t < T_1^{(1)}$, using (46) one has

$$\begin{aligned} U^{(2)}(t) &= u\exp\{(a-\frac{\sigma^2}{2})t+\sigma W_t\}+c\int_0^t \exp\{(a-\frac{\sigma^2}{2})(t-u)+\sigma(W_t-W_u)\}du \\ &= U^{(1)}(t). \end{aligned}$$

At $t = T_1^{(1)}$,

$$\begin{aligned} c\int_0^{T_1^{(1)}} \exp\{(a-\frac{\sigma^2}{2})(T_1^{(1)}-u) &+ \sigma(W_{T_1^{(1)}}-W_u\}du \geq c\int_0^{T_1^{(1)}} \exp\{(a-\frac{\sigma^2}{2})(T_1^{(1)}-u) \\ &+ \sigma(W_{T_1^{(1)}}-W_u\}du - X_1^{(1)}. \end{aligned}$$

Hence

$$U^{(2)}(t) \geq U^{(1)}(T_1^{(1)}).$$

For $T_1^{(1)} \leq t < T_2^{(1)}$, according to Lemma A.1

$$\begin{aligned} U^{(2)}(t) &= U^{(2)}(T_1^{(1)})\exp\{(a-\frac{\sigma^2}{2})(t-T_1^{(1)})+\sigma(W_t-W_{T_1^{(1)}}\} \\ &+ c\int_{T_1^{(1)}}^t \exp\{(a-\frac{\sigma^2}{2})(t-u)+\sigma(W_t-W_u)\}du \\ &\geq U^{(1)}(T_1^{(1)})\exp\{(a-\frac{\sigma^2}{2})(t-T_1^{(1)})+\sigma(W_t-W_{T_1^{(1)}}\} \\ &+ c\int_{T_1^{(1)}}^t \exp\{(a-\frac{\sigma^2}{2})(t-u)+\sigma(W_t-W_u)\}du \\ &= U^{(1)}(t) \end{aligned}$$

It follows by induction that $U^{(1)}(t) \leq U^{(2)}(t)$ for any $t$. Therefore, $\psi_1(u) > \psi_2(u)$, for any $u$. Analogously one can show that $\psi_{n+1}(u) \leq \psi_n(u)$, from which we obtain the result for arbitrary $m < n$. □

**Lemma A.2.** *If the interarrival time density satisfies an ODE with constant coefficients of the form*

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = \prod_{i=1}^n(\frac{d}{dt}+\beta_i)f_\tau(t) = 0,$$

*then the Laplace transform of the operator $\mathcal{L}^*(A)$ is itself an operator with*

$$\widehat{\mathcal{L}^*(A)}\psi = \prod_{i=1}^n(-\widehat{A+\beta_i})\psi = \prod_{i=1}^n\Big((-\widehat{A}+\beta_i)\widehat{\psi}\Big)+B, \tag{48}$$

*where $B$ is a constant.*

*Proof.* Recall that $A = (c + au)\frac{d}{du} + \frac{\sigma^2}{2}u^2\frac{d^2}{du^2}$. Using basic properties of the Laplace transform of derivatives, one has

$$\begin{aligned} \widehat{A\psi}(s) &= \frac{\sigma^2}{2}\frac{d^2}{ds^2}\left(s^2\widehat{\psi}(s)\right) - a\frac{d}{ds}\left(s\widehat{\psi}(s)\right) + c\left(s\widehat{\psi}(s) - \psi(0)\right) \\ &:= \widehat{A}\widehat{\psi}(s) - D_1, \end{aligned} \tag{49}$$

where $D_1 = c\psi(0)$ and the operator $\widehat{A}$ can be expanded as

$$\widehat{A} = \frac{\sigma^2}{2}s^2\frac{d^2}{ds^2} + \left(4\frac{\sigma^2}{2} - a\right)s\frac{d}{ds} + \left(2\frac{\sigma^2}{2} - a + cs\right). \tag{50}$$

For any $k \geq 2$, define

$$\widehat{A}^k\widehat{\psi}(s) = \widehat{A}(\widehat{A}^{k-1}\widehat{\psi}(s)), \quad \text{or} \quad \widehat{A}^k = \widehat{A}\widehat{A}^{k-1},$$

then one has recursively

$$\begin{aligned} \widehat{A^k\psi}(s) &= \widehat{A}\widehat{A^{k-1}\psi}(s) - D_k \\ &= \widehat{A}(\widehat{A}\widehat{A^{k-2}\psi}(s) - D_{k-1}) - D_k = \widehat{A}^2\widehat{A^{k-2}\psi}(s) - D_k \\ &= \dots \\ &= \widehat{A}^k\widehat{\psi}(s) - D_k, \end{aligned}$$

where $D_k = cA^{k-1}\psi(s)\mid_{s=0} = c^k\psi^{k-1}(0)$, leading to (48).

□

**Lemma A.3.** *The real parts of the complex conjugate solutions of* (25) *always lie outside the interval determined by* $r_1 = -1$ *and* $r_2 = -2 + \frac{2a}{\sigma^2}$*. The same is true for the other real solutions of* (25)*.*

*Proof.* The equation (25) is equivalent to

$$(\rho - r_1)(\rho - r_2) = \beta(1 - e^{\frac{2\pi i k}{n}}). \tag{51}$$

Consider the complex solutions $\rho = \alpha + ib$, where $b \neq 0$. Then the equation (51) can be written as

$$(\alpha - r_1 + ib)(\alpha - r_2 + ib) = \beta(1 - \cos(\frac{2\pi k}{n}) - i\sin(\frac{2\pi k}{n})).$$

The real part satisfies the equation

$$(\alpha - r_1)(\alpha - r_2) - b^2 = \beta(1 - \cos(\frac{2\pi k}{n})).$$

This implies

$$(\alpha - r_1)(\alpha - r_2) = b^2 + \beta(1 - \cos(\frac{2\pi k}{n})),$$

i.e. the product $(\alpha - r_1)(\alpha - r_2)$ is always positive. Therefore, $(\alpha - r_1)$ and $(\alpha - r_2)$ have the same sign. In other words, $\alpha$ is either bigger than both $r_1$ and $r_2$, or smaller than both, and the result follows. Note that for $b = 0$ the same is true, meaning that also the other real solutions are outside the interval determined by $r_1$ and $r_2$.

□

**Lemma A.4.** *Assume that for $i = 1, ..., 2n$, $y_i(s) = s^{r_i}\gamma_i(s)$ is a fundamental solution set of the homogeneous equation (22) with $\gamma_i(s)$ holomorphic functions, $\gamma_i(0) = 1$. For $m = 1, ..., 2n$, let*

$$\begin{cases} \overline{\omega} = \sum_{i=1}^{2n} r_i - n(2n-1), \\ \overline{\omega}_m = \sum_{i=1, i\neq m}^{2n} r_i - (n-1)(2n-1), \end{cases}$$

*Then there exist holomorphic functions $\widetilde{\gamma}$, $\widetilde{\gamma}_m, m = 1, ..., 2n$ such that the Wronskian determinants $W(s)$ and $W_m(s)$ (where $W_m$ is the determinant obtained from $W$ by replacing the $m$-th column by the column $(0, 0, \dots, 0, 1)^T \in \mathbb{R}^m$) can be written as*

$$W(s) = s^{\overline{\omega}}\widetilde{\gamma}(s), \quad W_m(s) = s^{\overline{\omega}_m}\widetilde{\gamma}_m(s).$$

*Furthermore, $\widetilde{\gamma}(0) \neq 0 \neq \widetilde{\gamma}_m(0)$ and for $m = 1, ...., 2n$,*

$$\frac{W_m(s)}{s^{2n}W(s)} = s^{-r_m-1}\frac{\widetilde{\gamma}_m(s)}{\widetilde{\gamma}(s)}$$

*where we denote $g_m = \frac{\widetilde{\gamma}_m(0)}{\widetilde{\gamma}(0)}$.*

*Proof.* Note that the $(k, l)$ entry of $W(s)$ can be written as

$$W_{kl} = s^{r_k-l+1}\gamma_{kl}(s)$$

where $\gamma_{kl}(s)$ is a holomorphic function. Then, one has

$$\begin{aligned} W(s) &= \sum \pm 1 \prod_{j=1}^{2n} W_{k_j l_j} \\ &= \sum \pm 1 \prod_{j=1}^{2n} s^{r_{k_j}-l_j+1}\gamma_{k_j l_j} \end{aligned}$$

where, as usual, the sum is over all permutations $k_j, l_j$ of the integers $1, 2, ...., 2n$ and the sign is determined by the signature of these permutations. The result follows by noting that

$$\begin{aligned} \log\left(\prod_{j=1}^{2n} s^{r_{k_j}-l_j+1}\right) &= \log(s)\sum_{j=1}^{2n}(r_{k_j} - l_j + 1) \\ &= \log(s)\left(\sum_{j=1}^{2n} r_j - \sum_{k=0}^{2n-1} k\right) \\ &= \log(s)\omega \end{aligned}$$

as claimed. A similar argument applies to the Wronskian determinant $W_m$. Since both $\widetilde{\gamma}_m(s)$ and $\widetilde{\gamma}(s)$ are analytic, the result follows. ☐

# References

J. Abate and W. Whitt. Asymptotics for $M/G/1$ low-priority waiting-time tail probabilities. *Queueing Systems Theory Appl.*, 25(1-4), 1997.

H. Albrecher, C. Constantinescu, G. Pirsic, G. Regensburger, and M. Rosenkranz. An algebraic operator approach to the analysis of Gerber-Shiu functions. *Insurance Math. Econom.*, 46(1):42–51, 2010.

S. Asmussen and H. Albrecher. *Ruin probabilities*, volume Second Edition of *Advanced Series on Statistical Science & Applied Probability*. World Scientific Publishing Co. Inc., River Edge, NJ, 2010.

N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.

J. F. Collamore. Random recurrence equations and ruin in a Markov-dependent stochastic economic environment. *Ann. Appl. Probab.*, 19(4):1404–1458, 2009.

C. Constantinescu and E. Thomann. Martingales for renewal jump-diffusion processes. *Preprint*, 2011.

M. V. Fedoryuk. *Asymptotic analysis*. Springer-Verlag, Berlin, 1993. Linear ordinary differential equations, Translated from the Russian by Andrew Rodick.

W. Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons Inc., New York, 1971.

A. Frolova, Y. Kabanov, and S. Pergamenshchikov. In the insurance business risky investments are dangerous. *Finance Stoch.*, 6(2):227–235, 2002.

J. Gaier and P. Grandits. Ruin probabilities in the presence of regularly varying tails and optimal investment. *Insurance Math. Econom.*, 30(2):211–217, 2002.

J. Gaier and P. Grandits. Ruin probabilities and investment under interest force in the presence of regularly varying tails. *Scand. Actuar. J.*, (4):256–278, 2004.

J. Gaier, P. Grandits, and W. Schachermayer. Asymptotic ruin probabilities and optimal investment. *Ann. Appl. Probab.*, 13(3):1054–1076, 2003.

H. K. Gjessing and J. Paulsen. Present value distributions with applications to ruin theory and stochastic equations. *Stochastic Process. Appl.*, 71(1): 123–144, 1997.

C. M. Goldie. Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.*, 1(1):126–166, 1991.

P. Grandits. A Karamata-type theorem and ruin probabilities for an insurer investing proportionally in the stock market. *Insurance Math. Econom.*, 34 (2):297–305, 2004.

C. Hipp and M. Plum. Optimal investment for insurers. *Insurance Math. Econom.*, 27(2):215–228, 2000.

V. Kalashnikov and R. Norberg. Power tailed ruin probabilities in the presence of risky investments. *Stochastic Process. Appl.*, 98(2):211–228, 2002.

C. Klüppelberg and U. Stadtmüller. Ruin probabilities in the presence of heavy-tails and interest rates. *Scand. Actuar. J.*, (1):49–58, 1998.

R. Kostadinova. Optimal investment for insurers when the stock price follows an exponential Lévy process. *Insurance Math. Econom.*, 41(2):250–263, 2007.

H. Nyrhinen. Large deviations for the time of ruin. *J. Appl. Probab.*, 36(3): 733–746, 1999.

H. Nyrhinen. Finite and infinite time ruin probabilities in a stochastic economic environment. *Stochastic Process. Appl.*, 92(2):265–285, 2001.

J. Paulsen. On Cramér-like asymptotics for risk processes with stochastic return on investments. *Ann. Appl. Probab.*, 12(4):1247–1260, 2002.

J. Paulsen. Ruin models with investment income. *Probab. Surv.*, 5:416–434, 2008.

J. Paulsen and H. K. Gjessing. Ruin theory with stochastic return on investments. *Adv. in Appl. Probab.*, 29(4):965–985, 1997.

S. Pergamenshchikov. Erratum to: "Ruin probability in the presence of risk investments" [Stochastic Process Appl. 116 (2006) 267–278]. *Stochastic Process. Appl.*, 119(1):305–306, 2009.

S. Pergamenshchikov and O. Zeitouny. Ruin probability in the presence of risky investments. *Stochastic Process. Appl.*, 116(2):267–278, 2006.

Q. Tang and G. Tsitsiashvili. Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks. *Stochastic Process. Appl.*, 108(2):299–325, 2003.

Q. Tang and G. Tsitsiashvili. Finite- and infinite-time ruin probabilities in the presence of stochastic returns on investments. *Adv. in Appl. Probab.*, 36(4): 1278–1299, 2004.

Q. Tang and L. Wei. Asymptotic aspects of the Gerber-Shiu function in the renewal risk model using Wiener-Hopf factorization and convolution equivalence. *Insurance Math. Econom.*, 46(1):19–31, 2010.

J. L. Teugels and G. Willmot. Approximations for stop-loss premiums. *Insurance Math. Econom.*, 6(3):195–202, 1987.

E. Thomann and E. Waymire. Contingent claims on assets with conversion costs. *Journal of Statistical Planning and Inference*, 113:403–417, 2003.

G. Wang and R. Wu. Distributions for the risk process with a stochastic return on investments. *Stochastic Process. Appl.*, 95(2):329–341, 2001.

L. Wei. Ruin probability of the renewal model with risky investment and large claims. *Sci. China Ser. A*, 52(7), 2009.

K. C. Yuen, G. Wang, and R. Wu. On the renewal risk process with stochastic interest. *Stochastic Process. Appl.*, 116(10):1496–1510, 2006.