

Tail Approximation for Reinsurance Portfolios of Gaussian-like Risks

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Abstract: We consider two different portfolios of proportional reinsurance of the same pool of risks. This contribution is concerned with Gaussian-like risks, which means that for large values the survival function of such risks is, up to a multiplier, the same as that of a standard Gaussian risk. We establish the tail asymptotic behavior of the total loss of each of the reinsurance portfolios and determine also the relation between randomly scaled Gaussian-like portfolios and unscaled ones. Further we show that jointly two portfolios of Gaussian-like risks exhibit asymptotic independence and their weak tail dependence coefficient is non-negative.

Key words: Gaussian-like risks; proportional reinsurance; asymptotic independence; weak tail dependence coefficient.

1 Introduction

In numerous insurance and financial situations the same source of risks impacts simultaneously different portfolios according to individual deterministic weights associated with those risks. For instance consider two big reinsurance companies that operate on the international level, and therefore happen to reinsure different proportions of the same risks. If the reinsurance treaty is a proportional one, then the total risk of each company for the proportional business is given by a linear combination of risks, arising from each portfolio of the direct insurer taking part in the reinsurance programme.

Throughout the paper $X_i, i \leq n$ will be independent random variables which alternatively are referred to as risks, reflecting our interest on insurance and finance applications. In a probabilistic setting the total loss amount of each reinsurance company can be modeled by Q_n and W_n , respectively with

$$Q_n = \sum_{i=1}^n \lambda_i X_i, \quad W_n = \sum_{i=1}^n \lambda_i^* X_i,$$

where X_i is the financial loss amount claimed from the i th direct insurer, and λ_i, λ_i^* are the proportionality factors of the risks being shared.

In a financial context, as for instance in that considered by Geluk et al. (2007), both Q_n and W_n model two portfolios with financial returns, where the risks X_1, \dots, X_n are the individual asset returns or risk factors and $\lambda_i, \lambda_i^*, i \leq n$ are the asset weights. Typically, the asset weights are assumed to sum to 1.

In concrete insurance and finance applications the distribution function of financial risks is not known. Essentially, this is not a major drawback, since often of interest is the quantification of the probability of large catastrophic risks, especially from the side of the reinsurer. In applications, departure from a Gaussian model is possible, however for inference a model with "Gaussian-like" features is of course preferable.

The main purpose of this article is to explore Gaussian-like risks i.e., risks that are similar to Gaussian ones in terms of the probability of producing large values. Specifically, we shall assume that for any risk $X_i, i \leq n$

$$P(X_i > u) \sim \mathcal{L}_i(u) u^{\alpha_i} \exp(-u^2/2), \quad u \rightarrow \infty, \quad (1)$$

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where $\mathcal{L}_i(\cdot), i \leq n$ are slowly varying functions at infinity i.e., for any t positive $\lim_{u \rightarrow \infty} \mathcal{L}_i(tu)/\mathcal{L}_i(u) = 1$. In other words

$$P(X_i > u) \sim \sqrt{2\pi} \mathcal{L}_i(u) u^{\alpha_i+1} \Psi(u), \quad u \rightarrow \infty, \quad (2)$$

with Ψ the survival function of a $N(0, 1)$ random variable; throughout this paper $f_1(x) \sim f_2(x)$ means asymptotic equivalence i.e., $f_1(x)/f_2(x) \rightarrow 1$ as $x \rightarrow \infty$.

Clearly, if $\alpha_i = -1$ and $\mathcal{L}_i(u) \rightarrow (2\pi)^{-1/2}$ as $u \rightarrow \infty$, then X_i is tail equivalent to a $N(0, 1)$ random variable. However, in general a Gaussian-like risk differs very strongly from a Gaussian one since α_i can take large negative values. It is therefore interesting to investigate the individual behavior of each portfolio consisting of Gaussian-like risks in terms of the probabilities of observing large losses. We shall investigate first the asymptotic behavior of $P(Q_n > u)$ for $u \rightarrow \infty$. In view of Lemma 8.6 in Piterbarg (1996)

$$P(X_1 + X_2 > u) \sim \mathcal{L}_1 \mathcal{L}_2 \sqrt{\pi} \frac{u^{\alpha_1 + \alpha_2 + 1}}{2^{\alpha_1 + \alpha_2 + 1}} \exp(-u^2/4), \quad u \rightarrow \infty \quad (3)$$

holds for any two Gaussian-like risks X_1, X_2 satisfying (1) with $\mathcal{L}_i(u) \equiv \mathcal{L}_i > 0, \forall u > 0, i = 1, 2$, which implies the tail asymptotic behavior of Q_2 for the case $\lambda_1 = \lambda_2 > 0$.

However, if $\lambda_1 \neq \lambda_2$ and, more generally, if risks obey (1), then the tail asymptotics of Q_2 cannot be established by simply using (3). The risks obeying (1) do not belong to the class of subexponential distributions (see Embrechts et al. (1997) or Foss et al. (2011) for the properties of this class). In fact those risks belong to the class of superexponential distributions, see Rootzén (1986,1987), Klüppelberg and Lindner (2005), or Geluk et al. (2007) for more details.

We note in passing that if X_1, X_2 are independent $N(0, 1)$ random variables, then $X_1 + X_2$ is a $N(0, 2)$ random variable, so (3) follows easily. Therefore, the appearance of $\exp(-u^2/4)$ in the general case in (3) is intuitively expected since we deal with "Gaussian-like" risks.

As it will be discussed below, special Gaussian-like risks relate to the random scaling of Gaussian risks.

Indeed, the random scaling is a common phenomena in various insurance models which incorporate inflation or deflation. In our framework, the random scaling of X_i 's will be modelled by non-negative random variables $S_i, i \leq n$ being independent of $X_i, i \leq n$. Under certain restrictions, it follows that the randomly scaled risk $S_i X_i$ is a Gaussian-like one, if X_i is a Gaussian like risk. This closure property together with the Gaussianity of X_i 's are crucial for extending (3) to Gaussian-like risks obeying (1). Further, the random scaling technique utilized in the proof of the main result leads to the derivation of the tail asymptotic behavior of Q_n if each risk S_i is bounded, and its survival function is regularly varying at its upper endpoint, see (4) below.

Our new result allows us to calculate the weak tail dependence coefficient $\bar{\chi}(Q_n, W_n)$. This measure of asymptotic independence introduced in Coles et al. (1999) is important for modelling of joint extremes.

The organization of the rest of the paper: we continue below with the formulation of the main results. Section 3 presents two applications. The first one establishes the asymptotic independence of both portfolios Q_n and W_n , whereas the second one derives the weak tail dependence coefficient $\bar{\chi}(Q_n, W_n)$. All the proofs are relegated to Section 4.

2 Main Results

In the following X_i 's are independent (but not identically distributed) risks with distribution functions $F_i, i \leq n$ and $S_i, i \leq n$ are independent non-negative risks with distribution function $G_i, i \leq n$. We shall write for short $X_i \sim F_i, S_i \sim G_i, i \leq n$. Further, we shall assume that $X_1, \dots, X_n, S_1, \dots, S_n$ are mutually independent. In the special

case $X_i \sim N(0, 1)$, $i \leq n$ and $\lambda_i, i \leq n$ are given constants

$$Q_n^* := \sum_{i=1}^n \lambda_i \sqrt{S_i} X_i \stackrel{d}{=} X_1 \sqrt{\sum_{i=1}^n \lambda_i^2 S_i} =: X_1 \sqrt{V_n},$$

where $\stackrel{d}{=}$ means equality of distribution functions.

For practical applications due to the time-value considerations of money random scaling of X_i 's by S_i 's is natural. If as above the X_i 's are normally distributed, then instead of considering the tail asymptotics of Q_n we can investigate that of V_n , and X_1 separately and then determine the tail asymptotics of Q_n^* . Indeed, by the fact that X_1 and V_n are independent, and the tail asymptotics of X_1 is known, in view of Hashorva et al. (2010), the tail asymptotic behavior of the portfolio of risks modeled by Q_n^* follows under certain assumptions on V_n which are satisfied if the G_i 's have a finite upper endpoint $\omega_i := \sup\{x : G_i(x) < 1\}$ and if $1 - G_i$ is regularly varying at ω_i . More specifically, we shall assume that $\omega_i = 1, i \leq n$ and

$$\lim_{x \rightarrow \infty} \frac{P(S_i > 1 - t/x)}{P(S_i > 1 - 1/x)} = t^{\gamma_i}, \quad \forall t > 0 \quad (4)$$

for each $i \leq n$ with some index $\gamma_i \in [0, \infty)$.

Our result on the tail behavior of V_n is surprising in that it links the tail asymptotic behavior of the aggregated risk with that of the products of the risks. The arithmetic-geometric mean inequality implies that

$$V_n \geq \prod_{i=1}^n S_i^{\lambda_i} =: V_n^*, \quad 1 \leq i \leq n,$$

provided that $\sum_{i=1}^n \lambda_i = 1$. Our first result below shows the surprising fact that V_n and V_n^* have the same tail asymptotic behavior.

Theorem 2.1. *Let $S_i \sim G_i, i \leq n$ be independent non-negative random variables satisfying (4). Then for any $\lambda_i > 0, i \leq n$ such that $\sum_{i=1}^n \lambda_i = 1$*

$$P\left(\sum_{i=1}^n \lambda_i S_i > u\right) \sim P\left(\prod_{i=1}^n S_i^{\lambda_i} > u\right) \sim \frac{\prod_{i=1}^n \lambda_i^{-\gamma_i} \Gamma(\gamma_i + 1) P(S_i > u)}{\Gamma(\sum_{i=1}^n \gamma_i + 1)}, \quad u \uparrow 1$$

holds, where $\Gamma(\cdot)$ is the Euler Gamma function.

Next, we show how by using this theorem, we can reduce the proof of the following Theorem 2.2 in an important particular case to random scaling of a portfolio of independent standard Gaussian variables. Under the assumptions of Theorem 2.1 we know the asymptotic behavior of $\widetilde{V}_n = \sum_{i=1}^n \lambda_i^2 S_i / \lambda^2, \lambda^2 = \sum_{i=1}^n \lambda_i^2$, and in particular we find that

$$P\left(\sqrt{\widetilde{V}_n} > 1 - 1/u\right) \sim P(\widetilde{V}_n > 1 - 2/u), \quad u \rightarrow \infty.$$

The distribution function of X_1 is in the Gumbel max-domain of attraction (MDA) with scaling function $w(x) = x$. We recall that a random variable Z with $P(Z > u) < 1, \forall u > 0$ is in the Gumbel MDA with some positive scaling function $w(\cdot)$ if

$$P(Z > u + s/w(u)) \sim \exp(-s)P(Z > u), \quad u \rightarrow \infty$$

holds for any $s \geq 0$, see e.g., Embrechts et al. (1997). Applying Theorem 3.1 of Hashorva et al. (2010) (see also Hashorva (2012)) we obtain thus

$$\begin{aligned} P(Z_n > u) &\sim P\left(X_1 \sqrt{\widetilde{V}_n} > u/\lambda\right) \sim \frac{1}{2} P\left(|X_1| \sqrt{\widetilde{V}_n} > u/\lambda\right) \\ &\sim \frac{1}{2} \Gamma\left(\sum_{i=1}^n \gamma_i + 1\right) P\left(\sqrt{\widetilde{V}_n} > 1 - \lambda^2/u^2\right) P(|X_1| > u/\lambda) \end{aligned}$$

$$\sim \Gamma\left(\sum_{i=1}^n \gamma_i + 1\right) P(\widetilde{V}_n > 1 - 2(\lambda/u)^2) \Psi(u/\lambda), \quad u \rightarrow \infty. \quad (5)$$

Since $\lim_{u \rightarrow \infty} P(\widetilde{V}_n > 1 - 1/u^2) = 0$ the above works only for $\alpha_i + 1 < 0, i \leq n$. Thus the above chain of asymptotic relations leads us to the main result of this paper in the particular case $\alpha_i < -1$, that is for the distributions possessing (1) with tails lighter than Gaussian. Next we state our main result for all values $\alpha_i \in \mathbb{R}, i \leq n$.

Theorem 2.2. *If $X_i, i \leq n$ are independent Gaussian-like risks satisfying (1) for some $\alpha_i \in \mathbb{R}, i \leq n$, then for any set of deterministic weights $\lambda_i > 0, i \leq n$ we have*

$$P(Q_n > u) \sim \frac{(\sqrt{2\pi})^{n-1} \prod_{j=1}^n [\lambda_j^{\alpha_j+1} \mathcal{L}_j(u)] u^{\alpha+n-1}}{\lambda^{2\alpha+2n-1}} \exp\left(-\frac{u^2}{2\lambda^2}\right) \quad (6)$$

as $u \rightarrow \infty$, where $\lambda^2 = \sum_{i=1}^n \lambda_i^2$, $\alpha = \sum_{i=1}^n \alpha_i$.

Remarks: a) In Theorem 2.2 we do not put any assumption on the lower asymptotic tail behavior of the risks. In the Gaussian mean-zero case such risks are symmetric about 0. If in Theorem 2.2 we assume that the Gaussian-like risks are symmetric about zero, then (6) can be easily adapted to the case that $\lambda_i \in \mathbb{R}, i \leq n$.

b) If $\mathcal{L}_i(\cdot)$ is a constant function, then as mentioned in the Introduction the risk X_i belongs to the class of superexponential distributions. The tail asymptotics of the convolution of identically distributed and independent superexponential risks is established in Rootzén (1987) and for more general risks in Klüppelberg and Lindner (2005). In the aforementioned papers the results are derived under several constraints on the probability density function of the risks, which we do not impose here. Hence both our results and Lemma 8.6 of Piterbarg (1996) do not follow from Rootzén (1987) or Klüppelberg and Lindner (2005).

d) In view of Theorem 2.1 and Theorem 2.2 the total loss of randomly scaled risks modeled by Q_n^* is a Gaussian-like risk if the S_i 's and X_i 's are independent and satisfy the assumptions of Theorem 2.1 and Theorem 2.2, respectively.

The proof of Theorem 2.2 in the general case is based on the following generalization of Lemma 8.6 of Piterbarg (1996) to random variables obeying (1).

Lemma 2.3. *If $X_i, i = 1, 2$ are two independent random variables such that*

$$P(X_i > u) \sim \mathcal{L}_i(u) u^{\alpha_i} \exp\left(-\frac{u^2}{2p_i^2}\right), \quad u \rightarrow \infty, \quad i = 1, 2 \quad (7)$$

for some $\alpha_i \in \mathbb{R}, p_i > 0$ and $\mathcal{L}_i(\cdot)$ are slowly varying functions at infinity, then as $u \rightarrow \infty$

$$P(X_1 + X_2 > u) \sim \frac{\sqrt{2\pi} p_1^{2\alpha_1+1} p_2^{2\alpha_2+1} \mathcal{L}_1(u) \mathcal{L}_2(u) u^{\alpha_1+\alpha_2+1}}{p^{2\alpha_1+2\alpha_2+3}} \exp\left(-\frac{u^2}{2p^2}\right), \quad (8)$$

with $p = \sqrt{p_1^2 + p_2^2}$.

Example 1. Consider X_1, X_2 two independent Gaussian-like risks which satisfy (1). Applying (8) with $p_1 = p_2 = 1, p = \sqrt{2}$ we obtain

$$P(X_1 + X_2 > u) \sim \sqrt{\pi} \mathcal{L}_1(u) \mathcal{L}_2(u) (u/2)^{\alpha_1+\alpha_2+1} \exp(-u^2/4)$$

as $u \rightarrow \infty$, which implies (3). In particular, when X_1, X_2 are independent $N(0, 1)$ random variables, then $X_1 + X_2$ is a $N(0, 2)$ random variable, and therefore its tail asymptotics is given by

$$P(X_1 + X_2 > u) = P\left(X_1 > u/\sqrt{2}\right) \sim \frac{1}{\sqrt{\pi}u} \exp(-u^2/4), \quad u \rightarrow \infty,$$

which follows also from (7) when $\mathcal{L}_1 = \mathcal{L}_2 = (2\pi)^{-1/2}$ and $\alpha_i = -1, i = 1, 2$.

3 Applications

A bivariate Gaussian random vector (X, Y) with $N(0, 1)$ marginals is specified completely by the correlation coefficient ρ . Although $\rho < 1$ can be very close to 1, still X and Y are asymptotically independent in the sense that

$$\lim_{u \rightarrow \infty} \frac{P(X > u, Y > u)}{P(X > u)} = 0. \quad (9)$$

Asymptotic independence is a nice property, closely related to joint asymptotic behavior of componentwise sample maxima (e.g., Resnick (1987)). For bivariate Gaussian samples the componentwise maxima are (using (9)) asymptotically independent. The asymptotic independence is a crucial property for the calculation of many indices related to extreme value statistics, finance and insurance applications. In our first application we show that the losses modeled by Q_n and W_n are asymptotically independent.

If Q_n and W_n have distribution functions H and H_* , respectively, then the asymptotic independence of Q_n and W_n means that

$$\chi_u(Q_n, W_n) := \frac{P(Q_n > t_u, W_n > t_u^*)}{P(Q_n > t_u)} \rightarrow 0, \quad u \rightarrow \infty,$$

with $t_u := H^{-1}(1 - 1/u)$, $t_u^* := H_*^{-1}(1 - 1/u)$, and H^{-1}, H_*^{-1} are the generalized inverses of H and H_* , respectively. In view of Theorem 2.2 both Q_n and W_n have distribution function in the Gumbel MDA. Utilizing the formula for the norming constants of Weibull-like distributions given on p. 317 of Mikosch (2009) and using again Theorem 2.2, it follows that with $\lambda = \sqrt{\sum_{i=1}^n \lambda_i^2} > 0$, $\lambda^* = \sqrt{\sum_{i=1}^n (\lambda_i^*)^2} > 0$

$$\frac{t_u}{\lambda} \sim \frac{t_u^*}{\lambda^*}, \quad u \rightarrow \infty. \quad (10)$$

Consequently, since for all u large

$$\chi_u(Q_n, W_n) = \frac{P(Q_n/\lambda > t_u/\lambda, W_n/\lambda > t_u^*/\lambda^*)}{P(Q_n/\lambda > t_u/\lambda)}$$

we shall assume without loss of generality that $\lambda_i, \lambda_i^*, i \leq n$ are positive and satisfy

$$\sum_{i=1}^n \lambda_i^2 = \sum_{i=1}^n (\lambda_i^*)^2 = 1. \quad (11)$$

The assumption (11) is reasonable since when all X_i 's are $N(0, 1)$ random variables, then Q_n and W_n are also $N(0, 1)$ distributed. Note that (11) implies

$$\varrho := \sum_{i=1}^n \lambda_i \lambda_i^* \in [0, 1].$$

Since both portfolios are supposed to be different, we shall assume below that

$$\varrho \in [0, 1). \quad (12)$$

Hence, by Theorem 2.2, (10) and (12) for any $\varepsilon > 0$ we obtain

$$\chi_u(Q_n, W_n) \leq \frac{P(\sum_{i=1}^n (\lambda_i + \lambda_i^*) X_i > 2t_u(1 + o(1)))}{P(W_n > t_u)} \rightarrow 0, \quad u \rightarrow \infty.$$

When asymptotic independence holds, as suggested by Coles et al. (1999) more insight on the strength of the joint tail behavior is obtained by calculating the weak tail dependence coefficient $\bar{\chi}(Q_n, W_n) := \lim_{u \rightarrow \infty} \bar{\chi}_u(Q_n, W_n)$ (supposing the limit exists), where

$$\bar{\chi}_u(Q_n, W_n) = \frac{\ln P(Q_n > t_u) + \ln P(W_n > t_u^*)}{\ln P(Q_n > t_u, W_n > t_u^*)} - 1.$$

Borrowing the idea of Piterbarg and Stamatovic (2005) (see also Deĭbicki et al. (2010)) we have for all large u

$$P(Q_n > t_u, W_n > t_u^*) \leq \inf_{a,b>0,a+b=1} P\left(\sum_{i=1}^n (a\lambda_i + b\lambda_i^*)X_i > t_u(1 + o(1))\right),$$

which implies using further Theorem 2.2

$$\begin{aligned} \limsup_{u \rightarrow \infty} \frac{\ln P(Q_n > t_u, W_n > t_u^*)}{t_u^2} &\leq \lim_{u \rightarrow \infty} \frac{\inf_{a,b \in \mathbb{R}, a+b=1} \ln\left(P\left(\sum_{i=1}^n (a\lambda_i + b\lambda_i^*)X_i > t_u(1 + o(1))\right)\right)}{t_u^2} \\ &= -\frac{1}{1 + \varrho}. \end{aligned} \quad (13)$$

Consequently,

$$\limsup_{u \rightarrow \infty} \bar{\chi}_u(Q_n, W_n) \leq \varrho.$$

Our last result shows that $\bar{\chi}(Q_n, W_n) = \varrho$ for two Gaussian-like portfolios Q_n and W_n .

Theorem 3.1. *Under the assumptions of Theorem 2.2, if further (11) and (12) are satisfied, then*

$$\bar{\chi}(Q_n, W_n) = \varrho \quad (14)$$

holds. Moreover, (14) still holds even if some λ_i, λ_i^* equal zero.

Remark: If we do not assume (11) in Theorem 3.1, then (14) is valid with $\varrho = \frac{\sum_{i=1}^n \lambda_i \lambda_i^*}{\sqrt{\sum_{i=1}^n \lambda_i^2 \sum_{i=1}^n (\lambda_i^*)^2}}$, provided that (12) holds.

4 Proofs

PROOF OF THEOREM 2.1. For all x large by the independence of S_1, \dots, S_n we may write (set $G_{i,x}(z) := G_i(1 - z/x), x, z \in (0, \infty)$)

$$\begin{aligned} &P\left(\sum_{i=1}^n \lambda_i S_i > 1 - 1/x\right) \\ &= \int_0^1 P(\lambda_1 S_1 > 1 - 1/x - \sum_{i=2}^n \lambda_i y_i) dG_2(y_2) \cdots dG_n(y_n) \\ &= \int_0^1 P(\lambda_1 S_1 > \lambda_1 - 1/x + \sum_{i=2}^n \lambda_i - \sum_{i=2}^n \lambda_i(1 - z_i/x)) dG_2(z_2) \cdots dG_n(z_n) \\ &= \int_0^\infty \cdots \int_0^\infty P(S_1 > 1 - (1 - \sum_{i=2}^n \lambda_i z_i)/(\lambda_1 x)) dG_{2,x}(z_2) \cdots dG_{n,x}(z_n) \\ &= \prod_{i=1}^n \bar{G}_i(1 - 1/x) \int_0^\infty \cdots \int_0^\infty \frac{P(S_1 > 1 - (1 - \sum_{i=2}^n \lambda_i z_i)/(\lambda_1 x))}{\bar{G}_1(1 - 1/x)} dG_{2,x}(z_2) \cdots dG_{n,x}(z_n) / \left(\prod_{i=2}^n \bar{G}_i(1 - 1/x)\right). \end{aligned}$$

Assume for simplicity that $n > 2$. By the assumption on G_i , for any $z_i > 0, i \leq n$

$$\frac{\bar{G}_{i,x}(z_i)}{\bar{G}_i(1 - 1/x)} = \frac{\bar{G}_i(1 - z_i/x)}{\bar{G}_i(1 - 1/x)} \rightarrow z_i^{\gamma_i}, \quad x \rightarrow \infty$$

and

$$\lim_{x \rightarrow \infty} \frac{P(S_1 > 1 - (1 - \sum_{i=2}^n \lambda_i z_i)/(\lambda_1 x))}{\bar{G}_1(1 - 1/x)} = \left(\max\left(0, 1 - \sum_{i=2}^n \lambda_i z_i\right)\right)^{\gamma_1},$$

which implies as $x \rightarrow \infty$

$$\begin{aligned} & P\left(\sum_{i=1}^n \lambda_i S_i > 1 - 1/x\right) \\ & \sim \prod_{i=1}^n \bar{G}_i(1 - 1/x) \lambda_1^{-\gamma_1} \prod_{i=2}^n \gamma_i \int_0^\infty \cdots \int_0^\infty \left(\max\left(0, 1 - \sum_{i=2}^n \lambda_i z_i\right)\right)^{\gamma_1} \prod_{i=2}^n z_i^{\gamma_i-1} dz_2 \cdots dz_n \\ & = \frac{1}{\Gamma\left(\sum_{i=1}^n \gamma_i + 1\right)} \prod_{i=1}^n \left(\lambda_i^{-\gamma_i} \Gamma(\gamma_i + 1) \bar{G}_i(1 - 1/x)\right). \end{aligned}$$

Applying Theorem 3.1 in Hashorva et al. (2010) we obtain as $x \rightarrow \infty$

$$\begin{aligned} \Gamma\left(\sum_{i=1}^n \gamma_i + 1\right) P\left(\prod_{i=1}^n S_i^{\lambda_i} > 1 - 1/x\right) & \sim \prod_{i=1}^n \left(P(S_i^{\lambda_i} > 1 - 1/x) \Gamma(\gamma_i + 1)\right) \\ & = \prod_{i=1}^n \left(P(S_i > 1 - 1/(\lambda_i x)) \Gamma(\gamma_i + 1)\right), \end{aligned}$$

hence the proof is complete.

PROOF OF THEOREM 2.2. In light of Lemma 2.3 for random variables $\lambda_i X_i$, $i = 1, \dots, n$ we have $p_i^2 = \lambda_i^2$ and correspondingly scaled \mathcal{L}_i 's. Thus for $n = 2$ we have proven that

$$P(Q_2 > u) = \frac{\sqrt{2\pi} \lambda_1^{\alpha_1+1} \lambda_2^{\alpha_2+1} \mathcal{L}_1(u) \mathcal{L}_2(u) u^{\alpha_1+\alpha_2+1}}{\theta_2^{2\alpha_1+2\alpha_2+3}} \exp\left(-\frac{u^2}{2\theta_2}\right) (1 + o(1))$$

as $u \rightarrow \infty$, with $\theta_2 = \lambda_1^2 + \lambda_2^2$. Now we proceed by induction assuming that

$$P(Q_k > u) \sim \frac{(\sqrt{2\pi})^{k-1} \prod_{j=1}^k \left[\lambda_j^{\alpha_j+1} \mathcal{L}_j(u)\right] u^{\alpha_1+\dots+\alpha_k+k-1}}{\theta_k^{2\alpha_1+\dots+2\alpha_k+2k-1}} \exp\left(-\frac{u^2}{2\theta_k}\right),$$

with $\theta_k = \lambda_1^2 + \dots + \lambda_k^2$ and $k > 2$. Considering that $Q_{k+1} = Q_k + \lambda_{k+1} X_{k+1}$ with Q_k being independent of X_{k+1} , the claim follows by a direct application of Lemma 2.3. \square

PROOF OF LEMMA 2.3. Let F_i , $i = 1, 2$ denote the distribution functions of X_1 and X_2 , respectively. Suppose without loss of generality that $p_1^2 + p_2^2 = 1$ and $p_1 \leq p_2$. Then for $c = 1.1$ and any $\varepsilon > 0$ we have

$$\begin{aligned} P(X_1 + X_2 > x, X_2 \leq (1 - cp_1)x) & \leq P(X_1 > cp_1 x) = O(x^{\alpha_1+\varepsilon} \exp(-c^2 x^2/2)), \\ P(X_1 + X_2 > x, X_2 > cp_2 x) & \leq P(X_2 > cp_2 x) = O(x^{\alpha_2+\varepsilon} \exp(-c^2 x^2/2)) \end{aligned} \quad (15)$$

as $x \rightarrow \infty$. Note that $0 < a := 1 - cp_1 < b := cp_2$; the first inequality follows from $p_1 \leq 1/\sqrt{2}$ and the second one follows from $p_1 + p_2 > 1$. Let us focus on the asymptotic behavior of the integral

$$I_x = \int_{ax}^{bx} \bar{F}_1(x-y) dF_2(y), \quad \bar{F}_i = 1 - F_i.$$

Pick small $h > 0$ and denote $h_k = kh/x$, $\Delta_k = [h_k, h_{k+1})$, where k is a positive integer. Then, bounding F_1 on intervals $[x - h_k, x - h_{k-1}]$ by its maximum and minimum values, respectively and then integrating in y we have

$$\sum_{k: \Delta_k \subset [ax, bx]} \bar{F}_1(x - h_{k-1})(F_2(h_k) - F_2(h_{k-1})) \leq I_x \leq \sum_{k: \Delta_k \cap [ax, bx] \neq \emptyset} \bar{F}_1(x - h_k)(F_2(h_k) - F_2(h_{k-1})).$$

Observe that there exist two positive functions A_1, A_2 decreasing to zero as $x \rightarrow \infty$ such that for $i = 1, 2$

$$\mathcal{L}_i(x) x^{\alpha_i} \exp(-x^2/2p_i^2)(1 - A_i(x)) \leq \bar{F}_i(x) \leq \mathcal{L}_i(x) x^{\alpha_i} \exp(-x^2/2p_i^2)(1 + A_i(x)), \quad \forall x > 0.$$

Similarly, there exist two positive functions B_1, B_2 decreasing to zero as $x \rightarrow \infty$ such that

$$1 - B_i(x) \leq \inf_{y \in [a, b]} \frac{\mathcal{L}_i(xy)}{\mathcal{L}_i(x)} \leq \sup_{y \in [a, b]} \frac{\mathcal{L}_i(xy)}{\mathcal{L}_i(x)} \leq 1 + B_i(x), \quad i = 1, 2.$$

Since $x^\alpha e^{-x^2/2q^2}$, $q > 0$ decreases for all sufficiently large x denoting

$$\gamma_2(x) = A_2(ax) + B_2(x) + A_2(ax)B_2(x), \quad r(x) = \frac{e^x - 1}{x}$$

we obtain

$$F_2(h_k) - F_2(h_{k-1}) \leq \mathcal{L}_2(x) \left[r(bh/p_2^2) h_k^{\alpha_2} e^{-h_k^2/2p_2^2} \frac{kh^2}{p_2^2 x^2} + 2 \gamma_2(x) (x) h_{k-1}^{\alpha_2} e^{-h_{k-1}^2/2p_2^2} \right].$$

In order to derive an estimation from below, note that for sufficiently large x

$$\frac{(2k-1)h^2}{2x^2 p_2^2} + \log \left(\frac{k-1}{k} \right)^{\alpha_2} \geq \frac{kh^2}{p_2^2 x^2} (1 - C/x^2)$$

for some $C > 0$ which does not depend on h for all sufficiently small h . Therefore

$$F_2(h_k) - F_2(h_{k-1}) \geq \mathcal{L}_2(x) \left[h_k^{\alpha_2} e^{-h_k^2/2p_2^2} \frac{kh^2}{p_2^2 x^2} r \left(\frac{kh^2(1 - C/x^2)}{p_2^2 x^2} \right) (1 - C/x^2) - 2\gamma_2(x) (x) h_{k-1}^{\alpha_2} e^{-h_{k-1}^2/2p_2^2} \right].$$

Thus we have

$$\begin{aligned} I_x &\leq \mathcal{L}_1(x) \mathcal{L}_2(x) \left[(1 + A_1(x/4))(1 + B_1(x/4))(1 + \gamma_2(x)) r(hb/p_2^2) \right. \\ &\quad \times \sum_{k: \Delta_k \cap [ax, bx] \neq \emptyset} (x - h_k)^{\alpha_1} \exp(-(x - h_k)^2/2p_1^2) h_k^{\alpha_2} e^{-h_k^2/2p_2^2} \frac{kh^2 p_2^2}{x^2} \\ &\quad + 2(1 + A_1(x/4))\gamma_2(x)(1 + B_1(x)) \\ &\quad \left. \times \sum_{k: \Delta_k \cap [ax, bx] \neq \emptyset} (x - h_k)^{\alpha_1} \exp(-(x - h_k)^2/2p_1^2) h_{k-1}^{\alpha_2} e^{-h_{k-1}^2/2p_2^2} \right] \end{aligned} \quad (16)$$

and

$$\begin{aligned} I_x &\geq \mathcal{L}_1(x) \mathcal{L}_2(x) \left[(1 - A_1(x/4))(1 - B_1(x/4))(1 - \gamma_2(x)) r(ha(1 - Cx^{-2})/p_2^2) \right. \\ &\quad \times \sum_{k: \Delta_k \subset [ax, bx]} (x - h_k)^{\alpha_1} \exp(-(x - h_k)^2/2p_1^2) h_k^{\alpha_2} e^{-h_k^2/2p_2^2} \frac{kh^2 p_2^2}{x^2} \\ &\quad - 2(1 + A_1(x/4))\gamma_2(x/4)(1 + B_1(x/4))(1 - Cx^{-2}) \\ &\quad \left. \times \sum_{k: \Delta_k \cap [ax, bx] \neq \emptyset} (x - h_k)^{\alpha_1} \exp(-(x - h_k)^2/2p_1^2) h_{k-1}^{\alpha_2} e^{-h_{k-1}^2/2p_2^2} \right]. \end{aligned} \quad (17)$$

The first sums in (16) and (17) differ from each other by two summands, so it is sufficient to estimate one of them.

Then the first sum in the right-hand side of (16) is equal to (set $h'_k = h_k/x = hk/x^2$)

$$\begin{aligned} I'_x &:= p_2^{-2} x^{\alpha_1 + \alpha_2 + 2} \sum_{k: \Delta_k/x \cap [a, b] \neq \emptyset} (1 - h'_k)^{\alpha_1} (h'_k)^{\alpha_2} \exp \left(-\frac{x^2(1 - h_k'^2)}{2p_1^2} - \frac{x^2 h_k'^2}{2p_2^2} \right) h'_k \frac{h}{x^2} \\ &\leq p_2^{-2} x^{\alpha_1 + \alpha_2 + 2} \int_a^b (1 - t + h/x^2)^{\alpha_1} t^{\alpha_2 + 1} \exp \left(-\frac{x^2}{2} \left(\frac{(1-t)^2}{p_1^2} + \frac{t^2}{p_2^2} \right) \right) dt, \end{aligned}$$

where we used the monotonicity of the involved functions. In order to obtain a lower bound for the first sum in (17)

replace $(1 - t + h/x^2)^{\alpha_1} t^{\alpha_2 + 1}$ by $(1 - t)^{\alpha_1} (t - h/x^2)^{\alpha_2 + 1}$. Next, Theorem 1.3 in Fedoryuk (1987) yields

$$I'_x = \sqrt{2\pi} p_1^{2\alpha_1 + 1} p_2^{2\alpha_2 + 1} x^{\alpha_1 + \alpha_2 + 1} e^{-x^2/2} (1 + O(x^{-2})), \quad x \rightarrow \infty.$$

We investigate below the second sums I_x'' and J_x'' on the right-hand side of (16) and (17), respectively. For any k , the k th summands in those sums are equal to the k th summands in the first sums multiplied by $x^2/(kh)^2$, which is not greater than b/h . Thus we obtain dividing right- and left- parts of (16) and (17) by

$$D(x) = \sqrt{2\pi} p_1^{2a_1+1} p_2^{2a_2+1} \mathcal{L}_1(x) \mathcal{L}_2(x) x^{\alpha_1+\alpha_2+1} e^{-x^2/2}$$

and letting $x \rightarrow \infty$, that

$$r(ha/p_2^2) \leq \liminf_{x \rightarrow \infty} \frac{I_x}{D(x)} \leq \limsup_{x \rightarrow \infty} \frac{I_x}{D(x)} \leq r(hb/p_2^2),$$

which by definition of $r(x)$ and the arbitrary choice of h establishes the asymptotic behavior of I_x . Clearly, in view of the fact that $\mathcal{L}_i(x/p) \sim \mathcal{L}_i(x)$, $i = 1, 2$ the proof in the case that $p_1^2 + p_2^2 = 1$ follows from (15). The general case of p_1, p_2 follows by re-scaling, and thus the proof is complete. \square

PROOF OF THEOREM 3.1. In view of (13) we need to estimate $P(Q_n > t_u, W_n > t_u^*)$ from below. We shall determine optimal $\delta_i(u)$, $i \leq n$ such that

$$P(Q_n > t_u, W_n > t_u^*) \geq P(X_i > \delta_i(u), i = 1, \dots, n). \quad (18)$$

In order to realize such a choice, consider the asymptotic behavior of the integral

$$\int_{\{\sum_{i=1}^n \lambda_i x_i \geq u, \sum_{i=1}^n \lambda_i^* x_i \geq u\}} e^{-\frac{1}{2} \|\mathbf{x}\|^2} d\mathbf{x} = u^n \int_{\{\sum_{i=1}^n \lambda_i s_i \geq 1, \sum_{i=1}^n \lambda_i^* s_i \geq 1\}} e^{-\frac{1}{2} u^2 \|\mathbf{s}\|^2} d\mathbf{s}.$$

In the spirit of the Laplace asymptotic method, we find the minimal value of $\|\mathbf{s}\|^2 = \sum_{i=1}^n s_i^2$ on the set $\{\mathbf{s} : \sum_{i=1}^n \lambda_i s_i \geq 1, \sum_{i=1}^n \lambda_i^* s_i \geq 1\}$. Since λ_i, λ_i^* are all non-negative, the minimum is attained at the boundary, that is, on the set $\{\mathbf{s} : \sum_{i=1}^n \lambda_i s_i = 1, \sum_{i=1}^n \lambda_i^* s_i = 1\}$. It follows that the point of minimum has components

$$s_i = \frac{\lambda_i + \lambda_i^*}{1 + \rho}, \quad i = 1, \dots, n.$$

Consequently, the minimal value of $\|\mathbf{s}\|^2$ on the integrating set equals $2/(1 + \rho)$. Setting now (write $z_u := \max(t_u, t_u^*)$)

$$\delta_i(u) = \frac{\lambda_i + \lambda_i^*}{1 + \rho} z_u$$

we have that (18) holds for any $u > 0$ and furthermore, by (1) and (9)

$$\begin{aligned} \log P(X_i > \delta_i(u), i = 1, \dots, n) &= \sum_{i=1}^n \log P(X_i > \delta_i(u)) \\ &\sim -z_u^2 \sum_{i=1}^n \frac{(\rho_i + \rho_i^*)^2}{2(1 + \rho)^2}, \quad u \rightarrow \infty \\ &= -\frac{z_u^2}{1 + \rho} \end{aligned}$$

and thus the claim follows using (10). \square

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