Tail dependence for two skew slash distributions

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Abstract: Coefficient of tail dependence measures the dependencies between extreme values. In this paper, the upper tail dependence coefficients of two classes of skew slash distributions are derived. The difference of tail dependence coefficients of the two types skew slash distributions sheds light on the model choice for random variables with asymptotic dependence.

Key words and phrases: Tail dependence coefficient; Skew slash distribution; Skew-normal distribution; Variance-mean mixture.

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1 Introduction

Let $\boldsymbol{X} = (X_1, X_2)^\top$ be a bivariate random vector with marginal distribution functions (dfs) F_1 and F_2, respectively. The upper tail dependence coefficient of $\boldsymbol{X}$ is defined by

$$\lambda_U = \lim_{u \uparrow 1} \mathbb{P}\left(F_1(X_1) \geq u | F_2(X_2) \geq u\right) \tag{1}$$

provided that the limit λ_U exists; see Nelsen (1999) and Embrechts et al. (2002). This quantity provides insight into the tendency for the distribution to describe joint extreme events since it measures the strength of dependence (or association) in the tails of a bivariate distribution. Generally, $\boldsymbol{X}$ is said to have asymptotic upper tail dependence if λ_U is positive. In particular, trivial values $\lambda_U = 1$ and $\lambda_U = 0$ correspond to full dependence and independence, respectively.

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Tail independence of bivariate normal distributions was firstly addressed by Sibuya (1960) (see also Embrechts et al. (2002)) while the tail dependence of symmetry t-distributions was established by Demarta and McNeil (2005). Their skew-versions were further considered by Banachewicz and van der Varrt (2008) and Fung and Seneta (2010). The skew t-distributions are more popular and useful since it provides tail dependence of some extent as well as the skewness and heavy tails compared with (skew) normal distributions. For more related studies see, e.g., Heffernan (2000), Fung and Seneta (2011), Padoan (2011), and references therein.

In recent years, the multivariate skew-slash distributions alternatively (see (2) and (3) below for two precise definitions) have received considerable attention both in theoretical studies for their numerous stochastic properties, and in applied studies for robust statistical modeling of datasets involving distributions with skewness and heavy tails, see, e.g., conditional distributions, moments and applying skews-lash distributions to fit AIS and glass-fiber data (Wang and Genton (2006)) and characteristics functions (Kim and Genton (2011)) for skew slash distributions (3), and parameters estimation procedure such as the EM based on MLE in Arslan (2009), MLE in Lachos et al. (2010), and empirical Bayes estimations in Zareifard and Khaledi (2013) for the skew slash distributions (2). For more details see, e.g., Genç (2013) and Punathumparambatha (2013), and references therein.

Recently, tail dependence has been discussed in financial applications related to market or credit risk; see, e.g., Schmidt (2005), Durante (2013). A generalized tail dependence measure, namely tail quotient correlation coefficient was proposed by Zhang (2008) where a new test statistics of tail independence was developed; see Wu et al. (2012) for more related studies. In this paper, we shall investigate the tail dependence coefficient for two classes of skew slash distributions. The first class is defined by the normal variance-mean method. Specifically, a random vector $\boldsymbol{X} = (X_1, X_2)^\top$ is called to be skew slash distributed with parameters $(\lambda, \boldsymbol{\theta}, \mathbf{R})$, denoted by $\boldsymbol{X} \sim SS(\lambda, \boldsymbol{\theta}, \mathbf{R})$, if $\boldsymbol{X}$ has the following stochastic representation (see Arslan (2008, 2009))

$$\boldsymbol{X} = \frac{\boldsymbol{\theta}}{V} + \frac{\boldsymbol{Z}}{\sqrt{V}}, \tag{2}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2)^\top \in \mathbb{R}^2$ and $V \sim Beta(\lambda, 1), \lambda > 0$ with probability density function (pdf) $f(x) = \lambda x^{\lambda-1}, x \in (0,1)$, independent of $\boldsymbol{Z} \sim N_2(\mathbf{0}, \mathbf{R})$, a bivariate normal distribution with mean $\mathbf{0}$ and correlation matrix $\mathbf{R}$ with correlation entry $\rho \in (-1,1)$. This skew slash distribution introduces randomness into the variance and mean of a normal distribution via a beta random variable so that it is more flexible and can provide useful asymmetric and heavy-tailed extensions of their symmetric counterparts ($\boldsymbol{\theta} = \mathbf{0}$) for robust statistical modeling of datasets. For more related studies on model (2) see, e.g., generalized hyperbolic skew t-distributions in Kjersti and Ingrid (2006), skew grouped t-distributions in Banachewicz and van der Varrt (2008) and skew t-distributions in Fung and Seneta (2011).

The second class of skew slash distributions is defined as the scale-mixed skew-normal distribution (Azzalini and Dalla Valle (1996)). A random vector $\boldsymbol{X}$ is called the second type skew slash distribution, denoted by $\boldsymbol{X} \sim ASS(\lambda, \boldsymbol{\theta}, \mathbf{R})$, if $\boldsymbol{X}$ is given by

$$\boldsymbol{X} = \frac{\boldsymbol{\mathcal{Z}}}{\sqrt{V}}, \tag{3}$$

where $V \sim Beta(\lambda, 1), \lambda > 0$, independent of $\boldsymbol{\mathcal{Z}} = (Z_1, Z_2)^\top \sim SN_2(\boldsymbol{\theta}, \mathbf{R})$, a bivariate skew normal distribution with pdf

$$2\phi_2(\boldsymbol{z}, \mathbf{R})\Phi(\boldsymbol{\theta}^\top \boldsymbol{z}),$$

where $\phi_2(\cdot, \mathbf{R})$ is the bivariate normal density function with mean $\mathbf{0}$ and correlation matrix $\mathbf{R}$, and $\Phi(\cdot)$ is the standard normal distribution function. For more related studies on model (3) see, e.g., Kim and Genton (2011), Lachos (2010) for other scaled positive variable V.

The goal of this paper is to establish the limit of the conditional distributions and to derive the upper tail dependence coefficient of $\boldsymbol{X}$ given by (2) and (3), respectively. Comparison with the findings of tail independence of bivariate normal (Embrechts et al. (2002)), skew-bivariate normal (Bortot (2010)); tail dependence of two skew-t distributions (Fung and Seneta (2010), Bortot (2010)), the tail dependence of the first class of skew slash distributions exist trivial values 0 or 1 for some special cases (Theorem 3.1), while the second class has wider region of tail dependence (Theorem 3.2).

The rest of the paper is organized as follows. The main results are provided in Section 3. All proofs are postponed to Section 4.

2 Preliminaries and notation

In this section, we first introduce some important functions with their asymptotic properties established in Lemma 2.1 and then give Lemma 2.2 for the distribution properties of the skew slash random vector $\boldsymbol{X}$ given by (2) via the normal variance-mean mixture.

Let $K_\tau(x;\omega)$ be the incomplete modified Bessel function of the third kind with index $\tau \in \mathbb{R}$ defined by

$$K_\tau(x;\omega) = \frac{1}{2}\int_x^\infty t^{\tau-1}\exp\left(-\frac{\omega}{2}\left(t+t^{-1}\right)\right)\,dt, \quad x \geq 0,\ \omega > 0. \tag{4}$$

It follows from (7.5) in Jones (2007) that for $\tau \in \mathbb{R}$

$$K_\tau(0;\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left(1+\frac{4\tau^2-1}{8\omega}+o\left(\frac{1}{\omega}\right)\right), \quad \omega \to \infty. \tag{5}$$

Define further $P_\tau(a;b)$ and $Q_\upsilon(x;a)$ respectively by

$$P_\tau(a;b) = \int_0^1 t^{\tau-1}\exp\left(-\frac{1}{2}\left(a^2t+\frac{b^2}{t}\right)\right)\,dt; \quad Q_\upsilon(x;a) = \int_{-\infty}^x\left(\int_0^a t^{\upsilon-1}e^{-(1+u^2)t}\,dt\right)du, \quad x \in I\!R, \tag{6}$$

where $\tau > 0, a, b \geq 0$ and $\upsilon \geq 1$. For simplicity, we write $\Gamma(\cdot)$ for the Euler gamma function.

The following result is about the asymptotic behaviors of $P_\tau(a;b)$ and $Q_\upsilon(x;a)$, respectively.

Lemma 2.1. *Let $P_\tau(a;b)$ and $Q_\upsilon(x;a)$ be those defined as in (6). Then, we have for $P_\tau(a;b)$ with $\tau > 0$*

$$P_\tau(a;b) = \begin{cases} \left(\frac{a^2}{2}\right)^{-\tau}\Gamma(\tau)(1+o(1)), & b=0, a\to\infty; \\ \frac{b^{\tau-1/2}}{a^{\tau+1/2}}\sqrt{2\pi}e^{-ab}\left(1+\frac{4\tau^2-1}{8ab}(1+o(1))\right), & b>0, a\to\infty; \\ 2\left(\frac{a}{b}\right)^{-\tau}K_\tau(0;\omega)(1+o(1)), & b\to 0, a\to\infty, ab\to\omega>0 \end{cases}$$

and for $Q_\upsilon(x;a)$ with $\upsilon \geq 1$ and $x \in I\!R$

$$Q_\upsilon(x;a) \to \Gamma(\upsilon)\int_{-\infty}^x (1+u^2)^{-\upsilon}\,du =: Q_\upsilon(x;\infty) > 0, \quad a \to \infty. \tag{7}$$

Recall that λ_U is equivalent to

$$\lambda_U = \lim_{x_1\to\infty}\mathbb{P}\left(X_2 \geq F_2^{-1}(F_1(x_1))|X_1=x_1\right) + \lim_{x_2\to\infty}\mathbb{P}\left(X_1 \geq F_1^{-1}(F_2(x_2))|X_2=x_2\right) \tag{8}$$

provided that the marginal distributions are continuous (cf. Nelsen (1999), p.11, 36). In the following we derive the marginal distribution and the conditional distribution of $\boldsymbol{X}$ given by (2).

Lemma 2.2. *For $\boldsymbol{X} \sim SS(\lambda,\boldsymbol{\theta},\mathbf{R})$ given by (2), let $f_2(\cdot)$ and $f_{1.2}(\cdot|x_2)$ denote the pdfs of X_2 and*

$X_{1.2} := (X_1|X_2 = x_2)$, *respectively. Then, with* $P_\tau(a;b)$ *given by* (6), *we have*

$$f_2(x_2) = \frac{\lambda e^{\theta_2 x_2}}{\sqrt{2\pi}} P_{\lambda+1/2}(|x_2|;|\theta_2|); \quad f_{1.2}(x_1|x_2) = \frac{e^{\beta(x_1-\rho x_2)}}{\sqrt{2\pi(1-\rho^2)}} \frac{P_{\lambda+1}\left(\sqrt{x_1'^2+x_2^2};\sqrt{\theta_1'^2+\theta_2^2}\right)}{P_{\lambda+1/2}(|x_2|;|\theta_2|)}, \tag{9}$$

where $\beta(1-\rho^2) = \theta_1 - \rho\theta_2, x_1'\sqrt{1-\rho^2} = x_1 - \rho x_2$ *and* $\theta_1'\sqrt{1-\rho^2} = \theta_1 - \rho\theta_2$. *Furthermore, for* $\boldsymbol{\theta} \neq \mathbf{0}$

$$\mathbb{E}e^{-sX_{1.2}} = \frac{P_{\lambda+1/2}\left(|x_2|;\sqrt{\theta_2^2+2\beta(1-\rho^2)s-(1-\rho^2)s^2}\right)}{P_{\lambda+1/2}(|x_2|;|\theta_2|)} e^{-\rho x_2 s}, \quad s \in \beta \pm \sqrt{\frac{\boldsymbol{\theta}^\top \mathbf{R}^{-1}\boldsymbol{\theta}}{1-\rho^2}}. \tag{10}$$

Remark 2.1. *Let* F_2 *be the df of* X_2 *for* $\boldsymbol{X} = (X_1, X_2)^\top$ *defined as in* (2)*.Then, using* (9) *and Lemma 2.1, we have as* $x_2 \to \infty$ *that*

$$1 - F_2(x_2) = \begin{cases} \left(\theta_2/x_2\right)^\lambda (1+o(1)), & \theta_2 > 0, \\ \left(\widetilde{\lambda}/x_2\right)^{2\lambda}(1+o(1)), & \theta_2 = 0, \\ \frac{\lambda}{2}\frac{|\theta_2|^{\lambda-1}}{x_2^{\lambda+1}} e^{-2|\theta_2|x_2}(1+o(1)), & \theta_2 < 0, \end{cases} \tag{11}$$

with

$$\widetilde{\lambda} = \left(\frac{2^{\lambda-1}\Gamma(\lambda+1/2)}{\sqrt{\pi}}\right)^{1/(2\lambda)}. \tag{12}$$

3 Main results

In this section, we provide the main results on the upper tail dependence coefficient λ_U of two skew slash distributions given by (2) and (3). The first result is about the upper tail dependence of the skew slash distributed random vector $\mathbf{X}$ defined by (2) via the normal variance-mean mixture model.

Theorem 3.1. *Let* $\boldsymbol{X} \sim SS(\lambda, \boldsymbol{\theta}, \mathbf{R})$ *be defined as in* (2), *and let* $T_{2\lambda+1}(\cdot)$ *be the student's t distribution function (df) with* $2\lambda+1$ *degrees of freedom. Then, with* $\widetilde{\lambda}$ *given by* (12), *we have*

(1). for $\theta_1 = \theta_2 = 0$,

$$\lambda_U = 2\left(1 - T_{2\lambda+1}\left(\sqrt{\frac{(2\lambda+1)(1-\rho)}{1+\rho}}\right)\right);$$

(2). for $\theta_1 > 0, \theta_2 > 0$, $\lambda_U = 1$;

(3). for $\theta_1 > 0, \theta_2 = 0$ or $\theta_1 = 0, \theta_2 > 0$,

$$\lambda_U = \int_0^1 \left(1 - \Phi\left(\widetilde{\lambda} u^{1/(2\lambda)}\right)\right)\, du - \frac{1}{2\lambda+1}\int_0^1 u\, d\left(1 - \Phi\left(\widetilde{\lambda} u^{1/(2\lambda)}\right)\right);$$

(4). for the remaining cases, $\lambda_U = 0$.

From (11) and Theorem 3.1, if both marginals posses power laws, i.e., $\theta_1, \theta_2 \geq 0$, then the skew slash random vector $\boldsymbol{X}$ has asymptotic upper tail dependence. Therefore, regular varying tails play an important role in the presence of tail dependence. Theorem 3.1 shows that tail dependence of the first class of skew slash distribution exists trivial values 0 or 1, which implies that it has extremal tail dependence (independence and full dependence), contrary to the second class of skew slash distributions showing that the tail dependence has nontrivial values.

Theorem 3.2. *Let $\boldsymbol{X} \sim ASS(\lambda, \boldsymbol{\theta}, \mathbf{R})$ be defined as in (3) and let $f_{2\lambda+1}(\cdot)$ be the probability density function (pdf) of student's t distribution with $2\lambda+1$ degrees of freedom. Then*

$$\begin{aligned}\lambda_U &= \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)}\left(\frac{1}{Q_{\lambda+1}(\mu_1;\infty)}\int_{z_0'}^{\infty} f_{2\lambda+1}(z) Q_{\lambda+3/2}\left(\frac{\theta_2\sqrt{\frac{1-\rho^2}{2\lambda+1}}z + (\theta_1+\rho\theta_2)}{\sqrt{1+z^2/(2\lambda+1)}};\infty\right) dz\right.\\ &\left.\quad + \frac{1}{Q_{\lambda+1}(\mu_2;\infty)}\int_{z_0}^{\infty} f_{2\lambda+1}(z) Q_{\lambda+3/2}\left(\frac{\theta_1\sqrt{\frac{1-\rho^2}{2\lambda+1}}z + (\theta_2+\rho\theta_1)}{\sqrt{1+z^2/(2\lambda+1)}};\infty\right) dz\right),\end{aligned}$$

where $Q_{\lambda+3/2}(\cdot;\infty)$ is given by (7) and

$$\mu_1 = \frac{\theta_1+\rho\theta_2}{\sqrt{1+\theta_2^2(1-\rho^2)}}, \qquad z_0 = \left(\left(\frac{\int_{-\infty}^{\mu_1}(1+u^2)^{-(\lambda+1)}\,du}{\int_{-\infty}^{\mu_2}(1+u^2)^{-(\lambda+1)}\,du}\right)^{1/(2\lambda)} - \rho\right)\sqrt{\frac{2\lambda+1}{1-\rho^2}};$$

$$\mu_2 = \frac{\theta_2+\rho\theta_1}{\sqrt{1+\theta_1^2(1-\rho^2)}}, \qquad z_0' = \left(\left(\frac{\int_{-\infty}^{\mu_2}(1+u^2)^{-(\lambda+1)}\,du}{\int_{-\infty}^{\mu_1}(1+u^2)^{-(\lambda+1)}\,du}\right)^{1/(2\lambda)} - \rho\right)\sqrt{\frac{2\lambda+1}{1-\rho^2}}.$$

4 Proofs

PROOF OF LEMMA 2.1 First we consider $P_\tau(a;b)$. We will treat the following three cases in turn: (1) $b=0, a\to\infty$; (2) $b>0, a\to\infty$; (3) $b\to 0, a\to\infty$ and $ab\to\omega>0$.

Case (1) as $b=0$ and $a\to\infty$. Using integration by substitution, we have

$$P_\tau(a;0) = \left(\frac{a^2}{2}\right)^{-\tau}\int_0^{a^2/2} t^{\tau-1}e^{-t}\,dt = \left(\frac{a^2}{2}\right)^{-\tau}\Gamma(\tau)(1+o(1)), \quad a\to\infty$$

since $\int_x^\infty t^{\tau-1}e^{-t}\,dt = x^{\tau-1}e^{-x}(1+o(1))$ as $x\to\infty$, the claim for $P_\tau(a;0)$ follows.

Case (2) as $b>0$ and $a\to\infty$. We rewrite $P_\tau(a;b)$ using $K_\tau(\cdot;\cdot)$ given by (4) as

$$P_\tau(a;b) = 2\left(\frac{a}{b}\right)^{-\tau}\left(K_\tau(0;ab) - K_\tau\left(\frac{a}{b};ab\right)\right). \tag{13}$$

Noting that

$$K_\tau\left(\frac{a}{b};ab\right) = \frac{1}{2}\left(\frac{ab}{2}\right)^{-\tau}\int_{a^2/2}^\infty t^{\tau-1}\exp\left(-\left(t+\frac{a^2b^2}{4t}\right)\right)\,dt$$

and

$$\exp\left(-\frac{a^2b^2}{4t}\right) = e^{-b^2/2}\sum_{n=0}^\infty\left(\frac{u}{u+1}\right)^n\left(\frac{b^2}{2}\right)^n, \quad u=\frac{2t}{a^2}-1,$$

we have

$$\int_{a^2/2}^\infty t^{\tau-1}\exp\left(-\left(t+\frac{a^2b^2}{4t}\right)\right)\,dt = \left(\frac{a^2}{2}\right)^\tau\exp\left(-\frac{a^2+b^2}{2}\right)\sum_{n=0}^\infty\left(\frac{b^2}{2}\right)^n d_n,$$

with

$$d_n = \frac{1}{n!}\int_0^\infty u^n(u+1)^{\tau-n-1}\exp\left(-\frac{a^2}{2}u\right)\,du =: U(n+1;\tau+1;a^2/2),$$

where U is the confluent hypergeometric function and $U(n+1;\tau+1;a^2/2) = (a^2/2)^{-n-1}(1+o(1))$ as $a\to\infty$ (cf. Chaudhry et al. (1996)). Hence,

$$K_\tau\left(\frac{a}{b};ab\right) = \frac{a^{\tau-2}}{b^\tau}\exp\left(-\frac{a^2+b^2}{2}\right)(1+o(1)).$$

This together with (5) yields that

$$\frac{K_\tau\left(\frac{a}{b};ab\right)}{K_\tau\left(0;ab\right)} = \sqrt{\frac{2}{\pi}}\frac{a^{\tau-3/2}}{b^{\tau-1/2}}\exp\left(ab-\frac{a^2+b^2}{2}\right)(1+o(1)), \tag{14}$$

which tends to zero as $a\to\infty$. Consequently, the claim for $P_\tau(a;b)$ as $b>0, a\to\infty$ follows.

Case (3) as $b\to 0, a\to\infty$ and $ab\to w>0$. The proof is similar to that of Case (2), and thus the details are omitted here.

Next, we consider $Q_\upsilon(x;a)$. Note that for all $x\in\mathbb{R}$

$$Q_\upsilon(x;a) = \Gamma(\upsilon)\int_{-\infty}^x(1+u^2)^{-\upsilon}\,du \;-\int_{-\infty}^x\int_a^\infty t^{\upsilon-1}e^{-(1+u^2)t}\,dtdu. \tag{15}$$

Further, recall that $\upsilon \geq 1$, and thus as $a \to \infty$

$$0 \leq \int_{-\infty}^{x} \int_{a}^{\infty} t^{\upsilon-1} e^{-(1+u^2)t}\, dtdu \leq \left(\int_{-\infty}^{x} (1+u^2)^{-\upsilon}\, du \right) \left(\int_{a}^{\infty} t^{\upsilon-1} e^{-t}\, dt \right) \to 0,$$

implying that

$$\int_{-\infty}^{x} \int_{a}^{\infty} t^{\upsilon-1} e^{-(1+u^2)t}\, dtdu \to 0, \quad a \to \infty.$$

Therefore, for all $x \in I\!R$ and $\upsilon \geq 1$,

$$Q_\upsilon(x; a) \to \Gamma(\upsilon) \int_{-\infty}^{x} (1+u^2)^{-\upsilon}\, du, \quad a \to \infty.$$

The proof is complete. □

PROOF OF LEMMA 2.2 Recall that $\boldsymbol{X}|(V = t) \sim N_2(\boldsymbol{\theta}/t, \mathbf{R}/t)$ with $t \in (0,1)$ given. It follows from the total probability formula that, the pdf of $\boldsymbol{X}$ defined as in (2), denoted by $f_{\boldsymbol{X}}(\cdot)$, is

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{\lambda e^{\boldsymbol{\theta}^\top \mathbf{R}^{-1} \boldsymbol{x}}}{2\pi\sqrt{1-\rho^2}} \int_0^1 t^{\lambda+1-1} \exp\left(-\frac{1}{2} \left(\boldsymbol{x}^\top \mathbf{R}^{-1} \boldsymbol{x} t + \frac{\boldsymbol{\theta}^\top \mathbf{R}^{-1} \boldsymbol{\theta}}{t} \right) \right) dt,$$

with $\boldsymbol{x} = (x_1, x_2)^\top \in I\!R^2$. Hence, the pdf of X_2, denoted by $f_2(\cdot)$, satisfies

$$f_2(x_2) = \frac{\lambda e^{\theta_2 x_2}}{\sqrt{2\pi}} \int_0^1 t^{\lambda+1/2-1} \exp\left(-\frac{1}{2} \left(x_2^2 t + \frac{\theta_2^2}{t} \right) \right) dt.$$

Consequently, the conditional density of $X_{1.2} := X_1 | X_2 = x_2$, denoted by $f_{1.2}(\cdot|x_2)$, is

$$f_{1.2}(x_1|x_2) = \frac{f_{\boldsymbol{X}}(\boldsymbol{x})}{f_2(x_2)} = \frac{e^{\beta(x_1 - \rho x_2)}}{\sqrt{2\pi(1-\rho^2)}} \frac{P_{\lambda+1}\left(\sqrt{x_1'^2 + x_2^2}; \sqrt{\theta_1'^2 + \theta_2^2} \right)}{P_{\lambda+1/2}(|x_2|; |\theta_2|)},$$

where $\beta(1-\rho^2) = \theta_1 - \rho\theta_2, x_1'\sqrt{1-\rho^2} = x_1 - \rho x_2, \theta_1'\sqrt{1-\rho^2} = \theta_1 - \rho\theta_2$. Therefore, we have with $s' = \sqrt{1-\rho^2}s$

$$\mathbb{E}e^{-sX_{1.2}} = e^{-\rho x_2 s} \mathbb{E}e^{-s'(X_{1.2} - \rho x_2)/\sqrt{1-\rho^2}}$$

and

$$\mathbb{E}e^{-s'(X_{1.2} - \rho x_2)/\sqrt{1-\rho^2}} = \frac{P_{\lambda+1/2}\left(|x_2|; |\theta_2'| \right)}{P_{\lambda+1/2}(|x_2|; |\theta_2|)},$$

with $\theta_2'^2 = \theta_1'^2 + \theta_2^2 - (\theta_1' - s')^2$ and s' satisfying $\theta_1'^2 + \theta_2^2 - (\theta_1' - s')^2 > 0$, i.e.,

$$\theta_2'^2 = \theta_2^2 + 2\beta(1-\rho^2)s - (1-\rho^2)s^2, \quad s \in \beta \pm \sqrt{\frac{\theta_1'^2 + \theta_2^2}{1-\rho^2}}.$$

The proof is complete. □

Proof of Theorem 3.1 For $\boldsymbol{\theta} = \mathbf{0}$, the skew slash random variable $\mathbf{X}$ is symmetry and has the same marginal distributions with regular varying tail index 2λ (see (11)), and thus the claim follows by Theorem 1 (i) of Abdous (2005). Next, we derive the remaining cases, i.e., $\boldsymbol{\theta} \neq \mathbf{0}$.

To this end, we need to derive the asymptotic distribution of $W(x_2)$ as $x_2 \to \infty$, where

$$W(x_2) := \begin{cases} x_2^{-1/2}\left(X_{1.2} - \left(\rho x_2 + \beta(1-\rho^2)|\theta_2|^{-1}\sqrt{x_2^2 + 2\lambda}\right)\right), & \theta_2 \neq 0; \\ x_2^{-2} X_{1.2}, & \theta_2 = 0. \end{cases}$$

For $\theta_2 \neq 0$, it follows from Lemma 2.2 that

$$\begin{aligned} \mathbb{E}e^{-sW(x_2)} &= \exp\left(\frac{\rho x_2 + \frac{\beta(1-\rho^2)}{|\theta_2|}\sqrt{x_2^2 + 2\lambda}}{\sqrt{x_2}} s\right) \mathbb{E}\exp\left(-\frac{s}{\sqrt{x_2}} X_{1.2}\right) \\ &= \exp\left(\frac{\beta(1-\rho^2)}{|\theta_2|}\sqrt{x_2 + \frac{2\lambda}{x_2}} s\right) \frac{P_{\lambda+1/2}\left(|x_2|; \sqrt{\theta_2^2 + \frac{2\beta(1-\rho^2)s}{\sqrt{x_2}} - \frac{(1-\rho^2)s^2}{x_2}}\right)}{P_{\lambda+1/2}(|x_2|; |\theta_2|)}, \end{aligned}$$

which, in view of Lemma 2.1, is asymptotically equal to

$$\begin{aligned} &\exp\left(\frac{\beta(1-\rho^2)}{|\theta_2|}\sqrt{x_2 + \frac{2\lambda}{x_2}} s\right) \left(\frac{\theta_2^2 + \frac{2\beta(1-\rho^2)s}{\sqrt{x_2}} - \frac{(1-\rho^2)s^2}{x_2}}{\theta_2^2}\right)^{\frac{\lambda}{2}} \exp\left(|\theta_2|x_2 - \sqrt{\theta_2^2 + \frac{2\beta(1-\rho^2)s}{\sqrt{x_2}} - \frac{(1-\rho^2)s^2}{x_2}} x_2\right) \\ &= \exp\left(\frac{\beta(1-\rho^2)}{|\theta_2|}\sqrt{x_2 + \frac{2\lambda}{x_2}} s\right) \exp\left(|\theta_2|x_2\left[1 - \left(1 + \frac{2\beta(1-\rho^2)s}{\theta_2^2\sqrt{x_2}} - \frac{(1-\rho^2)s^2}{\theta_2^2 x_2} x_2\right)^{1/2}\right]\right) \left(1 + O\left(\frac{1}{\sqrt{x_2}}\right)\right) \\ &= \exp\left(-\frac{(1-\rho^2)(\theta_1'^2 + \theta_2^2)}{2|\theta_2|^3} s^2 + O\left(\frac{1}{\sqrt{x_2}}\right)\right) \left(1 + O\left(\frac{1}{\sqrt{x_2}}\right)\right) \\ &\to \exp\left(-\frac{(1-\rho^2)(\theta_1'^2 + \theta_2^2)}{2|\theta_2|^3} s^2\right), \quad x_2 \to \infty, \end{aligned}$$

where $\theta_1' = (\theta_1 - \rho\theta_2)/\sqrt{1-\rho^2}$. Therefore, by the Laplace inverse transform, we have the following convergence in distribution (denoted by $\stackrel{d}{\to}$)

$$W(x_2) \stackrel{d}{\to} Z_1 \sim N\left(0, \frac{(1-\rho^2)(\theta_1'^2 + \theta_2^2)}{|\theta_2|^3}\right), \quad x_2 \to \infty. \tag{16}$$

For $\theta_2 = 0$, and thus $\theta_1 \neq 0$. It follows from Lemma 2.1 and Lemma 2.2 that as $x_2 \to \infty$

$$\mathbb{E}e^{-sW(x_2)} \to \frac{2(\sqrt{2\theta_1 s})^{\lambda+1/2} K_{\lambda+1/2}(0; \sqrt{2\theta_1 s})}{2^{\lambda+1/2}\Gamma(\lambda+1/2)},$$

which is the Laplace transform of θ_1/Y where $Y \sim \Gamma(1/2+\lambda, 1/2)$, a Gamma distributed random variable with shape and scale parameters $1/2+\lambda, 1/2$. Therefore

$$W(x_2) \xrightarrow{d} \frac{\theta_1}{Y}, \quad x_2 \to \infty. \tag{17}$$

Further, we need the asymptotic expression of the function $c(x_2) = F_1^{-1}(F_2(x_2))$. We have by Lemma 3.1 in Banachewicz and van der Vaart (2008) that

$$c(x_2) = \begin{cases} \frac{\theta_1}{\theta_2} x_2(1+o(1)), & \theta_1 > 0, \theta_2 > 0; \\ \frac{\theta_1}{\widetilde{\lambda}^2} x_2^2(1+o(1)), & \theta_1 > 0, \theta_2 = 0; \\ \left(\frac{2|\theta_2|}{\lambda+1}\right)^{1/\lambda} \frac{\theta_1}{|\theta_2|} x_2^{1+1/\lambda} \exp\left(\frac{2|\theta_2| x_2}{\lambda}\right)(1+o(1)), & \theta_1 > 0, \theta_2 < 0 \end{cases} \tag{18}$$

as $x_2 \to \infty$, where $\widetilde{\lambda}$ is given by (12).

Next, we give the proofs of assertions (2)–(4).

Assertion (2) as $\theta_1 > 0, \theta_2 > 0$. Using (16), (18) and $\beta(1-\rho^2) = \theta_1 - \rho\theta_2$, we have

$$\begin{aligned} & \lim_{x_2 \to \infty} \mathbb{P}\left(X_1 \geq F_1^{-1}(F_2(x_2)) | X_2 = x_2\right) \\ = & \lim_{x_2 \to \infty} \mathbb{P}\left(W(x_2) \geq \frac{c(x_2) - \left(\rho x_2 + \frac{\beta(1-\rho^2)}{|\theta_2|}\sqrt{x_2^2 + 2\lambda}\right)}{\sqrt{x_2}}\right) = \mathbb{P}(Z_1 \geq 0) = \frac{1}{2}. \end{aligned}$$

Similarly, $\lim_{x_1 \to \infty} \mathbb{P}\left(X_2 \geq F_2^{-1}(F_1(x_1)) | X_1 = x_1\right) = 1/2$. Therefore, in view of (8), we have

$$\lambda_U = \lim_{x_2 \to \infty} \mathbb{P}\left(X_1 \geq F_1^{-1}(F_2(x_2)) | X_2 = x_2\right) + \lim_{x_1 \to \infty} \mathbb{P}\left(X_2 \geq F_2^{-1}(F_1(x_1)) | X_1 = x_1\right) = 1$$

Assertion (3) as $\theta_1 > 0, \theta_2 = 0$ and $\theta_1 = 0, \theta_2 > 0$. For this, we only present the proof of $\theta_1 > 0, \theta_2 = 0$ since another case follows by the similar arguments. Using (17) and (18), we have

$$\lim_{x_2 \to \infty} \mathbb{P}\left(X_1 \geq F_1^{-1}(F_2(x_2)) | X_2 = x_2\right) = \lim_{x_2 \to \infty} \mathbb{P}\left(W(x_2) \geq \frac{c(x_2)}{x_2^2}\right) = \mathbb{P}\left(Y \leq \widetilde{\lambda}^2\right),$$

where $Y \sim \Gamma(1/2+\lambda, 1/2)$ and $\widetilde{\lambda}$ is defined by (12). Similarly

$$
\begin{aligned}
&\lim_{x_1\to\infty} \mathbb{P}\left(X_2 \geq F_2^{-1}(F_1(x_1))|X_1 = x_1\right)\\
= &\lim_{x_1\to\infty} \mathbb{P}\left(\frac{X_{2.1} - \left(\rho x_1 + \frac{\beta'(1-\rho^2)}{|\theta_1|}\sqrt{x_1^2+2\lambda}\right)}{\sqrt{x_1}} \geq \frac{\widetilde{\lambda}\sqrt{\frac{x_1}{|\theta_1|}} - \left(\rho x_1 + \frac{\beta'(1-\rho^2)}{|\theta_1|}\sqrt{x_1^2+2\lambda}\right)}{\sqrt{x_1}}\right)\\
= &\ \mathbb{P}\left(Z_1' \geq \frac{\widetilde{\lambda}}{\sqrt{|\theta_1|}}\right),
\end{aligned}
$$

where

$$
\beta'(1-\rho^2) = \theta_2 - \rho\theta_1, \quad Z_1' \sim N\left(0, \frac{(1-\rho^2)\boldsymbol{\theta}^\top\mathbf{R}^{-1}\boldsymbol{\theta}}{|\theta_1|^3}\right). \tag{19}
$$

Therefore, using integration by parts, we have

$$
\lambda_U = 1 - \Phi(\widetilde{\lambda}) + \mathbb{P}\left(Y \leq \tilde{\lambda}^2\right) = \int_0^1 \left(1 - \Phi(\widetilde{\lambda}u^{1/(2\lambda)})\right)\ du - \frac{1}{2\lambda+1}\int_0^1 u\ d\left(1 - \Phi(\widetilde{\lambda}u^{1/(2\lambda)})\right).
$$

Assertion (4) as $\theta_1\theta_2 < 0$ and $\theta_1 < 0, \theta_2 < 0$. Here, we only present the proof of $\theta_1 > 0, \theta_2 < 0$. The other cases follow by the similar arguments and thus are omitted here. Using (16), (18) and $x_2^{-1}c(x_2) \to \infty$, we have

$$
\lim_{x_2\to\infty} \mathbb{P}\left(X_1 \geq F_1^{-1}(F_2(x_2))|X_2 = x_2\right) = 0, \quad \lim_{x_1\to\infty} \mathbb{P}\left(X_2 \geq F_2^{-1}(F_1(x_1))|X_1 = x_1\right) = 0.
$$

Consequently, $\lambda_U = 0$ for $\theta_1 > 0, \theta_2 < 0$. The proof is complete. □

Proof of Theorem 3.2 Note that $\boldsymbol{X}|V = t$ is skew normal distributed with pdf $2\phi_2(\boldsymbol{x}; \mathbf{R}/t)\Phi(\sqrt{t}\boldsymbol{\theta}^\top\boldsymbol{x})$ with $t \in (0, 1)$ given. It follows from the total probability formula that the pdf of $\boldsymbol{X}$, denoted by $f_{\boldsymbol{X}}(\cdot)$, is

$$
\begin{aligned}
f_{\boldsymbol{X}}(\boldsymbol{x}) &= \frac{2\lambda}{(2\pi)^{3/2}|\mathbf{R}|^{1/2}}\int_0^1\int_{-\infty}^{\boldsymbol{\theta}^\top\boldsymbol{x}} t^{\lambda+3/2-1}\exp\left(-\frac{\boldsymbol{x}^\top\mathbf{R}^{-1}\boldsymbol{x}+u^2}{2}t\right)\ dudt\\
&= \frac{2\lambda}{(2\pi)^{3/2}|\mathbf{R}|^{1/2}}\frac{2^{\lambda+3/2}}{(\boldsymbol{x}^\top\mathbf{R}^{-1}\boldsymbol{x})^{\lambda+1}}\int_{-\infty}^{\frac{\boldsymbol{\theta}^\top\boldsymbol{x}}{\sqrt{\boldsymbol{x}^\top\mathbf{R}^{-1}\boldsymbol{x}}}}\int_0^{\frac{\boldsymbol{x}^\top\mathbf{R}^{-1}\boldsymbol{x}}{2}} t^{\lambda+3/2-1}\exp\left(-(1+u^2)t\right)\ dtdu\\
&= \frac{2\lambda}{(2\pi)^{3/2}|\mathbf{R}|^{1/2}}\frac{2^{\lambda+3/2}}{(\boldsymbol{x}^\top\mathbf{R}^{-1}\boldsymbol{x})^{\lambda+1}}Q_{\lambda+\frac{3}{2}}\left(\frac{\boldsymbol{\theta}^\top\boldsymbol{x}}{\sqrt{\boldsymbol{x}^\top\mathbf{R}^{-1}\boldsymbol{x}}}; \frac{\boldsymbol{x}^\top\mathbf{R}^{-1}\boldsymbol{x}}{2}\right), \quad \boldsymbol{x} \neq \mathbf{0}.
\end{aligned} \tag{20}
$$

Consequently, the pdf of X_i, denoted by $f_i(\cdot)$, is given by

$$
f_i(x) = \frac{\lambda}{\pi}\frac{2^{\lambda+1}}{|x|^{2\lambda+1}}Q_{\lambda+1}\left(\mu_i\text{sign}(x); x^2/2\right), \quad i = 1, 2, \tag{21}
$$

with

$$\mu_1 = \frac{\theta_1 + \rho\theta_2}{\sqrt{1+\theta_2^2(1-\rho^2)}}, \quad \mu_2 = \frac{\theta_2 + \rho\theta_1}{\sqrt{1+\theta_1^2(1-\rho^2)}}.$$

Hence, we have by Lemma 2.1

$$1 - F_2(x_2) = \frac{x_2}{2\lambda} f_2(x_2)(1+o(1)) = \frac{\Gamma(\lambda+1)}{\pi} \frac{2^\lambda}{x_2^{2\lambda}} \int_{-\infty}^{\mu_2} (1+u^2)^{-(\lambda+1)}\, du(1+o(1))$$

as $x_2 \to \infty$. Consequently, as $x_2 \to \infty$

$$c(x_2) = F_1^{-1}(F_2(x_2)) = \left(\frac{\int_{-\infty}^{\mu_1} (1+u^2)^{-(\lambda+1)}\, du}{\int_{-\infty}^{\mu_2} (1+u^2)^{-(\lambda+1)}\, du} \right)^{1/(2\lambda)} x_2(1+o(1)) \tag{22}$$

and the pdf of $X_1|X_2 = x_2$, denoted by $f_{1.2}(\cdot|x_2)$, satisfies

$$f_{1.2}(x_1|x_2) = \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)} \frac{f_{2\lambda+1}\left((x_1-\rho x_2)/s(x_2)\right)}{s(x_2)} \frac{Q_{\lambda+3/2}\left(\frac{\theta_1 x_1 + \theta_2 x_2}{\sqrt{x_1'^2 + x_2^2}}; \frac{x_1'^2 + x_2^2}{2} \right)}{Q_{\lambda+1}\left(\mu_2 \mathrm{sign}(x_2); \frac{x_2^2}{2}\right)},$$

where $f_{2\lambda+1}(\cdot)$ is the pdf of student's t with $2\lambda+1$ degrees of freedom and

$$x_1'\sqrt{1-\rho^2} = x_1 - \rho x_2, \quad s(x_2) = \sqrt{\frac{(1-\rho^2)x_2^2}{2\lambda+1}}.$$

Hence, we have by the dominated convergence theorem and Lemma 2.1 that

$$\begin{aligned} &\lim_{x_2\to\infty} \mathbb{P}\left(X_1 \geq F_1^{-1}(F_2(x_2))|X_2 = x_2\right) \\ = \;& \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)} \frac{1}{Q_{\lambda+1}(\mu_2;\infty)} \int_{z_0}^{\infty} f_{2\lambda+1}(z) Q_{\lambda+3/2}\left(\frac{\theta_1\sqrt{\frac{1-\rho^2}{2\lambda+1}}z + (\theta_2+\rho\theta_1)}{\sqrt{1+z^2/(2\lambda+1)}}; \infty \right) dz, \end{aligned} \tag{23}$$

where

$$z_0 = \lim_{x_2\to\infty} \frac{c(x_2) - \rho x_2}{s(x_2)} = \left(\left(\frac{\int_{-\infty}^{\mu_1} (1+u^2)^{-(\lambda+1)}\, du}{\int_{-\infty}^{\mu_2} (1+u^2)^{-(\lambda+1)}\, du} \right)^{1/(2\lambda)} - \rho \right) \sqrt{\frac{2\lambda+1}{1-\rho^2}}.$$

Similarly

$$\begin{aligned} &\lim_{x_1\to\infty} \mathbb{P}\left(X_2 \geq F_2^{-1}(F_1(x_1))|X_1 = x_1\right) \\ = \;& \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)} \frac{1}{Q_{\lambda+1}(\mu_1;\infty)} \int_{z_0'}^{\infty} f_{2\lambda+1}(z) Q_{\lambda+3/2}\left(\frac{\theta_2\sqrt{\frac{1-\rho^2}{2\lambda+1}}z + (\theta_1+\rho\theta_2)}{\sqrt{1+z^2/(2\lambda+1)}}; \infty \right) dz, \end{aligned} \tag{24}$$

with

$$z_0' = \left(\left(\frac{\int_{-\infty}^{\mu_2} (1+u^2)^{-(\lambda+1)} \, du}{\int_{-\infty}^{\mu_1} (1+u^2)^{-(\lambda+1)} \, du} \right)^{1/(2\lambda)} - \rho \right) \sqrt{\frac{2\lambda+1}{1-\rho^2}}.$$

The desired result follows by (23) and (24). □

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